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# BLIND PHASE EQUALIZATION

by

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# *Abstract*

The ability to separate a signal source from the distortion resulting from noise and the transmission medium is the goal in many information recovery and extraction situations. In the field of telecommunications, often a training sequence is used to construct a compensation or equalization filter. Unfortunately transmission of this training sequence is not always reliable or even available. A similar situation occurs in seismology, where it is not always possible to obtain or know the nature of the source wavelet or pulse. It is in these situations that a blind equalization technique is sought.

Blind equalization is a challenging problem because in the identification of the unknown system or of its inverse, the phase needs to be known. This phase may be mixed phase and therefore higher order spectra is required to obtain the non minimum phase elements. Complex cepstra is employed to produce linear solutions to the problem.

Initial approaches to the system identification involved the separation of the impulse response into minimum and maximum phase components. While such techniques have performed well, inefficiencies in the use of higher order spectra arise. An alternative approach involving the separation of the impulse response into the magnitude and phase components has also been suggested. In this approach, the equalization process may be separated into two parts each of which may even be strategically placed between the source and the receiver. Furthermore, this separation enables optimization of the amplitude and phase equalization functions to be performed.

An optimization of the separate amplitude and phase equalization functions is proposed in this work. Second order spectra are used to construct a zero phase amplitude equalizer. Higher order spectra are then optimized by using the complex conjugate symmetry inherent in an all pass phase structure to construct an improved blind phase equalizer. This optimization of the phase component results in a simplified structure for the phase equalizer. This amplitude and phase structure is demonstrated to produce an improvement in the performance of the blind equalizer.

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# *Dedication*

To my mother:

**Dr. Olasz-Argyelán Edith**

**von közép és felsö Homoród**

who taught me to dream...

To my beloved father:

**vitéz Olasz Jenö (Eugene)**

who taught me courage, heroism, and even love and whom I miss dearly...

To my companions:

Rip, Rudi, Zita, Cocoa, Tina, Lassie, Spitz and Treff

who watched over me and helped me study.

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# *List of Symbols & Acronyms*

| | |
|---|---|
| $r(\tau)$ | autocorrelation function |
| $\hat{C}(z_1,z_2,z_3)$ | complex logarithm of the trispectrum |
| $\hat{X}(z)$ | complex logarithm |
| $\hat{G}(z)$ | complex logarithm of the cumulant slice |
| $\hat{x}(k),\hat{h}(k),\hat{i}(k),\hat{o}(k)$ | complex cepstrums |
| $g(k)$ | cumulant slice |
| $x(k),h(k),i(k),o(k),s(k)$ | discrete time samples |
| $c(m,n,l)$ | fourth order cumulant |
| $E[\cdot]$ | expectation operator |
| ISI | intersymbol interference |
| $\delta(\cdot)$ | kronecker delta function |
| $\gamma$ | kurtosis |
| $\mu$ | mean |
| % | modulus operator |
| $A(k),B(k),D(k)$ | cepstral coefficients |
| $a_i,b_i,c_i,d_i$ | minimum and maximum phase poles and zeros |
| $q_k,p_k$ | number of minimum and maximum phase poles and zeros |
| $m(m,n,l)$ | forth order moment |
| $\hat{r}(k)$ | power cepstrum |
| $R(z)$ | power spectrum |
| $x,y$ | random variable sample |
| $\sigma$ | standard deviation |

| | |
|---|---|
| $\kappa$ | skewness |
| $\tau_k$ | time index |
| $\hat{c}(m,n,l)$ | tricepstrum |
| TEA | Tricepstrum Equalizer Algorithm |
| $\hat{g}(k)$ | tricepstrum slice |
| $C(z_1,z_2,z_3)$ | trispectrum |
| $G(z)$ | trispectrum slice |
| $X(z),H(z),I(z),O(z),S(z)$ | Z transforms |

# *Chapter 1*
# *Introduction*

## *1.1 The Communications Environment*

The ultimate goal in any communications system is the transmission of information from one location to another. Unfortunately the signal being transmitted is subject to varying degrees of distortion which affects the ability and ease by which the transmitted signal is recoverable. One source of distortion which is that resulting from the multiplicity of paths between the source and the receiver. These paths are referred to as the transmission channel and is described by an impulse response which represents a base band equivalent of a radio frequency (RF) channel. The propagation characteristics of each of the multipath components gives rise to phase shifts which causes the components to add constructively or destructively and causes fading or enhancement of the signal. In addition, if significant dispersion or delay distortion relative to a symbol interval occurs, then intersymbol interference (ISI) is observed. The received signal is also corrupted by noise which is usually assumed to be additive, white and Gaussian (AWGN). Finally, the nature of the channel may also be time varying which complicates the techniques used to recover the transmitted signal.

The impulse response of the channel is equivalently characterized by the discrete Fourier domain system transfer function. In the Z domain, the system transfer function is decomposed into a minimum and a maximum phase component corresponding to the location of the zeros with respect to the unit circle.

Minimum phase components are described by poles or zeros which are located inside the unit circle while maximum phase components are described by those poles or zeros which are located outside. Alternatively the system transfer function may also be described in terms of magnitude and phase corresponding to a minimum phase or zero phase part and an all-pass phase part. These factorizations are useful in the subsequent analysis and modeling of the channel in that modeling may be attempted in terms of the minimum and maximum phase components or alternatively in terms of amplitude and phase.

## *1.2 The Role of the Equalizer in the Receiver*

The main objective of the receiver is to recover the original transmitted signal sequence as precisely as possible. The most difficult distortion to compensate for is that which is induced by time dispersion. The term "equalizer" is used to describe devices which attempt to eliminate or reduce the distortion effects caused by the channel. An equalizer is therefore a type of receiving filter which compensates for time dispersion or equivalently the nonideal frequency response characteristics of the channel [1]. Similarly it may refer to an algorithm or device which is designed to deal with intersymbol interference [2].

Conventional adaptive equalizers require a training sequence during the sequences of initialization and periodic training. This is required because the system's transfer function is unknown. An estimate of the system can be made by sending a known sequence through the system. Unfortunately the transmission of this training sequence uses up available symbols and therefore

reduces the usable data rate and the available bandwidth. The implicit assumption is that the equalizer is able to adequately characterize the unknown system during the training period. This not guaranteed and the transmitter would not be aware of the success or failure of this training. As the channel is time varying, the equalizer must continually adapt to the changing environment which may not always be successful. Blind equalizers, also known as self-adaptive equalizers, converge without the need of a training sequence.

Both blind and non-blind traditional equalizers may be designed by modeling the inverse of the channel or by modeling the channel's impulse response directly and then constructing a compensating filter. Several approaches may be applied to the blind equalization problem. In the Bussgang equalizer approach [1][3][4], an iterative deconvolution procedure is used to determine the inverse channel filter. Some of the problems associated with this approach is that the performance surface is not well behaved and these schemes require much longer convergence periods during startup and reinitialization. The inverse model might be sensitive to noise especially near nulls in the channel's frequency response [4].

Another category of blind equalizer types consists of those equalizers based on the higher order spectra of the received signal [1][4][5][6]. These attempt to first model the channel directly and subsequently construct a compensating filter which forms the actual equalizer. This inverse filter may be designed using the zero forcing constraint which attempts to invert the channel. A compensating filter may also be designed under the minimum mean squared error constraint

which takes into account the spectral shape of the signal and the amount of additive noise.

The purpose of implementing an equalizer in a communications system is to recover information that has been distorted in some fashion. This suggests that the performance may be evaluated by comparing a system's input and output symbol sequences. The performance is considered to be better as the number of differences, considered as errors, between the two sequences is reduced. The bit error probability is used as a performance measure and is representative of the statistical nature of the simulation results.

## *1.3 Applications*

The use of higher order spectra to solve system identification and equalization problems is relatively new. The work which has been done so far may therefore be considered to be more of a theoretical approach rather than a practical one. It is the theoretical aspect of this approach which is examined in this work. A number of potential applications arise in the fields of speech coding, reflection seismology, image processing and in communications. These areas are discussed briefly in the following paragraphs.

Speech coding is thought to be a potential application where cumulants might offer a more efficient alternative to linear predictive coding. This alternative is suggested since it is felt that higher order statistics would potentially improve the speech data model accuracy [4].

Reflection seismology requires the removal of the source waveform from a seismogram. This is of extreme importance in the interpretation of the reflection data. Blind equalization or equivalently blind deconvolution has shown great promise in this field [7].

Image processing problems often appear in medical applications where the image information is often unavailable or limited. The reconstruction of the image with partial or no information is a problem which may be suited to the area of blind equalization. One potential application is that when phase content is more reliable than amplitude content and reconstruction is attempted from the phase only [8]. Higher order spectra allows the phase to be identified from the recorded image.

Blind equalizers based on higher order spectra find a number of potential applications in the field of communications. The areas include the telephone environment, multipoint data networks and high capacity line-of-sight digital radio [4][5]. In the telephone environment, channel equalization is required in high data rate transmission using modems. Ideally, blind equalizers based on higher order spectra would perform this channel equalization and may enable higher data rates to be used especially in the presence of colored line noise [5].

In a multipoint data network, a control unit is connected to several data terminals. A serious problem occurs when a particular unit is not able to link with the network until a training sequence is sent from the control unit. In large or heavily loaded networks with a large amount of data traffic, it is difficult to

ensure that all data links are continuously up. With blind equalization, the monitoring is facilitated as each link is independent and requires less time to establish a link. Thus the use of blind equalization would increase the overall performance of the multipoint network.

High capacity line-of-sight digital radio is also a good application for blind equalization. A line-of-sight microwave radio link has a wide-band, low noise channel. This channel exhibits time dispersive multipath fading which could potentially be corrected by the use of blind equalization even if the fading is severe [9]. It is even suggested that blind equalization especially those based on cepstra techniques, would perform effectively in conditions such as those found in ionospheric HF and in indoor communication system environments where channels are believed to be only slowly fading.

From the few applications referred to here, it is clear that blind equalization based on higher order spectra offers a potential solution to many important problems in a variety of fields.

## *1.4 Approaches to the System Identification Problem*

The estimation of the channel or of its inverse is an important element in the design of an equalizer. In blind equalization schemes, only the received signal is available to identify the unknown system or channel. It is possible to identify a linear system from output signals if the input signal to the system is an impulse. In this case the output signal is the impulse response of the system. The system

may also be identified from the received signal if the input to the system is white. Taking the autocorrelation of the received signal will result in the knowledge of the magnitude response of the system. This procedure corresponds to spectrum estimation in the frequency domain. Both the autocorrelation and spectrum estimation are second order statistics which are adequate to resolve minimum or maximum phase equivalent components but are not adequate to resolve mixed phase components. The system identification may also be affected by the presence of noise.

Higher order statistics may be applied to the received output signal sequence in order to identify the unknown system. This is possible if the input to the system is multidimensionally white in which case the cumulant at the output of the system represents the multidimensional impulse response of the system. The relationship between the impulse response and the cumulant is a nonlinear relationship and the extraction of the impulse response is the central aim in any higher order spectrum estimation technique.

A number of higher order spectra estimation techniques have been developed in recent years many of which are described in a tutorial paper by Mendel [4]. These techniques employ higher order statistics, specifically cumulants. Many of the techniques use the discrete Fourier transform of the cumulant known as polyspectra. These are similar to conventional spectra except that they are based on higher order time domain functions. Other techniques use the polycepstrum or the complex cepstrum and form an extension to the polyspectra based techniques which enables linear solutions to be obtained. The polycepstra refers

to the complex logarithm of the polyspectra while the complex cepstrum refers to the inverse discrete Fourier transform of the polycepstra.

Higher order spectra exhibit many properties which may be exploited to aid in the task of system identification. Perhaps the most significant property is that phase information is preserved [5]. In contrast second order statistics, such as the autocorrelation, are phase blind. Since the phase is unknown, the minimum phase component equivalents are assumed from a factorization of the spectrum. Another property of higher order statistics is that they are theoretically blind to Gaussian processes [4]. As such, they may be considered to be a measure of the deviation of the process from Gaussianity. As a result of this property, cumulants tend to be blind to the Gaussian noise which is often present in communication receivers, but not to the input sequence which must not be and generally is not Gaussian.

A number of techniques employing higher order statistics have been developed which use ARMA parametric models and require the solution of highly nonlinear systems of equations. In order to avoid these nonlinear problems, homomorphic filtering based on polycepstra techniques may be used [10][11]. These techniques make use of the complex cepstrum which may be obtained indirectly or directly from the cumulants. In the indirect approach, multidimensional Fourier transform (FFT) computations, logarithms and inverse multidimensional Fourier transformation are used. Alternatively in the direct approach, the complex cepstrum is obtained directly from the cumulants via cepstral and cumulant convolutional relationships. The direct method uses the method of least squares to solve for the complex cepstrum coefficients. The form

of these "cepstral equations" form the basis of the differences between the various polycepstrum based equalizers. This direct approach avoids the logarithm and phase unwrapping algorithms which is the disadvantage encountered in other homomorphic methods [12]. Once the cepstral coefficients are known the impulse response may be obtained by using well known recursive techniques [13].

## *1.5 Higher Order Spectra Based Equalizers*

The identification of the system's impulse response using the higher order spectra obtained from the output of the unknown system is approached by considering the factorization of the channel or equivalently the impulse response of the system itself. The channel may be decomposed into a minimum and a maximum phase term. The channel may also be decomposed into an amplitude and a phase representation. The amplitude part may be minimum phase resulting in a residual all-pass phase term. Alternatively the amplitude may be constructed to be zero phase leaving an all-pass phase term corresponding to the true phase of the system. These equivalent factorizations are presented as follows:

$$
\begin{aligned}
H(z) &= I(z)\,O(z) \\
&= \left\{I(z)\,O^*(1/z^*)\right\}\left\{\frac{O(z)}{O^*(1/z^*)}\right\} \\
&= \left\{H(z)\,H^*(1/z^*)\right\}^{1/2}\left\{\frac{H(z)}{H^*(1/z^*)}\right\}^{1/2}
\end{aligned}
\qquad (1.1)
$$

where:

$H(z)$ represents the system transfer function of the channel

$H^*(1/z^*)$ represents the reflection of the system's pole and zero locations

$I(z)$ represents the minimum phase components

$I^*(1/z^*)$ represents the maximum phase equivalents of $I(z)$

$O(z)$ represents the maximum phase components

$O^*(1/z^*)$ represents the minimum phase equivalents of $O(z)$

From the factorizations given, many different equalizer configurations are possible.

The Tricepstrum Equalizer Algorithm (TEA) method is a system identification procedure used to identify the channel's impulse response in terms of the minimum and maximum phase components [11][14]. This was the first cepstrum based equalizer to be presented. In this algorithm, a convolutional relationship between the cumulants and the complex cepstrum is used to obtain the minimum and maximum phase components of the impulse response. From the knowledge of the minimum and maximum phase components various equalizer structures such as a linear equalizer and a decision feedback type structure may be constructed [15][16].

If a diversity system is available, the Cross-Tricepstrum Equalizer Algorithm (CTEA) which is based on relating crosscumulants to cepstral coefficients may be used[17]. The method allows the simultaneous identification of independent nonminimum phase systems which are driven by a common input. The algorithm is therefore applicable to multiple-input single-output equalization models such as in the case of diversity. The number of signals or systems for

which the method may be applied corresponds to the order of the crosscepstrum. This further implies that the number of parameters to be estimated will increase by a multiple factor equal to the number of systems to be identified. Once the set of signal estimates is obtained a decision rule is used to select the final signal estimate. This method has shown only modest improvement over the TEA method depending on the channel but at the cost of an increased computational complexity.

The TEA method utilizes all the samples of the multidimensional polyspectrum, however, an alternative method is suggested by using only a single dimensional slice of the polyspectrum. This method may be referred to as the slice-TEA [18] and allows the development of a one dimensional function which results in less computation in determining the impulse response estimate. As in the basic TEA method, this method also determines the minimum and maximum phase components. This method is especially interesting in that the cumulants corresponding to the diagonal slice of the polyspectrum are thought to be the most phase sensitive.

Another approach to the equalization problem is suggested by considering the amplitude and phase parts problems separately. Thus far the amplitude and phase parts of the channel were modeled simultaneously in terms of minimum and maximum phase components and therefore it would be of interest to model these separately. When the channel amplitude and phase estimation processes are separated it is natural to also perform the amplitude and phase equalization separately. Amplitude equalization requires second order statistics only and

therefore does not require higher order statistically based methods. Also, many amplitude equalization techniques exist already.

The phase equalization may be accomplished by using one of two approaches. In the first approach a residual white sequence is obtained before the phase portion is identified. Cumulant estimation is therefore performed on a whitened sequence. In the second approach the phase equalization is independent of the amplitude equalization and does not require a unity power spectrum. This second approach allows separate amplitude and phase equalization, in that the phase equalization process does not require a white sequence or equivalently prior amplitude equalization. The advantage of having the equalization performed in two steps is that higher order spectra can be used to determine the phase portion exclusively. When the two functions are separated it is possible to perform pre-amplitude or pre-phase equalization at the terminal which ever is the most advantageous to the situation. An example of such a case is that of a base station which is stationary and has a higher processing capability versus a portable terminal which has limited processing capabilities and limited power availability. Another advantage of independent phase equalization is that sometimes the amplitude portion of the signal is not reliable and by knowing the phase of a MA system, it is possible to reconstruct the entire system by Hilbert transformation.

The difficulty in obtaining a phase estimate is that the phase part has a magnitude response which is unity for all frequencies and therefore passes all frequencies without any amplitude distortion. Such a filter, with a stable configuration, has all its poles located inside the unit circle and all the zeros

located outside the unit circle. The pole and zero locations are complex conjugate inverses of each other. The all-pass filter structure is therefore useful for phase equalization. When combined with an amplitude equalizer it ideally results in an overall linear phase response or constant group delay. Unfortunately second order statistics are not adequate to resolve the pole and zero locations of the all-pass structure used to represent the system's phase. Resolution of these locations is offered through the use of higher order statistics based on the use of cumulants which are not phase blind.

The Polycepstra and Prediction Equalization Algorithm (POPREA) is an approach to the equalization problem which is attempted in terms of minimum phase amplitude and residual phase system identification [19]. The amplitude portion is obtained by using power cepstra. The TEA method is used to solve for the residual phase component, however, only the maximum phase components are used to construct the phase equalizer. As a result, only some of the cepstral coefficients are actually used which is wasteful.

An alternative approach to system identification arises if the amplitude part is made to be zero phase. The phase part which remains is then the true phase of the system. The phase may be represented by an all-pass structure. TEA method may be used to obtained the phase of the system [8][20] by estimating the poles and zeros of the all-pass system. Although this is a solution to the problem, the performance of this phase-TEA method improves only somewhat over the TEA method's minimum and maximum phase estimation approach.

The Power Cepstrum and Tricoherence Equalizer (POTEA) is an equalizer which is based on the use of both the cepstrum of the power spectrum and the tricoherence [21]. The tricoherence is the complex cepstrum of the phase. In this algorithm the determination of the amplitude uses the power cepstrum based on second order spectra. The phase component is obtained from a system of equations using the tricoherence and involves a three dimensional convolution of the fourth order cumulants. This algorithm is very complex and requires significant amounts of computation, however, it is also reported to converge faster than the TEA algorithm.

## *1.6 Suggested Approach*

An alternative approach to IIR system estimation would be to design a new algorithm which would incorporate the constraint imposed by an all-pass system model. The procedure proposed in this work is to use complex cepstra in such a way as to exploit the complex conjugate inverse symmetry inherent in the all-pass structure. The result would improve the position accuracy of the pole and zero locations. This equalization scheme is expected to yield superior equalizer performance. Efficient and effective optimization between the numerator and the denominator of the all-pass structure is accomplished using the complex cepstrum of the fourth order cumulant which is also referred to as the tricepstrum. The complex cepstrum coefficients are obtained from the cumulant estimates directly using a least squares approach. It is possible to ensure that the filter has a unity amplitude by adding a constraint to the solution of the least squares problem. This forces complex conjugate symmetry between the

numerator and the denominator polynomials with the result of causing the number of parameters to be estimated to be reduced by one half. This approach of a joint optimization, exploiting the complex conjugate symmetry inherent in the all-pass structure, forms an original component of the proposed method.

The same type of analysis may be applied to the slice-TEA method. This method uses a trispectrum slice to determine minimum and maximum phase components. Selecting this slice results in cumulant averaging and yields a one dimensional function versus the three dimensional function given by the TEA method. Although this trispectrum slice is believed to contain the cumulant combinations which are the most sensitive to the system's phase, the performance of the slice-TEA method is inferior. Thus the best equalizer design would not be obtained using this approach.

Once the cepstral coefficients are obtained, the channel's impulse response may be reconstructed by inverse transformation or by a more desirable recursive method which would avoid any Fourier transform operation [13]. The all-pass phase channel model may then be obtained directly from these coefficients. Phase equalization may then proceed by inversion of this all-pass model, perhaps using the all-pass lattice.

## *1.7 Implementation of the Equalizer*

There are a number of methods by which the equalization process may be accomplished once the impulse response of the channel is obtained. Maximum

likelihood sequence estimators (MLSE) recover the unknown input signal sequence by effectively matching all possible resulting channel outputs to the observed output and selecting the best match in a minimum mean square error sense. Although this type of structure performs extremely well, the disadvantage is the high computational and storage requirements [2]. Another approach is that of mean square error filtering where the filter is designed to minimize the error between the predicted sequence provided by the equalizer and the original unobserved input sequence. In this method, either linear or decision feedback structures may be used. The decision feedback structure performs well in the presence of spectral nulls and is very effective in its ability to compensate for ISI, unfortunately however it suffers from error propagation [22]. The zero forcing criterion based approach attempts to invert the channel. In this method the complete cancellation of ISI is attempted. In this case noise enhancement from inverse filtered receiver noise may be a problem [23]. It is the zero forcing approach which is used in this work.

The effectiveness of the cepstrum based approaches when applied to equalization is demonstrated by comparison of the TEA method to the maximum likelihood, suboptimum LMS linear and nonlinear adaptation schemes which are not blind. The performance of the TEA method depends on the availability of a significant length of the received data, the number of cepstral coefficients estimated and the locations of the channel transfer function's singularities relative to the unit circle. From comparisons obtained from simulations using real and complex finite impulse response (FIR) channels, the probability of error achieved has been shown to be very close to that achieved by the maximum

likelihood method and better than the other linear and nonlinear LMS schemes [9].

The TEA method employing decision feedback equalization has also been examined and somewhat better performance has been indicated. Two decision feedback structures were constructed. In one scheme the feed forward part is set to correspond to the inverse of the channel's maximum phase component. In the second scheme the feed forward component is set to correspond to an all-pass component. In both structures the feed forward part is anticausal while the feedback portion consists of the remaining stable portion of the factorized transfer function. Of these two schemes, the second structure provided the best overall performance [15][16].

## *1.8 Summary of the Proposed Approach*

The approach which is investigated in this work is based on separating the amplitude and phase portions of the equalization into an amplitude and a phase component. The amplitude portion is implemented using a zero phase equalizer rather than a minimum phase one. Linear prediction filtering can be used to determine the amplitude component. Unfortunately, however, this method does not always whiten as well as one would desire. An amplitude equalizer based on the power cepstrum is therefore suggested for this implementation. This type of amplitude equalizer uses power cepstral coefficients and appears to result in the best performance.

The phase equalization is to be accomplished by using an all-pass structure. It has unity amplitude and does not affect the amplitude spectrum of the signal. An advantage of using this approach is that all-pass equalizers have simpler structures than general IIR filters of the same order [24]. In addition, the all-pass structure has complex conjugate symmetry between the numerator and the denominator which results in the complex cepstra coefficients having a complex conjugate symmetry. This feature allows exploitation of the complex conjugate symmetry to yield a solution which exhibits less complexity than previous methods. It is possible to constrain the symmetry in the all-pass structure as only the phase identification part is based on higher order spectra. This optimization reduces the variance of the estimates and thus implies a higher accuracy and achieves a faster convergence. This optimization is an original component to the work presented here. It is important because as the performance of the higher order spectra techniques is improved, so will the number of applications be increased to which the technique may be applied to.

Both the amplitude and the phase equalizer may be implemented under the zero forcing constraint. Under this criteria, a lattice filter is suggested for the amplitude equalizer while an all-pass lattice filter [25] is suggested for the phase equalizer. The lattice structure is modular which implies that sections can be added without the reoptimization of lower order coefficients. Both structures facilitate inverse filtering in that the structure lends itself naturally to inversion and time reversal. Stability of the filter is also easily verified by examination of the lattice coefficients.

## *1.9 Summary*

In summary, the potential advantages and justification of the proposed method would be to yield better blind equalization. An order of magnitude improvement over existing methods will be demonstrated. Separation of the amplitude and phase equalization components allows each individual part to be optimized in terms of computation, performance and even placement depending upon the application. The method would be applicable to systems with complex signals (inphase and quadrature) which is critical since complex equalization capability is necessary in any quadrature modulation system.

The disadvantage with higher order spectra methods appears to be the data length which is required to form the required cumulant estimates. The channel should be weakly or pseudo stationary over this data block. With the introduction of the suggested constraint and the use of cepstrums, this length requirement is reduced.

System identification using higher order spectra is discussed in chapter 2. Homomorphic systems and complex cepstra is introduced in chapter 3. Tricepstrum equalizer algorithm approaches are presented in chapter 4. This includes the original tricepstrum equalizer algorithm (TEA), the slice - tricepstrum equalizer algorithm (slice - TEA), as well as the proposed all-pass tricepstrum equalizer algorithm (all-pass TEA). The extension to complex data is also included. The equalizer structure using lattice filters is discussed in chapter 5. System simulation and performance results are investigated in chapter 6. Conclusions are summarized in chapter 7.

# *Chapter 2*
# *System Identification using Higher Order Spectra*

## *2.1 System Identification*

The extraction of the digital information from a received signal which has undergone transmission from the transmitter source is the goal in a communications receiver. The desired compensation is generally unknown because the channel represents a particular transmission path that the signal followed from the source to the receiver. This path may be time varying and generally time dispersive. Compensation for this channel often begins with first estimating it as an unknown system and is a procedure which constitutes a system identification problem. Either the actual channel or its inverse may be estimated.

System identification techniques have for the past number of years centered upon the ideas of spectral estimation. In the pursuit of spectral estimation a number of approaches have been developed based on conventional Fourier type methods which include the maximum likelihood, maximum entropy and minimum energy methods; signal modeling methods which employ MA, AR and ARMA parametric models; and harmonic decomposition methods that include Prony, Pisarenko, MUSIC, ESPRIT and Singular Value Decomposition [6]. Other techniques based on cross correlations require knowledge of the source symbol sequence.

The power estimation methods are based on second order statistics such as the autocorrelation function which provides a measure of how one sequence sample is correlated to another sample of the same sequence. These power estimation methods suffer from the limitation that phase relations between the various frequency components are suppressed. Accurate phase reconstruction can be achieved only if the system is correctly assumed to be either maximum phase or more commonly minimum phase. The phase reconstruction may also be affected by the presence of additive and especially coloured noise.

Higher order spectra, based on higher order statistics, provide a way in which more information can be extracted from a signal. Such information includes the measure of how Gaussian the process is, the identification, detection and characterization of nonlinearities in systems, and the identification of nonminimum phase systems. Furthermore, higher order statistics appear to suppress additive Gaussian noise which may even be coloured. These points [5] serve as the motivation to further investigate the area of higher order statistics.

## *2.2 Second Order Statistics*

System parameters of interest in many applications are often determined by using statistical analysis. Perhaps the most popular descriptors used to describe apparent random processes are those of mean and variance.

The mean is classified as a first order statistic and is defined as follows [26]:

$$\mu = E[x] \tag{2.1}$$

where:

$\mu$ = the mean

$x$ = a random variable sample

$E[\cdot]$ denotes an expectation operator.

The variance is considered to be a second order statistic and is defined as follows [28]:

$$\sigma^2 = E\left[(x - E[x])^2\right] = E\left[(x-\mu)^2\right] \tag{2.2}$$

The mean and variance parameters completely describe a process only if the process is truly Gaussian.

The expectation of two random variables, also referred to as joint statistics, may also be used to describe processes. Much information about the generating process may be revealed by the values of these estimates. For example, when two processes are independent, the expected values may be separated as [27]:

$$E[xy] = E[x]\,E[y] \tag{2.3}$$

and the two processes are said to be uncorrelated. Also two processes are said to be orthogonal if [27]:

$$E[xy] = 0 \tag{2.4}$$

Second order statistics are also useful when it is desirable to extract underlying periodic components from a signal. This is determined from the observation of spectral lines in the Fourier transform spectral representation of the

autocorrelation sequence. This spectrum also reveals the frequency content and distribution of a signal.

When a process is modified by a system, such as a signal source being modified by a channel, the spectrum can be used to reveal some information about the system from output observations alone when the statistics of the originating process are known. In the case of a discrete representation, the magnitude squared of the system's transfer function may be estimated. This is equivalent to the determination of the singularities of the unknown system. Unfortunately some ambiguities exist, such as whether the singularity is a maximum or a minimum phase component and also when two singularities form an all pass pair. In this latter case, the singular pairs are not observable. Thus the need arises for a way to resolve these components.

## *2.3 Higher Order Spectra: Moments and Cumulants*

Higher order spectra provide a method to obtain more information than that which can be obtained from traditional second order statistics alone. The higher order spectra exhibit properties that can be exploited to reveal information such as: the deviation from normality, phase estimation and knowledge which enables nonlinear properties to be detected and characterized [20][28].

The higher order spectra of a set of random variables are based on cumulants which are related to moments. The moment of order $k$ of a zero mean random variable $x$ is given by [4]:

$$m_k(\tau_1, \tau_2, \ldots, \tau_{k-1}) \equiv E\left[x(n)\, x(n+\tau_1) \ldots x(n+\tau_{k-1})\right] \tag{2.5}$$

where:

$n$ is the sample index

$k$ is the order

$x$ is the random variable

$E[\cdot]$ is the expectation operator.

In traditional methods, the minimum phase equivalent transfer function of a system may be obtained from output measurements when the input to the system is white. This is equivalent to having a source whose autocorrelation function is a single impulse. The same principle can be extended and applied to higher order spectra domains. The cumulants of white processes may be considered to be multidimensional impulse response functions with the corresponding polyspectrum being multidimensionally flat [4]. Furthermore, the system function obtained from the polyspectrum is not restricted to be minimum phase.

The relationship between the moment of a random process $x(t)$ and its cumulant is given by the following general expression [20]:

$$c_{k,x}(\tau_1, \tau_2, \ldots, \tau_{k-1}) = m_{k,x}(\tau_1, \tau_2, \ldots, \tau_{k-1}) - m_{k,G}(\tau_1, \tau_2, \ldots, \tau_{k-1}) \tag{2.6}$$

In this expression, the last term refers to a Gaussian random process which has the same second order statistics as the process $x(t)$. It is also evident from this expression that the cumulant is a measure of the deviation of the process from that of a Gaussian random process.

The cumulants for first, second and third orders are defined [4][20] and related to the moments as follows:

$$c_{1,x} = E\left[x(n)\right] = m_{1,x}$$
$$c_{2,x}(\tau_1) = E\left[x(n)\, x(n+\tau_1)\right] = m_{2,x}(\tau_1)$$
$$c_{3,x}(\tau_1, \tau_2) = E\left[x(n)\, x(n+\tau_1)\, x(n+\tau_2)\right] = m_{3,x}(\tau_1, \tau_2) \tag{2.7}$$

In the case of the fourth order cumulant, an additional component is required to produce a multidimensional impulse response function. The fourth order cumulant is given as follows:

$$\begin{aligned} c_{4,x}(\tau_1, \tau_2, \tau_3) &= cum\left(x(n), x(n+\tau_1), x(n+\tau_2), x(n+\tau_3)\right) \\ &= E\left[x(n)\, x(n+\tau_1)\, x(n+\tau_2)\, x(n+\tau_3)\right] \\ &\quad - E\left[x(n)\, x(n+\tau_1)\right] E\left[x(n+\tau_2)\, x(n+\tau_3)\right] \\ &\quad - E\left[x(n)\, x(n+\tau_2)\right] E\left[x(n+\tau_1)\, x(n+\tau_3)\right] \\ &\quad - E\left[x(n)\, x(n+\tau_3)\right] E\left[x(n+\tau_1)\, x(n+\tau_2)\right] \\ &= m_{4,x}(\tau_1, \tau_2, \tau_3) - c_{2,x}(\tau_1)\, c_{2,x}(\tau_2 - \tau_3) \\ &\quad - c_{2,x}(\tau_2)\, c_{2,x}(\tau_3 - \tau_1) - c_{2,x}(\tau_3)\, c_{2,x}(\tau_1 - \tau_2) \end{aligned} \tag{2.8}$$

So far the definitions of cumulants and moments have been restricted to processes with a zero mean. The expressions for the higher order spectra of nonzero mean random processes are obtained by applying the following substitution [4]:

$$x(n) \Leftrightarrow x(n) - E\left[x(n)\right] \tag{2.9}$$

In subsequent analysis, zero mean random processes will be assumed since the input signaling scheme is chosen to be symmetric about the origin.

