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Reforming Mathematics in Mathematics Education

by

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ABSTRACT

This dissertation is an interpretative study of teaching and learning mathematics in school. Based on findings from the Third International Mathematics and Science Study (TIMSS), it raises important mathematical and pedagogical issues that arise from sustained efforts to implement the National Council of Teachers of Mathematics (NCTM) Principles and Standards for School Mathematics (NCTM 2000).

Told as a series of nested stories that serve as both data and method, the dissertation uncovers the importance of landmarks that are new to the discourse of the reform of mathematics in mathematics education:

- the role of memory and stories. This study argues the centrality of children's coming to know the discipline of mathematics through the careful telling of key stories of mathematicians who have gone before. Such stories put students in touch with mathematics as a profoundly human enterprise that arises from people's deep longings to explore the world. Sharply contrasting the ubiquitous "story problems" of various trains that leave various stations traveling in various directions, such ancestral stories deliberately raise the same longings and desires in yet another human generation: what are the ever-mysterious contours of now-familiar landscapes such as counting, measuring, predicting, and exploring limits that mathematics was invented to resolve?
- the centrality of conversation in a mathematics classroom. Through dialogue, children and teachers gather, lay out and defend their thoughts, one with one another. The dissertation explores the "watering hole" of mathematical discourse: those places where students and teachers meet for rich mathematical conversation and hotly

contested debates. These are places where the "rightness" of answers are worked out within boundaries established by the actual discipline of mathematics, not as an anxious search for proper procedures.

- creating new mathematics. For many children and teachers, mathematics is a desolate sort of territory in which correct moves and answers are always known in advance, either by the teacher or by the people who wrote the keys at the back of the textbook. Genuine mathematical reform suggests that the classroom must, instead, be a deeply generative place in which the disposition to create new mathematics is carefully cultivated. Thought of in this way, mathematics loses its character as a series of preformulated problems to be solved, and gains a new one. It becomes a way of thinking in which new problems emerge when we learn to cover the ground of old territories in fruitful ways.

Dedicated to Pat

TABLE OF CONTENTS

LIST OF TABLES

ILLUSTRATIONS

Plates

Figures

CHAPTER ONE

Stalled At The Classroom Door

Introduction

Recently, professional mathematics and mathematics education organizations, concerned about the quality of mathematics instruction, have issued calls for reform. Not since the "new math" of the 1960's and 1970's has such an orchestrated change to mathematics curricula and pedagogy been issued (Almeida and Ernest 1996). For many, that is not good news. Few people remember the reforms of "new math" with fondness, but they find themselves caught once again in a call for fundamental changes in mathematics education. This time, unlike the reforms of the "new math" with its enthusiastic infusion of the study of sets, groups and other abstract mathematical structures, these new reforms involve an emphasis on problems (Barrow 1992, Davis 1996, English 1998, Ernest, 1991, 1994). They involve

> empowering individuals to be confident solvers and posers of mathematical problems embedded in social contexts... School mathematical knowledge must reflect the nature of mathematics as a social construction: tentative, growing by means of human creation and decision-making, and connected with other realms of knowledge, culture and social life. (Ernest 1991, 207)

Teachers are used to the roller coaster ride of reform—out with the old, in with the new. To the jaded, the call for reform in mathematics education seems to be just another in the long line of tried and soon-to-be-rejected educational reforms.

> Depending on the dominant concern (cynics might read 'fad') of any particular era, teachers have embraced new methodologies, techniques, and programs: phonics, look-and-say, whole language; the 'new math', manipulatives, computer assisted instruction, the spiral curriculum, open areas, cooperative learning, multi-aging. The names change from era to era, as do the dominant ideologies that inform whatever program is developed, implemented, tested—and ultimately rejected. Teachers talk about this phenomenon as 'the pendulum swing.' (Clifford and Friesen 1994, 4)

And indeed, our efforts to reform in mathematics education might again fail. We have been in this place before, wanting to demonstrate the living nature of the discipline. Calls for reform within mathematics education seem to indicate a desire to connect more strongly with the dynamic nature of the discipline of mathematics itself. This was the impetus for the change of the "new math" of the 1960's and 1970's. "Instead of the old-fashioned emphasis upon arithmetic, calculating interest rates, using logarithms, geometry and calculus," (Barrow 1992, 133) the founders of the "new math," a group of mathematicians known as the Bourbaki, were anxious to demonstrate that mathematics was a human creation and not a divine revelation.

Thirty years ago parents, who were comfortable with traditional mathematics and were often confused by the new approaches, soon discovered that their children were not as proficient with the mathematics they, themselves, knew. Unfamiliar with this "new math," and worried about children's apparent loss of basic skills, parents called for a return to more traditional, more familiar, ways. The idea that mathematics, itself might be changing was unsettling not only for parents but also for many other adults. The general public called for return to "the basics." It was not long before textbooks and teaching returned to more traditional, more familiar ground and the "experiment" with math reform was called a failure.

Our return to a more comfortable view of mathematics, textbooks and teaching satisfied us. But it was not long before news of North American students' deficiencies in mathematics started to surface. Barrow (1992) argues that traditional mathematics curricula and pedagogy have "not adapted to meet the challenges posed by new mathematics and its relationship to the external world and to the activities of mathematicians and computers" (145). What is evident to us now is that, as we returned to a more comfortable way of knowing and doing mathematics in schools, we became further divorced from the changing, dynamic discipline of mathematics and the technological advances that mathematics was creating. "Mathematics, in the common lay view, is a static discipline based on formulas taught in the school subjects of arithmetic, geometry, algebra, and calculus. But outside public view, mathematics continues to grow at a rapid rate" (Steen 1990, 1). Instead of working through the difficulties caused by the "new math", and there were many, we retreated. As a result, school mathematics and the mathematics that mathematicians recognize have become increasingly different disciplines. "School mathematics and the research mathematician's pure mathematics are wholly different areas of study" (Almedia and Ernest 1996, 1).

What Is Mathematics?

When I ask school children "What is mathematics?" they most frequently answer, numbers. It's about numbers. It's plussing and minusing, and timsing and dividing. But this definition does not belong only to schoolchildren.

> Ask this question of persons chosen at random, and you are likely to receive the answer 'Mathematics is the study of number.' With a bit of prodding as to what kind of study they mean, you may be able to induce them to come up with the description, 'the *science* of numbers.' But with that you will have obtained a description of mathematics that ceased to be accurate some two and a half thousand years ago! ...
>
> In fact, the answer to the question, 'What is mathematics?' has changed several times during the course of history. (Devlin 1997, 1)

Mathematics was the study of number up to 500 B.C. During this time Egyptian and Babylonian mathematics, dominated by arithmetic, formed the mathematical landscape. From 500 B.C. to 300 A.D. Greek mathematicians searched for ways to measure the Earth and the heavens. Earth measure and geometry were born and the mathematical landscape expanded and changed to include the study of shape. This was a time of great mathematics and great mathematicians—Pythagoras, Eudoxus, Euclid, Archimedes and Eratosthenes. During this time, Euclid introduced two new concepts to mathematics through geometry—the definition and the axiom. Even now, these form the bedrock of contemporary mathematics. "In fact, it was only with the Greeks that mathematics came into being as an area of study, and ceased being a collection of techniques for measuring, counting and accounting" (Devlin 1997, 2).

The mathematical landscape remained relatively unchanged until the middle of the seventeenth century. "Until Newton's great discoveries, it never occurred to 'scientists' that mathematics could be used to express basic principles about nature or the universe itself" (Motz and Weaver 1993, 125). Working independently, Newton, in England and Leibniz, in Germany both invented the calculus—the study of motion and change, "which led to one of the most famous controversies in the history of mathematics, and to the most heated intellectual rivalry between two nations" (Motz and Weaver 1993, 125).

> With the introduction of techniques to handle motion and change, mathematicians were able to study the motion of the planets and of falling bodies on earth, the workings of machinery, the flow of liquids, the expansion of gases, physical forces such as magnetism and electricity, flight, the growth of plants and animals, the spread of epidemics, the fluctuation of profits, and so on. After Newton and

> Leibniz, mathematics became the study of number, shape, *motion, change and shape* [sic]. (Devlin 1997, 2)

At first, mathematicians and physicists (who were one and the same during this time) directed their energies to the applications of calculus. But with time, another shift occurred in mathematics, and the mathematical landscape again expanded and changed. As mathematicians worked with the enormous power that the calculus provided, new types of mathematics were created. At the turn of this century mathematics had grown from three distinct categories—arithmetic, geometry and calculus—to twelve distinct categories.

> The explosion of mathematical activity that has taken place in the present century has been dramatic. In the year 1900, all the world's mathematics knowledge would have fitted [sic] into about eighty books. Today it would take perhaps 100,000 volumes to contain all known mathematics. (Devlin 1997, 3)

Today, mathematics includes almost seventy distinct categories, with some categories like algebra or topology, split into subfields. Entirely new categories of study in mathematics such as complexity theory and dynamical systems theory are being created. One of the reasons for this change seems to be the increasing use of computer technologies. "Nothing in recent times has had as great an impact on mathematics as computers..." (Inkpen 1997, 1).

> Not since the time of Newton has mathematics changed as much as it has in recent years. Motivated in large part by the introduction of computers, the nature and practice of mathematics have been fundamentally transformed by new concepts, tools, applications and methods. Like the telescope of Galileo's era that enabled the Newtonian revolution, today's computer challenges traditional views and forces re-examination of deeply held values. As it did three centuries ago in the transition from Euclidian proofs to Newtonian analysis, mathematics is undergoing a fundamental reorientation of procedural paradigms. (Steen 1990, 7)

The possibility that our understanding of the nature of mathematics itself (and not just the nature of teaching math) might be re-forming is difficult for most of us to comprehend. Introduced to mathematics through traditional teaching and textbooks, we came to know it as a fixed, logical, rational, absolute, objective, pure, abstract, and certainly unchanging discipline. This view was reinforced as we went about trying to solve "mainly unrelated routine mathematical tasks which involv[ed] the application of learnt procedures, and [we learned] that every task [had] a unique, fixed and objectively right answer, coupled with [teacher] disapproval and criticism of any failure to achieve this answer" (Ernest 1996, 1). Many of us have carried this view of mathematics with us into our adult world. For us, "mathematics is identified with a rote recitation of facts and a blind carrying out of procedures. Decades later this robotic mode of behavior kicks in whenever a mathematical topic arises. Countless people feel that if the answer or at least a recipe for finding it doesn't come to them immediately, they'll never get it" (Paulos 1991, 53).

The changes that are being called for in mathematics education this time appear to be forced by the changes that are occurring in the practice of mathematics. "Standard school practice, rooted in traditions that are several centuries old, simply cannot prepare students adequately for the mathematical needs of the twenty-first century" (Steen 1990, 2). Asking the question, What is mathematics? is important if we are to move forward with reforms this time. "Scholarly work in mathematics education has recently begun to look deeply at what mathematics is. This increased interest in what was once the purview of philosophers of mathematics grows from a recognition that both teaching and learning mathematics are intimately connected with doing mathematics" (Williams 1995, 184).

Currently, conversations involving the nature of mathematics do not seem to be a high priority for teachers and prospective teachers, those most responsible for bringing these reforms to life.

> More often than not, my efforts to discuss the nature of the subject matter have been regarded as irrelevant time-wasters. I must confess to a certain despair when faced with this sort of response, especially when it is manifested among pre-service and practicing teachers. In simplest terms, I hold little hope for any meaningful change in the teaching of mathematics until we are willing and able to interrogate

> earnestly the subject matter we are claiming to teach. A failure to do so, I fear, will compel us to reenact the same fragmenting and reductive practices that have recently come under harsh critique. (Davis 1996, 56)

Clearly, it is important for teachers and researchers to explore the nature of mathematics, itself, as they search for ways to improve the teaching of mathematics in school. However, as Davis (1996, 80) cautions, we need to be careful about how we ask the question, What is mathematics? "...In posing the question in those terms, there is an implication that we can somehow consider the body of knowledge as determinable, fixable, and separable from ourselves—as though we could somehow step outside of our mathematics".

Our conversations about the question "What is mathematics?" must go beyond simply discussing the nature of mathematics. Once we fix mathematics, separate it from ourselves, we lose sight of the fact that mathematical knowledge "emerges from our actions in the world and from our interactions with one another" (Davis 1996, 74). It is not preexistent, nor does it live in any one of us, yet it requires us. However, when our only experience of formal mathematics is through the schooled delivery of conclusions, then we come to know math as given—a determinable, static subject constituted by unchallengeable and unchanging truths. Teachers and the textbooks possess the facts. Their task is to transmit those facts to students. "This image of mathematical practice portrays mathematics as a dead subject—inquiry is unnecessary because our concepts have been formally defined in the 'right' way and our theorems demonstrated by linear and formal means" (Wilensky, 1993, 21).

"To do mathematics means to produce new mathematics" (King 1995, 34). But what exactly does that mean? From the time we enter formal schooling we are steeped in traditional school mathematics.

> All of us have endured a certain amount of classroom mathematics. We lasted, not because we believed mathematics worthwhile, nor because, like some collection of prevailing Darwinian creatures, we found the environment favorable. We endured because there was no other choice. Long ago someone had decided for us that mathematics was important for us to know and had concluded that, if the choice was ours, we would choose not to learn it. So we were compelled into a secondary school classroom fronted with grey chalkboards and spread with hard seats. A teacher who had himself once been compelled to this same place stood before us and day after day poured over us what he believed to be mathematics as ceaseless as a sea pours forth foam. (King 1992, 15-16)

As mathematics education re-forms itself, the question of what to teach and how to teach it is critical. If mathematics is not simply a closed and given axiomatic system but in fact a living discipline inspirited by ongoing questions, quarrels and conversations, then the pedagogy of mathematics is not an afterthought but a necessity. If mathematics lives in its continual re-forming, then how do we create a mathematical education that allows the young to experience the creation of mathematics? When "outside the closed circle of professional mathematicians, almost nothing is known of the true nature of mathematics or of mathematics research" (King 1992, 5), how do we begin to answer the calls for reform in mathematics education? What do we reform? What do we do differently?

Mathematics Education Reform Efforts

In the past decade, much rethinking has gone into mathematics educational reform in terms of curriculum, pedagogy, and epistemology (NTCM 1989, 1991, 1995; Grouws 1992). In 1989, the National Council of Teachers of Mathematics (NCTM) set forth the document *Curriculum and Evaluation Standards*, followed by *Professional Standards for Teaching Mathematics* in 1991 and *Assessment Standards* in 1995. These three documents, along with myriad support documents, were intended to provide recommendations to improve and reform mathematics education.

Although NCTM is a mathematics education research organization located in the United States, its call for reform was felt within the western provinces of Canada. In June 1995, the province of Alberta initiated a new mathematics curriculum. This curriculum, unlike any other before it, was the collaborative effort of Manitoba, Saskatchewan, Alberta, British Columbia, Yukon Territory and the Northwest

Territories. *The Common Curriculum Framework for K-12 Mathematics* (1995) was the first in a series of joint development projects in basic education. This curriculum, with its unmistakable relationship to the National Council of Teachers of Mathematics *Curriculum and Evaluation Standards* document, "identifies beliefs about mathematics, general and specific student outcomes and illustrative examples agreed upon by the six jurisdictions" (1).

Curriculum reform efforts by both the National Council of Teachers of Mathematics and the Western Canadian Protocol for Collaboration in Basic Education attempt to communicate clear, high expectations for students in mathematics. Organizations such as the Canadian Mathematical Society, the American Mathematical Association, the Canadian Forum for Mathematics Education, and the Pacific Institute of Mathematics have joined in the efforts to increase mathematical literacy for students. These reform efforts go beyond what traditional school mathematics has offered students.

> Traditional school mathematics picks very few strands (e.g., arithmetic, geometry, algebra) and arranges them horizontally to form the curriculum: first arithmetic, then simple algebra, then geometry, then more algebra, and finally—as if it were the epitome of mathematical knowledge—calculus. This layer-cake approach to mathematics education effectively prevents informal development of intuition along the multiple roots of mathematics. Moreover, it reinforces the tendency to design each course primarily to meet the

> prerequisites of the next course, making the study of mathematics largely an exercise in delayed gratification. To help students see clearly into their own mathematical futures, we need to construct curricula with greater vertical continuity, to connect the roots of mathematics to the branches of mathematics in the educational experience of children. (Steen 1990, 4)

> There is an endemic dissonance between the discourse about mathematics led by philosophers and logicians and picked up by educators and mathematicians and the actual practices of the creative mathematician.
> Recently, the discourse about mathematics has begun to change bringing the two views into greater harmony. Instead of viewing mathematics as part of the rationalist tradition in which truth and validity are primary, a new paradigm is emerging, a view of mathematics through an interpretative framework in which meaning making is primary. (Wilensky 1993, 20)

Currently, however, most reform efforts live only in this discourse. Despite some apparent, surface changes, neither the efforts of mathematics and mathematics education organizations, the publication of new curriculum documents, nor teachers' awareness of these documents, have fundamentally changed the nature of teaching and learning within the mathematics classroom (Borko et.al. 1992; Ernest 1991; Hoyles 1992; Lerman 1997; Schmidt et.al. 1996; Senger 1996; Thornton et.al. 1997; U.S. Department of Education, National Center for Education Statistics, 1996).

It appears that current reform efforts are stalled at the classroom door. Simon (1995) attributes this to the that fact that "traditional views of mathematics, learning and teaching have been so wide-spread that researchers studying teachers' thinking, beliefs, and decision making have had little access to teachers who understood and were implementing current reform ideas" (118). Research in this area seems to indicate that despite concentrated reform initiatives since 1989, "the mathematics classroom of today is not recognizably different from the classroom of one hundred years ago" (Wilensky 1995, 20). That this is particularly true in North America was one of the findings of the *Third International Mathematics and Science Study*.

The Third International Mathematics and Science Study

Background

In 1995, the International Association for the Evaluation of Educational Achievement (IEA), an association of universities, research institutes and ministries of education that conduct cooperative international research studies in education, conducted its largest and most comprehensive study, *The Third International*

Mathematics and Science Study (TIMSS)[1]. Its aim was to inform educators around the world about exemplary instructional practices and student outcomes in mathematics and science.

Forty-five countries participated in TIMSS, involving a half-million students at five different grade levels.

> TIMSS is significant not only because of its scope and magnitude, but also because of innovations in its design. In this international study the National Center for Education Statistics (NCES) combined multiple methodologies to create an information base that goes beyond simple student test score comparisons and questionnaires to

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[1] The Third International Mathematics and Science Study is comprised of many components: *Mathematics Achievement in the Primary School Years*; *Mathematics in the Middle School Years*; *Mathematics and Science Achievement in the Final Year of Secondary School; Characterizing Pedagogical Flow: An Investigation of Mathematics and Science Teaching in Six Countries; Mathematics Textbooks: A Comparative Study of Grade Eight Texts; Curriculum Frameworks for Mathematics and Science; Pursuing Excellence; The TIMSS Videotape Classroom Study: Methods and Preliminary Findings; Many Visions, Many Aims: A Cross-National Investigation of Curricular Intentions in School Mathematics;* and *Case Study Literature Review of Education Topics in Germany, Japan and the United States.* This list is not complete. It is intended to give the reader an idea of the scope and magnitude of this study. It is also intended to help the reader understand why authors I cite change but I still refer to the TIMSS study. In this paper, when I refer to TIMSS researchers, I intend the *Third International Mathematics and Science Study* in its entirety. When I refer to a particular research component of TIMSS, I make direct reference to it.

> examine the fundamental elements of schooling. Innovative research techniques include analyses of textbooks and curricula, video-tapes, and ethnographic case studies. (U.S. Department of Education 1996, 3)

> A rigorous quality control program ensured that the data were gathered from representative samples of comparable populations, that the instruments were not biased, and that the data collection and processing standards were of high quality. (Robitaille, Taylor and Orpwood 1997, 37)

TIMSS Achievement Results

The findings of Mathematics Achievement in the Primary School Years—Grades 3 and 4 were published in June 1997. Reporting on the results, the U.S. Department of Education, National Center for Education Statistics, 1997, Commissioner of the National Center for Education Statistics (NCES), stated that:

> In mathematics, seven countries score above the United States; six countries are similar; and 12 countries are below us. Our students' scores are below those of Japan, not significantly different from those of Canada, and are significantly higher than those of England. (U.S.

> Department of Education, National Center for Educational Statistics 1997, 1)

About 16,000 Canadian Grades 3 and 4 students participated in TIMSS. The achievement portion of the mathematics test required that students answer 102 questions. Of these 79 were multiple-choice questions and 23 were free-response questions.

> Over 89 percent of the items were considered suitable for the curricula studied by Canadian students, with B.C. rating the highest proportion as appropriate; and Newfoundland, the lowest. Analysis shows that the mean percent correct scores seem not to be affected much by either the selection of items used in calculating the scores or the proportion of items considered appropriate. (Robitaille, Taylor and Orpwood 1997, 4).

Grade 4 TIMSS Results

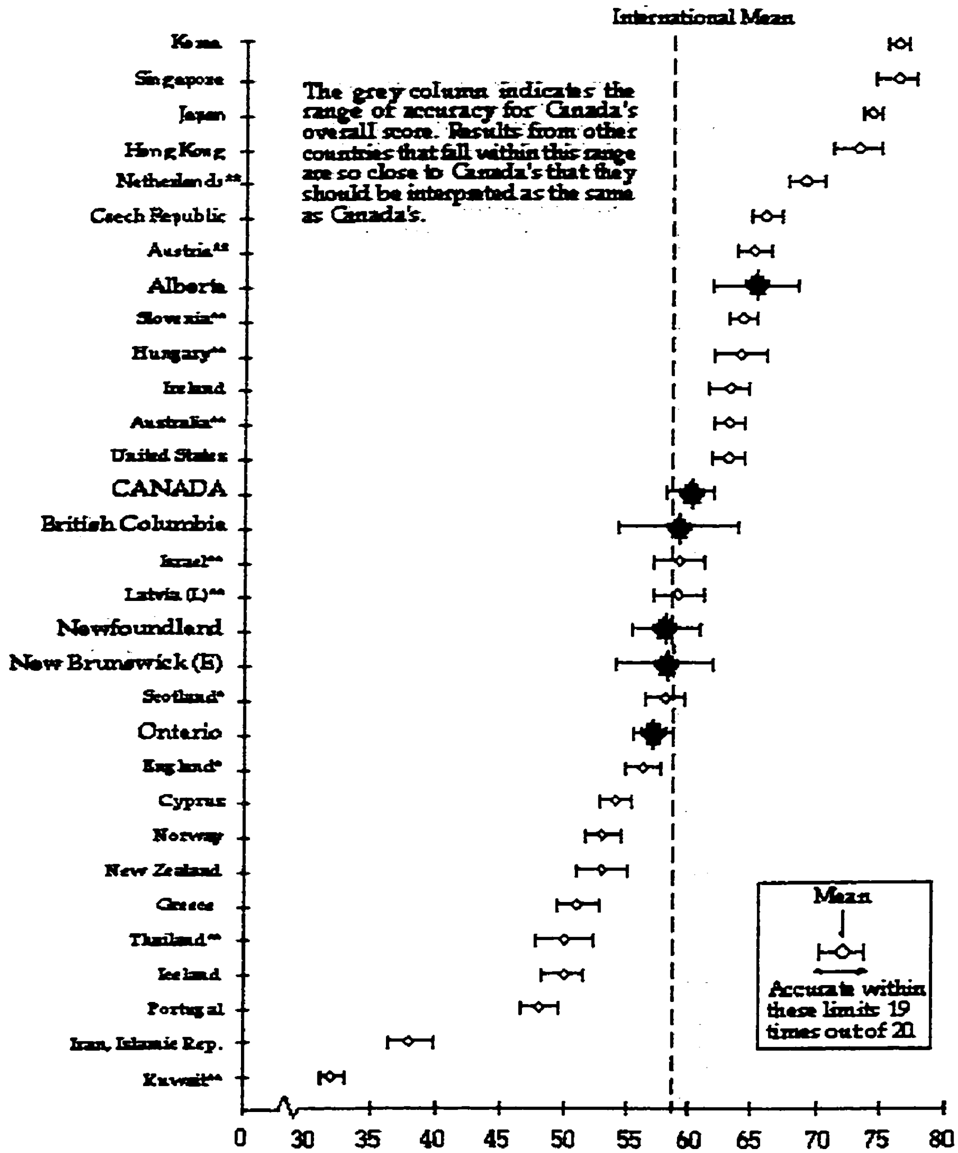

Figure 1.1
(Robitaille, Taylor, and Orpwood, 1997, 4)

Canadian fourth graders scored 532, three points above the international average of 529, and American students scored 545, sixteen points above the international average. Canadian students exceeded the international average in four of the six mathematics areas tested—geometry; whole numbers; patterns, relations, and functions; and data representation, analysis, and probability. In the two mathematical areas of measurement, estimation, and number sense and of fractions and proportionality did Canadian fourth graders score below the international average.

The findings of the Mathematics Achievement in the Middle School Years—Grades 7 and 8—indicate that eleven countries scored above Canada; four countries are similar; and twelve countries are below us. Canadian Grade 8 students attained a mean of 59 percent, four percentage points higher than the international mean, but significantly below the mean of Singapore, Korea, Japan and Hong Kong.

The TIMSS mathematics achievement test at the Grades 7 and 8 levels involved answering 151 items of which 128 were multiple-choice questions and 23 were free-response questions.

> Over 90 percent of the items were considered suitable for Canadian students, with B.C. rating the highest proportion as appropriate; and Ontario, the lowest. Analysis shows that the mean percent correct scores seem not to be affected much by either the selection of items used in calculating the scores or the proportion of items considered appropriate. (Robitaille, Taylor, and Orpwood 1997, 4)

Canadian eighth grade students exceeded or met the international average in all six of the six mathematics areas tested—fractions and number sense; geometry; algebra; data representation, analysis and probability; measurement; and proportionality.

