UNIVERSITY OF CALGARY

Refinements of Geodetic Boundary Value Problem Solutions

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE

DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF GEOMATICS ENGINEERING

CALGARY, ALBERTA

MAY, 2000

0-612-54776-0

ABSTRACT

This research investigates some problems on the refinements of the solutions of the geodetic boundary value problems. The main results of this research are as below:

Supplements to the Runge theorems are developed, which provide important guarantees for the approximate solutions of the gravity field, so that their guarantees are more sufficient.

A new ellipsoidal correction formula has been derived, which makes Stokes's formula error decrease from $O(e^2)$ to $O(e^4)$. Compared to other relative formulas, the new formula is very effective in evaluating the ellipsoidal correction from the known spherical geoidal heights. A new ellipsoidal correction formula is also given for the inverse Stokes/Hotine formulas.

The second geodetic boundary value problem (SGBVP) has been investigated, which will play an important role in the determination of high accuracy geoid models in the age of GPS. a generalized Hotine formula, the solution of the second spherical boundary value problem, and the ellipsoidal Hotine formula, an approximate solution of the second ellipsoidal boundary value problem, are obtained and applied to solve the SGBVP by the Helmert condensation reduction method, the analytical continuation method and the integral equation method.

Four models showing the local characters of the disturbing potential and other gravity parameters have been established. Three of them show the relationships among the disturbing density, the disturbing potential and the disturbing gravity. The fourth model gives the "multi-resolution" single-layer density representation of the disturbing potential. The important character of these models is that the kernel functions in these models decrease fast, which guarantees that the integrals in the models can be evaluated with high accuracy by using mainly the high-accuracy and high-resolution data in a local area, and stable solutions with high resolution can be obtained when inverting the integrals.

ACKNOWLEDGEMENTS

I wish to express my deepest gratitude to my supervisor, Dr. M.G. Sideris. His professional guidance, full support and constructive comments throughout the course of this research have been essential for the completion of this thesis.

My sincere thanks are extended to Dr. J.A.R. Blais, Dr. P. Wu and Dr. Y. Gao for teaching me some new knowledge.

I also wish to express my thanks to my former supervisors, Prof. Jiancheng Zhuo, Prof. H. T. Hsu, Prof Jiachun Meng and Prof. Zhenting Hou for their constant support of my studies.

Mrs. J. A. Helm and Mr. D. Provins are greatly thanked for helping me with English language.

Financial support for this research has been provided by NSERC grants of Dr. M.G. Sideris and University of Calgary scholarships. Some additional support was provided by the GEOIDE network of centres of excellence.

And last but not less important, my special thanks are given to my wife, my parents and my sister for their understanding and support during my studies.

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SYMBOLS	DEFINITION	PAGE
(X_P, Y_P, Z_P)	Rectangular coordinate of point P	7
$(r_P, \theta_P, \lambda_P)$	Spherical coordinate of a point P	7
l_{PQ}	Distance between point P and point Q	7
Ψ_{PQ}	Angle between the radius of P and Q	24
S_E	Topographic surface of the Earth	8
S_g	Geoid	9
S_t	Telluriod	11
S_e	Surface of the reference ellipsoid	10
S_B	Surface of the Bjerhammar sphere	30
S_M	Surface of the mean sphere	17
H (H^*)	Orthometric height (Normal height)	9,11
N (ζ)	Geoidal height (Height anomaly)	13
h	Geodetic height	10
a_e (b_e)	Semi major (minor) axis of S_e	10
R	Radius of S_M	14
e	First eccentricity of S_e	10
R_B	Radius of the Bjerhammar sphere	30
W	Gravity potential	7
U	Normal gravity potential	10
T	Disturbing potential	12
g	Gravity	8
γ	Normal gravity	10
δg	Gravity disturbance	12
Δg	Gravity anomaly	13
$S(\psi)$	Stokes's function	24
$H(\psi)$	Hotine's function	28
$i(S)$, $e(S)$ $I(S)$, $E(S)$	Sets of some points	47
$\aleph(O)$	A set of star-shaped surfaces	47
$\Im(S)$	A set of some surfaces	47
$H(S)$	A set of some harmonic functions	15
$H[S]$	A set of some harmonic functions	15
$\Re$	A set of some functions	48
$\\| \\|_S^N$	A norm in $\Re$	48
$F_{nP}(Q)$	A kernel function	161

0 Introduction

The main purposes of physical geodesy are the determination of the external gravity field and the geoid. Traditionally, these tasks are handled by solving the third geodetic boundary value problem in which the input data are gravity anomalies on the surface of the Earth. With the advancement of the gravimetric techniques, some new types of gravity data, such as the gravity disturbance data on the Earth's surface, airborne gravity data, satellite gravity data, etc., arise, and the accuracy and resolution of the data are improved constantly. So it becomes very important to utilize all these data for determining the high-resolution external gravity potential of the Earth. This research will discuss some aspects of refining the solutions of the geodetic boundary value problems (BVPs) to accommodate the developments of the gravimetric techniques.

In this chapter, we will briefly introduce the background of the research, the open problems to be treated here and the outline of this thesis.

0.1 Background and literature review

Since the days of G.G. Stokes (Stokes, 1849), Stokes's formula has been an important tool in the determination of the geoid. Rigorously, Stokes's formula is a solution of the third spherical BVP. The input data, which must be given on the geoid, are the gravity anomalies obtained from gravity and leveling observations. To apply Stokes's formula for the determination of the geoid, several schemes of transforming the disturbing potential, such as Helmert's condensation reduction and the analytical continuation method, have been employed (Moritz, 1980; Wang and Rapp, 1990; Sideris and Forsberg, 1991; Martinec and Vanièek, 1994; Vanièek and Martinec, 1994; Vanièek et al., 1999). To avoid the transformation of the disturbing potential, Molodensky et al.

(1962) and Brovar (1964) proposed respectively the integral equation methods to directly solve the third geodetic BVP. Similar to the analytical continuation method, the approximate solutions of Molodensky's and Brovar's methods are also expressed by Stokes's formula plus correction terms. In the past 35 years, further advances in the theory of the third geodetic BVP have been achieved. Some of these advances are the achievements of Molodensky et al. (1962), Moritz (1980), Cruz (1986), Sona (1995), Thông (1996), Yu and Cao (1996), Martinec and Grafarend (1997b), Martinec and Matyska (1997), Martinec (1998), Ritter (1998), Fei and Sideris (2000), etc., on the solution of the third ellipsoidal BVP. The resulting solutions of the third ellipsoidal BVP make the errors of the order of the Earth's flattening in the application of Stokes's formula decrease to the order of the square of the Earth's flattening.

The third geodetic BVP is based on gravity anomalies which can be obtained from gravity and leveling observations. A reason of employing the solution of the third geodetic BVP in the determination of the disturbing potential is that, in the past, gravity anomalies were the only disturbing gravity data that could be obtained accurately. M. Hotine at the end of the 1960's proposed a solution of the disturbing potential (Hotine's formula) which uses gravity disturbances as input data. The gravity disturbance is another kind of disturbing gravity, which can be evaluated from the gravity and the geodetic height of the observation point. Since the geodetic height could not be obtained directly by conventional survey techniques, Hotine and other authors (see, e.g., Sjoberg and Nord, 1992; Vanicek et al., 1991) had to employ an approximate geoidal height to obtain the gravity disturbance. Thanks to the advent of GPS techniques, the geodetic height can now be easily observed with very high accuracy. Consequently, the gravity disturbance can be easily obtained with a high accuracy. Therefore, for the second geodetic BVP, which is based on gravity disturbances, research parallel to what has been done for the third geodetic boundary value problem is very important in the era of GPS.

The third and second geodetic BVPs are all based on disturbing gravity data distributed over the globe. However, the gravity data over the oceans are hard to be obtained via conventional gravimetry. With the advent of satellite altimetry, the geoidal heights over the oceans can be measured with a high accuracy. From these geoidal height data, the disturbing potential outside the Earth's surface can be evaluated via the Poisson formula, the solution of the first spherical boundary value problem. Besides that, disturbing gravity can also be recovered from these geoidal height data. To invert these data into the disturbing gravity data, many methods have been proposed (Balmino et al., 1987; Zhang and Blais, 1995; Hwang and Parsons, 1995; Olgiati et al., 1995; Sandwell and Smith, 1996; Kim, 1996; Li and Sideris, 1997). One of the methods is to employ the inverse Hotine/Stokes formulas, which are directly derived from Poisson's formula. Similar to Stokes's formula, the Poisson formula and the inverse Hotine/Stokes formulas are all spherical approximation formulas. The application of these formulas will cause an error of the order of the Earth's flattening. To decrease the effect of the Earth's flattening on these formulas, Martinec and Grafarend (1997a) gave a solution of the first ellipsoidal boundary value problem while Wang (1999) and Sideris et al. (1999) proposed to add an ellipsoidal correction term to the spherical disturbing gravity recovered from altimetry data via the inverse Hotine/Stokes formulas.

The three boundary value problems discussed above are the basic boundary value problems. They only deal with a single type of gravity data (gravity anomalies, gravity disturbances or geoidal heights). To deal with multi-type data at the same time, many other geodetic BVPs, such as Bjerhammar's problem (Bjerhammar, 1964; Bjerhammar and Svensson, 1983; Hsu and Zhu, 1984), the mixed BVPs (Sanso and Stock, 1985; Mainville, 1986; Yu and Wu, 1998), the overdetermined BVPs (Rummel, 1989) and the two-boundary-value problem (Ardalan, 1999; Grafarend et al., 1999), etc., have been proposed. Bjerhammar's problem deals with the determination of a disturbing potential harmonic outside a sphere, called Bjerhammar sphere, from gravity data on or outside the Earth's surface. This disturbing potential can be simply represented by Stokes's formula (Bjerhammar, 1964) or a single-layer potential formula (Hsu and Zhu, 1984). The model parameters (the fictional gravity anomalies or the fictional single-layer densities) in these representations of the disturbing potential are obtained by means of the inversion of the gravity data. A basic question in Bjerhammar's problem is whether the disturbing potential, which is harmonic outside the Earth, can be approximated by the function harmonic outside the Bjerhammar sphere. This question is perfectly answered by Runge's theorems in physical geodesy (Moritz, 1980): the Runge-Krarup theorem (Krarup, 1969; Krarup, 1975) and the Keldysh-Lavrentiev theorem (Bjerhammar, 1975). Besides Bjerhammar's problem, the analytical continuation method for the geodetic boundary value problems also needs the guarantee of the Runge theorems. However, the Runge theorems only guarantee the disturbing potential can be approximated by the solution of Bjerhammar's problem. The derivatives of the disturbing potential are not involved in the theorems. Therefore the guarantee provided by the Runge theorems is not sufficient for the theory mentioned above since the geodetic problems usually involve the first-order derivative of the disturbing potential. It is thus valuable to give supplements to the Runge theorems so that they involve the derivatives of the disturbing potential.

Compared to satellite gravity data, the ground gravity data have better accuracy and resolution. They depict in detail the character of the gravity field. However, dense ground gravity data are only available in some local areas such as Europe and North America. In other areas and especially on the oceans, the best gravity data are those obtained from satellite measurements, which are globally producing gravity data with higher and higher accuracy and resolution. To solve the incomplete global coverage of accurate gravity measurements in the determination of the geoidal heights, Stokes's formula is modified so that the results can be evaluated from the input data in a local area (Vanicek and Sjoberg, 1991; Sjoberg and Nord, 1992; Gilliland, 1994; Vanicek and Featherstone, 1998). The important character of the modified Stokes formulas is that their kernel functions decay faster than the original Stokes function. The relationship models of the quantities of the anomalous gravity field established by kernel functions decaying fast are called the local relationship models (see Paul, 1991; Fei and Sideris, 1999). Another significance of these local relationship models is that we can obtain stable solutions with high resolution when we invert the integrals in the models. This property is very important to determine the parameters of the anomalous gravity field with high resolution by means of inversion of high-resolution gravity data.

0.2 Outline of the thesis

In this thesis, we will discuss some refinements of the solutions of the geodetic boundary value problems. The following is the outline of our work:

Chapter 1 is an introduction of some basic knowledge of the Earth's gravity field theory, which includes the definitions of the quantities of the gravity field, basic problems of the gravity field theory and their solutions, and some open problems.

In chapter 2, we give supplements to Runge-Krarup's theorem and Keldysh-Lavrentiev's theorem so that these two theorems involve the derivatives of the disturbing potential.

Chapter 3 discusses the ellipsoidal correction to Stokes's formula. The discussion includes a theoretical part from which a new ellipsoidal correction formula is developed, and a numerical test of the new ellipsoidal correction formula.

Chapter 4 discusses the ellipsoidal corrections to the inverse Hotine/Stokes formulas.

In chapter 5, we propose several approximate methods for solving the second geodetic boundary value problems. The work includes the generalized Hotine formula, the ellipsoidal correction to Hotine's formula and three methods for considering the effect of topographic mass in the application of Hotine's formula.

Finally, in chapter 6, we investigate the local character of the anomalous gravity field. Four local relationship models are established. Three of which show the local relationships among the disturbing potential, disturbing gravity and disturbing density. The fourth model is a "multi-resolution" representation of the disturbing potential, which is a generalization of the single-layer potential solution of Bjerhammar's problem.

Chapter 7 lists the major conclusions of this research and recommendations for further work.

1 Theoretical Background and Open Problems

This preparatory chapter is intended mainly to introduce the basic background knowledge of this research on the Earth's gravity field. Sections 1 to 4 review the basic concepts of the Earth's gravity field, the problems of physical geodesy, the methods for determining the Earth's gravity field and Runge's theorems in physical geodesy. Section 5 introduces some open problems on the refinements needed for the determination of the Earth's gravity field.

Like in most publication in the geodetic literature, this thesis is restricted to what can be called "classical physical geodesy": both the figure of the Earth and its gravity field are considered independent of time.

1.1 Basic concepts of the Earth's gravity field

To simplify the mathematics, one decomposes the Earth's gravity field into the sum of the normal gravity field and the anomalous gravity field. This section reviews the basic properties of the Earth's normal gravity field and anomalous gravity field, and the coordinate systems related to them.

1.1.1 The Earth's gravity potential

First of all, we give the definition of the **fundamental Earth-fixed rectangular coordinate system XYZ:** the origin O_E is at the Earth's centre of mass (the **geocentre**); the Z-axis coincides with the mean axis of rotation and points to the north celestial pole; the X-axis lies in the mean Greenwich meridian plane and is normal to the Z-axis; the Y-axis is normal to the XZ-plane and is directed so that the XYZ system is right-handed. The **rectangular coordinates** and the **spherical coordinates** of a point P are denoted by (X_P, Y_P, Z_P) and $(r_P, \theta_P, \lambda_P)$, respectively.

A basic quantity that describes the Earth's gravity field is the **gravity potential W**, which is defined as follows

$$W_P = V_P + V_{cP}, \tag{1.1.1}$$

where V_P is the **gravitational potential** defined by

$$V_P = G \int_{\tau_E} \frac{\rho(Q)}{l_{PQ}} dQ \tag{1.1.2}$$

where τ_E is the Earth's body, l_{PQ} is the distance between the computation point P and the moving point Q, $\rho(Q)$ is the **mass density** of the Earth at Q, G is the **Newtonian gravitational constant**

$$G=6.672\times10^{-11} m^3 s^{-2} kg^{-1},$$

and V_{cP} is the **potential of the centrifugal force** given by

$$V_{cP} = \frac{1}{2}\omega^2 (X_P^2 + Y_P^2) \tag{1.1.3}$$

where ω is the **angular velocity** of the Earth's rotation.

The gravity potential W satisfies the following relations:

$$\Delta W = \begin{cases} 2\omega^2 & \text{outside } S_E \\ -4\pi G\rho + 2\omega^2 & \text{inside } S_E \end{cases}, \tag{1.1.4}$$

where S_E is the **topographic surface,** the visible surface of the Earth, ω is the angular velocity of the Earth and Δ is the Laplacian operator.

The **gravity vector** $\underline{g}$ is the gradient of W:

$$\underline{g} = \text{grad } W = \begin{bmatrix} W_X \\ W_Y \\ W_Z \end{bmatrix}, \tag{1.1.5}$$

which consists of the gravitational force grad V and the centrifugal force grad V_{cP}.

The magnitude, or norm, of the gravity vector $\underline{g}$ is the **gravity** g:

$$g = \|\underline{g}\|; \tag{1.1.6}$$

the direction of $\underline{g}$, expressed by the unit vector

$$\underline{n} = g^{-1}\underline{g}, \tag{1.1.7}$$

is the **direction of the vertical,** or **plumb line.**

Both the gravity potential W and its first order derivative g are continuous in the space R^3 while the second order derivatives of W are discontinuous on the surface S_E.

The surfaces W=Const. are called **equipotential surfaces** or **level surfaces**. They are everywhere normal to the gravity vector. A particular one S_g of these surfaces,

$$W(X,Y,Z)=W_0=\text{Const.}$$

which approximately forms an average surface of the oceans, is distinguished by calling it the **geoid**.

The distance of a point to the geoid S_g along the plumb line is the **orthometric height** H.

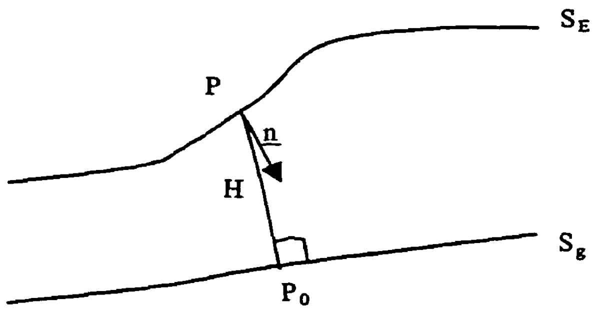

Figure 1.1 The orthometric height H

The **natural coordinates** of a point outside the geoid is the triplet (Φ, Λ, H), where Φ is the **astronomic latitude** defined as the angle between $\underline{g}$ and the equatorial plane and Λ the **astronomic longitude** defined as the angle between the local meridian plane and the mean Greenwich meridian plane.

1.1.2 Normal gravity field

The normal gravity field, a first approximation of the actual gravity field, is generated by an ellipsoid of revolution with its centre at the geocentre, called the **reference ellipsoid**. There are several reference ellipsoids. The most widely used reference ellipsoid is the **WGS-84 ellipsoid**, which is defined by the following parameters:

Major semi-axis a_e=6378137 m

Minor semi-axis b_e=6356752 m

Angular velocity ω=7292115×10^{-11} rad s^{-1}

Theoretical gravity Potential of the reference ellipsoid

U_0=62636860.8497 m^2 s^{-2}

Another important parameter is the first eccentricity e defined as

$$e = (1 - \frac{b_e^2}{a_e^2})^{\frac{1}{2}} \tag{1.1.8}$$

With the four quantities a_e, b_e, ω, U_0, the **normal gravity potential** U and the **normal gravity** γ outside (or on) the reference ellipsoid can be evaluated uniquely from closed formulas. For details, please see Heiskanen and Moritz (1967) and Guan and Ning (1981). U satisfies:

$$\Delta U = \begin{cases} 2\omega^2 & \text{outside } S_e \\ -4\pi G\rho_N + 2\omega^2 & \text{inside } S_e \end{cases} \tag{1.1.9}$$

where S_e is the **surface of the reference ellipsoid** and ρ_N is the **normal density**, which can not be determined uniquely by the four parameters.

Similar to the gravity potential W, the normal gravity potential U and its first order derivative γ are continuous in the space R^3 while the second order derivatives of U are discontinuous on the surface S_e.

The surfaces U=Const. are called **normal level surfaces** and the direction of the normal gravity vector $\underline{\gamma}$ is called the direction of the **normal vertical** or the **normal plumb line.**

The distance h of a point P to the reference ellipsoid is called the **geodetic height.**

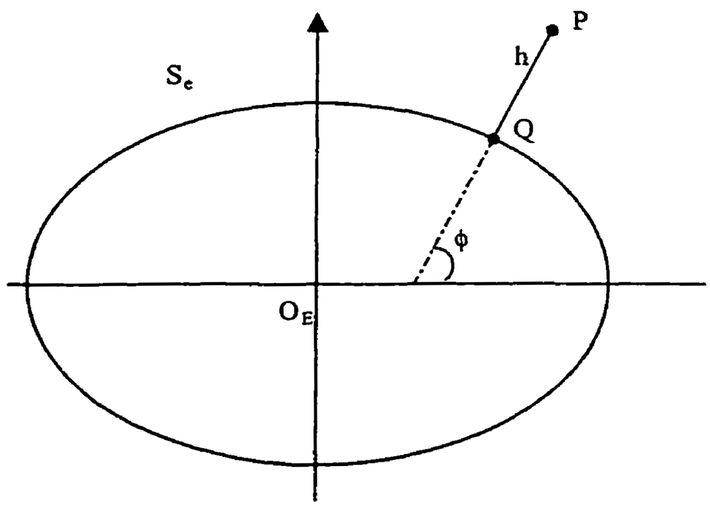

Figure 1.2 The reference ellipsoid S_e and the geodetic height h

The **geodetic coordinates** of a point is the triplet (ϕ, λ, h) where ϕ is the **geodetic latitude** defined as the angle between $\underline{\gamma}$ and the equator plane and λ is the **geodetic longitude,** which equals to Λ.

The **telluroid** S_t, the first approximation of the topographic surface, is defined as a surface, the points Q of which are in one to one correspondence with the points P of the topographic surface satisfying either

$$(\Phi, \Lambda, W)_P=(\phi, \lambda, U)_Q, \tag{1.1.10}$$

or

$$(\Phi, \Lambda, H)_P=(\phi, \lambda, h)_Q. \tag{1.1.11}$$

The distance of a point on S_t to S_e along the normal plumb is the **normal height** H^*.

1.1.3 Anomalous gravity field

The difference between the gravity potential W and the normal gravity potential U

$$T=W-U, \tag{1.1.12}$$

is called the **disturbing potential**. It can be considered as being produced by a **disturbing density** $\delta\rho$ ($\equiv \rho - \rho_N$) as follows:

$$T_P = G\int_{\tau_E} \frac{\delta\rho(Q)}{l_{PQ}} dQ \tag{1.1.13}$$

It can be proved that T satisfies the following conditions

$$\begin{cases} \Delta T_P = 0 & \text{P is outside } S_E \\ T_P = O(1/r_P) & r_P \to \infty \\ \text{T and its first order derivative are continuous outside and on } S_E \end{cases} \tag{1.1.14}$$

where the first condition is called the **harmonic condition** of T, the second condition is called the **regularity condition** of T and the third condition is called **the continuation condition** of T.

The **deflection of the vertical** Θ is the angle between the directions of the vertical and the normal vertical, which is very small.

The **disturbing gravity**, the difference between gravity and normal gravity, has two different definitions: One is the **gravity disturbance** δg defined as

$$\delta g(P) = g_P - \gamma_P, \tag{1.1.15a}$$

The other is the **gravity anomaly** Δg defined as

$$\Delta g(P) = g_P - \gamma_Q , \tag{1.1.15b}$$

where P and Q satisfy

$$(\Phi,\ \Lambda,\ H)_P=(\phi,\ \lambda,\ h)_Q. \tag{1.1.16}$$

The above relation shows that if P is on the geoid, Q is on the reference ellipsoid and if P is on the topographic surface, Q is on the telluroid.

The difference between the geoid and the reference ellipsoid can be expressed by the **geoidal height** N, which is defined as the geodetic height of a point of the geoid.

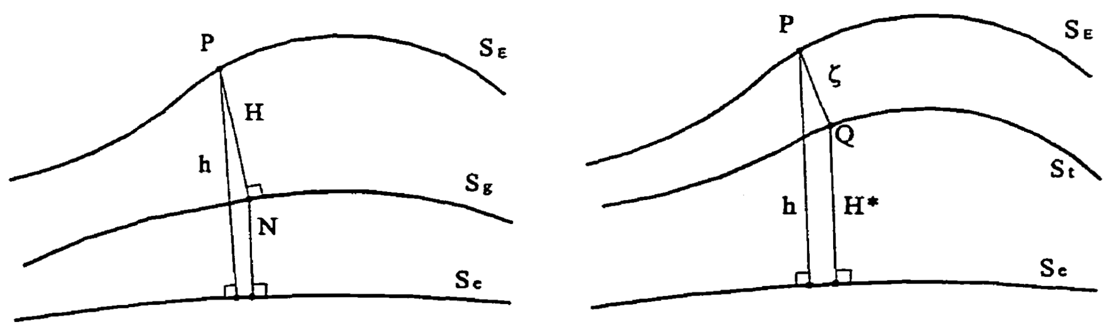

Figure 1.3 The geoidal height N and the height anomaly ζ

The difference between the topographic surface and the telluroid can be expressed by the **height anomaly** ζ, which is defined as the distance between a point P of the topographic surface and its corresponding point Q of the telluroid.

There exist the following approximate relations among the quantities of the anomalous gravity field (Heiskanen and Moritz, 1967; Moritz, 1980):

$$N = h - H \tag{1.1.17}$$

$$\zeta = h - H^{*} \tag{1.1.18}$$

$$\delta\zeta = N - \zeta = H^{*} - H = \frac{\bar{g} - \bar{\gamma}}{\bar{\gamma}} H \tag{1.1.19}$$

$$\frac{T_P}{\gamma_P} = \frac{T_P}{\gamma_Q} = \begin{cases} N & (\text{P is on } S_g \text{ and Q is on } S_e) \\ \zeta & (\text{P is on } S_E \text{ and Q is on } S_t) \end{cases} \tag{1.1.20}$$

$$\frac{\partial T}{\partial h} = -\delta g \tag{1.1.21}$$

$$\frac{\partial T}{\partial h} - \frac{1}{\gamma}\frac{\partial \gamma}{\partial h} T = -\Delta g \tag{1.1.22}$$

where $\frac{\partial}{\partial h}$ means the derivative along the normal plumb line, $\bar{g}$ is the mean value of g along the plumb line, and $\bar{\gamma}$ is the mean value of γ along the normal plumb line. Equation (1.1.20) is called the **Bruns formula** and equation (1.1.22) is called the **fundamental equation of physical geodesy.**

A spherical approximation of equation (1.1.22) is given as

$$\frac{\partial T}{\partial r} + \frac{2}{R} T = -\Delta g \tag{1.1.23}$$

where $\frac{\partial}{\partial r}$ means the derivative along the radial vector, R is the **mean radius** of the Earth defined as

$$R = \sqrt[3]{a_e^2 b_e} \tag{1.1.24}$$

1.2 Basic problems of physical geodesy

The main aims of physical geodesy are to determine the exterior gravity field and the geoid. Since the normal gravity field can be directly evaluated from simple closed formulas, the problems are converted to the determination of the disturbing potential T and the geoidal height N or height anomaly ζ, which are relatively small. The input data used are the quantities of the gravity field measured on the surface of the Earth or/and on surfaces at airplane or satellite altitudes. Since N and ζ can be directly evaluated from T by means of the Bruns formula and T satisfies (1.1.14), the basic problem of physical geodesy can be expressed by the geodetic boundary value problems and, if needed, the analytical continuation of the data.

For simplifying the description of the problems, we give some definitions before continuing the discussion:

Definition 1.1 For a closed surface S in the space R^3, let H(S) be the set of functions f satisfying:

$$\begin{cases} \Delta f_P = 0 & P \text{ is outside } S \\ f_P \to O(1/r_P) & P \to \infty \end{cases} \tag{1.2.1}$$

where r_P is the geocentric radius of P.

Definition 1.2 Let H[S] be the set of functions which belong to H(S) and have their first derivatives continuous on and outside S.

Definition 1.3 For a fixed point O in the space R^3, let H(O) be the set of functions f satisfying:

$$\begin{cases} \Delta f_P = 0 & P \neq O \\ f_P \to O(1/r_P) & P \to \infty \end{cases} \tag{1.2.2}$$

Examples:

1. For a fixed O in R^3, function $f_P = 1/l_{OP}$ belongs to H(O);

2. For a closed smooth surface S in R^3, the function

$$f_P = \int_S \frac{1}{l_{PQ}} dQ \tag{1.2.3}$$

belongs to H[S] while the function

$$\bar{f}_P = \int_S \frac{1}{l_{PQ}} dQ + \int_S \frac{\partial^2}{\partial r_Q^2} \left(\frac{1}{l_{PQ}}\right) dQ \tag{1.2.4}$$

belongs to H(S) but not to H[S];

3. According to (1.1.14), the disturbing potential T belongs to H[S_E].

1.2.1 Geodetic boundary value problems

Geodetic boundary value problems deal with the determination of the gravity potential on and outside the Earth's surface from the ground gravity data. They can be defined mathematically as finding the disturbing potential T satisfying:

$$\begin{cases} T \in H[S] \\ BT_P = f_P \quad P \text{ is on } S \end{cases} \tag{1.2.5}$$

where the boundary surface S is the topographic surface S_E outside which the mass density is zero and on which the input data f_P are given and B, which corresponds to f_P, is a zero or first order derivative operator or their combination. After a proper adjustment for the disturbing potential T, S can be the telluroid S_t, the geoid S_g, the reference ellipsoid S_e or the mean sphere S_M, where the **mean sphere** S_M is a sphere centred at the geocentre and with radius R.

According to the differences of the input data, there are various kinds of geodetic boundary value problems. In this subsection, we will introduce some geodetic boundary value problems that will be further investigated in the following chapters.

- **The third geodetic boundary value problem**

In this problem, the input data are the gravity potential W (or the orthometric height H or the normal height H^*) and the gravity g on S_E, which can be obtained via gravimetry and leveling, the output data are the topographic surface S_E (or the geodetic heights h or the geoidal height N) and the external gravity potential. Correspondingly, in (1.2.5), f_P is the gravity anomaly data Δg on S_E and B is a combination of the first and zero order derivative operators. The regularity condition of the third geodetic BVP is below

$$T_P = \frac{c}{r_P} + O(\frac{1}{r_P^3}) \qquad P \to \infty \qquad \text{(c is a constant)} \tag{1.2.5a}$$

This condition is stronger than the regularity condition of the disturbing potential (see (1.1.14)). It can be satisfied when the centre of the reference ellipsoid coincides with the geocentre (Heiskanen and Moritz, 1967). Furthermore, if the mass of the reference ellipsoid equals to the mass of the Earth, the constant c in (1.2.5a) equals to zero (Heiskanen and Moritz, 1967). So the regularity condition becomes

$$T_P = O(\frac{1}{r_P^3}) \qquad P \to \infty \tag{1.2.5b}$$

In the following, we suppose that T in the third geodetic BVP satisfies the regularity condition (1.2.5b). The mathematical expression of the third geodetic BVP is as follows

$$\begin{cases} T \in H[S_E] & \\ T_P = O(\frac{1}{r_P^3}) & P \to \infty \\ \frac{\partial}{\partial h}T_P - \frac{1}{\gamma_P}\frac{\partial \gamma_P}{\partial h}T_P = -\Delta g(P) & P \text{ is on } S_E \end{cases} \tag{1.2.6}$$

where $\frac{\partial}{\partial h}$ means the derivative along the normal plumb line.

Problem (1.2.6) is called **Molodensky's problem**. Since the normal plumb line is not normal to S_E, Molodensky's problem is an oblique derivative problem. After transforming the disturbing potential T, the Molodensky's problem can be converted into the **Stokes problem**, a normal derivative problem in which the boundary surface is the geoid.

- **The second boundary value problem**

In this case, the input data are the topographic surface of the Earth S_E (the geodetic heights h) and the gravity g on S_E, which can be obtained via gravimetry and GPS measurements, the output data are the gravity potential W on S_E (or the orthometric height H, the normal height H^* or the geoidal height N) and the external gravity potential.

Correspondingly, in (1.2.5), f_P is the gravity disturbance data δg on S_E and B is the first order derivative operator. That is

$$\begin{cases} T \in H[S_E] \\ \dfrac{\partial}{\partial h} T_P = -\delta g(P) \qquad \text{P is on } S_E \end{cases} \tag{1.2.7}$$

When the boundary surface is the geoid, the second boundary value problem is called the **Hotine problem.**

- **The first boundary value problem**

In this case, the input data are the topographic surface S_E and the gravity potential W on S_E (or the geoidal height N), the output data are the external gravity potential and the gravity on S_E. Correspondingly, in (1.2.5), f_P is the disturbing potential data T_0 on S_E, which can be obtained by leveling and GPS measurement on land or satellite altimetry over the ocean, and B is the identity operator. That is

$$\begin{cases} T \in H[S_E] \\ T_P = T_0(P) \qquad \text{P is on } S_E \end{cases} \tag{1.2.8}$$

The above problem is also called **Dirichlet's problem.** From the solution of above problem, we can also obtain the gravity anomaly or gravity disturbance on S_E (thus the gravity on S_E) via the following formulas:

$$\Delta g(P) = -\frac{\partial}{\partial h} T_P + \frac{1}{\gamma_P}\frac{\partial \gamma_P}{\partial h} T_P \tag{1.2.8a}$$

$$\delta g(P) = -\frac{\partial}{\partial h} T_P \tag{1.2.8b}$$

The determination of the gravity anomaly/disturbance on the geoid from the disturbing potential data on the geoid is called the **inverse Stokes/Hotine problem**.

1.2.2 Analytical downward continuation problems

Geodetic boundary value problems deal with data measured on the Earth's surface. With the advent of satellite and airborne gravity techniques, it becomes more and more important to investigate the methods that use data at satellite and airborne altitudes to determine the external disturbing potential. Since the input data are distributed on surfaces above the ground, we can call this kind of problem the analytical downward continuation problem.

The mathematical definition of the analytical downward continuation problems is to find a function T satisfying:

$$\begin{cases} T \in H[S_E] \\ BT_P = f_P \qquad P \text{ is on } S_{data} \end{cases} \tag{1.2.9}$$

where S_E is the topographic surface of the Earth, S_{data} is the surface on which the input data are given, and B, which corresponds to f_P, is a zero, first, or second order derivative operator or their combination.

1.3 Runge's theorems in physical geodesy

From its definition, we know that the disturbing potential T is harmonic only outside the Earth. Since the topographic surface is very complicated, T is a very complex function. To simplify the representation of the disturbing potential, we consider functions which are harmonic outside a spherical surface that lies completely inside the Earth. Can T be approximated by these functions? The possibility of such an approximation is guaranteed by Runge's theorem.

In physical geodesy, Runge's theorem has two forms (Moritz, 1980): the Runge-Krarup theorem and the Keldysh-Lavrentiev theorem.

- **Runge-Krarup's theorem**

Any function ϕ, harmonic and regular outside the Earth's surface S_E, may be uniformly approximated by functions ψ, harmonic and regular outside an arbitrarily given sphere S_B inside the Earth, in the sense that for any given small number $\varepsilon>0$, the relation

$$|\phi-\psi|<\varepsilon \tag{1.3.1}$$

holds everywhere outside and on any closed surface S_0 completely surrounding the Earth's surface.

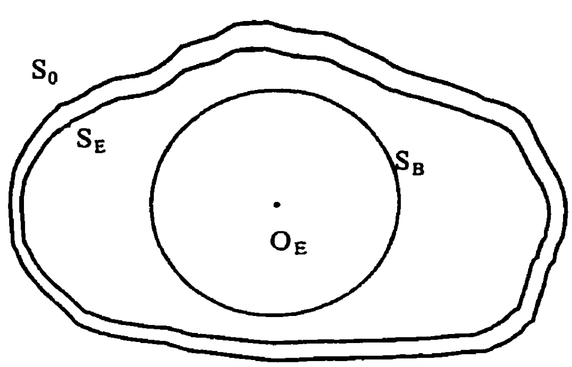

Figure 1.4 The relations of the surfaces S_0, S_E and S_B in Runge's theorem

- **Keldysh-Lavrentiev's theorem**

If the Earth's surface S_E is sufficiently regular (e.g. continuously differentiable), then any function ϕ, harmonic and regular outside S_E and continuous outside and on S_E, may be uniformly approximated by functions ψ, harmonic and regular outside an arbitrarily given sphere S_B inside the Earth, in the sense that for any given $\varepsilon>0$, the relation

$$|\phi-\psi|<\varepsilon \tag{1.3.2}$$

holds everywhere outside and on S_E.