Special terms describe the case when the lag between the samples are zero. This zero lag component for zero mean is described as follows [20]:

$\mu = c_1 =$ mean

$\sigma^2 = c_2(0) =$ variance

$\kappa = c_3(0,0) =$ skewness

$\gamma = c_4(0,0,0) =$ kurtosis

It is desirable to work with cumulants rather than with moments because cumulants exhibit certain properties that are useful. The property of the cumulant representing a multidimensional impulse response function is useful in the determination of the impulse response of a system. This property may be termed higher order whiteness [4]. In the case of a set of random variables being Gaussian, all higher order cumulants are identically zero. As such, higher order statistics are more robust to additive Gaussian noise, which may be either white or colored. Cumulants are able to distinguish non-Gaussian signals from Gaussian noise thereby boosting signal-to-noise ratios. In the case of statistically independent stationary random processes, the cumulant sum equals the sum of the cumulants of the individual random processes. This is not the case for higher order moments and thus this property allows the use of the cumulant as an operator [4].

## *2.4 Complex Data*

The type of data encountered in a communications system may be either real or complex. In order to increase data rates and spectral efficiency, complex data

systems are often employed. The received data is often described by the following convolutional model:

$$x(t) = \sum_{n} s(n)\, h(t-n) \tag{2.10}$$

where:

$x(t)$ represents the received signal

$s(t)$ represents the transmitted symbol sequence

$h(t)$ represents the impulse response of the channel

$t, n$ represent time samples

The real and imaginary components of the transmitted symbol sequence are generally independent and identically distributed. In the case of complex data, several possible ways exist to define the cumulant where the differences arise in the number of terms which are chosen to be conjugated [14]. Any potential definition for complex data must however also hold for real data.

The relationship between the fourth order cumulant of a process $x(t)$ and its moment is given by the following expression [9][4]:

$$\begin{aligned} c_x(\tau_1, \tau_2, \tau_3) = m_x(\tau_1, \tau_2, \tau_3) - r_x(\tau_1)\, r_x(\tau_3 - \tau_2) \\ - r_x(\tau_2)\, r_x(\tau_3 - \tau_1) - r_x(\tau_3)\, r_x(\tau_2 - \tau_1) \end{aligned} \tag{2.11}$$

From this expression it is observed that if the moment assumes a zero value, the cumulant will become dependent only upon the correlation terms. This situation is undesirable since second order statistics are phase blind.

If an ideal channel is assumed, the cumulant at the output of the system must then be the same as that of the source sequence. It follows then that the moment of the source sequence must therefore also be nonzero. In selecting the

appropriate cumulant definition, the fourth order moment at zero lag is considered. Adjacent samples of the source sequence are generally independent of each other and therefore the moment at zero lag is of prime importance to ensure a moment value which is nonzero.

The first possible definition for the fourth order moment uses only one conjugated term and is as follows:

$$m_x(\tau_1,\tau_2,\tau_3) = E\left[x^*(t)\, x(t+\tau_1)\, x(t+\tau_2)\, x(t+\tau_3)\right] \tag{2.12}$$

This moment at zero lag is therefore:

$$\begin{aligned} m_x(0,0,0) &= E\left[x^*(t)\, x(t)\, x(t)\, x(t)\right] \\ &= E\left[|x(t)|^2\, x^2(t)\right] \end{aligned} \tag{2.13}$$

The expression can be expanded in terms of the real and imaginary components as follows:

$$\begin{aligned} m_x(0,0,0) &= E\left[\left(x_R^2(t)+x_I^2(t)\right)\left(x_R^2(t)-x_I^2(t)+2j\, x_R(t)\, x_I(t)\right)\right] \\ &= E\left[x_R^4(t)\right]-E\left[x_I^4(t)\right]+2j\, E\left[x_R^3(t)\right]E\left[x_I(t)\right]+2j\, E\left[x_I^3(t)\right]E\left[x_R(t)\right] \\ &= 0 \end{aligned} \tag{2.14}$$

Since this definition of the moment yields a zero value, another better definition is sought.

The next possibility is that two of the terms are conjugated as follows:

$$m_x(\tau_1,\tau_2,\tau_3) = E\left[x^*(t)\, x(t+\tau_1)\, x^*(t+\tau_2)\, x(t+\tau_3)\right] \tag{2.15}$$

The moment at zero lag is then:

$$\begin{aligned} m_x(0,0,0) &= E\left[|x(t)|^4\right] = E\left[\left(x_R^2(t)+x_I^2(t)\right)^2\right] \\ &= E\left[x_R^4(t)\right]+E\left[x_I^4(t)\right]+2\,E\left[x_R^2(t)\right]E\left[x_I^2(t)\right] \\ &\neq 0 \end{aligned} \tag{2.16}$$

This result suggests that the definition may contain two conjugated terms and for complex harmonic processes, the only cumulant definition which yields a nonzero value is when two conjugated factors occur in the definition such that [4]:

$$\begin{aligned} c_{4,x}(\tau_1,\tau_2,\tau_3) &= cum\left(x^*(n), x(n+\tau_1), x^*(n+\tau_2), x(n+\tau_3)\right) \\ &= E\left[x^*(n)\, x(n+\tau_1)\, x^*(n+\tau_2)\, x(n+\tau_3)\right] \\ &\quad + E\left[x^*(n)\, x(n+\tau_1)\right] E\left[x^*(n+\tau_2)\, x(n+\tau_3)\right] \\ &\quad + E\left[x^*(n)\, x^*(n+\tau_2)\right] E\left[x(n+\tau_1)\, x(n+\tau_3)\right] \\ &\quad + E\left[x^*(n)\, x(n+\tau_3)\right] E\left[x(n+\tau_1)\, x^*(n+\tau_2)\right] \end{aligned} \tag{2.17}$$

Another interesting point is that all third order cumulants of complex harmonic signals are always zero thus making it necessary to work with fourth order cumulants [4].

The final possible definition for the moment is one which does not employ conjugates and is given as:

$$m_x(\tau_1,\tau_2,\tau_3) = E\left[x(t)\, x(t+\tau_1)\, x(t+\tau_2)\, x(t+\tau_3)\right] \tag{2.18}$$

The moment at zero lag is obtained as:

$$\begin{aligned} m_x(0,0,0) &= E\left[x^4(t)\right] \\ &= E\left[\left(x_R(t) + j\, x_I(t)\right)^4\right] \end{aligned} \tag{2.19}$$

This expression may be expanded as:

$$\begin{aligned} m_x(0,0,0) &= E\left[x_R^4(t)\right] + E\left[4j\, x_R^3(t)\, x_I(t)\right] - E\left[6\, x_R^2(t)\, x_I^2(t)\right] \\ &\quad - E\left[4j\, x_R(t)\, x_I^3(t)\right] + E\left[x_I^4(t)\right] \\ &= E\left[x_R^4(t)\right] + E\left[x_I^4(t)\right] - 6\, E\left[x_R^2(t)\, x_I^2(t)\right] \end{aligned} \tag{2.20}$$

$$m_x(0,0,0) \neq 0$$

When the signal is assumed to be independent and identically distributed on both the real and imaginary axis, the nonzero lag moment becomes:

$$m_x(\tau_1,\tau_2,\tau_3) = E[x(t)]\,E[x(t+\tau_1)]\,E[x(t+\tau_2)]\,E[x(t+\tau_3)] = 0 \tag{2.21}$$

If however the situation arises when $\tau_2 = \tau_3 = \tau$ for an assumed independent and identically distributed signal, the nonzero lag moment becomes:

$$m_x(\tau_1,\tau_2,\tau_3) = E[x(t)]\,E[x(t+\tau_1)]\,E[x^2(t+\tau)] = 0 \tag{2.22}$$

The moment is therefore nonzero only when $\tau_1 = \tau_2 = \tau_3 = 0$. This indicates that the process is white in the time domain. The moment for such an independent sequence may therefore be alternatively expressed as:

$$m_x(\tau_1,\tau_2,\tau_3) = \gamma\,\delta(\tau_1)\,\delta(\tau_2)\,\delta(\tau_3) \tag{2.23}$$

The fourth order cumulant represents a multidimensional impulse response as does this moment definition. The fourth order cumulant and the fourth order moments are therefore identical when the second order terms of the cumulant are equal to zero, i.e.:

$$c_x(\tau_1,\tau_2,\tau_3) = m_x(\tau_1,\tau_2,\tau_3) = \gamma\,\delta(\tau_1)\,\delta(\tau_2)\,\delta(\tau_3) = E[x(t)\,x(t+\tau_1)\,x(t+\tau_2)\,x(t+\tau_3)] \tag{2.24}$$

This definition is the simplest of the various forms.

The results here indicate that two possible definitions exist to describe the cumulant. The two possible choices consist of the definition containing two conjugated terms and the definition containing no conjugated terms.

## 2.5 *Cumulants and the Impulse Response*

The purpose for using the cumulant is to identify the impulse response of a finite dimensional, linear, time invariant system from output observations. Second order statistics are phase blind and therefore only contain spectrally equivalent minimum phase information. Cumulants on the other hand are phase sensitive which therefore enables a nonminimum phase impulse response to be recovered. This is possible since a relationship exists between the output cumulant and the impulse response even when the input is unknown. The input must be assumed to be stationary, non-Gaussian, independent and identically distributed. The development of this relationship begins with the definition of the cumulant given as [4]:

$$\begin{aligned} c(\tau_1,\tau_2,\tau_3) = {} & E\left[x(t)\,x(t+\tau_1)\,x(t+\tau_2)\,x(t+\tau_3)\right] \\ & - E\left[x(t)\,x(t+\tau_1)\right]\; E\left[x(t+\tau_2)\,x(t+\tau_3)\right] \\ & - E\left[x(t)\,x(t+\tau_2)\right]\; E\left[x(t+\tau_1)\,x(t+\tau_3)\right] \\ & - E\left[x(t)\,x(t+\tau_3)\right]\; E\left[x(t+\tau_1)\,x(t+\tau_2)\right] \end{aligned} \tag{2.25}$$

This expression may be simplified by considering the various terms separately. The first term is observed to be the fourth order moment of the output signal and is given by:

$$m_x(\tau_1,\tau_2,\tau_3) = E\left[x(t)\,x(t+\tau_1)\,x(t+\tau_2)\,x(t+\tau_3)\right] \tag{2.26}$$

The output signal may also be expanded in terms of the convolution between the input sequence and the impulse response as follows:

$$m_x(\tau_1,\tau_2,\tau_3) =$$

$$E\left[\sum_k h(k)\,s(t-k)\sum_l h(l+\tau_1)\,s(t-l)\sum_m h(m+\tau_2)\,s(t-m)\sum_n h(n+\tau_3)\,s(t-n)\right] \tag{2.27}$$

This expression may be rearranged since the input sequence is independent of the impulse response so that the expectation term can be taken into the summation term resulting in:

$$m_x(\tau_1,\tau_2,\tau_3) = \sum_k\sum_l\sum_m\sum_n h(k)\,h(l+\tau_1)\,h(m+\tau_2)\,h(n+\tau_3)\,E\left[s(t-k)\,s(t-l)\,s(t-m)\,s(t-n)\right] \tag{2.28}$$

In this expression it is observed that the expectation term is also the fourth order moment of the input sequence at various delays, i.e.:

$$m_s(k-l,k-m,k-n) = E\left[s(t-k)\,s(t-l)\,s(t-m)\,s(t-n)\right] \tag{2.29}$$

The moment may be written in terms of its corresponding cumulant in which case the expression becomes equal to:

$$\begin{aligned} m_s(k-l,k-m,k-n) = {} & c(k-l,k-m,k-n) + r_s(k-l)\,r_s(m-n) \\ & + r_s(k-m)\,r_s(l-n) + r_s(k-n)\,r_s(l-m) \end{aligned} \tag{2.30}$$

Evaluating the various terms results in:

$$\begin{aligned} & m_x(\tau_1,\tau_2,\tau_3) \\ & = \sum_k\sum_l\sum_m\sum_n h(k)\,h(l+\tau_1)\,h(m+\tau_2)\,h(n+\tau_3)\,\gamma_s\,\delta(k-l)\,\delta(k-m)\,\delta(k-n) \\ & + \sum_k\sum_l\sum_m\sum_n h(k)\,h(l+\tau_1)\,h(m+\tau_2)\,h(n+\tau_3)\,\sigma_s^4\,\delta(k-l)\,\delta(m-n) \\ & + \sum_k\sum_l\sum_m\sum_n h(k)\,h(l+\tau_1)\,h(m+\tau_2)\,h(n+\tau_3)\,\sigma_s^4\,\delta(k-m)\,\delta(l-n) \\ & + \sum_k\sum_l\sum_m\sum_n h(k)\,h(l+\tau_1)\,h(m+\tau_2)\,h(n+\tau_3)\,\sigma_s^4\,\delta(k-n)\,\delta(l-m) \end{aligned} \tag{2.31}$$

This expression may be further simplified to yield:

$$\begin{aligned} m_x(\tau_1,\tau_2,\tau_3) = {} & \gamma_s\sum_k h(k)\,h(k+\tau_1)\,h(k+\tau_2)\,h(k+\tau_3) \\ & + \sigma_s^4\sum_k\sum_m h(k)\,h(k+\tau_1)\,h(m+\tau_2)\,h(m+\tau_3) \\ & + \sigma_s^4\sum_k\sum_l h(k)\,h(l+\tau_1)\,h(k+\tau_2)\,h(l+\tau_3) \\ & + \sigma_s^4\sum_k\sum_l h(k)\,h(l+\tau_1)\,h(l+\tau_2)\,h(k+\tau_3) \end{aligned} \tag{2.32}$$

This expression expresses the moment of the received signal in terms of the impulse response of the system and of the second and fourth order statistics of the signal source.

The second order terms in the cumulant expression may be expressed in terms of the impulse response of the system and of the statistics of the signal source. This alteration begins by expanding the autocorrelation functions as a convolutional sum between the signal and the impulse response as follows:

$$\begin{aligned} r_x(\tau_1) &= E\left[x(t)\, x(t+\tau_1)\right] \\ &= E\left[\sum_k h(k)\, s(t-k) \sum_l h(l+\tau_1)\, s(t-l)\right] \\ &= \sum_k \sum_l h(k)\, h(l+\tau_1)\, E\left[s(t-k)\, s(t-l)\right] \\ &= \sum_k \sum_l h(k)\, h(l+\tau_1)\, \sigma_s^2\, \delta(k-l) \\ r_x(\tau_1) &= \sigma_s^2 \sum_k h(k)\, h(k+\tau_1) \end{aligned} \tag{2.33}$$

It similarly follows that:

$$r_x(\tau_2 - \tau_3) = E\left[x(t+\tau_2)\, x(t+\tau_3)\right] = \sigma_s^2 \sum_m h(m+\tau_2)\, h(m+\tau_3) \tag{2.34}$$

Substituting these expressions for each of the second order terms in the cumulant definition results in the following sets of equations:

$$\begin{aligned} &E\left[x(t)\, x(t+\tau_1)\right] E\left[x(t+\tau_2)\, x(t+\tau_3)\right] \\ &\quad = \sigma_s^4 \sum_k \sum_m h(k)\, h(k+\tau_1)\, h(m+\tau_2)\, h(m+\tau_3) \end{aligned} \tag{2.35}$$

$$\begin{aligned} &E\left[x(t)\, x(t+\tau_2)\right] E\left[x(t+\tau_1)\, x(t+\tau_3)\right] \\ &\quad = \sigma_s^4 \sum_k \sum_l h(k)\, h(l+\tau_1)\, h(k+\tau_2)\, h(l+\tau_3) \end{aligned} \tag{2.36}$$

$$E\left[x(t)\,x(t+\tau_3)\right]E\left[x(t+\tau_1)\,x(t+\tau_2)\right] = \sigma_s^4 \sum_k \sum_l h(k)\,h(l+\tau_1)\,h(l+\tau_2)\,h(k+\tau_3) \tag{2.37}$$

These equations comprising the second order terms in the cumulant definition along with the definition for the fourth order moment are substituted into the cumulant definition resulting in the following expression:

$$c(\tau_1,\tau_2,\tau_3) = \gamma_s \sum_k h(k)\,h(k+\tau_1)\,h(k+\tau_2)\,h(k+\tau_3) \tag{2.38}$$

In this simplification the second order terms are observed to vanish and thus, this simplification results in an important expression. This expression clearly shows that the cumulant of the output signal is related to the statistics of the source and the system's impulse response.

The definition relating the cumulant to the impulse response can also be equivalently expressed in the discrete frequency domain. The transformation begins as follows:

$$C(z_1,z_2,z_3) = \sum_{\tau_1}\sum_{\tau_2}\sum_{\tau_3} c(\tau_1,\tau_2,\tau_3)\,z_1^{-\tau_1}\,z_2^{-\tau_2}\,z_3^{-\tau_3} \tag{2.39}$$

Substituting for the cumulant in terms of the impulse response and interchanging the order of summation gives:

$$C(z_1,z_2,z_3) = \gamma_s \sum_k \sum_{\tau_1}\sum_{\tau_2}\sum_{\tau_3} h(k)\,h(k+\tau_1)\,h(k+\tau_2)\,h(k+\tau_3)\,z_1^{-\tau_1}\,z_2^{-\tau_2}\,z_3^{-\tau_3} \tag{2.40}$$

At this point the transform is taken with respect to each axis by considering the following transform relationship:

$$z^k\,H(z) = \sum_\tau h(\tau+k)\,z^{-\tau} \tag{2.41}$$

Therefore by taking the Z transform for each variable the following form is obtained:

$$C(z_1,z_2,z_3) = \gamma_s\,H(z_1)\,H(z_2)\,H(z_3)\sum_k h(k)\,(z_1z_2z_3)^k \tag{2.42}$$

Finally considering the final transform thus yields the desired result:

$$C(z_1, z_2, z_3) = \gamma_s \, H(z_1) \, H(z_2) \, H(z_3) \, H(1/z_1 \, 1/z_2 \, 1/z_3) \tag{2.43}$$

This expression relates the system function to the frequency domain representation of the fourth order cumulant which is also referred to as the trispectrum. It is also valid for complex data in the case where no conjugated terms are required in the cumulant definition.

In the case of complex data where two conjugated terms are used, the cumulant may also be expressed in the time domain as:

$$c(\tau_1, \tau_2, \tau_3) = \gamma_s \sum_k h^*(k) \, h(k+\tau_1) \, h^*(k+\tau_2) \, h(k+\tau_3) \tag{2.44}$$

In the frequency domain this expression becomes:

$$C(z_1, z_2, z_3) = \gamma_s \, H(z_1) \, H^*(z_2^*) \, H(z_3) \, H(1/z_1^* \, 1/z_2^* \, 1/z_3^*) \tag{2.45}$$

This relationship is valid for real data as well.

The accurate extraction of the impulse response from the cumulant is a very difficult and challenging problem as evidenced by the extensive and on going research in this area. It is remarkable and interesting that a very simple but theoretical closed form solution exists from which it is possible to determine the parameters of a MA model. Apriori knowledge of the order of the system is required in order to use this method which is referred to as the C(q,k) formula [4].

The method seeks to identify those points in the three dimensional cumulant space which correspond to the impulse response of the system. In order to achieve this objective appropriate lag values in the cumulant definition need to be selected.

The C(q,k) formula is derived for a FIR system using the cumulant definition in terms of the impulse response and by considering a number of interesting points. In this approach, a reduction in the number of unknown impulse response terms is attempted and used in order to express various cumulants in terms of various impulse response components.

A reduction in the number of unknown impulse response terms is achieved by considering that the first sample of the system's impulse response is always assumed to be unity i.e.:

$$h(0) = 1$$

Another major simplification is possible by noting that the cumulant vanishes for lags greater than the order of the impulse response of the system. This implies that nonzero elements appear in the summation only when the summation index is zero. This results in the elimination of the summation operation and yields a cumulant value which is the product of impulse response samples at various lags. This operation is expressed as follows:

$$\begin{aligned} c(\tau_1, \tau_2, \tau_3) &= \gamma_s \sum_k h(k)\, h(k+\tau_1)\, h(k+\tau_2)\, h(k+\tau_3) \\ &= \gamma_s\, h(0)\, h(\tau_1)\, h(\tau_2)\, h(\tau_3) \end{aligned} \tag{2.46}$$

The derivation of the C(q,k) formula therefore proceeds by assigning the following lag values:

$$\tau_1 = q$$
$$\tau_2 = 0$$
$$\tau_3 = k$$

which yields:

$$\begin{aligned} c(q,0,k) &= \gamma_s\, h(0)\, h(q)\, h(0)\, h(k) \\ &= \gamma_s\, h(q)\, h(k) \end{aligned} \tag{2.47}$$

The unknown terms in this expression are solved for by assigning the following lag values:

$$\tau_1 = q$$
$$\tau_2 = 0$$
$$\tau_3 = 0$$

which yields:

$$\begin{aligned} c(q,0,0) &= \gamma_s\, h(0)\, h(q)\, h(0)\, h(0) \\ &= \gamma_s\, h(q) \end{aligned} \tag{2.48}$$

Combining these two equations yields the C(q,k) formula as:

$$h(k) = \frac{c(q,0,k)}{c(q,0,0)} = \frac{c(q,0,k)}{c(-q,-q,-q)} \tag{2.49}$$

where because of symmetry the following holds [4]:

$$c(q,0,0) = c(-q,-q,-q) \tag{2.50}$$

The symmetry indicates that there are other cumulant values from which the impulse response can be obtained. The C(q,k) equation uses only a one dimensional slice of the output cumulant which indicates the weakness of the method in the sense that since the cumulant is only an estimate, no filtering or averaging occurs to reduce the estimation error.

The relationship between the impulse response of a system and its cumulant is unfortunately a nonlinear one. The principle of superposition, which is a requirement of a linear system, does not apply since, for example, a scaling of the impulse response does not result in a corresponding linear scaling of the cumulant. The application of homomorphic techniques to the higher order spectral methods allows linear methods to be applied to the nonlinear relationships resulting in a variety of interesting linear techniques.

## *2.6 Summary*

The use of higher order spectra enables one to identify an unknown system from the output of the system without knowledge of the input source. It is important to realize that the identification of the unknown system is based upon knowledge of the statistics of the input source sequence rather than knowledge of the actual symbol sequence. The system identified using higher order spectra will yield the true phase of the system and not just the equivalent minimum phase of the system as is the case when dealing with traditional second order statistically based methods.

The systems encountered in the communications environment, as well as in many other environments, are not limited to real data. Thus, the definition of the cumulant must be valid for both complex data as well as for real data. The extraction of the impulse response from the cumulant is very difficult and a number of methods using homomorphic techniques are introduced in the following two chapters.

# *Chapter 3*
# *Homomorphic Systems & Complex Cepstra*

## *3.1 Homomorphic Systems based on the complex logarithm*

The use of higher order spectra makes it possible to identify an unknown system using measurements obtained at the output of the system to be identified. The identification requires the knowledge of the statistics of the input source sequence rather than the knowledge of the actual symbol sequence. The fourth order cumulant has been shown to contain information about the impulse response of the unknown system. Unfortunately, the extraction of the impulse response is a nonlinear problem. It is therefore the intent here to describe the method by which a separation between the signal source and that of the impulse response is possible. Furthermore, this impulse response may then be recovered using a well known recursive technique [13].

The application of homomorphic techniques to higher order spectral methods provides a technique whereby the nonlinear relationship between the cumulant and the impulse response may be transformed into a linear one. A homomorphic system is a term which refers to those classes of nonlinear systems which obey a generalized principle of superposition [13]. In these systems, a translation is sought which transforms a previously nonlinear relationship, such as the one between the input and output components of a system, to that of a linear one. When such a transformation exists, the homomorphic system can be dealt with in

a similar manner to a linear system thereby facilitating linear filtering operations [13].

Perhaps the most extensively encountered operations to which homomorphic signal processing is applied are those of multiplication and convolution. In a multiplicative homomorphic system, the signal is comprised of the product of two or more component signals. Linear filtering is ineffective in its ability to distinguish these individual components even if they differ in frequency content. In this situation, applying a complex logarithm to the signal transforms the multiplied signal components into additive components which may then be filtered linearly. Inverse transformation of these individual components by the application of exponentiation restores the signal into its original domain.

In the convolutional homomorphic system, the signal is comprised of the time domain convolution of various signal components and corresponds to multiplication in the frequency domain. This is in contrast to a multiplicative homomorphic system where the convolution occurs in the frequency domain. In order to facilitate the ability to distinguish time domain convolved signals, a complex logarithm is applied to the Z transform of the signal, causing the signal components to now become additive. Inverse Z transforming the signal into the equivalent time domain results in the complex cepstrum representation. In this domain the addition relation is preserved and allows many filtering operations to be easily implemented. The complex cepstrum therefore refers to the inverse Fourier transform of the complex logarithm of the Fourier transform of the signal. This differs from the cepstrum which refers to the inverse Fourier transform of the logarithm of only the magnitude of the Fourier transform which

does not depend on the phase. The inverse system transformation by the Z transform, exponentiation and inverse Z transformation returns the signal to the original domain. In both convolutional and multiplicative homomorphic systems, the basic function used in the transformation is the complex logarithm. It is, however, the convolutional model which appears in the areas of seismology and in communications and is therefore the system of interest.

The usefulness of the system becomes apparent if the output of a convolutional system is defined as follows:

$$x(n) = s(n) * h(n) \tag{3.1}$$

where:

$x(n)$ is the output of the system

$s(n)$ is the unknown source sequence

$h(n)$ is the impulse response of the system

This is equivalently expressed in the Z domain as:

$$X(z) = S(z)\, H(z) \tag{3.2}$$

By applying the complex logarithm to this output sequence the decomposition into the source sequence and the system function components occurs as follows:

$$\hat{X}(z) = \ln\left[X(z)\right] = \hat{S}(z) + \hat{H}(z) \tag{3.3}$$

where

$$\hat{S}(z) = \ln S(z)$$

$$\hat{H}(z) = \ln H(z)$$

Transformation into the equivalent time domain yields the complex cepstrum and results in the following form:

$$\hat{x}(n) = \hat{s}(n) + \hat{h}(n) \tag{3.4}$$

The complex cepstrum of the source sequence and of the impulse response are thus observed to constitute additive components of the complex cepstrum of the system's output.

The ultimate goal in many signal processing applications is to recover the original sequence $s(n)$. This is generally accomplished by separating the signal component from the system function. In general these two components are not linearly separable, however in certain cases such as optics this is indeed the case. It will be shown that by applying homomorphic techniques to higher order spectra, a separation between the signal source and the impulse response is possible.

The homomorphic system is described mathematically with the forward transform representing the complex cepstrum given as:

$$\hat{x}(n) = Z^{-1}\left[\ln X(z)\right] \tag{3.5}$$

and the inverse transform given as:

$$x(n) = Z^{-1}\left[X(z)\right] = Z^{-1}\left[\exp \hat{X}(z)\right] \tag{3.6}$$

The basic operations just described are representative of a convolutional homomorphic system where the process of convolution is transformed into one of addition. This mapping operation may also be applied to higher order spectra to yield many interesting results.

## *3.2 Complex Logarithm, Complex Cepstrum and Differential Cepstrum*

In a convolutional homomorphic system the mapping or transformation of the operation of convolution into that of addition is accomplished by obtaining the complex cepstrum. The complex logarithm is a key element in the definition of the complex cepstrum. The complex logarithm for a convolutional homomorphic system is defined as follows [13]:

$$\hat{X}(z) = \ln X(z) = \ln |X(z)| + j \arg[X(z)] \tag{3.7}$$

where:

$X(z)$ is the Z-transform of an arbitrary sequence

$\hat{X}(z)$ represents the Z transform of the complex cepstrum

The problem with the use of this definition of the complex logarithm, is that the imaginary component corresponding to the phase or argument is not well defined. The argument must therefore not be restricted to only take on principal values. In order to ensure a continuous function, phase unwrapping is required [29].

The complex logarithm must have a convergent power series representation from which a valid transform can be extracted. The complex logarithm must therefore be representable in the following manner:

$$\hat{X}(z) = \ln [X(z)] = \sum_{n=-\infty}^{n=\infty} \hat{x}(n) z^{-n} \tag{3.8}$$

Stability is ensured by including the unit circle in the region of convergence. The sequence of coefficients of the power series then corresponds to the complex cepstrum.

The determination of the complex cepstrum may alternatively be obtained directly by differentiating the complex logarithm. This approach resolves the problems of the argument of the complex logarithm not being uniquely defined and discontinuous. The direct relationship between the complex cepstrum and the system's transform is established by differentiating the complex logarithm as follows:

$$\hat{X}_d(z) \equiv \frac{d\hat{X}(z)}{dz} = \frac{d}{dz} \ln X(z) = \frac{1}{X(z)} \frac{dX(z)}{dz} \tag{3.9}$$

where:

$\hat{X}(z)$ is the complex logarithm representation

$\hat{X}_d(z)$ is the differentiated complex logarithm

$X(z)$ is the Z transform of an arbitrary sequence

Multiplying through by $z\,X(z)$ yields the following:

$$z\,X(z)\,\hat{X}_d(z) = z\,\frac{dX(z)}{dz} = z\,X(z)\frac{d\hat{X}(z)}{dz} \tag{3.10}$$

The relationship between the complex cepstrum and the signal is obtained by transforming this equation into the time domain. In order to proceed further, the relationship between a time advanced derivative of a Z domain function and its time domain equivalent needs to be determined. This begins by considering first the Z transform definition given by:

$$X(z) = \sum_n x(n)\,z^{-n} \Leftrightarrow x(n) \tag{3.11}$$

The derivative of the Z domain function may be taken and yields the following:

$$\frac{d\,X(z)}{d\,z} = \sum_n x(n)(-n)z^{-n-1} \Leftrightarrow -(n-1)\,x(n-1) \tag{3.12}$$

From this form, an advanced Z transform relationship may be determined and yields a corresponding advanced time domain equivalent as:

$$z\,\frac{dX(z)}{dz} = \sum_n x(n)(-n)z^{-n} \Leftrightarrow -n\,x(n) \tag{3.13}$$

Similarly the advanced complex cepstrum is given by:

$$z\frac{d\hat{X}(z)}{dz} = \sum_{n} \hat{x}(n)(-n)z^{-n} \Leftrightarrow -n\,\hat{x}(n) \tag{3.14}$$

These transform relationships may now be used to transform the differentiated complex logarithm into the time domain to yield the complex cepstrum as follows:

$$-n\,x(n) = \left(-n\,\hat{x}(n)\right) * x(n) \tag{3.15}$$

This expression results in a convolutional relationship between a signal and its complex cepstrum. Expanding the convolution in terms of a series gives:

$$-n\,x(n) = -\sum_{k=-\infty}^{\infty} k\,\hat{x}(k)\,x(n-k) \tag{3.16}$$

Finally by rearranging the expression a bit further a relationship between a signal and its complex cepstrum is presented as follows:

$$x(n) = \sum_{k=-\infty}^{k=\infty} \frac{k}{n}\,\hat{x}(k)x(n-k), \qquad n \neq 0 \tag{3.17}$$

This equation results in a relationship between the signal's complex cepstrum $\hat{x}(n)$ and the signal represented by $x(n)$. The use of the complex logarithm is avoided and thus difficulties with phase ambiguities are avoided.