Grade 8 TIMSS Results

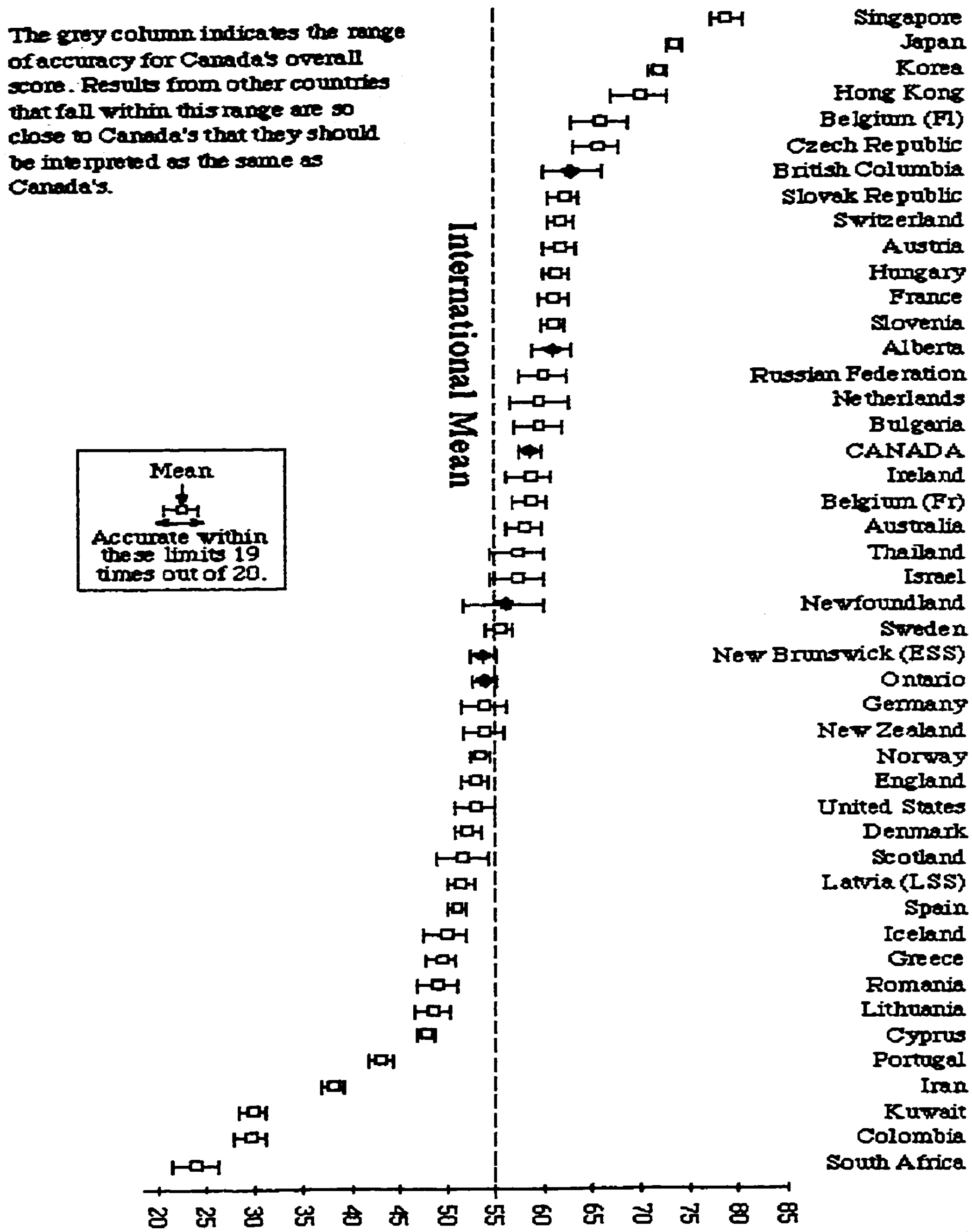

Figure 1.2
(Robitaille, Taylor, and Orpwood 1997, 4)

What Does It Mean?

The most obvious, and most widely discussed finding of TIMSS in the popular media is that Canadian and American students are lagging behind their Asian counterparts in mathematical achievement. "You see," I hear people say, "the Japanese are beating us. Our kids don't know enough math. We should stop with all these new ideas and go back to the basics." People such as E.D. Hirsch Jr., in the United States, and Dr. Joe Freedman, in Canada, have responded to the relative weakness of the North American showing by denouncing math reform efforts and calling for a return to "the basics" (Bracey 1998, Ireland 1998). Unfamiliar with the study itself, such critics of the current reform efforts point to North America's mediocre performance and blame the reform efforts. However, TIMSS findings indicate that 95 percent of U.S. eighth-grade students in North America have not been impacted by the reform initiatives while the higher achieving Japanese students have. Ironically, in light of North American criticism of reform efforts, "Japanese mathematics teaching more closely resembles the recommendations of the U.S. reform movement" (U.S. Department of Education, National Center for Education Statistics 1996, 70). The problem is not that new-fangled changes in North American approaches to teaching mathematics has failed students. In fact, the opposite seems to be the case: although "most U.S.…math teachers report familiarity with reform recommendations, …only a few apply the key points in their classrooms" (U.S. Department of Education, National Center for Education Statistics 1996, 70). Actual analysis of TIMSS findings demonstrates that

> the current push to see math and science classrooms return to the basics is based on 'a rash assumption unsupported by data.' The TIMSS data on what and how math and science are taught, the researchers say, 'are far from being a reflection of ill-conceived reforms. Instead, the empirical patterns observed reflect a widespread choice to focus on basics. (Lawton 1998, 1)

As we search for ways to improve mathematics education this time, it is important to remember the lessons we learned from the "new math" initiatives. We cannot afford to let the general public and uninformed critics gain popular support. Equipped with the extensive data that TIMSS provides, education researchers, mathematicians, politicians and teachers working together can effectively chart a course towards meaningful reforms. "TIMSS clearly and accurately provides a wealth of useful data and information on curriculum, instruction, teacher and student lives, and student achievement" (U.S. Department of Education, National Center for Education

Statistics1996, 4). TIMSS data from primary and middle schools[2] go a long way toward shedding light on the questions that lie at the heart of this dissertation: how do we begin to answer the calls for reform in mathematics education? What do we reform? What do we do differently? In particular, the data suggest that mathematics teaching in high-performing countries closely resembles reform initiatives recommended by the National Council of Teachers of Mathematics, initiatives hotly contested in the popular press and widely misunderstood both outside and inside the profession.

Searching For Explanations

A number of popular explanations are often mentioned as discussions arise about what factors contribute to North American students' weakness in mathematics. Some of these factors center on perceived problems with the children: Canadian and American kids watch too much TV; they are involved in too many extra-curricular activities; Asian students do far more homework. Some of the perceived problems lie in the classroom. It is not unusual to hear complaints about classes that are too large,

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[2] *The Mathematics and Science Achievement in the Final Year Of Secondary School*, IEA's Third International Mathematics and Science Study, was released in January 1998. Unlike the TIMSS study of *Mathematics Achievement in the Primary Years and Mathematics Achievement in the Middle School Years, the Mathematics and Science Achievement in the Final Year of Secondary School* involved a smaller sample of countries, which included Canada and the United States. The Asian countries did not participate in this study. Because of this fundamental difference in the design of the Secondary study, I will not include its data in my analysis.

teachers who lack education, or teaching styles that place too much emphasis on group work.

The richness of TIMSS data that was collected from students, teachers, and school principals makes "it possible to examine differences in current levels of performance in relation to a wide variety of variables associated with the classroom, school, and national contexts within which education takes place" (Mullis et.al. 1997, 10).

TIMSS Questionnaires

All students participating in TIMSS answered questions about their opinions, attitudes and interests in mathematics. Along with other questions, students reported on:

- whether they thought it was important to spend time doing mathematics, science, sports and having fun
- whether their mothers thought it was important to do mathematics, science, sports and having fun
- whether their friends thought it was important to do mathematics, science, sports and having fun
- how they spent their out-of-school time during the school week
- how they spend their leisure time on a normal school day
- how many hours they spent watching television and videos on a normal school day

Because teachers and the instructional approaches that they use are important in building students' mathematical understanding, all teachers participating in TIMSS completed questionnaires about their beliefs about math and about their teaching practices. Along with other questions teachers reported on:

- their academic qualifications and teaching experience
- their beliefs about mathematics and the way mathematics should be taught
- how their mathematics classes were organized
- what activities their students do in their mathematics learning
- how much homework they assign

Findings From The Questionaires

1. TV and extra-curricular activities

This is often cited as the major reason why students do not do well on their tests or assignments. The perception is that Canadian and American students watch more TV or participate in more extracurricular activities than their Asian counterparts. TIMSS found that "beyond the one to two hours of daily television viewing reported by close to the majority of eighth graders in all participating countries, the amount of television students watched was negatively associated with mathematics achievement" (4). "Fourth grade students in all countries also reported that they normally averaged an hour or two each school day watching television" (Mullis et.al., 1997, 5).

"In most countries, eighth graders reported spending as much out-of-school time each day in non-academic activities as they did in academic activities" (U.S. Department of Education, National Center for Education Statistics, 5). Besides watching television, Grade 4 students reported spending from one to two hours per day on extra curricular activities, such as sports.

2. Homework

Increase the amount of homework or increase the number of instructional hours are arguments that are frequently forwarded as ways to raise mathematical achievement.

Homework is a way of extending the school day and indirectly increasing instructional hours. TIMSS found that all students participating in the study typically reported spending approximately an hour each day on mathematics homework. "The relationship between amount of homework assigned and achievement was not straightforward. High-performing countries assigning relatively low levels of homework included Japan, the Czech Republic and Flemish-speaking Belgium" (Beaton et.al. 1997, 144). TIMSS found that Canadian and American teachers assign more homework and spend more class time discussing it than do their Asian colleagues.

The amount of time spent in mathematics classes varied from country to country. Many teachers, including teachers from Japan and Singapore, reported that students spent at least two hours to three and a half hours per week in class. Teachers from some countries, including Canada and the United States, reported that students spent three and a half to five hours per week in mathematics classes. TIMSS data revealed "no clear pattern between the number of in-class instructional hours and mathematics achievement either across or between countries" (Beaton et.al. 1996, 144).

3. Class Size

In North America, teachers and other educators readily blame large class sizes on decreased achievement levels. Concerned about huge classes of 30 or more students, teachers long for smaller classes. However, TIMSS found that there were significantly fewer students in each math class in North America than in the Asian countries. At the Grade 4 level, TIMSS found that on average Canadian and American classrooms have 24 students while Asian classrooms have greater than 30 with Singapore reported the highest number of students per classroom at 39. At the Grade 8 level, TIMSS found that there are typically fewer than 30 students in Canadian and American classrooms while most Asian countries reported classrooms of greater than 30 students, and Korea reported classes of more than 40 students. TIMSS researchers found that the four highest-performing countries at the fourth and eighth grade are among those with the largest mathematics classes. The numbers of students, in and of itself, was not a significant contributing factor to North American students' relatively poorer performance.

> Extensive research about class size in relation to achievement
> indicates that the existence of such a relationship is dependent on the

situation. Dramatic reductions in class size can be related to gains in achievement, but the chief effects of smaller classes often are in relation to teacher attitudes and instructional behaviors. (Beaton et.al. 1996, 151)

4. Teacher Education

The general public and critics of mathematics education are often quick to blame poor mathematical performance on lack of sufficient teacher education. The popular myth is that Canadian and in general North American teachers are not as well educated as their Asian colleagues. TIMSS found that the qualifications required for teaching certification were relatively uniform across countries. Canadian and American teachers have more college education than their colleagues do in all but a few TIMSS countries. The amount of teacher education, in and of itself, is not enough to account for lack of student achievement. Even (1993) contends that "good subject-matter preparation for teachers is necessary but not sufficient" (112).

5. Group Work

An emphasis on too much group work is sometimes cited as a reason why students do not do as well on individual measures of achievement. TIMSS found that placing students in small groups is an instructional strategy in many subject areas in North America, but it is not a common strategy in mathematics classrooms. TIMSS reported that small-group work was the least used instructional approach in both the primary and middle school years. Students in North American mathematics classrooms typically work together as one large group with the teacher directly instructing the whole class followed by students' working individually. The perception that group work is in some way contributing to lower achievement in the mathematics classroom is unfounded.

TIMSS findings clearly indicate that the comparative weakness of Canadian and American student achievement cannot be attributed to too much TV watching, too many extra curricular activities, lack of homework, large class sizes, lack of teacher education, or an increased emphasis on group work, despite critic's attempts to find explanations in these areas. Data from the questionnaires do not seem to identify any one factor that accounts for Canadian and American's students' achievement in mathematics.

TIMSS Videotape Study

In addition to the math assessments; school, teacher and student questionnaires; and curriculum analysis, the United States sponsored two additional parts of TIMSS which were carried out in Germany, Japan and the United States. One of these parts involved videotaping typical lessons taught to Grade 8 students in each of these countries. The tapes were analyzed to compare teaching techniques and the quality of instruction. The other part involved ethnographic case studies of key policy topics.

This part of the study involved "a team of 12 bilingual researchers spending three months in Germany, Japan or the United States observing classrooms, interviewing education authorities, principals, teachers, students and parents" (United States Department of Education 1996, 16). Analysis of the videotapes reveals important data, and sheds light on the current impasse in math reform initiatives. Although Canada did not participate in the Videotape Classroom Study, the findings from the United States can help us better understand the current state of mathematics teaching in Canada. Achievement of Canadian and American students was not significantly different, and curricular reforms in both countries have been dramatically impacted by the National Council of Teachers reform efforts and initiatives.

The TIMSS Videotape Classroom Study was conducted in a total of 231 classrooms: 100 in Germany, 50 in Japan and 81 in the United States. This part of the study had four goals:

- To provide a rich source of information regarding what goes on inside eighth-grade mathematics classes in three countries;
- To develop objective observational measures of classroom instruction to serve as quantitative indicators, at a national level, of teaching practices in the three countries;
- To compare actual mathematics teaching methods in the US and other countries with those recommended in current reform documents and with teachers' perceptions of those recommendations

- To assess the feasibility of applying videotapes methodology in future wider-scale national and international surveys of classroom instructional practices

(U.S. Department of Education, TIMSS Videotape Classroom Study 1996)

The data from the videotapes were analyzed according to five categories: the way the lessons are structured and delivered, the kind of mathematics that is taught, the kind of thinking students engage in during the lessons, and the way teachers view reform.

How Teachers Structure And Deliver Their Lessons

Before observing the mathematics classroom, researchers asked teachers to describe the goals that they had established for the lesson. Researchers found a significant difference between the stated goals of U.S. teachers and Japanese teachers. U.S. teachers' goals were to have students acquire particular skills, while Japanese teachers' goals were to have students understand a particular concept.

> Learning a skill, such as being able to solve a certain type of problem, or using a standard formula, was listed as the goal by about 60 percent of the U.S. teachers, compared with 27 percent of the Japanese teachers. Mathematical thinking, such as exploring, developing, and understanding concepts, or discovering multiple

solutions to the same problem, was described as the goal by 71% of the Japanese teachers, compared to 24% of U.S. teachers. (U.S. Department of Education 1996, 42).

This difference is evident in the transcripts from the classrooms that typically follow the following sequence of activities.

Table 1.1
Comparison of the steps typical of Eighth-Grade Mathematics Lessons In Japan, the U.S. and Germany

The emphasis on understanding is evident in the steps typical of Japanese eighth-grade mathematics lessons:
▪ teacher poses a complex thought-provoking problem ▪ students struggle with the problem ▪ various students present ideas or solutions to the class ▪ class discusses the various solution methods ▪ the teacher summarizes the class' conclusions ▪ students practice similar problems
In contrast, the emphasis on skill acquisition is evident in the steps common to most U.S. and German math lessons:
▪ teacher instructs students in a concept or skill ▪ teacher solves example problems with class ▪ students practice on their own while the teacher assists individual students

TIMMS, unpublished tabulations, Videotape Classroom Study, 1996, UCLA.

There is a strong correlation between teachers' stated goals and the type of work students do in-class. In the U.S. 96 percent of seatwork was devoted to practicing routine procedures that the teacher had demonstrated. A typical U.S. lesson is organized around acquisition and application. In the acquisition phase the teacher demonstrates how to solve a problem involving a particular skill. The goal is to clarify the procedural steps that are required so that students will be able to solve related problems on their own. In the application phase, students are assigned a worksheet or textbook pages, which reinforce the procedure that was demonstrated through the sample problem. The students work alone while the teacher circulates through the classroom assisting individual students. If a number of questions arise about a particular question, the teacher stops the class and works through the problem on the board with the whole class. The class typically ends with the teacher assigning any unfinished problems as homework.

In Japanese classes students follow quite a different script. The lesson typically focuses on one or two problems and the students are challenged to invent new solutions, proofs, or procedures without the teacher's direct instruction or intervention. After stating the problem or the nature of the investigation, the teacher generally asks students to work on the problem on their own for a few minutes. During this time the teacher circulates throughout the classroom assisting students by asking questions. After this the teacher asks students to work together to come up with possible solutions. Students work together for approximately 10 minutes. Some students are asked to come to the board and present their solutions to the class. The teacher and the rest of the students ask questions and request clarification, which the presenting students answer. The teacher then reviews each of the solutions and presents a follow-up problem. Again the teacher asks students to consider the problem individually first and then to work with peers. After approximately 30 minutes the teacher brings the class together again and reviews the various solutions. For homework the teacher either assigns a follow up problem to the one or two that were presented during the lesson or no homework. This type of lesson structure was observed in only one percent of U.S. classes.

The videotape transcripts indicate that when a lesson included a mathematical concept, it was usually simply stated in U.S. classrooms. This occurred in 78 percent of the U.S. math lessons and 17 percent of the Japanese lessons. It was much more common that concepts were developed, not simply stated, in Japanese classrooms. This occurred in 83 percent of Japanese lessons and only 22 percent of U.S. lessons. For example, a U.S. teacher might tell students that the Pythagorean theorem was $a^2 + b^2 = c^2$; whereas, a Japanese teacher would design the lesson in such a way that the students themselves derived the mathematical concept from their own struggle with a problem or investigation.

"These findings from the videotape study are corroborated by the TIMSS questionnaire findings" (U.S. Department of Education 1996, 43). Teachers were given questionnaires that asked them to select activities that were characteristics of their type of teaching. U.S. teachers generally selected activities that focused on computational skills. Japanese teachers selected activities that involved analyzing relationships, writing equations, explaining reasoning, and solving problems with no obvious solution.

This part of the study also examined the way that mathematical ideas and concepts were linked together. The *Curriculum and Evaluation Standards* and *The Common Curriculum Framework for K-12 Mathematics* state that a critical component to mathematical activity involves the ability to connect mathematical ideas to other concepts in mathematics. The videotape study found that 96 percent of Japanese teachers' lessons included explicit language to help students link concepts while this occurred in only 40 percent of U.S. lessons.

It is important to note that not all teachers who were videotaped taught in this way, but "what is striking, when viewing the videotapes across the two countries, is how many of the lessons appear consistent with these scripts" (Stigler and Hiebert, 1998, 8). The script of a typical U.S. lesson is familiar to most of us. The teacher demonstrates the procedure and the students reproduce the procedure. The questions that arise involve problems or misunderstanding of technique. Mathematics, in these classrooms, is the formulation of fixed, static, unchallengeable and unchanging truths. Because of this emphasis on "the right way", students seldom have the opportunity to experience the life of mathematics through negotiating meaning, constructing different representations, critiquing these representations, defending and debating possible solutions, and posing new problems. "If we deprive learners of this opportunity, we strip mathematics of its essential character and deprive them of real mathematical experience" (Wilensky 1993, 22).

The videotape script shows that a typical Japanese lesson is much more likely to invite students into mathematical ways of knowing and doing. Students are expected to engage and debate with each other as they progress through the messiness of creating, producing and defending their mathematical knowledge. "Clearly, Japanese students much more often engage in the type of mathematical thinking recommended by experts and the U.S. reform movement" (U.S. Department of Education, National Center for Education Statistics, 1996, 43).

The Kind Of Mathematics That Is Taught

There are significant differences in the kind of mathematics that is taught in the three countries. U.S. eighth-graders were learning material that was part of the seventh-grade curriculum in the other two countries. The U.S. eighth grade mathematics curriculum focuses more on arithmetic while the German and Japanese curriculum focuses on algebra and geometry. In addition, TIMSS researchers found that mathematics curricula in the U.S. is unfocused and consistently covers far more topics than is typical in the other countries.

Some of the differences in curricula might be attributed to fundamental differences between Japan and the United States when it comes to matters of who controls education. In Japan education standards are set and monitored by the Japanese Ministry of Education.

> The ministry develops national curricular guidelines that define education standards. In writing the curricular guidelines, no effort is made to define exactly what should be taught at each grade. Rather, the guidelines consist of general descriptions of what students are expected to accomplish during each year of schooling. The time and manner in which the material is presented in each classroom are

decided by the school administration or by the individual teacher.

(Stevenson, 1998, 6).

Mathematics teachers in Japan are familiar with the learning goals that are issued by the Ministry of Education. Japanese federal documents contain general goals and teachers work together to support and help each other understand how to implement the goals, improve their own pedagogy and improve the curriculum. "During their careers, Japanese teachers engage in a relentless, continuous process of improving their lessons to improve students' opportunities to achieve the learning goals. Small groups of teachers meet regularly, once a week for about an hour, to plan, implement, evaluate, and revise lessons collaboratively" (Stigler and Hiebert 1997, 9). Teachers are expected to work together developing lessons, observing and critiquing other teacher's lessons, and sharing their work with other teachers. This is in stark contrast to North American teachers who generally plan, teach and evaluate their lessons by themselves.

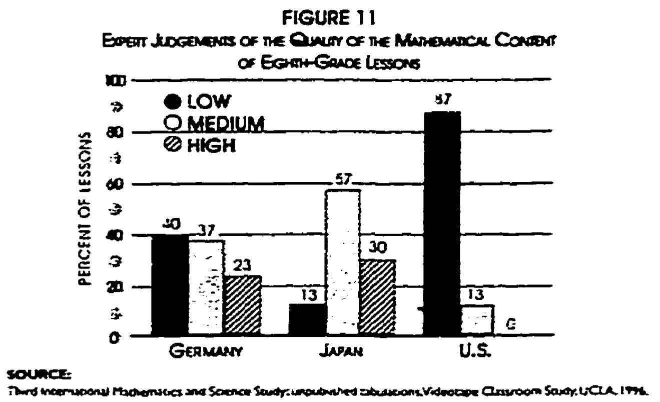
In the U.S., education is not a federal matter. Standard guidelines for education fall under state jurisdiction. "State education standards include content standards in core subjects, performance standards for students, and standards related to students' opportunities to learn" (Stevenson 1998, 7). Most of the 16,000 districts in the U.S. design their own curriculum or standards, which specifically address the broad guidelines issued by the individual states.

U.S. teachers reported that they seldom met with colleagues to plan lessons. Most U.S. teachers reported that they followed the textbook when deciding how to present a topic to their students and 95-99 percent of the teachers reported that they used textbooks in their lessons. "The question thus arises: Do U.S. mathematics textbooks add guidance and focus to help teachers cope with unfocused curricula? Unfortunately, the answer is 'no.' The splintered character of mathematics curricula is mirrored in the textbooks used by teachers and students" (Schmidt et.al. 1997, 4). U.S. textbooks are published for a national market even though education standards are set by the state. "Because there are no national guidelines, publishers have a wide degree of latitude to develop and market books that they believe will have the greatest sales" (Stevenson 1998, 7). U.S. textbooks tend to cover many topics, generally far more than a teacher can adequately cover in a year. Although mathematics curricula are different from state to state, textbook publishers do not publish differentiated content for the various states.

The Kind Of Thinking Students Engage In During The Lessons

The TIMMS videotape researchers asked three mathematics professors and one professor of mathematics education to examine the tapes and evaluate the quality of mathematics contained in the lessons. They were not actually allowed to view the videotapes, but were provided with a written summary of lessons from three countries: Germany, Japan and the U.S. Identifying words were altered so that the reviewers were unable to discern which country the transcript represented.

The following represents the findings of the review panelists.



Videotape Classroom Study 1996
Figure 1.3

87 percent of U.S lessons called for students to use the lowest level of mathematical reasoning, as compared to 13 percent of Japanese teachers' lessons. This finding suggests that a high quality of mathematical reasoning is probably a rare phenomenon in the U.S.

Since the 1960s, progressive ideologies have dominated mathematics education in North America. At the end of the 1960s the Mathematical Association published a report on primary mathematics endorsing a progressive philosophy (Ernest 1991). With its roots firmly grounded in developmental psychology, as opposed to behavioral psychology, the most important aspects of a mathematics education became: fostering student confidence, developing positive attitudes and self-esteem with regard to mathematics, and shielding the student from negative influences that might undermine these attitudes (Clifford and Friesen 1994, Ernest 1991). The pedagogy that developed from progressivism actively discouraged teachers from creating experiences that create dissonance and conflict for students. "Efforts to shield the child from these experiences mean that "children's 'errors' are not explicitly corrected, for fear of hurt and emotional damage" (Ernest 1991, 194).

A great deal of tension is created for teachers in attempting to reconcile progressive ideologies and their accompanying "child-centered" practices with their own understanding of and experiences with mathematics. For most North American teachers, mathematics is an absolute, fixed body of knowledge.[3] However, "a deep commitment to the ideals of progressive mathematics education can and frequently does co-exist with a belief in the objectivity and neutrality of mathematics, especially amongst mathematics teachers and educators" (Ernest, 1996, 5). Caught between two opposing forces, teachers attempt the fine art of juggling. But as the TIMSS videotape study shows, mathematical understanding and reasoning are jeopardized by failure to engage students in creating mathematical concepts and structures.

The Way Teachers View Reform

Ninety-five percent of U.S. teachers said that they were aware of the current ideas about teaching and learning mathematics. More than 75 percent of U.S. mathematics teachers stated that they were familiar the National Council of Teachers of Mathematics, *Curriculum and Evaluation Standards* and *Professional Teaching Standards*.