1.4 Some classical approaches for representing the gravity field

As we mentioned above, the determination of the gravity field of the Earth leads to the determination of the disturbing potential function T outside and on the Earth's surface. In the past, many approaches have been employed to process the different data for the determination of the disturbing potential. According to Moritz (1980), there are essentially two possible approaches to the determination of the gravity field: the model approach and the operational approach. Moritz (1980) wrote: "In the model approach, one starts from a mathematical model or from a theory and then tries to fit this model to reality, for instance by determining the parameters of the model from observation." In other words, in the model approach, we should first establish, from a theory, a model representing the disturbing potential by a set of parameters, called the model parameters, then determine the model parameters from observation, and finally evaluate the disturbing potential by using these parameters.

In this section, we will introduce some classical models for representing the gravity field. According to the difference of the model parameters, we further divide the models into direct parameter model and indirect parameter model.

1.4.1 Direct parameter model

In these approaches, the disturbing potential T is expressed directly as an analytic function of the observed gravity. In other words, the model parameters of the Earth's gravity field are the data directly measured or simply calculated from the observations. Usually, these gravity field models are directly obtained from solving the geodetic boundary value problems.

- **Stokes's formula**

The famous Stokes formula is an approximate solution of Stokes's problem (Heiskanen and Moritz, 1967), in which the mass density outside the geoid has been set to zero, and the gravity anomaly Δg on the geoid has already been evaluated by means of gravity reductions such as the remove-restore technique.

Since the geoid is approximated by the reference ellipsoid, Stokes's problem can be expressed mathematically by the following third ellipsoidal boundary value problem:

$$\begin{cases} T \in H[S_e] & \\ T_P = O(\frac{1}{r_P^3}) & P \to \infty \\ \frac{\partial}{\partial h} T_P - \frac{1}{\gamma_P} \frac{\partial \gamma_P}{\partial h} T_P = -\Delta g(P) & P \text{ is on } S_e \end{cases} \tag{1.4.1}$$

By neglecting the flattening of the ellipsoid S_e, we can get the spherical approximation solution of (1.4.1), the **general Stokes formula,** as follows

$$T_P = \frac{R}{4\pi}\int_\sigma S(P,Q)\Delta g_Q d\sigma \tag{1.4.2}$$

where σ is the **unit sphere,** R is the mean radius of the Earth, and the kernel function S(P,Q), the **general Stokes function,** is defined as

$$S(P,Q) = \frac{2R}{l_{PQ}} + \frac{R}{r_P} - 3\frac{Rl_{PQ}}{r_P^2} - \frac{R^2}{r_P^2}\cos\psi_{PQ}(5 + 3\ln\frac{r_P - R\cos\psi_{PQ} + l_{PQ}}{2r_P}) \tag{1.4.3}$$

where r_P is the radius of the computation point P, l_{PQ} is the distance between P and the moving point Q on S_M, and ψ_{PQ} is the angle between the radius of P and Q.

Let r_P=R, we obtain the **Stokes formula,** which is the classical formula for computing the geoidal height from the gravity anomaly, as follows

$$T_P = \frac{R}{4\pi}\int_\sigma S(\psi_{PQ})\Delta g_Q d\sigma \tag{1.4.4a}$$

$$N_P = \frac{R}{4\pi\gamma}\int_\sigma S(\psi_{PQ})\Delta g_Q d\sigma \tag{1.4.4b}$$

where the **Stokes function** $S(\psi_{PQ})$ is given as:

$$S(\psi_{PQ}) = [\sin\frac{\psi_{PQ}}{2}]^{-1} + 1 - 6\sin\frac{\psi_{PQ}}{2} - \cos\psi_{PQ}[5 + 3\ln(\sin\frac{\psi_{PQ}}{2} + \sin^2\frac{\psi_{PQ}}{2})] \tag{1.4.5}$$

- **Brovar's and Moritz's solutions for Molodensky's problem**

In Molodensky's problem, the gravity potential W (or the normal height H^*) and the gravity vector $\underline{g}$ are given on the topographic surface. Since the topographic surface can be approximated by the telluroid by properly linearizing, Molodensky's problem can be expressed mathematically by the following third boundary value problem (Moritz, 1980):

$$\begin{cases} T \in H[S_t] & \\ T_P = O(\frac{1}{r_P^3}) & P \to \infty \\ \frac{\partial}{\partial h}T_P - \frac{1}{\gamma_P}\frac{\partial \gamma_P}{\partial h}T_P = -\Delta g(P) & P \text{ is on } S_t \end{cases} \tag{1.4.6}$$

where S_t is the telluroid.

There are many methods for solving Molodensky's problem to get the formula for computing the disturbing potential on the telluroid from the gravity anomalies on the telluroid. Moritz (1980) introduced three term-wise equivalents in planar approximation series solutions: Molodensky's solution, Brovar's solution and Moritz' solution. These three solutions can be generally expressed as

$$T = \sum_{n=0}^{\infty} T_n \tag{1.4.7}$$

where T_0 is given by

$$T_0 = \frac{R}{4\pi}\int_{\sigma} S(\psi)\Delta g d\sigma \tag{1.4.8}$$

Here, we will give respectively the terms T_n of the Brovar solution and the Moritz solution.

Brovar's solution is obtained by directly solving an integral equation derived from equation (1.4.6). Its terms T_n (n>0) can be expressed as follows:

$$T_1 = \frac{R}{4\pi}\int_\sigma \mu_1 S(\psi)d\sigma \tag{1.4.9a}$$

$$T_2 = \frac{R}{4\pi}\int_\sigma \mu_2 S(\psi)d\sigma - \frac{R^2}{4\pi}\int_\sigma \frac{(H^* - H_P^*)^2}{l_0^3}\mu_0 d\sigma \tag{1.4.9b}$$

......

with

$$\mu_0 = \Delta g \tag{1.4.10a}$$

$$\mu_1 = \frac{R^2}{2\pi}\int_\sigma \frac{H^* - H_P^*}{l_0^3}\mu_0 d\sigma \tag{1.4.10b}$$

$$\mu_2 = \frac{R^2}{2\pi}\int_\sigma \frac{H^* - H_P^*}{l_0^3}\mu_1 d\sigma + \mu_0 \tan^2\beta \tag{1.4.10c}$$

......

where l_0 is the distance of the projections onto S_M of the moving point and the computation point and β is the terrain inclination angle at the computation point.

Moritz's solution is obtained by analytically continuing the gravity anomalies onto a point level surface (a level surface through the computation point) and applying the Stokes formula for these gravity anomalies. Its terms T_n (n>0) are as follows:

$$T_1(P) = \frac{R}{4\pi}\int_\sigma g_1 S(\psi)d\sigma \tag{1.4.11a}$$

$$T_2(P) = \frac{R}{4\pi}\int_\sigma g_2 S(\psi) d\sigma \tag{1.4.11b}$$

$\ldots\ldots$

with

$$g_0 = \Delta g \tag{1.4.12a}$$

$$g_1 = -(H - H_P)L(g_0) \tag{1.4.12b}$$

$$g_2 = -(H - H_P)^2 L[L(g_0)] \tag{1.4.12c}$$

$\ldots\ldots$

where the operator L is the vertical derivative operator defined as

$$L(f) = -\frac{f_P}{R} + \frac{R^2}{2\pi}\int_\sigma \frac{f - f_P}{l_0^3} d\sigma \tag{1.4.12d}$$

- **Hotine's formula**

The Hotine formula is an approximate solution of Hotine's problem, in which the mass density outside the geoid has been set to zero and the gravity disturbance δg on the geoid has already been evaluated by means of gravity reduction. Neglecting the small difference between the geoid and the reference ellipsoid, Hotine's problem can be expressed mathematically by the following second ellipsoidal boundary value problem:

$$\begin{cases} T \in H[S_e] \\ \dfrac{\partial}{\partial h} T_P = -\delta g(P) \qquad \text{P is on } S_e \end{cases} \tag{1.4.13}$$

The **Hotine formula**, which computes the geoidal height from the gravity disturbance, is as follows (Hotine, 1969):

$$N_P = \frac{R}{4\pi\gamma} \int_\sigma H(\psi_{PQ}) \delta g_Q d\sigma \tag{1.4.14}$$

where the **Hotine function** $H(\psi_{PQ})$ is given as:

$$H(\psi_{PQ}) = [\sin \frac{\psi_{PQ}}{2}]^{-1} - \ln[1 + (\sin \frac{\psi_{PQ}}{2})^{-1}] \tag{1.4.15}$$

- **Inverse Stokes/Hotine formulas**

Stokes's (Hotine's) formula is employed to evaluate the geoidal height from gravity anomalies (gravity disturbances). However, gravity data are hard to measure directly in ocean areas. With the advent of the satellite altimetry technique, geoidal heights can be measured directly with a high accuracy in ocean areas. The following **inverse Stokes/Hotine formulas** (Heiskanen and Moritz, 1967; Zhang,1993), which are the approximate solutions of the inverse Stokes/Hotine problem, are employed to compute the gravity in ocean areas from the geoidal height derived from satellite altimetry:

$$\Delta g_P = \frac{\gamma N_P}{R} + \frac{\gamma}{4\pi R} \int_\sigma M(\psi_{PQ})(N_P - N_Q) d\sigma \tag{1.4.16}$$

$$\delta g_P = -\frac{\gamma N_P}{R} + \frac{\gamma}{4\pi R} \int_\sigma M(\psi_{PQ})(N_P - N_Q) d\sigma \tag{1.4.17}$$

where the **Molodensky function** $M(\psi_{PQ})$ is given by

$$M(\psi_{PQ}) = \frac{1}{4 \sin^3(\psi_{PQ}/2)} \tag{1.4.18}$$

1.4.2 Indirect parameter approaches

In the direct parameter approaches, the disturbing potential can be directly computed from the measurements. However, the measurements must be the gravity anomalies, the gravity disturbances or the geoidal heights measured in the oceans. However, these data are only a part of the gravity data that can be measured via the current measuring techniques. We now have gravity gradiometer data, and gravity data measured at satellite and airborne altitudes. The problem of processing these data is the analytical downward continuation problem. In order to solve this problem, indirect parameter models are proposed. In these models, the parameters are intermediate parameters other than the data directly measured or simply calculated from the measurements, and may have no direct physical meanings. Usually, to determine these model parameters from the observations, one has to solve an integral equation of first kind or a normal equation. The advantage of these approaches is that all kinds of gravity data can be employed to determine the model parameters. Usually, in order to simplify the model so that the integral equation or the normal equation is simple, the disturbing potential T is supposed to be a function harmonic outside a spherical surface that lies completely inside the Earth. The validity of this assumption is guaranteed by Runge's theorem.

- **Spherical harmonic representation**

In this model, the model parameters are the set of spherical harmonic coefficients $\{C_{mn}, S_{mn}\}$ (Moritz, 1980). The disturbing potential T is expressed as

$$T(r,\theta,\lambda) = GM\sum_{n=0}^{\infty}\frac{R^n}{r^{n+1}}\sum_{m=0}^{n}P_{nm}(\cos\theta)[C^*_{nm}\cos m\lambda + S^*_{nm}\sin m\lambda] \qquad (1.4.19)$$

with

$$C^{*}_{nm} = C_{nm} - \overline{C}_{nm}; \quad S^{*}_{nm} = S_{nm} - \overline{S}_{nm} \tag{1.4.20}$$

where $\{\overline{C}_{nm}, \overline{S}_{nm}\}$ are the coefficients used in the computation of the normal gravity field, $\{P_{nm}\}$ are the Legendre functions and M is the total mass of the Earth.

- **Bjerhammar's representation**

In this model (Bjerhammar, 1964), the model parameters are the 'fictitious' gravity anomalies Δg^{*} on the surface of Bjerhammar's sphere S_B that lies completely inside the Earth. The disturbing potential is expressed as:

$$T_P = \frac{R_B}{4\pi} \int_{\sigma} S(P,Q) \Delta g^{*}_{Q} d\sigma \tag{1.4.21}$$

where S(P,Q) is the general Stokes function, R_B is the radius of S_B.

- **'Fictitious' single layer density representation**

The 'fictitious' single layer density representation of the disturbing potential, proposed by Hsu and Zhu (1984), is equivalent to but simpler in form than Bjerhammar's representation. In this model, the model parameters are the 'fictitious' single layer densities ρ^{*} on the surface of Bjerhammar's sphere S_B. The disturbing potential is expressed as:

$$T_P = \frac{R_B}{4\pi} \int_{\sigma} \frac{1}{l_{PQ}} \rho^{*}_{Q} d\sigma \tag{1.4.22}$$

1.5 Some open problems

In this section, we will introduce some open problems which require theoretical refinements and will be further discussed in the following chapters.

1.5.1 Insufficiency of Runge's theorem in physical geodesy

In the Moritz's solution for Molodensky's problem (section 1.4.1) and the approaches mentioned in section 1.4.2, to employ Stokes's formula or simplify the representation of the disturbing potential T harmonic outside the Earth, a function $\overline{T}$, harmonic down to a point level surface or the Bjerhammar sphere completely embedded in the Earth, is employed as an approximation of T. The validity of the approximation is justified by Runge's theorem (the Runge-Krarup theorem or the Keldysh-Lavrentiev theorem). However, in the geodetic boundary value problems (1.2.5) and the downward continuation problems (1.2.9), T satisfies not only the harmonicity condition (1.2.1) but also the boundary condition

$$BT_P = f_P \qquad \text{(P is on S)} \tag{1.5.1}$$

where S is the Earth's surface or the surface at the satellite or airborne altitude. So there is a need to prove that BT is approximated simultaneously by $B\overline{T}$ on S. In other words, a necessary condition under which the approaches mentioned in section 1.4.2 are valid is that:

(I). For any given $\varepsilon>0$, there exists a function $\overline{T}$, harmonic and regular outside the Bjerhammar sphere, satisfying

$$|T-\overline{T}|<\varepsilon \text{ and } |BT-B\overline{T}|<\varepsilon \tag{1.5.2}$$

everywhere outside and on the Earth's surface.

By solving the equations (1.2.5) and (1.2.9), we can get a $\overline{T}$ satisfying

$$|BT-B\overline{T}|<\varepsilon \tag{1.5.3}$$

When B is a zero-order derivative, Runge's theorem guarantees the condition (I). However, the data usually used in physical geodesy are gravity data (and even gradiometer data). So B must contain first or second-order derivatives. In this case, it is hard to get (I) directly from Runge's theorem or from the proof given in Moritz (1980). Indeed, when the geodetic boundary value problems (1.2.5) or the downward continuation problems (1.2.9) are properly-posed, it can be proved that (I) holds by means of Runge's theorem. However, the properly-posed problem of (1.2.5) is very complex and the problems (1.2.9) are improperly-posed. So for Moritz's method mentioned in section 1.4.1 and the methods mentioned in section 1.4.2, the guarantee provided by Runge's theorem is not sufficient.

In chapter 2, we will give supplements to the Runge-Krarup theorem and Keldysh-Lavrentiev theorem, respectively, so that they contain (I), thus supply a more sufficient guarantee to the methods mentioned.

1.5.2 Ellipsoidal correction problems

Stokes's formula is a classical formula in the theory of gravity field representation. At present, it is still the basic tool for computing the geoid from gravity anomaly data. Rigorously, Stokes's formula is a spherical approximation formula which holds only on a spherical reference surface, i.e. the input data (gravity anomalies) must be given on the sphere. However, gravity anomalies can only be observed on the Earth's topographic surface. These anomalies can be reduced to the geoid or to a local level surface via orthometric (or normal) heights. For example, in a remove-restore technique, the gravity anomalies are reduced onto a level surface via terrain reduction, and in Moritz's solution (Moritz, 1980) the gravity anomalies are analytically continued to the geoid (or a point level surface) via a Taylor series expansion. The geoid and the local level surface can be approximated respectively by the reference ellipsoid and the local reference ellipsoid (on which the normal potential equals the gravity potential of the local level surface). Since the flattening of the ellipsoid is very small (about 0.003), in practical computation the ellipsoid is treated as a sphere so that Stokes's formula can be applied on it. The error caused from neglecting the flattening of the ellipsoid is about 0.003N. This magnitude, amounting up to several tens of centimeters, is quite considerable now. So, it becomes very important to evaluate the effect of the flattening on the Stokes formula. In other words, we should investigate more rigorously the third ellipsoidal geodetic boundary value problem (1.4.1).

Similarly, the Hotine formula and the inverse Stokes/Hotine formula are also the spherical approximation formulas. In the application of these formulas for geodetic purposes, ellipsoidal corrections are needed to get results with higher accuracy.

In chapters 3, 4 and 5, we will give detailed investigations on the ellipsoidal corrections to the Stokes formula, the inverse Stokes/Hotine formula and the Hotine formula respectively.

1.5.3 GPS leveling and the second geodetic boundary value problem

In this subsection, we will discuss the relation between GPS and the geodetic boundary value problems.

- **GPS leveling problem**

The impact of the Global Positioning System (GPS) on control network surveying can hardly be overstated. In a short span of time, differential GPS technology for horizontal geodetic surveys has been adopted to completely replace conventional surveying techniques. The superb length accuracy, coupled with greater efficiency and increased productivity in the field, has revolutionized our field operations.

However, one sector of geodetic surveying has remained much the same. That is, the vertical control surveying. GPS is a three-dimensional system, and certainly provides height information. GPS data, whether collected and processed in a point position mode or in a differential mode, yield three-dimensional positions. These positions are usually expressed as Cartesian coordinates referred to the centre of the Earth. By means of a mathematical transformation, positions expressed in Cartesian coordinates are converted into geodetic latitude, longitude, and geodetic height. These heights are in a different height system than orthometric heights historically obtained with geodetic leveling. Topographic maps, not to mention the innumerable digital and analogue data sets, are based on orthometric heights.

GPS leveling is using GPS and other geodetic techniques (other than leveling) to produce the orthometric height so that GPS can completely replace spirit leveling. According to (1.1.17), to relate GPS height h to orthometric height H requires a high-resolution geoidal height model of comparable accuracy. In other words, the key problem of GPS leveling is to determine a high resolution and high accuracy geoid model.

- **Deficiency of the third geodetic boundary value problems in GPS leveling**

Stokes's and Molodensky's formulas are the classical methods for determining high-resolution gravimetric geoid models. The Stokes theory and the Molodensky theory solve the third geodetic boundary value problem. They produce respectively geoidal heights and height anomalies from gravity anomaly data Δg given on the geoid and the telluroid.

The gravity anomaly Δg is defined as

$$\Delta g_P = g_P - \gamma_Q \tag{1.5.4}$$

where P is on the geoid (Stokes's model) or on the topographic surface (Molodensky's model) and Q is the point corresponding to P on the reference ellipsoid or on the telluroid. If P is on the geoid, g_P (thus Δg_P) is obtained from the gravity observation via a gravity reduction by employing orthometric height H

$$\Delta g_P = g_{observed} + \frac{\partial g}{\partial H} H - g_B - \gamma_Q \tag{1.5.5}$$

where g_B is the refined Bouguer correction of the gravity observation. If P is on the topographic surface, γ_Q (thus Δg_P) is computed from the normal gravity formula by employing the normal height H^* of P

$$\Delta g_P = g_P - \gamma_0 + \frac{\partial \gamma}{\partial h} H^* \tag{1.5.6}$$

where γ_0 is the related normal gravity on the reference ellipsoid.

Therefore, to get the gravity anomaly Δg_P, we should know the gravity at P and the orthometric height H (or normal height H^*) at P. This means that the gravity anomalies consist of gravity data and leveling data.

A reason for using gravity anomalies as the input data to determine the geoidal heights was that before the advent of GPS, the gravity anomalies were the only disturbing gravity data that could be obtained via conventional survey techniques. The geodetic height, a basic parameter of the Earth's figure, was very difficult to be observed directly. Actually, in the past, determining the geodetic heights of points on the physical surface of the Earth was an important goal of geodesy. On the other hand, the orthometric heights can be measured with conventional geodetic leveling. So we can obtain the gravity anomaly data. Then from the third geodetic boundary value problem, we can obtain the geoidal height or height anomaly. Finally, the geodetic height can be simply approximated by the sum of the leveling height H (or H^*) and the geoidal height N (or ζ) obtained from the third geodetic boundary value problems.

So in the third geodetic boundary value problem the input data are gravity data g and leveling data H or H^* while the output data are the geoidal heights N or the height anomalies ζ and the geodetic heights h ($h=N+H$ or $h=\zeta+H^*$). We can call this problem the **Gravity+Leveling problem**. Obviously, using the geoidal heights or height anomalies obtained from the third geodetic boundary value problem to determine the orthometric heights or normal heights will encounter a logical problem.

In the practical application of Stokes's theory or Molodensky's theory, the orthometric heights H and the normal heights H^* are replaced approximately by the heights obtained from digital topography models in order to avoid costly leveling observations. Obviously, the replacement will cause an error in the gravity anomalies. According to (1.5.5) and (1.5.6), 1 m difference between these two height data will cause about 0.3mgal error on the gravity anomaly (the effect on the Bouguer correction is not included). In this case we cannot obtain the gravity anomalies with accuracy better than 0.3 mgal even if we can now obtain the gravity observations with 0.01mgal accuracy. Furthermore, we have from Stokes's formula

$$|\delta_N| \le \frac{R}{4\pi\gamma}\int_\sigma |\delta_{\Delta g}||S(\psi)|d\sigma \le \frac{0.3R}{4\pi\gamma}\int_\sigma |S(\psi)|d\sigma \approx 4.2\text{m} \tag{1.5.7}$$

This means that the 1m error in the orthometric heights may cause theoretically about 4.2m system error in geoidal heights. So without high accuracy leveling measurements, it is hard to obtain the geoidal heights with accuracy comparable to the accuracy of the geodetic heights obtained via GPS.

From the discussion above, we can conclude that it is difficult to solve the GPS leveling problem via the third geodetic boundary value problems.

- **Second geodetic boundary value problem**

Now we discuss the following second geodetic boundary value problem.

$$\begin{cases} T \in H[S] \\ \dfrac{\partial}{\partial h} T_P = -\delta g_P \qquad \text{P is on S} \end{cases} \tag{1.5.8}$$

where the boundary surface S is the topographic surface S_E of the Earth or the reference ellipsoid S_e. The input data δg is the gravity disturbance defined as:

$$\delta g_P = g_P - \gamma_P \tag{1.5.9}$$

When P is on the reference ellipsoid, δg_P can be obtained by replacing H by the geodetic height h in the equation (1.5.5). When P is on the topographic surface, δg_P can be obtained by substituting H^* by h in the equation (1.5.6). Above all, δg can be obtained from the measurements of the gravity and the geodetic height on the topographic surface.

So, in the second geodetic boundary value problem, the geodetic heights h of the points on the topographic surface, which replace the position of H in the third geodetic boundary value problem, are needed. However, in the past it was hard to directly measure h. This means that we could not easily obtain δg in the past. This is the reason why the second geodetic boundary value problem has not been fully investigated.

With the advent of GPS, the positions (thus the geodetic heights) of points on the topographic surface can be obtained. With gravity measurements on the topographic surface, we can obtain the gravity disturbances δg, another kind of disturbing gravity. So it is very important to investigate the boundary value problem corresponding to the gravity disturbances, the second geodetic boundary value problem. We can also call this problem **Gravity+GPS problem**.

Besides its input data being easier to obtain than the input data of the third boundary value problem, the second geodetic boundary value problem has two other advantages over the third geodetic boundary value problem:

- The second geodetic boundary value problem is in theory a fixed boundary surface problem. The boundary surface is the known topographic surface (obtained from GPS measurements). However, the third geodetic boundary value problem is a free boundary surface problem. The boundary surfaces are the topographic surface or the geoid, which are unknown and to be determined in the problem. In practical applications, these surfaces are approximated respectively by the telluroid (obtained from leveling measurements) and the reference ellipsoid.

- The boundary condition in the second geodetic boundary value problem is simpler than the boundary condition in the third geodetic boundary value problem which is obtained approximately via linearization.

At present, among the geodetic height h, the orthometric height H (or the normal height H^*) and the geoidal height N (or the height anomaly ζ), N (or ζ) can not be measured directly and the measurement of H (or H^*) is more complicated than the measure of h. N (or ζ) is a bridge between H (or H^*) and h. The following table 1.1 compares three basic methods of determining N (or ζ):

Table 1.1 Comparison of the basic methods of evaluating N (or ζ)

Method	Input Data	Mathematical Model	Output Data
GPS+Leveling	h and H (or H^*)	N (or ζ)=h-H (or H^*)	N (or ζ)
Gravity+Leveling	g and H (or H^*)	Stokes's model or Molodensky's model	N (or ζ)
Gravity+GPS	g and h	Second geodetic boundary value problem	N (or ζ)

From the above table, we see that the Gravity+GPS method is the unique basic method determining the geoidal height without leveling. It is very suitable to be used to solve the GPS leveling problem.

- **Brief summary**

  - The GPS leveling problem is proposed for replacing the conventional spirit leveling. The key problem is to determine a high resolution and high accuracy geoid model without using dense leveling data.

  - The third geodetic boundary value problem is not suitable for solving the GPS leveling problem since it needs dense leveling data as input data.

  - GPS surveying provides the important input data for the second geodetic boundary value problem, which makes the investigation and application of this problem possible. On the other hand, the second geodetic boundary problem also provides the possibility for solving the GPS leveling problem.

  - The second geodetic boundary problem is better than the third geodetic boundary value problem in terms of the accuracy of models and input data. The detailed investigation theoretically and practically of the second boundary value problem is very necessary not only for solving the GPS leveling problem but for the determination of a high accuracy and high-resolution gravity field model.

1.5.4 Local character of the gravimetric solutions

With the advance of the gravimetric techniques, the amount of global gravity data obtained on the Earth's surface and at aircraft or satellite altitudes is increasing, and the accuracy and resolution of the data are improved constantly. Compared to satellite gravity data, the ground gravity data have better accuracy and resolution. They depict in detail the character of the gravity field. However, dense ground gravity data can only be obtained in some local areas such as Europe and North America. In the other areas especially on the oceans, the best gravity data are those obtained from satellite gravity measurements, which are globally producing gravity data with higher and higher accuracy and resolution. So it becomes very important to utilize all these data for determining the high-resolution gravity potential outside the Earth and for researching the distribution of the Earth's density.

In the previous sections, we have introduced some basic models showing the relationships between the disturbing potential and the disturbing gravity data. These traditional models essentially involve an integral formula of the form:

$$Y(P) = \int_{\sigma} K(P,Q)X(Q)dQ \ , \tag{1.5.10}$$

where X and Y respectively represent the input and output data; σ is either the Earth's surface, or the geoid, or the surface of the Bjerhammar sphere, etc.; and the kernel $K(P,Q)$ satisfies the relationship

$$\lim_{l_{PQ}\to\infty} K(P,Q)/\frac{\partial}{\partial r_P} l_{PQ}^{-1} \neq 0, \tag{1.5.11}$$

where l_{PQ} is the distance between P and Q, and r_P is the radius vector of P.

Since $\frac{\partial}{\partial r_P} l_{PQ}^{-1}$ vanishes slowly as $l_{PQ} \to \infty$, it follows from (1.5.11) that K(P,Q) also vanishes slowly as $l_{PQ} \to \infty$. Therefore, we will encounter two problems in the application of the relation (1.5.10):

1. when computing Y from X in (1.5.10), the integral must be evaluated in a larger area, thus making the collection of data difficult;

2. when computing X from Y in (1.5.10), besides the need for data in a larger area, the stability of the solution X declines as its resolution increases, thus restricting the resolution of the solution X (Fei, 1994; Keller, 1995).

These two problems show that the model (1.5.10) doesn't reflect the local character of the input or output data. Here, the local character can be understood in that the value of output data at some point can be determined mainly by the values of the input data in a neighborhood of the point. From the following two examples, we can get further understanding about the two problems.

- In the Stokes formula, the basic formula for determining the exterior gravity field and the geoid from gravity anomalies, Y and X represent the disturbing potential T and the gravity anomaly Δg respectively, σ is the geoid, and K(P,Q) is the Stokes function $S(\psi)$. Since $S(\psi)$ decreases slowly when ψ increases, the gravity data at the points far away from the computation point still have very important effects on the evaluation. This means that we need a global gravity data set to compute the disturbing potential at a single point. However, the incomplete global coverage of the high accuracy and high-resolution ground gravity data precludes an exact evaluation of the disturbing potential using Stokes's formula. Instead, many modified formulas are used in practice, where only gravity data in the area around the computation point are needed in the integration. These modifications to Stokes's formula are also attractive due to the increase in computational efficiency that is offered by working with a smaller integration area. One of the modified Stokes formulas is the generalized Stokes scheme for geoid computation proposed by Vaníèek and Sjöberg (1991). In this scheme, the low-frequency information of the gravity field, which can be computed from a global geopotential model, is considered in the computation of the normal gravity field model so that the disturbing gravity field (T^M and Δg^M) does not contain the low frequency information. The modified Stokes formula is written as

$$T^M = \frac{R}{4\pi}\int_\sigma S^M(\psi)\Delta g^M d\sigma \tag{1.5.12}$$

where $S^M(\psi)$, the modified Stokes kernel function, has the series expansion

$$S^M(\psi) = S(\psi) - \sum_{n=2}^{M} \frac{2n+1}{n-1} P_n(\cos\psi) = \sum_{n=M+1}^{\infty} \frac{2n+1}{n-1} P_n(\cos\psi) \tag{1.5.13}$$

From table 1.2 showing the behaviours of kernel functions S and S^M, we see that S^M decreases faster than S when ψ increases, which guarantees the integration (1.5.12) can be done by using the gravity anomalies in a local area surrounding the computation point. It can be said that the modified Stokes formula (1.5.12) shows more local relationship between the disturbing potential and the gravity anomalies than Stokes's formula (1.4.4a).

Table 1.2 Behaviour of the Stokes function and the modified Stokes function (M=360)

ψ^0	$\|S^M(\psi^0)/S^M(0.5)\|$	$\|S(\psi^0)/S(0.5)\|$	ψ^0	$\|S^M(\psi^0)/S^M(0.5)\|$	$\|S(\psi^0)/S(0.5)\|$
0.5	1	1	**5.5**	2.3229784E-02	0.1056424206
1.0	0.3436138195	0.5166225365	**6.0**	2.0149055E-02	9.7206253E-02
1.5	0.1824867893	0.3532196332	**6.5**	1.7660911E-02	8.9963646E-02
2.0	0.1162550062	0.2703797466	**7.0**	1.5618097E-02	8.3664776E-02
2.5	8.1841479E-02	0.2199814108	**7.5**	0.0139171792	7.8125693E-02
3.0	6.1363823E-02	0.1859101097	**8.0**	0.0124836118	7.3207879E-02
3.5	0.0480484992	0.1612288120	**8.5**	1.1262468E-02	6.8804986E-02
4.0	3.8830902E-02	0.1424538348	**9.0**	1.0212487E-02	0.0648339704
4.5	3.2145647E-02	0.1276414551	**9.5**	9.3021389E-03	6.1229013E-02
5.0	0.0271191438	0.1156204690	**10**	8.5069642E-03	5.7937256E-02

- Now we discuss the problem of determining the model parameters of the 'fictitious' single layer density representation of the disturbing potential: determining ρ^* from the known gravity disturbances on or above the physical surface of the Earth. According to (1.4.22), the relation between the two kinds of data is as follows:

$$\delta g_P = \frac{G}{4\pi}\int_{S_B} \rho^*(Q)\frac{\partial}{\partial r_P}(-\frac{1}{l_{PQ}})dS \qquad \text{(P is on or above } S_E\text{)} \qquad (1.5.14)$$

Since S_B is completely inside S_E, the equation (1.5.14) is an integral equation of the first kind, which is improperly posed. In practice, we divide S_B into many blocks according to the resolution of the gravity data and suppose that ρ^* is constant in each block. Then the unknown function ρ^* becomes a vector $\{\rho^*_i\}$ with finite dimension and can be estimated by the least squares technique from the known gravity data. In more detail, we divide S_B into a set of grid elements $\{S_i\}$ by meridians and parallels. The size of the grid elements is chosen according to the resolution of the data. Thus equation (1.5.14) becomes

$$\sum_i \rho^*_i A_{iP} = \delta g_P \qquad (1.5.15)$$

with

$$A_{iP} = -\frac{\partial}{\partial r_P}(\frac{1}{l_{PQ_i}})\Delta S_i \qquad (1.5.16)$$

where Q_i is a point in S_i and ΔS_i is the area of S_i. From (1.5.15), we can estimate $\{\rho^*_i\}$ from $\{\delta g_P\}$ by the least squares technique. When the size of the grid elements $\{S_i\}$ is small, however, A_{iP} decreases slowly since $\frac{\partial}{\partial r_P} l_{PQ}^{-1}$ decreases slowly when the distance between P and S_i increases. Thus, the coefficient matrix of the normal equations is strongly correlated. This makes the least square solution unstable (see Fei, 1994; Keller, 1995). So from the 'fictitious' single layer density model (1.4.22), we can not obtain a stable high-resolution solution of the model parameters even if we have high-resolution gravity data on the topographic surface.

From the discussion above, we see that the establishment of the local relationships between the input data and the output data via fast decreasing kernel functions are very important for solving the incomplete global coverage of accurate gravity measurements and obtaining a stable high resolution solution of the gravity field model. In detail, the local relationship has two significances: one is that we can evaluate with a high accuracy the integrals in the models by using mainly high-accuracy and high-resolution data in a local area; the other is that we can get a stable solution with high resolution when we invert the integrals in the models because of the rapid decrease of the kernel function of the integrals. In chapter 5 and 6, we will give more detailed investigations on the local relationships.

2 Supplements to Runge's Theorems in Physical Geodesy

In physical geodesy, the approximations of the Earth's gravity field by means of Bjerhammar's representations and spherical harmonics series etc. all need the guarantee of Runge's theorem. Runge's theorem has two forms in physical geodesy: the Runge-Krarup theorem and Keldysh-Lavrentiev theorem. They guarantee that the disturbing potential $\overline{T}$ given in the approximating theories of physical geodesy can approximate arbitrarily well the actual disturbing potential T. They do not, however, deal with the problem of the radial derivative of T and $\overline{T}$. The data usually used are the first-order and second-order radial derivative of T, i.e. gravity data and gradiometer data. We need to know whether the derivatives of $\overline{T}$ can also approximate the derivatives of T simultaneously. In this sense, the guarantee of Runge's theorem for the approximation theories concerned is not sufficient. In this chapter, we will prove that for a given non-negative integer n and an arbitrary small positive constant ε, there exist $\overline{T}$ so that the k-order $(0 \le k \le n)$ radial derivatives of the difference between $\overline{T}$ and T are less than ε everywhere on and outside the Earth's surface (in this case, n=1) or a smooth surface S completely surrounding S_E. The Runge-Krarup theorem and Keldysh-Lavrentiev theorem (where k=0) are obviously special cases of the generalized theorems.

2.1 Preparations

In this subsection, we give some notations and some lemmas for the proofs of our final conclusions.

Definition 2.1 For a closed surface S in space R^3, let

$i(S) \equiv \{P\text{: P is inside S}\}$, $\quad I(S) \equiv \{P\text{: P is inside or on S}\}$,

$e(S) \equiv \{P\text{: P is outside S}\}$, $\quad E(S) \equiv \{P\text{: P is outside or on S}\}$.

Definition 2.2 For a given point O in space R^3, a closed surface S is called a **star-shaped surface** about O if O is inside S and there is only one intersection Q between S and the line OQ for every Q on S. The set of all the star-shaped surfaces about O is denoted by $\aleph(O)$. i.e.

$\aleph(O) \equiv \{S : S \text{ is a star surface about } O\}$.

Example of star-shaped surface: In the following figure, the surface S is a star-shaped surface about O_2, but is not a star-shaped surface about O_1.

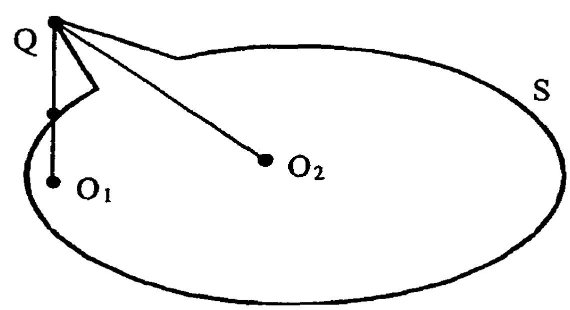

Figure 2.1 An example of star-shaped surface

Definition 2.3 For a given closed surface S, $\mathfrak{I}(S)$ is defined as the set of all the closed smooth surfaces completely surrounding S, i.e.