An alternative method to avoid the use of the complex logarithm is to use the differential cepstrum directly. The relationship between an arbitrary signal and it's differential cepstra may be determined by recalling that:

$$\hat{X}_d(z) = \frac{1}{X(z)}\frac{dX(z)}{dz} \tag{3.18}$$

By multiplying through by $X(z)$ and transforming to the time domain the following expression is obtained:

$$-(n-1)\,x(n-1) = x(n) * \hat{x}_d(n) \tag{3.19}$$

The convolutional part of this equation may be further expanded to yield the desired relationship between a signal and it's differential cepstra:

$$-(n-1)\,x(n-1) = \sum_{k=-\infty}^{\infty} \hat{x}_d(k)\,x(n-k) = \sum_{k=-\infty}^{\infty} x(k)\,\hat{x}_d(n-k) \tag{3.20}$$

The relationship between the differential cepstra and the complex cepstra may also be determined by recalling that:

$$\hat{X}_d(z) = \frac{d\,\hat{X}(z)}{dz} \tag{3.21}$$

By transforming to the time domain the expression becomes:

$$\hat{x}_d(n) = -(n-1)\,\hat{x}(n-1) \tag{3.22}$$

These expressions indicate that the differential cepstrum is closely connected to the complex cepstrum to the extent that the two forms are almost interchangeable with each other except for a time varying scaling factor [12].

## 3.3 *Complex Cepstrum of the Impulse Response*

The usefulness of the convolutional homomorphic system, based on the complex logarithm, is illustrated by considering the impulse response of the system. In the time domain representation, the impulse response of the communications channel may be represented in terms of a convolutional relationship between the minimum and maximum phase components expressed as:

$$h(n) = i(n) * o(n) \tag{3.23}$$

where:

$h(n)$ represents the impulse response of the channel

$i(n)$ represents the minimum phase components

$o(n)$ represents the maximum phase components

The minimum phase term refers to all the poles and zeros which are located within the unit circle in the Z plane while the maximum phase term refers to those zeros (and poles) which are located outside the unit circle [30].

The system's function in the Z domain may equivalently be expressed as a multiplication of the transformed minimum and maximum phase components as follows:

$$H(z) = I(z)\,O(z) \tag{3.24}$$

where:

$H(z)$ is the system function

$I(z)$ is the transformed minimum phase components of the system

$O(z)$ is the transformed maximum phase components of the system

The complex cepstrum is obtained by computing the complex logarithm of the system function and yields a linear relationship between the various components such that:

$$\hat{H}(z) = \hat{I}(z) + \hat{O}(z) \tag{3.25}$$

where:

$\hat{H}(z)$ is the complex logarithm of the system function

$\hat{I}(z)$ is the complex logarithm of the minimum phase components

$\hat{O}(z)$ is the complex logarithm of the maximum phase components

Transforming this equation into the time domain by applying an inverse Z transform results in the following linear relationship:

$$\hat{h}(n) = \hat{i}(n) + \hat{o}(n) \tag{3.26}$$

where:

$\hat{h}(n)$ refers to the complex cepstrum of the impulse response

$\hat{i}(n)$ refers to the complex cepstrum of the minimum phase components

$\hat{o}(n)$ refers to the complex cepstrum of the maximum phase components

This relationship is useful in that the complex cepstrum of the impulse response provides a form whereby the separation of the minimum and maximum phase components is possible.

Each of the minimum and maximum phase system components may be further expressed in terms of the individual pole and zero locations. Therefore the minimum phase component of the system function may be defined as [10][11]:

$$I(z) = \frac{\prod_{i=1}^{q_1}(1 - a_i z^{-1})}{\prod_{i=1}^{p_1}(1 - c_i z^{-1})}, \qquad |a_i| < 1,\ |c_i| < 1 \tag{3.27}$$

where:

$a_i$ is a minimum phase zero

$c_i$ is a minimum phase pole

$q_1$ is the number of minimum phase zeros

$p_1$ is the number of minimum phase poles

The maximum phase component of the system function may also be defined as [10][11]:

$$O(z) = \frac{\prod_{i=1}^{q_2}(1 - b_i z)}{\prod_{i=1}^{p_2}(1 - d_i z)}, \qquad |b_i| < 1, |d_i| < 1 \tag{3.28}$$

where:

$b_i$ is the inverse of a maximum phase zero

$q_2$ is the number of maximum phase zeros

$d_i$ is the inverse of a maximum phase pole

$p_2$ is the number of maximum phase poles

The maximum phase component is observed to contain maximum phase poles which result in system instability. They are included here only for completeness and will not cause a problem in further derivations.

The complex cepstrum of each of these terms is obtained by first taking the complex logarithm and then applying an inverse Z transform. The complex logarithm of the minimum phase component is therefore:

$$\hat{I}(z) = \ln I(z) = \sum_{i=1}^{q_1} \ln\left(1 - a_i z^{-1}\right) - \sum_{i=1}^{p_1} \ln\left(1 - c_i z^{-1}\right) \tag{3.29}$$

and the complex logarithm of the maximum phase component is:

$$\hat{O}(z) = \ln O(z) + \sum_{i=1}^{q_2} \ln\left(1 - b_i z\right) - \sum_{i=1}^{p_2} \ln\left(1 - d_i z\right) \tag{3.30}$$

Each of the logarithmic terms may be expanded into a power series since:

$$\ln(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n} \tag{3.31}$$

This expansion is done in order to facilitate the determination of the inverse Z transform. The expansions for the two minimum and maximum phase components are therefore:

$$\hat{I}(z) = -\sum_{i=1}^{q_1} \sum_{n=1}^{\infty} \frac{\left(a_i z^{-1}\right)^n}{n} + \sum_{i=1}^{p_1} \sum_{n=1}^{\infty} \frac{\left(c_i z^{-1}\right)^n}{n} \tag{3.32}$$

and

$$\hat{O}(z) = -\sum_{i=1}^{q_2} \sum_{n=1}^{\infty} \frac{\left(b_i z\right)^n}{n} + \sum_{i=1}^{p_2} \sum_{n=1}^{\infty} \frac{\left(d_i z\right)^n}{n} \tag{3.33}$$

Since the complex logarithm is expressed in terms of a power series, the inverse Z transform may be determined by inspection. Thus the inverse Z transform of an arbitrary function is simply the coefficient of the series, $f(k)$, and is obtained by inspection from the Z transform given by:

$$F(z) = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k} \tag{3.34}$$

The complex cepstrum for each of the minimum and maximum phase components is therefore:

$$\hat{i}(n) = -\sum_{i=1}^{q_1} \frac{a_i^{\,n}}{n} + \sum_{i=1}^{p_1} \frac{c_i^{\,n}}{n}, \quad n > 0 \tag{3.35}$$

and

$$\hat{o}(n) = -\sum_{i=1}^{q_2} \frac{b_i^{\,-n}}{-n} + \sum_{i=1}^{p_2} \frac{d_i^{\,-n}}{-n} = \sum_{i=1}^{q_2} \frac{b_i^{\,-n}}{n} - \sum_{i=1}^{p_2} \frac{d_i^{\,-n}}{n}, \quad n < 0 \tag{3.36}$$

The various regions of the complex cepstra are listed as follows:

$$\hat{h}(n) = Z^{-1}\{\ln H(z)\} = \begin{cases} 0, & n = 0, \text{ arbitrary scaling} \\ -\sum_{i=1}^{q_1} \frac{a_i^{\,n}}{n} + \sum_{i=1}^{p_1} \frac{c_i^{\,n}}{n}, & n > 0, \text{ minimum phase} \\ \sum_{i=1}^{q_2} \frac{b_i^{\,-n}}{n} - \sum_{i=1}^{p_2} \frac{d_i^{\,-n}}{n}, & n < 0, \text{ maximum phase} \end{cases} \tag{3.37}$$

From this list for the complex cepstra, it is observed that positive indexed cepstral coefficients correspond to minimum phase components while negative indexed cepstral coefficients correspond to the maximum phase components [7].

To facilitate further computation it is useful to define the minimum and maximum phase cepstral coefficients such that [10][11]:

$$A(n) = -n\,\hat{i}(n) = \sum_{i=1}^{q_1} a_i^n - \sum_{i=1}^{p_1} c_i^n, \qquad n > 0 \tag{3.38}$$

and

$$B(n) = -n\,\hat{o}(-n) = \sum_{i=1}^{q_2} b_i^n - \sum_{i=1}^{p_2} d_i^n, \qquad n > 0 \tag{3.39}$$

Thus by using a convolutional homomorphic system based on the complex logarithm, it is possible to express the complex cepstrum of the impulse response as a linear sum and difference of the complex cepstra of the minimum and maximum phase components corresponding to the system's pole and zero locations. This decomposition corresponds to the two regions of the interior and the exterior of the unit circle.

## *3.4 Impulse Response Coefficients from Cepstral Coefficients*

The impulse response coefficients are related to the cepstral coefficients through the difference equation developed earlier i.e. (eqn. 3.17):

$$x(n) = \sum_{k=-\infty}^{\infty} \frac{k}{n} \hat{x}(k)\, x(n-k), \qquad n \neq 0$$

This equation may be used to determine the minimum and maximum phase impulse coefficients individually. The minimum phase impulse response coefficients may be obtained by first noting that for causal minimum phase sequences both the cepstral coefficients and the impulse response coefficients are [13]:

$$x(n) = \hat{x}(n) = 0, \qquad n < 0 \tag{3.40}$$

The difference equation may therefore be modified for minimum phase systems to yield:

$$i(n) = \sum_{k=1}^{n} \frac{k}{n} \hat{i}(k)\, i(n-k), \qquad n > 0 \tag{3.41}$$

where

$i(n)$ refers to the minimum phase components

$\hat{i}(k)$ represents the minimum phase cepstral coefficients

A maximum phase sequence has no poles or zeros inside the unit circle and the impulse response coefficients are therefore [13]:

$$x(n) = \hat{x}(n) = 0, \qquad n > 0 \tag{3.42}$$

The difference equation is similarly modified for maximum phase systems to yield:

$$o(n) = \sum_{k=n}^{-1} \frac{k}{n}\ \hat{o}(k)\ o(n-k), \qquad n < 0 \tag{3.43}$$

where

$o(n)$ refers to the maximum phase components

$\hat{o}(k)$ represents the maximum phase cepstral coefficients

The impulse response coefficients both begin with unity as:

$$i(0) = o(0) = 1 \tag{3.44}$$

In conclusion, the impulse response of the system may be obtained in a recursive fashion from the complex cepstra of the system. Positive indexed cepstra correspond to the minimum phase terms while negative indexed cepstra correspond to the maximum phase terms of the impulse response.

## 3.5 *Complex Cepstra of Higher Order Spectra: The tricepstrum and cumulant relationship*

Higher order spectra which are also referred to as polyspectra are defined as the Fourier transform of higher order cumulants. The polyspectra is able to preserve the phase character of non-Gaussian parametric signals [11] even at the output of a system where only knowledge of the non-Gaussian distribution is required. Many procedures [4] have been developed using AR, MA and ARMA modeling techniques using cumulants in an attempt to identify the impulse response and

thus recover the original signal. These methods, however, require the solution of nonlinear systems of equations.

Convolutional homomorphic filtering, based on the complex logarithm, is an approach which may be applied to higher order spectra and is an alternative method to the problem of nonminimum phase system identification. In this method, separation of the impulse response of the system from the input source signal occurs when the input source signal is white or coloured but non-Gaussian. Furthermore, the solution in the cepstral domain yields a linear system of equations with the added feature that apriori knowledge of the model order is not required.

In convolutional homomorphic filtering involving higher order spectra, the complex logarithm is applied to the higher order spectra and inverse transformed to yield the complex cepstrum of the fourth order cumulant. In this method, an attempt is made to recover the true phase character of the impulse response by using the complex cepstra to identify the minimum and maximum phase components. The usefulness of the homomorphic system is based on establishing a direct relationship between the fourth order cumulant and its complex cepstrum. The complex cepstrum of the fourth order cumulant is termed the tricepstrum [14]. This relationship is determined by considering the fourth order cumulant and its discrete Fourier transform, the trispectrum:

$$c(m,n,l) = Z^{-1}\left[C(z_1,z_2,z_3)\right] \tag{3.45}$$

The tricepstrum is the complex cepstrum of the fourth order cumulant. A direct relationship between the cumulant and the tricepstrum is determined by

differentiating the complex logarithm of the trispectrum with respect to one of the Z variables as follows [11]:

$$\frac{d}{dz_1}\ln C(z_1,z_2,z_3)=\frac{d}{dz_1}\hat{C}(z_1,z_2,z_3)=\frac{1}{C(z_1,z_2,z_3)}\frac{d}{dz_1}C(z_1,z_2,z_3) \qquad (3.46)$$

The choice of the variable which the differentiation is done with respect to does not matter as all variables are interchangeable due to symmetry. Multiplying through by $z_1\,C(z_1,z_2,z_3)$ yields:

$$z_1\frac{dC(z_1,z_2,z_3)}{dz_1}=z_1\,C(z_1,z_2,z_3)\frac{d\hat{C}(z_1,z_2,z_3)}{dz_1} \qquad (3.47)$$

Transforming this expression into the time domain yields the relationship between the fourth order cumulant and its complex cepstrum or the tricepstrum. The transformation thus yields:

$$-m\,c(m,n,l)=\left(-m\,\hat{c}(m,n,l)\right)*c(m,n,l) \qquad (3.48)$$

where

$m,n,l$ are various sample numbers of the cumulant sequence on each axis

$c(m,n,l)$ represents the fourth order cumulant

$\hat{c}(m,n,l)$ represents the corresponding tricepstrum

This relationship is a three dimensional or triple convolutional equation and serves as the foundation from which many interesting solutions may be derived.

By further expansion, the convolutional relationship between the tricepstrum and the fourth order cumulant may be equivalently expressed in terms of a triple sum as follows:

$$m\,c(m,n,l)=\sum_{k=-\infty}^{\infty}\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}k\,\hat{c}(k,i,j)\,c(m-k,n-i,l-j) \qquad (3.49)$$

This equation may be simplified by determining which complex cepstra terms contribute to the convolutional sum. The complex cepstrum is nonzero only at integer values along each axis and along the main diagonal of the three

dimensional space [4]. This result may be shown by considering the cumulant at the output of a system in terms of the impulse response as follows:

$$c(m,n,l) = \gamma \sum_{k=-\infty}^{\infty} h(k)\, h(k+m)\, h(k+n)\, h(k+l) \tag{3.50}$$

Transforming this to the complex cepstra domain yields the complex cepstra as:

$$\begin{aligned} \hat{c}(m,n,l) = \ln \gamma\, \delta(m)\, \delta(n)\, \delta(l) + \hat{h}(m)\, \delta(n)\delta(l) + \hat{h}(n)\, \delta(m)\delta(l) \\ + \hat{h}(l)\, \delta(m)\delta(n) + \hat{h}(-m)\, \delta(m-n)\delta(n-l) \end{aligned} \tag{3.51}$$

It therefore follows that the complex cepstra are identical on each of the axis and along the common diagonal slice. Therefore the only contributions to the convolutional sum results from values along the $m$ axis when $k$ is not zero and along the diagonal axis $i = j = k$.

The convolutional equation may therefore be expressed in a fashion to indicate the valid contributions from the complex cepstrum as:

$$\begin{aligned} m\, c(m,n,l) = &\sum_{k=1}^{\infty} k\, \hat{c}(k,0,0)\, c(m-k,n,l) + \sum_{k=-1}^{-\infty} k\, \hat{c}(k,0,0)\, c(m-k,n,l) \\ &+ \sum_{k=1}^{\infty} k\, \hat{c}(k,k,k)\, c(m-k,n-k,l-k) \\ &+ \sum_{k=-1}^{-\infty} k\, \hat{c}(k,k,k)\, c(m-k,n-k,l-k) \end{aligned} \tag{3.52}$$

Further simplification yields:

$$\begin{aligned} m\, c(m,n,l) = &\sum_{k=1}^{\infty} k\, \hat{c}(k,0,0)\, c(m-k,n,l) - \sum_{k=1}^{\infty} k\, \hat{c}(-k,0,0)\, c(m+k,n,l) \\ &+ \sum_{k=1}^{\infty} k\, \hat{c}(k,k,k)\, c(m-k,n-k,l-k) \\ &- \sum_{k=1}^{\infty} k\, \hat{c}(-k,-k,-k)\, c(m+k,n+k,l+k) \end{aligned} \tag{3.53}$$

This form can be further simplified by identifying the individual complex cepstrum (tricepstrum) terms.

The complex cepstrum in terms of the minimum and maximum phase components of the impulse response may be expressed as:

$$\begin{aligned}\hat{c}(m,n,l) = \ln \gamma\, \delta(m)\,\delta(n)\,\delta(l) &+ \hat{i}(m)\,\delta(n)\delta(l) + \hat{o}(m)\,\delta(n)\delta(l) \\ &+ \hat{i}(n)\,\delta(m)\delta(l) + \hat{o}(n)\,\delta(m)\delta(l) \\ &+ \hat{i}(l)\,\delta(m)\delta(n) + \hat{o}(l)\,\delta(m)\delta(n) \\ &+ \hat{i}(-m)\,\delta(m-n)\delta(n-l) + \hat{o}(-m)\,\delta(m-n)\delta(n-l)\end{aligned} \tag{3.54}$$

This expression is useful because from it the complex cepstrum may be expressed in terms of the cepstral coefficients of the minimum and maximum phase components of the impulse response. The minimum phase cepstral coefficients represent a right hand sequence which may be defined such that:

$$A(k) = -k\,\hat{i}(k), \quad k > 0 \tag{3.55}$$

The maximum phase cepstral coefficients represent a left hand sequence which may also be defined as:

$$B(k) = -k\,\hat{o}(-k), \quad k > 0 \tag{3.56}$$

Substituting these cepstral coefficients into the previous expression yields:

$$\begin{aligned}\hat{c}(m,n,l) = -\frac{A(m)}{m}\,\delta(n)\delta(l) &+ \frac{B(-m)}{m}\,\delta(n)\delta(l) \\ -\frac{A(n)}{n}\,\delta(m)\delta(l) &+ \frac{B(-n)}{n}\,\delta(m)\delta(l) \\ -\frac{A(l)}{l}\,\delta(m)\delta(n) &+ \frac{B(-l)}{l}\,\delta(m)\delta(n) \\ +\frac{A(-m)}{m}\,\delta(m-n)\delta(n-l) &- \frac{B(m)}{m}\,\delta(m-n)\delta(n-l)\end{aligned} \tag{3.57}$$

The complex cepstrum expressed in this fashion may be used to further expand the convolutional equation. The complex cepstrum terms of interest are thus identified by substituting the appropriate variables to yield:

$$k\,c(k,0,0) = -A(k), \quad k > 0 \tag{3.58}$$

and

$$k\,c(k,k,k) = -B(k), \quad k > 0 \tag{3.59}$$

Similarly the terms corresponding to the negative axis are expressed as:

$$k\,c(-k,0,0) = -B(k), \quad k > 0 \tag{3.60}$$

and

$$k\,c(-k,-k,-k) = -A(k), \quad k > 0 \tag{3.61}$$

The cepstral coefficients just developed may now be substituted into the convolutional equation to yield the cepstral equation as follows:

$$\begin{aligned} -m\,c(m,n,l) = &\sum_{k=1}^{p} A(k)\left[c(m-k,n,l) - c(m+k,n+k,l+k)\right] \\ &+ \sum_{k=1}^{q} B(k)\left[c(m-k,n-k,l-k) - c(m+k,n,l)\right] \end{aligned} \tag{3.62}$$

where:

$p$ is the number of minimum phase cepstral coefficients

$q$ is the number of maximum phase cepstral coefficients

This equation provides a direct relationship between the cepstral coefficients *A(k)* and *B(k)* representing the minimum phase and maximum phase cepstral coefficients respectively and the fourth order cumulants $c(m,n,l)$.

The relationship between cumulants and cepstral coefficients is referred to as the cepstral equation. This equation forms the basis of the cepstral method and allows the cepstral coefficients to be calculated without the need of a logarithm or Fourier transform calculation.

## *3.6 Summary*

In this chapter the concept of a homomorphic system is introduced. Such a system is useful because it enables the transformation of a convolutional operation into one of addition. This transformation is achieved directly, thereby avoiding the complex logarithm and Fourier transformation operations. The impulse response may then be recovered in a recursive fashion from the complex cepstrum.

The homomorphic system transformation may also be applied to higher order spectra. This enables a recovery of the system's impulse response in terms of the minimum and maximum phase components. Other configurations will be explored in the following chapter.

# *Chapter 4*
# *Tricepstrum Equalizer Algorithm Approaches*

## *4.1 Cepstral Operation of Higher Order Spectra: The Tricepstrum Equalizer Algorithm (TEA)*

The relationship between the fourth order cumulant and the tricepstrum is used in the method known as the tricepstrum equalizer algorithm (TEA) to estimate directly the minimum and maximum phase components of the impulse response. This method was originally developed by Pan and Nikias [11] and by Hatzinakos and Nikias [9]. The derivation of the TEA method involves the determination of the complex cepstrum from the cumulant using the relationship between the cumulant and the unknown system's complex cepstrum and then the recovery of the impulse response from the complex cepstrum.

In the development of the TEA method, the objective is to identify the tricepstrum coefficients of the system function corresponding to the system's pole and zero locations. These pole and zero locations are identified as being either minimum or maximum phase components. The identification of the tricepstral coefficients begins by considering the definition of the fourth order cumulant in terms of the impulse response of the system. The fourth order cumulant is related to the impulse response in the following manner [4]:

$$c(\tau_1, \tau_2, \tau_3) = \gamma \sum_k h(k)\, h(k+\tau_1)\, h(k+\tau_2)\, h(k+\tau_3) \qquad (4.1)$$

where:

$\tau_i$ represents the time indices of the cumulant

$h(k)$ is the kth sample of the impulse response

$\gamma$ is the kurtosis

This equation may be equivalently expressed in the Z domain as [11]:

$$C(z_1, z_2, z_3) = \gamma\, H(z_1)\, H(z_2)\, H(z_3)\, H(1/z_1\, 1/z_2\, 1/z_3) \tag{4.2}$$

Taking the complex logarithm of the trispectrum results in the following:

$$\begin{aligned}\hat{C}(z_1, z_2, z_3) &= \ln C(z_1, z_2, z_3) \\ &= \ln\gamma + \ln H(z_1) + \ln H(z_2) + \ln H(z_3) + \ln H(1/z_1\, 1/z_2\, 1/z_3) \\ &= \ln\gamma + \hat{H}(z_1) + \hat{H}(z_2) + \hat{H}(z_3) + \hat{H}(1/z_1\, 1/z_2\, 1/z_3)\end{aligned} \tag{4.3}$$

The complex cepstrum of the system function may be further expanded into the minimum and maximum phase components. Thus the complex logarithm of the trispectrum in terms of the minimum phase and the maximum phase components may be expressed as:

$$\begin{aligned}\hat{C}(z_1, z_2, z_3) = \ln\gamma &+ \hat{I}(z_1) + \hat{O}(z_1) + \hat{I}(z_2) + \hat{O}(z_2) \\ &+ \hat{I}(z_3) + \hat{O}(z_3) + \hat{I}(1/z_1\, 1/z_2\, 1/z_3) + \hat{O}(1/z_1\, 1/z_2\, 1/z_3)\end{aligned} \tag{4.4}$$

where:

$\hat{I}(z)$ represents the complex logarithm of the minimum phase terms

$\hat{O}(z)$ represents the complex logarithm of the maximum phase terms

Each of the individual terms is actually a three dimensional function and when a term appears as a function of only one variable it represents only one axis in the cumulant space. When the three variables appear as a product, the representation implies the diagonal slice of the cumulant space. Taking the inverse Z transform yields the complex cepstrum of the cumulant or tricepstrum such that:

$$\hat{c}(m,n,l) = \ln \gamma\, \delta(m)\,\delta(n)\,\delta(l) + \hat{i}(m)\,\delta(n)\,\delta(l) + \hat{o}(m)\,\delta(n)\,\delta(l) + \hat{i}(n)\,\delta(m)\,\delta(l) + \hat{o}(n)\,\delta(m)\,\delta(l) + \hat{i}(l)\,\delta(m)\,\delta(n) + \hat{o}(l)\,\delta(m)\,\delta(n) + \hat{i}(-m)\,\delta(m-n)\,\delta(n-l) + \hat{o}(-m)\,\delta(m-n)\,\delta(n-l) \tag{4.5}$$

where

$\delta(n)$ represents a kronecker delta function

In the previous chapter the relationship between the tricepstrum and the fourth order cumulant was determined and is equivalently expressed in the triple convolutional relationship as:

$$m\,c(m,n,l) = \sum_{k=-\infty}^{\infty}\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} k\,\hat{c}(k,i,j)\,c(m-k,n-i,l-j) \tag{4.6}$$

From the previous determination of the tricepstrum, it is observed that the cepstral coefficients are defined only along the three axis slices and along the common diagonal slice. Furthermore the values of the cepstral coefficients are identical along each of these axis and as such, it is sufficient to express the triple convolutional relationship as a one dimensional expression. Substituting the complex cepstrum of the cumulant into this equation yields:

$$m\,c(m,n,l) = \sum_{k=-\infty}^{\infty} k\,\hat{i}(k)\,c(m-k,n,l) + \sum_{k=-\infty}^{\infty} k\,\hat{o}(k)\,c(m-k,n,l) + \sum_{k=-\infty}^{\infty} k\,\hat{i}(-k)\,c(m-k,n-k,l-k) + \sum_{k=-\infty}^{\infty} k\,\hat{o}(-k)\,c(m-k,n-k,l-k) \tag{4.7}$$

The minimum phase complex cepstra components are defined only for index values greater than zero and therefore it is sufficient to express the summation limits starting from a value of one. Similarly the maximum phase complex cepstra components are defined only on the negative axis corresponding to negative index values which again allows the summation limit to be reduced. Incorporating these limit changes results in:

$$m\,c(m,n,l) = \sum_{k=1}^{\infty} k\,\hat{i}(k)\,c(m-k,n,l) - \sum_{k=1}^{\infty} k\,\hat{o}(-k)\,c(m+k,n,l)$$
$$- \sum_{k=1}^{\infty} k\,\hat{i}(k)\,c(m+k,n+k,l+k) + \sum_{k=1}^{\infty} k\,\hat{o}(-k)\,c(m-k,n-k,l-k) \tag{4.8}$$

The minimum and maximum phase complex cepstra are more commonly expressed in terms of the cepstral coefficients. These cepstral coefficients are defined in terms of the minimum and maximum phase complex cepstra as well as the individual pole and zero locations as follows (eqn. 3.38):

$$A(k) = -k\,\hat{i}(k) = \sum_{i=1}^{q_1} a_i^k - \sum_{i=1}^{p_1} c_i^k, \quad k>0$$

and (eqn. 3.39)

$$B(k) = -k\,\hat{o}(-k) = \sum_{i=1}^{q_2} b_i^k - \sum_{i=1}^{p_2} d_i^k, \quad k>0$$

The last terms representing the pole and zero locations are obtained from the expansion of the complex logarithmic terms of the minimum and maximum phase components into a power series from which the inverse Z transform is easily determined. Thus substitution of these cepstral coefficients for each of the minimum and maximum phase complex cepstrum terms yields the following [9]:

$$-m\,c(m,n,l) = \sum_{k=1}^{\infty} A(k)\left[c(m-k,n,l) - c(m+k,n+k,l+k)\right]$$
$$+ \sum_{k=1}^{\infty} B(k)\left[c(m-k,n-k,l-k) - c(m+k,n,l)\right] \tag{4.9}$$

This equation is referred to as the tricepstrum convolutional equation. It is observed that both minimum and maximum phase cepstral coefficients are used to weight cumulants of various lags in order to predict a central cumulant. This system of equations allows the direct determination of the tricepstrum

represented by the cepstral coefficients without the use of the Fourier transform and of the complex logarithm. The advantage in this method is that phase unwrapping problems and methods are avoided.

## 4.2 *Complex TEA method*

Complex data systems consisting of an inphase and a quadrature channel are characteristic of digital communication systems. The inphase and quadrature channels although independent at the transmitter are not necessarily so at the receiver. Consequently cross talk would result from a channel having a complex impulse response. The implication of the complex data scenario is that the cumulant definition is affected. Two possible definitions for the cumulant are suggested. In the first case no conjugates are required resulting in the cumulant being defined as:

$$c(\tau_1,\tau_2,\tau_3)=\gamma\sum_k h(k)\,h(k+\tau_1)\,h(k+\tau_2)\,h(k+\tau_3) \tag{4.10}$$

In this case the TEA method is equivalent for both the real and complex cases. The difference arises in the cumulant estimation procedure as discussed earlier.

In the second case, two of the terms must be conjugated such that:

$$c(\tau_1,\tau_2,\tau_3)=\gamma\sum_k h^*(k)\,h(k+\tau_1)\,h^*(k+\tau_2)\,h(k+\tau_3) \tag{4.11}$$

This equation is equivalently expressed in the Z domain as:

$$C(z_1,z_2,z_3)=\gamma\,H(z_1)\,H^*(z_2^*)\,H(z_3)\,H^*(1/z_1^*\,1/z_2^*\,1/z_3^*) \tag{4.12}$$

Since an expression is sought which relates the cumulant to its complex cepstrum the complex logarithm of the trispectrum is taken and results in the following:

$$\begin{aligned}\hat{C}(z_1,z_2,z_3) &= \ln C(z_1,z_2,z_3)\\ &= \ln\gamma + \ln H(z_1) + \ln H^*(z_2^*) + \ln H(z_3) + \ln H^*(1/z_1^*\,1/z_2^*\,1/z_3^*)\\ &= \ln\gamma + \hat{H}(z_1) + \hat{H}^*(z_2^*) + \hat{H}(z_3) + \hat{H}^*(1/z_1^*\,1/z_2^*\,1/z_3^*)\end{aligned} \tag{4.13}$$

From this form the system function may be expanded into the minimum and maximum phase components yielding:

$$\begin{aligned}\hat{C}(z_1,z_2,z_3) &= \ln\gamma + \hat{I}(z_1) + \hat{O}(z_1) + \hat{I}^*(z_2^*) + \hat{O}^*(z_2^*)\\ &\quad + \hat{I}(z_3) + \hat{O}(z_3) + \hat{I}^*(1/z_1^*\,1/z_2^*\,1/z_3^*) + \hat{O}^*(1/z_1^*\,1/z_2^*\,1/z_3^*)\end{aligned} \tag{4.14}$$

Finally, by inverse Z transformation the desired tricepstrum is obtained with respect to the individual minimum and maximum cepstrum components as follows:

$$\begin{aligned}\hat{c}(m,n,l) &= \ln\gamma\,\delta(m)\,\delta(n)\,\delta(l) + \hat{i}(m)\,\delta(n)\,\delta(l) + \hat{o}(m)\,\delta(n)\,\delta(l)\\ &\quad + \hat{i}^*(n)\,\delta(m)\,\delta(l) + \hat{o}^*(n)\,\delta(m)\,\delta(l) + \hat{i}(l)\,\delta(m)\,\delta(n) + \hat{o}(l)\,\delta(m)\,\delta(n)\\ &\quad + \hat{i}^*(-m)\,\delta(m-n)\,\delta(n-l) + \hat{o}^*(-m)\,\delta(m-n)\,\delta(n-l)\end{aligned} \tag{4.15}$$

The complex cepstra identified so far is now used in the cumulant cepstral relationship developed earlier (eqn. 4.6) and given as:

$$m\,c(m,n,l) = \sum_{k=-\infty}^{\infty}\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} k\,\hat{c}(k,i,j)\,c(m-k,n-i,l-j)$$

Substituting the complex cepstra into this equation and recalling that the relationship reduces to a one dimensional relationship yields:

$$\begin{aligned}m\,c(m,n,l) &= \sum_{k=-\infty}^{\infty} k\,\hat{i}(k)\,c(m-k,n,l) + \sum_{k=-\infty}^{\infty} k\,\hat{o}(k)\,c(m-k,n,l)\\ &\quad + \sum_{k=-\infty}^{\infty} k\,\hat{i}^*(-k)\,c(m-k,n-k,l-k) + \sum_{k=-\infty}^{\infty} k\,\hat{o}^*(-k)\,c(m-k,n-k,l-k)\end{aligned} \tag{4.16}$$

The minimum phase complex cepstrum is defined only on the positive axis represented by positive index values. Similarly the maximum phase complex

cepstrum is defined for negative index values. These regions of support result in changes to the summation limits resulting in:

$$m\,c(m,n,l) = \sum_{k=1}^{\infty} k\,\hat{i}(k)\,c(m-k,n,l) - \sum_{k=1}^{\infty} k\,\hat{o}(-k)\,c(m+k,n,l)$$
$$-\sum_{k=1}^{\infty} k\,\hat{i}^{*}(k)\,c(m+k,n+k,l+k) + \sum_{k=1}^{\infty} k\,\hat{o}^{*}(-k)\,c(m-k,n-k,l-k) \tag{4.17}$$

The complex cepstrum of the cumulant is expressed in terms of the complex cepstrum of the minimum and maximum phase coefficients, however it is more common to express the relationship in terms of the cepstral coefficients and the pole and zero locations defined as follows (eqn. 3.38):

$$A(k) = -k\,\hat{i}(k) = \sum_{i=1}^{q_1} a_i^k - \sum_{i=1}^{p_1} c_i^k, \quad k>0$$

and (eqn. 3.39)

$$B(k) = -k\,\hat{o}(-k) = \sum_{i=1}^{q_2} b_i^k - \sum_{i=1}^{p_2} d_i^k, \quad k>0$$

The complex conjugate terms are also defined as:

$$A^{*}(k) = -k\,\hat{i}^{*}(k) = \sum_{i=1}^{q_1} \left(a_i^{*}\right)^k - \sum_{i=1}^{p_1} \left(c_i^{*}\right)^k, \quad k>0 \tag{4.18}$$

and

$$B^{*}(k) = -k\,\hat{o}^{*}(-k) = \sum_{i=1}^{q_2} \left(b_i^{*}\right)^k - \sum_{i=1}^{p_2} \left(d_i^{*}\right)^k, \quad k>0 \tag{4.19}$$

Substituting these cepstral coefficients for each of the minimum and maximum phase complex cepstrum terms results in the following:

$$-m\,c(m,n,l) = \sum_{k=1}^{\infty} A(k)\,c(m-k,n,l) \quad - \sum_{k=1}^{\infty} A^{*}(k)\,c(m+k,n+k,l+k)$$
$$- \sum_{k=1}^{\infty} B(k)\,c(m+k,n,l) \quad + \sum_{k=1}^{\infty} B^{*}(k)\,c(m-k,n-k,l-k) \tag{4.20}$$

This equation is the conjugated complex data equation for the TEA method. The equation is more complex than for the real case and for the case of unconjugated

complex data. It is observed that if the conjugated and unconjugated terms corresponding to the same term are equated the result will yield the unconjugated or real version of the TEA method developed previously.