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[3] As discussed earlier in an earlier section

> A great deal of effort has been invested in the reform of mathematics teaching in the U.S. in recent years. There is considerable agreement among experts about what good instruction should look like. The main goal of the reform is to create classrooms in which students are challenged to think deeply about mathematics and science, by discovering, understanding and applying concepts in new situations. For many years, Japanese mathematics educators have closely studied U.S. education reform recommendations, and attempted to implement these and other ideas in their own country. (U.S. Department of Education, National Center for Education Statistics 1996, 46)

With the amount of attention and emphasis that mathematical reform has received in the United States, it stands to reason that TIMSS researchers would find strong agreement between the reform initiatives and classroom practice. Teachers indicated that they were aware of reform initiatives and most stated that they were implementing reform recommendations in their lessons. But the findings suggest that they are not.

> When asked to evaluate to what degree the videotaped lesson was in accord with current ideas about teaching and learning mathematics, almost 75 percent of the teachers respond either "a lot" or "a fair amount." This discrepancy between teachers' beliefs and the TIMSS

> findings leads us to wonder how teachers themselves understand the key goals of the reform movement, and apply them in the classroom. (U.S. Department of Education, National Center for Education Statistics 1996, 46)

U.S. teachers were asked to view the videotapes of their lessons and indicate to researchers which aspects of their lesson, in particular, demonstrated current ideas about teaching and learning. Their answers indicate a strong awareness of instructional techniques: hands-on, use of manipulatives, real-world math and cooperative learning.

> Over 80 percent of the teachers in the study referred to something other than a focus on thinking, which is the central message of the mathematics reform movement. The majority of the teachers cited examples of hands-on math or cooperative learning, which are techniques included among the reform recommendations. However, these techniques can be used either with or without engaging students in real mathematical thinking. In fact, the videotape study observed many examples of these techniques being conducted in the absence of high-quality mathematical content. (U.S. Department of Education, National Center for Education Statistics 1996, 47)

Nineteen percent of the teachers stated that they believed that their lessons contained the type of mathematical thinking that was indicative of current ideas being forwarded in mathematical teaching and learning. TIMSS researchers found no indication of the type of mathematical thinking being called for in the reform documents when they analyzed the U.S. teachers' lessons. In contrast, the videotape study found that Japanese teaching more closely resembled the recommendations of the U.S. reform movement than did U.S. teaching.

The discrepancy between teacher's beliefs and their practice is not as unusual as it might first appear. Many research studies corroborate TIMSS findings. Hoyles (1992) points out research into the area of teacher thinking, content knowledge, and teacher beliefs "has shown evidence of inconsistencies between beliefs and beliefs-in-practice....This mismatch was thrown into relief when teachers were faced with an innovation..." (41). Numerous studies on teacher beliefs demonstrate that teacher beliefs are not always consistent with their practice (Borko 1992; Carpenter et.al. 1988; Cooney 1985; da Ponte 1994; Ernest 1996; Good 1990; Gorman 1991; Hoyles 1992; Lerman 1997; Raymond 1997; Senger 1996; Thompson 1985). And although TIMSS researchers found that many U.S. teachers report familiarity with reform recommendations, they found little evidence of an understanding of what is required to implement the reform initiatives.

Meaningful mathematical reform is still in its infancy in the United States. It is clear that teachers have not understood the intent of the reform initiatives and consequently have not been able to implement the necessary changes into their practice. To date, reform recommendations have been disseminated through documents issued from national organizations. If distributing written reports and establishing standards could change teaching, reform in the U.S. would be successful. If changing features of instruction led to increased achievement, then the increased use of manipulatives, cooperative groups and changes to current U.S. curricula and textbooks would lead to increased student achievement. If making teachers more accountable for increased achievement were all that were necessary, then U.S. students should have scored with the best.

North American efforts to improve mathematics education have focused on documentation, standards and accountability. These are all essential components of successful reform. Indeed, they formed the basis of changes mandated by the Japanese Ministry of Education. But however necessary they are, documentation, standards and accountability are not sufficient conditions to ensure that teachers and students increase the depth of their mathematical understanding. As TIMSS results clearly indicate, current reform initiatives ignore a fourth essential element: the processes of teaching and learning in classrooms.

It is very tempting to conclude that North American teachers should just teach more like their Asian counterparts. However, it would be folly to think that we could transplant Japanese methods in North American soil without considering the cultural milieu that support them. In Japan, teachers are expected to work together. They develop and refine lessons, critique each others' practice, seek and offer advice about how to improve. In North America, the classroom is the teacher's private domain. Teachers seldom work with one another for long periods of time. They rarely see one another teach, and it is almost unheard of for one teacher to critique another's lesson.

Larger cultural differences in the relative importance of group cohesion and individual autonomy that characterize Asian and North American societies work themselves out in such pointed, and significant, moments in teachers' lives in school.

Stigler and Hiebert (1997) point to the bleak implications of this difference: "our biggest long-term problem is not how we teach now but that we have no way of getting better. We have no mechanism built into the teaching profession that allows us to improve gradually over time " (10).

The Classroom: The Place Of Reform

Perhaps, more than any other research study, the findings from TIMSS shows us the current state of mathematics education. Although TIMSS was not designed to measure the effects of current mathematical reforms, it provides a lens through which we can see the working out of the current initiatives in the classroom. Even though the current reform initiatives have dominated documents and official discourse since 1989, it is clear that they are stalled outside the classroom door in far too many North American schools. Many teachers have been able to adopt the artifacts of the reforms, thinking that they were embracing the new ideas advocated by the various reform organizations. But however well intentioned and optimistic, far too few teachers actually encourage genuinely mathematical ways of thinking, knowing and creating.

As we try once again to move forward with important and necessary reforms to mathematics education, we need to remember the lessons of the "new math." This time, equipped with the extensive data that TIMSS provides, we have the information we need to be able to counter the arguments of uninformed critics. Some of those critics dismiss the relevance of North American students' achievement by insisting that social differences between the countries invalidate true comparison. Such people point to such factors as hours of homework, teacher preparation, or what they feel is an Asian emphasis on rote learning to the detriment of individual student development. The TIMSS data shows that these commonsense perceptions of the differences between Asian and American students and classrooms are incorrect. Others claim that international studies compare apples and oranges: Asian students drawn only from the academic elite and North American students drawn democratically from the wide spectrum of society. Analysis of the structure of the TIMSS study refutes that dismissal as well.

Nor can educators committed to fundamental reform of mathematics teaching let the popular power of uninformed criticism return us to a narrow and instrumental view of "the basics". It is exactly that turn back to "the basics" that has got us into the position we now find ourselves: "The majority of today's high school graduates—not to mention dropouts—still lack fundamental 'walking around' skills in quantitative literacy" (Steen 1997, xvi). As critics point to our international achievement results and blame the new reforms, we can confidently counter their attacks: reforms have not impacted North American classrooms in any substantial way.

It would help ease our burden if we could just import Japanese teaching and learning into our classrooms. But teaching is more than prescription. It is more than a collection of "individual features, such as using concrete materials, asking higher-order questions, or forming cooperative groups" (Stigler and Hiebert 1997, 8). Teaching is a cultural activity (Stigler and Hiebert 1999). It is situated in political, economic and social milieus that combine to work their way out in our classrooms. We have had enough add-ons already—that is part of our problem. The work we need to do is far more basic.

The classroom. This is the place where we need to focus our energies, attention and research to bring about meaningful reforms in mathematics education. The pedagogy of mathematics is not an afterthought but a necessity. This is the place where mathematics lives. This is the place that mathematics is re-formed. It is a messy place full of debating, negotiating, and presenting multiple representations about fundamental meanings of mathematical objects, operations and ideas. It is not the clean, sanitized picture presented in textbooks and proofs. It is a place where curricula gets created, worked out, and recreated anew.

> How can curriculum be so built that it will touch something deep that stirs teachers and students to animated living? How can a curriculum-as-plan be so built that it has the potential for a curriculum-as-lived which is charged with life? How can a curriculum be bult so invitingly that teachers and students extend a welcome hand? How can curriculm be so built with openings and open spaces that teachers and students come to in-dwell vitally? (Aoki 1989, 17)

As TIMSS clearly shows, teachers cannot create this place alone. Just writing documents and making teachers more accountable does not help them understand the intent of the new mathematics reforms. Despite the urgency and the pervasiveness of the documents and reform discourse, when teachers look at their own classrooms, they are still left wondering, what do we do differently? What do we reform? All of us who are interested in improving mathematics will have to work this out. We will have to redefine "the basics" so that we can identify the basic attributes of genuinely mathematical thinking, knowing and creating. We will need to create images of practice that best cultivate these attributes, and that speak in compelling ways to the context of North American classrooms. That is, we need to help teachers find answers to yet another question: "what does it look like when teachers and students engage with one another in deeply mathematical ways?"

The classroom. This is the place that this dissertation will take you. It is the place where teachers and students work together to re-form the mathematics in mathematics education. This is a story of what it looks like when teachers and students engage with one another in deeply mathematical ways.

Here You Do Things[4]

Here you do things.
As mighty as the solar system
as ancient as trees.

Ancient Greece a living friend
Far away China as close as can be.
Strange worlds not so different
than here.

Givers
and receivers
creators
and readers.

Geometric worlds
broken to pieces
recreated with
understanding.

Napier bones
wolf skulls
flying birds.

This year was filled
with my own wonders.

Margaret, age-12

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[4] Here You Do Things is an original poem composed by Margaret, a student in our class. She wrote this poem to Pat and me as a thank you gift at the end of the school year.

CHAPTER TWO

Finding New Ways

Starting Alone

"What you do is okay. But if you're going to teach like *that*, I can't help you. I don't know anyone who can."

That was fourteen years ago. I was just out of university. It was September and as a new teacher in my first teaching assignment, I was anxious to show the consultant that sat in my room that day, that I had the makings of a good teacher. It was her job to offer me assistance and advice as I started out not only in this classroom, in this school, but also in this profession. As a newcomer, I wanted to know if I was on the right path.

There are, for all of us, moments when time seems to freeze. Something occurs that is so charged with emotion and intensity that even though the background fades away, the thing itself stretches its icy tentacles into the vulnerable reaches of your brain. It's the unexpected.

I was just starting out. I was new to this place. I was hoping that the person who came into my classroom would help me recognize what was going on. Instead of clarification and direction, I heard that I would walk this path alone.

A Chance Meeting

I had just moved to a new school and was team teaching in a Grade 1/2 multiage classroom. Pat came into the classroom. She was a consultant. She wanted to do research. My teaching partner's friend had suggested that she contact us. I listened as she talked about what she had in mind.

"So what do you think?" my teaching partner, Chris[5] asked me after Pat left.

"It'll be okay," I replied, "but she'll have to teach with us. She's not going to sit there and watch us."

"Sharon, you're kidding. Pat won't do that. She's a high school teacher. She's a consultant." Chris laughed at my demand.

"No. I insist. Phone her and tell her."

I had no idea at that moment that this would be a new beginning for me. Pat agreed to my request. She came in, rolled up her sleeves, and taught with us. I don't know why I made that request. The memories of my first encounter with a consultant had faded and I was now very used to having consultants, researchers and preservice teachers come into my classroom to observe, "how I did things." I worked in a demonstration school that was connected to the university. My days were filled with observers. Somehow I heard something different in Pat's request.

Interpretive research begins with a different sense of *the* given.

Rather than beginning with an ideal of clarity, distinctness and

> methodological controllability and then rendering the given into the image of this ideal, it begins in the place where we actually start in being granted or given this incident in the first place. ... Interpretive research, too, suggests that these striking incidents make a claim on us and open up and reveal something to us about our lives together. (Jardine 1992, 55)

I remember looking forward to the days when Pat came into the classroom. At the end of the day's events, Chris, Pat and I would sit together, laughing and telling stories about the day as we planned our next day's agenda. Pat's presence in the classroom and in my life changed everything. As the days went on, I repeatedly found myself intrigued with her questions and observations. She noticed *those* children, the ones who disrupted the expected in the classroom. They were the children who did not fit, who could not—or would not—comply with the institutional demands of living the well-schooled life. She had no desire to analyze them or fix them. She wanted instead to understand how to go about making the classroom large enough to encompass them.

Pat had no way of knowing at that time, what I heard in her questions. During our months together, I made a promise to myself that I would not hold back. I would find a way to open myself to her questions to help her understand why I teach the way I do and why these children are essential to the way I think about teaching. Not because

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[5] To protect the anonymity of subjects, the names that I use in this dissertation are pseudonyms, except for Pat and David who keep their real names.

I wanted assurance or confirmation about the way that I did things, but because I wanted a fellow traveler. I wanted to journey with someone who also understood that pedagogy is not an afterthought, but a necessity.

Two years after meeting her, Pat and I began teaching together. Our first classroom was a multiage class of Grade One/Two students. Many of my colleagues were apprehensive about my teaching with Pat because she didn't have any "boxes", no "stuff" to teach with. She wasn't armed with the normal trappings of elementary school practice. She just brought herself. Would she be able to enter this place disarmed? What my colleagues saw as a lack of preparation, I saw the opportunity for beginning in a new place.

We began our journey together with fifty-some children. And even in those early days, we noticed those things that protruded above the surface of the commonplaces of classroom life. The space that opened because of the lack of stuff to fill each moment of the day allowed us time to attune ourselves differently to where we were. Together we learned to create a space for sustained dialogues with children. Together we learned to listen to the children. Together we learned how to create a space that was big enough, generous enough to include all of us. And together, altogether, all fifty-some of us, worked out what our next steps would be.

Bringing Ourselves To The Place

Our most recent school is on the edge of an old forest reserve. There are forty glorious acres of woods, ponds, lichens and wild flowers. Mosses that take a hundred years to form here in the foothills of the Rockies stretch along old fallen timbers. Squirrels scurry up and down scolding us for intruding into their territory. Some days, if we walk softly enough, we come upon a grazing deer. They come here to quench their thirst at the water's edge. If you look carefully you will notice the subtle contours of the forest floor. Here the folds now compacted by the ages, tell of receding ice fields. Back, way back, at the far edge of the forest, if you look carefully, you will notice the remains of a once bustling wagon trail. A trading route wound its way through this place. This forest is filled with memories and stories open to those who knew how to bring themselves to it.

We love to go into this forest with our students. But we needed to learn how to enter *this* place. This forest would not reveal its secrets when we went crashing through the trees (That's how it felt when the consultant came into my classroom that very first year that I started teaching.). As long as we walked its paths as though they were the paths in any forest, we could not know this place. It wasn't even enough to come and sit quietly. It took work. In order to learn who inhabits this place, it was necessary to open ourselves in such a way, so that we were receptive to what came to meet us. It was only then, when we had done *our* work, that this place began to share what was ordinarily hidden from view.

The classroom we created when we first started working together was like this forest. What opens to those who enter, what secrets it reveals, what stories it tells depends on how they enter. If they remain closed, even if they sit very still and listen very hard, they will not know this place. Visitors who enter to gaze at the lessons will not know this place, this classroom. It will not yield itself to them. They will comment "what you do is okay" but it's not the way they do things. They will leave having learned nothing of the place.

However, there is a way to bring yourself to the classroom so that it reveals itself to you. It requires that you begin as Pat first began, you roll up your sleeves and do some work. And as you do you will start to notice the patterns, the contours, this child here, and the stroppy one over there.

> When I first ventured into Sharon's world, it was enough to brandish swords of insight and method in hopes of retreating from the field, victorious, with completed thesis held high. But I got it wrong. The research problem, I came to understand, was not how to defeat or even charm known beasts into releasing their treasures. The research challenge was to learn how to *see* dragons, not quest after their subjugation. Looking, listening and learning on my journey called for what Bateson (1994, p.10) describes as a spiritual attentiveness, 'the modern equivalent of moving through life as a pilgrimage'. In order to make that pilgrimage, the deepest and most enduring

transformation of all ended up being the transformation of me"
(Clifford 1996, 46)

You open yourself to the place and in doing so the place opens itself to you. Entering a place in such a way so that "inner and outer reality flow seamlessly into each other, like the ever-merging surfaces of a Möbius strip, endlessly co-creating us and the world we inhabit" (Palmer 1998, 5). "We and the places we find ourselves co-emerge; we inhabit and enhabit one another" (Davis 1996, 132).

Our Classrooms

You are entering our classroom space. "By *space* I mean a complex of factors: the physical arrangement and feeling of the room, the conceptual framework that I build around the topic my students and I are exploring, the emotional ethos I hope to facilitate, and the ground rules that will guide our inquiry" (Palmer 1998, 73).

Our classroom space is composed of three different classrooms that have been shaped and created by Pat and me and some 250 students. In each of these classrooms we learned something different about finding new ways, learning what we needed to do differently and what we needed to re-form. In each of these classrooms we learned how to bring students together to create a classroom community that was filled with a deep longing to know and understand. "Good teachers also bring students into community with themselves and with each other—not simply for the sake of warm feelings, but to do the difficult things that teaching and learning require" (Palmer 1998, xvii). However, it's not enough to only say that we shaped and created these classrooms. These classrooms, these children have significantly shaped and created who we are today.

How you go to a place affects what it will show you of itself. Frighteningly, if you blunder in with your "stuff," the trappings of ordinary classroom practice, the worksheets, that textbook, those beautiful hand-painted counters, the place will seem to be precisely what is needed and best. "The way we treat a thing can sometimes change its nature" (Hyde 1983, xiii) or at least, how we treat a thing can show its nature.

Teaching Together

In the early years of my teaching career, I spent most of my time and energy learning how to make myself appear as a proper, ordinary (or at least appear to be doing proper and ordinary) elementary school teacher. There were crafts to make, bulletin boards to put up, and concerts to prepare for. And there was always all the flurry of hyper activity which surrounds the rituals associated with fall, Halloween, Christmas, Valentine's Day, St. Patrick's Day, Easter, spring and the end of term.

In schools, finding new ways of teaching and learning means going it alone. It means going in disguise. I had to work hard to appear ordinary. I stayed late into the evenings to make displays to decorate the hallway bulletin boards so that our room looked like every other room. It left me with precious little time to act upon the gnawing dissatisfaction that was eating away at me about the way that I taught math.

"A person who plays such a game denies, to all appearances, continuity with himself. But in truth that means that he holds on to this continuity with himself for himself and only withholds it from himself and only withholds it from those before whom he is acting (Gadamer 1995, 111)."

Now, starting again, no longer alone, disguise removed, I had space and time to consider how math might be different for me and for the students Pat and I taught. I wasn't sure what I had in mind and I didn't know how it would turn out. It was early in September, the time when teachers are busily mapping out long-range plans for the coming year.

"I don't like how I teach math," I announced after our first day of school with the children. "I want to try something different."

"Okay." Pat responded to my request with enthusiasm.

Convincing Pat was easy. Now we just needed to figure out *what* to do. We didn't know that our search would lead down complex and tangled trails through philosophy, mathematics, psychology and education. Knowing *what* to do differently was not going to be a simple undertaking. And we weren't far along when we realized that it also wouldn't be resolved in one year.

> It is impossible to divorce the question what we do from the question of where we are—or, rather, where we think we are. That no sane creature befouls its own nest is accepted as generally true. What we conceive to be our nest, and where we think it is, are therefore questions of the greatest importance. (Berry 1986, 51)

"We resisted the return to traditional images and practices that seem almost inevitably to accompany criticism of schools" (Clifford and Friesen 1993, 341). That was easy. But we also resisted "the fuzzy, feel-good legacy of much of what teachers [did] in the name of 'progressive' practice" (Clifford and Friesen 1993, 341; Jardine 1994a). We were searching for a way to think about mathematics practices that removed them from this dichotomous swing between traditionalism and progressivism. "Adopt a little of both," colleagues would advise. "I tend to the middle. You need both."

"That doesn't make sense," we would reply. "It would be a little schizophrenic don't you think? We won't get anywhere that way." We were disillusioned with and frustrated by the pendulum swings that dominated the educational landscape. We knew something about pendulums—they just swing back and forth, they don't go anywhere. They just fill time. Such is the nature of pendulums. We needed to find a different place—a place that that was strong enough to resist our being drawn back into the swing of the false dichotomies created by the traditionalist/progressivist arguments. These dichotomies kept us from bringing ourselves to this place. They kept us from learning how to inhabit this place—this classroom.

We turned to the mathematical education research community to help us with our search. Instead of clarity and direction, we found a confusing number of solutions to what researchers deemed either inherently difficult about learning mathematics or the child's failure to comprehend mathematics.

When analyzing the difficulties learners have in mathematics exercises, researchers often catalogue syntactic errors, rules that learners fail to follow such as: Johnny adds fractions by adding their numerators and denominators instead of making a common denominator. These educators prescribe more practice in applying these rules, or perhaps computer aided instruction programs which will help Johnny drill. Some researchers have begun to describe learner's difficulties as false theories or misconceptions, such as: Maggie thinks you can't subtract a bigger number from a smaller, or divide a smaller number by a bigger. The prescription offered here might be creating a simplified computer environment in which Maggie can play around with numbers, but is constrained to operations that are mathematically valid. In this way she will construct the true conception of, say, division instead of a misconception." (Wilensky 1993, 22)

There is something deeply disturbing about both of these formulations. They locate the difficulties of mathematics either in mathematics or in Johnny and Maggie. Adding fractions and subtracting whole numbers are simply given. The task of pedagogy is one of how to hand over such givenness so that Johnny and Maggie and their classmates do not mess it up.

If, however, we consider mathematics as a living discipline, its formulaic, axiomatic self-evidences become *precisely* what must be opened in order that we might get a glimpse of the roiling, living, originary work *from which* such equations might be 'genuinely drawn.' The real pedagogical work is not found in the handing on of self-evidences, but in the opening up of our access to the living resourcefulness, the living conversations and quarrels and controversies from which such self-evidences are genuinely drawn. The real pedagogical work is found in the effort to *get in on the conversation*. The danger of essentialism is that it hands us tradition in such a way that there is nothing left that needs to be said. Pedagogy is the work of *seeing through* the charm of such self-evidence, not in order to dispel tradition, history, language, but in order to *wake it up* to the fact that our children want in.

We soon learned that "faced with a strong demand to aim for deeper and more complex learning for children, teachers must develop new ways of teaching for which there are few available models" (Comiti and Ball, in Bauersfeld 1997, 612; Stigler and Hiebert 1999). Alone. I knew this place. " If you're going to teach like *that*, I can't help you. I don't know anyone who can."

Finding New Ways

Like Hansel and Gretel, setting out a path alone, we left behind everything that was familiar to us. We followed a path to *where*? We didn't know. In truth, there was no path. We had to create it. "Interpretation and understanding are creative acts"

(Smith 1994, 104). This is the venture that is necessary in all beginnings—the creation, the creative act.

> It seems evident that all this holds relevance for a conception of education—if education is conceived as a process of futuring, of releasing persons to become different, of provoking persons to repair lacks and to take action to create themselves. Action signifies beginnings or the taking of initiatives; and, in education, beginnings must be thought possible if authentic learning is expected to occur" (Greene 1988, 22).

Pat and I knew that in this beginning, that once we had decided to take this step there was no turning back. Unlike Hansel and Gretel, we didn't bother with the breadcrumbs. We couldn't hedge our bets by thinking we could go back. Our commitment had to be total. Experience told us that it takes too much time and energy to go in disguise. We also knew that we could only learn what we needed to learn when we opened ourselves to the place that we found ourselves. Even though there would be times filled with uncertainty and insecurity, times when we were unsure of our next steps, we would have to go on. We would have to learn to live with the unknowns that lurked in the depths of the forest. It was only then that the forest would reveal its secrets, its stories and its memories to us.

Had we known then, what we know now, that this way was long and hard, it would take ten years, three SSHRC grants and two Ph.D.'s, we might not have set out. But set out we did. And that has made all the difference.

A Story Of Our First Journey

Interpretative work is rooted in the particular instance. "Husserl showed that we never think or interpret 'in general' as a rhetorical activity that bears no necessary connection to the world at large" (Smith 1994, 108). Rather, thinking and interpreting require the particular, they cannot be worked out in the abstract. "Every consciousness is consciousness of something; every relation is a relation to something" (Gadamer 1995, 225).

This is our Grade 1/2 multiage classroom. It is one filled with fifty-some children. The room is a large double irregular shaped polygon. Tables and chairs fill the larger side of the room. The smaller side houses our classroom library and our common meeting area. We are all seated in our meeting place. I begin this particular math lesson:

A long, long time ago a young shepherd boy walked out into the field with his flock of sheep. He had the task of caring for his master's flock. It was his duty to make sure that he returned each evening with as many sheep as he set out with in the morning.

Now this was a very long time ago. Numbers hadn't been invented yet. And so the young boy used pebbles. He let one pebble represent one sheep. He kept all the pebbles in a pouch that he tied around his waist. At the end of the day he returned to the master's place and carefully removed the pebbles one-by-one as sheep-by- sheep entered their nighttime enclosure. If everything matched up, he would be allowed to continue to live and tend sheep for another day (such was the life of a young boy a long, long time ago).

Pat and I created this story as a way to introduce young children to the seemingly simple idea of one-to-one correspondence—one sheep matched with one pebble. "Only through a story was it possible to put aside what we knew or assumed or had memorized about the number system to think of a time when there was none. Only stories have the imaginal power to place us elsewhere" (Friesen, Clifford and Jardine 1998, 8). "Good mathematics ultimately comes from and returns to good stories—and the questions that bug you" (Casey and Fellows 1993, 1).

Pat and I wanted to move the story beyond the point of a simple one-to-one correspondence with a finite set of pebbles and sheep. If that were the only point to the telling this story then we would have been guilty of just "dressing up" and passing along already known math facts.