$\mathfrak{I}(S) \equiv \{S'\text{: S' is a closed smooth surface completely surrounding S}\}$.

Definition 2.4 Suppose that S is a closed surface, N is a non-negative integer, $\mathfrak{R}$ is the space formed by all n-order ($n \le N$) differentiable functions in E(S). Then the **norm** $\| \, \|_S^N$ in the space $\mathfrak{R}$ is defined as follows:

$$\|f\|_S^N \triangleq \max_{n=0,N} \sup_{\substack{P\in E(S)\\ v_P\in V_P}} \left| \frac{\partial^n}{\partial v_P^n} f(P) \right| \qquad (f \in \mathfrak{R}) \tag{2.1.1}$$

where V_P is the set of all the directions which pass through P.

Lemma 2.1. Suppose that S is a closed surface, O is in i(S) and $S_0 \in \aleph(O) \cap \mathfrak{I}(S)$. For any given point Q_0 on S_0, take a point Q_1 on the line OQ_0 such that

$$l_{OQ_1} = l_{OQ_0} - d_0 / 2 \tag{2.1.2}$$

where

$$d_0 = \inf\{l_{PQ} : P \in S_0, Q \in S\} \tag{2.1.3}$$

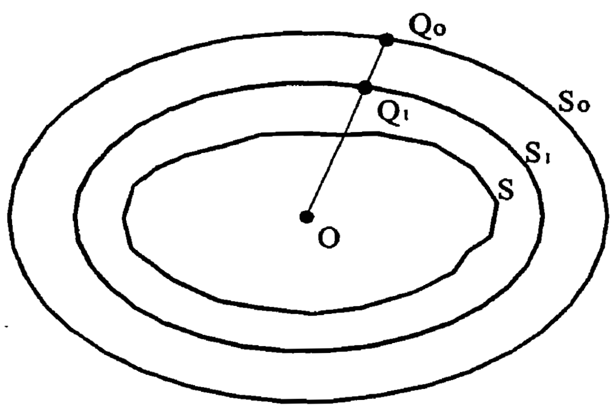

Figure 2.2 The relation among S, S_1 and S_0

Then the surface S_l composed by all Q_l satisfies (see figure 2.2):

(i) $S_l \in \aleph(O) \cap \Im(S)$,

(ii) $d \equiv \inf\{l_{PQ} : P \in E(S_0), Q \in I(S_l)\} > 0$,

(iii) $S_0 \in \Im(S_l)$.

Proof: (a). Since $S_0 \in \Im(S)$, we know from (2.1.2) and (2.1.3) that S_l is a closed smooth surface and $d_0 > 0$. From (2.1.3), we know that for any given point Q on or inside S and a point P on S_0, $l_{PQ} \geq d_0$. So from (2.1.2), we know that S_l completely surrounds S. Thus $S_l \in \Im(S)$. Now we prove that $S_l \in \aleph(O)$. That is to prove that for any given point P on S_l, only P is on line OP and S_l. If it is not true, then there must exist another point Q on line OP and S_l such that

$$l_{OQ} \neq l_{OP} \tag{2.1.4}$$

So from the definition of S_l, we know that there exist two points P' and Q' on S_0 and line OP' satisfying

$$l_{OQ'} = l_{OQ} + d_0/2 \ , \qquad l_{OP'} = l_{OP} + d_0/2$$

From (2.1.4), we see that P' and Q' are not the same point. This is contrary to $S_0 \in \aleph(O)$. So $S_l \in \aleph(O)$ and (i) holds.

(b). Since S_l and S_0 are closed surfaces, if d=0, then there must exist an intersection P of S_l and S_0. Consider P as the point on S_0; then from the definition of S_l, we know that there exists a point P_l on S_l and the line OP such that $P_l \neq P$. Since P is also on S_l, there exist two different points P_1 and P on line OP and S_1. This is contrary to $S_1 \in \aleph(O)$. So (ii) holds.

(c). (iii) can be obtained directly from (ii) and (2.1.2)#

Lemma 2.2. Suppose that $S_1 \in \aleph(O) \cap \Im(S)$ and there exist three real positive numbers a, b and c satisfying

$$0 < \sup_{Q \in S_1} \frac{l_{OQ} - b}{l_{OQ} + a} \le c \le \inf_{Q \in S_1} \frac{l_{OQ}}{l_{OQ} + a} < 1 \tag{2.1.5}$$

For every point Q_1 on S_1, take a set of points $\{Q_k, \overline{Q}_k\}_{k \ge 1}$ on the line OQ_1 so that

$$l_{OQ_k} = l_{OQ_1} c^{k-1} + (k-1)c^{k-1} a; \qquad l_{O\overline{Q}_k} = l_{OQ_k} c \tag{2.1.6}$$

For a fixed k, let S_k and $\overline{S}_k$ be the surfaces formed by all Q_k and $\overline{Q}_k$, respectively. Then

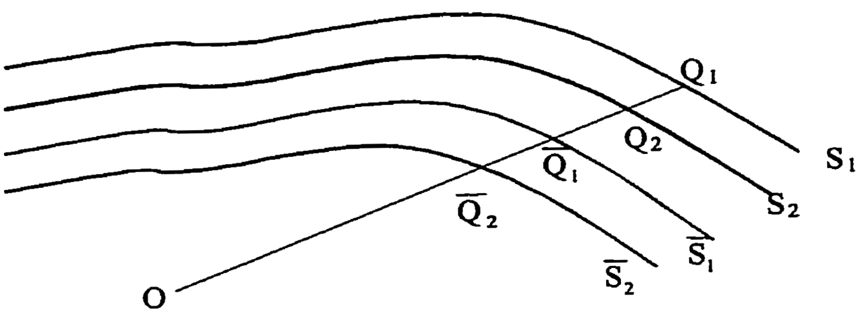

Figure 2.3 Relations among S_k and $\overline{S}_k$

(i). $\overline{Q}_k \in i(S_1)$ and $l_{Q_k\overline{Q}_k} < a + b \qquad (k \ge 1)$;

(ii). $\overline{S}_k$ is a closed surface containing O and

$$r_k \equiv \inf_{\overline{Q}_k \in \overline{S}_k} l_{O\overline{Q}_k} > 0; \tag{2.1.7}$$

(iii). $S_k \in \Im(\overline{S}_{k-1})$ where $\overline{S}_0 \equiv S$;

(iv). Let S_B be the spherical surface with O as its centre and r_B ($\equiv \frac{1}{2}\inf_{Q \in S_1} l_{OQ}$) as its radius. Then there exists an integer m such that $S_B \in \Im(\overline{S}_m)$.

Proof: (1). From (2.1.6) and the right hand side of (2.1.5), we obtain

$$\begin{aligned} l_{O\overline{Q}_k} &< l_{OQ_k} = l_{OQ_1}c^{k-1} + (k-1)c^{k-1}a = (l_{OQ_1} + a)c^{k-1} + (k-2)c^{k-1}a \\ &< l_{OQ_1}c^{k-2} + (k-2)c^{k-2}a = l_{OQ_{k-1}} < \cdots < l_{OQ_1} \end{aligned} \tag{2.1.8}$$

This means that Q_k and $\overline{Q}_k$ are on the line OQ_1. So from $S_1 \in \aleph(O)$, we obtain $\overline{Q}_k \in i(S_1)$. Furthermore, from the left side of (2.1.5), we have

$$1 - c \le \frac{a+b}{l_{OQ_1} + a} \tag{2.1.9}$$

It follows that

$$l_{Q_k\overline{Q}_k} = l_{OQ_k}(1-c) \le l_{OQ_1}\frac{a+b}{l_{OQ_1} + a} < a + b. \tag{2.1.10}$$

So (i) holds.

(2). Since O is an inner point of S, $\inf_{Q_1 \in S_1} l_{OQ_1} > 0$. It follows from (2.1.6) and the definition of $\overline{S}_k$ that (ii) holds.

(3). It is easy to see from the definition of S_k that S_k is a closed smooth surface containing O as an inner point. Now we prove that every $\overline{Q}_{k-1}$ is inside S_k. If this is not true, then there exists a $\overline{Q}_{k-1}$ which is on or outside S_k. Since O is inside S_k, there is an intersection Q of S_k and the line $O\overline{Q}_{k-1}$. That is

$$l_{OQ} \leq l_{O\overline{Q}_{k-1}} \tag{2.1.11}$$

However, from the definitions of S_k and $\overline{S}_{k-1}$, we know that there exists Q' on S_1 such that

$$l_{OQ} = l_{OQ'}c^{k-1} + (k-1)c^{k-1}a \tag{2.1.12}$$

$$l_{O\overline{Q}_{k-1}} = l_{OQ'}c^{k-1} + (k-2)c^{k-1}a \tag{2.1.13}$$

This is contrary to (2.1.11). So (iii) holds.

(4). From (2.1.6) and noting that c<1 and S_1 is bounded, it is easy to prove that there exists an integer m such that

$$\sup_{Q_m \in \overline{S}_m} l_{O\overline{Q}_m} < r_B \tag{2.1.14}$$

It follows that (iv) holds#

Lemma 2.3. Suppose that $\overline{S}$ is a closed surface and $S \in \Im(\overline{S})$. Then for any given $T \in H(\overline{S})$, there exist two bounded function X and Y on S such that

$$T(P) = \int_S \frac{X(Q)}{l_{PQ}} dS_Q - \int_S Y(Q) \frac{\partial}{\partial n_Q} \left(\frac{1}{l_{PQ}}\right) dS_Q \qquad (P \in e(S)) \qquad (2.1.15)$$

where n_Q is the outer direction of the normal of S at Q. Furthermore, if S is a spherical surface, then there exists a bounded function Z on S such that

$$T(P) = \int_S \frac{Z(Q)}{l_{PQ}} dS_Q \qquad (P \in e(S)) \qquad (2.1.16)$$

Proof: Since $T \in H(\overline{S})$ and $S \in \Im(\overline{S})$, T and $\frac{\partial}{\partial n_Q} T$ are continuous on S, thus bounded. For $Q \in S$, let

$$X(Q) = \frac{1}{4\pi} \frac{\partial}{\partial n_Q} T(Q), \qquad Y(Q) = \frac{1}{4\pi} T(Q) \qquad (2.1.17)$$

Then from Green's formula

$$T(P) = \frac{1}{4\pi} \int_S \frac{1}{l_{PQ}} \frac{\partial}{\partial n_Q} T(Q) dS_Q - \frac{1}{4\pi} \int_S T(Q) \frac{\partial}{\partial n_Q} \left(\frac{1}{l_{PQ}}\right) dS_Q \qquad (2.1.18)$$

we see that (2.1.15) holds.

If S is a spherical surface, then

$$\frac{\partial}{\partial n_Q} \left(\frac{1}{l_{PQ}}\right) = \frac{l_{O'P}^2 - l_{PQ}^2 - R^2}{2 l_{PQ}^3 R} \qquad (2.1.19)$$

where O' and R are respectively the centre and the radius of S. According to (1-89) in Heiskanen and Moritz (1967), we have

$$T(P) = \frac{1}{4\pi}\int_S T(Q)\frac{l_{O'P}^2 - R^2}{l_{PQ}^3 R} dS_Q \qquad (P \in e(S)) \tag{2.1.20}$$

Multiplying (2.1.20) by -½ and adding the product to (2.1.8), we obtain from (2.1.19) that

$$\frac{1}{2}T(P) = \frac{1}{4\pi}\int_S [\frac{\partial}{\partial n_Q}T(Q) + T(Q)\frac{1}{2R}]\frac{1}{l_{PQ}} dS_Q \tag{2.1.21}$$

Let

$$Z(Q) = [\frac{\partial}{\partial n_Q}T(Q) + T(Q)\frac{1}{2R}]\frac{1}{2\pi} \tag{2.1.22}$$

then (2.1.16) holds#

Lemma 2.4. Let $P_n(t)$ be n-degree Legendre polynomial. Then for $|t| \le 1$

(i). $|P_n(t)| \le 1$;

(ii). $|P_n'(t)| \le n^2$;

(iii). $|P_n''(t)| \le n^4$.

Proof: (1). (i) was stated in section 1.3 of Moritz (1990).

(2). From (1-86) of the appendix of Guan and Ling (1981), we have that for $n \ge 1$,

$$P_{n+1}'(t) - P_{n-1}'(t) = (2n+1)P_n(t) \tag{2.1.23}$$

Noting that

$$P_0(t)=1;\quad P_0'(t)=0;\quad P_1(t)=t;\quad P_1'(t)=1; \tag{2.1.24}$$

we obtain that

$$\left|P_{2n+1}'(t)\right|=\left|P_1'(t)+\sum_{k=1}^{n}[P_{2k+1}'(t)-P_{2k-1}'(t)]\right|\le 1+\sum_{k=1}^{n}(4k+1)=(n+1)(2n+1)\le(2n+1)^2 \tag{2.1.25}$$

$$\left|P_{2n}'(t)\right|=\left|P_0'(t)+\sum_{k=1}^{n}[P_{2k}'(t)-P_{2k-2}'(t)]\right|\le\sum_{k=1}^{n}4k=2(n+1)n\le(2n)^2 \tag{2.1.26}$$

This means that (ii) holds.

(3). From (2.1.23), we obtain

$$P_{n+1}''(t)-P_{n-1}''(t)=(2n+1)P_n'(t)$$

Then from the proof of (ii), we know that (iii) holds#

Lemma 2.5. Let O, P and Q be the three points in R^3; v_P and v_Q be the two directions through P and Q, respectively. For a constant c (0<c<1), take a point $\overline{Q}$ on the line OQ such that

$$l_{O\overline{Q}}=c\,l_{OQ} \tag{2.1.27}$$

Then

(i). $\left|\dfrac{\partial}{\partial v_Q}l_{OQ}\right|\le 1$

(ii). $\left|\frac{\partial}{\partial v_Q} l_{P\bar{Q}}\right| \le 1$; $\left|\frac{\partial}{\partial v_P} l_{P\bar{Q}}\right| \le 1$; $\left|\frac{\partial}{\partial v_P}\frac{\partial}{\partial v_Q} l_{P\bar{Q}}\right| \le N_{PQ}$

(iii). $\left|\frac{\partial}{\partial v_Q} t_{P\bar{Q}Q}\right| \le 2N_{PQ}$; $\left|\frac{\partial}{\partial v_P} t_{P\bar{Q}Q}\right| \le 2N_{PQ}$; $\left|\frac{\partial}{\partial v_P}\frac{\partial}{\partial v_Q} t_{P\bar{Q}Q}\right| \le 5N_{PQ}^2$

where

$$t_{P\bar{Q}Q} = \cos\angle P\bar{Q}Q \tag{2.1.28}$$

$$N_{PQ} = \frac{1}{l_{P\bar{Q}}} + \frac{1}{l_{O\bar{Q}}} \tag{2.1.29}$$

Proof: From the definition of the derivative and the knowledge of triangle functions, it can be proved that for any given two points P_0 and P_1 in space and two directions v_{P_0} and v_{P_1} respectively through P_0 and P_1 (see figure 2.4),

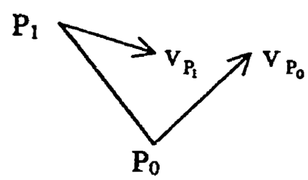

Figure 2.4 The directions v_{P_1} and v_{P_0}

$$\frac{\partial}{\partial v_{P_1}} l_{P_0P_1} = \cos(\overrightarrow{P_0P_1}, v_{P_1}) \tag{2.1.30}$$

$$\left|\frac{\partial}{\partial v_{P_0}} \cos(\overrightarrow{P_0P_1}, v_{P_1})\right| \le \frac{1}{l_{P_0P_1}} \tag{2.1.31}$$

Take a point P' in the direction $\overrightarrow{OP}$ (see figure 2.5) such that

$$l_{OP} = c l_{OP'} \tag{2.1.32}$$

and let $v_{\bar{Q}}$ be the direction through $\bar{Q}$ and parallel to v_Q. Then from (2.1.27) we obtain

$$l_{P\bar{Q}} = c l_{P'Q} \text{ and } \cos(\overrightarrow{P'Q}, v_Q) = \cos(\overrightarrow{P\bar{Q}}, v_{\bar{Q}}) \tag{2.1.33}$$

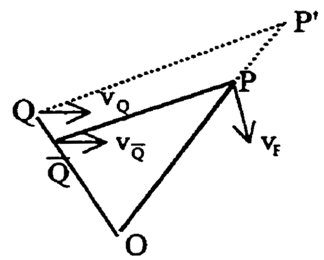

Figure 2.5 The relations between P and P', **and** v_Q and $v_{\bar{Q}}$

It follows from (2.1.30) and (2.1.31) that

$$\frac{\partial}{\partial v_Q} l_{OQ} = \cos(\overrightarrow{OQ}, v_Q) \tag{2.1.34}$$

$$\frac{\partial}{\partial v_P} l_{OP} = \cos(\overrightarrow{OP}, v_P) \tag{2.1.35}$$

$$\frac{\partial}{\partial v_Q} l_{P\bar{Q}} = c\cos(\overrightarrow{P'Q}, v_Q) = c\cos(\overrightarrow{P\bar{Q}}, v_{\bar{Q}}) \tag{2.1.36}$$

$$\frac{\partial}{\partial v_P} l_{P\bar{Q}} = c\cos(\overrightarrow{\bar{Q}P}, v_P) \tag{2.1.37}$$

$$\left|\frac{\partial}{\partial v_P}\frac{\partial}{\partial v_Q} l_{P\bar{Q}}\right| = \left|\frac{\partial}{\partial v_P}[c\cos(\overrightarrow{P\bar{Q}}, v_{\bar{Q}})]\right| \le \frac{c}{l_{P\bar{Q}}} \tag{2.1.38}$$

Therefore (i) and (ii) hold. Furthermore, from the definition of $t_{P\bar{Q}Q}$, we have

$$t_{P\bar{Q}Q} = \frac{l_{OP}^2 - c^2 l_{OQ}^2 - l_{P\bar{Q}}^2}{2c l_{OQ} l_{P\bar{Q}}} \tag{2.1.39}$$

Finally from (2.1.34-38) and noting that

$$\frac{\partial}{\partial v_Q} l_{OP} = 0, \quad \frac{\partial}{\partial v_P} l_{OQ} = 0, \quad \frac{\partial}{\partial v_P} \frac{\partial}{\partial v_Q} l_{OQ} = 0 \tag{2.1.40}$$

we see that (iii) holds#

2.2 Supplement to Runge-Krarup's theorem

In this section, we will give a supplement to Runge-Krarup's theorem. This is expressed as the following theorems.

Theorem 2.1 Let T, O_E and S_E be the disturbing potential, the centre and the surface of the Earth, respectively. Then for an arbitrary surface $S_0 \in \aleph(O_E) \cap \Im(S_E)$, an arbitrary point $\overline{O}$ inside S_0, an arbitrary non-negative integer N and an arbitrary real number $\varepsilon > 0$, there exists $\overline{T} \in H(\overline{O})$ satisfying

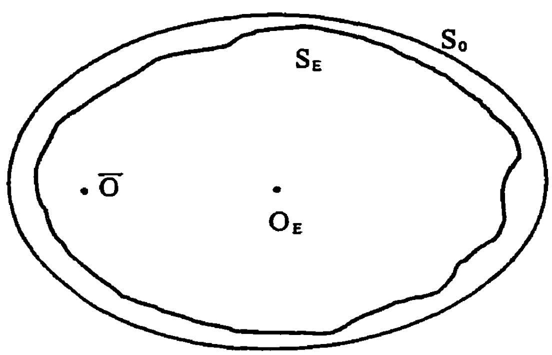

Figure 2.6 The relations between S_E, S_0, O_E and $\overline{O}$

$$\|T-\overline{T}\|_{S_0}^{N} < \varepsilon \tag{2.2.1}$$

This theorem shows that for any positive integer N, the n-degree ($0 \le n \le N$) derivatives of the disturbing potential T can be approximated on and outside a smooth star-shaped surface S_0 completely surrounding the Earth's surface by the n-degree derivatives of a function $\overline{T}$ which is harmonic everywhere except an inner point of S_0. Obviously, this gives a supplement to Runge-Krarup's theorem so that the derivatives of the disturbing potential are included. The following is a proof of the theorem.

Proof: Here we only prove the theorem in the case that N=1. For N>1, the proof is similar. Our proof consists of two steps.

Step 1: we will prove that there exists $T_B \in H(O_E)$ satisfying

$$\|T-T_B\|_{S_0}^{1} < \frac{\varepsilon}{2} \tag{2.2.2}$$

From Lemma 2.1, we know that for S_E, S_0 and O_E, there exists a surface S_I satisfying the conditions:

$$_{I} \in \quad (O_E \ \cap \Im(S_E \qquad S_0 \in \Im(S_I) \tag{2.2.3}$$

$$d \equiv \qquad _{PQ} : P \qquad _0), \quad \in I(S\ )\} > 0$$

Since S_I $\qquad\qquad$ $_E$, the centre of the Earth, as its interior point, the

r_Q of the points on S_I satisfy

$$0 < \inf_{Q \in S_I} r_Q \le \sup_{Q \in S_I} r_Q < +\infty \tag{2.2.5}$$

Take a, b satisfying

$$0 < b < \min(\frac{d}{2}, \inf_{Q \in S_1} r_Q)\,; \tag{2.2.6}$$

$$0 < a < \min(\frac{d}{2} - b, \frac{b \inf_{Q \in S_1} r_Q}{\sup_{Q \in S_1} r_Q}) \tag{2.2.7}$$

Let

$$c = 1 - \frac{a+b}{\sup_{Q \in S_1} r_Q + a} \tag{2.2.8}$$

Then it is easy to prove that

$$0 < \sup_{Q \in S_1} \frac{r_Q - b}{r_Q + a} \le c \le \inf_{Q \in S_1} \frac{r_Q}{r_Q + a} < 1 \tag{2.2.9}$$

$$a + b < \frac{d}{2} \tag{2.2.10}$$

So according to Lemma 2.2, for S_1, a, b and c given above, we can take a set of surfaces $\{S_k, \overline{S}_k\}_{k=1}^{m}$ containing O_E as their inner point, and a spherical surface S_B with centre at O_E and radius $r_B\ (= \frac{1}{2} \inf_{Q \in S_1} r_Q)$ so that they satisfy the conditions (i-iv) of Lemma 2.2. Since $T \in H(e(S_E))$, $S_1 \in \Im(S_E)$ and $S_0 \in \Im(S_1)$, it follows from Lemma 2.3 that there exist bounded functions X_1 and Y_1 and a positive constant M_1 such that

$$T(P) = \int_{S_1} \frac{X(Q)}{l_{PQ}} dS_{1Q} - \int_{S_1} Y(Q) \frac{\partial}{\partial n_Q} (\frac{1}{l_{PQ}}) dS_{1Q} \qquad (P \in E(S_0)) \tag{2.2.11}$$

$$\int_{S_1} |X(Q)| dS_{1Q} + \int_{S_1} |Y(Q)| dS_{1Q} < M_1 \tag{2.2.12}$$

For any given $Q_1 \in S_1$, take a $\overline{Q}_1$ on line OQ_1 such that

$$r_{\overline{Q}_1} = c r_{Q_1} \tag{2.2.13}$$

From Lemma 2.2, we know that

$$\overline{Q}_1 \in \overline{S}_1 \tag{2.2.14}$$

and

$$\overline{Q}_1 \in i(S_1) \tag{2.2.15}$$

$$r_{\overline{Q}_1} \geq r_1 \equiv \inf_{\overline{Q}_1 \in \overline{S}_1} r_{\overline{Q}_1} > 0 \tag{2.2.16}$$

$$l_{Q_1\overline{Q}_1} < a + b \tag{2.2.17}$$

From (2.2.4) and (2.2.10), we know that for $P \in E(S_0)$,

$$l_{P\overline{Q}_1} \geq d, \qquad l_{Q_1\overline{Q}_1} < \frac{d}{2} \tag{2.2.18}$$

Thus

$$\frac{l_{Q_1\overline{Q}_1}}{l_{P\overline{Q}_1}} < \frac{1}{2} \tag{2.2.19}$$

$$N_{PQ_1} = \frac{1}{l_{P\overline{Q}_1}} + \frac{1}{r_{\overline{Q}_1}} \leq \frac{1}{d} + \frac{1}{r_1} \equiv N_1 < +\infty \tag{2.2.20}$$

So according to Lemma 2.4, Lemma 2.5 and equation (1-81) of Heiskanen and Moritz (1967), we obtain that for $P \in E(S_0)$ and $Q_l \in S_l$

$$\left|\frac{1}{l_{PQ_l}}\right| = \left|\sum_{n=0}^{\infty} \frac{l^n_{Q_l\bar{Q}_l}}{l^{n+1}_{P\bar{Q}_l}} P_n(t_{P\bar{Q}_lQ_l})\right| \le \sum_{n=0}^{\infty} (\frac{1}{2})^n N_l < +\infty \tag{2.2.21}$$

$$\left|\frac{\partial}{\partial n_Q}(\frac{1}{l_{PQ_l}})\right| = \left|\sum_{n=0}^{\infty} \frac{\partial}{\partial n_Q}[\frac{l^n_{Q_l\bar{Q}_l}}{l^{n+1}_{P\bar{Q}_l}} P_n(t_{P\bar{Q}_lQ_l})]\right| \le \sum_{n=0}^{\infty} (\frac{1}{2})^n 2(n+1)^2 N_l^2 < +\infty \tag{2.2.22}$$

$$\left|\frac{\partial}{\partial v_P}(\frac{1}{l_{PQ_l}})\right| = \left|\sum_{n=0}^{\infty} \frac{\partial}{\partial v_P}[\frac{l^n_{Q_l\bar{Q}_l}}{l^{n+1}_{P\bar{Q}_l}} P_n(t_{P\bar{Q}_lQ_l})]\right| \le \sum_{n=0}^{\infty} (\frac{1}{2})^n 2(n+1)^2 N_l^2 < +\infty \tag{2.2.23}$$

$$\left|\frac{\partial}{\partial v_P}\frac{\partial}{\partial n_Q}(\frac{1}{l_{PQ_l}})\right| = \left|\sum_{n=0}^{\infty} \frac{\partial}{\partial v_P}\frac{\partial}{\partial n_Q}[\frac{l^n_{Q_l\bar{Q}_l}}{l^{n+1}_{P\bar{Q}_l}} P_n(t_{P\bar{Q}_lQ_l})]\right| \le \sum_{n=0}^{\infty} (\frac{1}{2})^n 4(n+1)^4 N_l^3 < +\infty \tag{2.2.24}$$

So the above four series are absolutely convergent when $P \in E(S_0)$ and $Q_l \in S_l$. It follows that for $\varepsilon_l = \dfrac{\varepsilon}{2(m+1)M_l}$, there exists a positive integer n_l such that

$$\sup_{\substack{P\in E(S_0)\\ Q_l\in S_l}} \left|\frac{\partial^i}{\partial v_P^i}\frac{\partial^j}{\partial n_Q^j}(\frac{1}{l_{PQ_l}}) - \frac{\partial^i}{\partial v_P^i}\frac{\partial^j}{\partial n_Q^j}\sum_{n=0}^{n_l}[\frac{l^n_{Q_l\bar{Q}_l}}{l^{n+1}_{P\bar{Q}_l}} P_n(t_{P\bar{Q}_lQ_l})]\right| \le \varepsilon_l \quad (\text{For } Q_l \in S_l) \tag{2.2.25}$$

where i, j=0 or 1. For $Q_l \in S_l$ and $P \neq \overline{Q}_l$, let

$$f_{lQ_l}(P) = \sum_{n=0}^{n_l} \frac{l^n_{Q_l\bar{Q}_l}}{l^{n+1}_{P\bar{Q}_l}} P_n(t_{P\bar{Q}_lQ_l}) \tag{2.2.26}$$

It can be proved that

$$f_{lQ_l} \in H(\overline{Q}_l) \tag{2.2.27}$$

Furthermore for $P \notin \overline{S}_1$, let

$$T_1(P) = \int_{S_1} X(Q_1) f_{1Q_1}(P) dS_Q - \int_{S_1} Y(Q_1) \frac{\partial}{\partial n_{Q_1}} [f_{1Q_1}(P)] dS_Q \tag{2.2.28}$$

Then

$$T_1 \in H(\overline{S}_1) \tag{2.2.29}$$

and

$$\|T - T_1\|^1_{S_0} = \max_{i=0,1} \sup_{\substack{P \in E(S_0) \\ v_P \in V_P}} \left| \int_{S_1} \frac{\partial^i}{\partial v^i_P} [\frac{1}{l_{PQ_1}} - f_{1Q_1}(P)] X_1(Q_1) dQ_1 \right.$$

$$\left. - \int_{S_1} \frac{\partial^i}{\partial v^i_P} \frac{\partial}{\partial n_Q} [\frac{1}{l_{PQ_1}} - f_{1Q_1}(P)] Y_1(Q_1) dQ_1 \right|$$

$$< \varepsilon_1 [\int_{S_1} |X_1(Q_1)| dQ_1 + \int_{S_1} |Y_1(Q_1)| dQ_1] \le \varepsilon_1 M_1 = \frac{\varepsilon}{2(m+1)} \tag{2.2.30}$$

For $S_2 \in \Im(\overline{S}_1)$, repeating the above work by replacing S_E, S_1, $\overline{S}_1$ and T by $\overline{S}_1$, S_2, $\overline{S}_2$ and T_1 respectively, we can obtain a $T_2 \in H(\overline{S}_2)$ such that

$$\|T_1 - T_2\|^1_{S_0} < \frac{\varepsilon}{2(m+1)} \tag{2.2.31}$$

Furthermore, for each k ($1 \le k \le m$), there exists a $T_k \in H(\overline{S}_k)$ such that

$$\|T_{k-1} - T_k\|^1_{S_0} < \frac{\varepsilon}{2(m+1)} \qquad (T_0 \equiv T) \tag{2.2.32}$$

So for $T_m \in H(\overline{S}_m)$, we have

$$\|T - T_m\|_{S_0}^1 < \sum_{k=1}^{m} \|T_{k-1} - T_k\|_{S_0}^1 < \frac{m\varepsilon}{2(m+1)} \tag{2.2.33}$$

From $S_B \in \Im(\overline{S}_m)$ and Lemma 2.3, we know that there exists a function Z(Q) on S_B and a positive constant N_B such that

$$T_m(P) = \int_{S_m} \frac{Z(Q)}{l_{PQ}} dS_{mQ} \qquad (P \in E(S_0)) \tag{2.2.34}$$

$$\int_{S_m} |Z(Q)| dS_{mQ} < N_B \tag{2.2.35}$$

According to the definitions of S_B and S_0, we know that

$$r_Q = r_B = \frac{1}{2} \inf_{Q' \in S_1} r_{Q'} \qquad (Q \text{ on } S_B) \tag{2.2.36}$$

$$r_P \geq \inf_{Q' \in S_1} r_{Q'} + d \qquad (P \in E(S_0)) \tag{2.2.37}$$

So for $P \in E(S_0)$ and Q on S_B,

$$\frac{r_Q}{r_P} \leq \frac{\frac{1}{2} \inf_{Q' \in S_1} r_{Q'}}{\inf_{Q' \in S_1} r_{Q'} + d} < \frac{1}{2} \tag{2.2.38}$$

Let $t = \cos\angle POQ$ and note that $\frac{1}{r_P} < \frac{1}{d}$, we have from Lemma 2.4 and (2.1.30) and (2.1.31) that

$$\left|\frac{1}{l_{PQ}}\right| = \left|\sum_{n=0}^{\infty}\left(\frac{r_Q}{r_P}\right)^n \frac{1}{r_P} P_n(t)\right| \le \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n \frac{1}{d} < +\infty \tag{2.2.39}$$

$$\left|\frac{\partial}{\partial v_P}\left(\frac{1}{l_{PQ}}\right)\right| = \left|\sum_{n=0}^{\infty}\frac{\partial}{\partial v_P}\left[\left(\frac{r_Q}{r_P}\right)^n \frac{1}{r_P} P_n(t)\right]\right| \le \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n (n+1)^2 \frac{1}{d^2} < +\infty \tag{2.2.40}$$

So there exists a positive integer n_B such that

$$\sup_{\substack{P\in E(S_0)\\ Q_i\in S_i\\ v_P\in V_P}}\left|\frac{\partial^i}{\partial v_P^i}\left(\frac{1}{l_{PQ}}\right) - \frac{\partial^i}{\partial v_P^i}\sum_{n=0}^{n_B}\left[\left(\frac{r_Q}{r_P}\right)^n \frac{1}{r_P} P_n(t)\right]\right| \le \frac{\varepsilon}{2(m+1)N_B} \tag{2.2.41}$$

For $P \ne O_E$, let

$$f_{BQ}(P) = \sum_{n=0}^{n_B}\left[\left(\frac{r_Q}{r_P}\right)^n \frac{1}{r_P} P_n(t)\right] \qquad \text{(Q is on } S_B\text{)} \tag{2.2.42}$$

$$T_B(P) = \int_{S_B} Z(Q) f_{BQ}(P) dS_{BQ} \tag{2.2.43}$$

Obviously, $T_B \in H(O_E)$, and from (2.2.34-35) and (2.2.41-43) we obtain

$$\|T_m - T_B\|_{S_0}^{1} < \frac{\varepsilon}{2(m+1)N_B} N_B = \frac{\varepsilon}{2(m+1)} \tag{2.2.44}$$

Thus from (2.2.33)

$$\|T - T_B\|_{S_0}^{1} \le \|T - T_m\|_{S_0}^{1} + \|T_m - T_B\|_{S_0}^{1} < \frac{\varepsilon m}{2(m+1)} + \frac{\varepsilon}{2(m+1)} = \frac{\varepsilon}{2} \tag{2.2.45}$$

Thus we have finished the work of the first step.

Step 2: In the following, we will prove the conclusion of the theorem 2.1.

Since $\overline{O}$ is inside S_0 and $S_0 \in \aleph(O_E)$, we know that there is no intersection between S_0 and the line $O_E\overline{O}$. That is

$$d_1 \equiv \inf\{l_{PQ} : P \in S_0, Q \text{ is on the line } O_E\overline{O}\} > 0 \qquad (2.2.46)$$

Draw two closed surfaces S' and S" satisfying respectively the following conditions:

$$Q' \in S' \quad \text{if and only if} \quad \inf\{l_{Q'Q} : Q \text{ is on the line } O_E\overline{O}\} = d_1/2 \qquad (2.2.47)$$

$$Q'' \in S'' \quad \text{if and only if} \quad \inf\{l_{Q''Q} : Q \text{ is on the line } O_E\overline{O}\} = d_1/4 \qquad (2.2.48)$$

Obviously, S' and S" are smooth surfaces surrounding completely the line $O_E\overline{O}$ and being surrounded completely by S_0, and

$$S' \in \Im(S'') \cap \aleph(\overline{O}) \qquad (2.2.49)$$

Since $T_B \in H(O_E)$ and O_E is inside S", we have

$$T_B \in H(S'') \qquad (2.2.50)$$

Repeating the work done in the first step by using S", S', T_B and $\overline{O}$ to replace respectively S_E, S_0, T and O_E, and noting (2.2.49) and (2.2.50), we can see that there exists $\overline{T} \in H(\overline{O})$ satisfying

$$\left\| T_B - \overline{T} \right\|_{S'}^{1} < \frac{\varepsilon}{2} \qquad (2.2.51)$$

Since S' is inside S_0, it follows from (2.1.1) that

$$\|T_B - \overline{T}\|_{S_0}^1 \le \|T_B - \overline{T}\|_{S'}^1 < \frac{\varepsilon}{2} \tag{2.2.52}$$

Therefore we obtain from (2.2.45) that

$$\|T - \overline{T}\|_{S_0}^1 \le \|T - T_B\|_{S_0}^1 + \|T_B - \overline{T}\|_{S_0}^1 < \varepsilon \tag{2.2.53}$$

Thus we have finished the proof of the theorem 2.1#

2.3 Supplement to the Keldysh-Lavrentiev theorem

In the above section, we gave a supplement to the Runge-Krarup theorem so that it is also valid for the derivatives of the disturbing potential. Now we will give a supplement to the Keldysh-Lavrentiev theorem. This is expressed as the following theorem 2.2.