## *4.3 Cepstral Operations on Slices of Higher Order Spectra: The Slice Tricepstrum Equalizer Algorithm (slice-TEA)*

The tricepstrum equalizer algorithm developed in the previous section makes use of all the samples in the polyspectrum. This results in a three dimensional convolutional relationship from which, because of axis symmetry, it is possible to reduce the problem to a one dimensional one. Another approach is to only use the diagonal slice of the polyspectrum and is an approach which differs from the TEA method in that a one dimensional sequence is defined rather than a three dimensional sequence. This approach is known as the slice-tricepstrum equalizer algorithm (slice-TEA) and was developed by Ashebeli et al. [18]. The slice-TEA method attempts to determine the minimum and maximum phase components of the system's impulse response by applying the cepstral operations, derived from homomorphic techniques, to a diagonal slice of the fourth order spectrum of the output signal.

The slice is defined by considering the fourth order spectrum and its relationship to the system function described as (eqn. 4.2):

$$C(z_1, z_2, z_3) = \gamma\, H(z_1)\, H(z_2)\, H(z_3)\, H(1/z_1\, 1/z_2\, 1/z_3)$$

The method utilizes the diagonal slice of the fourth order spectrum corresponding to $z_1 = z_2 = z_3 = z$. This implies that the corresponding

relationship between the fourth order spectrum (trispectrum) and the system function be given by:

$$G(z) = C(z,z,z) = \gamma\, H^3(z)\, H\left(1/z^3\right) \tag{4.21}$$

Although the system function could be determined from this expression by utilizing a root finding algorithm, the use of the complex cepstrum operations provides an alternative approach. The cepstrum operations employ the method of least squares which yields a solution of the roots in a minimum mean square error sense versus an iterative approach. The cumulant slice expression is obtained by transforming the tricepsturm slice into the time domain as follows:

$$\begin{aligned} g(k) &= Z^{-1}\left[G(z)\right] = Z^{-1}\left[C(z,z,z)\right] \\ &= \sum_{n,l} c(k-n-l,n,l) \end{aligned} \tag{4.22}$$

Since a relationship is sought between the cumulant slice and the complex cepstrum of the minimum and maximum phase components of the system's impulse response, the complex cepstrum of the cumulant slice needs to be determined. The complex cepstrum is determined by first obtaining the complex logarithm of the diagonal slice as follows:

$$\begin{aligned} \hat{G}(z) &= \ln G(z) \\ &= \ln \gamma + \hat{H}^3(z) + \hat{H}\left(1/z^3\right) \\ &= \ln \gamma + 3\,\hat{H}(z) + \hat{H}\left(1/z^3\right) \end{aligned} \tag{4.23}$$

The system transfer function may be further expanded in terms of the minimum and maximum phase components to yield the following:

$$\hat{G}(z) = \ln \gamma + 3\hat{I}(z) + 3\hat{O}(z) + \hat{I}\left(1/z^3\right) + \hat{O}\left(1/z^3\right) \tag{4.24}$$

The complex cepstrum is obtained from the complex logarithm by inverse Z transformation. The last two terms of the expression are observed to be expressed in terms of $z^{-3}$. The inverse Z transform of this expression is determined by considering the Z transform relation expressed as follows:

$$X(z) = \sum_n x(n)\, z^{-n}$$
$$X(z^{-3}) = \sum_n x(n)\, z^{3n} \qquad (4.25)$$
$$= \sum_n x(-n/3)\, \delta(n\%3)\, z^{-n}$$

where

$n\%3$ represents a modulo 3 operation

Thus performing the inverse Z transform on the complex logarithm yields the complex cepstrum of the slice as:

$$\hat{g}(n) = \ln \gamma\, \delta(n) + 3\,\hat{i}(n) + 3\,\hat{o}(n) + \hat{i}(-n/3)\, \delta(n\%3) + \hat{o}(-n/3)\, \delta(n\%3) \qquad (4.26)$$

where the terms containing the modulo operation represent nonzero samples only when the index is a multiple of three.

It remains now to develop a direct relationship between the complex cepstrum and the cumulant slice. This relationship may be determined by differentiating the expression for the complex cepstra of the trispectrum slice such that:

$$\frac{d\,\hat{G}(z)}{dz} = \frac{d \ln G(z)}{dz} = \frac{1}{G(z)} \frac{d\,G(z)}{dz} \qquad (4.27)$$

Multiplying through by $z\,G(z)$ and rearranging yields:

$$G(z)\, z \frac{d\,\hat{G}(z)}{dz} = z \frac{d\,G(z)}{dz} \qquad (4.28)$$

Finally applying the inverse transformation to this equation results in the desired convolutional relationship as:

$$g(n) * n\, \hat{g}(n) = n\, g(n) \qquad (4.29)$$

or equivalently:

$$\sum_{m=-\infty}^{\infty} m\, \hat{g}(m)\, g(n-m) = n\, g(n) \qquad (4.30)$$

At this point it is desirable to relate the complex cepstra to the actual minimum and maximum phase cepstral coefficients. This relationship is determined by substituting the expression relating the minimum and maximum phase cepstral coefficients to the slice's cepstral coefficients into the convolutional relationship just developed. This substitution is as follows:

$$\begin{aligned} n\,g(n) = & \sum_{m=-\infty}^{\infty} m \ln\gamma\,\delta(m)\,g(n-m) \\ & + \sum_{m=-\infty}^{\infty} m\,3\,\hat{i}(m)\,g(n-m) + \sum_{m=-\infty}^{\infty} m\,3\,\hat{o}(m)\,g(n-m) \\ & + \sum_{m=-\infty}^{\infty} m\,\hat{i}(-m/3)\,\delta(m\%3)\,g(n-m) + \sum_{m=-\infty}^{\infty} m\,\hat{o}(-m/3)\,\delta(m\%3)\,g(n-m) \end{aligned} \tag{4.31}$$

Since the fractionated last two terms are undesirable the following variable substitution can be made:

$$k = -m/3 \quad \Rightarrow \quad m = -3k \tag{4.32}$$

which then results in:

$$\begin{aligned} n\,g(n) = & \sum_{m=-\infty}^{\infty} m\,3\,\hat{i}(m)\,g(n-m) + \sum_{m=-\infty}^{\infty} m\,3\,\hat{o}(m)\,g(n-m) \\ & - \sum_{k=-\infty}^{\infty} k\,3\,\hat{i}(k)\,g(n+3k) - \sum_{k=-\infty}^{\infty} k\,3\,\hat{o}(k)\,g(n+3k) \end{aligned} \tag{4.33}$$

Grouping the minimum and maximum phase components together and changing the variables yields:

$$\begin{aligned} n\,g(n) = & \sum_{m=-\infty}^{\infty} m\,\hat{i}(m)\left[3g(n-m) - 3g(n+3m)\right] \\ & + \sum_{m=-\infty}^{\infty} m\,\hat{o}(m)\left[3g(n-m) - 3g(n+3m)\right] \end{aligned} \tag{4.34}$$

Since the minimum phase complex cepstrum is not defined for negative index values and the maximum phase complex cepstrum does not exist for positive index values the limits over which the summation is performed may be reduced such that:

$$n\,g(n) = \sum_{m=1}^{\infty} m\,\hat{i}(m)\left[3g(n-m) - 3g(n+3m)\right] - \sum_{m=1}^{\infty} m\,\hat{o}(-m)\left[3g(n+m) - 3g(n-3m)\right] \tag{4.35}$$

Finally it remains only to define the cepstral coefficients in a consistent manner as before (eqn. 3.55) such that:

$$A(k) = -k\,\hat{i}(k), \quad k > 0$$

and (eqn. 3.56)

$$B(k) = -k\,\hat{o}(-k), \quad k > 0$$

Substituting in these definitions for the actual cepstral coefficients yields:

$$-n\,g(n) = \sum_{k=1}^{\infty} A(k)\left[3g(n-k) - 3g(n+3k)\right] - \sum_{k=1}^{\infty} B(k)\left[3g(n+k) - 3g(n-3k)\right] \tag{4.36}$$

This equation forms the basis of the slice-TEA algorithm where the cepstral coefficients are used in a recursive fashion to yield the minimum and maximum phase components of the impulse response.

## *4.4 Complex slice-TEA Method*

The complex data scenario consisting of an inphase and quadrature part impacts the form of the slice-TEA method for complex data. The complex data structure affects the spectral slice definition in that two cases are possible. The first case requires no conjugates to appear in the spectral slice definition (eqn. 4.21) as follows:

$$G(z) = \gamma\,H^3(z)\,H\left(1/z^3\right)$$

The use of this form yields the same result as for the real case. The difference between the real and complex case would arise in the cumulant estimation of the output data.

The second case under consideration requires that two of the terms be conjugated such that:

$$G(z) = \gamma\, H^2(z)\, H^*\left(z^*\right)\, H^*\left(\left(1/z^*\right)^3\right) \tag{4.37}$$

The desired tricepstrum of the slice is obtained by first taking the complex logarithm of the expression yielding:

$$\begin{aligned}\hat{G}(z) &= \ln G(z) \\ &= \ln \gamma + 2 \ln H(z) + \ln H^*\left(z^*\right) + \ln H^*\left(\left(1/z^*\right)^3\right) \\ &= \ln \gamma + 2\, \hat{H}(z) + \hat{H}^*\left(z^*\right) + \hat{H}^*\left(\left(1/z^*\right)^3\right)\end{aligned} \tag{4.38}$$

From this expression, the system function is expanded into its minimum and maximum phase components such that:

$$\hat{G}(z) = \ln \gamma + 2\, \hat{I}(z) + 2\, \hat{O}(z) + \hat{I}^*\left(z^*\right) + \hat{O}^*\left(z^*\right) + \hat{I}^*\left(\left(1/z^*\right)^3\right) + \hat{O}^*\left(\left(1/z^*\right)^3\right) \tag{4.39}$$

Applying the inverse Z transform to this expression yields the tricepstrum of the slice as follows:

$$\begin{aligned}\hat{g}(n) = \ln \gamma\, \delta(n) &+ 2\, \hat{i}(n) + 2\, \hat{o}(n) + \hat{i}^*(n) + \hat{o}^*(n) \\ &+ \hat{i}^*(-n/3)\, \delta(n\%3) + \hat{o}^*(-n/3)\, \delta(n\%3)\end{aligned} \tag{4.40}$$

The delta function appearing in the last two terms represents nonzero samples only when $n$ is a multiple of three.

This expression for the tricepstrum of the slice may now be used in the previously developed relationship between the tricepstrum and the cumulant slice (eqn. 4.30) which is repeated as follows:

$$n\,g(n) = \sum_{m=-\infty}^{\infty} m\,\hat{g}(m)\,g(n-m)$$

Substituting this into the expression for the tricepstrum slice now proceeds as follows:

$$\begin{aligned} n\,g(n) = & \sum_{m=-\infty}^{\infty} m \ln \gamma\,\delta(m)\,g(n-m) \\ & + \sum_{m=-\infty}^{\infty} m\,2\,\hat{i}(m)\,g(n-m) + \sum_{m=-\infty}^{\infty} m\,2\,\hat{o}(m)\,g(n-m) \\ & + \sum_{m=-\infty}^{\infty} m\,\hat{i}^{*}(m)\,g(n-m) + \sum_{m=-\infty}^{\infty} m\,\hat{o}^{*}(m)\,g(n-m) \\ & + \sum_{m=-\infty}^{\infty} m\,\hat{i}^{*}(-m/3)\,\delta(m\%3)\,g(n-m) + \sum_{m=-\infty}^{\infty} m\,\hat{o}^{*}(-m/3)\,\delta(m\%3)\,g(n-m) \end{aligned} \tag{4.41}$$

This equation contains undesirable fractionally indexed terms which may be eliminated by the following variable substitutions (eqn. 4.32):

$$k = -m/3 \quad \Rightarrow \quad m = -3k$$

Applying the variable change results in:

$$\begin{aligned} n\,g(n) = & \sum_{m=-\infty}^{\infty} m\,2\,\hat{i}(m)\,g(n-m) + \sum_{m=-\infty}^{\infty} m\,2\,\hat{o}(m)\,g(n-m) \\ & + \sum_{m=-\infty}^{\infty} m\,\hat{i}^{*}(m)\,g(n-m) + \sum_{m=-\infty}^{\infty} m\,\hat{o}^{*}(m)\,g(n-m) \\ & + \sum_{k=-\infty}^{\infty} -3k\,\hat{i}^{*}(k)\,g(n+3k) + \sum_{m=-\infty}^{\infty} -3k\,\hat{o}^{*}(k)\,g(n+3k) \end{aligned} \tag{4.42}$$

Since the minimum phase complex cepstrum is not defined for negative index values and the maximum phase complex cepstrum is not defined for positive index values, the following expression results:

$$\begin{aligned} n\,g(n) = & \sum_{m=1}^{\infty} m\,2\,\hat{i}(m)\,g(n-m) - \sum_{m=1}^{\infty} m\,2\,\hat{o}(-m)\,g(n+m) \\ & + \sum_{m=1}^{\infty} m\,\hat{i}^{*}(m)\,g(n-m) - \sum_{m=1}^{\infty} m\,\hat{o}^{*}(-m)\,g(n+m) \\ & - \sum_{m=1}^{\infty} 3m\,\hat{i}^{*}(m)\,g(n+3m) + \sum_{m=1}^{\infty} 3m\,\hat{o}^{*}(-m)\,g(n-3m) \end{aligned} \tag{4.43}$$

Further rearranging and grouping results in:

$$n\,g(n) = \sum_{m=1}^{\infty} m\,2\,\hat{i}(m)\,g(n-m) - \sum_{m=1}^{\infty} m\,2\,\hat{o}(-m)\,g(n+m)$$
$$+\sum_{m=1}^{\infty} m\,\hat{i}^{*}(m)\left[g(n-m) - 3g(n+3m)\right] \qquad (4.44)$$
$$-\sum_{m=1}^{\infty} m\,\hat{o}^{*}(-m)\left[g(n+m) - 3g(n-3m)\right]$$

The final step remaining is to characterize the tricepstrum slice in terms of the individual pole and zero locations of the system's impulse response. These individual pole and zero locations are defined in terms of cepstral coefficients (eqn. 3.55) as follows:

$$A(k) = -k\,\hat{i}(k), \quad k > 0$$

and (eqn. 3.56)

$$B(k) = -k\,\hat{o}(-k), \quad k > 0$$

Substituting these cepstral coefficients for each of the minimum and maximum phase complex cepstra results in the following desired form:

$$-n\,g(n) = \sum_{k=1}^{\infty} 2\,A(k)\,g(n-k) - \sum_{k=1}^{\infty} 2\,B(k)\,g(n+k)$$
$$+\sum_{k=1}^{\infty} A^{*}(k)\left[g(n-k) - 3g(n+3k)\right] \qquad (4.45)$$
$$-\sum_{k=1}^{\infty} B^{*}(k)\left[g(n+k) - 3g(n-3k)\right]$$

This equation represents the slice-TEA for complex data with the assumption that the cumulant is estimated using two conjugated terms. It is observed that if the conjugated terms of the cepstral coefficients are equated to the unconjugated equivalents, the result will yield the same solution as for the unconjugated or real case.

## *4.5 Representation in terms of Magnitude and Phase*

An alternative to the representation of the system transfer function in terms of its minimum and maximum phase components is the equivalent magnitude and phase representation. This equivalent representation suggests an alternative method to system identification in terms of magnitude and phase. The decomposition into the magnitude and phase components is demonstrated in the following factorization:

$$H(z) = \underbrace{\left[H(z)\,H^*(1/z^*)\right]^{1/2}}_{\text{magnitude}} \quad \underbrace{\left[\frac{H(z)}{H^*(1/z^*)}\right]^{1/2}}_{\text{phase}} \tag{4.46}$$

Using this approach the magnitude and phase identification and ultimately the equalization can be treated independently. This is advantageous because it allows flexibility in the equalizer configuration. The optimization of each component is then possible in terms of computational requirements and equalizer placement resulting in an improvement in performance.

The independence assumption of the magnitude and phase functions allows one to consider the two components separately. In the case of the magnitude identification this approach is suggested as second order statistics are adequate to recover the magnitude information. Since the magnitude component is composed of all the minimum and maximum phase equivalents it is not necessary to distinguish between them. Thus it is not necessary to use higher order statistics in order to recover magnitude information.

Knowledge of higher order statistics is required in order to identify the phase component of the system since second order statistics are not adequate to identify mixed phase systems. The phase term is seen to display inverse complex conjugate symmetry between the numerator and denominator terms which is a defining characteristic of the all-pass structure. All-pass filters are therefore used to implement phase filters as all frequencies will be passed without any amplitude distortion. Since second order statistics are not adequate to resolve the pole and zero locations of a system, higher order statistics which are not phase blind offer a solution to this problem. Finally applying higher order statistics to just the phase term should also reduce some of the variance experienced in methods which effectively determine both the magnitude and phase jointly as in algorithms which identify maximum and minimum phase components.

## *4.6 Whitening Filters*

Whitening is a term used to indicate a process which uncorrelates signal samples in the time domain [1]. In such a sequence the value of any given sample is uncorrelated with past and future values of the sequence. In the frequency domain this corresponds to the adjustment or equalization of the magnitude of the spectral components so that the spectrum appears flat. Since all frequencies are present, the signal appears "white" hence the term whitening. The whitening process corresponds to amplitude equalization.

A stationary stochastic data sequence may be whitened by the use of a prediction error filter. The design of such a filter may be approached from the idea of modeling the data so that the output of the designed filter corresponds to an error process [1]. The performance of the whitening filter is dependent upon the order of the filter. As this model order is increased the correlation between successive signal samples of the output is reduced until a white error process is achieved. This principle is also applied in linear predictive coding for transmission and reception of digitized speech.

An alternative approach to the design of a whitening filter is to use convolutional homomorphic techniques applied to second order statistics. Second order statistics are adequate to identify the minimum phase equivalents required in amplitude equalization. In this alternative method a relationship is sought between the autocorrelation function and its complex cepstrum or equivalently the power cepstrum. From the power cepstrum the equivalent minimum and maximum phase components are easily obtained [13]. The use of this direct relationship between the autocorrelation function and the power cepstrum avoids the use of the complex logarithm and the Fourier transform. As a result phase unwrapping problems and aliasing are avoided.

The power cepstrum and autocorrelation relationship is established by considering the autocorrelation and its transform shown as:

$$R(z) = Z[r(\tau)] \tag{4.47}$$

The power cepstrum is defined as the inverse discrete Fourier transform of the complex logarithm of the discrete spectrum expressed as follows:

$$\hat{r}(m) = Z^{-1}[\ln R(z)] = Z^{-1}\left[\hat{R}(z)\right] \tag{4.48}$$

The power cepstrum may also be obtained directly from the autocorrelation function and as such avoids the use of the complex logarithm and the discrete Fourier transform with the associated phase unwrapping problems and aliasing. This relationship is developed by differentiating the complex logarithm of the spectrum as follows:

$$\frac{d}{dz}\ln R(z) = \frac{d}{dz}\hat{R}(z) = \frac{1}{R(z)}\frac{d}{dz}R(z) \tag{4.49}$$

Multiplying through by $z\,R(z)$ yields:

$$z\frac{d}{dz}R(z) = z\,R(z)\frac{d}{dz}\hat{R}(z) \tag{4.50}$$

It remains to transform this equation into the time domain which will result in:

$$-k\,r(k) = \left(-k\,\hat{r}(k)\right) * r(k) \tag{4.51}$$

where:

$k$ is the lag value between samples

$r(k)$ is the autocorrelation function

$\hat{r}(k)$ is the power cepstrum

This equation thus yields a relationship between a sequence's power cepstrum and its autocorrelation.

The convolutional equation may be written as a convolutional sum as follows:

$$k\,r(k) = \sum_{n=-\infty}^{\infty} n\,\hat{r}(n)\,r(k-n) \tag{4.52}$$

It remains to obtain an expression for the power cepstrum in terms of the system function and its minimum and maximum phase components. This expression is obtained by first considering the discrete power spectrum which is defined at the output of a system fed by a white input as:

$$R(z) = \sigma^2\,H(z)\,H^*\left(1/z^*\right) \tag{4.53}$$

The system transfer function written in terms of minimum and maximum phase components which are just complex conjugate inverses of each other and may be expressed as:

$$H(z) = I(z)\, O(z) \tag{4.54}$$

The power cepstrum is obtained by first applying the complex logarithm to the power spectrum as follows:

$$\begin{aligned} \hat{R}(z) &= \ln R(z) = \ln\left[\sigma^2\, H(z)\, H^*\left(1/z^*\right)\right] \\ &= \ln\left[\sigma^2 I(z)\, I^*\left(1/z^*\right) O(z)\, O^*\left(1/z^*\right)\right] \\ &= \ln \sigma^2 + \ln I(z) + \ln I^*\left(1/z^*\right) + \ln O(z) + \ln O^*\left(1/z^*\right) \\ &= \ln \sigma^2 + \hat{I}(z) + \hat{I}^*\left(1/z^*\right) + \hat{O}(z) + \hat{O}^*\left(1/z^*\right) \end{aligned} \tag{4.55}$$

The inverse Z transform of this expression yields the desired power cepstrum as:

$$\hat{r}(n) = \ln \sigma^2\, \delta(n) + \hat{i}(n) + \hat{o}(n) + \hat{i}^*(-n) + \hat{o}^*(-n) \tag{4.56}$$

It is observed that for every minimum phase pole or zero of the system's transfer function, a corresponding maximum phase pole or zero exists. Conversely for every maximum phase pole or zero appearing, a corresponding minimum phase pole or zero is observed. Furthermore it is not possible to determine whether a particular component results from an actual system pole or zero or from a reflection about the unit circle.

Since the whitening filter to be constructed is based on the relationship between the power cepstrum and the autocorrelation, it remains to substitute the power cepstrum expression into the previously obtained convolutional relationship as:

$$\begin{aligned} k\, r(k) = &\sum_{n=-\infty}^{\infty} n\, \hat{i}(n)\, r(k-n) + \sum_{n=-\infty}^{\infty} n\, \hat{o}(n)\, r(k-n) \\ &+ \sum_{n=-\infty}^{\infty} n\, \hat{i}^*(-n)\, r(k-n) + \sum_{n=-\infty}^{\infty} n\, \hat{o}^*(-n)\, r(k-n) \end{aligned} \tag{4.57}$$

The minimum phase sequence is a right hand sequence while the maximum phase sequence is a left hand sequence. The limits of summation may therefore be reduced to yield:

$$k\,r(k) = \sum_{n=1}^{\infty} n\,\hat{i}(n)\,r(k-n) - \sum_{n=1}^{\infty} n\,\hat{o}(-n)\,r(k+n) - \sum_{n=1}^{\infty} n\,\hat{i}^{*}(n)\,r(k+n) + \sum_{n=1}^{\infty} n\,\hat{o}^{*}(-n)\,r(k-n) \tag{4.58}$$

This expression may be altered further so that the minimum and maximum phase components and their equivalents are expressed in terms of the pole and zero locations. This is done by substituting the minimum and maximum phase terms by the following cepstral coefficients (eqn. 3.55) defined as:

$$A(k) = -k\,\hat{i}(k), \quad k > 0$$

and (eqn. 3.56)

$$B(k) = -k\,\hat{o}(-k), \quad k > 0$$

This substitution yields:

$$-k\,r(k) = \sum_{n=1}^{\infty} A(n)\,r(k-n) - \sum_{n=1}^{\infty} B(n)\,r(k+n) - \sum_{n=1}^{\infty} A^{*}(n)\,r(k+n) + \sum_{n=1}^{\infty} B^{*}(n)\,r(k-n) \tag{4.59}$$

This expression may be simplified by regrouping terms to give:

$$-k\,r(k) = \sum_{n=1}^{\infty} \left(A(n) + B^{*}(n)\right) r(k-n) - \sum_{n=1}^{\infty} \left(A^{*}(n) + B(n)\right) r(k+n) \tag{4.60}$$

From this expression it is clearly observed that the sum of the cepstral coefficients specifies the amplitude of the system as stated by Petropulu & Nikias [8]. A new amplitude cepstral coefficient representing the minimum phase equivalent may be thus be defined such that:

$$S(k) = A(k) + B^{*}(k), \quad k > 0 \tag{4.61}$$

Substituting this definition into the equation above yields:

$$-k\,r(k) = \sum_{n=1}^{\infty} S(n)\,r(k-n) - \sum_{n=1}^{\infty} S^{*}(n)\,r(k+n) \tag{4.62}$$

This equation forms the whitening filter where the unknown cepstral coefficients are solved for by using the method of least squares. The impulse response of the minimum phase equivalent is then obtained by recursion using these amplitude or minimum phase equivalent cepstral coefficients.

## *4.7 Phase Equalization: Phase-TEA and Phase slice-TEA methods*

The result of cascading the transmission channel with the amplitude equalizer yields ideally a system which can be modeled as an all-pass system. Since an all-pass transfer function has a unity power spectrum, second order spectra is not adequate to resolve pole and zero locations of the system. Higher order spectra offers a solution to this problem and with the application of homomorphic techniques, desirable linear solutions are obtained.

The TEA and slice-TEA methods both use higher order statistics to determine the minimum and maximum phase components of the system's impulse response. Knowledge of these components is considered to be adequate information to construct the phase of the system. Nikias and Liu [31] have shown that the subtraction of the maximum phase cepstral coefficients from the minimum phase cepstral coefficients will give the phase cepstral coefficients.

The identification of the phase begins by considering the phase of the system transfer function given as:

$$H_{ap}(z) = \left[\frac{H(z)}{H^*(1/z^*)}\right]^{1/2} = \left[\frac{I(z)\,O(z)}{I^*(1/z^*)\,O^*(1/z^*)}\right]^{1/2} \tag{4.63}$$

where

$H(z)$ represents the system transfer function

$H^*(1/z^*)$ represents the complex conjugate inverse of the system function

From this expression the complex logarithm of the phase is determined such that:

$$\begin{aligned}\hat{H}_{ap}(z) &= \ln H_{ap}(z) \\ &= \frac{1}{2}\ln I(z) + \frac{1}{2}\ln O(z) - \frac{1}{2}\ln I^*(1/z^*) - \frac{1}{2}\ln O^*(1/z^*) \\ &= \frac{1}{2}\hat{I}(z) + \frac{1}{2}\hat{O}(z) - \frac{1}{2}\hat{I}^*(1/z^*) - \frac{1}{2}\hat{O}^*(1/z^*)\end{aligned} \tag{4.64}$$

The complex cepstrum of the system's phase is determined by inverse Z transformation to yield:

$$\hat{h}_{ap}(k) = \frac{1}{2}\hat{i}(k) + \frac{1}{2}\hat{o}(k) - \frac{1}{2}\hat{i}^*(-k) - \frac{1}{2}\hat{o}^*(-k) \tag{4.65}$$

The complex cepstra of the minimum and the maximum phase components are commonly expressed in terms of the minimum and maximum phase cepstral coefficients (eqn. 3.55) defined as:

$$A(k) = -k\,\hat{i}(k), \quad k > 0$$

and (eqn. 3.56)

$$B(k) = -k\,\hat{o}(-k), \quad k > 0$$

Substitution of these cepstral coefficients into the previous expression results in:

$$-k\,\hat{h}_{ap}(k) = \frac{1}{2}A(k) - \frac{1}{2}B(-k) + \frac{1}{2}A^*(-k) - \frac{1}{2}B^*(k), \quad k > 0 \tag{4.66}$$

At this point a symmetry between the minimum and maximum phase cepstral coefficients is observed. Grouping of the positive indexed terms and of the negative indexed terms leads to a definition of the phase cepstral coefficient given by:

$$D(k) = A(k) - B^*(k), \quad k > 0 \tag{4.67}$$

Substituting the cepstral coefficient of the phase for the minimum and maximum phase cepstral coefficients yields:

$$-k\,\hat{h}_{ap}(k)=\frac{1}{2}D(k)-\frac{1}{2}D^{*}(-k),\qquad k>0 \tag{4.68}$$

It is thus shown that the phase of the system is fully described by the phase cepstral coefficients. It is therefore possible to construct a phase equalizer from the minimum and maximum phase cepstral coefficients using the TEA or slice-TEA methods. An equalizer constructed in this manner shall conveniently be referred to as the "phase-TEA" method.

## *4.8 Phase Equalization: All-pass TEA cepstral equation derivation*

The phase of a system may be obtained by using either the TEA or slice-TEA methods. In both of the methods, the subtraction of the minimum from the maximum phase cepstral coefficients will yield the phase cepstral coefficients from which the phase may be calculated. This approach is suboptimum because the inherent symmetry of the all-pass structure is not taken into account. A method is therefore proposed which applies a constraint to ensure that the complex conjugate symmetry between the numerator and denominator is maintained. This will better model the all-pass structure with an anticipation of an improvement in the positional accuracy of the pole and zero locations. Correspondingly there is an expectation of superior equalizer performance.