> When tradition becomes master, it does so in such a way that what it transmits is made so inaccessible that it rather becomes concealed. Tradition takes what has come down to us and delivers it over to self-

> evidence; it blocks our access to those primordial 'sources' from which the concepts and categories handed down to us have been in part quite generously drawn. Indeed, it makes us forget that they have had such an origin and makes us suppose that the necessity of going back to these sources is something which we need not even understand. (Heidegger 1962, 43)

Too often, in elementary schools, incredible care, energy and attention are paid to making everything smooth, effortless, and fun. "Teachers act as if student interest will be generated only by diversions outside of mathematics" (Stigler and Hiebert 1999, 89). However, mathematics, dispensed as math facts "dressed up" to fit the theme of the month severs mathematics from it origins and its relationships. It turns mathematics into a commodity that is consumed and produced, rather than a "world into which we ourselves are drawn, a world which we do not and cannot 'own,' but must rather somehow 'inhabit' in order to understand it" (Jardine, Friesen and Clifford 2000, 4).

David Jardine (1994) talks about witnessing a classroom in which the teacher has placed math facts on a teddy bear's tummy. These cute, laminated math facts are tacked to the classroom wall. Jardine sites this example, not to find fault or lay blame with the teacher but as an "interpretative opportunity" to consider how such activity offers "no resistance and [demands] no real work" (Jardine 1994b, 264). If as a teacher you can dress up even hard and cold little math facts like 5+3=__ to make them slide down easier, like some sugar-coated pill, then the difficulties of learning mathematics will be removed or at least the children will have so much fun that they won't notice it going down.

Our purpose was not to "dress up the facts" to make them more palatable, but to invite the mathematics of this place to show itself. We wanted to learn to inhabit this place. We wanted the students to know and understand how the residue of things, often long since forgotten, remains in our modern world. The use of stones for tallying still bears traces to its origins. The root of our word 'calculate' derives from the Latin *calculus* meaning a 'pebble.' We also saw, in this story, possibilities looming on the horizon opening into matters of the finite and infinite—the paradoxes of Zeno and Cantor. What happens when you add one and one and one and one…? Can you count to infinity? How many is that? How big is infinity? Could the shepherd really count an infinite number of sheep? And what if he was able to and then he got one more sheep? What's infinity plus one?

"The world of our everyday experience is finite. We can't exactly say where the boundary line is, but beyond the finite, in the realm of the transfinite, things are different" (Casey and Fellows 1993, 116). So as the children began to pair each sheep with a pebble and each pebble with a tally and each tally with a numeral, they were creating the same line of thinking that Cantor used as he set about exploring the paradoxical infinity.

> What Cantor then set out to do was create exact notions of what it means for an infinity to be equal to, greater than, or less than another infinity. The resulting arithmetic of the infinite, or *transfinite*

arithmetic, was a dramatic and controversial departure from the past attitudes to actual infinities by mathematicians who had regarded them as a concept for the theologians.

Cantor's transfinite arithmetic is very simple: two infinite sets are equal if they can be put in one-to-one correspondence with each other. Sets which can be matched to each other in this sense are then said to have the same *cardinality*.

(Barrow 1992, 206)

We asked, is there a way to put the set of all counting numbers {1,2,3,4, ...} into one-to-one correspondence with the set {sheep, 1, 2, 3, 4,...}? And we learned that yes there is.

$$\begin{array}{cccccc} \{ & 1, & 2, & 3, & 4, & \ldots\} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \\ \{ & \text{sheep}, & 1, & 2, & 3, & \ldots\} \end{array}$$

During the days that the story was created and recreated, we investigated various counting systems. Working together and separately, we moved from a system of pebbles and tallying by ones to a system employing two numbers (one and two, because one and one makes two, of course). But what if you have *only* two numbers to count with? Can you only count to two? What is a binary number system? Is it good for anything? These questions invited fertile conversations and further investigation, which led to connections between this primitive counting system and the underlying structure of the most powerful microprocessors in the world today.

During this time a deepening sense of number emerged in our classroom. Our tale, with its simple beginnings, was no longer so simple. "It is amazing how rich a range of topics there are which can be explored at a variety of levels, with ever more sophisticated questions yielding increasingly deeper insights and connections" (Friesen and Stone 1996, 9). "Good problems lead to more problems—and if the domain is rich enough, students can start with the seed problem and proceed to make the domain their own" (Schoenfeld 1994, 18). Mathematics is more than finding and solving problems. "Mathematics is more generative-the central activity being *making* new mathematics. In so doing, it fosters a culture of design and exploration-designing new representations of mathematics and encouraging critique of those designs" (Wilensky 1996).

It was during one of these critiques that James, one of our Grade 2 students, rose up to his knees. Rubbing his hands together he proclaimed, "But you can make five by two and one and one and one. And you can make five by one and two and two."

The rest of the children caught James's excitement. Another space had opened. Just how many ways were there to make five? What if you were not limited to ones and twos?

> The space that James opened for us all was larger than just his
> particular questions. It was as if we had come with him over a rise
> and that just these few particular steps, taken seriously and followed,

> had opened up a huge horizon of possibilities around all of us. And it was not simply that we now had new territories to traverse. We also now came to understand territories already traversed in a new way." (Friesen, Clifford, and Jardine 1998, 9)

Within good mathematical explorations there is no one "right" way to proceed. There were many ways. An intriguing web of connections, interconnections and crossroads await those who long to know this territory. "To recognize the role of perspective and vantage point, to recognize at the same time that there are always multiple perspectives and multiple vantage points, is to recognize that no accounting, disciplinary or otherwise, can ever be finished or complete" (Greene 1988, 128). This is the ontological unfinished character of a living place—a living discipline. This is a feature of the mathematical territory. (Wilensky 1996, 1993)

But for many, they see only a chaotic confusion of branching forever, with nothing solid at the bottom. For them, this is not a desirable place to be. They locate the subjectivities of this place in the people themselves and so strive to protect the place. They long for clarity and certainty in an uncertain world—yearning for the right path, the right technique that will reveal everything. Too often schools, responding to this impulse, focus solely on transmitting a "right" path.

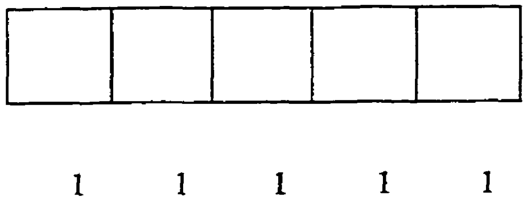
> But in throwing out the 'bathwater' of error, they lose the 'baby if the learner never enters the messy process of negotiating meaning, constructing different representations and critiques of these

> representations. If we deprive learners of this opportunity, we strip mathematics of its essential character and deprive them of real mathematical experience. We also deprive them of respect. Mathematicians throughout history have constructed many different meanings for mathematical concepts and argued their relative merits. If mathematicians of distinction needed to go through this process in order to make sense of mathematics, why do we expect that the learner will take our conceptions on faith? We respect the learner by viewing her as a mathematician in a community which is still negotiating the meaning of a new concept (Wilensky 1993, 22).

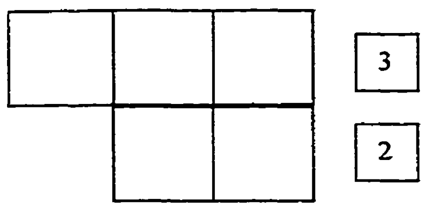
Sometimes schools counter the impulse of the "right way" with its opposite. Now everyone has their *own* "right" path. In wanting to spare the student possible confusion and error, they insist that each student can construct her own mathematics. The student is no longer accountable to anyone but herself. Every student has her own path. "That is, we are *all* producers and consumers of knowledge, and the whole known world is at the formative disposal of our knowing" (Jardine, Friesen and Clifford 2000, 3). And now the idea of mathematics being a place makes no sense anymore.

The Facts Of This Place

Caught up in James's question, "Just how many ways are there to make five?" some of the children colored in five squares in a row.



These children decided that they could represent these by 1 + 1 + 1 + 1 + 1. However, when they grouped the five squares like this:



those who knew the mathematical designations wrote 3 + 2 = 5. But other children wrote 1 + 2 + 2 = 5 for this very same figure.

> It might seem trivial that we have now discovered that 3 + 2 = 5, but what isn't so trivial is that, as a memorized "math fact," 3 + 2 = 5 bears no memory or trace of how it is possible, of how it came to be, such that, if you forget this fact, you're lost. And, even if you simply memorize and remember this fact, you have no way to go on, since it also carries no memory or trace of directionality and place. (Friesen, Clifford and Jardine 1998, 11).

It's "hard to get at the information encoded in mathematical formulas because little or nothing in the actual patterns of the stark symbols on the printed page offers [the students] any clue to the formula's meaning" (Peterson 1990, 10).

Once out of elementary school, these same cold math facts, now no longer placed (or rather out of place) on teddy bear tummies, fill pages upon pages of math texts. Now is the time children learn to do "real" math. The hard stuff—fractions, ratio, algebra, trigonometry, geometry, calculus and so begins the

> litany of definition/theorem/proof chanted day in and day out. This image of mathematical practice portrays mathematics as a dead subject—inquiry is unnecessary because our concepts have been formally defined in the 'right' way and our theorems demonstrated by linear and formal means." (Wilensky 1993, 21)

Students faced with the burden of memorizing multiplication tables, struggling to calculate the age of a farmer who is twice as old as his son will be in six years if the farmer is now three times as old as the son, or pondering how long it takes a slowly leaking conical vessel to drain, are left with the feeling that mathematics is an unchanging body of knowledge that must be painstakingly and painfully passed on from generation to generation. Students, who have no choice but to endure such mathematics practice, pronounce a chilling damnation on mathematics itself, "When you go home you're just blank. They wouldn't explain it. They'd just say 'sit down and do the sheet.' When you know what is ahead of you, you end up with a bad attitude about it, day after day. I hated math." (Aaron - age 13)

But it's not only the children who go "blank". Teachers, too, go "blank". Knowing mathematics as belonging only to the realm of memorized technique, they develop a shallow notion of mathematical understanding as performance rather than understanding as searching for what is *under*. Their mathematics is flat—a two dimensional place. "A body without a shadow: that's as good a description as any of the flatness of much of the institutional surface" (Clifford and Friesen 1999, 58).

> Mathematics becomes akin to a tourist attraction, something to look at but never enter into, open up, and learn to live with. And we, in turn, become akin to curricular tourists ready to be momentarily entertained and amused. However, since we just see the thin, tarted-up, presentable surface of things, we along with our children, become equally subject to boredom [and] frustration..." (Jardine 1994b, 265).

And here is the depressing consequence of all of this. Mathematics, itself goes "blank." Everything is now completely coherent. You are "blank" marking time in the pendulum swing. It's teacher-centered or child-centered. You are caught in the insanity of *Mad Minute Math* [6] or alternatively, putting in time tarting-up math facts, because that it all you *can* do. The research impulse then is to revive teacher-expert-manipulator or revive the child. There is no alternative.

So the task of research, and particularly of this research, is to "fill in the blank"—not with the *right* answer or just *any* answer, but with understanding, searching for what is under the surface of appearances, and noticing those things that protruded above the surface of the commonplaces. It requires that teachers and students enter into dialogues with each other about their mathematical learning and understanding and difficulties because they now have *something* (it's not a blank) to have a conversation about.

Kay (1995) insists, "difficulty should be sought out as a spur to delving more deeply into an interesting area. An educational system that tries to make everything easy and pleasurable will prevent much important learning from happening." But it's not just a matter of making things difficult for difficulty's sake. The fact is mathematics *is* difficult and messy. It's not the sanitized picture we see in textbooks and proofs" (Wilensky 1993, 20). It is filled with arguments, paradoxes, controversies, scandals and murders (Davis and Hersh 1998; Devlin 1997; Kasner and Newman 1989; Motz and Weaver 1993; Pappas 1997). But most children know nothing of these. Instead

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[6] Mad Minute Math is a program designed to test children's math skills and increase their calculation speed. They are tested in computation: addition, subtraction, multiplication, or division, and they generally have 60 seconds to answer as many questions as possible.

they are indoctrinated into the still unchallenged Euclidean myth that mathematical knowledge is

> certain, objective, and eternal. Even now, it seems that most educated people believe in the Euclid myth. Up to the middle or late nineteenth century, the myth was unchallenged. Everyone believed it. It has been the major support for metaphysical philosophy, that is, for philosophy which sought to establish some a priori certainty about the nature of the universe." (Davis and Hersch 1998, 325).

Mathematics is supposed to lead us to certainty. The long chains of reasoning are supposed to get you from here to there without a misstep or wrong turn. What we found out in our particular example is that questions and uncertainties are necessary to the life of mathematics, itself. It is a feature of the territory.

"The logic we are living out is centuries old" (Berman 1983, 23). It winds its way through Descartes, Aquinas, Plato to Euclid. Devlin (1997) contends that the logical road has led to a dead end. It is this myth that has gone mostly unchallenged in mathematics education research even today (Battista 1999; Davis 1996; Dowling 1998; Ernst 1991; Hersh 1997; Schneider 1994; Sierpinska, et.al. 1993; Wilensky 1993). While critiques of rationalist and positivist traditions "have made serious inroads into the hegemony of the dominant epistemology, the calls for interpretive frameworks have largely focused on the social sciences and to a lesser degree on the natural sciences. To a great extent, mathematics has still escaped the full glare of this critique" (Wilensky 1993, 30).

However, left unexplored, unopened, uninterpreted, unresearched, the "stubborn particulars" (Jardine 1994b) inherent in this place will be seen only as something to weed out, to be rid out and smoothed over. Their obstinate traces will continue to be understood as difficulties residing in the student, the teacher, or the practice, rather than as a messy feature of the place itself. "The character of mathematical knowledge, is inextricably interwoven with its genesis-both its historical genesis and its development in the mathematical learner" (Wilensky 1996).

Mathematics is something that lives in its "being handed along (Gadamer 1989, 284). It is something we inherit. We can't individually construct it nor can we passively submit to it. Mathematics comes to us through the generations of mathematicians that explored its contours, created its ways, and mapped its paths. Its legacy involves learning how to hear Zeno's paradox of Achilles and the Tortoise anew in a six year old's question, as Kathy wonders aloud, "What happens when I divide each of the squares into halves. And each of those halves into halves. Could I ever divide them up so small that I could reach zero?" Or recognize the voices of Newton and Leibnitz in a similar wondering by six year old James, "So what happens if I divided the squares so that I had a half, a quarter, an eighth, and a sixteenth and then add them all up together. Would they add up to one?"

Passing on this legacy involves learning how to give students "direct contact with 'the great chain of being,' so that they can internally generate the structures needed to hold powerful ideas" (Kay 1995). The real pedagogical work is found in the effort to *get in on the conversation.* To wake it up to the fact that our children want in.

But it's not only the children who want in. We need to wake up to the fact that mathematics wants in, it wants to be taken up as an inheritance because if it is not, then it just becomes a "blank"—pages upon pages of "blank."

This legacy also involves finding ways to live generously in this territory, to embrace its ambiguities, and to learn its ways. As you enter this place, you don't have to turn your back on Euclid knowing how his myth was passed on and continues to be passed on through the generations. The process of understanding is the handing on. It is the process of handing down. You now have a place where you can situate Euclid in a larger environment thereby understanding what Euclid's axioms are good for and recognize that within the place of mathematics there are other good fors and they may be completely contradictory. This involves the ability to look

> critically at the role of *shame* in the mathematical community. Listening to learners and fostering an environment in which it becomes safe for mathematical learners to express their partial understandings results in a dismantling of the culture of shame which paralyzes learners-preventing them from proposing the tentative conjectures and representations necessary to make mathematical progress. In doing so, it parts company with the literature on misconceptions which highlights the gulf between expert and novice. [This way of knowing and doing mathematics] stresses the continuity

> between expert and novice understanding, noticing that even expert mathematicians have had to laboriously carve out small areas of well connected clarity from the generally messy terrain." (Wilensky 1996)

Mathematics, itself, as a living discipline is constituted by its partial understandings. Our partial understandings in learning the ways of this territory are a feature of the place. They don't belong solely to a person. Anything that is living is not wholely worked out, it is not complete, it is always partial. When you are standing in mathematics, you are standing in a living place. It's moving. It's alive.

CHAPTER THREE

The Children Want In

The re-formed mathematics classroom is a dynamic place brought to life through conversations and dialogue. Making meaning in mathematics involves "persons-in-conversation" (Driver, et.al. 1994). The National Council of Teachers of Mathematics (1989, 1991, 1995, 2000) has repeatedly stressed the importance of conversation and dialogue within the mathematics classroom but as TIMSS (1996) found, in many classrooms, learning mathematics is a silent individual activity. Finding ways to break this silent spell requires putting forth new images of classroom life. Learning to describe the complexities of classroom life means capturing its dynamics, the connected collective movement of conversation as understandings emerge in the seemingly disparate and unconnected approaches, dialogues, attempts, and arguments. "In [a] conversation, all of the participants are oriented toward deepening their understanding of the issue at hand" (Davis 27, 1996). "To conduct a conversation means to allow oneself to be conducted by the subject matter to which the partners in the dialogue are oriented" (Gadamer 1995, 367). In this way a mathematical conversation is not the same as a traditional classroom discussion.

> The goal of [a] discussion is more toward the articulation of preformulated ideas, and so the subjects endeavor to exert some measure of control over the subject matter. The emphasis in the discussion is placed on the subjects' conceptual differences rather than on achieving a consensus. (Davis 1996, 27)

In conversations the participants do not attempt to control the subject matter, but rather are deeply engaged in attempting to understand the issue at hand. The subject (mathematics) participates in the conversation—mathematics speaks. It is no longer a silent "blank". The "circle of seekers" (Palmer 1998, 107) that have gathered to explore this new situation are not even aware that a conversation is taking place but only know ***that*** a conversation has taken place "when understandings have changed, when a new commonsense has been established—when self and other have been altered—then it has happened" (Davis 1996, 28).

In our classrooms, mathematics lives in these day-to-day details of its being worked out through conversation. Through conversations we create not only our understanding of mathematics but also we gain an understanding about what it means for mathematics to be a living discipline. But herein lies a difficulty. "The idea that one can be aware that one is in a conversation is in some ways self-contradictory; it presumes an awareness of one's self and one's subjectivity. It is precisely this detached, observer-like awareness that must be set aside in order to allow a conversation in the first place" (Davis 1996, 28). Knowing that a conversation has taken place is always, out of necessity, something that occurs after the fact. It is the outcome, the destination, that determines whether a conversation has taken place, when new now commonly-held understandings can be proclaimed.

To show how mathematical understandings emerge in our classroom, we must let you in on the conversation. To assist you, I have taken an ordinary classroom mathematical exploration and the various dialogues that were involved in coming to understand the mathematical territory that the exploration opened for us. I will freeze some of the individual instances of talk, lingering with each for a few moments, to show: what sometimes lurks beneath the surface of students' frustrations and struggles, how mathematical ideas cohere, how understandings ripple, how a new commonsense comes into being and how new mathematical territories open. In doing this, in freezing the talk, you might tend to lose sight of the dynamic, the fluid, often messy, meandering back and forth flow of conversation. Throughout this chapter you will need to hold the tension of the seemingly disparate instances of talk together to see how conversation works, to witness how we arrived at a new shared understanding of the mathematical exploration we undertook over the course of several weeks.

Paradox In The Classroom

Mathematics shows us a way to hold such a tension. It is called paradox. "Paradox is another name for that tension, a way of holding opposites together that creates an electric charge that keeps us awake" (Palmer 1998, 73-74).

> Paradoxes have played a dramatic role in intellectual history, often foreshadowing revolutionary developments in science, mathematics and logic. Whenever, in any discipline, we discover a problem that cannot be solved within the conceptual framework that supposedly should apply, we experience shock. The shock may compel us to discard the old framework and adopt a new one. It is to this process of intellectual molting that we owe the birth of many of the ideas in mathematics and science. (Wilensky 1993, 68).

The tension created by paradox has a long history within mathematics. The Greek philosopher Zeno of Elia, who lived about 450 BCE invented several famous paradoxes through which he intended to show that there is something extremely mysterious about motion. One of these involved an arrow in flight. Zeno contented that at every instant of time the arrow was somewhere, in some place or position, and therefore, could not at any instant be in motion. At any instant, the arrow is indistinguishable from an arrow at rest. Zeno concluded that if the arrow is at rest at every instant, then it is always at rest. All motion is an illusion.

Zeno posed this paradox, not to argue that an arrow cannot move, but to challenge the belief that time consists of a succession of discrete instants, a challenge that the Greeks themselves were not able to meet. In fact, the paradox of the Arrow was left unresolved for approximately 2000 years. "Indeed, truly satisfactory resolutions to [this paradox were] not found until the end of the nineteenth century, when mathematicians finally came to grips with the mathematically infinite" (Devlin 1997, 76).

Zeno was right to believe that at any particular instant the arrow is at a particular position. He was also right in believing that there is no intrinsic difference between an arrow being at rest at a particular instant of time and being in motion at that instant. His mistake was in concluding that motion was thus impossible. Motion is not the sequential accumulation of incremental bits. "The key to finding the value of the series was to shift attention from the process of adding the individual terms to the identification and manipulation of the overall *pattern*" (Devlin 1997, 76).

> Zeno himself didn't have a proper solution to the paradox, nor did he seek one. The paradox suited his philosophy perfectly. He was a member of the Eleatic school of thought, whose founder, Parmenides, held that the underlying nature of the universe was changeless and immobile. (Seife 2000, 45).

In this chapter I want to show how this paradox works itself out in our classroom, how the individual instances of talk create the movement of a mathematical conversation. I want you to understand that "to ask good questions, deflect answers, and connect students in dialogue" (Palmer 1998, 134) is not enough. Taken up only by themselves, these unconnected moments of talk remain discrete fragments, isolated anecdotes, that even when added together cannot provide a sense of the whole, of the conversation. Instead of giving up altogether and just announcing, "Well you just had to be there...", I want to find a way to make visible the "movement of showing" (Heidegger 1962, 1) that underwrites the work that we as students, teachers and researchers do in coming to understand. Palmer (1998) calls this "the skill of lifting up and reframing what my students are saying... But I [need] to wait for the moment when my students [can] experience it as their own, as a way of naming a discovery that they [have] made for themselves but [are] not yet able to put into words" (134-135).

Gathering up the seemingly discrete fragments, the tangential meanderings, the scattered filaments of talk, accomplishes three important things: "we [gather] up the elements of our dialogue and [give] them coherence, we [build] a bridge to our next topic; and we [do] it all in a way that [makes] students full participants" (Palmer 1998, 135). In this way we shift attention away from the process of adding the individual instances of talk to the identification of the movement created with the overall conversation. "By holding the tension of opposites, we hold the gateway to inquiry open, inviting students into a territory in which we all can learn" (Palmer 1998, 85).

To see how this works itself out, you will enter our Grade 7 classroom. It is a big double room, filled with 60 adolescents, 15 tables and 17 computers. Computers and tables are organized in such a way that every table grouping has access to at least one computer. We do not have a designated space for whole group lessons and conversations. When we need to all come together, the students bring their chairs into one of the areas of the classroom.

The Exploration Begins

The class starts with a mathematical exploration called *Triangles Got Legs!* (Sabinin and Stone 1999).

To introduce the situation, I draw a line segment AB on the whiteboard.

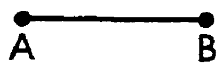

Figure 3.1

I explain that this forms one side of a triangle ABC. Point C of the triangle is missing. The students need to find this third point.

I explain that for some of their choices of the third point C, the side AB would be the longest side of the triangle and for other points; it would not be the longest side of the triangle. As I draw the following figures on the white board, I suggest that these might be some of the possible choices for point C:

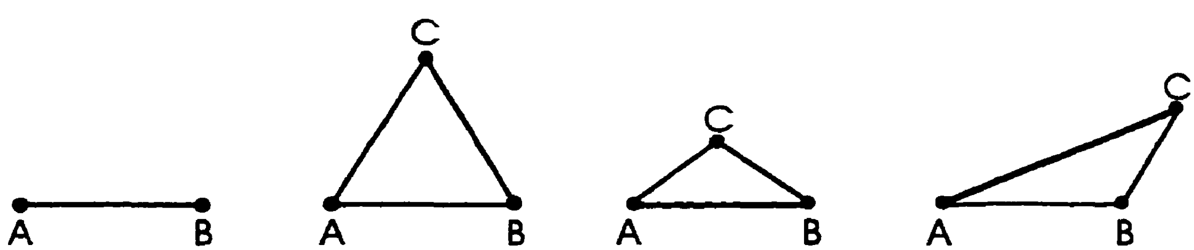

Figure 3.2

I explain that the purpose of this exploration is find for which of the points C is AB the longest side of the triangle. I suggest that as a strategy, a way of approaching this exploration, that students might want to try to identify those points C that "work" by shading in all the points in the plane that made AB the longest side of the triangle ABC.

The students are given time to think and ask questions about the task ahead of them for a few minutes before getting down to work.[7] As they begin, in their own ways either by themselves or with the support of a small group, to formulate their initial tentative conjectures Pat, David and I join them. As I approach this particular group I hear:

> *"We're confused. We really don't understand this problem. The points are there, the ones you showed us. You gave us all the answers when you gave us the question. What's the point?"*

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[7] In our classroom, students choose how they will proceed with the exploration. We don't assign students to groups. They choose with whom they do their work. Some students choose to work on their own, at least initially, when they are working on explorations of this nature; others prefer to begin with the company and support of another person. Students also choose what materials they want to work with as they make their way. Finding ways to both formulate your emerging understanding of the territory that the exploration opens and representing that understanding are both essential elements for communicating that understanding to others.

The problem and my question threw this group of students into an unfamiliar landscape. They were asking: "*Is that all there is?*" The initial exploration we did together as we set the parameters of the problem seemed, to these students to *be* the answer.

"*So what's,*" they asked, "*the point of doing this question at all?*"

"*This is so dumb,*" I hear one of the students say.

It is possible to dismiss these students' complaints, but I believe that it is important to stop here. I believe that under their complaints, so full of frustration, lurks a problem of practice that dominates school mathematics. Left undisturbed it will continue to plague our best efforts to reform our mathematics classrooms.