Theorem 2.2. Let T, O_E and S_E be the disturbing potential, the centre and the surface of the Earth, respectively. If S_E is smooth and $S_E \in \aleph(O_E)$, then for an arbitrary point $\overline{O}$ inside S_E and an arbitrary real number $\varepsilon > 0$, there exists $\overline{T} \in H(\overline{O})$ satisfying:

$$\|T - \overline{T}\|_{S_E}^1 < \varepsilon \tag{2.3.1}$$

This theorem shows that the n-degree (n=0,1) derivatives of the disturbing potential T can be approximated on and outside the Earth's surface S_E by the n-degree derivatives of a function $\overline{T}$ which is harmonic outside an inner point of S_E if S_E is a smooth star-shaped surface. In this theorem, the degree of the derivative is less than 2. This is because the second (or higher) degree derivatives of the disturbing potential are not continuous on the Earth's surface. Obviously, it gives a supplement to Keldysh-Lavrentiev's theorem so that the first order derivatives of the disturbing potential are included. The following is a proof of the theorem.

Proof: According to (1.1.13), the disturbing potential can be expressed as

$$T(P) = G\int_{\tau} \frac{\delta\rho(Q)}{l_{PQ}} d\tau \tag{2.3.2}$$

where τ is the space surrounded by S_E, $\delta\rho$ is the disturbing density function of the Earth and G is the Newton gravitation constant.

Let

$$\rho_{max} = \max\{\rho(Q) : Q \in \tau\}, \tag{2.3.3}$$

$$r_{max} = \max\{r_Q : Q \in S_E\}, \tag{2.3.4}$$

and

$$d_3 = \min\{1, \frac{\varepsilon}{8\pi G\rho_{max}(r_{max}^2 + 1)}\}, \tag{2.3.5}$$

$$d_4 = (d_3)^3 \tag{2.3.6}$$

For every Q $(r_Q, \theta_Q, \lambda_Q)$ on S_E, take a point $\overline{Q}$ $(r_{\overline{Q}}, \theta_Q, \lambda_Q)$ satisfying the following condition (see figure 2.7)

$$r_{\overline{Q}} = r_Q - d_4 \tag{2.3.7}$$

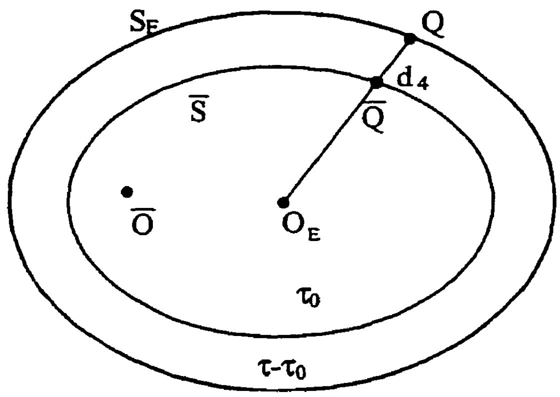

Figure 2.7 The relation between S_E and $\overline{S}$

Obviously, since $S_E \in \aleph(O_E)$, the surface $\overline{S}$ formed by all points $\overline{Q}$ is completely surrounded by S_E. Thus $S_E \in \Im(\overline{S})$. Let τ_0 be the space surrounded by $\overline{S}$ and $\tau_1 = \tau - \tau_0$. Then the volume of τ_1 is as follows:

$$V(\tau_1) = \int_{\tau_1} d\tau_{1Q} = \int_0^{2\pi}\int_0^{\pi}\int_{r_{\overline{Q}}}^{r_Q} r^2 dr d\cos\theta_Q d\lambda_Q = \int_0^{2\pi}\int_0^{\pi} \frac{r_Q^3 - r_{\overline{Q}}^3}{3} d\cos\theta_Q d\lambda_Q$$
$$= \int_0^{2\pi}\int_0^{\pi} \frac{[r_Q - r_{\overline{Q}}][r_Q^2 + r_Q r_{\overline{Q}} + r_{\overline{Q}}^2]}{3} d\cos\theta_Q d\lambda_Q$$
$$\le \int_0^{2\pi}\int_0^{\pi} d_4 r_{max}^2 d\cos\theta d\lambda = 4\pi r_{max}^2 d_4 \tag{2.3.8}$$

For an arbitrary given point P on or outside S_E ($P \in E(S_E)$), draw a sphere τ_P with the centre at P and a radius of d_3. Let $\tau_2 = \tau_1 \cap \tau_P$. Then from (2.3.2), we have for any given $v_P \in V_P$ that

$$T(P) = G\int_{\tau_0} \frac{\rho(Q)}{l_{PQ}} d\tau + G \int_{\tau_1-\tau_2} \frac{\rho(Q)}{l_{PQ}} d\tau + G\int_{\tau_2} \frac{\rho(Q)}{l_{PQ}} d\tau \tag{2.3.9}$$

$$\frac{\partial}{\partial v_P} T(P) = G\int_{\tau_0} \rho(Q) \frac{\partial}{\partial v_P}[\frac{1}{l_{PQ}}] d\tau + G \int_{\tau_1-\tau_2} \rho(Q) \frac{\partial}{\partial v_P}[\frac{1}{l_{PQ}}] d\tau + G\int_{\tau_2} \rho(Q) \frac{\partial}{\partial v_P}[\frac{1}{l_{PQ}}] d\tau \tag{2.3.10}$$

Let

$$T_0(P) = G\int_{\tau_0} \frac{\rho(Q)}{l_{PQ}} d\tau \tag{2.3.11}$$

Then from (2.3.9) and (2.3.10) and noting (2.1.30), we obtain that

$$|T(P) - T_0(P)| \le G \int_{\tau_1-\tau_2} \left|\frac{\rho(Q)}{l_{PQ}}\right| d\tau + G\int_{\tau_2} \left|\frac{\rho(Q)}{l_{PQ}}\right| d\tau \le G\rho_{max}\{ \int_{\tau_1-\tau_2} \frac{1}{l_{PQ}} d\tau + \int_{\tau_2} \frac{1}{l_{PQ}} d\tau\} \tag{2.3.12}$$

$$\begin{aligned}\left|\frac{\partial}{\partial v_P} T(P) - \frac{\partial}{\partial v_P} T_0(P)\right| &\le G \int_{\tau_1-\tau_2} \left|\rho(Q) \frac{\partial}{\partial v_P}[\frac{1}{l_{PQ}}]\right| d\tau + G\int_{\tau_2} \left|\rho(Q) \frac{\partial}{\partial v_P}[\frac{1}{l_{PQ}}]\right| d\tau \\ &\le G\rho_{max}\{ \int_{\tau_1-\tau_2} \frac{1}{l_{PQ}^2} d\tau + \int_{\tau_2} \frac{1}{l_{PQ}^2} d\tau\}\end{aligned} \tag{2.3.13}$$

From the definition of τ_2, we know that

$$l_{PQ} \ge d_3 \quad \text{for } Q \in \tau_1 - \tau_2 \tag{2.3.14}$$

It follows from (2.3.6) and (2.3.8) that

$$\int_{\tau_1-\tau_2} \frac{1}{l_{PQ}} d\tau \le \frac{1}{d_3} \int_{\tau_1-\tau_2} d\tau \le \frac{1}{d_3} V(\tau_1) \le \frac{4\pi r_{max}^2 d_4}{d_3} = 4\pi r_{max}^2 d_3^2 \tag{2.3.15}$$

$$\int_{\tau_1-\tau_2} \frac{1}{l_{PQ}^2} d\tau \le \frac{1}{d_3^2} \int_{\tau_1-\tau_2} d\tau \le \frac{1}{d_3^2} V(\tau_1) \le \frac{4\pi r_{max}^2 d_4}{d_3^2} = 4\pi r_{max}^2 d_3 \tag{2.3.16}$$

Since $\tau_2 \subset \tau_P$, we have

$$\int_{\tau_2} \frac{1}{l_{PQ}} d\tau \le \int_{\tau_P} \frac{1}{l_{PQ}} d\tau = \int_0^{2\pi}\int_0^{\pi}\int_0^{d_3} \frac{1}{r} r^2 dr d\cos\theta d\lambda = 2\pi d_3^2 \tag{2.3.17}$$

$$\int_{\tau_2} \frac{1}{l_{PQ}^2} d\tau \le \int_{\tau_P} \frac{1}{l_{PQ}^2} d\tau = \int_0^{2\pi}\int_0^{\pi}\int_0^{d_3} \frac{1}{r^2} r^2 dr d\cos\theta d\lambda = 4\pi d_3 \tag{2.3.18}$$

So from (2.3.12) and (2.3.13), we have

$$|T(P) - T_0(P)| \le G\rho_{max}[4\pi r_{max}^2 d_3^2 + 2\pi d_3^2] \tag{2.3.19}$$

$$\left|\frac{\partial}{\partial v_P} T(P) - \frac{\partial}{\partial v_P} T_0(P)\right| \le G\rho_{max}[4\pi r_{max}^2 d_3 + 4\pi d_3] \tag{2.3.20}$$

It follows from (2.3.5) that

$$\|T - T_0\|_{S_E}^1 \le G\rho_{max}[4\pi r_{max}^2 d_3 + 4\pi d_3] < \frac{\varepsilon}{2} \tag{2.3.21}$$

From the definition (2.3.11) of T_0, we know that $T_0 \in H(\overline{S})$. Substituting respectively T, S_E, S_0, N and ε by T_0, $\overline{S}$, S_E, 1 and $\frac{\varepsilon}{2}$ in the theorem 2.1, we see that, for the $\overline{O}$ inside S_E, there exists $\overline{T} \in H(\overline{O})$ satisfying

$$\|T_0 - \overline{T}\|_{S_E}^1 < \frac{\varepsilon}{2} \tag{2.3.22}$$

It follows from (2.3.21) that

$$\|T - \overline{T}\|_{S_E}^1 < \varepsilon \tag{2.3.23}$$

Thus we have finished the proof of the theorem#

2.4 Discussion

In the preceding sections, we gave two theorems (Theorem 2.1 and Theorem 2.2), which deal with not only the disturbing potential but also with its derivatives. In the following, we will discuss the conditions in the theorems and give an example on how the theorems are applied.

2.4.1 Conditions in the theorems

In these two theorems, the smooth star-shaped surface (see the definition 2.2) plays a very important role. In theorem 2.2, the Earth's surface S_E is supposed to be a smooth star-shaped surface about the Earth's centre O_E. In theorem 2.1, the surface S_0, on and outside which the disturbing potential T or its up to n-degree derivatives are approximated, is supposed to be a smooth star-shaped surface about O_E. In other words, S_E (or S_0) should be smooth, and for a point Q on S_E (or S_0), all the other points on the line O_EQ are inside S_E (or S_0). For physical geodesy purposes, it is acceptable to adjust slightly the figure of the Earth so that its surface is a star-shaped surface about the geocentre. In fact, in the application of physical geodesy theories, the surface S_E is supposed to be a plane, a spherical surface or a smooth fitting surface. Therefore, we will always suppose that the Earth's surface S_E is a smooth star-shaped surface about the geocentre. In this case, the condition of theorem 2.2 is satisfied and the surface S_0 in the theorem 2.1 can be taken as close to S_E as we want. Furthermore, we can merge the two theorems into the following theorem:

Theorem 2.3. For a point $\overline{O}$ inside S_E, an arbitrary given positive integer N, a star-shaped surface S_0 completely surrounding S_E and an arbitrary positive ε, there exists a function $\overline{T}$, harmonic everywhere except $\overline{O}$, such that

$$\left\|T-\overline{T}\right\|_{S_0}^{N} < \varepsilon \quad \text{and} \quad \left\|T-\overline{T}\right\|_{S_E}^{1} < \varepsilon \tag{2.4.1}$$

2.4.2 An application in Moritz's solution for Molodensky's problem

In the following, we will give an example to show how theorem 2.3 is applied to the solution of the geodetic BVP.

As we know Molodensky's problem deals with the determination of the disturbing potential from gravity anomalies on the surface of the Earth. Unlike a level surface, which can be approximated by an ellipsoidal surface, the Earth's surface is very complex. Stokes's formula can not be applied to the gravity anomalies on that surface. Therefore, Moritz (1980) proposed the analytical continuation method, in which the disturbing potential T and the gravity anomaly Δg are analytically continued from S_E onto a point level surface S_P (see figure 2.8).

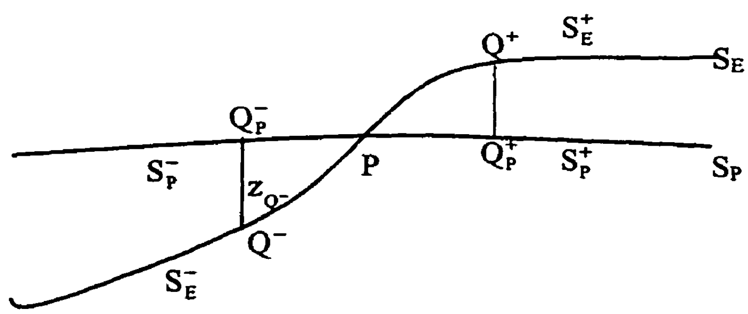

Figure 2.8 The topographic surface S_E and the point level surface S_P

According to our theorems, for an arbitrarily small positive constant ε, an arbitrarily large positive integer N and a surface S_0 sufficiently close to S_E, there exists a function $\overline{T}$ satisfying that

1. $\overline{T} \in H(O)$
2. $|T_Q - \overline{T}_Q| < \varepsilon$ (for Q on or outside S_E)
3. $|\Delta g_Q - \Delta \overline{g}_Q| < \varepsilon$ (for Q on or outside S_E)
4. $\left|\frac{\partial^n}{\partial z^n}\Delta g_Q - \frac{\partial^n}{\partial z^n}\Delta \overline{g}_Q\right| < \varepsilon$ (for $0 \le n \le N$ and Q on or outside S_0)

From the conditions 3 and 4, when the distance z_Q of a point Q on S_E to S_P is small enough, $\Delta\overline{g}$ on S_P has the following relations with Δg on S_E:

$$\Delta g_{Q^-} \approx U_{Q^-}(\Delta g_{Q_P^-}) \approx U_{Q^-}(\Delta \overline{g}_{Q_P^-}) \tag{2.4.2}$$

$$\Delta g_{Q^+} \approx \Delta \overline{g}_{Q^+} \approx U_{Q^+}(\Delta \overline{g}_{Q_P^+}) \tag{2.4.3}$$

where U is the continuation operator (see Moritz, 1980)

$$U_Q = \sum_{n=0}^{N} z_Q^n \frac{1}{n!}\frac{\partial^n}{\partial z_Q^n} \quad \text{with } z_Q = H_Q - H_P \tag{2.4.4}$$

Here, the first relation of (2.4.2) and the second relation of (2.4.3) are obtained from a Taylor expansion, the second relation of (2.4.2) is guaranteed by condition 4 and the first relation of (2.4.3) is guaranteed by condition 3.

After we get $\Delta\overline{g}$ from Δg by resolving (2.4.2) and (2.4.3), we can use $\Delta\overline{g}$ in the Stokes formula to get $\overline{T}_P$ since $\overline{T}$ satisfies the condition 1 and S_P is a level surface. Finally, from the condition 2, we know that $\overline{T}_P$ equals approximately to T_P.

2.5 Chapter summary

In this chapter, we proved theorem 2.1 and theorem 2.2 as supplements to the Runge-Krarup theorem and the Keldysh-Lavrentiev theorem, respectively, so that they are valid for the derivatives of the disturbing potential as well as the disturbing potential itself.

The conditions about the Earth's surface in our theorems, which are a little bit different than those in the Runge-Krarup theorem and the Keldysh-Lavrentiev theorem, are acceptable for geodesy purposes.

Besides Moritz's analytical continuation method for Molodensky's theory, the indirect parameter approaches introduced in subsection 1.4.2 all need the guarantee of theorem 2.1 or theorem 2.2 when they deal with gravity or gravity gradiometer data.

3 A New Method for Computing the Ellipsoidal Correction for Stokes's Formula

In this chapter, we will discuss the ellipsoidal correction problem for Stokes's formula and derive a new solution.

Stokes's formula, an approximate solution of the Stokes problem, has been playing a key role in the determination of the geoid from gravity anomaly data. Rigorously, Stokes's problem is a geoidal boundary value problem. That is that gravity anomalies are given on the geoid and the disturbing potential is supposed to be harmonic outside the geoid. Since the difference between the geoid and the reference ellipsoid is very small, we can treat Stokes's problem as an ellipsoidal boundary value problem. In other words, the Stokes problem can be described mathematically as determining a disturbing potential function T satisfying (1.4.1)

Various approaches have been proposed to solve the above ellipsoidal boundary value problem (Molodensky et al., 1962; Moritz, 1980; Cruz, 1986; Sona, 1995; Thông, 1996; Yu and Cao, 1996; Martinec and Grafarend, 1997; Martinec and Matyska, 1997; Martinec, 1998; Ritter, 1998). Usually, they can be divided into two main approaches: One directly represents the disturbing potential T in terms of an ellipsoidal harmonic series, which is rigorous but very complicated because it requires the introduction of Legendre functions of the second kind (Sona, 1995). Another one regards Stokes's formula as the first approximation of the solution of (1.4.1) and pushes the approximation up to the term of $O(e^2)$, where e is the first eccentricity of the ellipsoid. This term, called the ellipsoidal correction, is expressed in terms of closed integral formulas, such as the solutions described in Molodensky et al. (1962), Moritz (1980) and Martinec and Grafarend (1997).

In the following sections, we will give a new closed integral formula for computing the ellipsoidal correction. A brief comparison of the ellipsoidal corrections given in Molodensky et al. (1962), Moritz (1980), Martinec and Grafarend (1997) and this chapter and a numerical test for the new formula will also be given.

3.1 Derivation of the ellipsoidal correction

In the following, to solve equation (1.4.1), we will (a) establish an integral equation, and (b) solve the integral equation to get Stokes's formula plus its ellipsoidal correction.

3.1.1 Establishment of the integral equation

According to Moritz (1980), for an arbitrary point P_0 given inside S_e, the generalized Stokes function

$$S(P,P_0)=\frac{2}{l_{PP_0}}+\frac{1}{r_P}-\frac{3l_{PP_0}}{r_P^2}-\frac{r_{P_0}}{r_P^2}\cos\psi_{PP_0}\left[5+3\ln\frac{r_P-r_{P_0}\cos\psi_{PP_0}+l_{PP_0}}{2r_P}\right] \qquad (3.1.1)$$

satisfies

$$\begin{cases}\Delta S(P,P_0)=0 & (\text{P is outside } S_e)\\ \lim\limits_{P\to\infty} S(P,P_0)=0 & \\ S(P,P_0)\text{ is continuously differentiable on and outside } S_e & \end{cases} \qquad (3.1.2)$$

From Green's second identity (Heiskanen and Moritz, 1962), we obtain that

$$\int_{S_e} T(Q)\frac{\partial S(Q,P_0)}{\partial h_Q}dS_{eQ}=\int_{S_e} S(Q,P_0)\frac{\partial T(Q)}{\partial h_Q}dS_{eQ} \qquad (3.1.3)$$

It follows from the third formula in (1.4.1) that

$$\int_{S_e} T(Q)[-\frac{\partial S(Q,P_0)}{\partial h_Q}+\frac{1}{\gamma_Q}\frac{\partial \gamma_Q}{\partial h_Q}S(Q,P_0)]dS_{eQ}=\int_{S_e}\Delta g(Q)S(Q,P_0)dS_{eQ} \tag{3.1.4}$$

In (1.4.1), letting P be the moving point Q on S_e and differentiating the function $S(Q, P_0)$ along the normal vertical at Q, we obtain

$$\begin{aligned}\frac{\partial S(Q,P_0)}{\partial h_Q}&=-\frac{2}{l_{QP_0}^2}\frac{\partial l_{QP_0}}{\partial h_Q}+\frac{1}{r_Q^3}[6l_{QP_0}\frac{\partial r_Q}{\partial h_Q}-3r_Q\frac{\partial l_{QP_0}}{\partial h_Q}-r_Q\frac{\partial r_Q}{\partial h_Q}+3r_{P_0}\cos\psi_{QP_0}\frac{\partial r_Q}{\partial h_Q}]\\&-\frac{3r_{P_0}\cos\psi_{QP_0}}{r_Q^2(r_Q-r_{P_0}\cos\psi_{QP_0}+l_{QP_0})}[\frac{\partial l_{QP_0}}{\partial h_Q}+\frac{\partial r_Q}{\partial h_Q}-\frac{\partial(r_{P_0}\cos\psi_{QP_0})}{\partial h_Q}]\\&+\frac{1}{r_Q^3}[2r_{P_0}\cos\psi_{QP_0}\frac{\partial r_Q}{\partial h_Q}-r_Q\frac{\partial(r_{P_0}\cos\psi_{QP_0})}{\partial h_Q}][5+3\ln\frac{r_Q-r_{P_0}\cos\psi_{QP_0}+l_{QP_0}}{2r_Q}]\\&=-\frac{2}{l_{QP_0}^2}\frac{\partial l_{QP_0}}{\partial h_Q}+K_1(Q,P_0)\end{aligned} \tag{3.1.5}$$

where

$$\begin{aligned}K_1(Q,P_0)&=\frac{1}{r_Q^3}[6l_{QP_0}\frac{\partial r_Q}{\partial h_Q}-3r_Q\frac{\partial l_{QP_0}}{\partial h_Q}-r_Q\frac{\partial r_Q}{\partial h_Q}+3r_{P_0}\cos\psi_{QP_0}\frac{\partial r_Q}{\partial h_Q}]\\&-\frac{3r_{P_0}\cos\psi_{QP_0}}{r_Q^2(r_Q-r_{P_0}\cos\psi_{QP_0}+l_{QP_0})}[\frac{\partial l_{QP_0}}{\partial h_Q}+\frac{\partial r_Q}{\partial h_Q}-\frac{\partial(r_{P_0}\cos\psi_{QP_0})}{\partial h_Q}]\\&+\frac{1}{r_Q^3}[2r_{P_0}\cos\psi_{QP_0}\frac{\partial r_Q}{\partial h_Q}-r_Q\frac{\partial(r_{P_0}\cos\psi_{QP_0})}{\partial h_Q}][5+3\ln\frac{r_Q-r_{P_0}\cos\psi_{QP_0}+l_{QP_0}}{2r_Q}]\end{aligned} \tag{3.1.6}$$

Following Moritz (1980), when P_0 goes to P (the projection of P_0 on S_e) from the inner of S_e, then

$$\int_{S_e} -T(Q)\frac{2}{l_{QP_0}^2}\frac{\partial l_{QP_0}}{\partial h_Q}dS_e = \int_{S_e} 2T(Q)\frac{\partial}{\partial h_Q}(\frac{1}{l_{QP_0}})dS_e \rightarrow -4\pi T(P) + \int_{S_e} -T(Q)\frac{2}{l_{QP}^2}\frac{\partial l_{QP}}{\partial h_Q}dS_e \tag{3.1.7}$$

and

$$\int_{S_e} T(Q)K_1(Q,P_0)dS_e \rightarrow \int_{S_e} T(Q)K_1(Q,P)dS_e \tag{3.1.8}$$

$$\int_{S_e} T(Q)\frac{1}{\gamma_Q}\frac{\partial \gamma_Q}{\partial h_Q}S(Q,P_0)dS_e \rightarrow \int_{S_e} T(Q)\frac{1}{\gamma_Q}\frac{\partial \gamma_Q}{\partial h_Q}S(Q,P)dS_e \tag{3.1.9}$$

$$\int_{S_e} \Delta g(Q)S(Q,P_0)dS_e \rightarrow \int_{S_e} \Delta g(Q)S(Q,P)dS_e \tag{3.1.10}$$

So for any given point P on S_e, we obtain by letting $P_0 \rightarrow P$ in (3.1.4) that

$$4\pi T(P) - \int_{S_e} T(Q)K(Q,P)dS_e = \int_{S_e} \Delta g(Q)S(Q,P)dS_e \tag{3.1.11}$$

where

$$K(Q,P) = \frac{\partial S(Q,P)}{\partial h_Q} - \frac{1}{\gamma_Q}\frac{\partial \gamma_Q}{\partial h_Q}S(Q,P) \tag{3.1.12}$$

Equation (3.1.11) is the integral equation that will be used for determining T on S_e.

3.1.2 Determination of the geoidal height

Denoting the projection of the surface element dS_{eQ} onto the unit sphere σ by $d\sigma_Q$, we have

$$dS_e = r_Q^2 \sec\beta_Q d\sigma_Q \tag{3.1.13}$$

where β_Q is the angle between the radius vector of Q and the surface normal of the surface S_e at point Q. Then, for any given point P on S_e, (3.1.11) becomes

$$4\pi T(P) - \int_\sigma T(Q)K(Q,P)r_Q^2 \sec\beta_Q d\sigma_Q = \int_\sigma \Delta g(Q)S(Q,P)r_Q^2 \sec\beta_Q d\sigma_Q \tag{3.1.14}$$

With b_e the semiminor axis and e the first eccentricity of the reference ellipsoid, and θ_P and θ_Q respectively the complements of the geocentric latitudes of P and Q, we have

$$r_P = b_e[1 + \frac{1}{2}e^2(1-\cos^2\theta_P) + O(e^4)] \tag{3.1.15a}$$

$$r_Q = b_e[1 + \frac{1}{2}e^2(1-\cos^2\theta_Q) + O(e^4)] \tag{3.1.15b}$$

$$l_{QP} = 2b_e \sin\frac{\psi_{QP}}{2}[1 + \frac{1}{4}e^2(2-\cos^2\theta_Q - \cos^2\theta_P) + O(e^4)] \tag{3.1.15c}$$

$$r_Q^2 \sec\beta_Q = b_e^2[1 + e^2(1-\cos^2\theta_Q) + O(e^4)] \tag{3.1.15d}$$

Furthermore, from Molodensky et al. (1962), we have

$$\frac{\partial r_Q}{\partial h_Q} = 1 + O(e^4) \tag{3.1.16a}$$

$$\frac{\partial l_{QP}}{\partial h_Q} = \sin\frac{\psi_{QP}}{2}[1 - \frac{1}{4}e^2(3\cos^2\theta_Q + \cos^2\theta_P - \frac{(\cos\theta_Q - \cos\theta_P)^2}{\sin^2\frac{\psi_{QP}}{2}}) + O(e^4)] \tag{3.1.16b}$$

$$\frac{\partial(r_P\cos\psi_{QP})}{\partial h_Q} = e^2(\cos\theta_Q\cos\theta_P - \cos^2\theta_Q\cos\psi_{QP}) + O(e^4) \tag{3.1.16c}$$

$$-\frac{1}{\gamma_Q}\frac{\partial\gamma_Q}{\partial h_Q} = \frac{2}{b_e}[1 - e^2(\cos^2\theta_Q - \frac{1}{2}) + O(e^4)] \tag{3.1.16d}$$

It then follows from (3.1.1) that

$$S(Q,P)r_Q^2\sec\beta_Q = b_e[S(\psi_{QP}) + e^2 f_1(\psi_{QP},\theta_Q,\theta_P) + O(e^4)] \tag{3.1.17}$$

where $S(\psi_{QP})$ is the Stokes function

$$S(\psi_{QP}) = \frac{1}{\sin\frac{\psi_{QP}}{2}} - 6\sin\frac{\psi_{QP}}{2} + 1 - \cos\psi_{QP}[5 + 3\ln(\sin\frac{\psi_{QP}}{2} + \sin^2\frac{\psi_{QP}}{2})] \tag{3.1.18}$$

and

$$f_1(\psi_{QP},\theta_Q,\theta_P) = \frac{1}{2}S(\psi_{QP})(\sin^2\theta_P) + (\cos^2\theta_Q - \cos^2\theta_P)(3\sin^2\frac{\psi_{QP}}{2} - 2) \tag{3.1.19}$$

So

$$\begin{aligned}-\frac{1}{\gamma_Q}\frac{\partial\gamma_Q}{\partial h_Q}S(Q,P)r_Q^2\sec\beta_Q &= \frac{2}{b_e}[1 - e^2(\cos^2\theta_Q - \frac{1}{2}) + O(e^4)]b_e[S(\psi_{QP}) \\ &\quad + e^2 f_1(\psi_{QP},\theta_Q,\theta_P) + O(e^4)] \\ &= 2S(\psi_{QP}) + e^2 f_2(\psi_{QP},\theta_Q,\theta_P) + O(e^4)\end{aligned} \tag{3.1.20}$$

where

$$\begin{aligned}f_2(\psi_{QP},\theta_Q,\theta_P) &= S(\psi_{QP})(2\sin^2\theta_Q - \cos^2\theta_P) \\ &\quad + 2(\cos^2\theta_Q - \cos^2\theta_P)(3\sin^2\frac{\psi_{QP}}{2} - 2)\end{aligned} \tag{3.1.21}$$

From (3.1.5), we obtain

$$r_Q^2\sec\beta_Q\frac{\partial S(Q,P)}{\partial h_Q} = -2S(\psi_{QP}) + 1 + 3\cos\psi_{QP} + e^2 f_3(\psi_{QP},\theta_Q,\theta_P) + O(e^4) \tag{3.1.22}$$

where

$$
\begin{aligned}
f_3(\psi_{QP},\theta_Q,\theta_P) = & -\frac{(\cos\theta_Q-\cos\theta_P)^2}{8\sin^3\frac{\psi_{QP}}{2}} + \frac{3(\cos\theta_Q\cos\theta_P-\cos^2\theta_Q)}{4\sin^2\frac{\psi_{QP}}{2}} \\
& + \frac{1}{\sin\frac{\psi_{QP}}{2}}[-\frac{25}{8}\cos^2\theta_Q+\frac{5}{8}\cos^2\theta_P+3\cos\theta_Q\cos\theta_P] \\
& + [\frac{137}{8}\cos^2\theta_Q-\frac{73}{8}\cos^2\theta_P-\frac{13}{2}\cos\theta_Q\cos\theta_P] \\
& + \sin\frac{\psi_{QP}}{2}[\frac{57}{4}\cos^2\theta_Q-\frac{21}{4}\cos^2\theta_P-3\cos\theta_P\cos\theta_Q] \\
& + \sin^2\frac{\psi_{QP}}{2}[-\frac{125}{4}\cos^2\theta_Q+\frac{73}{4}\cos^2\theta_P]-6\sin^3\frac{\psi_{QP}}{2}\cos^2\theta_Q \\
& + \frac{3\cos\psi_{QP}}{8(1+\sin\frac{\psi_{QP}}{2})}[\frac{(\cos\theta_Q-\cos\theta_P)^2}{\sin\frac{\psi_{QP}}{2}}-4\cos\theta_Q\cos\theta_P+4\cos^2\theta_Q \\
& \qquad -(3\cos^2\theta_Q+\cos^2\theta_P)\sin\frac{\psi_{QP}}{2}-8\cos^2\theta_Q\sin^2\frac{\psi_{QP}}{2}] \\
& + 3\ln(\sin\frac{\psi_{QP}}{2}+\sin^2\frac{\psi_{QP}}{2})[\cos\psi_{QP}(2\cos^2\theta_Q-\cos^2\theta_P)-\cos\theta_Q\cos\theta_P]
\end{aligned}
\tag{3.1.23}
$$

Therefore, from (3.1.12), (3.1.14), (3.1.17), (3.1.20) and (3.1.22), we obtain

$$
\begin{aligned}
& 4\pi T(P)-\int_\sigma T(Q)(1+3\cos\psi_{QP})d\sigma_Q - e^2\int_\sigma T(Q)[f_2(\psi_{QP},\theta_Q,\theta_P)+f_3(\psi_{QP},\theta_Q,\theta_P)]d\sigma_Q \\
& = b_e\int_\sigma \Delta g(Q)S(\psi_{QP})d\sigma_Q + e^2 b_e\int_\sigma \Delta g(Q)f_1(\psi_{QP},\theta_Q,\theta_P)d\sigma_Q + O(e^4)
\end{aligned}
\tag{3.1.24}
$$

The second condition in (1.4.1) means that the disturbing potential T does not contain the spherical harmonics of degrees one and zero. So we have

$$\int_\sigma T(Q)d\sigma_Q = \int_\sigma T(Q)P_0(\cos\psi_{QP})d\sigma_Q = 0 \tag{3.1.25}$$

$$\int_\sigma T(Q)\cos\psi_{QP}d\sigma_Q = \int_\sigma T(Q)P_1(\cos\psi_{QP})d\sigma_Q = 0 \tag{3.1.26}$$

Thus (3.1.24) becomes

$$\begin{aligned}4\pi T(P) = b_e\int_\sigma \Delta g(Q)S(\psi_{QP})d\sigma_Q + e^2\{b_e\int_\sigma \Delta g(Q)f_1(\psi_{QP},\theta_Q,\theta_P)d\sigma_Q \\ + \int_\sigma T(Q)[f_2(\psi_{QP},\theta_Q,\theta_P) + f_3(\psi_{QP},\theta_Q,\theta_P)]d\sigma_Q\} + O(e^4)\end{aligned} \tag{3.1.27}$$

Let

$$T(P) = T_0(P) + e^2T_1(P) + O(e^4) \tag{3.1.28}$$

Then from (3.1.27), we obtain

$$T_0(P) = \frac{b_e}{4\pi}\int_\sigma \Delta g(Q)S(\psi_{QP})d\sigma_Q \tag{3.1.29}$$

$$T_1(P) = \frac{1}{4\pi}\{b_e\int_\sigma \Delta g(Q)f_1(\psi_{QP},\theta_Q,\theta_P)d\sigma_Q + \int_\sigma T_0(Q)f_0(\psi_{QP},\theta_Q,\theta_P)d\sigma_Q\} \tag{3.1.30}$$

where

$$\begin{aligned}f_0(\psi_{QP},\theta_Q,\theta_P) &= f_2(\psi_{QP},\theta_Q,\theta_P) + f_3(\psi_{QP},\theta_Q,\theta_P) \\ &= -\frac{(\cos\theta_Q - \cos\theta_P)^2}{8\sin^3\frac{\psi_{QP}}{2}} + \frac{3(\cos\theta_Q\cos\theta_P - \cos^2\theta_Q)}{4\sin^2\frac{\psi_{QP}}{2}}\end{aligned}$$

$$+\frac{1}{\sin\frac{\psi_{QP}}{2}}[2-\frac{41}{8}\cos^2\theta_Q-\frac{3}{8}\cos^2\theta_P+3\cos\theta_Q\cos\theta_P]$$

$$+[-8+\frac{169}{8}\cos^2\theta_Q-\frac{9}{8}\cos^2\theta_P-\frac{13}{2}\cos\theta_Q\cos\theta_P]$$

$$+\sin\frac{\psi_{QP}}{2}[-12+\frac{105}{4}\cos^2\theta_Q+\frac{3}{4}\cos^2\theta_P-3\cos\theta_Q\cos\theta_P]$$

$$+\sin^2\frac{\psi_{QP}}{2}[20-\frac{181}{4}\cos^2\theta_Q+\frac{9}{4}\cos^2\theta_P]-6\sin^3\frac{\psi_{QP}}{2}\cos^2\theta_Q$$

$$+\frac{3\cos\psi_{QP}}{8(1+\sin\frac{\psi_{QP}}{2})}[\frac{(\cos\theta_Q-\cos\theta_P)^2}{\sin\frac{\psi_{QP}}{2}}-4\cos\theta_Q\cos\theta_P+4\cos^2\theta_Q$$

$$-(3\cos^2\theta_Q+\cos^2\theta_P)\sin\frac{\psi_{QP}}{2}-8\cos^2\theta_Q\sin^2\frac{\psi_{QP}}{2}]$$

$$-3\ln(\sin\frac{\psi_{QP}}{2}+\sin^2\frac{\psi_{QP}}{2})\,[\cos\theta_Q\cos\theta_P+(2-4\cos^2\theta_Q)\cos\psi_{QP}] \tag{3.1.31}$$

According to the Bruns formula, we obtain

$$N(P)=N_0'(P)+e^2N_1'(P)+O(e^4) \tag{3.1.32}$$

with the spherical geoidal height

$$N_0'(P)=\frac{b_e}{4\pi\gamma_P}\int_\sigma\Delta g(Q)S(\psi_{QP})d\sigma_Q \tag{3.1.33}$$

and its ellipsoidal correction

$$N_1'(P)=\frac{1}{4\pi}\{\frac{b_e}{\gamma_P}\int_\sigma\Delta g(Q)f_1(\psi_{QP},\theta_Q,\theta_P)d\sigma_Q+\int_\sigma N_0'(Q)f_0(\psi_{QP},\theta_Q,\theta_P)d\sigma_Q\} \tag{3.1.34}$$

where f_1 is defined by (3.1.19) and f_0 is obtained from (3.1.31), (3.1.21) and (3.1.23).