An all-pass system is defined as having the numerator being the complex conjugate inverse of the denominator. In the case of an all-pass transfer function the phase of a system's function is represented as follows (eqn. 4.63):

$$H_{ap}(z)=\left[\frac{H(z)}{H^{*}(1/z^{*})}\right]^{1/2}=\left[\frac{I(z)\,O(z)}{I^{*}(1/z^{*})\,O^{*}(1/z^{*})}\right]^{1/2}$$

where

$H(z)$ represents the system transfer function

$H^*(1/z^*)$ represents the complex conjugate inverse of the system function

The objective of this approach is therefore to apply cepstral operations to the phase of the system function. This begins by considering the cumulant of the phase and its relationship to the impulse response given by:

$$c_{ap}(\tau_1,\tau_2,\tau_3)=\gamma\sum_{k=-\infty}^{\infty}h_{ap}(k)\,h_{ap}(k+\tau_1)\,h_{ap}(k+\tau_2)\,h_{ap}(k+\tau_3) \tag{4.69}$$

This relationship is valid whenever real data is assumed or whenever the unconjugated complex data definition is assumed.

Transforming the expression into the Z domain results in:

$$C_{ap}(z_1,z_2,z_3)=\gamma\,H_{ap}(z_1)\,H_{ap}(z_2)\,H_{ap}(z_3)\,H_{ap}(1/z_1\,1/z_2\,1/z_3) \tag{4.70}$$

Substituting in the all-pass system function yields:

$$C_{ap}(z_1,z_2,z_3)$$

$$=\gamma\left[\frac{H(z_1)}{H^*(1/z_1^*)}\right]^{1/2}\left[\frac{H(z_2)}{H^*(1/z_2^*)}\right]^{1/2}\left[\frac{H(z_3)}{H^*(1/z_3^*)}\right]^{1/2}\left[\frac{H(1/z_1\,1/z_2\,1/z_3)}{H^*(z_1^*z_2^*z_3^*)}\right]^{1/2} \tag{4.71}$$

The complex cepstrum is determined by first taking the complex logarithm to yield:

$$\hat{C}_{ap}(z_1,z_2,z_3)=\ln C_{ap}(z_1,z_2,z_3)$$

$$=\ln\gamma+\hat{H}_{ap}(z_1)+\hat{H}_{ap}(z_2)+\hat{H}_{ap}(z_3)+\hat{H}_{ap}(1/z_1\,1/z_2\,1/z_3)$$

$$=\ln\gamma+\frac{1}{2}\hat{H}(z_1)-\frac{1}{2}\hat{H}^*(1/z_1^*)+\frac{1}{2}\hat{H}(z_2)-\frac{1}{2}\hat{H}^*(1/z_2^*)$$

$$+\frac{1}{2}\hat{H}(z_3)-\frac{1}{2}\hat{H}^*(1/z_3^*)+\frac{1}{2}\hat{H}(1/z_1\,1/z_2\,1/z_3)-\frac{1}{2}\hat{H}^*(z_1^*z_2^*z_3^*) \tag{4.72}$$

Further expansion of the system function into the minimum and maximum phase components gives:

$$\begin{aligned}\hat{C}_{ap}(z_1,z_2,z_3) = \ln\gamma &+ \frac{1}{2}\hat{I}(z_1) - \frac{1}{2}\hat{I}^*(1/z_1^*) + \frac{1}{2}\hat{O}(z_1) - \frac{1}{2}\hat{O}^*(1/z_1^*) \\ &+ \frac{1}{2}\hat{I}(z_2) - \frac{1}{2}\hat{I}^*(1/z_2^*) + \frac{1}{2}\hat{O}(z_2) - \frac{1}{2}\hat{O}^*(1/z_2^*) \\ &+ \frac{1}{2}\hat{I}(z_3) - \frac{1}{2}\hat{I}^*(1/z_3^*) + \frac{1}{2}\hat{O}(z_3) - \frac{1}{2}\hat{O}^*(1/z_3^*) \\ &+ \frac{1}{2}\hat{I}(1/z_1\,1/z_2\,1/z_3) - \frac{1}{2}\hat{I}^*(z_1^*z_2^*z_3^*) \\ &+ \frac{1}{2}\hat{O}(1/z_1\,1/z_2\,1/z_3) - \frac{1}{2}\hat{O}^*(z_1^*z_2^*z_3^*)\end{aligned} \tag{4.73}$$

Inverse Z transformation of this expression yields the desired complex cepstrum of the phase and is:

$$\begin{aligned}\hat{c}_{ap}(m,n,l) = \ln\gamma\,&\delta(m)\,\delta(n)\,\delta(l) \\ &+ \frac{1}{2}\hat{i}(m)\,\delta(n)\,\delta(l) + \frac{1}{2}\hat{o}(m)\,\delta(n)\,\delta(l) \\ &- \frac{1}{2}\hat{i}^*(-m)\,\delta(n)\,\delta(l) - \frac{1}{2}\hat{o}^*(-m)\,\delta(n)\,\delta(l) \\ &+ \frac{1}{2}\hat{i}(n)\,\delta(m)\,\delta(l) + \frac{1}{2}\hat{o}(n)\,\delta(m)\,\delta(l) \\ &- \frac{1}{2}\hat{i}^*(-n)\,\delta(m)\,\delta(l) - \frac{1}{2}\hat{o}^*(-n)\,\delta(m)\,\delta(l) \\ &+ \frac{1}{2}\hat{i}(l)\,\delta(m)\,\delta(n) + \frac{1}{2}\hat{o}(l)\,\delta(m)\,\delta(n) \\ &- \frac{1}{2}\hat{i}^*(-l)\,\delta(m)\,\delta(n) - \frac{1}{2}\hat{o}^*(-l)\,\delta(m)\,\delta(n) \\ &+ \frac{1}{2}\hat{i}(-m)\,\delta(m-n)\,\delta(n-l) + \frac{1}{2}\hat{o}(-m)\,\delta(m-n)\,\delta(n-l) \\ &- \frac{1}{2}\hat{i}^*(m)\,\delta(m-n)\,\delta(n-l) - \frac{1}{2}\hat{o}^*(m)\,\delta(m-n)\,\delta(n-l)\end{aligned} \tag{4.74}$$

The complex cepstrum of the phase having been determined may now be used in the convolutional relationship between the complex cepstra and the cumulant. The cepstral relationship (eqn. 4.6) is recalled as being:

$$m\,c(m,n,l) = \sum_{k=-\infty}^{\infty}\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} k\,\hat{c}(k,i,j)\,c(m-k,n-i,l-j)$$

The complex cepstrum of the phase is substituted into the cepstral relationship to yield:

$$\begin{aligned} m\,c(m,n,l) = & \sum_{k=-\infty}^{\infty} \frac{k}{2}\hat{i}(k)\,c(m-k,n,l) + \sum_{k=-\infty}^{\infty} \frac{k}{2}\hat{o}(k)\,c(m-k,n,l) \\ & - \sum_{k=-\infty}^{\infty} \frac{k}{2}\hat{i}^{*}(-k)\,c(m-k,n,l) - \sum_{k=-\infty}^{\infty} \frac{k}{2}\hat{o}^{*}(-k)\,c(m-k,n,l) \\ & + \sum_{k=-\infty}^{\infty} \frac{k}{2}\hat{i}(-k)\,c(m-k,n-k,l-k) \\ & + \sum_{k=-\infty}^{\infty} \frac{k}{2}\hat{o}(-k)\,c(m-k,n-k,l-k) \\ & - \sum_{k=-\infty}^{\infty} \frac{k}{2}\hat{i}^{*}(k)\,c(m-k,n-k,l-k) \\ & - \sum_{k=-\infty}^{\infty} \frac{k}{2}\hat{o}^{*}(k)\,c(m-k,n-k,l-k) \end{aligned} \tag{4.75}$$

Further simplification may be brought about by changing the summation limits to correspond to the regions of support of the minimum and maximum phase complex cepstra thus yielding:

$$\begin{aligned} m\,c(m,n,l) = & \sum_{k=1}^{\infty} \frac{k}{2}\hat{i}(k)\,c(m-k,n,l) - \sum_{k=1}^{\infty} \frac{k}{2}\hat{o}(-k)\,c(m+k,n,l) \\ & + \sum_{k=1}^{\infty} \frac{k}{2}\hat{i}^{*}(k)\,c(m+k,n,l) - \sum_{k=1}^{\infty} \frac{k}{2}\hat{o}^{*}(-k)\,c(m-k,n,l) \\ & - \sum_{k=1}^{\infty} \frac{k}{2}\hat{i}(k)\,c(m+k,n+k,l+k) \\ & + \sum_{k=1}^{\infty} \frac{k}{2}\hat{o}(-k)\,c(m-k,n-k,l-k) \\ & - \sum_{k=1}^{\infty} \frac{k}{2}\hat{i}^{*}(k)\,c(m-k,n-k,l-k) \\ & + \sum_{k=1}^{\infty} \frac{k}{2}\hat{o}^{*}(-k)\,c(m+k,n+k,l+k) \end{aligned} \tag{4.76}$$

The complex cepstra of the minimum and maximum phase components are commonly expressed in terms of the minimum and maximum phase cepstral coefficients (eqn. 3.55) given as:

$$A(k) = -k\,\hat{i}(k), \quad k > 0$$

and (eqn. 3.56)

$$B(k) = -k\,\hat{o}(-k), \quad k > 0$$

Further replacement of the minimum and maximum phase complex cepstra terms by the minimum and maximum phase cepstral coefficients just defined yields:

$$\begin{aligned} -m\,c(m,n,l) = &\sum_{k=1}^{\infty}\frac{1}{2}A(k)\,c(m-k,n,l) - \sum_{k=1}^{\infty}\frac{1}{2}B(k)\,c(m+k,n,l) \\ &+\sum_{k=1}^{\infty}\frac{1}{2}A^{*}(k)\,c(m+k,n,l) - \sum_{k=1}^{\infty}\frac{1}{2}B^{*}(k)\,c(m-k,n,l) \\ &-\sum_{k=1}^{\infty}\frac{1}{2}A(k)\,c(m+k,n+k,l+k) \\ &+\sum_{k=1}^{\infty}\frac{1}{2}B(k)\,c(m-k,n-k,l-k) \\ &-\sum_{k=1}^{\infty}\frac{1}{2}A^{*}(k)\,c(m-k,n-k,l-k) \\ &+\sum_{k=1}^{\infty}\frac{1}{2}B^{*}(k)\,c(m+k,n+k,l+k) \end{aligned} \tag{4.77}$$

In this equation a symmetry of the various coefficients is observed. This suggests a regrouping of the terms such that:

$$\begin{aligned} -m\,c(m,n,l) = &\sum_{k=1}^{\infty}\frac{1}{2}\left(A(k) - B^{*}(k)\right)c(m-k,n,l) \\ &-\sum_{k=1}^{\infty}\frac{1}{2}\left(A^{*}(k) - B(k)\right)c(m-k,n-k,l-k) \\ &+\sum_{k=1}^{\infty}\frac{1}{2}\left(A^{*}(k) - B(k)\right)c(m+k,n,l) \\ &-\sum_{k=1}^{\infty}\frac{1}{2}\left(A(k) - B^{*}(k)\right)c(m+k,n+k,l+k) \end{aligned} \tag{4.78}$$

This regrouping suggests a new cepstral coefficient in terms of the phase which may be defined as:

$$D(k) = A(k) - B^{*}(k), \quad k > 0 \tag{4.79}$$

Replacing the minimum and maximum phase cepstral coefficients by the phase cepstral coefficient into the previous expression yields:

$$\begin{aligned} -m\,c(m,n,l) = &\sum_{k=1}^{\infty}\frac{1}{2}D(k)\,c(m-k,n,l) \\ &-\sum_{k=1}^{\infty}\frac{1}{2}D^{*}(k)\,c(m-k,n-k,l-k) \\ &+\sum_{k=1}^{\infty}\frac{1}{2}D^{*}(k)\,c(m+k,n,l) \\ &-\sum_{k=1}^{\infty}\frac{1}{2}D(k)\,c(m+k,n+k,l+k) \end{aligned} \tag{4.80}$$

This equation may be rearranged to yield:

$$\begin{aligned} -m\,c(m,n,l) = &\sum_{k=1}^{\infty}\frac{1}{2}D(k)\left[c(m-k,n,l)-c(m+k,n+k,l+k)\right] \\ &-\sum_{k=1}^{\infty}\frac{1}{2}D^{*}(k)\left[c(m-k,n-k,l-k)-c(m+k,n,l)\right] \end{aligned} \tag{4.81}$$

This expression forms the all-pass TEA method and is valid for real data or for complex data under the assumption of no conjugated terms in the definition of the cumulant.

## *4.9 Complex All-pass TEA method*

The all-pass TEA method will be somewhat different for the case of complex data where the cumulant as a function of the impulse response is:

$$c_{ap}(\tau_1,\tau_2,\tau_3) = \gamma \sum_{k=-\infty}^{\infty} h_{ap}^{*}(k)\,h_{ap}(k+\tau_1)\,h_{ap}^{*}(k+\tau_2)\,h_{ap}(k+\tau_3) \tag{4.82}$$

The trispectrum is related to the system transfer function in the following manner:

$$C_{ap}(z_1,z_2,z_3) = \gamma\,H_{ap}(z_1)\,H_{ap}^{*}(z_2^{*})\,H_{ap}(z_3)\,H_{ap}^{*}(1/z_1^{*}\,1/z_2^{*}\,1/z_3^{*}) \tag{4.83}$$

Substituting in the all-pass system function yields:

$$C_{ap}(z_1,z_2,z_3)$$
$$= \gamma \left[\frac{H(z_1)}{H^*(1/z_1^*)}\right]^{1/2}\left[\frac{H^*(z_2^*)}{H(1/z_2)}\right]^{1/2}\left[\frac{H(z_3)}{H^*(1/z_3^*)}\right]^{1/2}\left[\frac{H^*(1/z_1^*\,1/z_2^*\,1/z_3^*)}{H(z_1z_2z_3)}\right]^{1/2} \tag{4.84}$$

The next step is to determine the complex cepstrum by first taking the complex logarithm to yield:

$$\begin{aligned}\hat{C}_{ap}(z_1,z_2,z_3) &= \ln C_{ap}(z_1,z_2,z_3)\\ &= \ln\gamma + \hat{H}_{ap}(z_1) + \hat{H}_{ap}^*(z_2^*) + \hat{H}_{ap}(z_3) + \hat{H}_{ap}^*(1/z_1^*\,1/z_2^*\,1/z_3^*)\\ &= \ln\gamma + \frac{1}{2}\hat{H}(z_1) - \frac{1}{2}\hat{H}^*(1/z_1^*) + \frac{1}{2}\hat{H}^*(z_2^*) - \frac{1}{2}\hat{H}(1/z_2)\\ &\quad + \frac{1}{2}\hat{H}(z_3) - \frac{1}{2}\hat{H}^*(1/z_3^*) + \frac{1}{2}\hat{H}^*(1/z_1^*\,1/z_2^*\,1/z_3^*) - \frac{1}{2}\hat{H}(z_1z_2z_3)\end{aligned} \tag{4.85}$$

Since it is desirable to express the system function in terms of the complex logarithm of the minimum and maximum phase components, the following expansion results in:

$$\begin{aligned}\hat{C}_{ap}(z_1,z_2,z_3) = \ln\gamma &+ \frac{1}{2}\hat{I}(z_1) - \frac{1}{2}\hat{I}^*(1/z_1^*) + \frac{1}{2}\hat{O}(z_1) - \frac{1}{2}\hat{O}^*(1/z_1^*)\\ &+ \frac{1}{2}\hat{I}^*(z_2^*) - \frac{1}{2}\hat{I}(1/z_2) + \frac{1}{2}\hat{O}^*(z_2^*) - \frac{1}{2}\hat{O}(1/z_2)\\ &+ \frac{1}{2}\hat{I}(z_3) - \frac{1}{2}\hat{I}^*(1/z_3^*) + \frac{1}{2}\hat{O}(z_3) - \frac{1}{2}\hat{O}^*(1/z_3^*)\\ &+ \frac{1}{2}\hat{I}^*(1/z_1^*\,1/z_2^*\,1/z_3^*) - \frac{1}{2}\hat{I}(z_1z_2z_3)\\ &+ \frac{1}{2}\hat{O}^*(1/z_1^*\,1/z_2^*\,1/z_3^*) - \frac{1}{2}\hat{O}(z_1z_2z_3)\end{aligned} \tag{4.86}$$

Inverse Z transformation of this expression yields the desired complex cepstrum of the phase and is:

$$
\begin{aligned}
\hat{c}_{ap}(m,n,l) = \ln \gamma\, \delta(m)\, \delta(n)\, \delta(l) \\
&+\frac{1}{2}\hat{i}(m)\, \delta(n)\, \delta(l) + \frac{1}{2}\hat{o}(m)\, \delta(n)\, \delta(l) \\
&-\frac{1}{2}\hat{i}^{*}(-m)\, \delta(n)\, \delta(l) - \frac{1}{2}\hat{o}^{*}(-m)\, \delta(n)\, \delta(l) \\
&+\frac{1}{2}\hat{i}^{*}(n)\, \delta(m)\, \delta(l) + \frac{1}{2}\hat{o}^{*}(n)\, \delta(m)\, \delta(l) \\
&-\frac{1}{2}\hat{i}(-n)\, \delta(m)\, \delta(l) - \frac{1}{2}\hat{o}(-n)\, \delta(m)\, \delta(l) \\
&+\frac{1}{2}\hat{i}(l)\, \delta(m)\, \delta(n) + \frac{1}{2}\hat{o}(l)\, \delta(m)\, \delta(n) \\
&-\frac{1}{2}\hat{i}^{*}(-l)\, \delta(m)\, \delta(n) - \frac{1}{2}\hat{o}^{*}(-l)\, \delta(m)\, \delta(n) \\
&+\frac{1}{2}\hat{i}^{*}(-m)\, \delta(m-n)\, \delta(n-l) + \frac{1}{2}\hat{o}^{*}(-m)\, \delta(m-n)\, \delta(n-l) \\
&-\frac{1}{2}\hat{i}(m)\, \delta(m-n)\, \delta(n-l) - \frac{1}{2}\hat{o}(m)\, \delta(m-n)\, \delta(n-l)
\end{aligned}
\tag{4.87}
$$

The complex cepstrum of the phase having been determined may now be used in the convolutional relationship between the complex cepstra and the cumulant. The cepstral relationship (eqn. 4.6) is recalled as being:

$$
m\, c(m,n,l) = \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} k\, \hat{c}(k,i,j)\, c(m-k, n-i, l-j)
$$

The complex cepstrum of the phase is substituted into the cepstral relationship to yield:

$$
\begin{aligned}
m\,c(m,n,l) = &\sum_{k=-\infty}^{\infty} \frac{k}{2}\,\hat{i}(k)\,c(m-k,n,l) + \sum_{k=-\infty}^{\infty} \frac{k}{2}\,\hat{o}(k)\,c(m-k,n,l) \\
&- \sum_{k=-\infty}^{\infty} \frac{k}{2}\,\hat{i}^{*}(-k)\,c(m-k,n,l) - \sum_{k=-\infty}^{\infty} \frac{k}{2}\,\hat{o}^{*}(-k)\,c(m-k,n,l) \\
&+ \sum_{k=-\infty}^{\infty} \frac{k}{2}\,\hat{i}^{*}(-k)\,c(m-k,n-k,l-k) \\
&+ \sum_{k=-\infty}^{\infty} \frac{k}{2}\,\hat{o}^{*}(-k)\,c(m-k,n-k,l-k) \\
&- \sum_{k=-\infty}^{\infty} \frac{k}{2}\,\hat{i}(k)\,c(m-k,n-k,l-k) \\
&- \sum_{k=-\infty}^{\infty} \frac{k}{2}\,\hat{o}(k)\,c(m-k,n-k,l-k)
\end{aligned}
\tag{4.88}
$$

Simplification may be brought about by changing the summation limits to correspond to the regions of support of the minimum and maximum phase complex cepstra thus yielding:

$$
\begin{aligned}
m\,c(m,n,l) = &\sum_{k=1}^{\infty} \frac{k}{2}\,\hat{i}(k)\,c(m-k,n,l) - \sum_{k=1}^{\infty} \frac{k}{2}\,\hat{o}(-k)\,c(m+k,n,l) \\
&+ \sum_{k=1}^{\infty} \frac{k}{2}\,\hat{i}^{*}(k)\,c(m+k,n,l) - \sum_{k=1}^{\infty} \frac{k}{2}\,\hat{o}^{*}(-k)\,c(m-k,n,l) \\
&- \sum_{k=1}^{\infty} \frac{k}{2}\,\hat{i}^{*}(k)\,c(m+k,n+k,l+k) \\
&+ \sum_{k=1}^{\infty} \frac{k}{2}\,\hat{o}^{*}(-k)\,c(m-k,n-k,l-k) \\
&- \sum_{k=1}^{\infty} \frac{k}{2}\,\hat{i}(k)\,c(m-k,n-k,l-k) \\
&+ \sum_{k=1}^{\infty} \frac{k}{2}\,\hat{o}(-k)\,c(m+k,n+k,l+k)
\end{aligned}
\tag{4.89}
$$

The complex cepstra of the minimum and maximum phase components are commonly expressed in terms of the minimum and maximum phase cepstral coefficients (eqn. 3.55) given as:

$$A(k) = -k\,\hat{i}(k), \qquad k > 0$$

and (eqn. 3.56)

$$B(k) = -k\,\hat{o}(-k), \quad k > 0$$

Further replacement of the minimum and maximum phase complex cepstra terms by the minimum and maximum phase cepstral coefficients just defined yields:

$$\begin{aligned} -m\,c(m,n,l) = &\sum_{k=1}^{\infty}\frac{1}{2}A(k)\,c(m-k,n,l) - \sum_{k=1}^{\infty}\frac{1}{2}B(k)\,c(m+k,n,l) \\ &+\sum_{k=1}^{\infty}\frac{1}{2}A^{*}(k)\,c(m+k,n,l) - \sum_{k=1}^{\infty}\frac{1}{2}B^{*}(k)\,c(m-k,n,l) \\ &-\sum_{k=1}^{\infty}\frac{1}{2}A^{*}(k)\,c(m+k,n+k,l+k) \\ &+\sum_{k=1}^{\infty}\frac{1}{2}B^{*}(k)\,c(m-k,n-k,l-k) \\ &-\sum_{k=1}^{\infty}\frac{1}{2}A(k)\,c(m-k,n-k,l-k) \\ &+\sum_{k=1}^{\infty}\frac{1}{2}B(k)\,c(m+k,n+k,l+k) \end{aligned} \tag{4.90}$$

In this equation a symmetry is observed which suggests a regrouping of the terms such that:

$$\begin{aligned} -m\,c(m,n,l) = &\sum_{k=1}^{\infty}\frac{1}{2}\left(A(k) - B^{*}(k)\right)c(m-k,n,l) \\ &-\sum_{k=1}^{\infty}\frac{1}{2}\left(A(k) - B^{*}(k)\right)c(m-k,n-k,l-k) \\ &+\sum_{k=1}^{\infty}\frac{1}{2}\left(A^{*}(k) - B(k)\right)c(m+k,n,l) \\ &-\sum_{k=1}^{\infty}\frac{1}{2}\left(A^{*}(k) - B(k)\right)c(m+k,n+k,l+k) \end{aligned} \tag{4.91}$$

This regrouping suggests a new cepstral coefficient in terms of the phase may be defined as:

$$D(k) = A(k) - B^{*}(k), \quad k > 0 \tag{4.92}$$

Replacing the minimum and maximum phase cepstral coefficients by the phase cepstral coefficient into the previous expression yields:

$$
\begin{aligned}
-m\,c(m,n,l) = &\sum_{k=1}^{\infty}\frac{1}{2}D(k)\,c(m-k,n,l) \\
&-\sum_{k=1}^{\infty}\frac{1}{2}D(k)\,c(m-k,n-k,l-k) \\
&+\sum_{k=1}^{\infty}\frac{1}{2}D^{*}(k)\,c(m+k,n,l) \\
&-\sum_{k=1}^{\infty}\frac{1}{2}D^{*}(k)\,c(m+k,n+k,l+k)
\end{aligned}
\tag{4.93}
$$

This equation may be rearranged to yield:

$$
\begin{aligned}
-m\,c(m,n,l) = &\sum_{k=1}^{\infty}\frac{1}{2}D(k)\left[c(m-k,n,l)-c(m-k,n-k,l-k)\right] \\
&+\sum_{k=1}^{\infty}\frac{1}{2}D^{*}(k)\left[c(m+k,n,l)-c(m+k,n+k,l+k)\right]
\end{aligned}
\tag{4.94}
$$

This expression forms the all-pass TEA method and is valid for complex data under the assumption of conjugated terms in the definition of the cumulant. The all-pass structure is constructed by obtaining the numerator term from the phase cepstral coefficient by recursion. The denominator term is then the complex conjugate time reversal of the numerator polynomial.

## *4.10 Introduction to the Method of Least Squares*

The method of least squares may be used to solve problems which have a form similar to the cepstral equations. This type of problem contains a number of equations and a solution is sought which minimizes the sum of the squared errors. The error refers to an estimation error and is defined as the difference between a desired response and the prediction of that response obtained by the weighting of data. The least squares solution may be performed on systems which are over or underdetermined. Special techniques such as singular value

decomposition (SVD) and total least squares would have to be performed on underdetermined systems. The method of least squares is often used in prediction problems where past values are used to predict future values.

The least squares solution described here follows Haykin [1] and is obtained by minimizing the sum of squared errors. The error is given by:

$$e(i) = d(i) - \mathbf{w}^H \mathbf{u}(i) \tag{4.95}$$

where:

$e(i)$ refers to a single sample error

$d(i)$ refers to a desired response

$i$ refers to a sample index

$\mathbf{w}$ refers to the filter tap weights

$\mathbf{u}(i) = [u_1(i)\, u_2(i) \ldots u_M(i)]^T$ represents data samples

The sum of magnitude squared errors may be expressed as follows:

$$\xi(w_1, \ldots w_M) = \sum_{i=1}^{N} |e(i)|^2 \tag{4.96}$$

where:

$w$ refers to the filter tap weights

$M$ refers to the order of the system

$N$ is the number of equations

The sum of squared errors may be more conveniently expressed in matrix form in terms of the estimation error vector as:

$$\xi(\mathbf{w}) = \boldsymbol{\varepsilon}^H \boldsymbol{\varepsilon} \tag{4.97}$$

where

$$\varepsilon = [e(1)\, e(2) \ldots e(N)]^H \tag{4.98}$$

The error vector may also be expressed as the difference between the desired response vector and the filter output as:

$$\boldsymbol{\varepsilon} = \mathbf{d} - \boldsymbol{\Psi}\mathbf{w} \tag{4.99}$$

where

$\mathbf{d} = \left[d(1)\, d(2) \ldots d(N)\right]^H$ is the desired response vector

$\boldsymbol{\Psi} = \left[\mathbf{u}(1)\, \mathbf{u}(2) \ldots \mathbf{u}(N)\right]^H$ is the input data matrix

$\mathbf{u}(i) = \left[u_1(i)\, u_2(i) \ldots u_M(i)\right]^H$ is the input data vector

The minimization is accomplished by differentiating the sum of squared errors with respect to each of the tap weights of the filter. This operation is shown as follows:

$$\frac{\partial \xi}{\partial \mathbf{w}} = -2\boldsymbol{\Psi}^H\mathbf{d} + 2\boldsymbol{\Psi}^H\boldsymbol{\Psi}\mathbf{w} = -2\boldsymbol{\Psi}^H\boldsymbol{\varepsilon} \tag{4.100}$$

When this gradient vector is zero, the least squares solution is obtained and the resulting expression is referred to as the deterministic normal equation which is expressed as:

$$\boldsymbol{\Psi}^H\boldsymbol{\Psi}\hat{\mathbf{w}} = \boldsymbol{\Psi}^H\mathbf{d} \tag{4.101}$$

where $\hat{\mathbf{w}}$ is the least squares solution vector.

The minimum value of the solution given by the least squares method is assured if the second derivative of the error performance, the Hessian matrix, is determined to be positive semidefinite. This is a necessary condition because when the gradient is zero, the solution may correspond to a minimum or a maximum value. When the condition is satisfied such that:

$$\nabla(\nabla\varepsilon) > 0 \tag{4.102}$$

the error surface curves upward in all directions and a minimum solution is obtained. The Hessian matrix is obtained as follows:

$$\mathbf{H} = \frac{\partial^2 \varepsilon}{\partial \mathbf{w}^2} = \nabla(\nabla\varepsilon) = \frac{\partial}{\partial \mathbf{w}}\left(2\,\Psi^H(\Psi\,\mathbf{w} - \mathbf{d})\right) = 2\,\Psi^H\Psi \tag{4.103}$$

If the Hessian matrix is positive semidefinite, which is ensured since the correlation matrix is symmetric, then the least squares method yields the minimum solution value.

## *4.11 Solving the Cepstral Equations using Least Squares*

In the cepstra equation various cumulant samples at a number of lags are weighted by cepstral filter coefficients to yield the central cumulant at the corresponding sample values on each axis. This corresponds to a linear prediction of the central cumulant sample. These cepstral coefficients are determined using the method of least squares. In this arrangement, the data set is comprised of cumulants at a number of lags. The error to be minimized refers to the difference between the set of weighted cumulant values and the desired central cumulant. The weighting filter consists of the actual cepstral coefficients. The number of equations is dependent upon the range of the cumulant lags chosen and must be made sufficiently large enough to ensure that the resulting system of equations is overdetermined. This method may be applied to all the cepstral equations such as the TEA, phase TEA, all-pass TEA, slice-TEA and even the whitening.

The least squares solution may operate on complex data directly or an equivalent solution may be obtained using only real numbers. This is achieved by replacing every complex data element by the equivalent real and imaginary components.

This is achieved by rewriting the cepstral equations in terms of the equivalent inphase and quadrature components. Once the expression is obtained, separation into the two parts of the inphase and quadrature components results in two separate equations. This has the benefit of optimizing the solution in certain cases such as in the all-pass TEA method. A reduction in the number of unknown variables occurs as duplication from conjugated and unconjugated complex variables is avoided.