A Problem Of Practice

All too often, teachers and researchers read student's complaints as trouble with the student or trouble with the teacher. Within educational discourse these are the most commonly available alternatives. Read as trouble with the student, these complaints are turned back on the students themselves. Their frustrations, their feelings of dislocation, are often read as failure: failure to engage with the initial problem, failure to understand the problem, failure to have the proper attitude, failure of motivation. Read as trouble with the teacher, the problem becomes one of technique: failure to properly clarify the initial problem, failure to provide enough guidance, failure to provide enough examples.

The TIMSS Video Tape Study (1996) helps to show us how mathematics education gets caught into a dichotomy of blame—blaming the student or blaming the teacher. This video study involved 81 U.S. mathematics classrooms. It revealed that teaching in the U.S. is focused for the most part on a very narrow band of procedural skills. The teacher carefully instructs the students in a concept or a skill by solving one or more example problems with the class. "Whether students are in rows working individually or sitting in groups, whether they have access to the latest technology or are working only with paper and pencil, they spend most of their time acquiring isolated skills through repeated practice" (Stigler and Hiebert 1999, 10-11). "Mathematical concepts are acquired by 'absorbing' teacher and textbook communications" (Battista 1999).

"In traditional mathematics instruction, every day is the same: the teacher shows students several examples of how to solve a certain type of problem and then has them practice this method in class and in homework" (Battista 1999). Being well conditioned in this method, students readily equate mathematical understanding with knowing how to follow the instructions by diligently duplicating the teacher's prescribed method over a number of examples. "The National Research Council has dubbed the 'learning' produced by such instruction as 'mindless mimicry mathematics.' Instead of understanding what they are doing, students parrot what they have seen and heard" (Battista 1999).

By the time these Grade 7 students came to us, they are very familiar with this method.

> *We have to get up early in the morning. We get there and expect something. Why don't they pay the textbook to teach us? When I was in grade five we had sheets of math and when we finished and [the teacher] wouldn't have enough, so we would have to make up our own sheets. It was so boring.* (Amber - age 12)

And even though Pat and I had taught these same students in Grade 6, some of them still had difficulty letting go of the dependency created by this method of teaching. By suggesting some possible places to begin our *Triangles Got Legs* investigation, it appears that I have thrown some of the students back into a style of teaching that still lurked in the shadows of their mathematical experience. The students seemed to read these initial possibilities as instructions rather than as potential beginning places. And we heard the old complaints resurface, the complaints that often filled our classroom the previous year:

"I used to be good at math and I would still be good at math if you would just tell me whether I should add, subtract, multiply or divide."

"What do all these problems have to do with math?"

"What a stupid question."

By turning the exploration over to them, they were lost because this was not the way we *do* mathematics in our classroom now. Mathematics, for them, is not taught and learned this way.

But for these students, *doing* mathematics is still too easily associated with ritual and method that needs to be rehearsed, page after endless page, sheet after sheet.

When I was in Grade 5 I hated math. We got sheets of paper with questions. If we finished one sheet we got another. (Pam - age 12)

This ritualized method does not belong only to grade school mathematics but appears to dominate the landscape of mathematical pedagogy right through schooling.

> For the most part my student teachers do not want to be problem solvers. They want to know the problems, be told how to solve them, and exercise that facility with endless variations on the theme, which require only recall and patience with tedious calculation. Many of them have been given very positive reinforcement for this type of activity in school. Mistaking this anal-retentive activity for the real thing, they have drawn the conclusion that they are good at, and enjoy, mathematics. Faced with the necessity to explore problems they have not been trained to solve, they are often frustrated, unsuccessful, and feel they are somehow not being properly taught if I will give them neither the method nor the answer. (M.G. Stone, email to the author, February 1998)

> I'd like to be able to report that this stress on computation ends when students reach college. But, alas, even in calculus, linear algebra, and differential equations courses (course required by many different

> majors) there is the same mind-numbing tendency toward routine computational problems. (Paulos 1991, 55)

Most non-mathematicians share the view that mathematics is a question of knowing *what* to do, and view as suspect any attempts to teach otherwise (Dawson 1995). Even people who recognize the intellectual value of problem solving and critical thinking in a wide range of situations often think that within mathematics to solve problems means learning to solve *long lists* of problems (Paulos 1991; Schoenfeld 1992,1994; Stigler and Hiebert 1999). "This is the Truth; now do 400 identical problems" (Paulos 1991, 53).

Mathematical thinking and understanding cannot be reduced to the sum total of memorized procedures. Mathematical thinking and understanding involves "seeking solutions, not just memorizing procedures; exploring patterns, not just memorizing formulas; and formulating conjectures, not just doing exercises" (Schoenfeld 1992). Mathematical thinking, knowing and creating are not the accumulation of these incremental bits. Many teachers and students believe that doing these discrete exercises will add up (Schoenfeld 1992, 1994). And so students repeat procedure[8] after procedure, diligently adding to their growing mathematical toolkit. It doesn't take long however, before the sheer number of procedures overwhelms them. "For most students, school mathematics is an endless sequence of memorizing and forgetting facts and procedures that make little sense to them" (Battista 1999). Memorized bits such as: "always put the bigger number on top of the little number," "cross multiply and divide," "invert and multiply," "a negative times a negative equals positive," "*is* is the numerator and *of* is the denominator" are but a few of the refrains that students learn to chant as they wander through their pages of math "blanks". Along the way they also memorize procedures—procedures for single digit whole numbers, two digit whole numbers, ratios, percents, proportions, scale, rate, interest, fractions, integers and the list goes on and on. Unable to recall which chant goes with what procedure, what procedure belongs with what problem many students find themselves frozen, afraid to take the next step. They don't remember what to do. And so they do exactly what survival training tells them to do, they stand still in one place until someone comes to

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8 Although it is important that students know how to execute mathematical procedures reliably and efficiently, knowledge of procedures involves much more than simple execution. Students must know when to apply them, why they work, and how to verify that they give correct answers; they also must understand concepts underlying a procedure and the logic that justifies it. Procedural knowledge also involves the ability to differentiate those procedures that work from those that do not and the ability to modify them or create new ones.

rescue them and they bewail that fact that they are lost. And so their teacher comes to the rescue.

It is a familiar trap and it dominates many North American mathematical classrooms which leads to the cycle of blame—blaming the students or blaming the teachers, and it forms the basis of much mathematics education research (Hiebert 1999, Russell 2000). The purpose of this chapter is not to enter into the seemingly endless exhausting either-or debate that this dichotomy opens. Rather I want to ask a different type of question of this situation.

> How can we escape the grip of either-or thinking? What would it look like to "think the world together," not to abandon discriminatory logic where it serves us well but to develop a more capacious habit of mind that supports the capacity for connectedness on which good teaching depends? (Palmer 1998, 62)

Being Lost

I want to take up the students' complaints of frustration in another way. Rather than hearing the students' wails as a cry for rescue, I hear and interpret their words as something that is true of all beginnings. Being lost *is* the first step in all new mathematical investigations and explorations. Being part of a living discipline means that you are dropped into a conversation that is centuries old. The students' words can be heard as their desire to be brought into that conversation.

They [the teachers] must memorise everything from university and recite it in front of the class. They aren't including us. We want in. Bring us into the conversation. There is something we want. (Tyson - age 12)

Learning how to be a part of that already on-going conversation means that you have to begin by figuring out where you are. In the beginning, you are lost. And so you start by asking questions and testing the ground.

> But the art of testing is the art of questioning. For we have seen that to question means to lay open, to place in the open. As against the fixity of opinions, questioning makes the object and all its possibilities fluid... Dialectic consists not in trying to discover the weakness of what is said, but in bringing out its real strength. It is not the art of arguing (which can make a strong case out of a weak one) but the art of thinking (which can strengthen the objects by referring to the subject matter). (Gadamer 1995, 367).

"So what's the point?" heard differently as, "Where am I?" ***is*** the question that needs to be addressed and first and foremost in its address, is addressing us. The students "***want in***," and in that desire, they want to know how to join in. It is now possible to hear the student's complaints, "I'm confused. I really don't understand this problem." more generously than the students intended them.

Pat and I were well aware that at the time these words were uttered, these students wanted to be rescued. They also wanted us to know that they thought that this was a "dumb problem." But both of us understand teaching as the point of access to something beyond the teacher.

> Every academic discipline has such "grains of sand" through which its world can be seen. So why do we keep dumping truckloads of sand on our students, blinding them to the whole, instead of lifting up a grain so they can learn to see for themselves? Why do we keep trying to cover the field when we can honor the stuff of the discipline more profoundly by teaching less of it at a deeper level? (Palmer 1998, 122)

We wanted the students to see that being lost, confused and uncertain was not a problem that belonged solely to them as students as they ventured into the terrain that this new problem opened up but it was a feature of a living discipline. In "lifting up this grain of sand" we could show the students that they weren't the only ones who "want in" mathematics also "wants in" to the conversation. Mathematics needs these students to join in so that it can continue as a living discipline. If the young cannot find a way to join in the conversations, then mathematics will remain "blank" and eventually it will die. So it is vital to open a space in which things can now move.

Asking questions of the territory that we have inherited and now find ourselves inhabiting is essential to how we begin to make our way in this new place. Learning how to take the first steps requires letting go—letting go of the lure of certainty, of a "right" path created by the ritualized method of demonstration and rehearsal. It requires letting go of knowing exactly *what* to do. It begins by asking questions like these: "Do I have enough information to begin? I tried, and I couldn't do it; does that mean I can't do it? Does it mean that the problem is unsolvable? Has anyone else ever tried this before? How do I know that what I am doing is leading me somewhere?"

In wanting these students to let go, to enter this mathematical space, I found that I needed to learn more about the territory that I was asking them to enter. Some days, I found myself having to face my own ignorance. "I don't know," I'd have to admit. "Wait, I'll try to find out." I'd rush home at night, madly dig through my books, search the internet, email a friend (one who wouldn't think I was stupid). "God, she doesn't know that! What's she doing teaching mathematics?" I could hear the scorn from the silent corners of my study walls. "If you're going to teach like *that*, I can't help you. I don't know anyone who can." The consultant's words uttered so many years ago had found their way into the private walls of my study and came back to haunt me. In times of uncertainty I found myself caught in the myth that mathematics is something you do by yourself and you do it quickly and you get it right. There is a penalty for not knowing.

> *That's not learning. It's punishment. It's like every morning when I got up to go to school, it was like punishment. The only reason we want to get it right is so that we don't have to do any more. If you don't get it done, you just get more.* (Aaron, age 13)

The students, Pat and I all had to learn to let go of the trauma created by the years of conditioning that living this myth created. There was work to be done. Just as Pat had done when she first came into my classroom as a researcher at Beacon Elementary School, we had to once again roll up our sleeves and get to work. We had to learn how to join in the mathematical conversation. In hearing these students' complaints differently, Pat and I entered into an interpretative space that requires that we begin "with a different sense of *the* given... it begins in the place where we actually start in being granted or given this incident in the first place" (Jardine 1992, 55).

Mathematical understanding is not something that we can learn by observing someone else do it correctly; rather it is something that we must undergo. It *is* messy, filled with first steps. This *is* what beginning a mathematical investigation is all about. This is what learning is about. It is not the accumulation of all of those carefully prescribed precise discrete procedures because "mathematics is messy and not the clean picture we see in textbooks and proofs" (Wilensky 1993, 20). "If we deprive learners of this opportunity, we strip mathematics of its essential character and deprive them of real mathematical experience" (ibid, 22).

All Over The Place

> *"Get over here. How are we supposed to figure out the points for C? We have all these triangles. Look at all these places C can be. C is all over the place. C can be anything.*

How am I supposed to make sense of all the places C could be?

It would take all day."

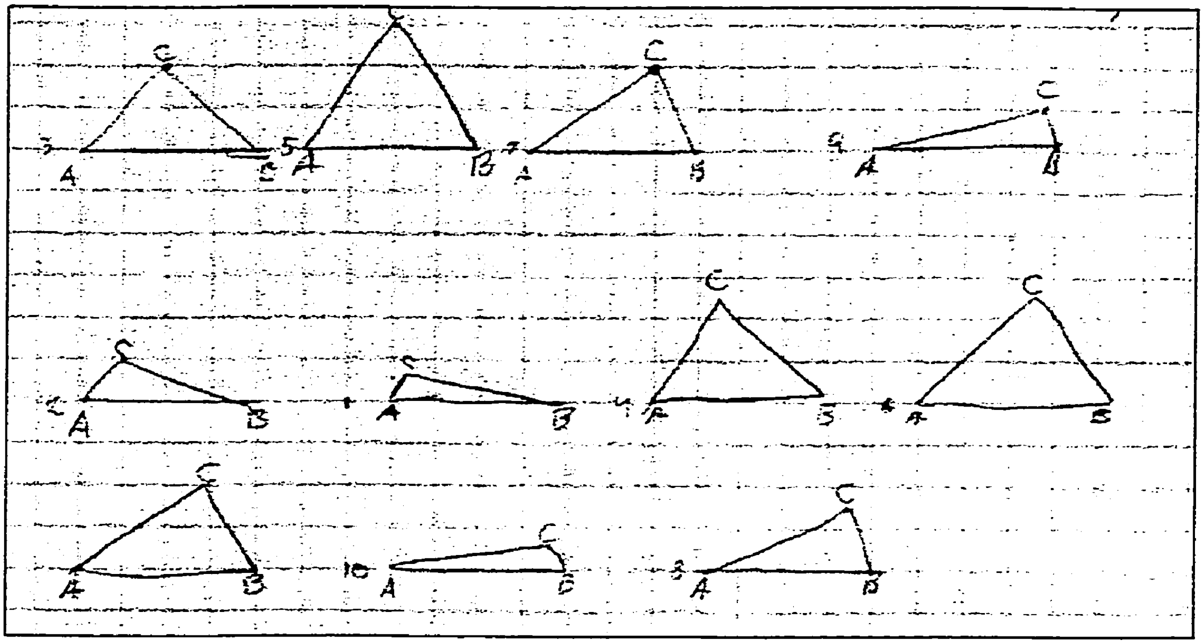


Figure 3.3 Student Work

I glance down at the paper, as I listen to these students' wails of anguish. An assortment of triangles covers their page. Throwing precision to the wind, while keeping line segment AB constant; these students have started to play with some of the places point C can reside. The constraints proscribed by the initial problem statement impose these boundaries upon the students and therefore define but do not limit the region in which they make conduct their play. As they move point C around above line segment AB, the space that opens invites exploration. It is full of possibilities.

In beginning to see some of the possibilities, they are overwhelmed by the space that has opened. *"Look at all these places C can be. C is all over the place. C can be anything. How am I supposed to make sense of all the places C could be? It would* *take all day."* Interpreted as words of turmoil, a rescue is in order. *"Help me, I'm drowning."* Resisting the temptation to rescue, which would throw me back into the either-or dichotomy I described earlier in this chapter, requires that I hear and interpret these students' words differently. Finding ways to hear differently moves me into the unexpected places in my own research. I know that in a very real sense, the students are right. It *would* take all day, if they continued in the way that they are proceeding. They need help, not a rescue, but some guidance—another "grain of sand lifted up" (Palmer 1998, 122).

First, the students need to see that their initial steps are correct; the space, that their initial ventures into this problem open, is large. But it is not *"all over the place"* as they say. They are right in asking for someone to help point to the markers that they have found to show them which ones to attend to at this point.

"I see that you have kept line segment AB constant in all your triangles. What happens if you draw only one line segment AB and then you plot your C points all from that one line segment?

A Place To Play

Beginning to focus on their words: *"It would take all day"* an interpretive space had opened, we had stumbled onto a fundamental aspect of a good mathematical investigation—the invitation to play. Like the "play of light, the play of the waves, the play of gears or parts of machinery, the interplay of limbs, the play of forces, the play of gnats, even a play on words" (Gadamer 1995, 103), the type of play which suspends time for awhile, this is exactly the space that good investigations open and an aspect of mathematics that I want the students to know. It is a space large enough that it *could* even take all day. But in order to take up this conversation, to hear these words as an invitation, the students need to put aside the accumulated weight of experience that tells them that mathematics is repetitive drudgery full of speed, right answers and punishment. "Mathematics may be the only discipline that bases its instruction on hundreds of exercises of five minutes or less" (Dawson 1995).

In selecting this particular investigation, I sought out a problem that was rich enough, open enough, generous enough, so that it revealed something of the ways into the mathematical landscape. I wanted the students to learn to play, to see that playing is essential and that their movement within the play space is demanded. Mathematics and mathematical thinking, knowing and understanding are created in its being played out.

Play *(paidia)* and education *(paideia)*, "both terms arise from an original reference to the activity of the child *(pais)*, an echo of which can be heard in the word "pedagogy" *(paidagogos)*" (Davis 1996, 212). If mathematics is not simply a closed and given axiomatic system but in fact a living discipline inspirited by ongoing questions, quarrels and conversations, then play and the pedagogy of mathematics is not an afterthought but a necessity. If mathematics lives in its continual re-forming, then we need to create a mathematical education that allows the young to experience the creation of mathematics.

Play "is a phenomenon that has tended to be shallowly understood and, in consequence, almost universally scorned by mathematics teachers" (Davis 1996, 214).

Understood as something frivolous, play, as opposed to work, off-task behavior, is not welcomed into most mathematics classrooms. But play is exactly what is needed. It is only play that can entice us to the type of repetition and rehearsal that is needed to learn how to inhabit the mathematical landscape and how to create new mathematics.

"The movement of play has no goal that brings it to an end; rather, it renews itself in constant repetition" (Gadamer 1995, 103). This is not the same type of repetition that is the hallmark of traditional mathematics teaching, that anxiety-inducing, mind-numbing repetitive pacing shaped by rote recitation and *Mad Minute Math*. I'm not talking about the repetitive pacing of the polar bears at the Calgary Zoo, that back and forth motion that wears one single muddy path inside the boundary of their limited confines. Once freed from that enclosure it is possible to understand repetition in a different way. "Within the repetition itself, there is movement (play), so that each act of repetition is indeed a new (informed and transformed) act. It is thus that play sustains itself" (Davis 1996, 300). Each instance creates the next new instant, it is both recursive and iterative, it is what keeps the play, the mathematics, alive.

Coming Together As A Group

As a whole group, we come together to talk about our initial conjectures. Some of the students speak about their feelings of dislocation, of being overwhelmed.

" I looked at this problem and I got so mad and frustrated. I only wanted the answer. I wanted to yell, 'Give me the answer.' You just said, 'I'm showing you how.' In Grades 2 - 5 math it was all rules and doing questions. I liked it because I knew how to do it all. There was only one way and I knew it. But this problem doesn't have just one way." (Tiffany - age 12)

I ask the group with their numerous triangles to talk to their classmates about their conjectures. As the group puts up their paper with all the triangles a murmur of voices goes through the room. A number of the students talk about what it is like to be stuck.

"Getting stuck is okay. Before, I'd just be called 'dumb.' There was a penalty for not knowing how to proceed so I just kept my mouth shut and I didn't tell anyone." (Maria - age 12)

"I'm not afraid to admit that I don't understand something anymore because I don't want to do more of what I don't understand." (Mary - age 12)

Together we look at the question again, we look at all the triangles in front of us and together we start to plan some ways to proceed. I want the students to understand that "the essence of the *question* is to open up possibilities and keep them open" (Gadamer 1995, 299) but at the same time, I do not want them to read this openness as being *anything* and *all over the place.*

One of the students from this group points to the triangles that they have drawn on their paper, which is now taped on the white board for the rest of the class to see.

"We drew all these triangles. All of them are right. We thought that C could be anywhere."

A number of students from the class comment that this looks somewhat like the place that their group is stuck. I think it is important for the students to both recognize that they are stuck and to know that being stuck is another fundamental aspect of doing mathematics—"another grain of sand". It really can't be avoided. It is one of the ways that you know that you are in mathematical territory. I talk with the students about what happens to me when I get stuck. I let them know that sometimes I just stare at the page in front of me, sometimes I get tense because I can't seem to make progress, sometimes I feel frustrated because nothing seems to work, and sometimes I don't even realize that I am stuck until I'm well into a place before I suddenly realize it is a dead-end. I tell them that one of things that I have learned is that the worst thing that I can do is to stop doing anything at all. One of the things that I do is to read the question again carefully ***not*** because I have read it inadequately the first time, but rather, it is often the case that the question only really makes sense after I have played with it for a while and start to recognize the territory. In this way I can now read it more thoughtfully because I have had some experience with the question.

I ask the students to return to the original problem with me. We read it again as we look at all the triangles in front of us. "*Are you just looking at all the individual triangles*?" I ask the students. "*Try looking for an overall pattern.*"

"*Is that why you asked us to keep only one line segment AB and then work the points C from that*?" asks one of the students from this group. I ask the student to come forward to the whiteboard to show us all how she might go about doing this. The various members of the group pick up felt markers. They begin to superimpose one triangle on top of another. As they do this other members of the class drift off into their groups again. They are ready to re-enter the question once again.

Looking For Boundaries

> *"Look, these are points that C can be. $\overline{AB}$ is still the longest side at all of these points. We know that the triangle cannot be an obtuse triangle because then the other line BC will be longer than the AB line. Look we tried this: $\overline{AB}$ is 35 mm. If we make an obtuse triangle then AC is 20 mm and CB is 44 mm. See it doesn't work, so it can't be an obtuse triangle."*

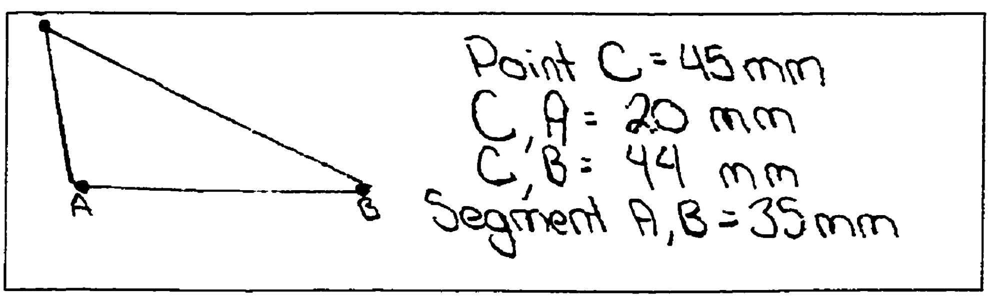


Figure 3.4 Student Work

These students have started to test the boundaries imposed by the structure of this investigation. "The *bounds* are the limits that separate this place from that place; the marking of bounds is the first step in transforming a space into a place" (Davis 1996, 168). Creating the boundary around learning "keeps us focused on the subject at hand. Within those boundaries, students are free to speak, but their speaking is always guided toward the topic" (Palmer 1998, 74). This group recognizes that point C "can't be just anything," it can't be just anywhere, but in their exploration they are recognizing that they are not just filling in the bounds. This group is waking up the bounds and in doing so the boundedness of the territory is becoming visible to them. They are starting to define the places that might hold point C. The task is somewhat like that of determining the boundary, that jagged, fuzzy edge, of a forest. In defining this place, matters of measurement are important. The length of a line now is not just the idea of length as in the earlier investigations, but the actual length. Rulers come out and drawn lines are measured: 45mm, 20 mm, 44 mm, 35mm.

This group is onto something and they just want to continue. From across the room I hear:

> *"Come here. We think we have something. We have found that all the points inside an equilateral triangle would have to work as points for C. Now we could flip the triangle and it would still be the same. But, we think we got it. Is it right?"*

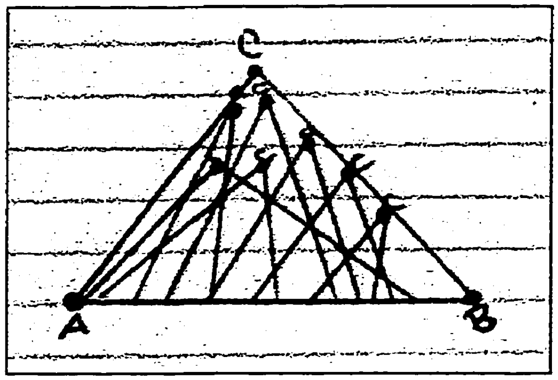


Figure 3.5 Student Work

I look down at these students' page. They have drawn an equilateral triangle and then placed a number of points inside that triangle. On closer examination, I see that they had also changed the lengths of AB. So instead of searching for the boundary, awakening the boundary as the previous group had done, this group fixed the boundary and then filled it in by making all their points fit inside of it.

I turned their question back to them. *"How would you know that you are right?"* I know that they want me to just say whether their answer is right or not. I know that they do not want to me to turn the question back to them, to ask them to convince themselves that their answer is right. However, I want students to understand that

> advances in mathematics happen through the negotiation of a community of practitioners. Moreover, the development of mathematical proofs is not linear, but rather follows the "zig-zag" path of example, conjecture, counter-example, revised conjecture or revised definition of the terms referred to in the conjecture. In this

view, mathematical meaning is not given in advance by a transcendent world, nor is it stipulated in an arbitrary way by conventions of language; rather, mathematics is constructed by a community of practitioners and given meaning by the practices, needs, uses and applications of that community. (Wilensky 1993, 37).

In this sense living mathematical questions are produced through the negotiation of a community of practitioners "during [such] production[s] the student[s] progressively work out [their] statement[s] through an intensive argumentative activity functionally intermingled with the justification of [their] choices" (Bartolini 2000).

This shows us how to read a concern of Palmer's (1998) regarding pedagogy and, I suggest especially mathematics pedagogy which

> centers on a teacher who does little more than deliver conclusions to students. It assumes that the teacher has all the knowledge and the students have little or none that the teacher must give and the students must take, that the teachers sets all the standards and the students must measure up.
>
> In reaction to this scenario, a pedagogy based on an antithetical principal has arisen: students and the act of learning are more important than teachers and the act of teaching. The student is regarded as a reservoir of knowledge to be tapped, students are

> encouraged to teach each other, the standards of accountability emerge from the group itself, and the teacher's role varies from facilitator to co-learner to necessary evil.
>
> As the debate swings between the teacher-centered model, with its concern for rigor, and the student-centered model, with its concern for active learning, some of us are torn between the poles. We find insights and excesses in both approaches, and neither seems adequate to the task. The problem, of course, is that we are caught in yet another either-or. (Palmer 1998, 116).