Now we further discuss the first term of N_1':

$$N_{11}'(P) = \frac{b_e}{4\pi\gamma_P} \int_\sigma \Delta g(Q) f_1(\psi_{QP}, \theta_Q, \theta_P) d\sigma_Q \tag{3.1.35}$$

From (3.1.19), (3.1.33), (3.1.25) and (3.1.26), we have that

$$N_{11}'(P) = \frac{\sin^2\theta_P}{2} N_0'(P) - \frac{b_e}{4\pi\gamma_P} \int_\sigma \Delta g(Q) \frac{\cos^2\theta_Q (1 + 3\cos\psi_{QP})}{2} d\sigma_Q \tag{3.1.36}$$

From sections 1-11 to 1-14 of Heiskanen and Moritz (1967), we can obtain that

$$\begin{aligned}\frac{\cos^2\theta_Q(1+3\cos\psi_{QP})}{2} = \frac{1}{6} + \frac{\sqrt{5}}{15}\overline{R}_{20}(\theta_Q,\lambda_Q) + \cos\theta_P[\frac{3\sqrt{7}}{35}\overline{R}_{30}(\theta_Q,\lambda_Q) + \frac{3\sqrt{3}}{10}\overline{R}_{10}(\theta_Q,\lambda_Q)] \\ + \sin\theta_P\cos\lambda_P[\frac{\sqrt{42}}{35}\overline{R}_{31}(\theta_Q,\lambda_Q) + \frac{\sqrt{3}}{10}\overline{R}_{11}(\theta_Q,\lambda_Q)] \\ + \sin\theta_P\sin\lambda_P[\frac{\sqrt{42}}{35}\overline{S}_{31}(\theta_Q,\lambda_Q) + \frac{\sqrt{3}}{10}\overline{S}_{11}(\theta_Q,\lambda_Q)]\end{aligned} \tag{3.1.37}$$

We now represent Δg by the spherical harmonic expansion

$$\Delta g(Q) = G\sum_{n=2}^{\infty}(n-1)\sum_{m=0}^{n}[\delta J_{nm}\overline{R}_{nm}(\theta_Q,\lambda_Q) + \delta K_{nm}\overline{S}_{nm}(\theta_Q,\lambda_Q)] \tag{3.1.38}$$

where G is the mean gravity, $\delta J_{nm}, \delta K_{nm}$ are the fully normalized geopotential coefficients of the disturbing potential and $\overline{R}_{nm}(\theta_Q,\lambda_Q), \overline{S}_{nm}(\theta_Q,\lambda_Q)$ are the fully normalized Legendre harmonics.. Then from the orthogonality of the harmonics $\overline{R}_{nm}(\theta_Q,\lambda_Q)$ and $\overline{S}_{nm}(\theta_Q,\lambda_Q)$, we obtain

$$N'_{11}(P) = \frac{\sin^2\theta_P}{2} N'_0(P) - \frac{Gb_e}{\gamma_P}[\frac{\sqrt{5}}{15}\delta J_{20} + \cos\theta_P \frac{3\sqrt{7}}{35}\delta J_{30}$$

$$+ \sin\theta_P \cos\lambda_P \frac{\sqrt{42}}{35}\delta J_{31} + \sin\theta_P \sin\lambda_P \frac{\sqrt{42}}{35}\delta K_{31}] \tag{3.1.39}$$

So, finally, the ellipsoidal correction is expressed as

$$N'_1(P) = N'_{11}(P) + \frac{1}{4\pi}\int_\sigma N'_0(Q) f_0(\psi_{QP},\theta_Q,\theta_P) d\sigma_Q \tag{3.1.40}$$

Equations (3.1.32), (3.1.33), (3.1.39), (3.1.40) and (3.1.31) are the formulas for computing the ellipsoidal geoidal height with an accuracy of the order of $O(e^4)$.

Rigorously, equation (3.1.33) is not the same as the standard Stokes's formula because the semiminor axis b_e is used instead of the mean radius R. According to the definition of R (see (1.1.24)),

$$b_e = R(1 - \frac{1}{3}e^2 + O(e^4))$$

Therefore from (3.1.32), (3.1.33), (3.1.39) and (3.1.40), we obtain

$$N(P) = N_0(P) + e^2 N_1(P) + O(e^4) \tag{3.1.41}$$

$$N_0(P) = \frac{R}{4\pi\gamma_P}\int_\sigma S(\psi_{QP})\Delta g(Q) d\sigma \tag{3.1.42}$$

$$N_1(P) = N_{11}(P) + N_{12}(P) \tag{3.1.43}$$

where

$$N_{11}(P) = [\frac{\sin^2\theta_P}{2} - \frac{1}{3}]N_0(P) - \frac{GR}{\gamma_P}[\frac{\sqrt{5}}{15}\delta J_{20} + \cos\theta_P \frac{3\sqrt{7}}{35}\delta J_{30}$$

$$+\sin\theta_P \cos\lambda_P \frac{\sqrt{42}}{35}\delta J_{31} + \sin\theta_P \sin\lambda_P \frac{\sqrt{42}}{35}\delta K_{31}] \qquad (3.1.44)$$

$$N_{12}(P) = \frac{1}{4\pi}\int_\sigma N_0(Q) f_0(\psi_{QP}, \theta_Q, \theta_P) d\sigma_Q \qquad (3.1.45)$$

and f_0 is still defined by (3.1.31).

Thus we finally obtain the formulas for computing the ellipsoidal geoidal height, which is expressed as the standard Stokes formula (3.1.42) plus an ellipsoidal correction (3.1.43).

3.2 A brief comparison of Molodensky's method, Moritz's method, Martinec and Grafarend's method and the method in this chapter

The following Tables 3.1, 3.2 and 3.3 show the differences of the solutions in Molodensky et al. (1962), Moritz (1980), Martinec and Grafarend (1997) and our development.

Notes:

(i) The method used in this chapter follows the method used in Molodensky et al. (1962). The difference is that instead of using the general Stokes function S(P, P_0)

Table 3.1 Differences of the solutions in Molodensky et al. (1962), Moritz (1980), Martinec and Grafarend (1997) and this work

	Molodensky et al.	Moritz	Martinec & Grafarend	This work
Regularity condition	$T(P)=O(\frac{1}{r_P^3})$ $(r_P\to\infty)$	$T(P)=O(\frac{1}{r_P^3})$ $(r_P\to\infty)$	$T(P)=\frac{c}{r_P}+O(\frac{1}{r_P^3})$ $(r_P\to\infty)$	$T(P)=O(\frac{1}{r_P^3})$ $(r_P\to\infty)$
Boundary condition	$\frac{\partial T(P)}{\partial h_P}-\frac{1}{\gamma_P}\frac{\partial\gamma_P}{\partial h_P}T(P)=-\Delta g(P)$	$\frac{\partial T(P)}{\partial h_P}-\frac{1}{\gamma_P}\frac{\partial\gamma_P}{\partial h_P}T(P)=-\Delta g(P)$	$\frac{\partial T(P)}{\partial h_P}+\frac{2}{b}T(P)=-\Delta g(P)$	$\frac{\partial T(P)}{\partial h_P}-\frac{1}{\gamma_P}\frac{\partial\gamma_P}{\partial h_P}T(P)=-\Delta g(P)$
Ellipsoidal correction	$N_1(P)=\frac{a_e}{4\pi\gamma_P}\bullet[\int_\sigma\Delta g(Q)f_{M1}(Q,P)d\sigma+\int_\sigma\chi(Q)f_{M2}(Q,P)d\sigma+\int_\sigma T_0(Q)f_{M3}(Q,P)d\sigma]$ where $T_0(P)=\int_\sigma\Delta g(Q)S(\psi_{QP})d\sigma$ $\chi(P)=\int_\sigma T_0(Q)f_{M3}(Q,P)d\sigma$	$N_1(P)=\overline{N}_{11}(P)+\frac{R}{4\pi\gamma_P}\int_\sigma\Delta g^1(Q)S(\psi_{QP})d\sigma$ where $\overline{N}_{11}(P)=\frac{(1-3\sin^2\phi)}{4}\bullet\frac{R}{4\pi\gamma_P}\int_\sigma\Delta g(Q)S(\psi_{QP})d\sigma$ and $\Delta g^1(P)$ is given in Table 3.2	$N_1(P)=\frac{b_e}{4\pi\gamma_P}\bullet\int_\sigma\Delta g(Q)f_{MG}(Q,P)d\sigma$	$N_1(P)=N_{11}(P)+\frac{1}{4\pi}\int_\sigma N_0(Q)\,f_0(Q,P)\,d\sigma$ where $N_{11}(P)$ is given by (3.1.44) and $N_0(P)=\frac{R}{4\pi\gamma_P}\bullet\int_\sigma\Delta g(Q)S(\psi_{QP})d\sigma$
Kernel functions	f_{M1}, f_{M2} and f_{M3} are given in Table 3.3.	$S(\psi_{QP})$ is the Stokes function.	f_{MG} is given in Table 3.3.	f_0 is given by (3.1.31).

Table 3.2 The definition of Δg^1 in Moritz (1980)

$$\Delta g^1(Q)=\frac{1}{R}\sum_{n=2}^{\infty}\sum_{m=0}^{n}[G_{nm}R_{nm}(\phi_Q,\lambda_Q)+H_{nm}S_{nm}(\phi_Q,\lambda_Q)]$$

where $R_{nm}(\phi_Q,\lambda_Q), S_{nm}(\phi_Q,\lambda_Q)$ are Legendre surface harmonics and

$$G_{nm}=\kappa_{nm}A_{n-2,m}+\lambda_{nm}A_{nm}+\mu_{nm}A_{n+2,m}$$

$$H_{nm}=\kappa_{nm}B_{n-2,m}+\lambda_{nm}B_{nm}+\mu_{nm}B_{n+2,m}$$

where A_{nm}, B_{nm} are the coefficients of the spherical harmonic expansion of the disturbing potential T and

$$\kappa_{nm}=-\frac{3(n-3)(n-m-1)(n-m)}{2(2n-3)(2n-1)}$$

$$\lambda_{nm}=\frac{n^3-3m^2n-9n^2-6m^2-10n+9}{3(2n+3)(2n-1)}$$

$$\mu_{nm}=-\frac{(3n+5)(n+m+2)(n+m+1)}{2(2n+5)(2n+3)}$$

Table 3.3 The definitions of the kernel functions in Molodensky et al. (1962) and Martinec and Grafarend (1997)

$$f_{M1}(Q,P)=\frac{\cos^2\theta_Q-3\cos\theta_P}{4}S(\psi_{QP});\quad f_{M2}(Q,P)=\frac{3}{8\pi}S(\psi_{QP});$$

$$f_{M3}(Q,P)=\frac{1}{8\pi}\left[\frac{4-5\cos^2\theta_P}{\sin\frac{\psi_{QP}}{2}}-\frac{(\cos\theta_Q-\cos\theta_P)^2}{4\sin^3\frac{\psi_{QP}}{2}}\right];$$

$$\begin{aligned}f_{MG}(Q,P)=\ &\sin\bar\theta_P(\cos\bar\theta_P\sin\psi_{QP}\cos\psi_{QP}\cos\alpha_{QP}-\sin\bar\theta_P\cos^2\psi_{QP}\cos^2\alpha_{QP}\\&+\sin\bar\theta_P\sin^2\alpha_{QP})K_1(\cos\psi_{QP})+(1-\sin^2\bar\theta_P\sin^2\alpha_{QP})K_2(\cos\psi_{QP})\\&-\sin\bar\theta_P\cos\alpha_{QP}(\cos\bar\theta_P\sin\psi_{QP}-\sin\bar\theta_P\cos\psi_{QP}\cos\alpha_{QP})K_3(\cos\psi_{QP})-K_4(\cos\psi_{QP})\end{aligned}$$

where $\bar\theta_P$ is the reduced latitude of P and K_i (i=1,2,3,4) are given as follows:

$$K_1(t)=\sum_{j=3}^{\infty}\frac{2j-1}{(j-2)^2(2j+1)}\frac{dP_j(t)}{dt};\quad K_2(t)=\sum_{j=2}^{\infty}\frac{(j+1)^2(2j+1)}{(j-1)^2(2j+3)}P_j(t);$$

$$K_3(t)=\sum_{j=3}^{\infty}\frac{j(2j-1)}{(j-2)^2(2j+1)}P_j(t);\quad K_4(t)=\sum_{j=2}^{\infty}\frac{(j+1)(2j+1)}{(j-1)^2}P_j(t).$$

(see (3.1.1)), Molodensky et al. (1962) used the function $1/l_{PP_0}$ as the kernel function of the equation (3.1.3).

(ii) The regularity condition used in Molodensky et al. (1962), Moritz (1980) and this chapter is the same as that used in the derivation of Stokes's formula (see Heiskanen and Moritz, 1967). This condition is stronger than that used in Martinec and Grafarend (1997).

From the derivations in section 2, we see that the regularity condition in (3.0.1) is used to make (3.1.25) and (3.1.26) hold so that we can get (3.1.27) from (3.1.24). However, if we substitute $S(Q,P_0)$ in (3.1.3) by $S(Q,P_0)-1/r_Q$, which is also harmonic outside S_e according to Moritz (1980), then the term $\int_\sigma T(Q)d\sigma_Q$ will disappear in an equation corresponding to (3.1.24). What we still need to do is to make (3.1.26) hold. Obviously, this can be guaranteed by the more general regularity condition used in Martinec and Grafarend (1997). The spherical geoidal height will be given by the general Stokes formula (see Heiskanen and Moritz, 1967) and the ellipsoidal correction will be somewhat different than that given in section 2.

At present time, the mass of the Earth can be estimated very accurately. By properly selecting the normal gravity field, we can easily make the disturbing potential T satisfy the regular condition in (3.0.1). So the difference between the two regularity conditions is not a key problem.

(iii) The boundary condition used in Martinec and Grafarend (1997) is somewhat different than the boundary condition used in Molodensky et al. (1962), Moritz (1980) and this chapter. The difference is

$$\frac{T(Q)}{b_e}[e^2 \cos 2\theta_Q + O(e^4)] \tag{3.2.1}$$

From the derivations in subsection 3.1, we see that the effect of this difference on the ellipsoidal correction is

$$\frac{1}{4\pi}\int_\sigma N_0(Q)S(\psi_{QP})\cos 2\theta_Q d\sigma_Q \tag{3.2.2}$$

Martinec (1998) has shown that the term above has a small impact on the ellipsoidal correction because it is characterized by an integration kernel with a logarithmic singularity at $\psi_{QP}=0$ and it can be neglected in cm geoid computation if a higher-degree reference field is introduced as a reference potential according to the numerical demonstration given by Cruz (1986).

(iv) All four solutions express the ellipsoidal geoidal height by the spherical geoidal height N_0 given by Stokes's formula plus the ellipsoidal correction N_1 given by closed integral formulas. The relative errors of the solutions are $O(e^4)$.

(v) In Molodensky et al. (1962), to evaluate N_1 at a single point from Δg, we need two auxiliary data sets T_0 and χ. First we integrate Δg to get auxiliary data set T_0; then we integrate T_0 to get another auxiliary data set χ; finally, we obtain N_1 from integrating Δg, T_0 and χ. That is:

$$\Delta g \longrightarrow T_0 \longrightarrow \chi \longrightarrow N_1$$

The auxiliary data sets T_0 and χ, except for T_0 at the computation point which can be further used to compute the final ellipsoidal geoidal height, are useless after computing N_1. So the solution in Molodensky et al. (1962) is very computation-intensive even through the kernel functions f_{M1}, f_{M2} and f_{M3} are simple analytical functions.

(vi) In Moritz (1980), only one auxiliary data set Δg^1 is needed and the kernel function is also a simple analytical function, but Δg^1 is only expressed by an infinite summation of the coefficients A_{nm}, B_{nm} of the spherical harmonic expansion of the disturbing potential T.

The coefficients A_{nm}, B_{nm}, however, are what we want to know. They are not the coefficients $\delta J_{nm}, \delta K_{nm}$ of the spherical harmonic expansion of the spherical approximation disturbing potential T_0 corresponding to N_0, which can be computed from the gravity data by means of the spherical approximation formulas.

In the practical evaluation, Δg^1 is approximately computed using truncated spherical harmonic coefficients $\{\delta J_{nm}, \delta K_{nm}\}_{n=2}^{n_{max}}$.

(vii) In Martinec and Grafarend (1997), no auxiliary data set is needed. We can directly integrate Δg to obtain N_1:

$$\Delta g \longrightarrow N_1$$

So this solution is much simpler than that in Molodensky et al. (1962) and Moritz (1980). However, the simplicity is obtained by complicating the kernel function in its solution. From table 3.3, we see that the kernel function f_{MG} contains the series of Legendre polynomials and their derivatives, so it is obviously more complicated than the kernel functions in Molodensky et al. (1962) and Moritz (1980).

(viii) In this chapter, one auxiliary data set N_0 is needed to evaluate N_1 from Δg. We first integrate Δg to get N_0, the 'auxiliary' data set; then we obtain N_1 from N_0 and the first 3 degree harmonic coefficients plus an integral about N_0. That is:

$$\Delta g \longrightarrow N_0 \longrightarrow N_1$$

Like in Molodensky et al. (1962), the kernel function in the integral of this solution is a simple analytical function. So this solution is simpler than the solution of Molodensky et al. (1962) in the sense that only one auxiliary data set is needed for the evaluation of N_1 from Δg.

Because of the need of an auxiliary data set, it seems that this solution is more complex than the solution in Martinec and Grafarend (1997). However, the 'auxiliary' data N_0 are nothing else but the spherical geoidal heights, which are already available in many areas of the world, such as in Europe and North America. When we evaluate N_1 in such areas, this solution is simpler than the solution in Martinec and Grafarend (1997) in the sense that we can directly evaluate N_1 from N_0 with a simple analytical function.

This solution is similar to the solution in Moritz (1980). They both need an auxiliary data set. Their kernel functions are simple analytical functions and have the same degree of singularity at the origin. However, the auxiliary data set N_0 in this solution is much simpler than the auxiliary data set Δg^1 in Moritz (1980) in the sense that:

(a) N_0 can be computed directly from gravity anomaly data by means of Stokes's formula;

(b) N_0 can also be computed approximately from the geopotential model

$$N_0(P) = R\sum_{n=2}^{\infty}\sum_{m=0}^{n}[\delta J_{nm}\overline{R}_{nm}(\theta_P,\lambda_P)+\delta K_{nm}\overline{S}_{nm}(\theta_P,\lambda_P)] \tag{3.2.3}$$

Obviously, this formula is simpler than that used for computing Δg^1 (see Table 3.2) and N_0 is less sensitive to high degree coefficients than Δg^1 is; and

(c) N_0 is already available globally with resolutions of less than 1 degree and locally with higher resolutions.

3.3 Practical computation of the ellipsoidal correction

In the above section, we obtained the Stokes formula (3.1.42) and its ellipsoidal correction (3.1.43). The ellipsoidal correction term N_1 consists of two components: N_{11} and N_{12}. The component N_{11} is a simple analytical function about the spherical geoidal height N_0 and the first three degree spherical harmonic coefficients of the disturbing potential. It is easy to be evaluated from equation (3.1.44). The component N_{12}, called the integral term, is expressed by a closed integral formula (3.1.45). In this section, we will further discuss the ellipsoidal correction formula (3.1.43).

3.3.1 Singularity of the integral term in the ellipsoidal correction formula

The kernel function f_0 of the integral term is singular at $\psi_{QP}=0$ (see figure 3.1) because it contains the factors

$$\frac{(\cos\theta_Q-\cos\theta_P)^2}{\sin^3\frac{\psi_{QP}}{2}};\ \frac{\cos\theta_Q-\cos\theta_P}{\sin^2\frac{\psi_{QP}}{2}};\ \frac{1}{\sin\frac{\psi_{QP}}{2}};\ \ln(\sin\frac{\psi_{QP}}{2}+\sin^2\frac{\psi_{QP}}{2}) \qquad (3.3.1)$$

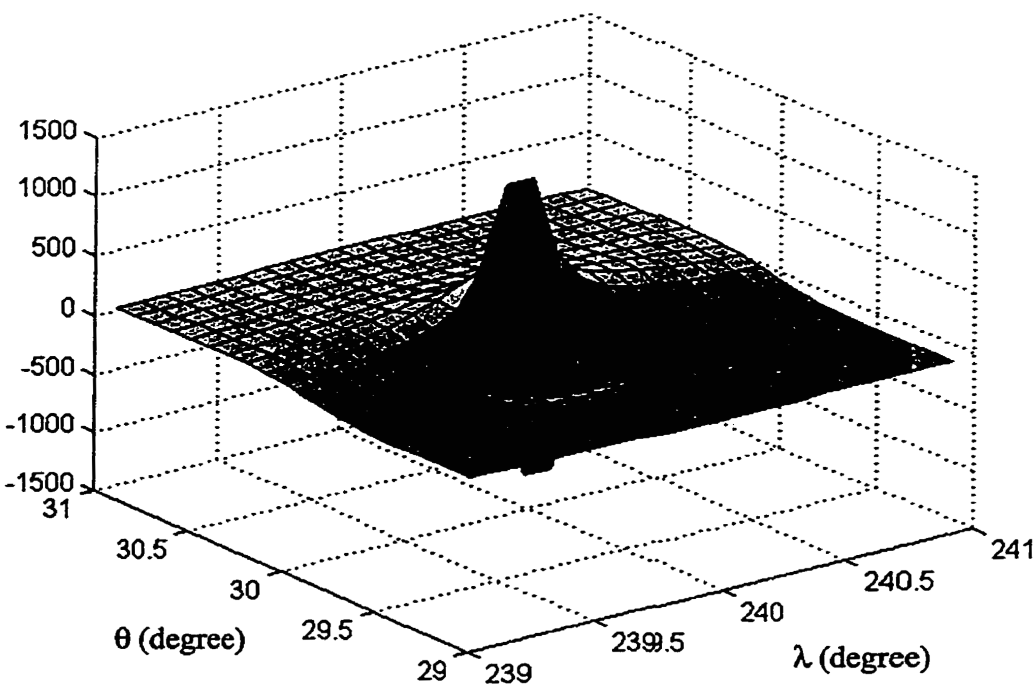

Figure 3.1 Behavior of kernel function f_0 of the ellipsoidal correction to Stokes's formula in the neighborhood of ($\theta_P=30^0$, $\lambda_P=240^0$)

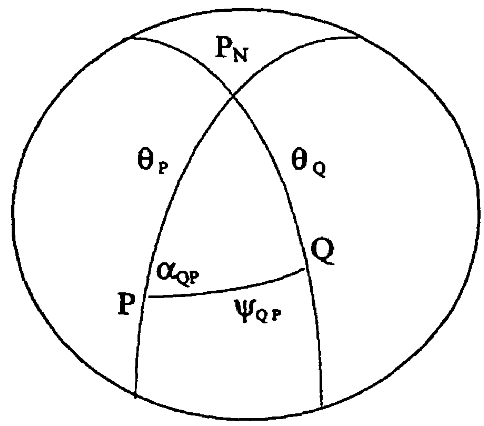

Figure 3.2 Spherical triangle

From the spherical triangle of figure 3.2, we have

$$\cos\theta_Q = \cos\psi_{QP}\cos\theta_P + \sin\psi_{QP}\sin\theta_P\cos\alpha_{QP} \tag{3.3.2}$$

So

$$\cos\theta_Q - \cos\theta_P = 2\sin\frac{\psi_{QP}}{2}[\cos\frac{\psi_{QP}}{2}\cos\alpha_{QP}\sin\theta_P - \sin\frac{\psi_{QP}}{2}\cos\theta_P] \tag{3.3.3}$$

It then follows from (3.1.31) that

$$f_0(\psi_{QP},\theta_Q,\theta_P) \approx \frac{4 - 5\cos^2\theta_P - (\sin\theta_P\cos\alpha_{QP})^2 - \frac{3}{2}\sin 2\theta_P\cos\alpha_{QP}}{\psi_{QP}} \quad (\psi_{QP} << 1) \tag{3.3.4}$$

This means that the kernel function f_0 has the same degree of singularity at $\psi_{QP} = 0$ as the Stokes function $S(\psi_{QP})$. So the integral component in (3.1.43) is a weakly singular integral and the singularity can be treated by the method used for Stokes's integral (see Heiskanen and Moritz, 1967, or the following subsection).

3.3.2 Method for applying the ellipsoidal correction formula

In the ellipsoidal correction formula (3.1.43), the first term N_{11} is easy to compute from N_0 and the first 3-degree harmonic coefficients. The term $N_{12}(P)$ is a global integral formula. The input data is the "spherical" geoidal height data. Since the high-resolution and high-accuracy "spherical" geoidal height data are only given in some local areas and the kernel function has same degree of singularity at the computing point P as the Stokes function, we will use the following well known method to evaluate the term $N_{12}(P)$.

From the definitions (3.1.31) of the kernel function f_0 of $N_{12}(P)$, we know that, like the Stokes function, f_0 quickly decreases when ψ_{QP} goes from 0 to π. Therefore, in practical evaluation of the integral term, we divide σ into two parts: σ_{near} and σ_{far}, where the area σ_{near} is usually a square area containing the computation point P as its centre. Since the kernel function is larger over σ_{near}, the integral over σ_{near} should be carefully computed using a high resolution and high accuracy spherical geoid model obtained from the ground gravity data by means of Stokes's formula (3.1.42) if a high accuracy geoid is required. The area σ_{far} is far from the computation point P, so the kernel function is relatively small over σ_{far}. Therefore, in the computation of the integral over σ_{far}, we can use the spherical geoidal height data N_0 computed from a global geopotential model. In detail, we express $N_{12}(P)$ as

$$N_{12}(P) = N_{far}(P) + N_{near}(P) \tag{3.3.5}$$

where

$$N_{far}(P) = \frac{e^2}{4\pi}\int_{\sigma_{far}} N_0(Q) f_0(\psi_{QP}, \theta_Q, \theta_P) d\sigma_Q \tag{3.3.6}$$

$$N_{near}(P) = \frac{e^2}{4\pi}\int_{\sigma_{near}} N_0(Q) f_0(\psi_{QP}, \theta_Q, \theta_P) d\sigma_Q \tag{3.3.7}$$

- **The formula for computing $N_{far}(P)$ from a global geoid model**

The evaluation of $N_{far}(P)$ is performed using a finite summation:

$$N_{far}(P) = \frac{e^2}{4\pi}\sum_{i,j} N_{ij}^0 f_{ij}^0 \sigma_{ij} \tag{3.3.8}$$

where σ_{ij} are the surface elements bordered by meridians and parallels both separated by c degrees and $\{\sigma_{ij}\} = \sigma_{far}$; N_{ij}^0 is the mean value of the spherical geoidal height N_0 for σ_{ij}; f_{ij}^0 is the mean value of the kernel function f_0 for σ_{ij}; and

$$\sigma_{ij} = \frac{\pi^2}{180^2} c^2 \sin\theta_i \tag{3.3.9}$$

where θ_i is the mid-latitude of σ_{ij}, and c, the side length (degree) of the grids in far-zone, is related to the resolution of the global geoid model.

- **The formula for computing $N_{near}(P)$ from a local geoid model**

The evaluation of $N_{near}(P)$ is also performed using a finite summation:

$$N_{near}(P)=\frac{e^2}{4\pi}\sum_{\substack{i,j\\ \sigma_{ij}\neq\sigma_P}} N_{ij}^0 f_{ij}^0 \sigma_{ij} + \frac{e^2}{4\pi}\int_{\sigma_P} N_0(Q) f_0(\psi_{QP},\theta_Q,\theta_P) d\sigma_Q \qquad (3.3.10)$$

where σ_{ij} are the surface elements bordered by meridians and parallels both separated by d degrees and $\{\sigma_{ij}\}=\sigma_{near}$; σ_P is the surface element containing the computing point P (the origin); N_{ij}^0 is the mean value of the spherical geoidal height N_0 for σ_{ij}; f_{ij}^0 is the mean value of the kernel function f_0 for σ_{ij}; and

$$\sigma_{ij}=\frac{\pi^2}{180^2} d^2 \sin\theta_i \qquad (3.3.11)$$

where θ_i is the mid-latitude of σ_{ij}, and d, the side length (degree) of the grids in near-zone, is related to the resolution of the local geoid model.

According to (3.3.4), in a small neighborhood of the computing point P, the kernel function f_0 can be expressed as:

$$f_0(\psi_{QP},\theta_Q,\theta_P)\approx\frac{2-\cos^2\theta_P-(\sin\theta_P\cos\alpha_{QP})^2-\frac{3}{2}\sin 2\theta_P\cos\alpha_{QP}}{\psi_{QP}} \qquad (3.3.12)$$

So we can obtain

$$\frac{1}{4\pi}\int_{\sigma_P} N_0(Q)f_0(\psi_{QP},\theta_Q,\theta_P)d\sigma_Q \approx \frac{(3-\cos^2\theta_P)}{4}\frac{\pi d}{360}N_0(P) \tag{3.3.13}$$

Thus (3.3.10) should be rewritten as

$$N_{near}(P) = \frac{e^2}{4\pi}\sum_{\sigma_{ij}\neq\sigma_P} N_{ij}^0 f_{ij}^0 \sigma_{ij} + \frac{(3-\cos^2\theta_P)}{4}\frac{\pi d}{360}N_0(P)e^2 \tag{3.3.14}$$

In σ_{near}, since the kernel function f_0 is larger than in σ_{far}, we have to use a high resolution and high accuracy geoid model for the evaluation of the integral if the required accuracy of the ellipsoidal correction is high.

3.3.3 A numerical test for the ellipsoidal correction formula

In this subsection, we will apply the ellipsoidal correction formula to the computation of the US geoid.

- **The input data**

In the test for computing the ellipsoidal correction, a global geoid model and a high-resolution and high-accuracy local geoid model are need.

The global geoid model used for computing N_{far} is the geoidal height grid computed at 1 degree spacing with the EGM96 spherical harmonic potential coefficient set complete to degree and order 360 (for more information about EGM96, please see its official webpage: http://cddisa.gsfc.nasa.gov/926/egm96/egm96.html).

The local geoid model used for computing N_{near} is the 2 arc minute geoidal height grid for the conterminous United States (GEOID96) (Smith and Milbert, 1999), computed from about 1.8 million terrestrial and marine gravity data.

- **Results and discussion**

  - **The size of the area σ_{near}**

    Figure 3.3 gives the relationship between the ellipsoidal correction N_1 and the side length of the square area σ_{near}. It shows that a global geoid model with a resolution of 1 degree is sufficient for the computation of the integral if the required accuracy is of the order of 1 cm.

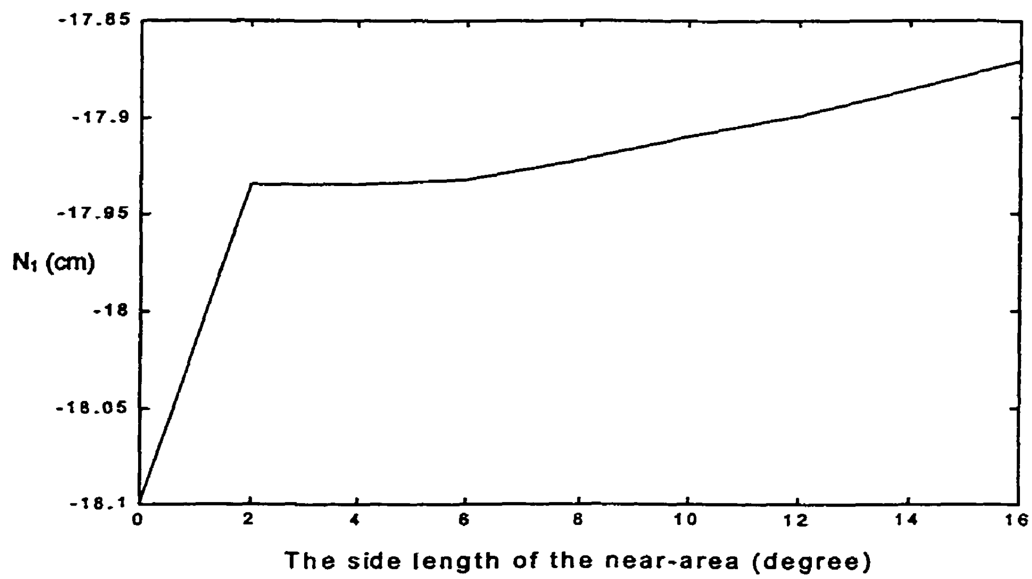

Figure 3.3 N_1 (cm) at P(45N, 240E) with different side length of the area σ_{near}

- **The contributions of N_{11} and N_{12}**

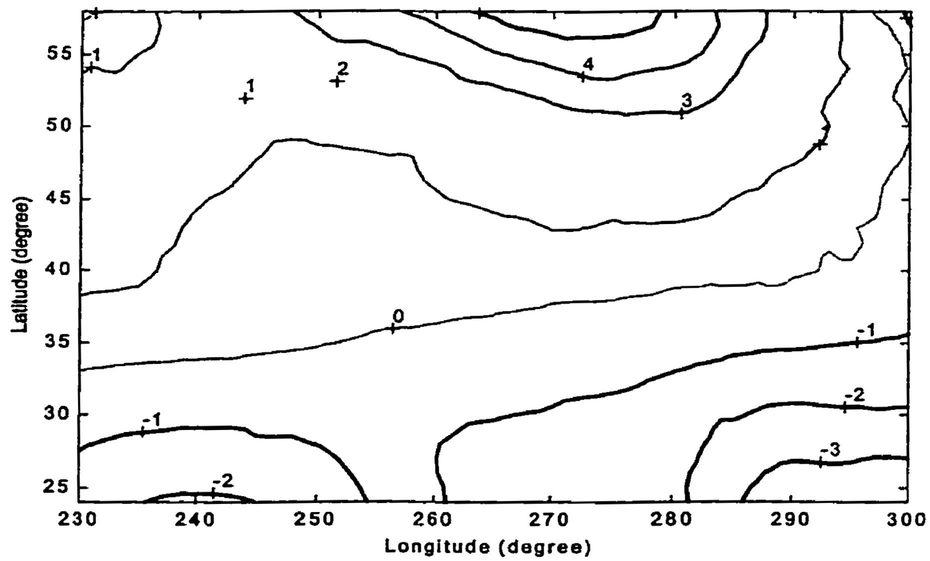

Figure 3.4 The contribution (cm) of the term N_{11} in the ellipsoidal correction N_1

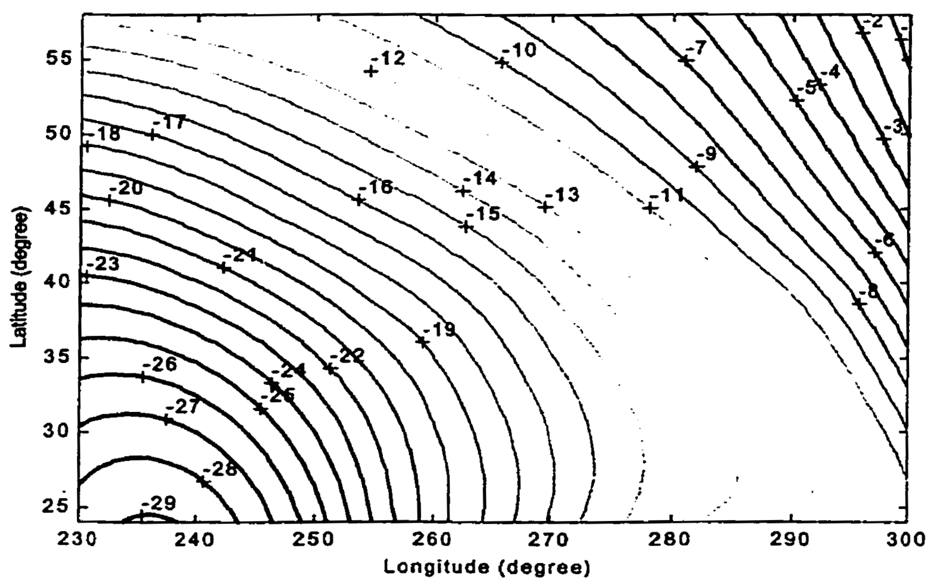

Figure 3.5 The contribution (cm) of the term N_{12} in the ellipsoidal correction N_1

Figure 3.4 and Figure 3.5 are respectively the maps of the term N_{11} and the integral term N_{12} in US (24^0-58^0N, 230^0-300^0E). In this area, N_{11} ranges from –3.8cm to 5.6cm and N_{12} ranges from –28cm to -2cm. So both N_{11} and N_{12} are important in the computation of N_1.