The solution of the cepstral coefficients using the TEA method is the first solution to consider. The application of the least squares approach to this problem begins by considering the equation of the TEA method for complex data with conjugates in the cumulant definition given by (eqn. 4.20):

$$-m\,c(m,n,l) = \sum_{k=1}^{p} A(k)\,c(m-k,n,l) - \sum_{k=1}^{p} A^*(k)\,c(m+k,n+k,l+k)$$
$$-\sum_{k=1}^{q} B(k)\,c(m+k,n,l) + \sum_{k=1}^{q} B^*(k)\,c(m-k,n-k,l-k)$$

where:

$p$ is the number of minimum phase cepstra coefficients

$q$ is the number of maximum phase cepstra coefficients

This equation is rewritten so that it is separated into its inphase and quadrature components. This yields the following expression for the inphase part of the TEA method:

$$
\begin{aligned}
-m\,\mathrm{Re}\big(c(m,n,l)\big) = &\sum_{k=1}^{p}\mathrm{Re}\big(A(k)\big)\Big[\mathrm{Re}\big(c(m-k,n,l)\big)-\mathrm{Re}\big(c(m+k,n+k,l+k)\big)\Big]\\
&-\sum_{k=1}^{p}\mathrm{Im}\big(A(k)\big)\Big[\mathrm{Im}\big(c(m-k,n,l)\big)+\mathrm{Im}\big(c(m+k,n+k,l+k)\big)\Big]\\
&-\sum_{k=1}^{q}\mathrm{Re}\big(B(k)\big)\Big[\mathrm{Re}\big(c(m+k,n,l)\big)-\mathrm{Re}\big(c(m-k,n-k,l-k)\big)\Big]\\
&+\sum_{k=1}^{q}\mathrm{Im}\big(B(k)\big)\Big[\mathrm{Im}\big(c(m+k,n,l)\big)+\mathrm{Im}\big(c(m-k,n-k,l-k)\big)\Big]
\end{aligned}
\tag{4.104}
$$

and the corresponding quadrature part:

$$
\begin{aligned}
-m\,\mathrm{Im}\big(c(m,n,l)\big) = &-\sum_{k=1}^{p}\mathrm{Re}\big(A(k)\big)\Big[\mathrm{Im}\big(c(m+k,n+k,l+k)\big)-\mathrm{Im}\big(c(m-k,n,l)\big)\Big]\\
&+\sum_{k=1}^{p}\mathrm{Im}\big(A(k)\big)\Big[\mathrm{Re}\big(c(m+k,n+k,l+k)\big)+\mathrm{Re}\big(c(m-k,n,l)\big)\Big]\\
&+\sum_{k=1}^{q}\mathrm{Re}\big(B(k)\big)\Big[\mathrm{Im}\big(c(m-k,n-k,l-k)\big)-\mathrm{Im}\big(c(m+k,n,l)\big)\Big]\\
&-\sum_{k=1}^{q}\mathrm{Im}\big(B(k)\big)\Big[\mathrm{Re}\big(c(m-k,n-k,l-k)\big)+\mathrm{Re}\big(c(m+k,n,l)\big)\Big]
\end{aligned}
\tag{4.105}
$$

The slice-TEA method results in the following expression (eqn. 4.45):

$$
\begin{aligned}
-n\,g(n) = &\sum_{k=1}^{p}2A(k)\,g(n-k)-\sum_{k=1}^{q}2B(k)\,g(n+k)\\
&+\sum_{k=1}^{p}A^{*}(k)\big[g(n-k)-3g(n+3k)\big]\\
&-\sum_{k=1}^{q}B^{*}(k)\big[g(n+k)-3g(n-3k)\big]
\end{aligned}
$$

where:

$p$ is the number of minimum phase cepstra coefficients

$q$ is the number of maximum phase cepstra coefficients

Separation into the real and imaginary parts yields:

$$\begin{aligned}
-n\,\mathrm{Re}(g(n)) = &\sum_{k=1}^{p}\mathrm{Re}(A(k))\left[3\,\mathrm{Re}(g(n-k)) - 3\,\mathrm{Re}(g(n+3k))\right] \\
&-\sum_{k=1}^{p}\mathrm{Im}(A(k))\left[\,\mathrm{Im}(g(n-k)) + 3\,\mathrm{Im}(g(n+3k))\right] \\
&-\sum_{k=1}^{q}\mathrm{Re}(B(k))\left[3\,\mathrm{Re}(g(n+k)) - 3\,\mathrm{Re}(g(n-3k))\right] \\
&+\sum_{k=1}^{q}\mathrm{Im}(B(k))\left[\,\mathrm{Im}(g(n+k)) + 3\,\mathrm{Im}(g(n-3k))\right]
\end{aligned} \tag{4.106}$$

and

$$\begin{aligned}
-n\,\mathrm{Im}(g(n)) = &\sum_{k=1}^{p}\mathrm{Re}(A(k))\left[3\,\mathrm{Im}(g(n-k)) - 3\,\mathrm{Im}(g(n+3k))\right] \\
&+\sum_{k=1}^{p}\mathrm{Im}(A(k))\left[\,\mathrm{Re}(g(n-k)) + \mathrm{Re}(g(n+3k))\right] \\
&-\sum_{k=1}^{q}\mathrm{Re}(B(k))\left[3\,\mathrm{Im}(g(n+k)) - \mathrm{Im}(g(n-3k))\right] \\
&-\sum_{k=1}^{q}\mathrm{Im}(B(k))\left[\,\mathrm{Re}(g(n+k)) + 3\,\mathrm{Re}(g(n-3k))\right]
\end{aligned} \tag{4.107}$$

The all-pass TEA method yields the following equation (eqn. 4.94):

$$\begin{aligned}
-m\,c(m,n,l) = &\sum_{k=1}^{p}\frac{1}{2}D(k)\left[c(m-k,n,l) - c(m-k,n-k,l-k)\right] \\
&+\sum_{k=1}^{p}\frac{1}{2}D^{*}(k)\left[c(m+k,n,l) - c(m+k,n+k,l+k)\right]
\end{aligned}$$

where:

$p$ is the number of minimum phase cepstra coefficients

Separation into the real and imaginary parts yields:

$$-m\,\mathrm{Re}(c(m,n,l)) = \sum_{k=1}^{p}\frac{1}{2}\mathrm{Re}(D(k))\,\left[\mathrm{Re}(c(m-k,n,l)) - \mathrm{Re}(c(m-k,n-k,l-k))\right.$$
$$\left. + \mathrm{Re}(c(m+k,n,l)) - \mathrm{Re}(c(m+k,n+k,l+k))\right]$$
$$-\sum_{k=1}^{p}\frac{1}{2}\mathrm{Im}(D(k))\,\left[\mathrm{Im}(c(m-k,n,l)) - \mathrm{Im}(c(m-k,n-k,l-k))\right.$$
$$\left. - \mathrm{Im}(c(m+k,n,l)) + \mathrm{Im}(c(m+k,n+k,l+k))\right] \tag{4.108}$$

and

$$-m\,\mathrm{Im}(c(m,n,l)) = \sum_{k=1}^{p}\frac{1}{2}\mathrm{Re}(D(k))\,\left[\mathrm{Im}(c(m-k,n,l)) - \mathrm{Im}(c(m-k,n-k,l-k))\right.$$
$$\left. + \mathrm{Im}(c(m+k,n,l)) - \mathrm{Im}(c(m+k,n+k,l+k))\right]$$
$$+\sum_{k=1}^{p}\frac{1}{2}\mathrm{Im}(D(k))\,\left[\mathrm{Re}(c(m-k,n,l)) - \mathrm{Re}(c(m-k,n-k,l-k))\right.$$
$$\left. - \mathrm{Re}(c(m+k,n,l)) + \mathrm{Re}(c(m+k,n+k,l+k))\right] \tag{4.109}$$

In all the equations listed the least squares method is used by assigning the cumulants on the right hand side of the equation to the data matrix, the cumulant on the left hand side of the equation to the desired vector and finally the unknown cepstral coefficients as the unknown vector to be determined. One equation is generated for each combination of integer values of $m,n$ and $l$ selected. Therefore the number of equations is given by:

$$\text{number of equations} = (\text{range number})^3 * 2 \tag{4.110}$$

where the range number refers to the number of integer values of the parameter $m$ selected. Each of the remaining parameters generally and for simplicity take on the same range of values. The number of equations is observed to increase as the range number increases. The range must be chosen large enough to ensure an adequate number of independent equations. A value of two was used in these simulations.

The number of unknown cepstral coefficients may be arbitrarily set to the order of the system. In the case of the TEA and slice-TEA methods, the number of unknowns corresponding to the real and imaginary parts is therefore $2(p+q)$ corresponding to the real and imaginary parts while in the all-pass TEA method the number of unknowns is only $2p$. The number of unknowns in the all-pass TEA method is therefore less than in the TEA and slice-TEA methods.

## *4.12 Conclusion*

The treatment of higher order spectra as a convolutional homomorphic system has been shown to lead to a relationship between the cumulant of a system and the complex cepstrum of the impulse response. As a result, the use of the complex logarithm has additionally transformed a previously nonlinear relationship between a system's impulse response and its associated cumulant into a linear one. This linear relationship allows one to use the method of least squares in order to obtain the unknown complex cepstrum of the impulse response and hence the impulse response.

The application of higher order spectral techniques to determine the phase of the system exclusively yields to the introduction of a phase equalizer. The phase equalizer may be optimized by further consideration of the symmetry inherent in an all-pass structure which is representative of the phase. This leads to a computationally more efficient phase equalizer which will be shown to yield

superior performance. The implementation of the equalizer part will be addressed in the next chapter.

# *Chapter 5*
# *Equalization Structure*

## *5.1 Approaches to Equalization*

In the communications environment the term equalization is used to describe a device which attempts to equalize or compensate for channel induced distortion. The distortion is a result of time dispersion which occurs from multipath propagation and results in a set of multiple copies of the original signal at different time delays. This corresponds to a frequency selective distortion in the frequency domain. An equalizer is used when distortionless transmission is desired and is necessary in the case of severe intersymbol interference. Larger intersymbol interference results in a greater scattering of the received signal samples relative to the transmitted signal points. The purpose of equalization is to attempt to compensate for the channel induced time dispersive distortion.

A variety of approaches to the equalization problem exist. The approach depends on whether the estimate of the channel is available or whether the actual inverse is modeled. Most traditional adaptive equalizer structures fall into this latter category. Such traditional equalizers would include linear equalizers and equalizers with decision feedback [32]. The Bussgang family of blind equalizers [4][1] is also classified in this category as these are based on an iterative deconvolution procedure.

In the other main category of equalizer design approaches, an estimate of the model of the system is available. With the knowledge or estimate of the channel an equalizer may be constructed based on several philosophies. Some of the design approaches would be those based on the zero forcing criterion, the minimization of the mean square error and also maximum likelihood sequence estimation.

Perhaps the most obvious approach to equalization is to attempt to invert the channel corresponding to a deconvolution operation. In the ideal case, convolving the received signal with the inverted channel estimate should yield a received signal with no channel distortion. Reducing the multiple signal copies to a single component is referred to as zero forcing [2]. Unfortunately in the nonideal case compensation may be incomplete. One source of inaccuracy arises when a finite length equalizer is used when an infinite length one is required. Another source arises because the channel estimate or its inverse estimate will not be perfect. In addition inverting a channel which has a frequency null will give large values resulting in noise enhancement.

Another common approach to equalization results by attempting to minimize the mean square of the error between the actual symbol transmitted and the estimate of the symbol at the output of the equalizer [2]. Using this type of equalizer an estimate of the transmitted symbol is made which takes into account additive noise. In the case of no noise the equalizer will tend to a zero forcing equalizer. In contrast, such as in a high noise environment, the equalizer will tend towards a matched filter. A compromise is therefore made between the noise level and the signal distortion resulting from ISI [33].

Maximum likelihood equalization is another approach to the equalization problem [33]. This method requires a channel model through which all possible input sequences are passed. The outputs corresponding to each input sequence are compared to the actual output. The output sequence which is closest to the actual received signal is selected as the maximum likelihood sequence. This approach requires a good channel model estimate and a good algorithm to increase the efficiency of the sequence estimation.

The use of higher order spectra enables the estimation and hence identification of a nonminimum phase channel impulse response to be made. One of the interesting properties of cepstral coefficients is that the channel estimate and its inverse have cepstral coefficients of opposite signs [14]. As a result, the zero forcing criterion, which attempts to invert the channel, is directly and easily implemented. From this inverse estimate, a direct form filter structure or an equivalent lattice structure may then be used to implement the zero forcing equalizer. The use of the lattice structure allows the inverse filter to be constructed very easily.

## 5.2 *Lattice Filters*

The realization of linear time-invariant discrete systems is often accomplished using a direct form structure however an equivalent lattice structure may also be used. The lattice structure is suggested as it exhibits properties and symmetries which simplify the realization of the system's equalizer. The description of the lattice filter follows from Proakis [20][34].

The lattice filter combines forward and backward prediction into one structure. Forward linear prediction deals with the prediction of future values of a stationary random process from past observed values. The past observed values are weighted by prediction coefficients and combined linearly. The linearly predicted value in direct form is expressed as:

$$\hat{x}(n) = -\sum_{i=1}^{m} \alpha_m(i)\, x(n-i) \tag{5.1}$$

where:

$x(n)$ corresponds to the input data samples

$\hat{x}(n)$ corresponds to the output prediction samples

$\alpha_m(i)$ represents the forward prediction coefficients

$i, n$ are the sample indexes

$m$ is a filter element

$M$ is the order of the filter

Linear prediction may be viewed as a type of linear filtering where the input sequence, consisting of the observed samples, is weighted and summed to yield an output equal to the forward prediction error sequence [20]. The difference between the observed current sample value and the predicted value is the forward prediction error which in direct form is expressed as:

$$\begin{aligned} f_m(n) &= x(n) - \hat{x}(n) \\ &= \sum_{i=0}^{m} \alpha_m(i)\, x(n-i)\,; \qquad \alpha_m(0) = 1 \end{aligned} \tag{5.2}$$

where

$f_m(n)$ represents the forward prediction error samples.

The backwards prediction filter operates in reverse order to that of the forward predictor. This implies that future data samples are used to predict the current data sample. Thus the data sequence $x(n), x(n-1),\ldots, x(n-m+1)$ is used to predict the value of $x(n-m)$. The direct form of the backward prediction is expressed as:

$$\hat{x}(n-m) = -\sum_{i=0}^{m-1} \beta_m(i)\, x(n-i) \tag{5.3}$$

where:

$\beta_m(i)$ represents the backward prediction coefficients

The backward prediction error is the difference between the observed value and the estimated value and is given by:

$$\begin{aligned} g_m(n) &= x(n-m) - \hat{x}(n-m) \\ &= \sum_{i=0}^{m} \beta_m(i)\, x(n-i); \quad \beta_m(m) = 1 \end{aligned} \tag{5.4}$$

The transfer function associated with the forward predictor is the reciprocal or reverse polynomial of the backward predictor for stationary signals. In the time domain this is expressed as:

$$\beta_m(k) = \alpha_m^*(m-k), \quad k = 1,\ldots m \tag{5.5}$$

In the frequency domain this relationship may be equivalently expressed as:

$$B_m(z) = z^{-m}\, A_m^*\left(1/z^*\right) \tag{5.6}$$

where the forward prediction system function is given by:

$$A_m(z) = \sum_{i=0}^{m} \alpha_m(i)\, z^{-i} \tag{5.7}$$

and the backward prediction system function is also given by:

$$B_m(z) = \sum_{i=0}^{m} \beta_m(i)\, z^{-i} \tag{5.8}$$

These two system transfer functions are observed to form a matched filter pair.

The lattice structure is illustrated in Figure 5.1 and consists of a series of stages which are connected in cascade. The number of stages equals the order of the direct form realization. A typical lattice element or stage is illustrated in Figure 5.2. Each stage in the form of a lattice element contains a reflection coefficient indicated by the parameter $K_i$. Each lattice stage results in two outputs corresponding to forward prediction and backward prediction.

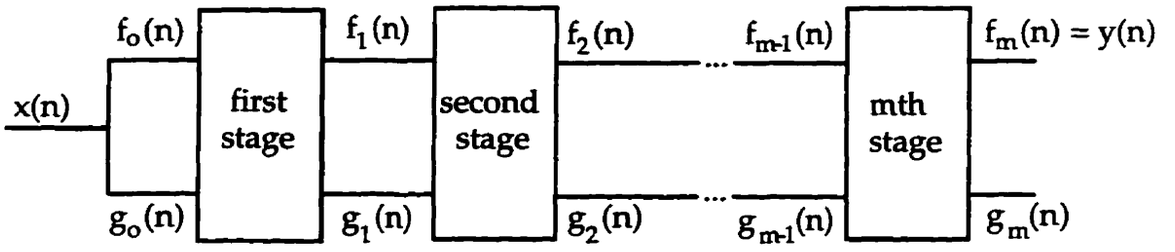


Figure 5.1 - Lattice Filter Structure

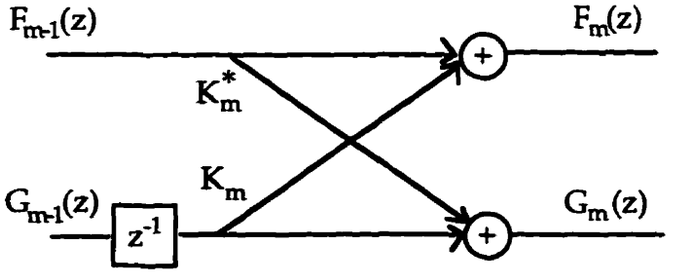


Figure 5.2 - Lattice Element

One of the advantages of using a lattice structure is that when the order of the filter needs to be increased, additional stages may be appended without affecting the previous calculations and results [1]. In order to append these stages the forward prediction system function needs to be written in an order update recursive form. This form is obtained by determining first the next forward prediction error in the lattice from Figure 5.2 as:

$$F_m(z) = F_{m-1}(z) + K_m\, z^{-1}\, G_{m-1}(z) \tag{5.9}$$

The forward prediction system function is obtained by dividing this equation by the input $X(z)$ such that:

$$A_m(z) = \frac{F_m(z)}{X(z)} = \frac{F_{m-1}(z)}{X(z)} + K_m\, z^{-1}\, \frac{G_{m-1}(z)}{X(z)} \tag{5.10}$$

This function may be simplified by noting that:

$$B_{m-1}(z) = \frac{G_{m-1}(z)}{X(z)} = z^{-(m-1)}\, A^*_{m-1}\left(1/z^*\right) \tag{5.11}$$

Using these relationships, the forward prediction system function is therefore obtained as:

$$\begin{aligned} A_m(z) &= A_{m-1}(z) + K_m\, z^{-1}\, B_{m-1}(z) \\ &= A_{m-1}(z) + K_m\, z^{-m}\, A^*_{m-1}\left(1/z^*\right) \end{aligned} \tag{5.12}$$

The backward prediction system function may also be expressed in an order update recursive form by considering the expression for the backward prediction error given by:

$$G_m(z) = z^{-1}\, G_{m-1}(z) + K^*_m\ F_{m-1}(z) \tag{5.13}$$

Dividing this equation by the input $X(z)$ yields the backward prediction system function as follows:

$$\begin{aligned} B_m(z) = \frac{G_m(z)}{X(z)} &= z^{-1}\, \frac{G_{m-1}(z)}{X(z)} + K^*_m\, \frac{F_{m-1}(z)}{X(z)} \\ &= z^{-1}\, B_{m-1}(z) + K^*_m\, A_{m-1}(z) \end{aligned} \tag{5.14}$$

Finally writing the backward prediction system function in terms of the forward prediction system function yields:

$$B_m(z) = z^{-m}\, A^*_m\left(1/z^*\right) = z^{-m}\, A^*_{m-1}\left(1/z^*\right) + K^*_m\, A_{m-1}(z) \tag{5.15}$$

The frequency domain symmetry between the forward and backward prediction filters and the configuration of these within the lattice structure will be shown to be especially useful for inverting general filters and particularly all-pass filters.

## 5.3 *Lattice Reflection Coefficients*

The lattice filter is characterized by a set of reflection coefficients corresponding to each stage in the lattice structure. These reflection coefficients may also be calculated from the direct form realization in a recursive manner. In addition if the magnitude of the reflection coefficients are all less than unity such that:

$$|K_i| < 1; \qquad i = 1, 2 \ldots M \tag{5.16}$$

then all the roots of the forward prediction system function are located inside the unit circle. The forward prediction filter is therefore minimum phase and its inverse is ensured to be stable [1].

The series of reflection coefficients are determined by first considering the highest order reflection coefficient. This coefficient is obtained from the direct form by setting:

$$K_M = \alpha_M(M) \tag{5.17}$$

Using this reflection coefficient, a lower degree polynomial is computed using the following recursive equation:

$$\alpha_{m-1}(k) = \frac{\alpha_m(k) - K_m\, \alpha_m(m-k)}{1 - |K_m|^2} \tag{5.18}$$

From this new lower degree polynomial the corresponding reflection coefficient is determined since:

$$K_{m-1} = \alpha_{m-1}(m-1) \tag{5.19}$$

The process is continued in a step down fashion until all the reflection coefficients have been determined.

## 5.4 *IIR Systems - Inverse and All-pass Structures*

In order to implement an equalizer under the zero forcing criterion, it is necessary to invert the lattice filter. Fortunately this operation is facilitated since a system's lattice structure and its corresponding inverse lattice structure are both characterized by the same set of reflection coefficients [34]. The poles of the IIR lattice filter are derived from an equivalent FIR structure described by an all-zero system function. The inverse lattice filter structure may therefore be represented by an IIR system having an all-pole system function. The IIR structure is obtained from the FIR structure by interchanging the roles of the input and output quantities as well as redefining the input and output of Figure 5.1 as [34]:

$$x(n) = f_M(n) \quad \text{or} \quad X(z) = F_M(z) \tag{5.20}$$

and

$$y(n) = f_0(n) \quad \text{or} \quad Y(z) = F_o(z) \tag{5.21}$$

The resulting structure is illustrated in Figure 5.3.

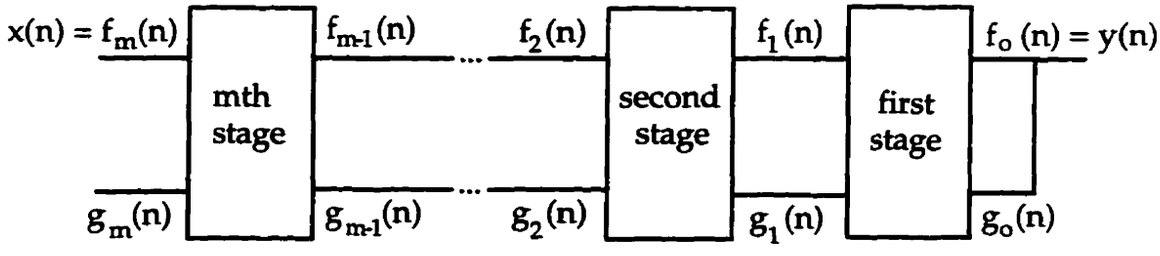


Figure 5.3 - Inverse Lattice Filter Structure

This rearranging of the input and output quantities also introduces feedback terms and implies that the forward prediction errors now need to be computed in

a descending order. Expressing the forward prediction error in descending order yields:

$$F_{m-1}(z) = F_m(z) - K_m\, z^{-1}\, G_{m-1}(z); \quad m = M, M-1, \ldots 1 \tag{5.22}$$

The backward prediction error remains unchanged and is therefore the same in both lattice structures. Both lattice structures are also characterized by the same reflection coefficients. The forward prediction error coefficients however occur in reverse order to each other. The resulting inverse lattice element is illustrated in Figure 5.4.

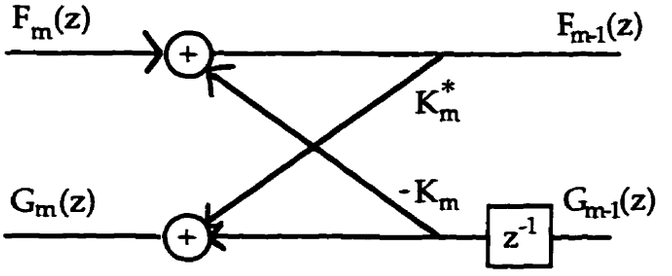


Figure 5.4 - Inverse Lattice Filter Element

The two lattice structures of Figures 5.2 and 5.4 are thus observed to differ only in the interconnections within the lattice elements. This simple relationship between the two structures allows the inverse lattice filter element to be obtained from the lattice filter in a very simple manner.

The all-pole lattice also provides the basic structure necessary to implement IIR systems which contain both poles and zeros [34]. In an all-pass filter the poles and zeros are complex conjugate inverses of one another with respect to the unit circle in the Z-plane. The transfer functions of the forward and backward prediction filters are also related in a similar fashion. The forward prediction filter contains zeros located inside the unit circle and is termed a minimum phase

system [13]. The backward prediction filter on the other hand contains zeros located outside the unit circle and is termed a maximum phase system [13]. The inverse lattice structure is observed to yield a simple implementation of an all-pass filter when the IIR output of the mth order pole zero ladder is considered.

From the structure for an all-pole lattice element in Figure 5.4, the following set of equations can be obtained:

$$\begin{aligned} F_{m-1}(z) &= F_m(z) - z^{-1} K_m G_{m-1}(z) \\ G_m(z) &= K_m^* F_{m-1}(z) + z^{-1} G_{m-1}(z) \end{aligned} \tag{5.23}$$

From the structure diagram of the all-pole lattice illustrated in Figure 5.3, the following relationships may also be obtained:

$$\begin{aligned} Y(z) &= F_0(z) = G_0(z) \\ X(z) &= F_M(z) \end{aligned} \tag{5.24}$$

From these equations the system function of the forward prediction error corresponding to an all-pole IIR system is given by:

$$H_a(z) = \frac{Y(z)}{X(z)} = \frac{F_0(z)}{F_M(z)} = \frac{1}{A_M(z)} \tag{5.25}$$

It can be seen that the filter corresponding to this transfer function may be used to invert the general FIR function. The backward prediction system function of the system is therefore given by:

$$H_b(z) = \frac{G_M(z)}{Y(z)} = \frac{G_M(z)}{F_0(z)} = \frac{G_M(z)}{G_0(z)} = B_M(z) = z^{-M} A_M^*\left(1/z^*\right) \tag{5.26}$$

An all-pass system function may then be obtained by combining the forward prediction system function and the backward prediction system function as a product such that:

$$
\begin{aligned}
H_{ap}(z) &= H_a(z)\, H_b(z) \\
&= \frac{Y(z)}{X(z)} \cdot \frac{G_M(z)}{Y(z)} = \frac{G_M(z)}{X(z)} \\
&= \frac{B_M(z)}{A_M(z)} = \frac{z^{-M} A_M^*(1/z^*)}{A_M(z)}
\end{aligned}
\tag{5.27}
$$

This result indicates that the lattice structure may be used to implement an all-pass filter. The all-pass filter structure serves to act as a phase filter and thus may be also used for phase equalization.

## 5.5 *Inverse Filtering of Maximum Phase Components*

The impulse response of a channel is generally characterized by a combination of minimum and maximum phase components. The channel is therefore not restricted to be minimum phase but may also be maximum phase or mixed phase. The difficulty encountered with mixed phase and maximum phase systems is that upon inversion these terms become unstable unless they are allowed to be noncausal. When a causal maximum phase system is allowed to be anticausal the system becomes stable. Although noncausal systems are not realizable by physical devices they are possible when processing off line because one has access to future as well as past inputs. The treatment of the maximum phase system begins by considering the convolution of the maximum phase system given by:

$$y(n) = h(n) * x(n) \tag{5.28}$$

where

$h(n)$ is the impulse response containing maximum phase poles only

$x(n)$ is the sequence to be filtered

$y(n)$ is the output of the filter

This relationship is equivalently expressed in the Z domain as:

$$Y(z) = H(z)\,X(z) \tag{5.29}$$

The maximum phase system may be converted to an equivalent minimum phase system by reflecting all the maximum phase components into the interior of the unit circle. This operation is given by:

$$Y^*(1/z^*) = H^*(1/z^*)\,X^*(1/z^*) \tag{5.30}$$

This expression corresponds to a time reversal in the time domain and is equivalently expressed in the discrete time domain as:

$$y^*(-n) = h^*(-n) * x^*(-n) \tag{5.31}$$

This convolution is now stable and therefore may be performed. To obtain the actual non time reversed output, the output must be again time reversed and unconjugated to yield the original convolution. One must be careful to deal only with the maximum phase components otherwise instability from the time reversed minimum phase components will result. It is thus observed that maximum phase components may be inverted easily with the introduction of a delay.

This procedure used to invert the maximum phase components is not required when dealing with the all-pass phase structure as time reversal and inversion of the filter is not required. In the case of an all-pass structure the denominator is constrained so that it contains only minimum phase components such that:

$$H_{ap}(z) = \frac{A^*(1/z^*)}{A(z)} \tag{5.32}$$

The inverse of this all-pass filter yields an unstable system due to the maximum phase term in the denominator as follows:

$$\frac{1}{H_{ap}(z)} = \frac{A(z)}{A^*(1/z^*)} \tag{5.33}$$

When the process of time reversal and inversion are combined, the original all-pass structure is obtained as follows:

$$\frac{1}{H^*_{ap}(1/z^*)} = \frac{A^*(1/z^*)}{A(z)} = H_{ap}(z) \tag{5.34}$$

The convolution of the time reversed data with the original all-pass filter may now be performed yielding a time reversed output. To complete the procedure the output must be reversed in time as before. Inverse filtering using an all-pass structure is thus easily performed.

## *5.6 Summary*

In this chapter it has been shown that a system equalizer may be implemented using a lattice filter structure. The use of the lattice structure is suggested as the inverse filter of the system required for equalization may be easily determined from the system's model. It has also been demonstrated that phase equalization may be implemented using an all-pass lattice structure. Finally, stability of the filter may also be easily checked from the magnitude of the reflection coefficients. From these observations it appears that the use of the lattice filter structure is of benefit in the equalizer implementation required in this application.

# *Chapter 6*
# *System Simulation & Performance Results*

## *6.1 Introduction*

The availability of the computer allows a number of theoretical ideas to be verified and investigated through the use of computer simulation. This system simulation is also of benefit because insight into performance can be obtained without the need to build costly and complicated real time systems. When it comes time to actually implement a system the computer simulation serves as a guide to the actual system development. Another advantage to the simulation process is that system parameters may be easily verified without redesigning the complete system and therefore system parameters may be more easily optimized.

The extraction of the digital information from a received signal which has undergone transmission from the transmitter source is the goal in a communications system. The compensation for the transmission channel is generally unknown because it represents a particular transmission path followed from the source to the receiver. This path may be time varying and generally time dispersive. Compensation for this channel often begins with first estimating it as an unknown system. Either the actual channel or its inverse may be estimated. The algorithms investigated here attempt to estimate the channel and therefore constitute a system identification problem.

The simulation of the various blind equalization schemes is comprised of a transmitter section consisting of two level Pulse Amplitude Modulation (PAM) symbol generation for real systems and four level Quadrature Amplitude Modulation (QAM) symbol generation for complex systems. The transmitted signal is then convolved with a transmission channel followed by the addition of noise. The transmission channel represents the environment of the system and is characterized by a base band equivalent of an actual channel at RF frequencies. The channel used here has been selected from previous work [9] and is given as:

$$h(n) = 0.2197\ \delta(n) - 0.747\ \delta(n-1) + 0.6085\ \delta(n-2) + 0.1533\ \delta(n-3) \quad (6.1)$$

The zeros of this channel are located at -0.2, 1.8+j0.5 and 1.8-j0.5. The noise is added after convolution with the channel and is assumed to be zero mean, Gaussian and statistically independent from the input sequence. Other assumptions in this simulation include perfect carrier recovery and frequency offset estimation. Sampling of the received sequence occurs at the symbol period with no bandwidth assumptions. Accurate estimation of the transmission channel in the presence of noise is of considerable importance in the performance of the blind equalizer.

At the receiver, cumulants are determined from blocks of data from which cepstral coefficients are determined indirectly by the method of least squares. The estimate of the channel's impulse response is determined in a recursive fashion from this set of cepstral coefficients. The equalizer is determined under the zero forcing criterion from the estimated impulse response of the channel. The zero-forcing equalizer structure facilitates inverse filtering in that an actual inverse of the channel is attempted. This structure is not ideal in the case of severe fading as the inversion process results in noise enhancement. Noise

enhancement was not handled in this work. In addition the frequency selective fades of the channel's response were not severe enough to cause a problem. The receiver structure presented consists of three basic parts. These parts consist of the cumulant estimation of the received data blocks, the channel estimation part which comprises the particular algorithm and the equalization implementation part.

The ultimate goal in a digital communications system is to recover the transmitted signal as accurately as possible. The performance is most often measured by obtaining probability of error values for various noise levels. These values will of course depend upon the type of environment described by the transmission channel. The probability of error is compared with respect to the ratio of energy in each bit to the noise power spectral density and is designated as the Bit Error Rate (BER). Equivalently this ratio may be thought of as the signal to noise ratio of the signal at the receiver.

The performance of the blind equalization algorithms is dependent upon the selection of various parameters. These parameters of interest are those of the block length of the received data and the model order of the channel estimate. Cumulant estimates are made from the received data and it follows that a longer block length will result in better cumulant estimates [11]. The channel must be slowly fading or equivalently stationary over the block length [9]. Shorter block lengths are hence more desirable since this would allow the channel to vary more rapidly. However, an accurate channel estimation becomes more difficult to achieve on these shorter lengths. Longer block lengths also require longer

processing times. The effect of block length therefore needs to be investigated to yield a suitable compromise between these two conflicting features.

The model order of the channel to be estimated is also a parameter of interest. If the model order is smaller than the actual channel order then fewer coefficients need to be estimated which results in estimates with smaller variances that are biased [18]. On the other hand, larger model orders result in larger variances and more calculations which may be necessary to accurately estimate the channel. This chapter details the performance of the system under the various scenarios to be described where the model order, block length and noise levels are to be varied.

In system simulation it is important to verify the correct operation of the system. As such, sections of the system need to be isolated and the performance of each is verified by inputting ideal values and observing the effect at the output. The use of ideal cumulant estimates and an ideal channel are useful in the process of the verification of the processing scheme. The performance of the blind equalization scheme is also dependent upon the input parameters to the algorithm as well as the type of environment characterized. The input parameters consist of the model order of the channel estimate as well as the block length of the received signal.