In a teacher-centered classroom, a question like, *"Is it right?"* receives a quick yes or no response from the teacher. In a student-centered classroom, this same question gets turned back to the students as: *"What do you think?"* or *"Everyone has their own right answer. You are all right."* And "whiplashed, with no way to hold the tension, we fail to find a synthesis that might embrace the best of both" (Palmer 1998, 116).

Inviting Mathematics In

There is another way to understand the question, *"Is it right?"* This way points to the discipline of mathematics, itself. Heard differently, *"Is it right?"* provides the opening that invites mathematics into the discussion.

> A subject-centered classroom is characterized by the fact that the third thing has a presence so real, so vivid, so vocal, that it can hold teacher and students alike accountable for what they say and do. In such a classroom, there are no inert facts. The great thing is so alive that teacher can turn to student or student to teacher, and either can make a claim on the other in the name of that great thing. Here, teacher and students have a power beyond themselves to contend with—the power of a subject that transcends our self-absorption and refuses to be reduced to our claims about it. (Palmer 1998, 117)

In such a classroom all conversations are three-way. They are about something and that something has something to say. In a mathematics classroom it means striving "to understand what a person says, to come to an understanding about the subject matter, not to get inside another person and relive his experiences" (Gadamer 1995, 383).

"How do you know you are right?" or *"Show me that it is right."* are the questions that mathematics begs of the students.

Mathematicians come together in conferences, through their journals and in lecture halls to show others what they have discovered and why and how it is "right" or true. This is the way the community of mathematicians creates the discipline of mathematics (Davis 1996; Davis and Hersh 1998; Ernest 1991; Thurston 1998; Tymoczko 1998; Wilensky 1993). *"I will show you the line of thinking that I have* *created and I will show you why it is so and if you can find no fault with any of the steps that I go through, then it must be true"* the individual presenting the mathematical proof reasons. The final arbiter of the truth is the mathematical community of practicing mathematicians.[9]

> What distinguishes mathematics from other disciplines is the certainty that is obtained through the rigor of proofs. But in fact proofs are not the source of mathematical certainty. They are a technique used by mathematicians to create a uniform procedure for verification of mathematical knowledge. The technique consists of 'linearizing' the complex structure that constitutes the ideas in a mathematical argument. By means of this linearization, mathematical proofs can be checked line by line, each line either an axiom or derived from previous lines by accepted rules of inference.
>
> But the hegemony of the standard style of communicating mathematics (definition/theorem/proof) constitutes a failure to come to terms with the mind of the learner. We attempt to teach formal logical structures in isolation from the experiences that can connect

[9] This account of proof as a means of persuading the mathematical community is gaining acceptance in the mathematical community; however, I do acknowledge that it is contentious. For those who hold that mathematics is a system of absolute truths, independent of human construction or knowledge—then mathematical proofs are external and eternal.

> those structures to familiar ideas. The result is that the idea too often remains 'abstract' in the mind of the student, disconnected, alien and separate, a pariah in the society of agents. (Wilensky 1993, 70-71)

Once the students have entered into the living community of mathematical practice the point is not to cast them to the place of procedural technique—definition/theorem/proof. *"Is this right?"* now needs to be taken up with all the seriousness that the mathematical community demands. Students need to come before their peers to explain the rightness of their thinking. This is not easy for students to do initially. They feel self-conscious. They don't want to show their partial conjectures for fear of embarrassment, for fear that the topic of their conjectures is them and not math.

The mathematics textbooks that students have used to date do not help to ease the students' fears. The mathematics of textbooks is clean and precise.

> If learners believe that the mathematics as presented is a true picture of the way the mathematics is actually discovered and understood, they can be quite discouraged. Why is their thinking so messy when others' is so clean and elegant? They conclude that clearly mathematics must be made only for those born to it and not available to mere mortals. Mathematical discourse is not a form of persuasion continuous with daily discourse, but is instead in some special

> province all its own, a purely formal phenomenon. These mathematical learners are deprived of the experience of struggling for a good definition, and the knowledge that mathematical truths are arrived at by a process of successive refinement not in a linear and logically inexorable fashion… It is difficult to challenge old ideas, or to formulate new ones, in the absence of a culture that supports the floundering, messy process of mathematical exploration. (Wilensky 1993, 72-73)

It is important for the teacher to create a climate in which students are freely encouraged to bring their partial understandings and conjectures forward to the rest of the class because this is the climate that mathematics requires. Students should, therefore, be expected to talk about and justify their thinking and the work they create; their mathematical understanding. The students want in, they want to have access to the conversation of mathematics, so they need a culture that will support their gaining access to the ways of knowing mathematics. In such a classroom, students muster up the courage, come before their classmates, to explain and justify why their conjecture is right.

We all gather together as these students recreate their solution on the whiteboard for their classmates.

"We drew an equilateral triangle. We knew that an equilateral triangle has all sides the same length. We have found that all the points inside an equilateral

triangle would have to work as points for C. Now we could flip the triangle and it would still be the same."

They wait for their classmates' responses.

"That kinda looks like the solution we got. But ours is a bit different."

A student holds up his group's solution.

"Here, can I show you what we did?"

The student comes up to the whiteboard and puts his group's solution next to the first one and then explains:

> *"The points for C' that work are all less than 60°. Look, we have shaded in all the green dots in the middle and all the places of the green dots and all the places in-between all the green dots work. This is because if a green dot is chosen to be C it will be shorter than AB because they are less than 60°. All the red dots including the ones on the equilateral line will not work considering the fact they are 60° or over. So all the points on the side or on the sides of the equilateral triangle work."*

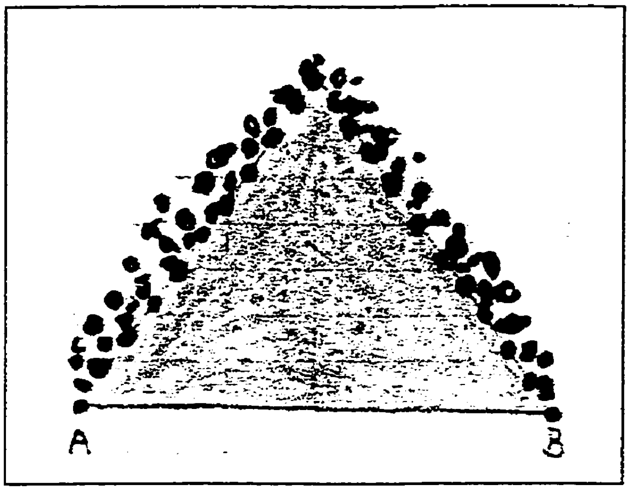


Figure 3.6 Student Work

Again, the lines of an equilateral triangle are sharply drawn. Green dots appear inside of these lines and red discrete dots mark the outside. This group also claims that any points C that are situated outside the boundary line "*will not work considering the fact they are 60° or over.*"

The two groups are very pleased that they have come to the same conclusion. They are all smiles; as far as they are concerned the problem is solved. You can almost hear their thoughts, "Phew! We solved that one."

They turn to their classmates.

A hand goes up.

"*I think there is a problem,*" Ian suggests. You can see furrows start to appear on the group members' brows as they look back at their drawing. "*The first group kept on moving the length of the AB line. See take a look at all their AB lines. That doesn't make sense. I don't think that is allowed. You are supposed to keep AB the same.*"

Small conversations and arguments erupt about the classroom as students talk with each other about whether this breaches the conditions imposed by the problem. Some students start to dig through the papers on the table next to them to find the problem. They go over the problem, reading it to each other.

You can hear, "*That's right. You have to keep AB the same,*" erupt around the room.

"*Your solution is not right,*" several students challenge.

The group looks at each other as if to say, "Okay which one of you said that this was right." They look back at the solutions on the whiteboard, "*But they got the same solution,*" Linda, a member of the first group counters while pointing to the other solution on the whiteboard "*and they didn't change the length of AB. So it doesn't matter that we changed AB it's still right.*"

The class is silent. I can almost hear the thinking as I see the students going over the drawings and the solutions again. Yeah, everything seems to check out. Quod erat demonstrandum (QED), I can almost hear these students' mathematical colleagues announce as students around the room nod their approval.

"I think they are right, Mrs. Friesen. I want to show you what our group did and even though our way of doing the problem is different, we think we have the solution." Pam comes to the front of the class and puts up her group's work .

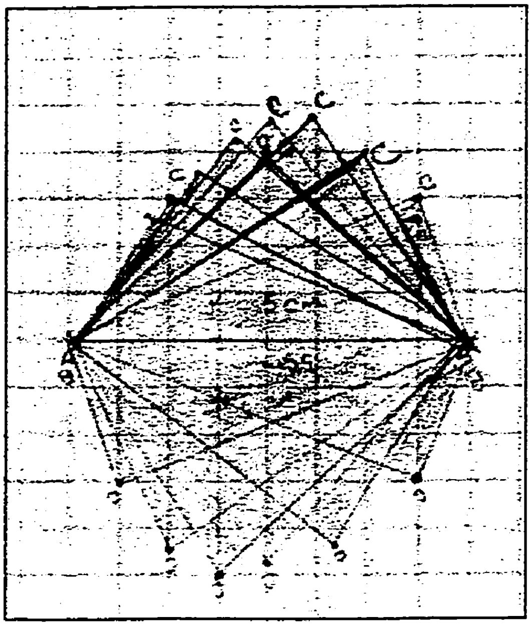


Figure 3.7 Student Work

"The points always have to be inside points A and B. They also have to be shorter in height than the length of the AB line," Pam goes on to explain.

I can see what the students are doing and thinking as they are going about their solutions, but there is something important that they are missing. I'm not sure how to go about helping them to see what they are currently unable to see. I clumsily ask, *"But show me again, how it is that you determined that AB was the longest?"* The students at the whiteboard look confused by my question. Paul hands them a meter stick.

"What do we need that for?" one of the students asks. A student in the group grabs the meter stick and measures each of the three lines of the equilateral triangle. The other members of the group look on. *"They are all the same,"* the student

announces to the class. By this time Aaron had moved himself right in front of the students.

"Are you sure the problem asked you to find a triangle that had all sides the same length?" I try to encourage the students to look again.

"Yes. And we knew that an equilateral triangle would work because all the sides are the same length," answered Myra.

"They are all sixty degrees," said Evelyn. *"So they are all the same length."*

I watch Aaron, who has now left his chair and is walking to the front of the class. He takes the meter stick from Paul. There are a number of different discussions happening around the classroom as students argue with each other trying to convince each other that they solved the problem by pointing to the properties of their equilateral triangles. They pay little attention to Aaron as he lays the meter stick on the line segment AB. I watch Aaron as he carefully measures line segment AB and then places his finger on the meter stick at point B.

I notice that David has his eyes fixed on Aaron too. Aaron seems totally engrossed with the measurements he is taking and is oblivious to the attention that David and I are giving him. He removes the meter stick from the paper and with his finger firmly clutching the measured point B puts the end that he had on point A onto point B on the piece of paper. He then swings the other end of the meter stick up to measure the length of the line BC. As he swings the meter stick up into position the meter stick passes through an arc before it comes to rest at point C. Aaron stops. He seems to notice something. He looks around and now notices David and I watching him. He also notices that several classmates have stopped their conversations and are also looking at what he is doing.

Aaron looks down at the place he had put his finger. He again measures line segment AB and once again places his finger on point B on the meter stick. Then again places the end of the meter stick that had been on point A onto point B and swings the meter stick up to point C. Once again he stops, but this time very briefly. He swings the meter stick down the arc again and then up again. Many students have stopped their discussions and are watching Aaron. *"Aaron, can you explain what you are you noticing?"*

"Look. I think the points can be outside of this line." Aaron sounds very confused as he points to the part of the arc that falls outside of the line. *"That really doesn't make sense."*

"He's right, but that can't be." Paul reluctantly agrees with Aaron. Some of the students look confused and it seems as though everyone in the class starts to talk at once.

"Wait! That has to be." Tyson leaves his chair in the classroom and comes up to the board. He takes the meter stick from Aaron and begins to trace out the arc of the meter stick. *"I've got it!"* He looks at the members of his group.

"Let's go. I've got it!"

A number of the students come up to join Aaron who now once again has the meter stick in his hand. Everything feels very chaotic. The groups reform now in many different ways. Many students temporarily join Tyson and his group. Some students go back to their previous groups. Still other students stay with Aaron and coax him into remeasuring the length of the line segment AB again. They want him to repeat what he had done that had caused Tyson to see something beyond what they have been able to see. Everyone is either in a state of excitement or bewilderment. Some students understand the break through that Aaron's movement has made possible.

Everyone seems to want to get into the place that Aaron has stumbled into. Some students huddle together in tight groups as they work with the problem anew, others start rummaging through the classroom cupboards for string, still others ask if they can go out to their lockers "*to get something.*"

Searching For The Vesica Piscis[10]

"I think that the C point would 'work' as long as it is not in a direct straight line with point A or point B."

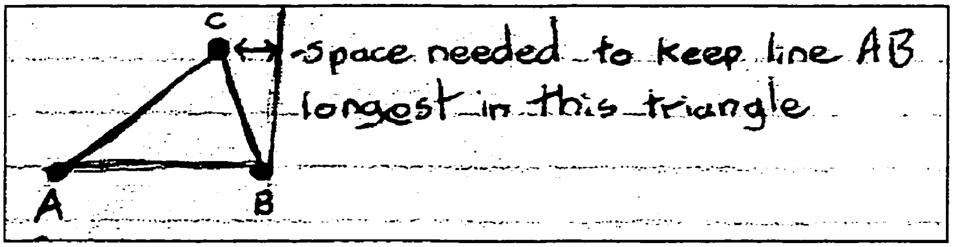


Figure 3.8 Student Work

[10] A figure composed of two equal and symmetrically placed circular arcs. It is also known as the fish (piscis) bladder (vesica). It is formed by the intersection of two unit circles whose centers are offset by a unit distance.

"Look C can't be too far above $\overline{AB}$ *, and C can't be directly above or to the outside of point A or B. I know that all the points for C are in the shaded area. The result is that* $\overline{AB}$ *is the longest side. Once C is outside the shaded area,* $\overline{AB}$ *is no longer the longest side. So it must look something like this."*

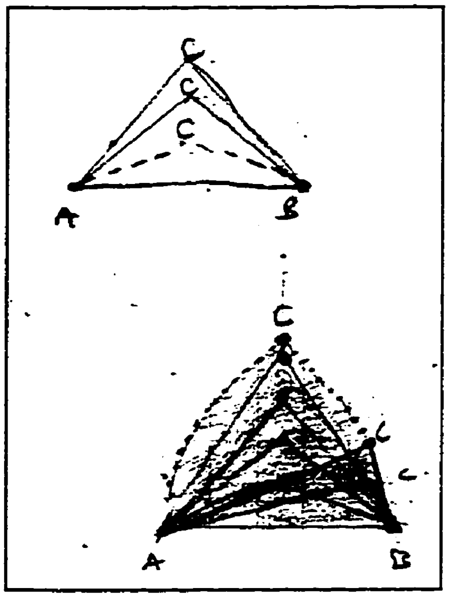


Figure 3.9 Student Work

This group of students has gone back to their original drawings and has now extended a somewhat perpendicular line above point B. They seem to know that the points for C must fall into an area just outside of the equilateral triangle but they seem uncertain at first about how to go about determining where these points might be. One of the members of the group has redrawn the equilateral triangle and has started to place some points C onto it. However, a group of students using a string that they have found in the classroom take the paper and start to work just below his triangle.

Using the string, they measure the length of the line AB. Placing one end on point B they start to measure off the points C as they bring the string up to point C.

Another group comes over to join this group and there are now about 20 students all standing around this table watching as Tyler marks off the points C. The students start to express wonder and amazement as a curved line starts to emerge.

Finding The Vesica Piscis

"C can be anywhere within the area that the circles intersect. The point where C can be located keeping AB the longest are within the area where the radius of A and the radius of B overlap. To keep the line segment AB the longest, point C must be placed within the intersecting area of both radiuses (sic)*."*

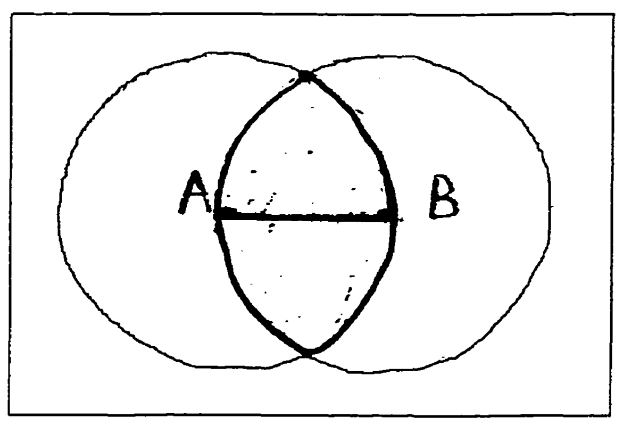


Figure 3.10 Student Work

To create this shape, Ian and Drew retrieved their geometry sets from their lockers. They used the line AB as the length of the radius of two circles of which point A is the center of one circle and point B is the center of the other circle. I wanted to know what Ian and Drew had seen that led them to understand line AB as a radius and not just the length of the leg of a triangle. They could not explain what they had seen or why they thought that they could find the points for C with a compass but they knew that the line segment AB could be both the base of a triangle and the radius of a circle. As we stood talking about the two intersecting circles in front of us Christopher spoke up, very confused, *"But I thought we were learning about triangles. What do circles have to do with triangles?"*

With his eyes fixed on the paper, Paul gasped in disbelief, *"Hey, that means that the sides of your triangle are not straight. They are curved. Can angles be curved? Can a triangle have curved angles?"*

A Place That Draws Us In

The leap that the students had made through the conversation that we had had as we worked through our exploration of triangles gave birth to a gestalt and a whole new area of kinship, a whole new world of relations, opened between triangles and circles. In being able to see the line segment AB as both the base of a triangle and the radius of a circle they made a connection between circles and triangles, which cannot happen when it's only about triangles, unless you draw every possible triangle. They already knew that that would take all day.

The puzzlement and questions that Christopher forwarded as we looked at the two identical intersecting circles opened another mathematical place, another beginning. Christopher's question pulls us deeper into the mathematical space of

Euclidean constructions, the work of the Pythagoreans, Babylonians and Greeks. Paul's question *"Can triangles have curved angles?"* opens the territory of non-Euclidean geometry and the work of Bolyai and Lobachevsky.

The students had found the vesica piscis. Working together they had learned something about the nature of mathematical thinking, knowing and creating. The space we now inhabited pulled us into "*its* question, *its* respose, *its* regard. Therefore, first is the question posed, not *by us* but *to us*" (Jardine, Friesen and Clifford 2000, 6).

Our next steps lie before us. We had more mathematical territory to traverse. But for this moment we sat back and enjoyed the thrill of understanding that we laboured so hard to bring forth. On this occasion of genuine learning, we laughed. For it seems that whenever learning is truly educational, when it occurs in a way that transforms our experience or sharpens our powers of reason and observation, we are most happy and satisfied.

A Necessary Feature Of The Mathematical Landscape

What has become clearer to me is the centrality of conversation and dialogue as a necessary feature of mathematical practice. Working together we deepened our understanding of geometry. Together we learned that each of one of us is capable of comprehending much more than we had initially realized. And in working in this way we all learned how mathematics is practiced—how to help keep it "open for the future" (Gadamer 1989, xxiv).

We learned that mathematics is created through a practice of inquiry. Mathematics is not static discipline the point of which is a piling on of "facts". In mathematics, consuming and producing memorized procedures do not add up to mathematical understanding (Russell 2000; Stigler and Hiebert 1999; Video tape classroom study 1996) "The critically important point is that *mathematical thinking* consists of a lot more than knowing facts, theorems, techniques, etc." (Schoenfeld 1992).

Posing and solving problems lives at the heart of mathematics. Introducing students to mathematical problems and investigations, "involves renewed effort to focus on: seeking solutions, not just memorizing procedures; exploring patterns, not just memorizing formulas; and formulating conjectures, not just doing exercises" (Schoenfeld 1992).

Mathematics is a dynamic discipline. Students need to "study mathematics as an exploratory, dynamic, evolving discipline rather than as a rigid, absolute, closed body of laws to be memorized" (National Research Council 1989, 84). This image of mathematics is very different from images of traditional school mathematics. In learning how to inhabit and explore the mathematics, students are disciplined by the bounds that mathematics itself places on the territory—ways of conduct aimed at satisfying the human desire to know and understand. That is, students who learn to recognize the mathematical space they are in understand what it means

> to keep it susceptible to be taken up and transformed anew and, it must be emphasized, to keep ourselves open to being transformed in

> our traversing its terrain and meeting our own ancestors in that terrain. In such a sojourn, we risk becoming someone who bears the marks of having undergone such an adventure. We run the risk of coming to bear the marks of becoming *experienced* in mathematics in that wonderfully ecological sense [of] coming to 'know your way around.' (Jardine, Friesen and Clifford 2000, 13)

Students who learn mathematics in this way learn ways of a generative culture. They learn what it means to create mathematics.

CHAPTER 4

Anh Linh's Shapes

"It's a poor sort of memory that only works backwards," the Queen remarked

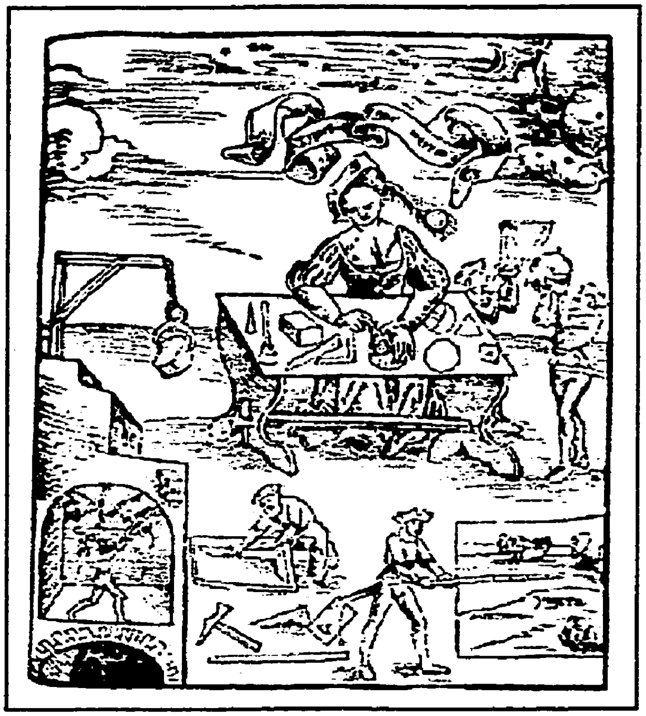
Lewis Carroll, *Through the Looking Glass*

The Story Begins With Anh Linh At Work

She sits at the end of one of the tables in our classroom. Her long dark hair falls onto her paper as she methodically calculates then meticulously measures each new line. Placing her ruler across the two points that she has calculated and measured, she ever so carefully draws the first light pencil line. Then checking to ensure the accuracy of the line, Anh Linh draws the second, now darker line over the first line. She removes the ruler from the paper and critically analyzes her work. "Good, it's good," she seems to say. And then she repeats the process, recursively adding the next and then the next line to the geometrical drawing.

Sometimes a smile of intense satisfaction crosses her face. Sometimes fellow students come by to inquire about her work. "Wow Anh Linh, that is so beautiful," they say as they admire the emerging form. Anh Linh smiles and then goes back to calculating, measuring and drawing. Each line is precise. Each calculation is exact.

Pat and I also watch Anh Linh as she works on this construction. Images of Basle's (1583) *Margarita philosophica* (in Lawlor 1982, 7) come to mind in which geometry is depicted as a contemplative practice, "personified by an elegant and refined woman, for geometry functions as an intuitive, synthesizing, creative, yet exact activity of mind associated with the feminine principle" (Lawlor 1982, 7). Deeply immersed in the traditions of geometrical ways of knowing and doing that have "arisen within our human space through human activity" (Husserl 1970, 355), Anh Linh has come upon "an inner logic so profound that every critical piece of it [contains] the information necessary to reconstruct the whole" (Palmer 1998, 123).



Margarita philosophica

It Also Begins With The Pythagorean Theorem...

Pat and I learned the stories of the mystical Pythagoras and his disciples when we first set out on this journey together in our irregularly shaped classroom with fifty-some Grade 1/2 children. Now here we were, once again telling the secrets of these early mathematicians and their quest to unite numbers and shapes to fifty-some Grade 8 children. These students were just as enchanted by the stories of these ancient radicals as the younger children had been. "Good mathematics ultimately comes from and returns to good stories—and the questions that bug you" (Casey and Fellows 1993, 1)—stories that have the power to open an engaging mathematical space in which compelling mathematical explorations invite and entice both the novice and the expert mathematician (Friesen and Stone 1996). In this space, right angle triangles are so much more than finding the length of the hypotenuse using the handy formula—a theorem that stills bears the Pythagorean name.

Invoking a 3-4-5 triangle and unfolding its beauty and simplicity necessitates the story of a man, an outcaste. How else can we let the students know that this simple formula carries with it the weight of history? It stands the test of time. It still stands as a pillar in trigonometry. This act of measurement is a fundamental one that reaches back to Ancient Egypt. Using a rope knotted into 12 sections stretched out to form a 3-4-5 triangle, rope-stretchers reclaimed and reestablished the boundaries of land and set order to the watery chaos created by the annual flooding of the Nile.

Reaching back in time, the Pythagorean theorem is one of the earliest theorems to known ancient civilizations. There is evidence that the Babylonians knew about this proportional relationship some 1000 years before Pythagoras (Siefe 2000, 29). *Plimpton 322*, a Babylonian mathematical tablet which dates back to 1900 B.C., contains a table of Pythagorean triples—3-4-5, 5-12-13, 7-24-25... The *Chou-pei*, an ancient Chinese text, also provides evidence that the Chinese knew about the

Pythagorean theorem many years before Pythagoras discovered and proved it (Joseph 1991).