- **The contribution of the ellipsoidal correction**

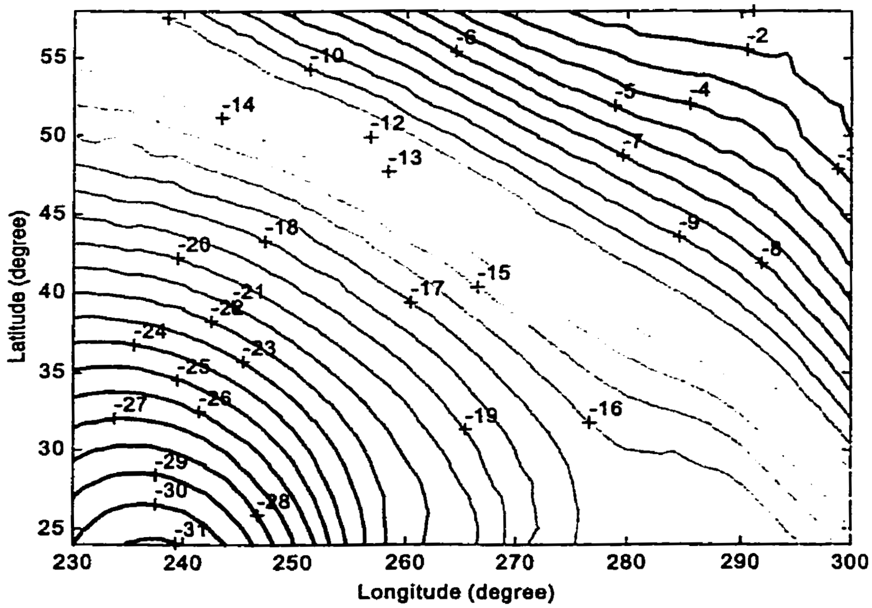

Figure 3.6 The contribution (cm) of the ellipsoidal correction

The above figure 3.6 is the map of the ellipsoidal correction in US. In this area, the ellipsoidal correction ranges from –31 cm to –1 cm.

3.4 Chapter summary

In this chapter, we investigated the ellipsoidal geodetic boundary value problem and gave a solution for the ellipsoidal geoidal heights.

- The solution, as those already available, generalizes the Stokes formula from the spherical boundary surface to the ellipsoidal boundary surface by adding an ellipsoidal correction to the Stokes formula. It makes the error of geoidal height decrease from $O(e^2)$ to $O(e^4)$, which can be neglected for most practical purposes.

- The ellipsoidal correction N_1 involves the spherical geoidal height N_0 and a kernel function which is a simple analytical function that has the same degree of singularity at the origin as the Stokes function.

- The solution is simpler than the solutions in Molodensky et al. (1962) and Moritz (1980). It is also simpler than the solution in Martinec and Grafarend (1997) when evaluating the ellipsoidal correction N_1 in an area where the spherical geoidal height N_0 has already been evaluated.

- A numerical test for the ellipsoidal correction formula shows that:

  - The effect of the flattening of the ellipsoid should be taken into account in the computation of the geoid when the required accuracy is better than the decimetre level.

  - The new ellipsoidal correction formula, which uses the spherical geoidal height data as its input data, is an effective formula.

- For the computation of the ellipsoidal correction with accuracy of the order of 1cm, a global geoid model with a resolution of 1 degree is sufficient.

- For more accurate ellipsoidal correction computation, a detailed local 'spherical' geoid model in the computation area is needed for the computation in the near-zone.

- The contribution of the ellipsoidal correction ranges from –31 cm to –1 cm in the conterminous United States.

4 Ellipsoidal Corrections to the Inverse Hotine/Stokes Formulas

The satellite altimetry technique provides direct measurements of sea surface heights with respect to the reference ellipsoid, the geometrical reference surface for elevations. Since 1973, a series of altimetry satellites such as SKYLAB, GEOS-3, SEASAT, GEOSAT, ERS-1, TOPEX, etc., have been launched and have collected data over the oceans. Owing to instrument improvement, geophysical and environmental correction improvement and radial orbit error reduction, the precision of satellite altimetry measurements has improved from the 3-metre to the 2-centimetre level. The resolution of satellite altimeter data along the tracks has also come down from 70 km to 20 km or less (see Zhang, 1993). Tremendous amounts of satellite altimeter data with very high precision have been collected since the advent of the satellite altimetry. After subtracting the dynamic sea surface topography, satellite altimetry can provide an estimation of the geoidal height N in ocean areas with a level of precision of about 10cm (Rummel and Haagmans, 1990). These geoidal height data can be used to recover the gravity disturbances and gravity anomalies over the oceans.

Papers reporting results on recovering the gravity information from satellite altimeter data, and in some cases, a review of prior work, include those of Zhang and Blais (1995), Hwang and Parsons (1995), Olgiati et al. (1995), Sandwell and Smith (1996) and Kim (1997). The models employed for recovering the gravity information from the satellite altimeter data are mainly the spherical harmonic expansion of the disturbing potential, the Hotine/Stokes formulas and the inverse Hotine/Stokes formulas. The gravity disturbances/anomalies obtained via these models may be called the spherical gravity disturbances/anomalies since these models are valid under the spherical approximation. In these models, the input and output data are supposed to be given on a sphere, the mean sphere. The geoidal height N (disturbing potential T) from altimetry and the gravity disturbance/anomaly $\delta g/\Delta g$ to be computed from N refer to the geoid which is very close to the reference ellipsoid S_e. They satisfy the following relations:

$$T \in H[S_e] \tag{4.0.1}$$

$$T(P) = O(\frac{1}{r_P^3}) \qquad \text{(P is at infinite)} \tag{4.0.2}$$

$$\frac{\partial}{\partial h_P} T(P) = -\delta g(P) \qquad \text{(P is on } S_e\text{)} \tag{4.0.3}$$

$$\Delta g(P) = \delta g(P) + \frac{1}{\gamma_P}\frac{\partial \gamma_P}{\partial h_P} \qquad \text{(P is on } S_e\text{)} \tag{4.0.4}$$

$$T(P) = \gamma_P N(P) \qquad \text{(P is on } S_e\text{)} \tag{4.0.5}$$

where r_P is the radius of point P and $\frac{\partial}{\partial h_P}$ is the derivative along the ellipsoidal normal direction of P.

The maximum difference between S_e and the geoid is about 100m, so we can treat the data given on the geoid as data on the reference ellipsoid. The relative error caused by doing so is about 10^{-4}. However, the relative error of substituting the reference ellipsoid by the mean sphere surface is about 3×10^{-3}. The effects of this error on the gravity anomaly and gravity disturbance, which are also called the effects of the Earth's flattening, may reach about 0.3 mGal. When the aim of the satellite altimetry is to recover the gravity information with accuracy less than 1 mGal, the effects of the Earth's flattening should be considered.

In order to reduce the effects of the Earth's flattening on the gravity anomaly, Wang (1999) proposed to add an ellipsoidal correction term to the spherical gravity anomaly recovered from the altimetry data via the inverse Stokes formula. The ellipsoidal correction is expressed by integral formulas and in series of spherical harmonic expansions. In the integral formulas, an auxiliary function χ is needed for computing the ellipsoidal correction Δg^{I} from the disturbing potential T, that is:

$$T \xrightarrow{\text{global integral}} \chi \xrightarrow{\text{global integral}} \Delta g^{I}$$

In this chapter, we will derive new ellipsoidal correction formulas to the spherical gravity disturbances/anomalies respectively. These ellipsoidal correction formulas consist of two parts: a simple function part and an integral part. The input data are the disturbing potentials and the spherical gravity disturbances/anomalies, which are already computed from altimetry data in some ocean areas with a high accuracy or are computed approximately from the Earth geopotential.

4.1 Formulas of the ellipsoidal corrections to the spherical gravity disturbance and the spherical gravity anomaly

In this section, we will

(a) establish an integral equation, which shows the relation between the geoidal heights and the gravity disturbances on the reference ellipsoid;

(b) solve the integral equation to get the formula of the ellipsoidal correction to the inverse Hotine formula (the spherical gravity disturbance); and

(c) derive the formula of the ellipsoidal correction to the inverse Stokes formula (the spherical gravity anomaly) from the result of (b);

4.1.1 Establishment of the integral equation

It can be proved that for an arbitrarily point P_0 given inside S_e, the function

$$F(Q,P_0) \equiv \frac{r_Q^2 - r_{P_0}^2}{r_{P_0} l_{QP_0}^3} = 2\frac{\partial}{\partial r_{P_0}}(\frac{1}{l_{QP_0}}) + \frac{1}{r_{P_0}}(\frac{1}{l_{QP_0}}) \tag{4.1.1}$$

satisfies

$$F(Q,P_0) \in H[S_e] \tag{4.1.2}$$

According to Green's second identity (Heiskanen and Moritz, 1962), we obtain that for an arbitrary function $V \in H[S_e]$,

$$\int_{S_e} V(Q)\frac{\partial F(Q,P_0)}{\partial h_Q} dS_{eQ} = \int_{S_e} \frac{\partial}{\partial h_Q} V(Q) F(Q,P_0) dS_{eQ} \tag{4.1.3}$$

Let V in (4.1.3) be the disturbing potential T. Then we obtain from (4.0.3)

$$\int_{S_e} T(Q)\frac{\partial F(Q,P_0)}{\partial h_Q} dS_{eQ} = -\int_{S_e} \delta g(Q) F(Q,P_0) dS_{eQ} \tag{4.1.4}$$

We denote the ellipsoidal coordinates and the spherical coordinates of a point P by (u_P, β_P, λ_P) and (r_P, θ_P, λ_P) respectively. From section 1-20 of Heiskanen and Moritz (1967), we know that

$$V_a(P) \equiv \frac{Q_0(i\frac{u_P}{E})}{Q_0(i\frac{b_e}{E})} \in H[S_e] \tag{4.1.5}$$

and for Q on S_e,

$$V_a(Q)=1 \tag{4.1.6}$$

From sections 2-7, 2-8 and 2-9 of Heiskanen and Moritz (1967), we have

$$Q_0(i\frac{u_P}{E}) = -i\tan^{-1}\frac{E}{u_P} \tag{4.1.7}$$

$$\frac{\partial V_a}{\partial h_P}(P) = -\sqrt{\frac{u_P^2+E^2}{u_P^2+E^2\sin^2\beta_P}}\frac{\partial V_a}{\partial u_P}(P) \tag{4.1.8}$$

and

$$\tan^{-1}\frac{E}{b_e} = \frac{E}{b_e}[1-\frac{1}{3}(\frac{E}{b_e})^2+O((\frac{E}{b_e})^4)] = e'[1-\frac{1}{3}e'^2+O(e'^4)] \tag{4.1.9}$$

So for Q on S_e,

$$\frac{\partial V_a}{\partial h_Q}(Q) = -\sqrt{\frac{b_e^2+E^2}{b_e^2+E^2\sin^2\beta_Q}}\frac{\frac{E}{b_e^2+E^2}}{\tan^{-1}\frac{E}{b_e}} = -\frac{1}{a_e\sqrt{1+e'^2\sin^2\beta_Q}}\frac{1}{1-\frac{1}{3}e'^2+O(e'^4)}$$

$$= -\frac{1}{a_e}[1+e'^2(\frac{1}{3}-\frac{1}{2}\sin^2\beta_Q)+O(e'^4)] \tag{4.1.10}$$

Since

$$a_e = R(1+\frac{1}{6}e^2);\ e'^2 = e^2+O(e^4) \tag{4.1.11}$$

and

$$\sin^2\beta_Q = \frac{\tan^2\beta_Q}{1+\tan^2\beta_Q} = \frac{a_e^2\cos^2\theta_Q}{b_e^2\sin^2\theta_Q + a_e^2\cos^2\theta_Q} = \cos^2\theta_Q[1+e^2\sin^2\theta_Q + O(e^4)]$$
$$= \cos^2\theta_Q + O(e^2) \qquad (4.1.12)$$

equation (4.1.10) can be rewritten as

$$\frac{\partial V_2}{\partial h_Q}(Q) = -\frac{1}{R}[1+e^2(\frac{1}{6}-\frac{1}{2}\cos^2\theta_Q)+O(e^4)] \qquad (4.1.13)$$

It then follows from (4.1.3), (4.1.5) and (4.1.6) that

$$\int_{S_e}\frac{\partial F(Q,P_0)}{\partial h_Q}dS_{eQ} = -\int_{S_e}\frac{1}{R}[1+e^2(\frac{1}{6}-\frac{1}{2}\cos^2\theta_Q)+O(e^4)]F(Q,P_0)dS_{eQ} \qquad (4.1.14)$$

For a given P on S_e, we obtain from (4.1.4) minus (4.1.14) multiplied by T(P) that

$$\int_{S_e}[T(Q)-T(P)]\frac{\partial F(Q,P_0)}{\partial h_Q}dS_{eQ}$$
$$= \int_{S_e}\{-\delta g(Q)+\frac{T(P)}{R}[1+e^2(\frac{1}{6}-\frac{1}{2}\cos^2\theta_Q)+O(e^4)]\}F(Q,P_0)dS_{eQ} \qquad (4.1.15)$$

According to the properties of the single-layer potential and note (4.1.1), we obtain by letting $P_0 \rightarrow P$ in (4.1.15) and neglecting the quantities of the order of $O(e^4)$ that

$$\int_{S_e}[T(Q)-T(P)]M(Q,P)dS_{eQ} = 4\pi\{-\delta g(P)+\frac{T(P)}{R}[1+e^2(\frac{1}{6}-\frac{1}{2}\cos^2\theta_P)]\}\cos(r_P,h_P)$$
$$+\int_{S_e}\{-\delta g(Q)+\frac{T(P)}{R}[1+e^2(\frac{1}{6}-\frac{1}{2}\cos^2\theta_Q)]\}F(Q,P)dS_{eQ} \qquad (4.1.16)$$

where

$$F(Q,P) \equiv \frac{r_Q^2 - r_P^2}{r_P l_{QP}^3} \tag{4.1.17}$$

and

$$M(Q,P) \equiv \frac{\partial}{\partial h_Q} F(Q,P) = \frac{2r_Q}{r_P l_{QP}^3} \frac{\partial r_Q}{\partial h_Q} - \frac{3(r_Q^2 - r_P^2)}{r_P l_{QP}^4} \frac{\partial l_{QP}}{\partial h_Q} \tag{4.1.18}$$

The kernel functions M(Q,P) and F(Q,P) are singular when Q→P. Their singularities for Q→P will be discussed in section 4.2.1.

Equation (4.1.16) is the integral equation from which the inverse Hotine formula and its ellipsoidal correction will be obtained.

4.1.2 Inverse Hotine formula and its ellipsoidal correction

Denoting the projection of the surface element dS_{eQ} onto the unit sphere σ by $d\sigma_Q$, we have

$$dS_e = r_Q^2 \sec\beta_Q d\sigma_Q \tag{4.1.19}$$

where β_Q is the angle between the radius vector of Q and the surface normal of the surface S_e at point Q. With R the mean radius and e the first eccentricity of the reference ellipsoid, and θ_P and θ_Q respectively the complements of the geocentric latitudes of the points P and Q on S_e, we have

$$r_P = R[1 + \frac{1}{2}e^2(\sin^2\theta_P - \frac{2}{3}) + O(e^4)] \tag{4.1.20a}$$

$$r_Q = R[1 + \frac{1}{2}e^2(\sin^2\theta_Q - \frac{2}{3}) + O(e^4)] \tag{4.1.20b}$$

$$l_{QP} = 2R\sin\frac{\psi_{QP}}{2}[1+\frac{1}{4}e^2(\sin^2\theta_Q+\sin^2\theta_P-\frac{4}{3})+O(e^4)] \tag{4.1.20c}$$

$$r_Q^2\sec\beta_Q = R^2[1+e^2(\sin^2\theta_Q-\frac{2}{3})+O(e^4)] \tag{4.1.20d}$$

Furthermore, from Molodensky et al. (1962), we have

$$\frac{\partial r_Q}{\partial h_Q} = \cos(r_Q,h_Q) = 1+O(e^4) \tag{4.1.21a}$$

$$\frac{\partial l_{QP}}{\partial h_Q} = \sin\frac{\psi_{QP}}{2}[1-\frac{1}{4}e^2(3\cos^2\theta_Q+\cos^2\theta_P-\frac{(\cos\theta_Q-\cos\theta_P)^2}{\sin^2\frac{\psi_{QP}}{2}})+O(e^4)] \tag{4.1.21b}$$

$$-\frac{1}{\gamma_Q}\frac{\partial\gamma_Q}{\partial h_Q} = \frac{2}{R}[1+e^2(\frac{5}{6}-\cos^2\theta_Q)+O(e^4)] \tag{4.1.21c}$$

It then follows from (4.1.17) and (4.1.18) that

$$F(Q,P)r_Q^2\sec\beta_Q = f(\psi_{QP},\theta_Q,\theta_P)[e^2+O(e^4)] \tag{4.1.22}$$

$$M(Q,P)r_Q^2\sec\beta_Q = \frac{M(\psi_{QP})}{R}[1+e^2(\frac{1}{2}\cos^2\theta_P-\frac{1}{6})+O(e^4)] \tag{4.1.23}$$

where

$$f(\psi_{QP},\theta_Q,\theta_P) = \frac{1}{8}\frac{\sin^2\theta_Q-\sin^2\theta_P}{\sin^3\frac{\psi_{QP}}{2}} \tag{4.1.24a}$$

$$M(\psi_{QP}) = \frac{1}{4\sin^3\frac{\psi_{QP}}{2}} \tag{4.1.24b}$$

Let

$$\delta g(Q) = \delta g^0(Q)+\delta g^1(Q)e^2+O(e^4) \tag{4.1.25}$$

Inserting (4.1.19), (4.1.22), (4.1.23) and (4.1.25) into (4.1.16) and neglecting the quantities of order of $O(e^4)$, we obtain

$$\begin{aligned}
&\int_\sigma [T(Q)-T(P)]\frac{M(\psi_{QP})}{R}[1+e^2(\frac{1}{2}\cos^2\theta_P-\frac{1}{6})]d\sigma_Q \\
&\quad =4\pi\{-\delta g^0(P)+\frac{T(P)}{R}+e^2[-\delta g^1(P)+\frac{T(P)}{R}(\frac{1}{6}-\frac{\cos^2\theta_P}{2})]\} \\
&\quad -e^2\int_\sigma[\delta g^0(Q)-\frac{T(P)}{R}]f(\psi_{QP},\theta_Q,\theta_P)d\sigma_Q
\end{aligned} \tag{4.1.26}$$

From (3.3.2), we have

$$\begin{aligned}
\sin^2\theta_Q-\sin^2\theta_P &=\cos^2\theta_P-\cos^2\theta_P\cos^2\psi_{QP}-\sin^2\theta_P\sin^2\psi_{QP}\cos^2\alpha_{QP} \\
&\quad -2\cos\theta_P\cos\psi_{QP}\sin\theta_P\sin\psi_{QP}\cos\alpha_{QP} \\
&=4\sin^2\frac{\psi_{QP}}{2}\cos^2\frac{\psi_{QP}}{2}[\cos^2\theta_P-\sin^2\theta_P\cos^2\alpha_{QP}] \\
&\quad -2\cos\theta_P\cos\psi_{QP}\sin\theta_P\sin\psi_{QP}\cos\alpha_{QP}
\end{aligned} \tag{4.1.27}$$

Noting that

$$\int_0^{2\pi} d\alpha_{QP}=2\pi;\quad \int_0^{2\pi}\cos\alpha_{QP}d\alpha_{QP}=0;\quad \int_0^{2\pi}\cos^2\alpha_{QP}d\alpha_{QP}=\pi \tag{4.1.28}$$

we obtain that

$$\begin{aligned}
\int_\sigma \frac{\sin^2\theta_Q-\sin^2\theta_P}{\sin^3\frac{\psi_{QP}}{2}}d\sigma &=\int_0^\pi\int_0^{2\pi}\frac{\sin^2\theta_Q-\sin^2\theta_P}{\sin^3\frac{\psi_{QP}}{2}}4\sin\frac{\psi_{QP}}{2}d\sin\frac{\psi_{QP}}{2}d\alpha_{QP} \\
&=16\pi(2\cos^2\theta_P-\sin^2\theta_P)\int_0^1(1-x^2)dx
\end{aligned}$$

$$= 16\pi(2\cos^2\theta_P - \frac{2}{3}) \tag{4.1.29}$$

It follows from (4.1.24) and (4.1.26) that

$$\delta g^0(P) = \frac{T(P)}{R} - \frac{1}{4\pi R}\int_\sigma [T(Q) - T(P)]M(\psi_{QP})d\sigma_Q \tag{4.1.30}$$

$$\delta g^1(P) = \delta g_1^1(P) + \delta g_2^1(P) \tag{4.1.31}$$

where

$$\delta g_1^1(P) = (\frac{\cos^2\theta_P}{2} - \frac{1}{6})\delta g^0(P) \tag{4.1.31a}$$

$$\delta g_2^1(P) = -\frac{1}{4\pi}\int_\sigma \delta g^0(Q) f(\psi_{QP}, \theta_Q, \theta_P) d\sigma_Q \tag{4.1.31b}$$

Formula (4.1.30) is the inverse Hotine formula, from which the spherical gravity disturbance is computed, and (4.1.31) is the ellipsoidal correction for the inverse Hotine formula.

4.1.3 Inverse Stokes formula and its ellipsoidal correction

According to (4.0.4) and noting (4.1.21c), (4.1.30) and (4.1.31), we have that

$$\begin{aligned}\Delta g(P) &= \delta g(P) + (\frac{1}{\gamma_P}\frac{\partial\gamma_P}{\partial h_P})T(P) \\ &= \delta g^0(P) - \frac{2T(P)}{R} + e^2[\delta g^1(P) - \frac{2T(P)}{R}(\frac{5}{6} - \cos^2\theta_P)] + O(e^4)\end{aligned} \tag{4.1.32}$$

Let

$$\Delta g(P) = \Delta g^0(P) + \Delta g^1(P)e^2 + O(e^4) \tag{4.1.33}$$

then

$$\Delta g^0(P) = \delta g^0(P) - \frac{2T(P)}{R}$$
$$= -\frac{T(P)}{R} - \frac{1}{4\pi R}\int_\sigma [T(Q) - T(P)]M(\psi_{QP})d\sigma_Q \tag{4.1.34}$$

$$\Delta g^1(P) = \delta g^1(P) - \frac{2T(P)}{R}(\cos^2\theta_P - \frac{1}{6})$$
$$= (\frac{\cos^2\theta_P}{2} - \frac{1}{6})\delta g^0(P) - \frac{1}{4\pi}\int_\sigma \delta g^0(Q)f(\psi_{QP},\theta_Q,\theta_P)d\sigma_Q - \frac{2T(P)}{R}(\frac{5}{6} - \cos^2\theta_P)$$
$$= \Delta g_1^1(P) + \Delta g_2^1(P) \tag{4.1.35}$$

where

$$\Delta g_1^1(P) = -\frac{T(P)(2 - 3\cos^2\theta_P)}{R} + (\frac{\cos^2\theta_P}{2} - \frac{1}{6})\Delta g^0(P) \tag{4.1.35a}$$

$$\Delta g_2^1(P) = -\frac{1}{2\pi R}\int_\sigma T(Q)f(\psi_{QP},\theta_Q,\theta_P)d\sigma_Q - \frac{1}{4\pi}\int_\sigma \Delta g^0(Q)f(\psi_{QP},\theta_Q,\theta_P)d\sigma_Q \tag{4.1.35b}$$

The formula (4.1.34) is the inverse Stokes formula, from which the spherical gravity anomaly is computed, and (4.1.35) is the ellipsoidal correction for the inverse Stokes formula.

4.2 Practical considerations for the integrals in the formulas

In the above section, we obtained the closed formulas (4.1.31) and (4.1.35) of the ellipsoidal corrections δg^1 and Δg^1 respectively to the inverse Hotine formula (4.1.30) (the spherical gravity disturbance δg^0) and the inverse Stokes formula (4.1.34) (the spherical gravity anomaly Δg^0) from the basic integral equation (4.1.16). Formula (4.1.31) (formula (4.1.35)) is expressed as a sum of a simple analytical function and an integral about δg^0 (Δg^0 and T). Obviously, the first part of δg^I (Δg^I) is easy to be computed from δg^0 (Δg^0 and T). In the following, we will give detailed discussions on the integral parts (4.1.31b) and (4.1.35b).

4.2.1 Singularities

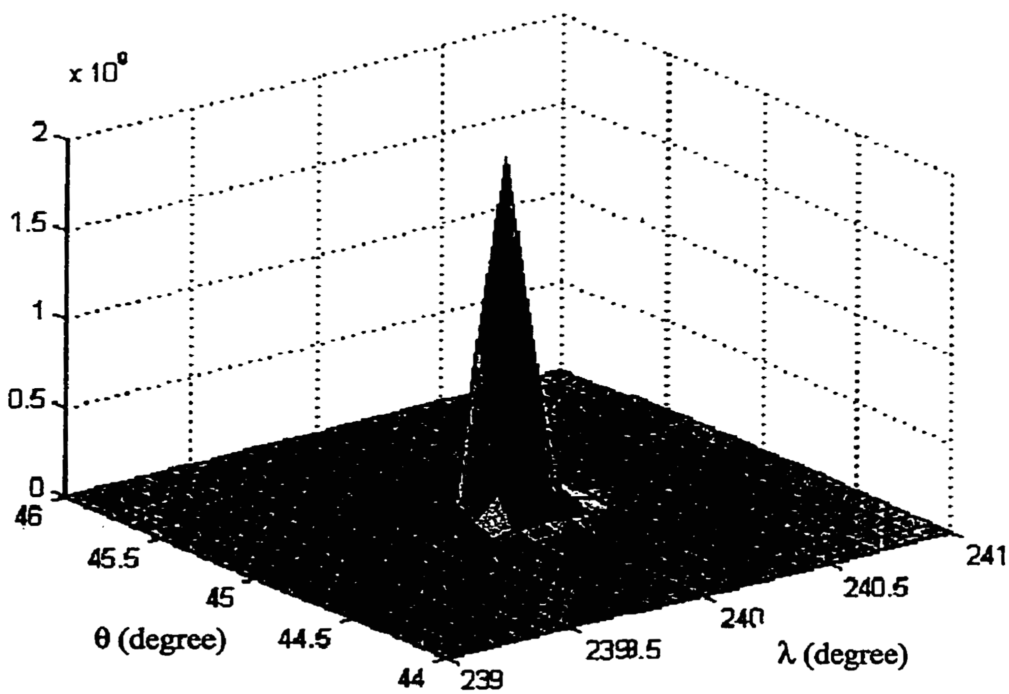

Figure 4.1 Behavior of kernel function M of the inverse Hotine/Stokes formulas in the neighborhood of (θ_P=45^0, λ_P=240^0)

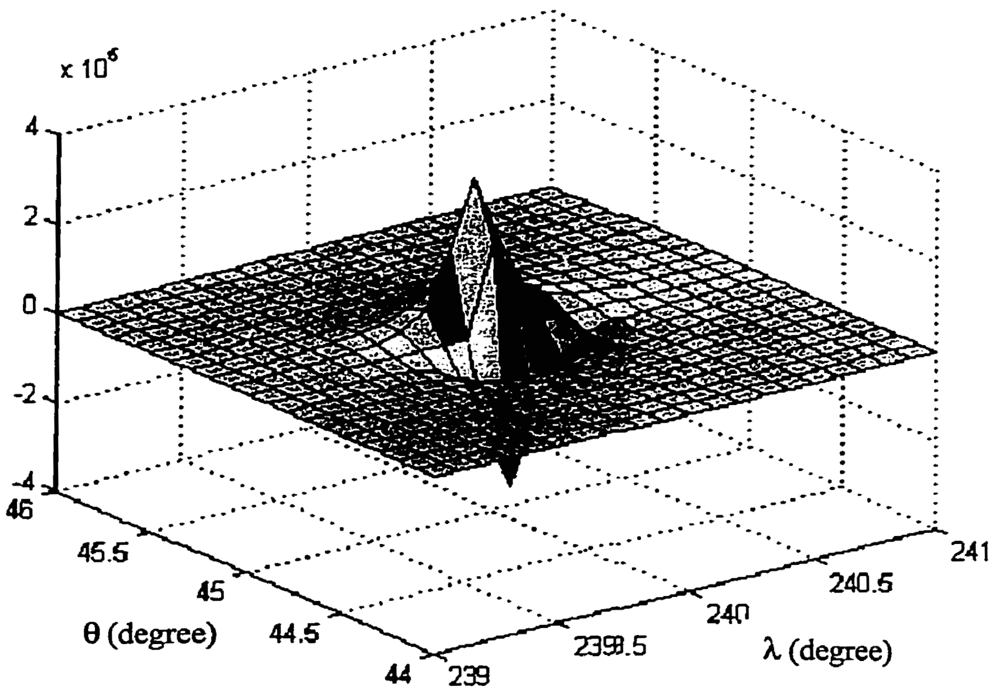

Figure 4.2 Behavior of kernel function f of the ellipsoidal corrections to the inverse Hotine/Stokes formulas in the neighborhood of (θ_P=45^0, λ_P=240^0)

The integrals in formulas (4.1.16), (4.1.30), (4.1.31b), (4.1.34) and (4.1.35b) are singular because their kernel functions M(Q, P), F(Q, P) and $M(\psi_{QP})$, $f(\psi_{QP}, \theta_Q, \theta_P)$ are singular when Q→P or $\psi_{QP} \to 0$ (see figures 4.1 and 4.2).

The singularity of the integral in the inverse Stokes (or Hotine) formula (4.1.34) (or (4.1.30)) has been discussed in many references, such as Heiskanen and Moritz (1967), Bian and Dong (1991) and Zhang (1993). Here we discuss the singularities of the integrals in (4.1.16), (4.1.31b) and (4.1.35b).

According to (4.1.22), we know that the integral on the left hand side of (4.1.16) and the integrals in the inverse Stokes formula (4.1.30) and the inverse Stokes formula (4.1.34) have the same form. So the integral on the left hand side of (4.1.16) can be treated with the same method used in processing the inverse Stokes (Hotine) formula.

Similarly, according to (4.1.23), the integral on the left hand side of (4.1.16) and the integrals in (4.1.31b) and (4.1.35b) have the same form. Therefore, in the following, we only discuss the method to treat the singularity of the integral in (4.1.31b).

Obviously, we only need to consider the integral in the innermost spherical cap area σ_0 with the centre at the computation point P and a radius ψ_0 which is so small that the spherical cap area can be treated as a plane. That is we discuss the following integral

$$\overline{\delta g}(P)=\frac{1}{4\pi}\int_{\sigma_0}\delta g^0(Q)f(\psi_{QP},\theta_Q,\theta_P)d\sigma_Q \tag{4.2.1}$$

From (4.1.27) and (4.1.24a), and noting that σ is a unit sphere, we have

$$\begin{aligned}\overline{\delta g}(P)=&\frac{1}{4\pi}\int_{\psi_{QP}=0}^{\psi_0}\int_{\alpha_{QP}=0}^{2\pi}\delta g^0(Q)\frac{1}{8\sin^3\frac{\psi_{QP}}{2}}\{4\sin^2\frac{\psi_{QP}}{2}\cos^2\frac{\psi_{QP}}{2}[\cos^2\theta_P-\sin^2\theta_P\cos^2\alpha_{QP}]\\&-2\cos\theta_P\cos\psi_{QP}\sin\theta_P\sin\psi_{QP}\cos\alpha_{QP}\}\sin\psi_{QP}d\alpha_{QP}d\psi_{QP}\\=&\frac{1}{4\pi}\int_{l_{QP}=0}^{l_0}\int_{\alpha_{QP}=0}^{2\pi}\delta g^0(Q)\{(1-\frac{l_{QP}^2}{4})[\cos^2\theta_P-\sin^2\theta_P\cos^2\alpha_{QP}]\\&-\frac{1}{l_{QP}}(1-\frac{l_{QP}^2}{2})(1-\frac{l_{QP}^2}{4})^{\frac{1}{2}}\sin 2\theta_P\cos\alpha_{QP}\}d\alpha_{QP}dl_{QP}\end{aligned} \tag{4.2.2}$$

where

$$l_0=2\sin\frac{\psi_0}{2}. \tag{4.2.3}$$

For Q in σ_0, we expand $\delta g^0(Q)$ into a Taylor series at the computation point P:

$$\delta g^0(Q) = \delta g^0(P) + x\delta g_x^0(P) + y\delta g_y^0(P) + \cdots \tag{4.2.4}$$

where the rectangular coordinates x, y are defined by

$$x = l_{QP}\cos\alpha_{QP}; \quad y = l_{QP}\sin\alpha_{QP} \tag{4.2.5}$$

so that the x-axis points north, and

$$\delta g_x^0(P) = \frac{\partial \delta g^0}{\partial x}(P); \quad \delta g_y^0(P) = \frac{\partial \delta g^0}{\partial y}(P) \tag{4.2.6}$$

The Taylor series (4.2.4) may also be written as

$$\delta g^0(Q) = \delta g^0(P) + [\delta g_x^0(P)\cos\alpha_{QP} + \delta g_y^0(P)\sin\alpha_{QP}]l_{QP} + \cdots \tag{4.2.7}$$

Inserting this into (4.2.2), performing the integral with respect to α_{QP} first, noting (4.1.28) and neglecting the quantities of $O(l_0^2)$, we have

$$\overline{\delta g}(P) = \frac{l_0}{4}[\delta g^0(P)(3\cos^2\theta_P - 1) + \delta g_x^0(P)] \tag{4.2.8}$$

We see that the effect of the innermost spherical cap area on the integral (4.1.31b) depends, to a first approximation, on $\delta g^0(P)$ and $\delta g_x^0(P)$. The value of $\delta g_x^0(P)$ can be obtained by numerical differentiation δg^0.

4.2.2 Input data

In (4.1.31b) and (4.1.35b), the input data are respectively δg^0, and Δg^0 and T. These data are available only in some ocean areas. Here we give some modification on the input data.

According to (4.1.25) and (4.1.33), we have

$$\delta g^0(Q) = \delta g(Q) - \delta g^1(Q)e^2 + O(e^4) \tag{4.2.9}$$

$$\Delta g^0(Q) = \Delta g(Q) - \Delta g^1(Q)e^2 + O(e^4) \tag{4.2.10}$$

In addition, the disturbing potential T(P) on the reference ellipsoid can be expressed as

$$T(P) = T^0(P) + e^2 T^1(P) \tag{4.2.11}$$

where $T^0(P)$ is the spherical approximation of T(P). Since δg^1 should be multiplied by e^2 before it is added to δg^0, we obtain by inserting the above formulas into the integrals in (4.1.31) and (4.1.34) respectively and neglecting the quantities of order of $O(e^2)$ that

$$\delta g_2^1(P) = \frac{1}{4\pi}\int_\sigma \delta g(Q) f(\psi_{QP}, \theta_Q, \theta_P) d\sigma_Q \tag{4.2.12}$$

$$\Delta g_2^1(P) = \frac{1}{4\pi}\int_\sigma [\Delta g(Q) + \frac{2T^0(Q)}{R}] f(\psi_{QP}, \theta_Q, \theta_P) d\sigma_Q \tag{4.2.13}$$

where δg is the gravity disturbance which can be computed approximately from the global geopotential models, Δg and T^0 are the gravity anomaly and the spherical disturbing potential which are already available globally with the resolutions of less than 1 degree and locally with higher resolutions.

4.2.3 Spherical harmonic expansions of the integrals

In the following, we will expand $\delta g_2^1(P)$ and $\Delta g_2^1(P)$ into series of spherical harmonics so that they can be computed from the global geopotential models.