## *6.2 Performance Using Ideal Cumulant Estimates*

The performance of the blind equalization schemes based on higher order spectra is dependent upon the ability to obtain good cumulant estimates as cumulant estimates are used exclusively to obtain the estimates of the channel. If perfect cumulant estimates would be available then the validity of the various algorithms can be determined by examining how well the channel estimate is made. Cumulant estimates are obtained from the received data sequence by using temporal averages in place of the defined ensemble averages. The true cumulant value is defined in terms of the impulse response (eqn. 3.51) as follows [4]:

$$c(m,n,l) = \gamma \sum_i h(i)\, h(i+m)\, h(i+n)\, h(i+l)$$

where:

$\gamma$ is the kurtosis

$h(i)$ is an impulse response sample

Perfect cumulant estimates are obtained by replacing the received data with the impulse response since the temporal averages of the cumulant then become deterministic. Thus perfect cumulant estimates are obtained when the received signal is replaced by the channel's impulse response.

The equalizer constructed from the cumulants may then be convolved with the actual channel to yield a residual ISI sequence. Perfect equalization would result when this sequence is just an impulse, however, the ability to achieve this is impaired by the presence of noise and by any truncation effects arising from both channel inversion as well as model order selection.

The performance of the system is expected to improve as the residual ISI values tend towards zero. This performance measure is particularly useful when perfect cumulant estimation is assumed since data is not transmitted and therefore bit errors cannot be determined. The residual ISI values using the perfect cumulant estimates yielded values that are close to zero. This therefore verifies that the impulse response estimates are being determined correctly. This result also indicates that the performance of blind equalizers is affected significantly by the ability to make good cumulant estimates.

## *6.3 System Identification*

The blind equalizer algorithms presented here attempt to estimate the transmission channel. It follows that the accuracy of this estimate can serve as a measure of the performance of the system identification. The channel estimates may be obtained for each block and compared to the actual channel. Each block differs only by the noise value as well as by the transmitted data sequence. The mean and standard deviation of the channel estimates are determined statistically to give an idea of the performance of the channel estimation. A comparison of the impulse response estimates using the TEA and slice-TEA methods is illustrated in Table 1 and Table 2 for various block lengths. For comparison purposes model orders using a value of two have been selected with a bit energy to noise power spectral density ratio of 30 dB.

| Table 1 - TEA Method block length system identification comparisons | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 128 symbols | | 256 symbols | | 512 symbols | | 1024 symbols | |
| true IR | mean | stdev | mean | stdev | mean | stdev | mean | stdev |
| 0.2197 | 0.1549 | 0.1067 | 0.1757 | 0.0833 | 0.1891 | 0.0576 | 0.1930 | 0.0394 |
| -0.7470 | -0.6150 | 0.1007 | -0.6659 | 0.0680 | -0.6866 | 0.0405 | -0.6991 | 0.0264 |
| 0.6085 | 0.6835 | 0.0997 | 0.6722 | 0.0573 | 0.6682 | 0.0390 | 0.6631 | 0.0285 |
| 0.1533 | 0.0749 | 0.1200 | 0.1140 | 0.0923 | 0.1221 | 0.0683 | 0.1347 | 0.0431 |

| Table 2 - slice-TEA Method block length system identification comparisons | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 128 symbols | | 256 symbols | | 512 symbols | | 1024 symbols | |
| true IR | mean | stdev | mean | stdev | mean | stdev | mean | stdev |
| 0.2197 | 0.1841 | 0.1673 | 0.1828 | 0.1279 | 0.1818 | 0.1256 | 0.1700 | 0.0953 |
| -0.7470 | -0.6254 | 0.1753 | -0.6229 | 0.1338 | -0.6261 | 0.0948 | -0.6302 | 0.0772 |
| 0.6085 | 0.4960 | 0.2349 | 0.5462 | 0.2016 | 0.6407 | 0.1528 | 0.6874 | 0.0778 |
| 0.1533 | 0.0553 | 0.1730 | 0.0374 | 0.1301 | 0.0362 | 0.1346 | 0.0339 | 0.1036 |

The comparisons between the various entries in Table 1 show that as the block length of the data is increased, the estimate of the channel's impulse response is improved with a corresponding reduction in the variances of these estimates. In Table 2, the noticeable improvement occurs in the reduction of the variance as the block length is increased. The estimates using the TEA method are observed to be much better than those obtained from the slice-TEA method.

## *6.4 Bit Error Probability Performance*

The performance of an equalizer is ultimately evaluated by it's ability to recover transmitted data bits. This performance is measured by obtaining probability of error values for various environments described by the transmission channel. The probability of error is displayed with respect to the ratio of the energy in each bit to the noise power spectral density. The ratio may be equivalently thought of as the signal to noise ratio of the signal at the receiver.

The probability of error is obtained by transmitting a sequence of data through the communications system and attempting to recover it at the receiver through the use of equalization. The recovered data is compared to the original data and errors in the data bits are then counted. These errors are then compared to the total number of transmitted data bits. The process is repeated for various noise levels. In order to achieve a correct comparison between the original data sequence and the recovered data sequence, a time delay and a phase rotation needs to be applied to the recovered or estimated data sequence. This time delay and phase rotation is required in any blind equalization scheme.

The bit error probability values will depend most significantly upon the block length of the received data. Throughout all the simulations the block length of the received data is varied in order to illustrate the effect of varying this parameter. It is expected that an increase in block length will improve the performance of the algorithm.

## *6.5 Significance of Model Order Estimation*

In any system identification method in order to properly model an unknown system it is desirable to have an estimate of the model order of the unknown system of interest. The successful modeling of the unknown system will be dependent somewhat on the model order estimate used. An underestimation of this parameter results in a biased model estimate. This yields a migration of the pole locations towards the unit circle and a smoothing of the impulse response. An overestimation of the model order results in high variances as more model parameters need to be estimated. Model artifacts can appear which may result from the noise being modeled rather than the actual channel impulse response. The high variances may also result in instability which forces pole locations outside the unit circle.

The performance of the equalizer is affected by the selection of the model order estimate. The performance is also affected by the block length of the received data and therefore the optimum model order estimate for one block length may not be the optimum choice for another block length. The effect on performance of the model order estimate with different data block lengths is compared in Figures 6.1 through 6.10 using the TEA and slice -TEA methods. Figures 6.1 through 6.4 compare the model order estimates for various block lengths using the TEA method while Figures 6.5 through 6.8 use the slice-TEA method. The estimated model order is varied from two through to ten. Real data is used for these simulations which implies a PAM symbol generation. An improvement in performance is observed when the bit error probability yields a lower value implying fewer errors.

Figures 6.1 through 6.4 compare the BER performance of the TEA method for various model order estimates for block lengths of 128, 256, 512 and 1024 symbols respectively. The optimal model order estimate appears to be a value of four followed by an order of six and then an order of two. Thus the trade off between the accuracy attainable using a model order with a large number of parameters to be estimated and that of a model order which has an inadequate number of parameters to describe the model accurately is observed. The actual channel's impulse response is comprised of two maximum phase components and one minimum phase component yielding a system order of three. It therefore appears that a slight over estimation of the model order is preferable to that of an underestimation of the model order.

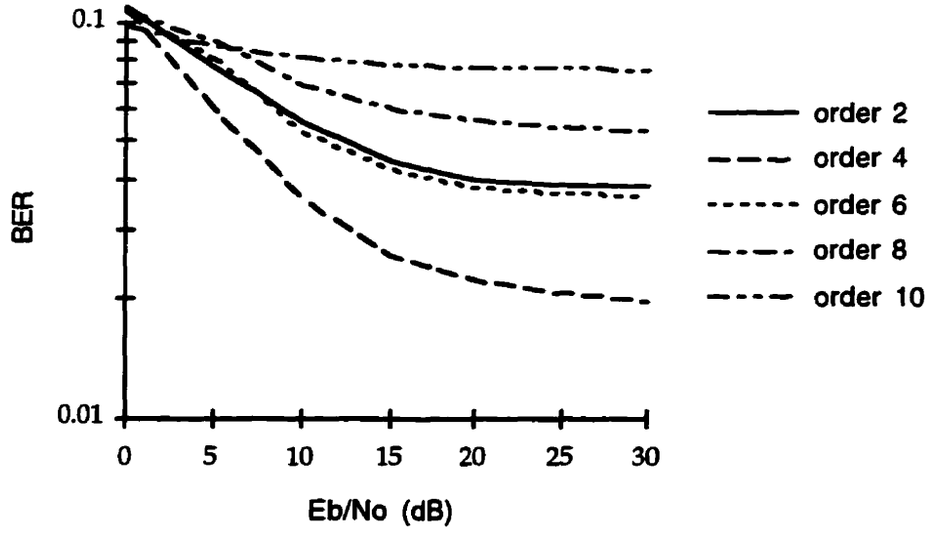


Figure 6.1 - 128 symbol TEA BER order comparison

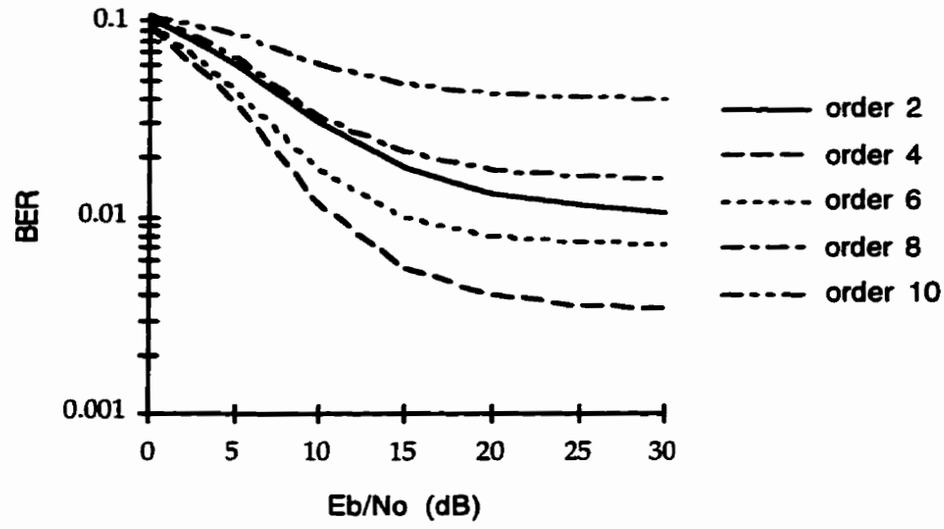


Figure 6.2 - 256 symbol TEA BER order comparison

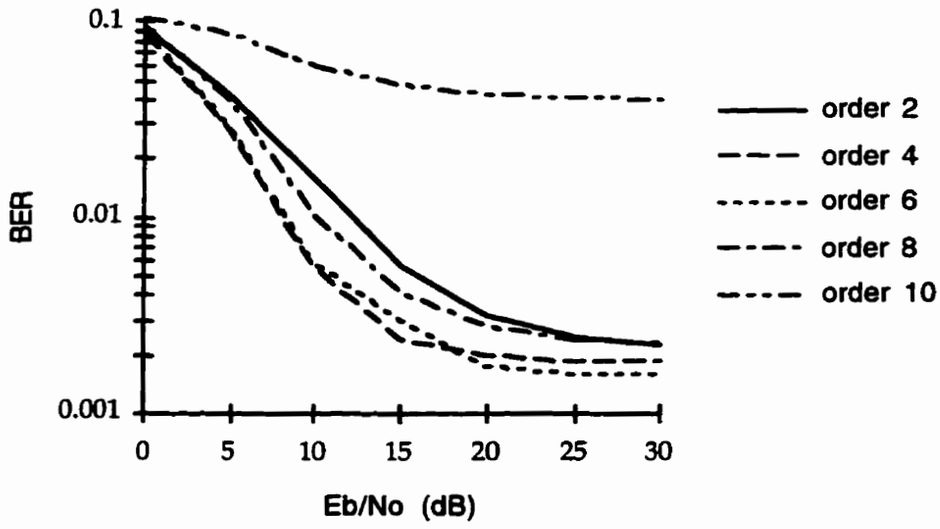


Figure 6.3 - 512 symbol TEA BER order comparison

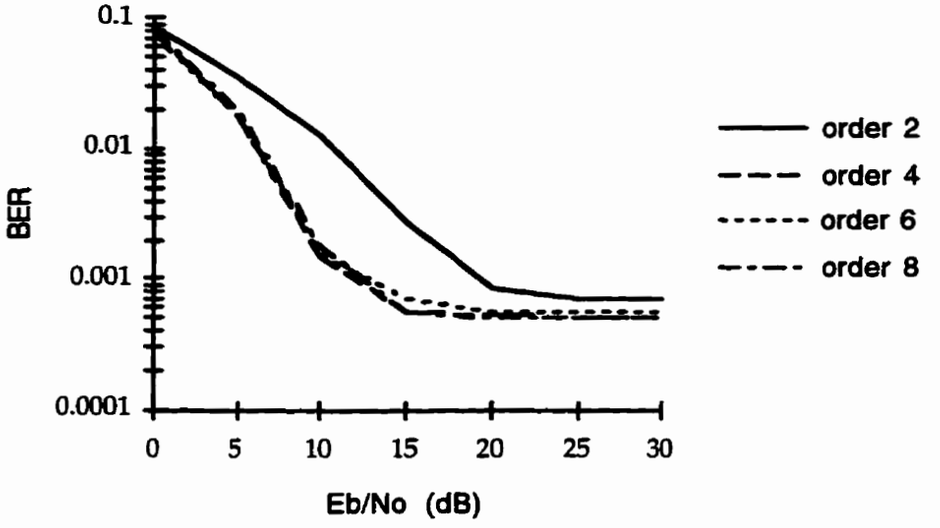


Figure 6.4 - 1024 symbol TEA BER order comparison

Figures 6.5 through 6.8 compare the performance of various model orders using the slice-TEA method with block lengths of 128, 256, 512 and 1024 symbols respectively. The BER comparisons differ from the previous TEA method in that the performance curves are much flatter and are not as good. The performance of the slice-TEA method is not as affected by changes in the model order estimate. The best overall performance is obtained using a model order estimate of two very closely followed by a value of four.

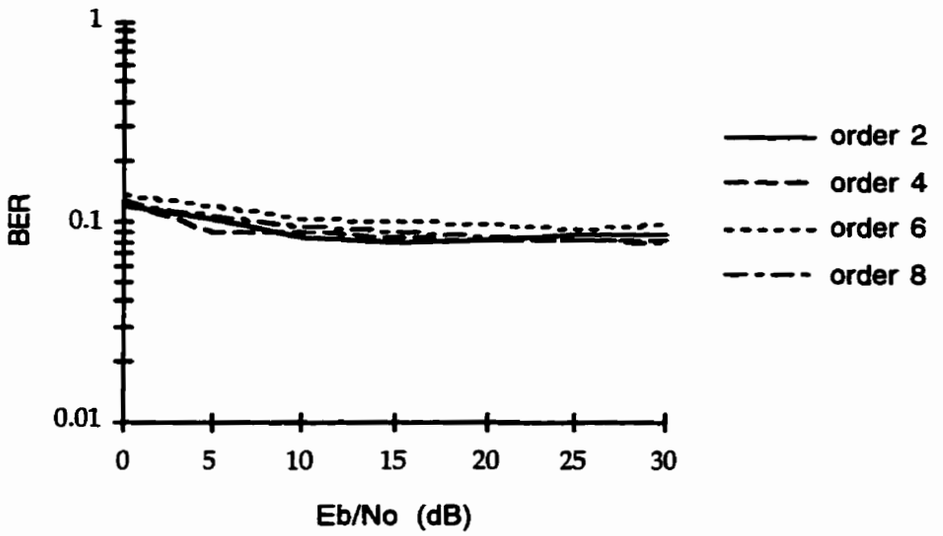


Figure 6.5 - 128 symbol slice-TEA BER order comparison

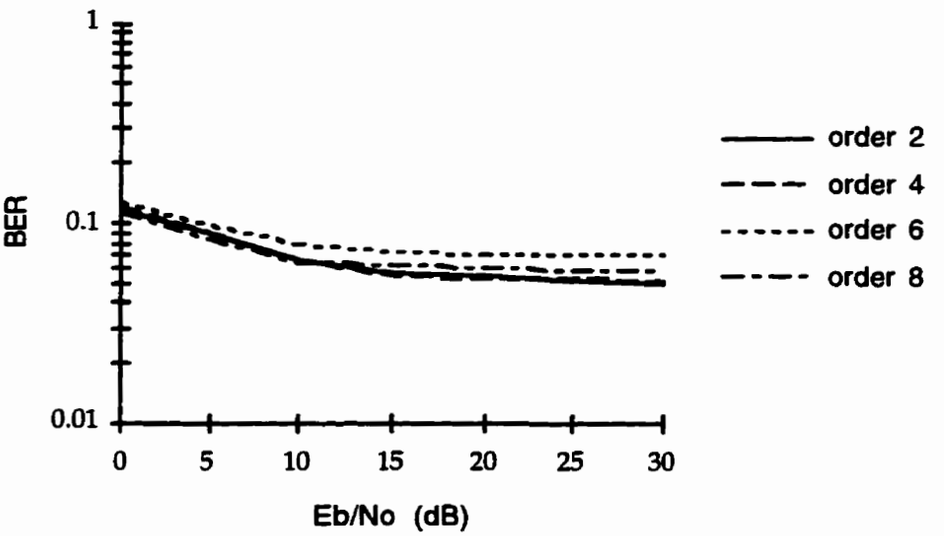


Figure 6.6 - 256 symbol slice-TEA BER order comparison

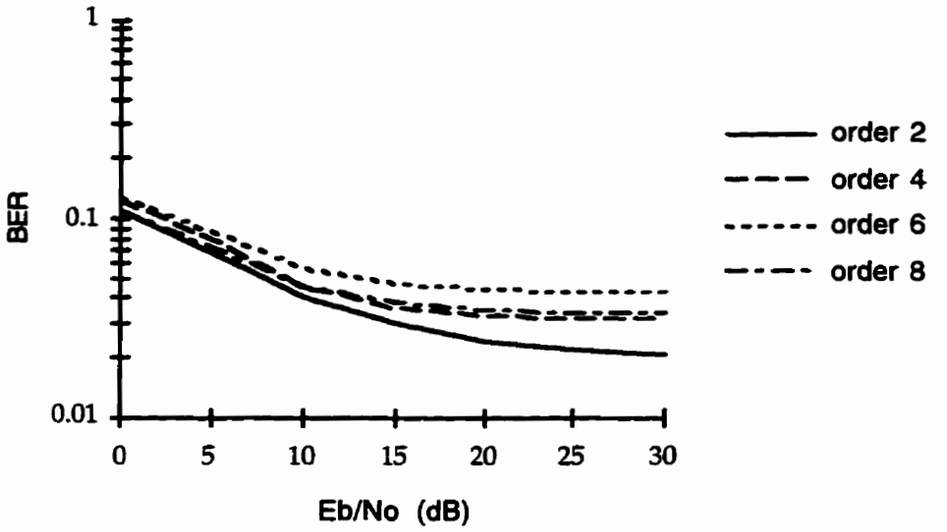


Figure 6.7 - 512 symbol slice-TEA BER order comparison

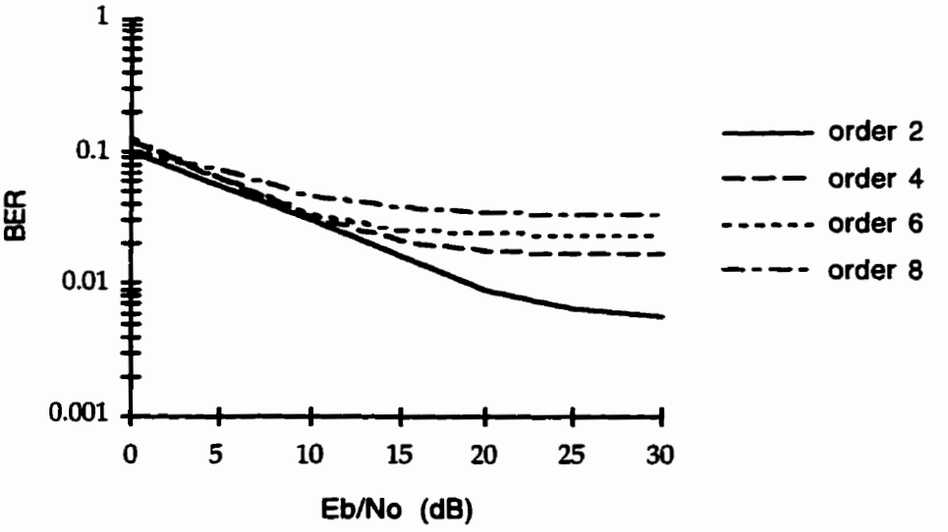


Figure 6.8 - 1024 symbol slice-TEA BER order comparison

In Figures 6.9 and 6.10 the performance of the various block lengths is compared using the TEA and slice-TEA methods. From all these figures it is observed that

longer block lengths are more desirable with the greatest improvement occurring between the block lengths of 128 to 256 symbols. Improvement occurs less dramatically for larger block lengths.

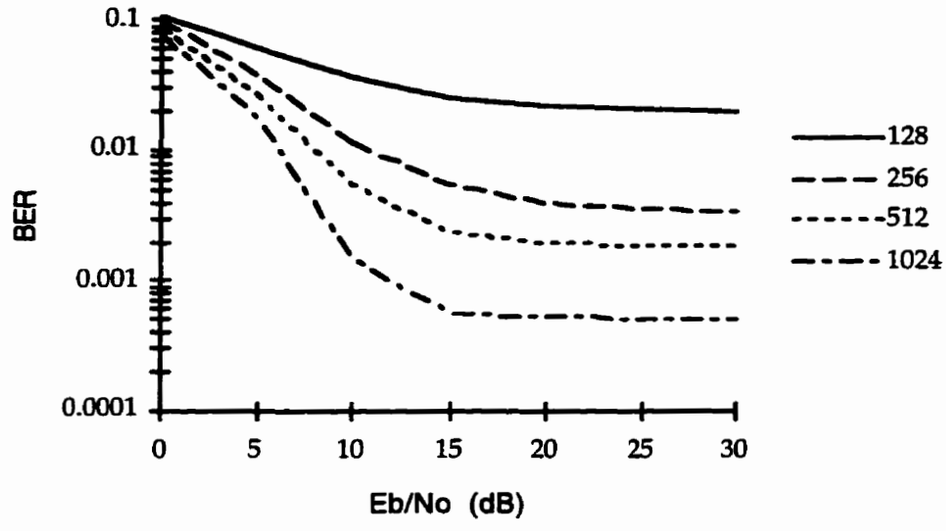


Figure 6.9 - TEA BER block length comparison

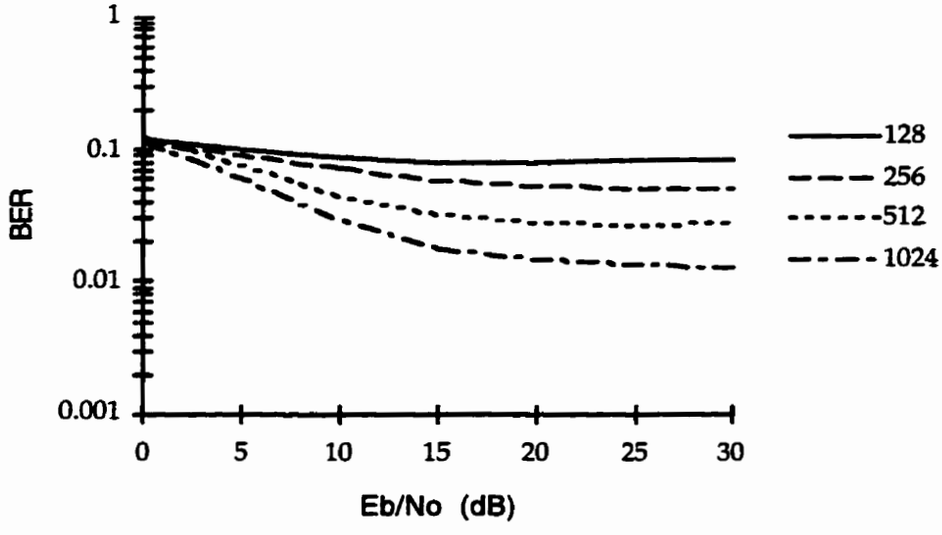


Figure 6.10 - slice-TEA BER block length comparison

## *6.6 Examination of Complex Cumulant Definitions*

Signaling schemes such as Quadrature Amplitude Modulation (QAM) and Quadrature Phase Shift Keying (QPSK) are typical digital modulation schemes which consist of a quadrature and an inphase part and therefore describe a complex data structure. The way in which this complex data is handled by the fourth order cumulant is an important concern. Two complex cumulant definitions are possible depending on whether or not the inphase and quadrature components are independent of each other. If the two components are not independent and identically distributed then the cumulant is defined as (eqn. 2.16):

$$\begin{aligned} c(m,n,l) = {} & E\left[x^*(t)\, x(t+m)\, x^*(t+n)\, x(t+l)\right] \\ & - E\left[x^*(t)\, x(t+m)\right]\, E\left[x^*(t+n)\, x(t+l)\right] \\ & - E\left[x^*(t)\, x^*(t+n)\right]\, E\left[x(t+m)\, x(t+l)\right] \\ & - E\left[x^*(t)\, x(t+l)\right]\, E\left[x^*(t+n)\, x(t+m)\right] \end{aligned}$$

where:

$x(t)$ is the received signal

$m,n,l$ correspond to various lags of the signal

On the other hand, if the inphase and quadrature components are independent of each other then the following simplified definition may be used for the cumulant and is (eqn. 2.24) given as:

$$c(m,n,l) = E\left[x(t)\, x(t+m)\, x(t+n)\, x(t+l)\right]$$

The difference between the two definitions is in the number of terms that are conjugated and in the addition of the second order terms.

The particular definition of the cumulant used has an effect on the performance of the equalizer algorithm. This effect is observed by comparing the performance of the two definitions using the TEA and the slice-TEA methods. The performance is evaluated by comparing the two approaches for various model orders and for various data block lengths. Figures 6.11 through 6.18 compare the two definitions using a model order estimate of two.

In Figures 6.11 through 6.14 the BER performance curves are presented for the conjugated and non-conjugated versions of the TEA method. In all four figures the conjugated versions of the cumulant definition appear to exhibit better performance. It is observed however that as the block length is increased the difference in the performance is not as great. This seems to suggest that as the block sequences become longer the inphase and quadrature components become more independent which is an essential requirement in the unconjugated cumulant version.

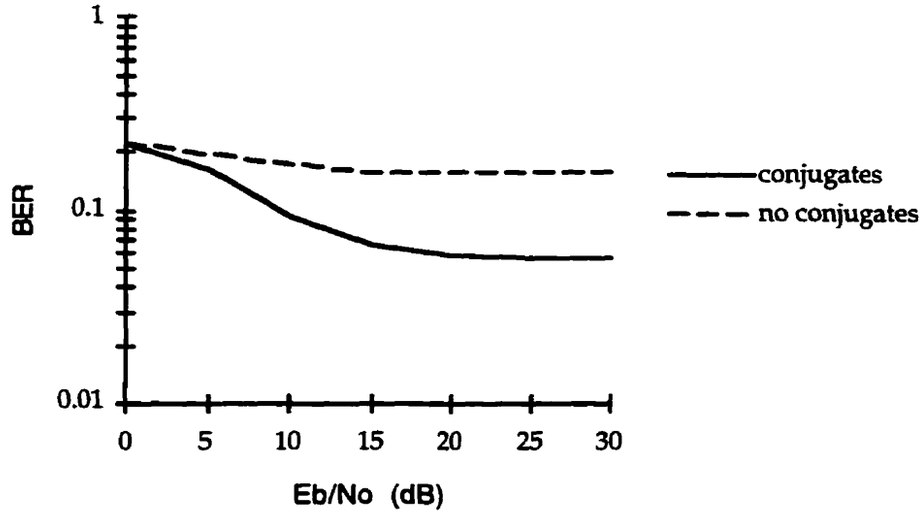


Figure 6.11 - 128 symbol TEA BER complex definition comparison

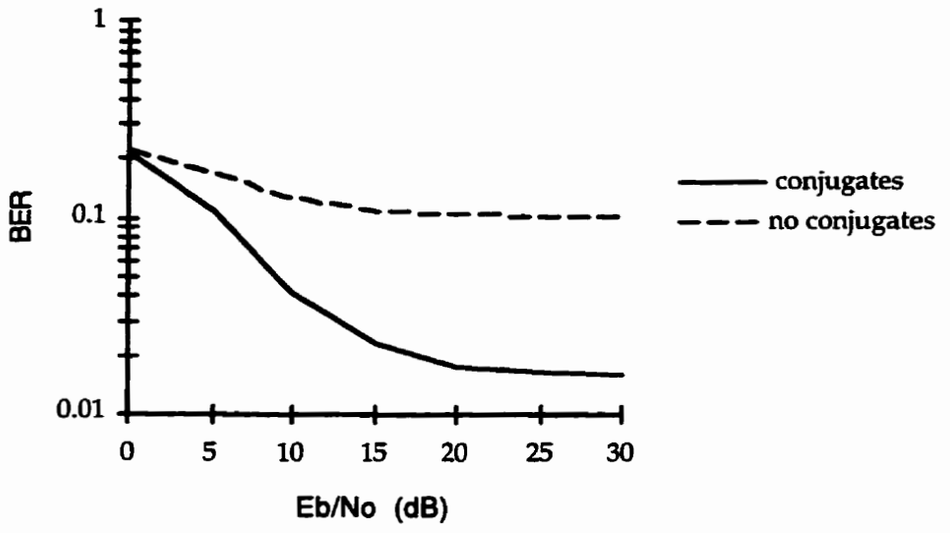


Figure 6.12 - 256 symbol TEA BER complex definition comparison

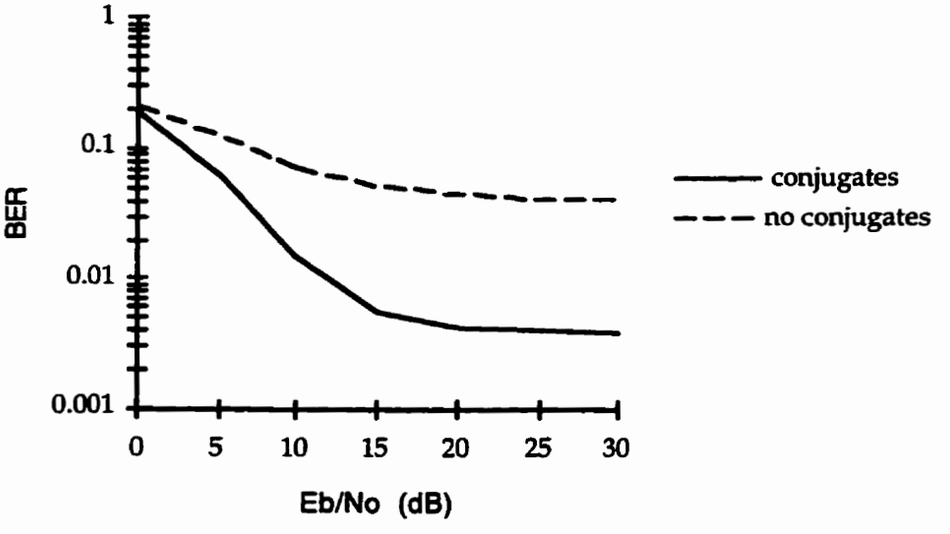


Figure 6.13 - 512 symbol TEA BER complex definition comparison

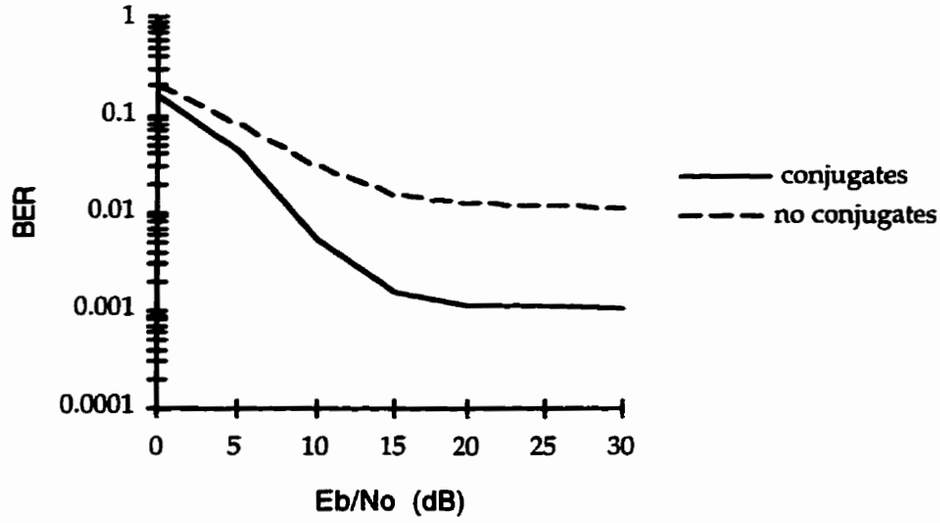


Figure 6.14 - 1024 symbol TEA BER complex definition comparison

In Figures 6.15 through 6.19 the BER performance of the conjugated and non-conjugated versions of the slice-TEA method are presented for various block lengths. In all figures the conjugated definitions of the cumulant exhibit superior performance.