And It Begins With An Exploration...

"Draw a right angle triangle. Any sized right angle triangle. Using only triangles that are similar to and/or congruent with your original, I want you to explore the properties of right angle triangles."

My instructions were very simple. The story had already charmed the students and generously bounded the territory of the exploration. I provided these few directions to start our mathematical journey, and then we all began.

What a strange place to be teaching like *this*. We were in the heart of East Calgary. These students scored in the lowest quartile in the entire province. Our colleagues told us that what these students needed were "the basics."

"Make them memorize their basic facts."

"Give them real life problems. You know problems like calculating how much change they need to give someone. Or how much money they will need to earn to buy groceries. Or how much material they will need to purchase in order to make the items that their customers desire."

We seldom entered into the exhaustive debates that these well-intended comments opened. "What if this is not the way that mathematics **exists**, as object either produced or consumed, either individually or collectively" (Jardine, Friesen, and

Clifford 2000, 3)? Having endured seven years of *consuming* and *producing* mathematics, these students were very clear about their regard for math. *"We HATE math." "It's boring." "We are never going to need it." "We'll just get a calculator."* These students who were bored and turned off almost from their earliest days in school, who could not (or would not) read, who knew far too little mathematics, who would stop taking science as soon as they could get away with it, who dropped out of school at worrisome rates. It is with these students that we now taught like *this*.

It Also Began The Year Before...

It began last year. Having made the decision to move to this school, Pat and I knew that if we were to make a difference to these students, we would have to work with them for longer than one year. And we would need to keep them together for long blocks of uninterrupted time throughout the day. And we would also need to teach them all the core academic subjects. This seems like a strange request when everything about the structure of junior high school works against this type of organization, this type of connection and connectedness. But the administrators were receptive and supportive of our request, eager to see what differences this would make to how these students learned.

We needed this type of structure in all the core academic subjects, and in mathematics we needed it to break free from the spell that mathematics is about the quick method, the quick answer, the one right algorithm, the boring repetitive math that they hated. We wanted to connect students meaningfully with the discipline of mathematics in all its wondrous complexity rather than reducing it to more memorized formulas and computation or more real life problems of consumption and production. We knew that "to decide whether a math statement is true, it is not sufficient to reduce the statement to marks on paper and to study the behaviour of the marks. Except in trivial cases, you can decide the truth of a statement only by studying its meaning and its context in the larger work of mathematical ideas" (Dyson 1996, 801). What we wanted to do was to present the idea that mathematics contained a landscape of possibilities.

"By teaching this way, we do not abandon the ethic that drives us to cover the field—we honor it more deeply" (Palmer 1998, 123). We learn how to 'inhabit' such a mathematical landscape. Teaching in this way requires nurturing. The cultivation of this place is not simply a recapitulation of the old, like plowing the same old furrow again and again. "Teaching from the microcosm, we exercise responsibility toward both the subject and our students by refusing merely to send data 'bites' down the intellectual food chain" (Palmer 1998, 123). We were working more like the rope-stretchers of Ancient Egypt taking time and care to bring order to the newly fertile landscape so that we might find ways to draw new boundaries upon fertile ground. At times we would take out our string with the 12 evenly spaced knots and draw out 3-4-5 triangles. At other times, changing our perspective, we would open our rope stretcher's triangle revealing a circle with 12 evenly spaced knots linking us to the perfect, endless, infinite and to time itself. "By diving deep into the particularity, these students [were] developing an understanding of the whole" (Palmer 1998, 123). Working in this way with these students, we began to show them that the cultivation of mathematics necessitates the creation of the new in the midst of the old. Such cultivation requires creation and recreation. It is a fruitful space, a space that "bears" something, births something and contains the conditions to take care of what is thus "birthed."

In this space, with these students, we asked:

> What if mathematics is much more a world into which we ourselves are drawn, a world which we do not and cannot 'own,' but must rather somehow 'inhabit' in order to understand it? What if we cannot own mathematics (either individually or collectively), not because it is some object independent of us and our (individual or collective) ownerships, but because **it is not an object at all**? What if, instead of production and consumption, the **world** of mathematics (as a **living, breathing, contested, human discipline** that has been handed to us) needs our memory, our care, our intelligence, our work, the 'continuity of [our] attention and devotion' (Berry 1977,32) and understanding if it is to remain hale and healthy and whole? (Jardine, Friesen and Clifford 2000, 4)

Deeply committed to finding new approaches, we struggled to find ways to help our students 'inhabit' mathematics. From our first beginnings we worked with mathematical explorations—the stories and fruitful spaces that they opened knowing that working in this way would "bear" something if we cared properly for it. A full year had now elapsed and we were seeing some of the fruits of our care. It was a full year ago that I told these students the story of four spiders that started crawling from four corners of a six meter by six meter square. As I remember, each spider began to pursue the spider on its right, moving towards the center of the square at a constant rate of one centimeter per second. I embellished the story as I went along so that the students would be intrigued by the exploration that the story opened. Would these spiders ever meet, and if so how long would it take and what would their paths look like? (Pappas 1989, 228; Holding 1991, 119) Through this exploration I intended to introduce the students to the ideas of area, ratio, similarity and limits.

The students, however, became entranced by the pursuit curve—the path that an object takes when pursuing another object. They couldn't believe their eyes that these straight lines produced curves. We never did calculate the time it would take the spiders to meet. Instead, the beautiful curve that emerged as the students worked so captivated them that they spent their time drawing and redrawing the path produced by the four spiders. Beauty and wonder are not attributes that any students, especially these students would associate with mathematics. However, here they were, describing these four congruent logarithmic spirals as beautiful, awesome, magical.

In The Presence Of The Past: Anh Linh, The Pythagorean Theorem, The Exploration, The Year Before...

Now, one full year later, Anh Linh called forward the pursuit curve and the beautiful logarithmic spiral as she explored the 3-4-5 triangle. However, she was not content to stay within the confines of the exploration. She began the exploration by creating a series of right angle triangles much like this:

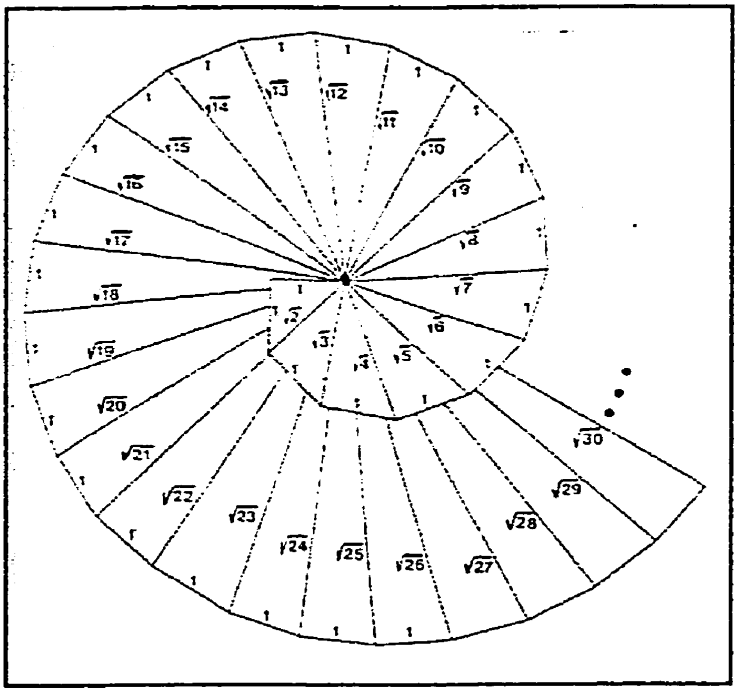


Figure 4.1 spiral using right angle triangles (Pappas 1989, 99)

From these sketches she drew this (Figure 4.2) logarithmic spiral. As Pat and I gazed upon this incredible piece of work, each point meticulously measured, each line precisely drawn, we could barely believe that this work came from a twelve-year-old child.

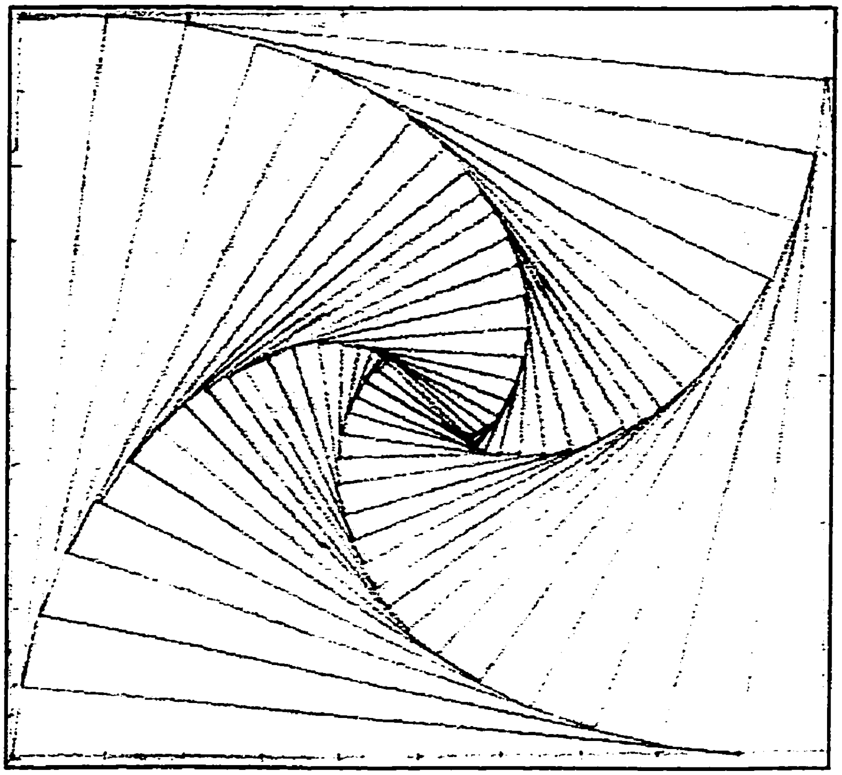


Figure 4.2 Anh Linh's spiral

Anh Linh was on to something else. There was something in the spiral that still called to her, something still unresolved. She wrote:

> I began with right angle triangles. I saw a spiral when I started to put them together. I knew this shape. I remembered the spider's path. I saw the spider's path in the right angle triangles and I wanted to know if these were the same. I thought that my shapes might to be similar in some way. I wasn't sure in what way they would be similar. I wanted to see what would happen.

The path formed by the pursuit curve that she had experienced last year, had a similarity known as self-similarity. By rotation, the curve can be made to match any scaled copy of itself. In the figure below I have shown how the angle between the radius from the origin and the tangent to the curve is constant.

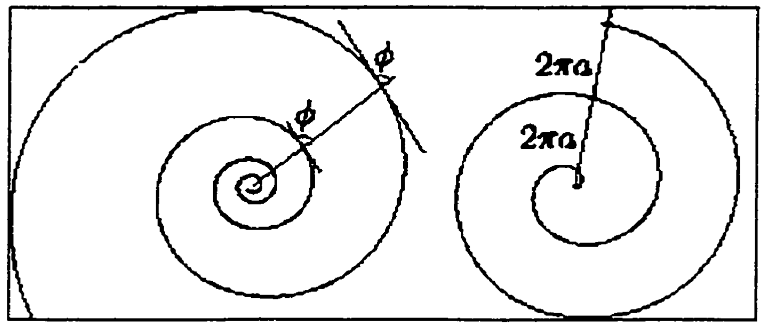


Figure 4.3 Angle between the radius from the origin and the tangent to the curve is constant

This curve is known as the logarithmic spiral, the equiangular spiral and the growth spiral. Growing larger, this spiral exhibits expansive growth in the form of seashells and hurricanes. It results from the play of a square with the transcendental ϕ ratio—1.6180339...

Getting lost in the exploration, Anh Linh decided to create another logarithmic spiral next to the one that she had just created.

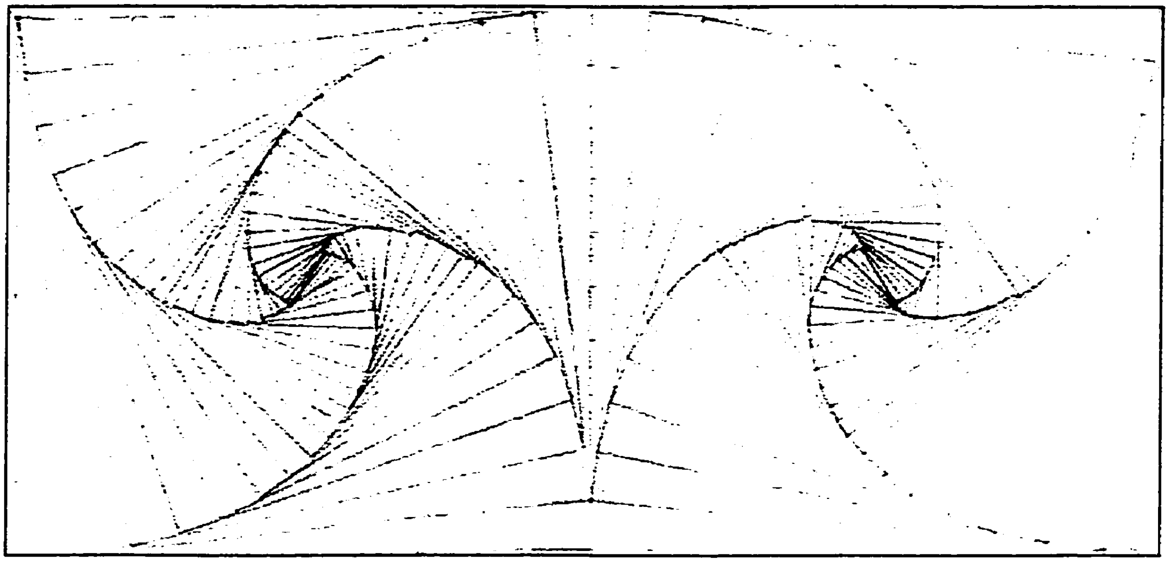
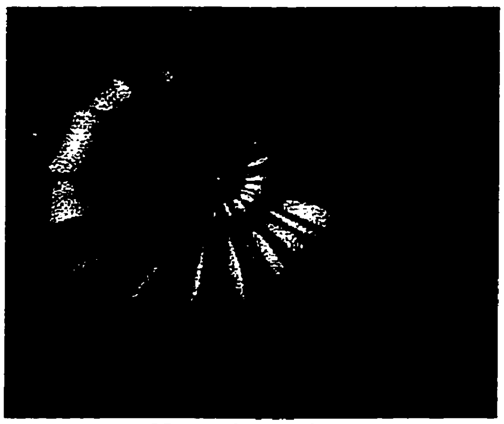
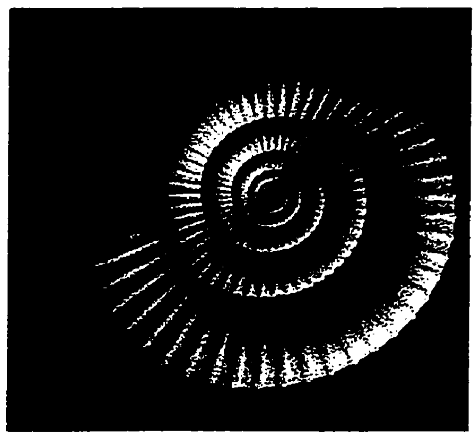


Figure 4.4 Anh Linh's reflected spirals

As Anh Linh continued with her exploration, we all became intrigued with the natural forms this shape reminded us of and we started to examine naturally occurring logarithmic spirals.



Natural Spiral - 1
http://www.notam.uio.no/%7Eoyvindha/loga.html



Natural Spiral - 2
http://www.notam.uio.no/%7Eoyvindha/loga.html

Sometimes what at first seems unrelated, not similar, on closer inspection bears family resemblance. This shape was deeply familiar—a figure that the "Greek mathematicians called the *gnomon* and the type of growth based upon it. 'A gnomon is any figure which, when added to an original figure, leaves the resultant figure similar to the original" (Lawson 1982, 64).

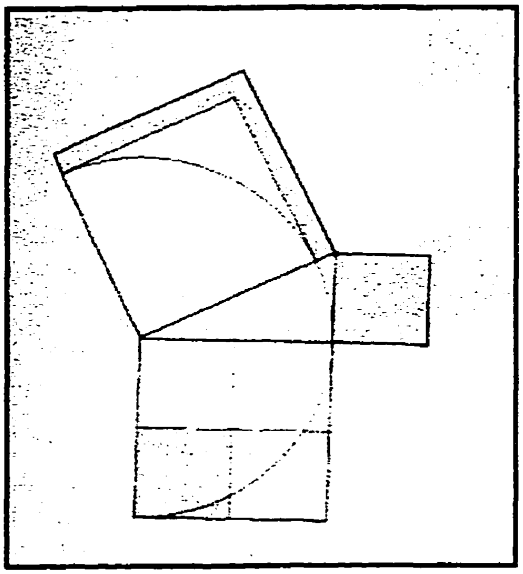


Figure 4.5 Relationship between gnomon and Pythagorean formula

"This method of figuring the gnomon shows its relationship to the Pythagorean formula $a^2 + b^2 = c^2$. Shown here is the gnomonic increase from the square surface area of 4 to the square of 5, where the gnomon of the larger square 5 is equal to 1/4 of the initial square of 4" (Lawson 1982, 65).

Anh Linh's quest to understand these dynamic spirals continued. When we saw her drawing of four tessellated, symmetrical patterns, we were awed. To produce such a stunning beautiful piece of work by hand certainly required contemplation and exactitude beyond what we could have ever hoped for. And for us, this would have been enough, but not for Anh Linh.

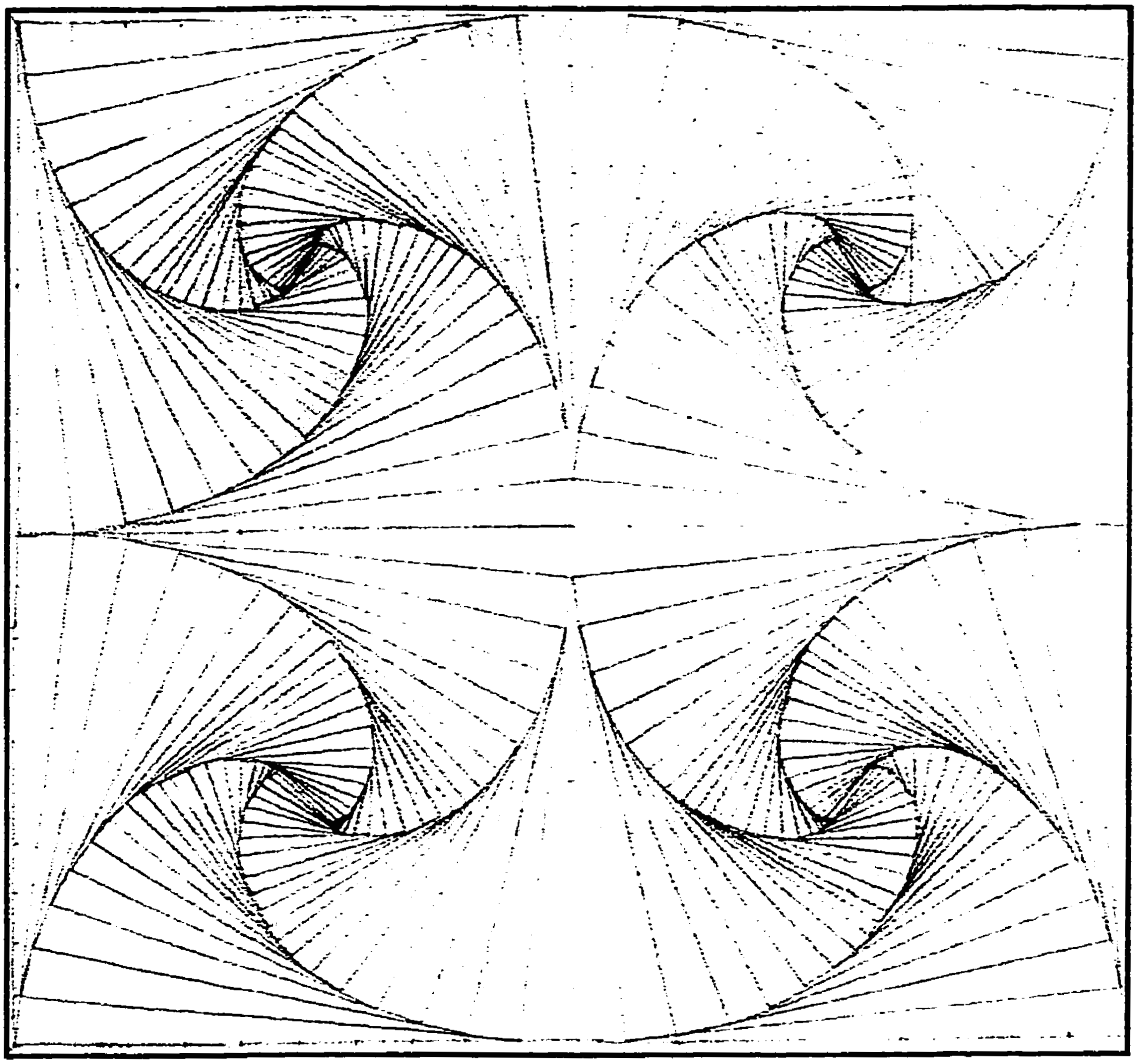


Figure 4.6 Anh Linh's rotated spirals

She continued to ask questions of this beautiful form and its symmetry, and each new question led us all deeper into this exploration. Spiral doodles started to appear all over the classroom—on notebooks, scraps of paper, borders on assignments. Some students started to create a variety of spirals using the Logo program we had in the classroom. They learned the power of variables. Creating the following set of commands:

```
TO POLYGON :SIDE :ANGLE :AMT
IF :SIDE>300 [STOP]
FD :SIDE
RT :ANGLE
POLYGON (:SIDE + :AMT) :ANGLE :AMT
END
```

produced this spiral:

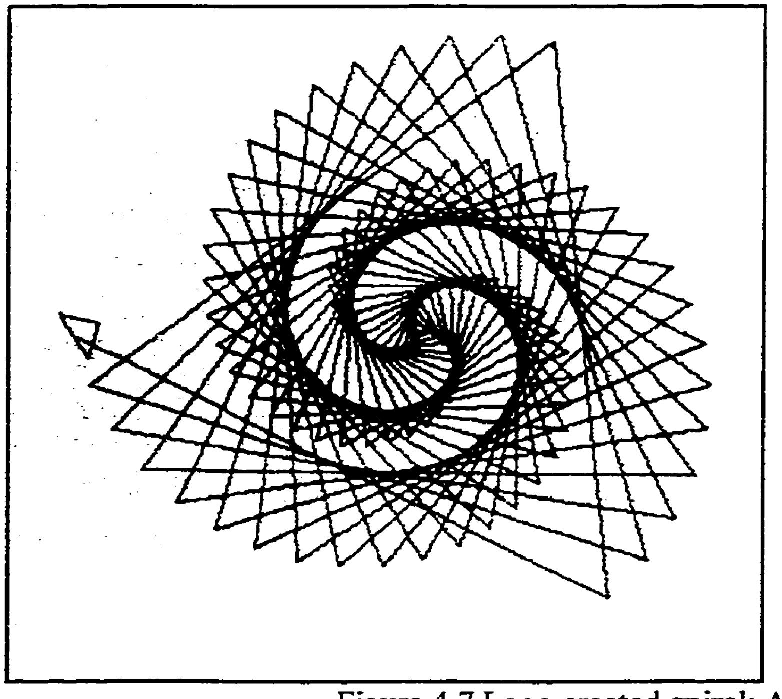
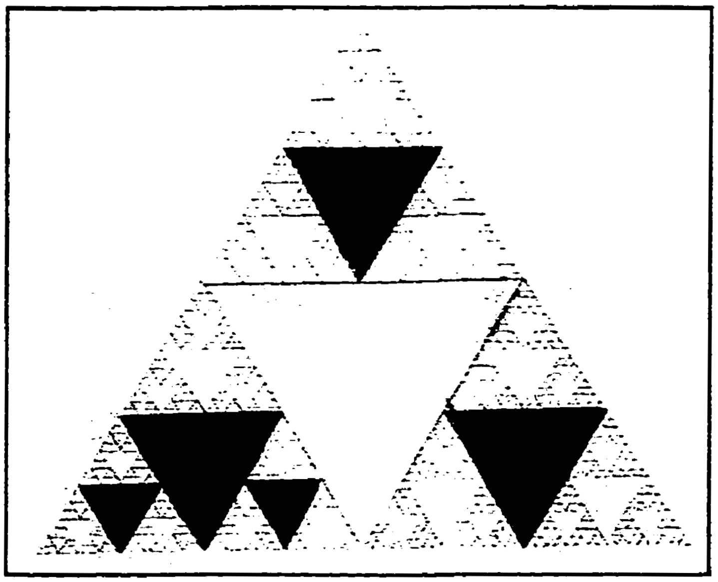
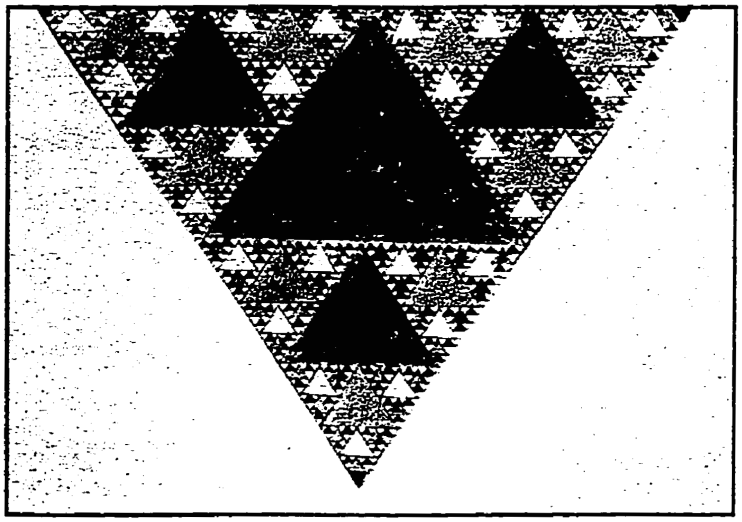


Figure 4.7 Logo created spiral: Arthur - age 13

Our work with Logo led us into the area of recursion and iteration—fractals. We saw a level of care, concern and questioning that we had never before witnessed in this group of students. Their fractals were exquisite. Each calculation and line was exact. The students understood that the slightest variation would dramatically affect the outcome.