According to section 2-14 of Heiskanen and Moritz (1967), under the spherical approximation, we have

$$\delta g(\theta,\lambda) = \frac{1}{R}\sum_{n=2}^{\infty}(n+1)T_n(\theta,\lambda) \tag{4.2.14}$$

where $T_n(\theta, \lambda)$ is Laplace's surface harmonics of the disturbing potential T:

$$T_n(\theta,\lambda) = \sum_{m=0}^{n}[c_{nm}R_{nm}(\theta,\lambda) + d_{nm}S_{nm}(\theta,\lambda)] \tag{4.2.15}$$

Let

$$\delta g(\theta,\lambda)\cos^2\theta = \frac{1}{R}\sum_{n=2}^{\infty}(n+1)X_n(\theta,\lambda) \tag{4.2.16}$$

From (4.1.34) and their definitions, we know that integrals $\delta g_2^1(P)$ and $\Delta g_2^1(P)$ are equal. According to (1-102) of Heiskanen and Moritz (1967), we have from (4.2.12) that

$$\begin{aligned}\Delta g_2^1(P) = \delta g_2^1(P) &= \frac{1}{4\pi}\int_\sigma \frac{\delta g(Q)\cos^2\theta_Q - \delta g(P)\cos^2\theta_P}{8\sin^3\frac{\psi_{QP}}{2}}d\sigma_Q - \frac{\cos^2\theta_P}{4\pi}\int_\sigma \frac{\delta g(Q)-\delta g(P)}{8\sin^3\frac{\psi_{QP}}{2}}d\sigma_Q \\ &= \frac{1}{2R}[-\sum_{n=2}^{\infty}n(n+1)X_n(\theta_P,\lambda_P) + \sum_{n=2}^{\infty}n(n+1)T_n(\theta_P,\lambda_P)\cos^2\theta_P]\end{aligned} \tag{4.2.17}$$

Let

$$\sum_{n=2}^{\infty}(n+1)T_n(\theta,\lambda)\cos^2\theta \equiv \sum_{n=2}^{\infty}(n+1)\sum_{m=0}^{n}[\delta E_{nm}R_{nm}(\theta,\lambda)+\delta F_{nm}S_{nm}(\theta,\lambda)] \tag{4.2.18}$$

$$\sum_{n=2}^{\infty}n(n+1)T_n(\theta,\lambda)\cos^2\theta \equiv \sum_{n=2}^{\infty}n(n+1)\sum_{m=0}^{n}[\delta G_{nm}R_{nm}(\theta,\lambda)+\delta H_{nm}S_{nm}(\theta,\lambda)] \tag{4.2.19}$$

From (A11) of Wang (1999) (Note: there is a printing error in that formula) and (4.2.15), we know that

$$\begin{aligned} T_n(\theta,\lambda)\cos^2\theta = \sum_{m=0}^{n}\{c_{nm}[\alpha_n^m R_{n+2m}(\theta,\lambda)+\beta_n^m R_{nm}(\theta,\lambda)+\gamma_n^m R_{n-2m}(\theta,\lambda)] \\ + d_{nm}[\alpha_n^m S_{n+2m}(\theta,\lambda)+\beta_n^m S_{nm}(\theta,\lambda)+\gamma_n^m S_{n-2m}(\theta,\lambda)]\} \end{aligned} \tag{4.2.20}$$

where

$$\alpha_n^m = \frac{(n-m+1)(n-m+2)}{(2n+1)(2n+3)} \tag{4.2.21}$$

$$\beta_n^m = \frac{2n^2-2m^2+2n-1}{(2n+1)(2n-1)} \tag{4.2.22}$$

$$\gamma_n^m = \frac{(n+m)(n+m-1)}{(2n+1)(2n-1)} \tag{4.2.23}$$

Therefore

$$\begin{aligned} \begin{Bmatrix}\delta E_{nm}\\ \delta F_{nm}\end{Bmatrix} &= \sum_{k=2}^{\infty}\frac{k+1}{n+1}N_n^m\int_{\sigma_0}T_k(\theta,\lambda)\cos^2\theta\begin{Bmatrix}R_{nm}(\theta,\lambda)\\ S_{nm}(\theta,\lambda)\end{Bmatrix}d\sigma_0 \\ &= \frac{n-1}{n+1}\alpha_{n-2}^m\begin{Bmatrix}c_{n-2m}\\ d_{n-2m}\end{Bmatrix}+\beta_n^m\begin{Bmatrix}c_{nm}\\ d_{nm}\end{Bmatrix}+\frac{n+3}{n+1}\gamma_{n+2}^m\begin{Bmatrix}c_{n+2m}\\ d_{n+2m}\end{Bmatrix} \end{aligned} \tag{4.2.24}$$

$$\begin{Bmatrix}\delta G_{nm}\\ \delta H_{nm}\end{Bmatrix}=\sum_{k=2}^{\infty}\frac{k(k+1)}{n(n+1)}N_n^m\int_{\sigma_0}T_k(\theta,\lambda)\cos^2\theta\begin{Bmatrix}R_{nm}(\theta,\lambda)\\ S_{nm}(\theta,\lambda)\end{Bmatrix}d\sigma_0$$

$$=\frac{(n-2)(n-1)}{n(n+1)}\alpha_{n-2}^m\begin{Bmatrix}c_{n-2m}\\ d_{n-2m}\end{Bmatrix}+\beta_n^m\begin{Bmatrix}c_{nm}\\ d_{nm}\end{Bmatrix}+\frac{(n+2)(n+3)}{n(n+1)}\gamma_{n+2}^m\begin{Bmatrix}c_{n+2m}\\ d_{n+2m}\end{Bmatrix} \quad (4.2.25)$$

So we obtain from (4.2.17) that

$$\Delta g_2^1(P)=\delta g_2^1(P)=\frac{1}{R}\sum_{n=2}^{\infty}\sum_{m=0}^{n}[\delta A_{nm}R_{nm}(\theta_P,\lambda_P)+\delta B_{nm}S_{nm}(\theta_P,\lambda_P)] \qquad (4.2.26)$$

where

$$\begin{Bmatrix}\delta A_{nm}\\ \delta B_{nm}\end{Bmatrix}=-\frac{(n-1)(n-m-1)(n-m)}{(2n-1)(2n-3)}\begin{Bmatrix}c_{n-2m}\\ d_{n-2m}\end{Bmatrix}$$

$$+\frac{(n+3)(n+m+1)(n+m+2)}{(2n+5)(2n+3)}\begin{Bmatrix}c_{n+2m}\\ d_{n+2m}\end{Bmatrix} \qquad (4.2.27)$$

Thus we express $\delta g_2^1(P)$ and $\Delta g_2^1(P)$ by a series of spherical harmonics. The input data $\{c_{nm},d_{nm}\}$are the spherical harmonic coefficients of the disturbing potential.

4.3 Chapter summary

This chapter gives the ellipsoidal corrections $\delta g^1(P)$ and $\Delta g^1(P)$ to the inverse Hotine formula $\delta g^0(P)$ (the spherical gravity disturbance) and the inverse Stokes formula $\Delta g^0(P)$ (the spherical gravity anomaly) respectively.

- By adding the ellipsoidal corrections to their spherical solutions, the error of the gravity disturbance and the gravity anomaly decreases from $O(e^2)$ to $O(e^4)$, which can be neglected for most practical purposes.

- $\delta g^1(P)$ is expressed as a sum of a simple analytical function about $\delta g^0(P)$ and an integral about δg^0. In the practical computation of the integral, the input data δg^0 can be substituted by the gravity disturbance δg, which can be approximately computed from global geopotential models. The integral part of $\delta g^1(P)$ can also be computed directly from global geopotential models via formula (4.2.26).

- $\Delta g^1(P)$ is expressed as a sum of a simple analytical function about $\Delta g^0(P)$ and T(P) and an integral about Δg^0 and T. In the practical computation of the integral, the input data Δg^0 and T(P) can be substituted respectively by the gravity anomaly Δg and the spherical disturbing potential T^0, which are already available globally with resolutions of less than 1 degree and locally with higher resolutions. The integral part of $\Delta g^1(P)$ can also be computed directly from the global geopotential models via formula (4.2.26).

- Like the ellipsoidal correction to gravity anomaly given in Wang (1999), the ellipsoidal correction $\Delta g^1(P)$ is also be computed from an auxiliary data Δg^0 (or Δg). However, the kernel function in the formula of computing Δg^0 (or Δg) is simpler than the kernel function in the formula of computing the auxiliary data χ used in Wang (1999) and Δg^0 (or Δg) is already available globally with the resolutions of less than 1 degree and locally with higher resolutions. Therefore the ellipsoidal correction $\Delta g^1(P)$ given in this chapter is more effective.

5 Solutions to the Second Geodetic Boundary Value Problem

In chapter 1, we discussed the definition and the significance of the second geodetic boundary value problem (SGBVP). In this chapter, we will give some approximate solutions of this problem.

By definition, the SGBVP is an oblique derivative problem and its boundary surface is the very complicated topographical surface of the Earth. Similar to solving the third geodetic boundary value problem, we can directly solve the SGBVP by an integral equation method or convert this problem into a normal derivative problem, such as the spherical boundary value problem or the ellipsoidal boundary value problem, by properly adjusting the disturbing potential.

In this chapter, we will first investigate the second spherical boundary value problem and the second ellipsoidal boundary value problem, then apply the solutions of these two normal derivative problems to solve the SGBVP. Three approximate solutions of the SGBVP and a brief comparison of these solutions will be given.

5.1 Second spherical boundary value problem

In this section, we will discuss the second spherical boundary value problem. It can be defined mathematically as finding a function T such that

$$\begin{cases} T \in H[S_M] \\ \dfrac{\partial}{\partial r} T_P = -\delta g(P) \qquad \text{P is on } S_M \end{cases} \tag{5.1.1}$$

where $\frac{\partial}{\partial r}$ means the derivative along the radial vector and S_M is the mean sphere.

5.1.1 Generalized Hotine formula

Since T is harmonic outside S_M, it can be shown that $r\frac{\partial}{\partial r}T$ is also harmonic outside S_M. Following Heiskanen and Moritz (1967) and noting the second condition in (5.1.1), we have for P outside S_M

$$r_P\frac{\partial}{\partial r}T(P) = -\frac{R}{4\pi}\int_\sigma \frac{r_P^2-R^2}{l_{PQ}^3}R\delta g(Q)d\sigma \qquad \text{(Q is on } S_M\text{)} \qquad (5.1.2)$$

Therefore

$$\frac{\partial}{\partial r}T(P) = -\frac{R}{4\pi r_P}\int_\sigma \frac{r_P^2-R^2}{l_{PQ}^3}R\delta g(Q)d\sigma \qquad \text{(Q is on } S_M\text{)} \qquad (5.1.3)$$

Integrating the above formula from r_P to ∞ along the radial direction of P and noting the regularity condition in (5.1.1), we have for P outside S_M

$$T(P) = \frac{R}{4\pi}\int_\sigma H(P,Q)\delta g(Q)d\sigma \qquad \text{(Q is on } S_M\text{)} \qquad (5.1.4)$$

where

$$H(P,Q) = \int_{r_P}^{\infty} \frac{r_{P'}^2-R^2}{l_{PQ}^3}\frac{R}{r_{P'}}dr_{P'} \qquad \text{(Q is on } S_M\text{)} \qquad (5.1.5)$$

In the following, we further evaluate the integral in (5.1.5). According to the definition of the integral, the point P' in the above integral satisfies $\psi_{PQ} = \psi_{P'Q}$. Let

$$r_P = x_0\,;\quad r_{P'} = x\,;\quad \psi_{PQ} = \psi\,;\quad X = R^2 + x^2 - 2Rx\cos\psi \tag{5.1.6}$$

then

$$\begin{aligned}\int_{r_P}^{\infty} \frac{r_{P'}^2\text{-}R^2}{l_{PQ}^3}\frac{R}{r_{P'}}dr_{P'} &= R\int_{x_0}^{\infty} \frac{x^2\text{-}R^2}{x(R^2+x^2-2Rx\cos\psi)^{\frac{3}{2}}}dx \\ &= R\{\int_{x_0}^{\infty}\frac{x}{X\sqrt{X}}dx - R^2\int_{x_0}^{\infty}\frac{1}{xX\sqrt{X}}dx\}\end{aligned} \tag{5.1.7}$$

According to Wang et al. (1985),

$$\begin{aligned}R\int_{x_0}^{\infty}\frac{x}{X\sqrt{X}}dx &= -R\frac{4R^2-4Rx\cos\psi}{4R^2\sin^2\psi\sqrt{x^2-2Rx\cos\psi+R^2}}\Bigg|_{x_0}^{\infty} \\ &= \frac{\cos\psi}{\sin^2\psi} + \frac{R - r_P\cos\psi_{PQ}}{l_{PQ}\sin^2\psi_{PQ}}\end{aligned} \tag{5.1.8}$$

We denote

$$a = R\cos\psi\,;\quad b = R\sin\psi \tag{5.1.9}$$

Then

$$\begin{aligned}\int_{x_0}^{\infty}\frac{1}{xX\sqrt{X}}dx &= \int_{x_0}^{\infty}\frac{1}{x[(x-a)^2+b^2]\sqrt{(x-a)^2+b^2}}dx \\ &= \int_{x_0-a}^{\infty}\frac{1}{(x+a)[x^2+b^2]\sqrt{x^2+b^2}}dx\end{aligned}$$

$$= \int_{x_0-a}^{\infty} \frac{x-a}{(x^2-a^2)[x^2+b^2]\sqrt{x^2+b^2}} dx$$

$$= \int_{x_0-a}^{\infty} \frac{x-a}{(a^2+b^2)\sqrt{x^2+b^2}} [\frac{1}{(x^2-a^2)} - \frac{1}{x^2+b^2}] dx$$

$$= \frac{1}{(a^2+b^2)} [\int_{x_0-a}^{\infty} \frac{1}{(x+a)\sqrt{x^2+b^2}} dx - \int_{x_0-a}^{\infty} \frac{x-a}{(x^2+b^2)\sqrt{x^2+b^2}} dx]$$

$$\begin{aligned} &= \frac{1}{(a^2+b^2)} \{\int_{x_0}^{\infty} \frac{1}{x\sqrt{(x-a)^2+b^2}} dx \\ &\qquad - \int_{x_0-2a}^{\infty} \frac{x}{[(x+a)^2+b^2]\sqrt{(x+a)^2+b^2}} dx\} \end{aligned} \tag{5.1.10}$$

Obviously, $a^2+b^2=R^2$. Setting $X_1=(x+a)^2+b^2$, then we have from Wang et al. (1985) that

$$\begin{aligned} \int_{x_0}^{\infty} \frac{1}{xX\sqrt{X}} dx &= \frac{1}{R^2} [\int_{x_0}^{\infty} \frac{1}{x\sqrt{X}} dx - \int_{x_0-2R\cos\psi}^{\infty} \frac{x}{X_1\sqrt{X_1}} dx] \\ &= \frac{1}{R^2} \{[-\frac{1}{R} \ln(\frac{\sqrt{X}+R}{x} + \frac{-2R\cos\psi}{2R})]_{x_0}^{\infty} + [\frac{4Rx\cos\psi+4R^2}{4R^2\sin^2\psi\sqrt{X_1}}]_{x_0-2R\cos\psi}^{\infty}\} \\ &= \frac{1}{R^3} \{\ln(1-\cos\psi) - \ln(\frac{l_{PQ}+R}{r_P} - \cos\psi) \\ &\qquad - \frac{\cos\psi}{\sin^2\psi} + \frac{r_P\cos\psi + R - 2R\cos^2\psi}{l_{PQ}\sin^2\psi}\} \end{aligned} \tag{5.1.11}$$

It follows from (5.1.5), (5.1.7) and (5.1.8) that

$$H(P,Q) = \frac{2R}{l_{PQ}} - \ln \frac{l_{PQ}+R-r_P\cos\psi_{PQ}}{r_P(1-\cos\psi_{PQ})} \tag{5.1.12}$$

Thus we obtain formula (5.1.4) plus formula (5.1.12) for computing T of the points outside S_M. Furthermore, letting P go to the surface S_M in (5.1.4), i.e. letting $r_P \to R$, and noting that T is continuous onto S_M, we obtain

$$T(P) = \frac{R}{4\pi}\int_\sigma H(\psi_{PQ})\delta g(Q)d\sigma \tag{5.1.13}$$

where

$$H(\psi_{PQ}) = \frac{1}{\sin(\psi_{PQ}/2)} - \ln(1 + \frac{1}{\sin(\psi_{PQ}/2)}) \tag{5.1.14}$$

This means that (5.1.4) and (5.1.12) also hold on S_M. Therefore the solution of (5.1.1) can be represented by (5.1.4) and (5.1.12).

The formula (5.1.13) plus (5.1.14) is Hotine's formula, which represents the disturbing potential on a sphere using gravity disturbances on the sphere. The other expression of Hotine's formula for N can be obtained directly via Bruns' formula

$$N_P = \frac{R}{4\pi\gamma}\int_\sigma H(\psi_{PQ})\delta g(Q)d\sigma \tag{5.1.15}$$

Obviously, formula (5.1.4) plus (5.1.12) generalizes Hotine's formula from the surface S_M to its external space. So it can be called the generalized Hotine formula.

5.1.2 Discussion

- The solution (5.1.4), the unique solution of the second spherical boundary value problem (5.1.1), is a generalization of Hotine's formula. It is similar to the generalized Stokes formula (2-16) in Heiskanen and Moritz (1967) and its kernel function H(P,Q) is simpler than the generalized Stokes function.

- From the derivation of the generalized Hotine formula, we know that the condition $T(P) = O(1/r_P^3)$, which is required in the derivation of Stokes's formula, is not needed. This means that in the SGBVP, the reference ellipsoid is not required to satisfy that its mass equals to the mass of the Earth and its centre coincides with the centre of mass of the Earth.

- To apply Hotine's formula to the SGBVP, the disturbing potential should be transformed so that it is harmonic outside S_M and the gravity disturbances should be reduced from S_E onto S_M. The distance δr between the two surfaces S_E and S_M can also be obtained from GPS measurements. Since δr is large than the geodetic heights h, the distance between S_E and S_M, in most areas, the transform of the disturbing potential and the reduction of gravity disturbances may cause a big error. A method to avoid the big error is transforming the disturbing potential so that it is harmonic outside S_e and reducing the gravity disturbances from S_E onto S_e and then solving the second ellipsoidal boundary value problem.

5.2 Ellipsoidal corrections to Hotine's formula

Hotine's formula gives a method for evaluating the geoidal heights from the gravity disturbances. As we discussed above, however, the input data and the output data in Hotine's formula are on a sphere with radius R. An adjustment of the anomalous gravity field such as the gravity reduction or the analytical continuation of gravity disturbances is needed to apply Hotine's formula to solve the SGBVP. To avoid the big error caused by the adjustment, it is better to adjust the anomalous gravity field so that the disturbing potential is harmonic outside the reference ellipsoid S_e and the gravity disturbance data are given on S_e. Thus we obtain the second ellipsoidal boundary value problem. The mathematical definition is as follows:

$$\begin{cases} T \in H[S_e] \\ \dfrac{\partial}{\partial h} T_P = -\delta g(P) \qquad \text{P is on } S_e \end{cases} \tag{5.2.1}$$

where $\dfrac{\partial}{\partial h}$ means the derivative along the normal plumb line and S_e is the surface of the reference ellipsoid.

Unlike the second spherical boundary value problem, the second ellipsoidal boundary value problem has no exact closed solution like Hotine's formula. An approach for approximately expressing the solution of the problem is regarding Hotine's formula as its first approximation and extending the approximation up to the term of $O(e^2)$, called the ellipsoidal correction to Hotine's formula. In the following subsections, we will give a formula of computing the ellipsoidal correction.

5.2.1 Establishment of the integral equation

In this section, we will establish an integral equation by means of (5.2.1), which will be employed to obtain the final solution of (5.2.1).

It can be proved that for an arbitrarily point P_0 given inside S_E, the general Hotine function

$$H(P, P_0) = \frac{2r_{P_0}}{l_{PP_0}} - \ln \frac{l_{PP_0} + r_{P_0} - r_P \cos\psi_{PP_0}}{r_P(1 - \cos\psi_{PP_0})} \tag{5.2.2}$$

satisfies

$$H(P,P_0) \in H[S_e] \qquad \text{(for the fixed point } P_0) \tag{5.2.3}$$

So from (5.2.1) and Green's second identity (Heiskanen and Moritz, 1967), we obtain that

$$\int_{S_e} T(Q)\frac{\partial H(Q,P_0)}{\partial h_Q}dQ = -\int_{S_e} \delta g(Q)H(Q,P_0)dQ \tag{5.2.4}$$

In (5.2.2), letting P be the moving point Q on S_e and differentiating $H(Q, P_0)$ along the normal plumb line at Q, we get

$$\begin{aligned}\frac{\partial H(Q,P_0)}{\partial h_Q} &= -\frac{2r_{P_0}}{l_{QP_0}^2}\frac{\partial l_{QP_0}}{\partial h_Q} + \frac{1}{r_Q(1-\cos\psi_{QP_0})}\left[\frac{\partial r_Q}{\partial h_Q} - \frac{\partial(r_Q\cos\psi_{QP_0})}{\partial h_Q}\right] \\ &\quad - \frac{1}{l_{QP_0}+r_{P_0}-r_Q\cos\psi_{QP_0}}\left[\frac{\partial l_{QP_0}}{\partial h_Q} - \frac{\partial(r_Q\cos\psi_{QP_0})}{\partial h_Q}\right] \\ &= -\frac{2r_{P_0}}{l_{QP_0}^2}\frac{\partial l_{QP_0}}{\partial h_Q} + K_1(Q,P_0)\end{aligned} \tag{5.2.5}$$

where

$$\begin{aligned}K_1(Q,P_0) &= \frac{1}{r_Q(1-\cos\psi_{QP_0})}\left[\frac{\partial r_Q}{\partial h_Q} - \frac{\partial(r_Q\cos\psi_{QP_0})}{\partial h_Q}\right] \\ &\quad - \frac{1}{l_{QP_0}+r_{P_0}-r_Q\cos\psi_{QP_0}}\left[\frac{\partial l_{QP_0}}{\partial h_Q} - \frac{\partial(r_Q\cos\psi_{QP_0})}{\partial h_Q}\right]\end{aligned} \tag{5.2.6}$$

It is easy to prove that when P_0 goes to P (the projection of P_0 on S_E) from the inner of S_E,

$$\int_{S_e} T(Q)\frac{2r_{P_0}}{l_{QP_0}^2}\frac{\partial l_{QP_0}}{\partial h_Q}dQ = -\int_{S_e} 2T(Q)r_{P_0}\frac{\partial}{\partial h_Q}(\frac{1}{l_{QP_0}})dQ \to 4\pi T(P)r_{P_0} + \int_{S_e} T(Q)\frac{2r_{P_0}}{l_{QP}^2}\frac{\partial l_{QP}}{\partial h_Q}dQ \quad (5.2.7)$$

and

$$\int_{S_e} T(Q)K_1(Q,P_0)dQ \to \int_{S_e} T(Q)K_1(Q,P)dQ \quad (5.2.8)$$

$$\int_{S_e} \delta g(Q)H(Q,P_0)dQ \to \int_{S_e} \delta g(Q)H(Q,P)dQ \quad (5.2.9)$$

So for any given point P on S_E, we obtain by letting $P_0 \to P$ in (5.2.4) that

$$4\pi T(P)r_P - \int_{S_e} T(Q)K(Q,P)dQ = \int_{S_e} \delta g(Q)H(Q,P)dQ \quad (5.2.10)$$

where

$$K(Q,P) = \frac{\partial H(Q,P)}{\partial h_Q} \quad (5.2.11)$$

Equation (5.2.10) is the integral equation that will be used for determining T on S_e.

5.2.2 Determination of the geoidal height

Denoting the projection of the surface element dQ onto the unit sphere σ by $d\sigma_Q$, we have

$$dQ = r_Q^2 \sec\beta_Q d\sigma_Q \quad (5.2.12)$$

where β_Q is the angle between the radial vector and the normal of Q on S_e. Then, for any given point P on S_E, (5.2.10) becomes

$$4\pi T(P)-\int_\sigma T(Q)K(Q,P)\frac{r_Q^2}{r_P}\sec\beta_Q d\sigma_Q=\int_\sigma \delta g(Q)H(Q,P)\frac{r_Q^2}{r_P}\sec\beta_Q d\sigma_Q \qquad (5.2.13)$$

With b_e the semiminor axis and e the first eccentricity of the reference ellipsoid, and θ_P and θ_Q respectively the complements of the geocentric latitudes of P and Q, we have similar formulas with those given in chapter 3:

$$r_P=b_e[1+\frac{1}{2}e^2(1-\cos^2\theta_P)+O(e^4)] \qquad (5.2.14a)$$

$$r_Q=b_e[1+\frac{1}{2}e^2(1-\cos^2\theta_Q)+O(e^4)] \qquad (5.2.14b)$$

$$l_{QP}=2b_e\sin\frac{\psi_{QP}}{2}[1+\frac{1}{4}e^2(2-\cos^2\theta_Q-\cos^2\theta_P)+O(e^4)] \qquad (5.2.14c)$$

$$r_Q^2\sec\beta_Q=b_e^2[1+e^2(1-\cos^2\theta_Q)+O(e^4)] \qquad (5.2.14d)$$

$$\frac{\partial r_Q}{\partial h_Q}=1+O(e^4) \qquad (5.2.14e)$$

$$\frac{\partial l_{QP}}{\partial h_Q}=\sin\frac{\psi_{QP}}{2}[1-\frac{1}{4}e^2(3\cos^2\theta_Q+\cos^2\theta_P-\frac{(\cos\theta_Q-\cos\theta_P)^2}{\sin^2\frac{\psi_{QP}}{2}})+O(e^4)] \qquad (5.2.14f)$$

$$\frac{\partial(r_Q\cos\psi_{QP})}{\partial h_Q}=\cos\psi_{QP}+e^2(\cos\theta_Q\cos\theta_P-\cos^2\theta_Q\cos\psi_{QP})+O(e^4) \qquad (5.2.14g)$$

It then follows from (5.2.1) that

$$H(Q,P)\frac{r_Q^2}{r_b}\sec\beta_Q=b_e[H(\psi_{QP})+e^2f_1(\psi_{QP},\theta_Q,\theta_P)+O(e^4)] \qquad (5.2.15)$$

where $H(\psi_{QP})$ is the Hotine function

$$H(\psi) = \frac{1}{\sin(\psi/2)} - \ln\left(1 + \frac{1}{\sin(\psi/2)}\right) \tag{5.2.16}$$

and

$$f_1(\psi_{QP}, \theta_Q, \theta_P) = H(\psi_{QP})(\sin^2\theta_Q - \frac{1}{2}\sin^2\theta_P) \tag{5.2.17}$$

From (5.2.11) and (5.2.5), we obtain

$$K(Q,P)\frac{r_Q^2}{r_P}\sec\beta_Q = e^2 f_2(\psi_{QP}, \theta_Q, \theta_P) + O(e^4) \tag{5.2.18}$$

where

$$\begin{aligned} f_2(\psi_{QP}, \theta_Q, \theta_P) = & -\frac{(\cos\theta_Q - \cos\theta_P)^2}{8\sin^3\frac{\psi_{QP}}{2}} + \frac{(\cos\theta_P - 3\cos\theta_Q)(\cos\theta_P - \cos\theta_Q)}{8\sin^2\frac{\psi_{QP}}{2}} \\ & + \frac{5\cos^2\theta_Q - \cos^2\theta_P}{8\sin\frac{\psi_{QP}}{2}} \\ & + \frac{1}{8(1+\sin\frac{\psi_{QP}}{2})}\left[-\frac{(\cos\theta_Q - \cos\theta_P)^2}{\sin^2\frac{\psi_{QP}}{2}} + \frac{4\cos\theta_Q\cos\theta_P - 4\cos^2\theta_Q}{\sin\frac{\psi_{QP}}{2}}\right. \\ & \left. - 5\cos^2\theta_Q + \cos^2\theta_P\right] \end{aligned} \tag{5.2.19}$$

Therefore from (5.2.13), we obtain

$$\begin{aligned} & 4\pi T(P) - e^2\int_\sigma T(Q) f_2(\psi_{QP}, \theta_Q, \theta_P) d\sigma_Q \\ & = b_e\int_\sigma \delta g(Q) H(\psi_{QP}) d\sigma_Q + e^2 b_e \int_\sigma \delta g(Q) f_1(\psi_{QP}, \theta_Q, \theta_P) d\sigma_Q + O(e^4) \end{aligned} \tag{5.2.20}$$

Let

$$T(P) = T_0(P) + e^2 T_1(P) + O(e^4) \tag{5.2.21}$$

Then from (6.1.27), we obtain

$$T_0(P) = \frac{b_e}{4\pi}\int_\sigma \delta g(Q) H(\psi_{QP}) d\sigma_Q \tag{5.2.22}$$

$$T_1(P) = \frac{1}{4\pi}\{b_e \int_\sigma \delta g(Q) f_1(\psi_{QP}, \theta_Q, \theta_P) d\sigma_Q + \int_\sigma T_0(Q) f_2(\psi_{QP}, \theta_Q, \theta_P) d\sigma_Q\} \tag{5.2.23}$$

According to the Bruns formula, we obtain from (5.2.21-23) the geoidal height

$$N(P) = N_0'(P) + e^2 N_1'(P) + O(e^4) \tag{5.2.24}$$

with the spherical geoidal height

$$N_0'(P) = \frac{b_e}{4\pi\gamma_P}\int_\sigma \delta g(Q) H(\psi_{QP}) d\sigma_Q \tag{5.2.25}$$

and its ellipsoidal correction

$$N_1'(P) = \frac{1}{4\pi}\{\frac{b_e}{\gamma_P}\int_\sigma \delta g(Q) f_1(\psi_{QP}, \theta_Q, \theta_P) d\sigma_Q + \int_\sigma N_0'(Q) f_2(\psi_{QP}, \theta_Q, \theta_P) d\sigma_Q\} \tag{5.2.26}$$

where f_1 and f_2 are defined respectively by (5.2.17) and (5.2.19).

From the definition of the mean radius R of the Earth (see (1.1.24)), we have

$$b_e = R(1-\frac{e^2}{3}) \tag{5.2.27}$$

It follows from (5.2.24-26) that

$$N(P) = N_0(P) + e^2 N_1(P) + O(e^4) \tag{5.2.28}$$

$$N_0(P) = \frac{R}{4\pi\gamma_P}\int_\sigma H(\psi_{QP})\delta g(Q)d\sigma \tag{5.2.29}$$

$$N_1(P) = \frac{1}{4\pi}\{\frac{R}{\gamma_P}\int_\sigma \delta g(Q) f_0(\psi_{QP},\theta_Q,\theta_P)d\sigma_Q + \int_\sigma N_0'(Q) f_2(\psi_{QP},\theta_Q,\theta_P)d\sigma_Q\} \tag{5.2.30}$$

where

$$f_0(\psi_{QP},\theta_Q,\theta_P) = H(\psi_{QP})(\sin^2\theta_Q - \frac{1}{2}\sin^2\theta_P - \frac{1}{3}) \tag{5.2.31}$$

Equation (5.2.28) plus equations (5.2.29), (5.2.30), (5.2.31) and (5.2.19) are the formulas for computing the ellipsoidal geoidal height with accuracy of the order of $O(e^4)$. We can call (5.2.28) the **ellipsoidal Hotine formula** and N_1 the **ellipsoidal correction** for Hotine's formula (5.2.29).

5.2.3 Practical considerations on the ellipsoidal correction

In the above subsection, we obtained the Hotine formula (5.2.29) and its ellipsoidal correction (5.2.30). The ellipsoidal correction N_1 is a sum of two integrals about the gravity disturbance δg and the spherical geoidal height N_0. Here we give some details on the evaluation of the integrals of the ellipsoidal correction N_1.

(i) The kernel functions f_0 and f_2 of the integrals are singular at $\psi_{QP}=0$ because they contain the factors

$$\frac{(\cos\theta_Q-\cos\theta_P)^2}{\sin^3\frac{\psi_{QP}}{2}},\ \frac{\cos\theta_Q-\cos\theta_P}{\sin^2\frac{\psi_{QP}}{2}},\ \frac{1}{\sin\frac{\psi_{QP}}{2}} \tag{5.2.32}$$

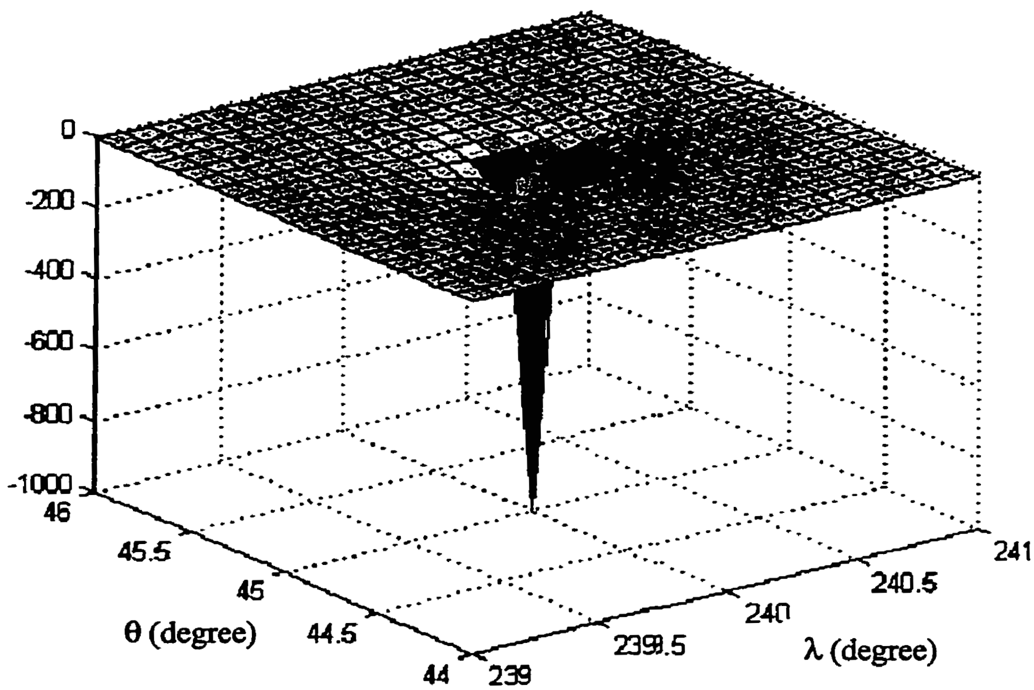

Figure 5.1 Behavior of kernel function f_0 of the ellipsoidal correction to Hotine's formula in the neighborhood of (θ_P=45^0, λ_P=240^0)

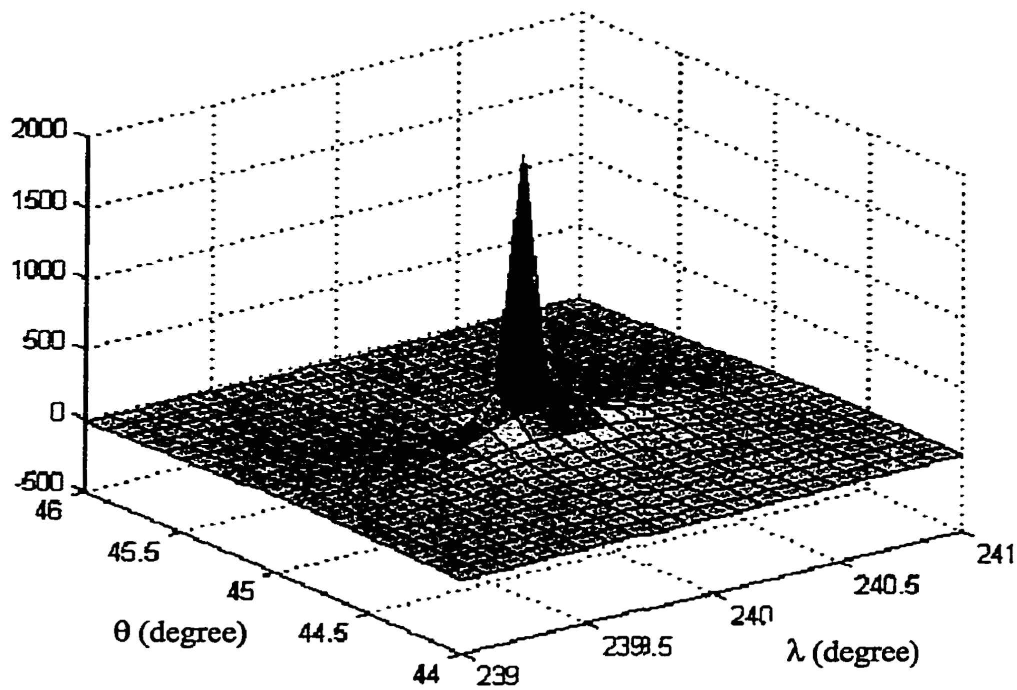

Figure 5.2 Behavior of kernel function f_2 of the ellipsoidal correction to Hotine's formula in the neighborhood of (θ_P=45^0, λ_P=240^0)

From (3.3.3), we have that for $\psi_{QP} << 1$,

$$f_0(\psi_{QP}, \theta_Q, \theta_P) \approx \frac{1}{\psi_{QP}}(\sin^2\theta_P - \frac{2}{3}) \tag{5.2.33}$$

$$f_2(\psi_{QP}, \theta_Q, \theta_P) \approx \frac{\cos^2\theta_P + (\sin\theta_P \cos\alpha_{QP})^2 - \frac{1}{2}\sin 2\theta_P \cos\alpha_{QP}}{\psi_{QP}} \tag{5.2.34}$$

Thus the kernel functions f_0 and f_2 have the same degree of singularity at $\psi_{QP} = 0$ as the Stokes function $S(\psi_{QP})$ and the Hotine function $H(\psi_{QP})$. So the integrals in (5.2.30) are weakly singular integrals and the singularity can be treated by the method used for Stokes's integral (see Heiskanen and Moritz, 1967).