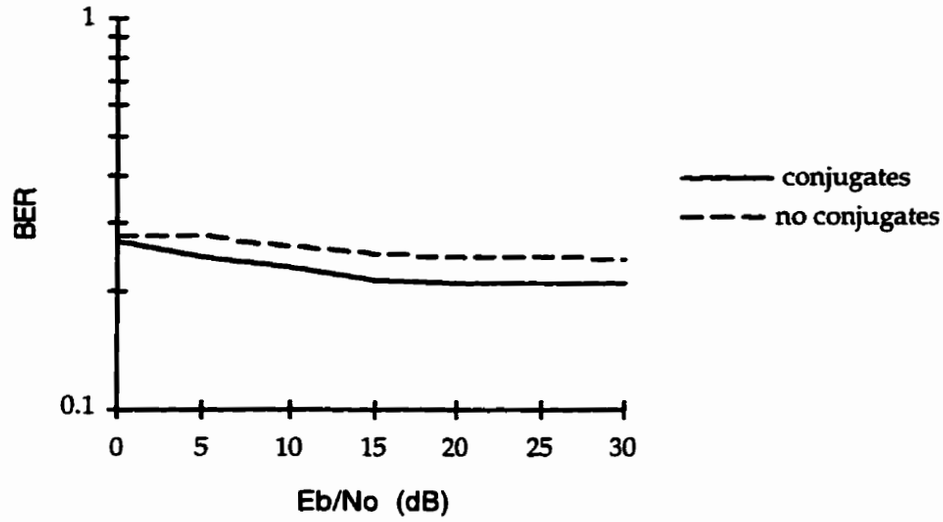


Figure 6.15 - 128 symbol slice-TEA BER complex definition comparison

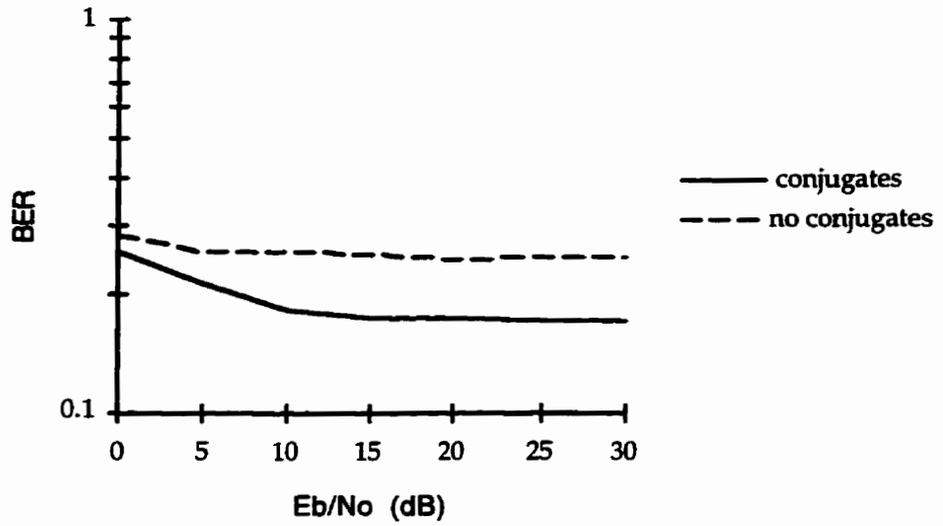


Figure 6.16 - 256 symbol slice-TEA BER complex definition comparison

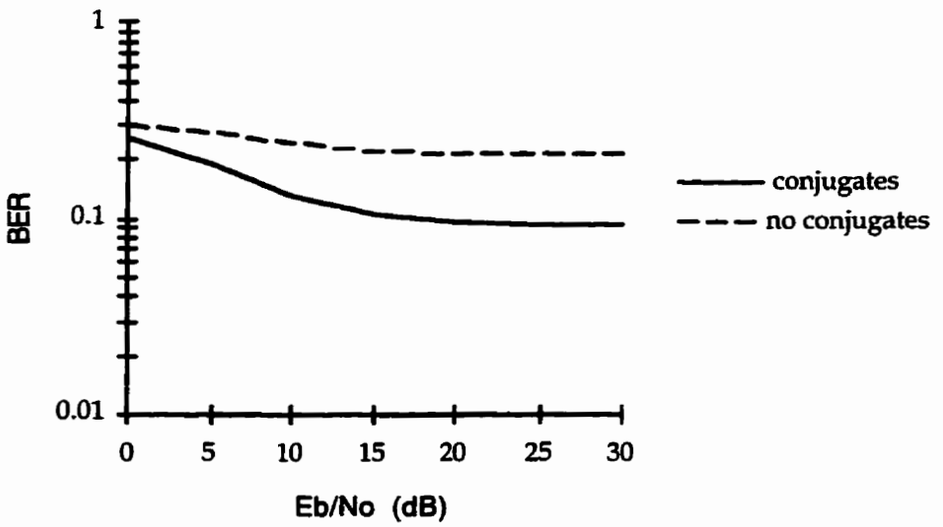


Figure 6.17 - 512 symbol slice-TEA BER complex definition comparison

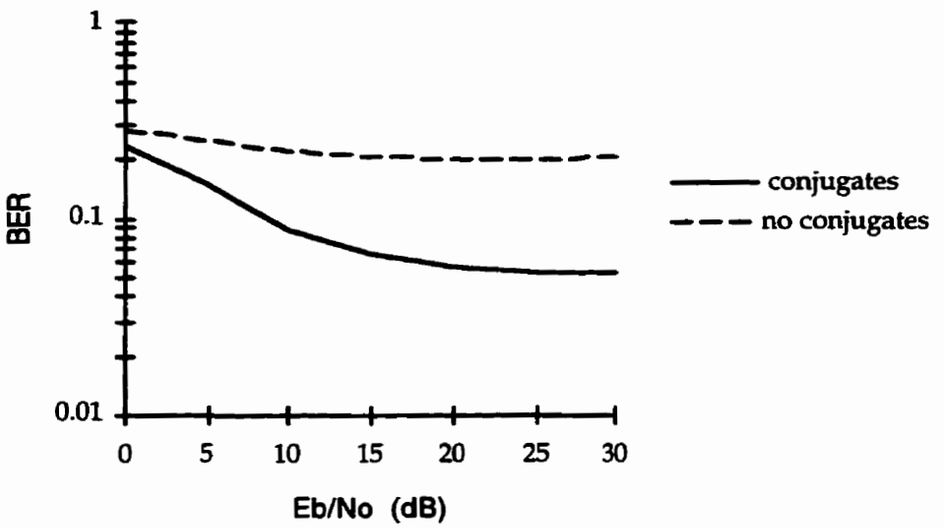


Figure 6.18 - 1024 symbol slice-TEA BER complex definition comparison

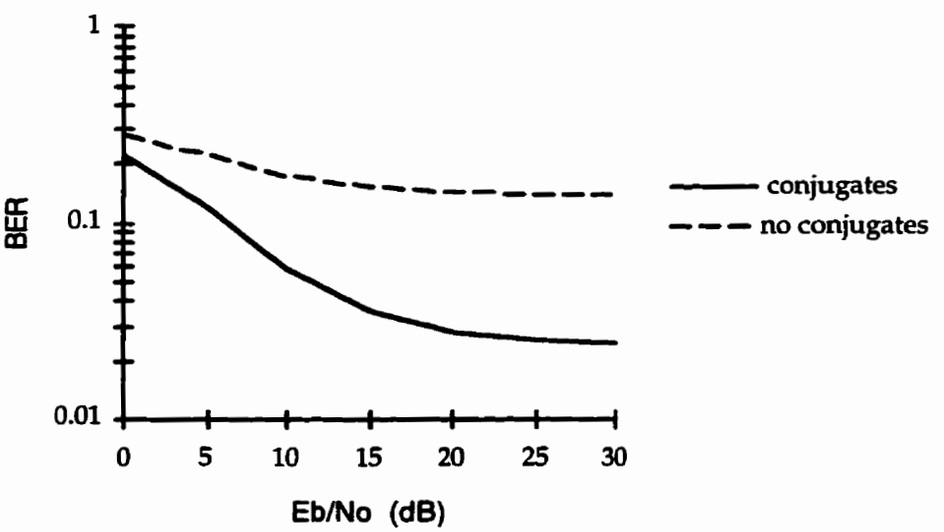


Figure 6.19 - 2048 symbol slice-TEA BER complex definition comparison

In both the TEA and slice-TEA methods, it is observed that an improved performance, indicated by a lower BER, is achieved when using conjugated versions of the cumulants.

## *6.7 Amplitude & Phase decomposition*

In the previous methods, such as the TEA and slice-TEA methods, cumulants estimated from the data at the receiver are used to determine the minimum and maximum phase components of the system's impulse response. An attempt to improve upon the performance of the blind equalization schemes presented so far is sought by the alternative approach of the identification of the system in terms of the amplitude and phase components. The decomposition into these two components allows higher order spectral methods to be applied to the phase component of the system exclusively. The identification of the zero phase amplitude component requires the use of second order statistics which are adequate to resolve minimum and maximum phase equivalent components. This decomposition into the amplitude and phase components results in a somewhat simplified structure for the equalizer.

The first concern to deal with is that of the amplitude equalization part. Amplitude equalization may be carried out using a number of whitening techniques such as predictive filtering. The method suggested here is to use the power cepstra to determine the minimum and maximum phase equivalent components. The performance of this amplitude equalizer is compared to one obtained using perfect cumulant estimates where the TEA method is used to obtain the phase portion. This amplitude and phase combination is referred to as the phase-TEA method. The comparisons between the amplitude equalizers using ideal and actual cumulants appear in Figures 6.20 through 6.23 for various block lengths. The figures illustrate that the amplitude equalizer's performance using ideal cumulants is very close to that of the equalizer using actual cumulant

estimates. The performance of the amplitude equalization using actual estimates improves as the block length is increased.

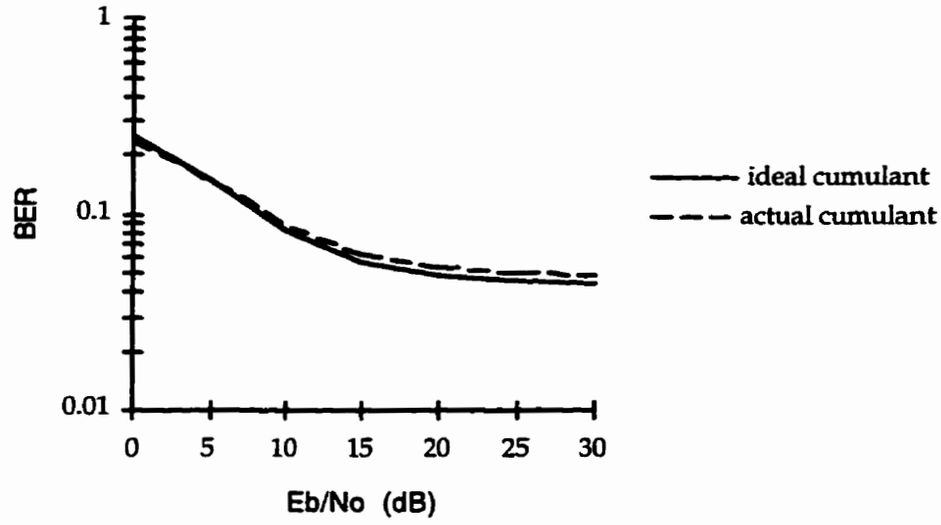


Figure 6.20 - 128 symbol Amplitude Equalization BER comparison

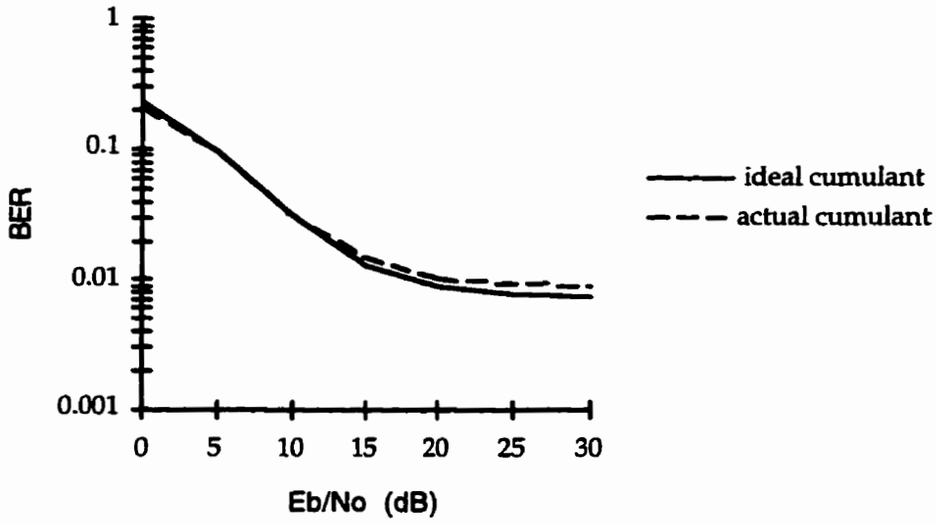


Figure 6.21 - 256 symbol Amplitude Equalization BER comparison

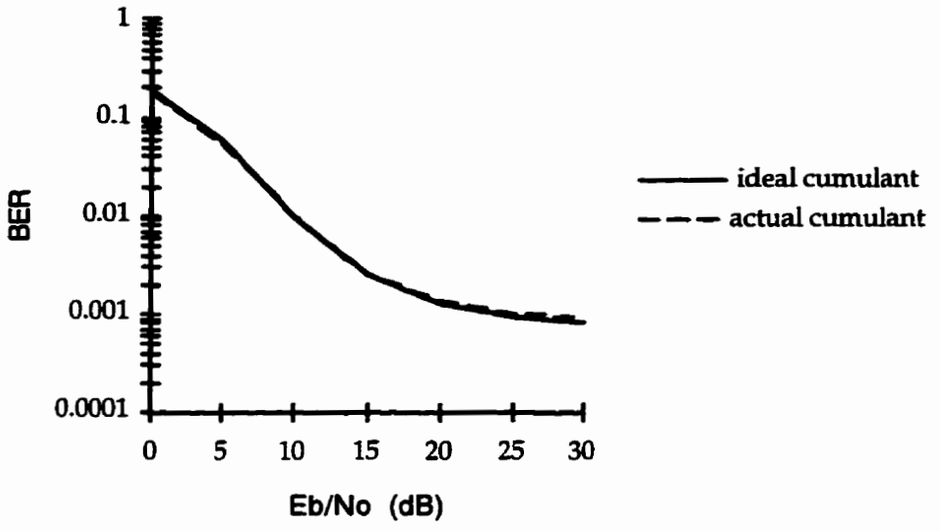


Figure 6.22 - 512 symbol Amplitude Equalization BER comparison

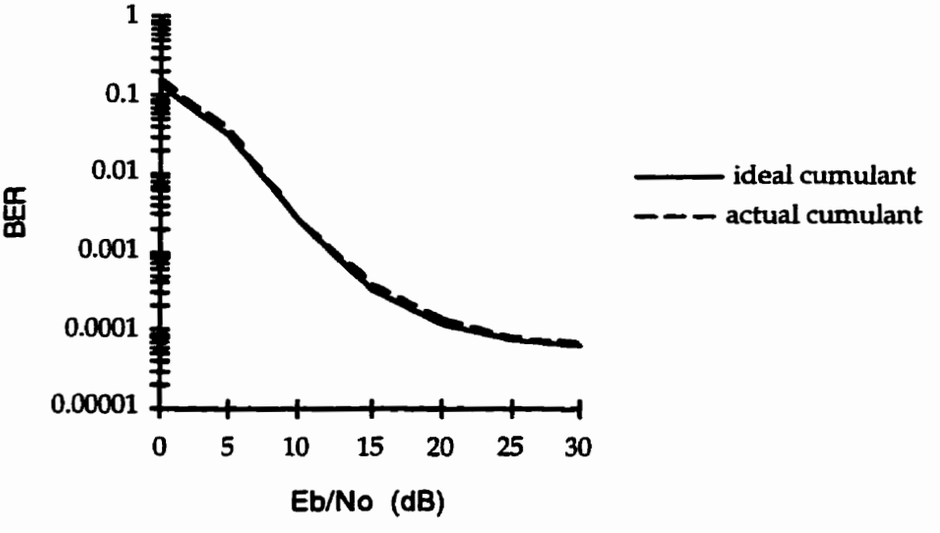


Figure 6.23 - 1024 symbol Amplitude Equalization BER comparison

## *6.8 Amplitude & Phase decomposition using a constrained TEA algorithm*

The TEA method is based on using cumulants estimated from the received data at the receiver and attempts to determine the minimum and maximum phase components separately. As suggested earlier, an alternative approach to the system identification and hence to equalization is to separate the system to be identified into its amplitude and phase part. It is understood that the phase part is not a residual phase resulting from causal minimum phase equivalent amplitude equalization. The identification of the amplitude component requires only the use of second order statistics which are adequate to resolve minimum phase equivalent components. This decomposition into the amplitude and all-pass phase component also results in a somewhat simplified structure for the phase equalizer.

The comparison of the BER performance of the TEA method to that of the proposed whitening and all-pass TEA methods for various block lengths is illustrated in Figures 6.24 through 6.27. In all figures the performance of an equalizer is also illustrated which has been constructed using perfect cumulant estimates. This is equivalent to the construction of an equalizer using a known channel and yields an ideal bound. In analyzing the performance curves, the most significant observation is that the performance is dependent upon the block length. The longer block length allows a better cumulant estimate to be made and is indicated as the performance of the various equalizers tend towards the bound as the block length is increased. The performance of the phase-TEA versus the TEA method is also improves as the block length is increased. In all

figures it is observed that the equalizer scheme utilizing whitening and the all-pass TEA method out performs the other equalization schemes. The improvement in performance is quite significant in that an improvement of approximately an order of magnitude is observed at lower noise levels.

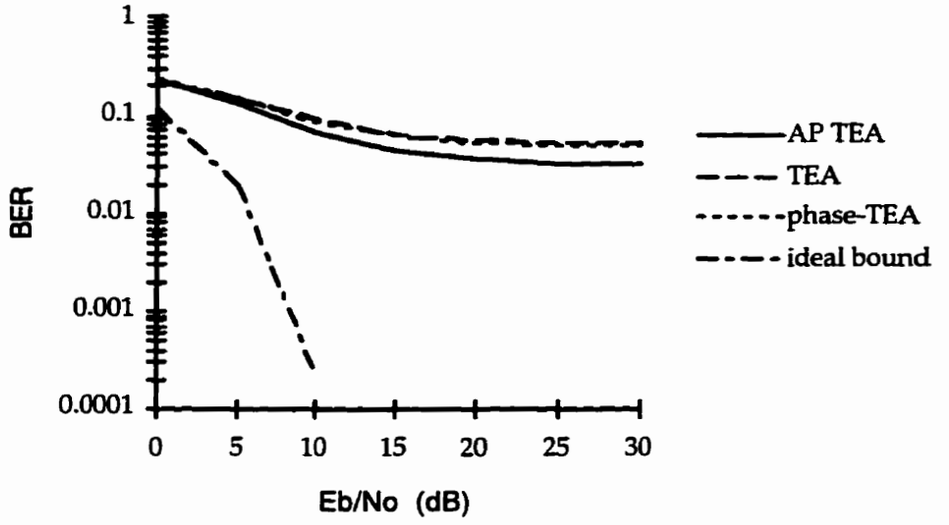


Figure 6.24 - 128 symbol TEA based methods BER comparison

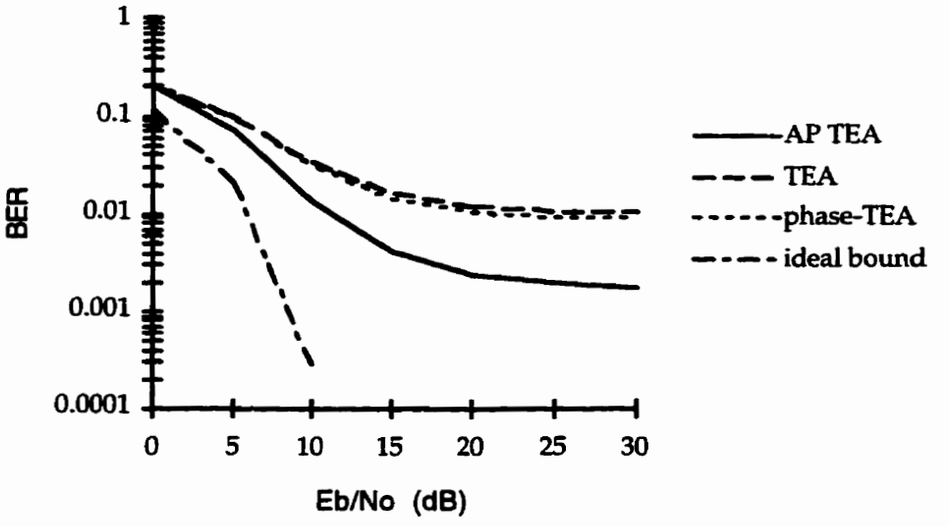


Figure 6.25 - 256 symbol TEA based methods BER comparison

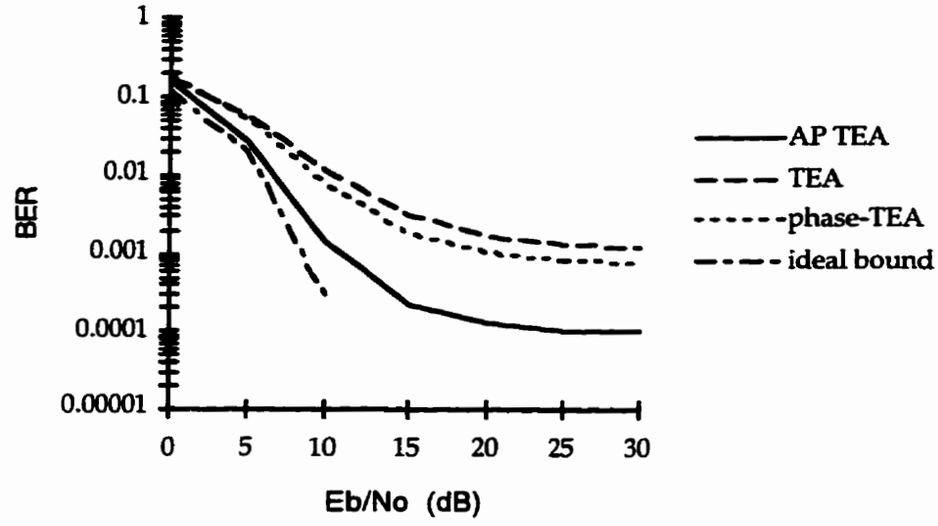


Figure 6.26 - 512 symbol TEA based methods BER comparison

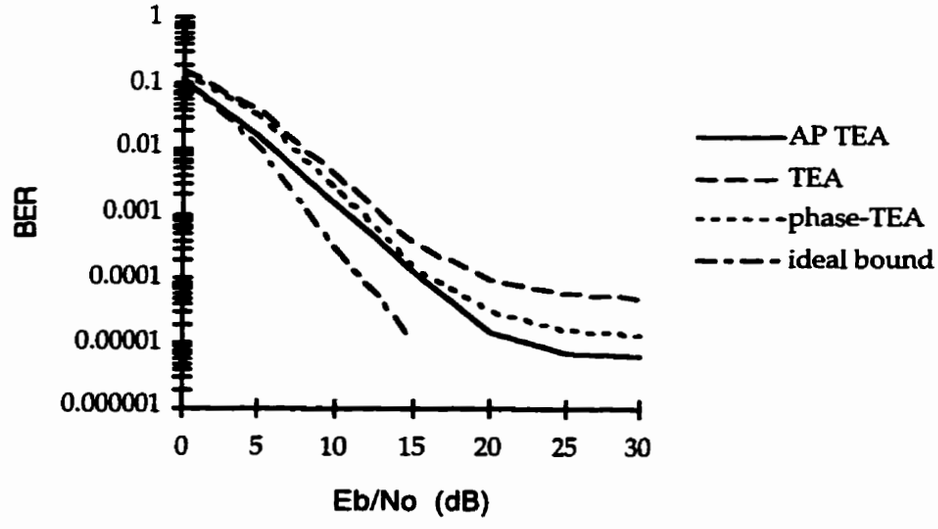


Figure 6.27 - 1024 symbol TEA based methods BER comparison

In Figure 6.27 it is further observed that the BER performance of the various methods begins to converge. The performance at lower noise levels is however still better.

## *6.9 Summary*

In this chapter, the performance of the various blind equalizers based on higher order spectra using homomorphic techniques is examined. The performance is evaluated via computer simulation to produce bit error probabilities representing the number of errors probable during the recovery of the signal at the receiver. The results indicate that performance is dependent upon the ability to obtain good cumulant estimates. Better estimates are achieved with longer data block lengths which then requires a more stationary channel.

In the simulations presented, a flattening out of the performance curves is observed. This effect implies a floor or a limit on the performance which can be expected from the algorithm as measured by the bit error probability. This effect is observed to result from an inability to obtain good cumulant estimates. When perfect cumulant estimates are used this flattening of the BER performance curves does not occur. The cumulant estimate is improved as longer block lengths of data are employed and is observed to yield an improved performance limit but at the expense of an increase in computational requirements.

The decomposition of the channel into amplitude and phase components rather than into minimum and maximum phase components allows an alternative

approach to system identification. The amplitude part is equalized using an equalizer based on second order statistics while higher order spectra is used to determine the phase part. Both the amplitude and phase equalizer portions are able to handle non-causal systems (i.e. maximum phase zeros) by using a technique of data inversion as discussed earlier in section 5.5.

In the proposed all-pass TEA method, the symmetry of the all-pass structure is exploited to yield a constrained solution. The performance of this approach is found to be significantly better than that of the previous methods with an additional reduction in the number of computations required.

The computational complexity of the blind equalization schemes based on the use of higher order spectra is of interest in an actual implementation. Each of the higher order spectra based blind equalization algorithms contain certain operations that are common to all methods. These consist of the cumulant estimation, least squares solution and inverse channel filtering. Only the cumulant estimation part has been separated out. In order to calculate the computational complexity of the all-pass TEA method the following variables along with the value used are defined as follows:

$L = 2$ is the range of cumulant indexes

$p = 2$ is the number of cepstral coefficients

$N = 1024$ is the block length

$M = 2p$ is the number of variables in the least squares problem

$K$ is the number of equations in the least squares solution

The number of computations (additions and multiplications) required for the cumulant estimation is given by:

$$\text{cumulant ops} = (L+p)(32N-8) + 8N + \frac{15}{24}\left(2(L+p)+1\right)^3 - 2$$
$$= 139,686$$

The all-pass TEA method requires the following number of computations in addition to the cumulant estimation:

$$\text{all-pass TEA ops} = 2M^2K + 2MK - 2M^3 - 2M + 8.5p^2 - 14.5p + 16pN + 8$$
$$= 42,624$$

where $K = 2(2L+1)^3 = 250$

The amplitude method requires the following number of computations in addition to the cumulant estimation:

$$\text{amplitude ops} = 2M^2K + 2MK - 2M^3 - 2M + 8.5p^2 - 14.5p + 32pN + 8$$
$$= 65,813$$

where $K = 2(2L+1) = 10$

The combined computational complexity of the amplitude and all-pass TEA for a block length of 1024, excluding the cumulant estimation part, is therefore given as:

$$\text{amplitude and all-pass TEA ops} = 108,437$$

and with the addition of the cumulant estimation the total combined computational complexity becomes:

$$\text{total ops} = 248,123$$

It is observed that the cumulant estimation comprises a significant portion of the computational requirements. In addition, the cumulant estimation also has the greatest effect upon the performance of the algorithm.

# *Chapter 7*
# *Conclusions*

Blind equalization may not just be a luxury but a necessity in many of the circumstances where a training sequence is not available. In a communications system these situations arise in multipoint environments, military situations and in those environments where the training sequence is unreliable because it cannot be found easily. Blind equalizers based on the use of higher order spectral techniques are able to identify non-minimum phase systems. Through the use of the complex cepstrum, homomorphic methods enable linear solutions to be obtained. Using these methods, the use of the Fourier transform and complex logarithm is not required and thus phase unwrapping issues are avoided.

In a communications system, an equalizer is required in a frequency selective environment in an attempt to achieve distortion free transmission. The success of the equalization results in increased data rates. This is desirable in order to utilize available spectrum efficiently. The presence of training sequences reduces useful information transfer due to increased transmission overhead. The complexity of the receiver is reduced by eliminating the training sequence which is possible only by using blind equalization techniques.

Until now the cepstral based blind equalization techniques, such as the TEA and slice-TEA methods, identified the unknown system in terms of the minimum and maximum phase components. In contrast, the proposed equalizer involves the

characterization of the system in terms of the amplitude and phase components. This separation allows optimization of the various equalization processes. It also allows the possibility to separate the equalization functions between the transmitter and receiver which may be of benefit in an actual implementation. The amplitude equalizer component is determined from second order spectra. This is advantageous because second order spectra is adequate to determine amplitude components and also results in less variance. Higher order spectra is used to determine the phase equalizer component. A phase equalizer, when implemented as an all-pass structure, yields a complex conjugate symmetry between its numerator and denominator polynomials. This special structure is used to obtain an optimized solution to the identification of the phase component.

The proposed amplitude and all-pass phase blind equalizer is shown to yield superior performance compared to the TEA and slice-TEA cepstra based blind equalization methods. Superior performance is indicated by fewer bit errors and implies a more reliable transmission. Performance comparisons are made by simulating a digital communications environment using a 4 level QAM modulation scheme. The transmission channel used characterizes a frequency selective channel typical of a radio environment. Performance is evaluated by determining bit error probabilities for various noise levels. A very significant performance improvement increase of an order of magnitude is observed for many of the block lengths.

The performance of the higher order spectra based methods is hindered by the ability to estimate the cumulant. Longer block lengths of data are required to

improve the cumulant estimate. This situation is undesirable because the longer block lengths require an environment which is more stationary. In addition, the computational requirements to obtain a cumulant estimate will increase with longer block lengths. The estimation of the cumulants could possibly be achieved by using neural nets which may result in a reduced computation requirement. The investigation of this suggests a direction for future work.

As a final note, higher order spectral estimation is a field which shows a lot of promise in system identification. Second order spectra estimation is limited in its ability to extract information from a system. It can only be used to obtain amplitude and minimum phase equivalent information. It is also affected by the presence of noise. On the other hand, higher order spectra may be used to obtain much more information from the system and it is not affected to the same degree by the presence of noise. The ability to obtain phase information is very important to the description of the system and cannot be overstated. The proposed all-pass TEA equalizer method provides a significant improvement in performance and provides a framework from which more improvements and applications may be thought of.

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