Sierpinski Triangle: Tuyen - age 13



Sierpinski Triangle: Simon - age 13

We were experiencing what it meant to create mathematics. We were beginning to understand how creating new mathematics begins with asking questions. Sometimes a question that is easy to ask is impossible to answer. Sometimes a question that sounds difficult turns out to be something you already know, just dressed up to look different. Sometimes the question leads, not to an answer, but to another question. And for these questions, the answers are not in the back of the book. It's the posing of questions that kept calling us on to new possibilities, wondering what might be around the next corner, helping us to understand that mathematics is not finished, it's work in progress, it's a "**living, breathing, contested, human discipline** that has been handed to us" (Jardine, Friesen and Clifford 2000, 4).

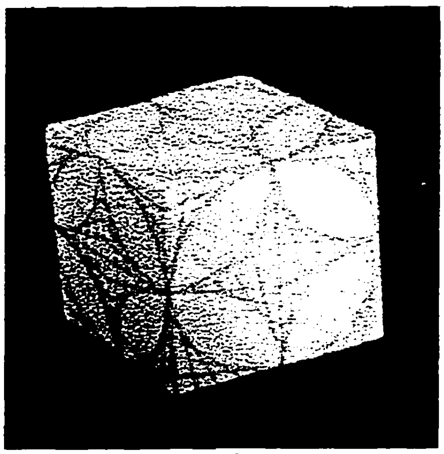
Working In 3D

Working on the two dimensional plane was intriguing and engaging, but what about 3D? Our questions were quite playful as we started, "I wonder what would happen if..." "I wonder if the symmetries that we had found on the 2D plane would hold as we tiled them onto the surfaces of a solid."

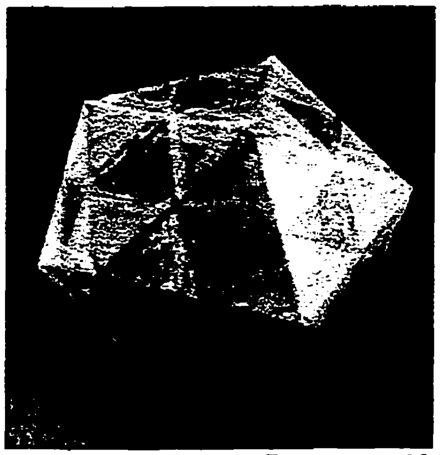
We decided to begin by tiling the surfaces of regular solids known as Platonic solids: tetrahedron, icosahedron, dodecahedron, octahedron, and cube with the various symmetrical designs that we had constructed. What better place to try out our emerging understandings than on such perfectly symmetrical solids. Each of our geometric models began as a flat design. We not only had to determine the shapes of the sides we needed to construct in order to create the transition from the two dimensional net to the three dimensional solid, we also needed to figure out how to place our designs on the two dimensional plane so they it would be perfectly symmetrical in three dimensions. The two dimensional pattern gives few clues as to what you will see and feel when it takes shape in three dimensions.

> The flat designs represent the possibility of infinite repetition but only a fragment of this infinity can be captured on a sheet of paper. On the surface of a three-dimensional object, infinite repetition of design can be realized with only a finite number of figures—the pattern on a solid has neither beginning nor end" (Schattschneider and Walker 1982, 16).

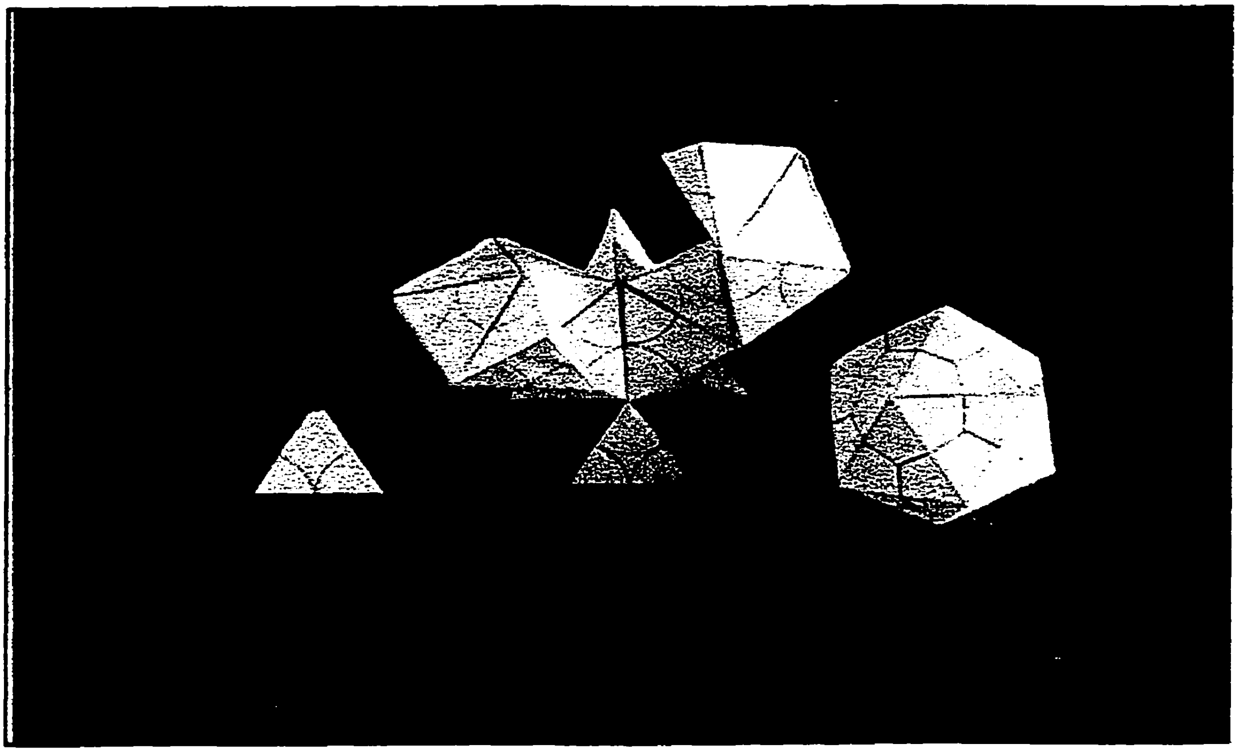
Creating the nets for each of the solids was fairly challenging, but determining how to draw the designs onto the surfaces so that when the edges came together the illusion of infinity was produced, was exigent. "Contrary to the impression given by most textbooks, the discovery of new forms and new ideas is rarely the product of the predictable evolution" (Schattschneider and Walker 1982, 8). After many attempts the student's solids began to take shape.



Cube: Tuyen - age 13



Icosahedron: Trung - age 13



Excrescence: Simon and Rajit - age 13

But it was Anh Linh who really pushed our thinking. It was Anh Linh and her love for the logarithmic spiral that pushed us into to the frontiers of mathematics itself.

Starting with the cube, Anh Linh drew the curves on each of the six faces. Upon assembling the cube she discovered that the designs did not flow. The symmetry was broken. How could symmetry be lost on this perfectly symmetrical solid?

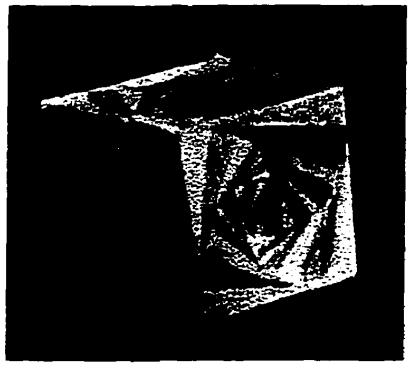


Figure 4.13 Anh Linh's Cube

Believing that she had made an error, she drew another cube. This time she transformed the spirals by reflecting them. However, upon putting the net together, she discovered that the problem was not solved. The pattern of the curve had broken the symmetry of the perfectly symmetrical cube—Greek symbol of earth. The act of reflection had not solved the problem. How could that be? What would work? "I want to find out why the symmetry breaks," Anh Linh wrote. *"I am going to see if I can make the symmetry work on any of the other solids. If I can, then maybe I will know why it doesn't work on the cube."*

Creating the curves on four equilateral triangles, Anh Linh started on her consuming quest to understand more about symmetry. She created the tetrahedron—the symbol of fire.

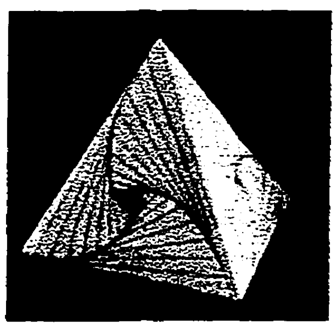


Figure 4.14 Anh Linh's Tetrahedron

It didn't work. The symmetry didn't hold. Anh Linh wrote: "In this shape I noticed that the pattern (curves of pursuit) didn't match on all the faces. The symmetry breaks along the edge. I also found out that you can use the curve of pursuit on any platonic solid. I didn't know that when I started."

Intrigued by her new discovery and undaunted by her disappointment, Anh Linh took on the challenge of the octahedron—the symbol of air.

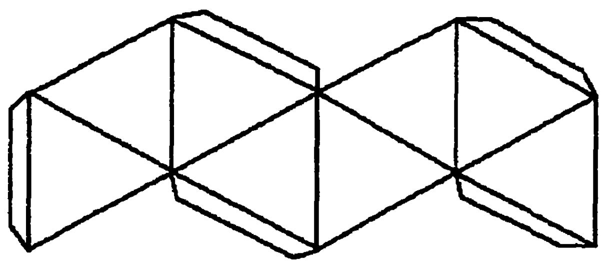


Figure 4.15 Octahedron net

Once again, working on the two dimensional equilateral triangles, Anh Linh meticulously measured and drew what we all now called "Anh Linh's curves."

Magic—*"It was like magical,"* Anh Linh later wrote. As Anh Linh folded the edges of these eight equilateral triangles together form and design came together, symmetry held and infinity emerged from the finite.

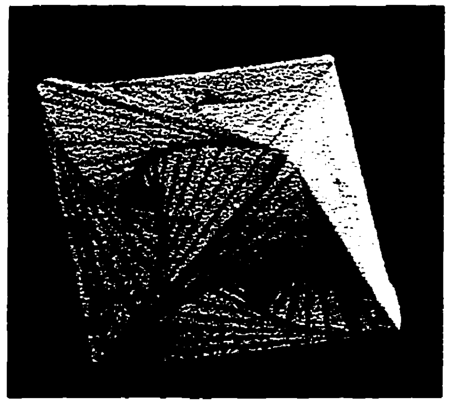


Figure 4.16 Anh Linh's Octahedron

What was it about the octahedron that was different from the tetrahedron or the cube? Everyone in the classroom was now involved in Anh Linh's problem—including Pat and me. Was there a solution? *"If there is, I don't know it,"* Anh Linh wrote. *"There might be an easy way to figure this out, but I don't know it. I will draw an icosahedron. It's faces are also triangles."*

For Anh Linh, as for all of us, we thought that the solution might be in the shape of the faces themselves. The tetrahedron did not work. But it was small—it only had four faces. Perhaps there was something in the number of faces. The octahedron had eight faces. Why should the symmetry hold with eight faces and not with four faces? They were both even numbers. But so was six for that matter—the number of faces on the cube. The solution had to be in the shape. Maybe there was something in the shape of the triangle that held the key to this problem. It had three vertices. The cube had four. Maybe there was something in that. Maybe there was something in the odd and the even. Like the ancient Pythagoras, we went looking for a connection between shapes and numbers.

Anh Linh continued drawing. Her next shape was the beautiful, perfectly symmetrical solid icosahedron, representing the Greek symbol for water. Upon each of its twenty identical equilateral triangle surfaces, Anh Linh drew the logarithmic spiral.

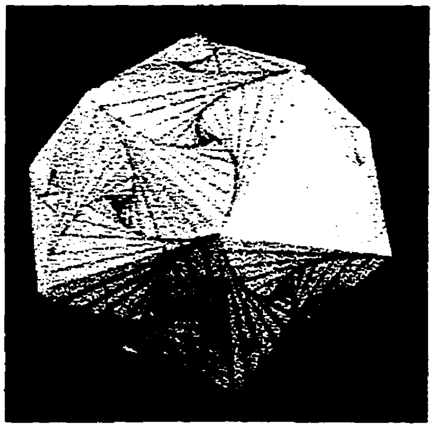


Figure 4.17 Anh Linh's Icosahedron

As she brought each of the five vertices of the solid together, she discovered, as did we all that symmetry was lost. But why? There had to be a solution.

It would be easy to conclude that we were just involved in solving the problem posed by Anh Linh's shapes. But that is not really what was happening—at least not all that was happening. Mathematics is not just a problem solving activity. We were involved in something far more fundamental—far more "basic" to mathematics. We were caught up in a generative act "the central activity being *making* new mathematics" (Wilensky, 1996). It was consuming for all of us. We noticed the students puzzling with the various shapes, trying to put them together in different ways, trying the dodecahedron, looking again at previously failed symmetries whenever they found breaks in their normal day-to-day studies. Pat and I puzzled along with them.

While driving home from school one day along the busy, accident-riddled Deerfoot freeway I had a flash of insight. I suddenly knew a direction to take that might hold the key to Anh Linh's shapes. I pulled over to the side of the road and frantically dug through my books for a piece of paper and a pen or a pencil. That's it. Flatten the shape. Step on it. Make a graph. Not the normal school type graph—a statistical graph, but a network, that type of graph.

We had been playing with networks earlier in the year. I had read the students the story *Superperson Saves the Monster* (Casey and Fellows 1993, 51). It is a zany story about three characters: Gertrude the goose, Monster and Superperson. Now suddenly on this freeway, driving home from school this story seemed to somehow to hold the key to Anh Linh's shapes. "Sometimes ideas are often born unexpectedly—from complexity, contradiction, and, more than anything else, perspective" (Negroponte 1996).

"Look at the vertices," Anh Linh's shapes seemed to call. As I flattened each of the shapes, about their vertex points, I noticed that the vertices and edges came together in a pattern of odds and evens. The tetrahedron—three, the cube—three, the octahedron—four, the icosahedron—five. There it was. I could hardly wait to get back to school the next day. I needed to let the class know that the Superperson story might hold the key. Upon revisiting the story, the students saw it too. "I don't think I need to make a decahedron," wrote Anh Linh. "It has an odd number of edges at the vertices."

I still had some reservations. How could we be sure that we were right? I packed up all of Anh Linh's shapes and took them to a mathematician at the university. I told him the story of Anh Linh's shapes and showed him how we had come to a solution. *"Does this make sense to you, Albert?"* I asked.

"Let's see." Albert drew a number of sketches on the chalkboard in his office. *"Yes, I believe you and your class are on to something,"* he said. *"The direction you have chosen seems to be a good one."*

"But are we right?" I wanted to know.

"I don't know," he said. *"But it looks like you are in an exciting and productive place. This is all new mathematics. There are people here who know more than I do about this area. You are creating mathematics."*

We began our exploration with Euclidean geometry but as we searched for a solution to the problem of determining symmetry we found ourselves in a very different space—a geometrical space that had more questions than answers. It seemed as though we had left the deeply familiar Euclidean geometry behind and were pushing at the very frontiers of mathematics itself—graph theory. It was an exploration that drew us in. "It [pulled] us into *its* question, *its* repose, *its* regard. Therefore, first is the question posed, not *by us* but *to us*" (Jardine, Friesen and Clifford 2000, 6)? We were consumed by the questions that kept presenting themselves, that kept calling to us from Anh Linh's shapes.

Where was Pythagoras? Did we leave him behind? Or are we in a place that required Pythagoras? Were we standing in the "long and twisted entrails of all the interdependencies that gave rise to [what was] being manifest, just here, just now" (Jardine, Friesen and Clifford 2000, 7)? Did Pythagoras, in his explorations and eccentricity know he was preparing a place which could give "birth" to this new mathematics? A place that could support Anh Linh's quest. A place large enough for all her classmates and her teachers. A place that required us all and all of us. "Mathematics *is*, in some sensible sense, all the actual human, bodily work which is required if it is to remain hale and healthy, if it is to continue as a living practice which we desire to pass on, in some form, to our children" (Jardine, Friesen and Clifford 2000, 12).

CHAPTER FIVE

To Teach Like This

"What we teach and how we teach is why we score the way we do" (Schmidt 2000) was the finding of the Third International Math and Science Study (TIMSS). This multiyear research and development project with its five components: curriculum analyses, achievement tests, questionnaires, case studies and a video study, assessed the current state of mathematics education. The findings for North America were disheartening.

In the past, many critics have attempted to place the blame for schoolchildren's poor performance on achievement tests on a variety of factors external to schooling. However, analysis of TIMSS data suggests that schooling itself is largely responsible. It points to the classroom identifying *it* as the source of the problem. The data from TIMSS questionnaires indicated that the majority of North American teachers were familiar with current NCTM (1989) reform recommendations. Many teachers claimed to be implementing these reforms in their classrooms. However, analysis of TIMSS videotape data revealed a big discrepancy between what teachers said they did and what they actually did.

"It may come as a surprise, but the video study of TIMSS marked the first time we have collected a fully recorded, representative sample of teaching" (Hiebert 1997).

Even though Canada did not participate in the videotape study the lesson scripts from the United States classrooms are deeply familiar. Analysis of the videotape study data revealed that the current NCTM (1989) reform initiatives to improve mathematics teaching have not impacted U.S. classrooms in any significant way. Clearly teachers have not understood the intent of the reform initiatives and consequently have not been able to implement the necessary changes into their practice.

In the face of overwhelming evidence about what is wrong with North American mathematics education, it is tempting to reach for a quick fix—simply transplant the superior teaching strategies and curricula of those countries that scored better than North America. But this would be short sighted. Teaching is a complex activity. It has deep cultural roots. To ignore this is to head down a path that would leave us "without a mechanism for steadily improving the way we teach" (Hiebert 1997). And wouldn't be long before we would find ourselves in this place again wondering, what should we do differently? What do we reform?

> Information on teaching is essential for understanding what happens when policies are implemented. There are many stories in this country about the failed process of educational reform: recommendations are proposed, no changes in outcomes are observed, complaints about the recommendations mount, committees meet and propose new recommendations, often reversing the thrust of the old ones. The entire process plays out without ever checking whether the initial

> recommendations were enacted. There is no way to know whether the old recommendations should be changed. (Hiebert 1997)

What is clearly evident is that mathematics education research will have to move into the classroom to learn how to reform mathematics education. Research initiatives are needed, initiatives that look deeply and thoughtfully into classroom practice to come up

> with new ideas--new ways of teaching, new curriculum materials, new ways of organizing schools. Generating new ideas depends on the creative acts of the human mind. Research, by itself, is no substitute. Of course, the research process can place people in a position to see things in a new way and imagine new possibilities, but it is the individual's interpretation, not the research evidence alone, that generates the new ideas. (Hiebert 1999, 8)

The work of my research, of this doctoral dissertation, is one such initiative. Understanding the "inherent creativity of interpretation, the pivotal role of language in human understanding, and the interplay of part and whole in the process of interpretation" (Smith 1994, 104), I looked deeply into the classroom, into my own classroom practice, to examine closely what it looks like when teachers and students engage in deeply mathematical ways in order to understand what needs to be reformed in mathematics education. What I learned has implications for teaching, learning and further research work.

New ways of teaching are necessary; ways of teaching that are fundamentally and philosophically different from the current scripts that underlie much of North American pedagogy. This has implications for both the education of new teachers and practicing teachers. Working together, teachers and researchers will have to roll up their sleeves, to create the new images of practice that are needed. This will require forming new types of working relationships between schools and universities, and between teachers and researchers.

> Teaching is unlikely to improve through researchers' developing innovations in one place and then prescribing them for everyone. Innovations can spread around the country, but only by trying them out and adjusting them again and again as they encounter different kinds of classrooms. (Stigler and Hiebert 1999, 134-135).

Just as Pat, David and I did, it will require working together in the classroom, learning how to focus on the teaching act, paying close attention to those things that protrude above the surface of the commonplaces of classroom life, those things that disrupt our ordinary taken-for-granted assumptions about what constitutes good teaching and learning—those things and those children, who we now work hard to eliminate or remediate.

And as my research revealed, one of the commonplaces that needs to be studied is the mathematics of mathematics education, the mathematics that has been aptly described as "a mile wide and an inch deep" (Schmidt, McKnight and Raizen 1997). A mathematics education that focuses on endless disembodied numbers and operations, memorized procedures and fragmented topics, has more devastating consequences than poor achievement results. Many students emerge from twelve to sixteen years of such mathematics with deep scars. Far too many of these students believe that they are incapable of understanding mathematics. Far too few people recognize that the shadows that they have come to know as mathematics bear little resemblance to mathematics itself.

> In Malcolm's words, "If you change the way mathematics is taught, you'll be surprised at who can learn mathematics. The idea of fitting the subject to the audience is real uncharted territory." What we do know is that memorizing formulas doesn't make anyone literate. (Steen 1997, xxv).

How Do You Learn To Teach Like This?

"How do you learn to teach like this?" is an old familiar question. It is also a difficult question to pose as it has too many answers. There is one way of answering this question that leads directly into the search for better teaching tips and techniques.

> [But] there are no formulas for good teaching, and the advice of experts has but marginal utility. If we want to grow in our practice, we have two primary places to go: to the inner ground from which good teaching comes and to the community of fellow teachers from whom we can learn more about ourselves and our craft. (Palmer 1998, 141).

Learning to teach like this requires:

1. seeking out and living in the presence of those people who think and live like this. They are the colleagues who know and understand that teaching and learning is a way of being. They are the colleagues who refuse the institutional barriers imposed by walls that place one teacher in one classroom. They are the colleagues who open their classrooms, themselves and their teaching to others knowing that imitating, repeating and practicing are essential to learning to teach like this. They are the ones who understand that ancestry and memory are not just part of subject disciplines but a part of the stream of the teaching profession. This is the practice of teaching that teachers need to enter. Learning to teach like this "depends on shared practice and honest dialogue among the people who do it" (Palmer 1998, 144).

2. refusing the company of those who will not support the journey of learning to teach like this, the ones who demean and are cynical. This is a very delicate professional issue. Wood (2000) discusses the difficulty teachers encounter and the ways in which innovative initiatives get derailed. With no way to

understand the desire to search for something different, something more from teaching than improved techniques, some colleagues will attempt to derail and intimidate those who choose this way of teaching. "The issue of collegial jealousy and how it can become a barrier to innovative teaching may be one of the most important challenges we educators face" (Wood 200, 7).

3. clearing a territory. Learning to teach like this requires that a space be opened. The clutter of activities that currently fills the days of far too many classrooms are intrusions into this way of teaching. Going deeply into a teaching and learning space requires an openness with yourself, your colleagues and your students. This is soul work. You can't hide it away. There can be no disguises. To begin to teach like this you need to ask yourself what you want to be answerable to and surround yourself with those who will help you. Finding colleagues who expect it, invite it and provide a generous place is essential to learning to teach like this.

4. taking heart that the world will support such work, even when schools will not. It is very easy to get discouraged when you start to work like this. "I have been forced to ask myself whether the pessimists are right. If they are, integrity would require me to stop peddling false hope about the renewal of teaching and learning" (Palmer 1998, 164). But as Pat and I learned, you will find colleagues in schools and in places other than school. Mathematicians who love their art will recognize this way of working. Poets and writers who love their art will

recognize this way of working. Artists who love their art will recognize this way of working. You will find fellow travelers both inside and outside of the institutions. They will be the ones who share your vision and your passion.

5. understanding this is life work. Learning to teach like this is not about how much you know. Rather, it is about knowing that there is much to know and that that your work, as a teacher and a learner, is about finding out. Not knowing is not a project, but a condition of knowing a lot. It involves the realization that you will never know enough, not because you haven't learned it well, but because there is so much to learn and know. It will mean that you will find yourself needing to learn more mathematics, more physics, more philosophy, more literature, more art…

To Teach Like this

I stepped from plank to plank

So slow and cautiously;

The stars about my head I felt,

And my feet the sea.

I knew not but the next

Would be my final inch,—

This gave me that precarious gait

Some call experience.

Emily Dickinson

For me this research experience involved learning how to teach and learn in new ways. Working together with mathematicians, Pat and I found ways to create deeply engaging mathematical explorations for the students we taught. And together, we found and created the ways into those explorations, the stories, which connected us with those who had prepared this landscape for us. Learning how to create mathematical explorations and documenting the work that we do is in its infancy. This dissertation is the beginning of such work.

Learning how to listen to the students to hear what they say, finding different ways to interpret the words we hear, is essential. Only in this way will researchers continue to examine and understand the ordinary commonplace scripts that underwrite current educational theorizing and philosophy. As researchers and teachers work together, they will have to keep a constant focus on student learning to ensure that they do not get so caught up in the massive task of the reforms themselves that they lose sight of the reason why they are doing them.

Finding new ways to help and support practicing teachers re-form and deepen their understanding of mathematics will require forging new working relationships between teachers and mathematicians and between mathematics departments and mathematics education. If mathematics is in fact a living discipline inspirited by ongoing questions, quarrels and conversations, then the pedagogy of mathematics is not an afterthought but a necessity. Teaching like this requires that we begin the task of re-forming the mathematics of mathematics education.

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