(ii) From the definitions (2.31) of f_0, we know that like the Stokes function, f_0 quickly decreases when ψ_{QP} goes from 0 to π. Therefore, in the practical evaluation of the integral component, we divide σ into two parts: σ_{near} and σ_{far}, where the area σ_{near} is usually a spherical cap containing the computation point P as its centre. Since the kernel function is larger over σ_{near}, the integral over σ_{near} should be carefully computed using a high resolution and high accuracy spherical geoid model obtained from the ground gravity data by means of Stokes's formula (2.42) if a high accuracy geoid is required. The area σ_{far} is far from the computation point P, so the kernel function is relatively small over σ_{far}. Therefore, in the computation of the integral over σ_{far}, we can use the spherical geoidal height data N_0 computed from a global geopotential model.

5.3 Treatment of the topography in Hotine's formula

In the preceding sections, we have given the solutions of the spherical and ellipsoidal boundary value problems: Hotine's formula (5.1.13) and the ellipsoidal Hotine formula (5.2.28). In these two formulas, the input data δg should be given on the surface S_M of the mean sphere (or the reference ellipsoid S_e) and there is no mass outside S_M (or S_e). However, in the SGBVP, we can only have gravity disturbances on the topographic surface S_E and the mass densities outside S_e (or S_M) are not zero. So before employing the Hotine formula to solve the SGBVP, we should adjust the anomalous gravity field to convert the SGBVP into the spherical or ellipsoidal boundary value problem. In this section, we will introduce two methods of adjusting the anomalous gravity field: Helmert's condensation reduction and the analytical continuation approach. The solutions of these two methods are expressed as the sum of Hotine's formula and a correction term. This correction term, which reflects the effect of the mass above the sphere or ellipsoid, is called the topographic correction. We will also introduce an integral equation method that gives an approximate solution of the SGBVP by solving it directly. The solution of the integral equation method is also expressed as the sum of Hotine's formula and a topographic correction. Finally, we will give a brief comparison of the three topographic corrections.

5.3.1 Helmert's condensation reduction

Helmert's condensation reduction is a very classical method of accounting for the topographic masses in Stokes's theory (Sideris, 1990). Here we use its basic spirit to handle the effect of the topographic masses with Hotine's formula.

- **Basic Steps**

Similar to its application in Stokes's theory, the basic steps of Helmert's condensation reduction are as follows:

(a) remove all masses above the reference ellipsoid S_e;
(b) lower the station from the observation point P on the topographic surface S_E to the point P_0 on the ellipsoid S_e;
(c) restore the masses condensed on a layer on the ellipsoid S_e with density $\sigma=\rho h$.

- **Formulas**

From this procedure, we can compute δg_0^H on the ellipsoid via

$$\delta g_0^H = \delta g_P - A_P + A_{P_0}^C = \delta g_P + \delta A \tag{5.3.1}$$

where the superscript H denotes Helmet's reduction, δg_P is the free-air gravity disturbance at P, A_P is the attraction of the topography at P and $A_{P_0}^C$ is the attraction of the condensed topography at P_0.

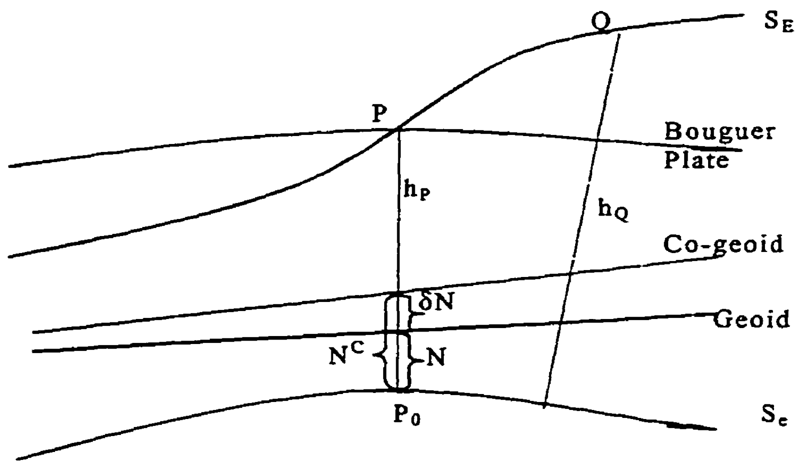

Figure 5.3 The geometry of Helmert's condensation reduction

Obviously, the attraction change δA is not the only change associated with this reduction. Due to the shifting of masses, the potential changes as well by an amount called the indirect effect on the potential, given by the following equation:

$$\delta T = T_{P_0} - T_{P_0}^{C} \tag{5.3.2}$$

where T_{P_0} is the potential of the topographic masses at P_0 and $T_{P_0}^{C}$ is the potential of the condensed masses at P_0. Due to this potential change, the use of Hotine's formula with δg_0^H produces not the geoid but a surface called the co-geoid. Thus, the final expression giving the geoidal heights can be written as

$$N_P = \frac{R}{4\pi\gamma}\int_\sigma H(\psi_{P_0Q_0})\delta g_0^H(Q)d\sigma + \frac{\delta T(P)}{\gamma} = N^c(P) + \delta N(P) \tag{5.3.3}$$

where N^c is the co-geoidal height and δN is the indirect effect on the geoid. In planar approximation, using geodetic heights h instead of orthometric heights H in Sideris (1990), δT and δA can be formulated using the vertical derivative operator **L** (see (1.4.12d)). The potential change is

$$\delta T = -\pi G\rho h_P^2 - 2\pi G\rho \sum_{k=1}^{\infty} \frac{1}{(2k+1)!} L^{2k-1} h^{2k+1} \tag{5.3.4}$$

and the attraction change is equal to the terrain correction c:

$$\delta A_P = c_P = 2\pi G\rho \sum_{k=1}^{\infty} \frac{1}{(2k)!} L^{2k-1} (h - h_P)^{2k} \tag{5.3.5}$$

where G denotes Newton's gravitational constant and ρ is the density of the topography, which is supposed to be known.

If we just consider the first terms of (5.3.4) and (5.3.5), then

$$\delta N(P) = -\pi G\rho h_P^2 \gamma^{-1} \tag{5.3.6}$$

and

$$\delta A_P = \pi G\rho (L(h - h_P)^2)_P = \frac{1}{2} G\rho R^2 \int_\sigma \frac{(h_Q - h_P)^2}{l_{P_0Q_0}^3} d\sigma \tag{5.3.7}$$

- **Discussion**

  - The method described above is similar to that used in the Stokes theory except that the orthometric height H used in the Stokes theory is substituted by the geodetic height h here.

  - The density of the topographic masses is assumed to be known in the solution.

  - An ellipsoidal correction discussed in section 5.2 should be added to the final solution above if a high accuracy geoid model is required.

5.3.2 Analytical continuation method

In this section, we will use the Moritz's method (Moritz 1980) used in Molodensky's problem to solve the SGBVP. In this method, no density assumptions are required and to satisfy the condition of Hotine's formula, a potential $\overline{T}$ harmonic outside the geocentre and satisfying the theorem 3 in chapter 2 will be employed.

- **Analytical continuation of the gravity disturbance**

Let P be a point on the topographic surface S_E, at which the height anomaly ζ is wanted, and Q be a point on S_E at which the gravity disturbance δg is given. The gravity disturbance of $\overline{T}$ at the ellipsoid S_P through P is denoted by $\delta\overline{g}$ (see the following figure).

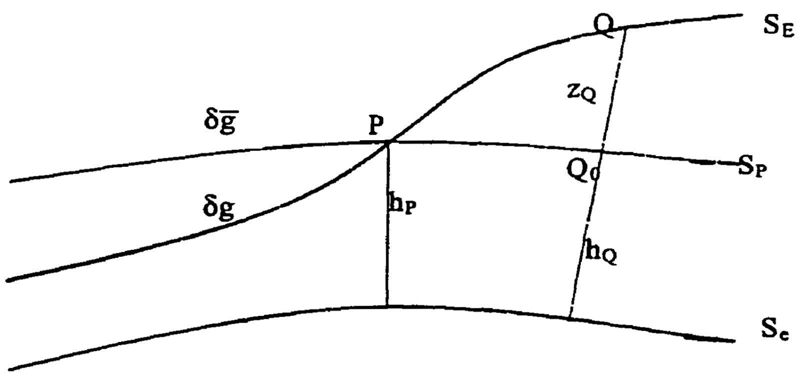

Figure 5.4 The geometry of the analytical continuation method

Similar to the work done in section 2.4.2, we can obtain the relation between δg and $\delta\overline{g}$ as follows

$$\delta g_Q = \sum_{n=0}^{\infty} \frac{1}{n!} z_Q^n \frac{\partial^n \delta \overline{g}_{Q_0}}{\partial z_Q^n} \qquad z_Q = h_Q - h_P \tag{5.3.8}$$

By inverting the above equation, we obtain

$$\delta \overline{g}_{Q_0} = \sum_{n=0}^{\infty} g_n(Q), \; g_0(Q) = \delta g_Q, \; g_n(Q) = -\sum_{m=1}^{n} z_Q^m (L_m g_{n-m})_Q \tag{5.3.9}$$

where L is the vertical derivative operator.

- **Applying Hotine's formula**

Since $\delta\overline{g}$ is given on the ellipsoid S_P, ζ at point P can be obtained by Hotine's formula

$$\zeta_P = \frac{R_P}{4\pi\gamma_P} \int_\sigma H(\psi_{PQ}) \delta \overline{g}_Q d\sigma_Q = \sum_{n=0}^{\infty} \frac{R_P}{4\pi\gamma_P} \int_\sigma H(\psi_{PQ}) \delta g_n(Q) d\sigma_Q \tag{5.3.10}$$

Considering only the first two terms of the above infinite series, we have

$$\zeta_P = N_0(P) + \frac{3h_P}{R} N_0(P) + \mathbf{H}(g_1)_P \tag{5.3.11}$$

where

$$N_0(P) = \frac{R}{4\pi\gamma} \int_\sigma H(\psi_{PQ}) \delta g_Q d\sigma_Q \tag{5.3.12}$$

The Hotine operator **H** is defined by

$$\mathbf{H}(f)_P = \frac{R}{4\pi\gamma}\int_\sigma H(\psi_{PQ}) f_Q d\sigma_Q \tag{5.3.13}$$

and

$$g_1(Q) = -(h_Q - h_P)L(\delta g)_Q = \frac{(h_Q - h_P)\delta g_Q}{R} - \frac{R^2}{2\pi}\int_\sigma \frac{(h_Q - h_P)(\delta g_{Q'} - \delta g_Q)}{l^3_{Q_0 Q'_0}} d\sigma_{Q'} \tag{5.3.14}$$

- **Discussion**

  - The method described above is identical to the method given by Moritz (1980) in the third geodetic boundary value problem except that the orthometric height is employed here.

  - An ellipsoidal correction discussed in section 5.2 should be added to the final solution above if the geodetic height h is employed in the solution.

5.3.3 The integral equation method

In this section, we will give an integral equation method of solving the second geodetic boundary value problem

$$\begin{cases} T \in H[S_E] \\ \dfrac{\partial}{\partial h} T_P = -\delta g(P) \qquad P \text{ is on } S_E \end{cases} \tag{5.3.15}$$

where S_E is the topographic surface of the Earth. The generalized Hotine formula will be employed to establish two integral equations from which the height anomaly ζ will be obtained.

- **Establishment of the integral equations**

It can be proved that for any continuous function δg^* on S_E, the function

$$T(P) \triangleq \frac{1}{4\pi R}\int_{S_E} H(P,Q)\delta g^*(Q)dS_E \quad \text{(P is on and outside } S_E) \tag{5.3.16}$$

satisfies the first condition of (5.3.15). Thus we can represent the disturbing potential T by the above formula provided that there exists δg so that T satisfies the boundary condition in (5.3.15). Differentiating (5.3.16), we obtain

$$\frac{\partial}{\partial r_P}T(P) = \frac{1}{4\pi R}\int_{S_E}[\frac{\partial}{\partial r_P}H(P,Q)]\delta g^*(Q)dS_E \quad \text{(P is outside } S_E) \tag{5.3.17}$$

From the definition (5.1.12) of the generalized Hotine function, we have that

$$\frac{\partial}{\partial r_P}H(P,Q) = \frac{r_Q(r_Q^2 - r_P^2)}{r_P l_{PQ}^3} = 2r_Q\frac{\partial}{\partial r_P}l_{PQ} + \frac{r_Q}{r_P l_{PQ}} \tag{5.3.18}$$

Letting P go to S_E, we obtain from the boundary condition and the properties of the single layer potential (see Heiskanen and Moritz, 1967) that

$$-\delta g_P = -\frac{r_P}{R}\delta g^*(P)\cos\beta_P + \frac{1}{4\pi R}\int_{S_E}\frac{r_Q(r_Q^2 - r_P^2)}{r_P l_{PQ}^3}\delta g^*(Q)dS_E \tag{5.3.19}$$

where β_P is the angle between the radius vector of P and the normal of S_E at point P.

Denoting $\delta g^*(Q)\sec\beta_Q$ and the projection of the surface element dS_E onto the unit sphere σ by $\mu(Q)$ and $d\sigma_Q$, respectively, we have from (5.3.16) and (5.3.19) that

$$T(P)=\frac{1}{4\pi R}\int_{\sigma} r_Q^2 H(P,Q)\mu(Q)d\sigma_Q \quad \text{(P is on and outside } S_E \text{ and Q is on } S_E\text{)} \quad (5.3.20)$$

$$\delta g_P=\frac{r_P}{R}\mu(P)\cos^2\beta_P+\frac{1}{4\pi}\int_{\sigma}\frac{r_Q^3(r_P^2-r_Q^2)}{Rr_P l_{PQ}^3}\mu(Q)d\sigma_Q \quad \text{(P and Q are on } S_E\text{)} \quad (5.3.21)$$

- **An approximate solution of the integral equations**

In the following, we will now solve the above two integral equations to get the formulas for computing the height anomalies on the topographic surface S_E from the gravity disturbances on S_E.

Denoting the distances of P and Q to S_M by δr_P and δr_Q respectively (see figure below), we have

$$r_P=R+\delta r_P=R[1+\frac{\delta r_P}{R}] \quad (5.3.22)$$

$$r_Q=R+\delta r_Q=R[1+\frac{\delta r_Q}{R}] \quad (5.3.23)$$

$$l_{PQ}^2=4R^2\sin^2\frac{\psi_{PQ}}{2}[1+\frac{\delta r_P+\delta r_Q}{R}+\frac{\delta r_P\delta r_Q}{R^2}+\frac{(\delta r_P-\delta r_Q)^2}{4R^2\sin^2\frac{\psi_{PQ}}{2}}] \quad (5.3.24)$$

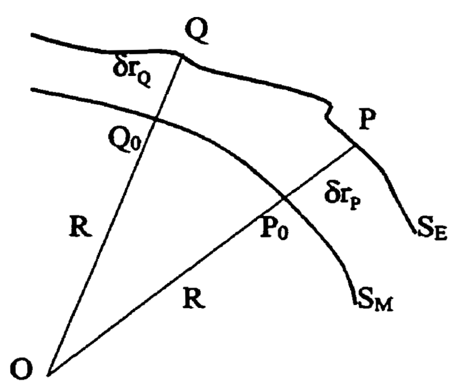

Figure 5.5 The geometry of the integral equation method

Let

$$f_{PQ} = \frac{\delta r_P - \delta r_Q}{2R\sin\frac{\psi_{PQ}}{2}}, \quad k_1 = \sup_{P,Q\in S_T} |f_{PQ}| \tag{5.3.25}$$

and

$$k_2 = \sup_{P,Q\in S_T} \left|\frac{\delta r_P}{R}\right| \tag{5.3.26}$$

Obviously

$$k_2 < \frac{25\text{km}}{6371\text{km}} < 0.004 \tag{5.3.27}$$

In the sequel, we suppose that

$$k_1 < 1 \tag{5.3.28}$$

and neglect the quantities equal to or less than the order of k_2^2 and $k_2 f_{PQ}^2$.

From (5.3.24) and noting (5.3.27) and (5.3.28), we have

$$l_{PQ} = 2R\sin\frac{\psi_{PQ}}{2}[1 + \frac{1}{2}\frac{\delta r_P + \delta r_Q}{R} + \frac{1}{2}f_{PQ}^2 + O(f_{PQ}^4)] \tag{5.3.29}$$

$$\frac{1}{l_{PQ}} = \frac{1}{2R\sin\frac{\psi_{PQ}}{2}}[1 - \frac{1}{2}\frac{\delta r_P + \delta r_Q}{R} - \frac{1}{2}f_{PQ}^2 + O(f_{PQ}^4)] \tag{5.3.30}$$

$$\frac{1}{l_{PQ}^3} = \frac{1}{8R^3\sin^3\frac{\psi_{PQ}}{2}}[1 - \frac{3}{2}\frac{\delta r_P + \delta r_Q}{R} - \frac{3}{2}f_{PQ}^2 + O(f_{PQ}^4)] \tag{5.3.31}$$

It follows that

$$r_Q^2 H(P,Q) = R^2[H(\psi_{PQ}) + \frac{2\delta r_Q}{R} H(\psi_{PQ}) - \frac{(1+2\sin\frac{\psi_{PQ}}{2})f_{PQ}^2}{2\sin\frac{\psi_{PQ}}{2}(1+\sin\frac{\psi_{PQ}}{2})} + O(\frac{f_{PQ}^4}{\sin\frac{\psi_{PQ}}{2}})] \quad (5.3.32)$$

$$\frac{r_Q^3(r_P^2 - r_Q^2)}{Rr_P l_{PQ}^3} = \frac{\delta r_P - \delta r_Q}{4R\sin^3\frac{\psi_{PQ}}{2}}[1 + 2\frac{\delta r_Q - \delta r_P}{R} - \frac{3}{2}f_{PQ}^2 + O(f_{PQ}^4)] \quad (5.3.33)$$

Furthermore, let

$$\mu(P) = \mu_0(P) + \mu_1(P) \quad (5.3.34)$$

$$T(P) = T_0(P) + T_1(P) \quad (5.3.35)$$

where

$$\mu_0(P) = \delta g_P \quad (5.3.36)$$

$$T_0(P) = \frac{R}{4\pi}\int_\sigma H(\psi_{PQ})\delta g_Q d\sigma_Q \quad (5.3.37)$$

Inserting equations (5.3.22), (5.3.33), (5.3.34) and (5.3.36) into (5.3.21) and equations (5.3.32), (5.3.35) and (5.3.37) into (5.3.20) and neglecting the terms equal to or less than the order of $k_2\mu_1$, we obtain

$$\mu_1(P) = \delta g_P[\tan^2\beta_P - \frac{\delta r_P}{R}] + \frac{\sec^2\beta_P}{4\pi}\int_\sigma \frac{\delta r_Q - \delta r_P}{4R\sin^3\frac{\psi_{PQ}}{2}}\delta g_Q d\sigma_Q$$

$$+\frac{1}{2\pi}\int_\sigma \frac{f_{PQ}^2}{\sin\frac{\psi_{PQ}}{2}}\delta g_Q d\sigma_Q - \frac{3}{8\pi}\int_\sigma \frac{\delta r_Q - \delta r_P}{4R\sin^3\frac{\psi_{PQ}}{2}} f_{PQ}^2 \delta g_Q d\sigma_Q$$

$$+O\left(\frac{1}{4\pi}\int_\sigma \frac{\delta r_Q - \delta r_P}{4R\sin^3\frac{\psi_{PQ}}{2}} f_{PQ}^4 \delta g_Q d\sigma_Q\right) \tag{5.3.38}$$

$$T_1(P) = \frac{R}{4\pi}\int_\sigma H(\psi_{PQ})[\mu_1(Q) + \frac{2\delta r_Q}{R}\delta g_Q]d\sigma_Q$$

$$-\frac{R}{4\pi}\int_\sigma \frac{(1+2\sin\frac{\psi_{PQ}}{2})f_{PQ}^2}{2\sin\frac{\psi_{PQ}}{2}(1+\sin\frac{\psi_{PQ}}{2})}\delta g_Q d\sigma_Q + O\left(\frac{R}{4\pi}\int_\sigma \frac{f_{PQ}^4}{\sin\frac{\psi_{PQ}}{2}}\delta g_Q d\sigma_Q\right) \tag{5.3.39}$$

In the above two equations, neglecting respectively the terms of the order of

$$\left|\frac{1}{4\pi}\int_\sigma \frac{\delta r_Q - \delta r_P}{4R\sin^3\frac{\psi_{PQ}}{2}} f_{PQ}^4 \delta g_Q d\sigma_Q\right| \tag{5.3.40}$$

$$\left|\frac{R}{4\pi}\int_\sigma \frac{f_{PQ}^4}{\sin\frac{\psi_{PQ}}{2}}\delta g_Q d\sigma_Q\right| \tag{5.3.41}$$

we obtain

$$\mu_1(P) = \delta g_P[\tan^2\beta_P - \frac{\delta r_P}{R}] + \frac{\sec^2\beta_P}{4\pi}\int_\sigma \frac{\delta r_Q - \delta r_P}{4R\sin^3\frac{\psi_{PQ}}{2}}\delta g_Q d\sigma_Q$$

$$+\frac{1}{2\pi}\int_\sigma \frac{f_{PQ}^2}{\sin\frac{\psi_{PQ}}{2}}\delta g_Q d\sigma_Q - \frac{3}{8\pi}\int_\sigma \frac{\delta r_Q - \delta r_P}{4R\sin^3\frac{\psi_{PQ}}{2}} f_{PQ}^2 \delta g_Q d\sigma_Q \tag{5.3.42}$$

$$T_1(P)=\frac{R}{4\pi}\int_\sigma H(\psi_{PQ})[\mu_1(Q)+\frac{2\delta r_Q}{R}\delta g_Q]d\sigma_Q$$

$$-\frac{R}{4\pi}\int_\sigma \frac{(1+2\sin\frac{\psi_{PQ}}{2})f_{PQ}^2}{2\sin\frac{\psi_{PQ}}{2}(1+\sin\frac{\psi_{PQ}}{2})}\delta g_Q d\sigma_Q \tag{5.3.43}$$

Let

$$\delta g_1(P)\doteq\mu_1(P)+\frac{2\delta r_P}{R}\delta g_P=\delta g_P[\tan^2\beta_P+\frac{\delta r_P}{R}]+\frac{\sec^2\beta_P}{4\pi}\int_\sigma \frac{\delta r_Q-\delta r_P}{4R\sin^3\frac{\psi_{PQ}}{2}}\delta g_Q d\sigma_Q$$

$$+\frac{\sec^2\beta_P}{2\pi}\int_\sigma \frac{f_{PQ}^2}{\sin\frac{\psi_{PQ}}{2}}\delta g_Q d\sigma_Q-\frac{3\sec^2\beta_P}{8\pi}\int_\sigma \frac{\delta r_Q-\delta r_P}{4R\sin^3\frac{\psi_{PQ}}{2}}f_{PQ}^2\delta g_Q d\sigma_Q \tag{5.3.44}$$

Then from Bruns' formula, we finally obtain from (5.3.35) that

$$\zeta(P)=\zeta_0(P)+\zeta_1(P) \tag{5.3.45}$$

and

$$\zeta_0(P)=\frac{R}{4\pi\gamma_P}\int_\sigma H(\psi_{PQ})\delta g_Q d\sigma_Q \tag{5.3.46}$$

$$\zeta_1(P)=\frac{R}{4\pi\gamma_P}\int_\sigma H(\psi_{PQ})\delta g_1(Q)d\sigma_Q-\frac{R}{4\pi}\int_\sigma \frac{(1+2\sin\frac{\psi_{PQ}}{2})f_{PQ}^2}{2\sin\frac{\psi_{PQ}}{2}(1+\sin\frac{\psi_{PQ}}{2})}\delta g_Q d\sigma_Q \tag{5.3.47}$$

Equations (5.3.45), (5.3.46), (5.3.47) and (5.3.44) are the formulas for computing the height anomalies on the topographic surface S_E, which take the effect of the topography into account. The term δg_1 defined by (5.3.44) can be called the topography correction to the gravity disturbance and ζ_1 defined by (5.3.47) the topographic correction to the height anomaly.

- **Discussion**

Here we will give some discussion on the approximate formulas for computing the height anomalies on the topographic surface S_E.

1. The solution is based on the second geodetic boundary value problem with the topographic surface S_E as its boundary surface. The method for approximately solving the problem is similar with that used by Brovar (Moritz 1980) in Molodensky's problem.

2. In the derivation of the formulas (5.3.45), (5.3.46), (5.3.47) and (5.3.44), we made several assumptions:

   a. To make the equations (5.3.29-31) hold, we suppose (5.3.28) holds;
   b. To make the solutions (5.3.44) and (5.3.47) valid, we suppose that the integral

$$\frac{1}{4\pi}\int_{\sigma}\left|\frac{\delta r_Q - \delta r_P}{4R\sin^3\frac{\psi_{PQ}}{2}}\delta g_Q\right|d\sigma_Q \qquad (5.3.48)$$

   exists.

Since

$$f_{PQ} \rightarrow \tan\beta'_P \qquad \text{when } Q \rightarrow P \tag{5.3.49}$$

where β'_P is the inclination angle of the topographic surface along the direction $\overrightarrow{PQ}$, the condition (5.3.28) can be satisfied when

$$\sup_{P\in S_E}\left|\tan^2\beta_P\right| < 1 \text{ or } \sup_{P\in S_E}\left|\beta_P\right| < 44.856^0 \tag{5.3.50}$$

Now we investigate the method for handling the integral (5.3.48). Obviously, we only need to consider the integral in the innermost spherical cap area σ_0 with the centre at the computation point P and a radius ψ_0, which is so small that the spherical cap area σ_0 can be treated as a plane. That is we discuss the following integral:

$$\delta g^0(P) \triangleq \frac{1}{4\pi}\int_{\sigma_0}\left|\frac{\delta r_Q - \delta r_P}{4R\sin^3\frac{\psi_{PQ}}{2}}\delta g_Q\right| d\sigma_Q \tag{5.3.51}$$

For Q in σ_0, we expand δr_Q into a Taylor series at the computation point P:

$$\delta r_Q = \delta r_P + x\delta r_{xP} + y\delta r_{xP} + (x^2\delta r_{x^2P} + xy\delta r_{xyP} + y^2\delta r_{y^2P})\cdots \tag{5.3.52}$$

where the rectangular coordinates x, y are defined by

$$x = l_{PQ}\cos\alpha_{QP}; \quad y = l_{PQ}\sin\alpha_{QP} \tag{5.3.53}$$

so that the x-axis points north, and

$$\delta r_{xP} = (\frac{\partial\delta r}{\partial x})_P; \delta r_{yP} = (\frac{\partial\delta r}{\partial y})_P; \delta r_{x^2P} = (\frac{\partial^2\delta r}{\partial x^2})_P; \delta r_{xyP} = (\frac{\partial^2\delta r}{\partial x\partial y})_P; \delta r_{y^2P} = (\frac{\partial^2\delta r}{\partial y^2})_P \tag{5.3.54}$$

The Taylor series (5.3.52) may also be written as

$$\delta r_Q - \delta r_P = (\delta r_{xP} \cos\alpha_{QP} + \delta r_{yP} \sin\alpha_{QP}) l_{PQ}$$

$$+ (\delta r_{x^2P} \cos^2\alpha_{QP} + \delta r_{xyP} \cos\alpha_{QP} \sin\alpha_{QP} + \delta r_{y^2P} \sin^2\alpha_{QP}) l_{PQ}^2 \cdots \qquad (5.3.55)$$

Inserting this into (5.3.51), performing the integration with respect to α_{QP} first, noting (4.1.29) and neglecting the quantities of $O(l_0^2)$, we have

$$\delta g^0(P) = \left| \frac{\delta g_P l_0}{2} [\delta r_{x^2P} + \delta r_{y^2P}] \right| \qquad (5.3.56)$$

where

$$l_0 = 2R \sin\frac{\psi_0}{2} \qquad (5.3.57)$$

So, when the topographic surface S_E is smooth enough (its radius vector is differentiable at least twice), the integral (5.3.48) exists and can be handled via the method discussed above.

In practice, the topographic surface can be obtained via mathematical fit from the GPS measurements. So it can be selected to satisfy the above two conditions.

3. In the derivation of the formulas, the quantity δr_P is the distance of P on the topographic surface S_E onto the surface of the mean sphere S_M. This quantity can also be obtained from GPS measurements. In the formulas, the effect of the flattening of the Earth is included in the topographic correction.

5.3.4 Comparison of the three methods

In this subsection, we will discuss briefly the relationships among the three methods for the computation of the geoidal height (or height anomaly) taking the topography into account.

- **Relationship between Helmert's condensation reduction and the analytical continuation**

Now we suppose that δg is linearly related with the elevation h. That is

$$\delta g_P = a + 2\pi G\rho h_P \tag{5.3.58}$$

Furthermore, we suppose that h/R is small enough so that the terms containing it can be neglected. So from (5.3.14), we have

$$g_1(Q) = G\rho R^2 \int_\sigma (h_Q - h_P) \frac{h_Q - h_{Q'}}{l^3_{P_0Q'_0}} d\sigma_{Q'} \tag{5.3.59}$$

It follows from (5.3.7) that

$$\begin{aligned}(\delta A - g_1)_Q &= \frac{1}{2} G\rho R^2 \int_\sigma \frac{(h_Q - h_{Q'})^2 - 2(h_Q - h_P)(h_Q - h_{Q'})}{l^3_{P_0Q'_0}} d\sigma_{Q'} \\ &= \pi G\rho \frac{R^2}{2\pi} \int_\sigma \frac{h^2_{Q'} - h^2_Q}{l^3_{P_0Q'_0}} d\sigma_{Q'} - 2\pi G\rho h_P \frac{R^2}{2\pi} \int_\sigma \frac{h_{Q'} - h_Q}{l^3_{P_0Q'_0}} d\sigma_{Q'} \\ &= \pi G\rho (Lh^2)_Q - 2\pi G\rho h_P (Lh)_Q \end{aligned} \tag{5.3.60}$$

Since

$$\mathbf{HL} = -\gamma^{-1}\mathbf{I} \tag{5.3.61}$$

where I is the unit operator, we obtain that

$$\mathbf{H}(g_1)_P = \mathbf{H}(\delta A)_P - \pi G\rho h_P^2\gamma^{-1} \tag{5.3.62}$$

So from (5.3.11), we have

$$\begin{aligned}\zeta_P &= N_0(P) + \mathbf{H}(\delta A)_P - \pi G\rho h_P^2\gamma^{-1} \\ &= N^C(P) + \delta N(P)\end{aligned} \tag{5.3.63}$$

Comparing with (5.3.33), we can conclude that Helmert's condensation reduction and the analytical continuation are equivalent to each other when δg is linearly related with the elevation h and h/R is zero (planar approximation).

- **Relationship between the solutions of the analytical continuation and the integral equation method**

Here we discuss briefly the relationship between solutions of the analytical continuation and the integral equation method. We use the geodetic height h to replace δr and suppose that h and β_P are small enough so that the terms containing h/R, $\tan^2\beta_P$ and f_{PQ}^2 can be neglected in (5.3.44) and (5.3.47). Thus we obtain

$$\zeta(P) = \zeta_0(P) + \zeta_1(P) \tag{5.3.64}$$

where

$$\zeta_0(P) = N_0(P) = \frac{R}{4\pi\gamma}\int_S H(\psi_{PQ})\delta g_Q dS_Q \tag{5.3.65}$$

and

$$\zeta_1(P) = \frac{R}{4\pi\gamma}\int_\sigma H(\psi_{PQ})\delta g_1(Q)d\sigma_Q \tag{5.3.66}$$

with

$$\delta g_1(P) = \frac{R^2}{2\pi}\int_\sigma \frac{h_Q \delta g_Q - h_P \delta g_Q}{l^3_{P_0Q_0}} d\sigma_Q \tag{5.3.67}$$

The gravity correction term δg_1 can be rewritten as

$$\begin{aligned}\delta g_1(P) &= \frac{R^2}{2\pi}\int_\sigma \frac{h_Q \delta g_Q - h_P \delta g_P}{l^3_{P_0Q_0}} d\sigma_Q - \frac{R^2}{2\pi}\int_\sigma \frac{h_P \delta g_Q - h_P \delta g_P}{l^3_{P_0Q_0}} d\sigma_Q \\ &= -\mathbf{L}(hg)_P + g_1(P)\end{aligned} \tag{5.3.68}$$

where g_1 is defined by (5.3.14). It follows from (5.3.61) and (5.3.66) that

$$\zeta_1(P) = \mathbf{H}(\delta g_1)_P = \mathbf{H}(g_1)_P + \frac{h_P \delta g_P}{\gamma} \tag{5.3.69}$$

where the second term of the right hand side of above formula can be neglected since it is also a small quantity as the quantities containing h/R.

So from (5.3.64) and (5.3.11) we can conclude that under the assumption that h and β_P are small enough so that the terms containing h/R, $\tan^2\beta_P$ and f^2_{PQ} can be neglected in (5.3.44) and (5.3.47), the analytical continuation method and the integral equation method are equivalent to each other.

- **Brief summary**

In this subsection, we compared the three methods of evaluating the topography correction. We have the following suggestions:

1. In the area that the topography satisfies that S_E is smooth enough (its radius vector is differentiable at least twice) and its inclination angles are less than 44^0, we can

employ the integral equation method to evaluate the topographic correction to the geoidal height (or height anomaly) for it is rigorous under these assumptions.

2. In the area that the inclination angles and the elevations of the topography are very small, we can employ the analytical continuation method to evaluate the topographic correction to the geoidal height (or height anomaly) for it is equivalent to the integral equation method and has a simpler expression.

3. Furthermore, in the area that the mass densities of the topography are known and the gravity disturbance δg is linearly related with the elevation h, we can employ Helmert's condensation reduction to evaluate the topographic correction to the geoidal height (or height anomaly) for it is equivalent to the analytical continuation method and has a simpler expression.

5.4 Chapter summary

In this chapter, we investigated in detail the solutions to the SGBVP. We first obtained the generalized Hotine formula and the ellipsoidal Hotine formula respectively from solving the second spherical boundary value problem and the second ellipsoidal boundary value problem. Then we applied the Hotine formulas to solve the SGBVP by the Helmert condensation reduction and the analytical continuation method. We also gave an integral equation method for directly solving the SGBVP. A brief comparison of the solutions of the three methods shows that the integral equation solution needs fewest assumptions but is formulated complicatedly and under some assumptions, these three solutions are equivalent to each other.

6 Local Character of the Anomalous Gravity Field

In section 1.5.4, we discussed the significance of the local character of the anomalous gravity field. In this chapter, we will establish some integral models showing the local relationships of the quantities of the anomalous gravity field by means of kernel functions having some properties of wavelet.

6.1 Definition and properties of the basic kernel

In this section, we will define the basic kernel function for the models, from which other kernel functions can be obtained, and discuss its properties.

Definition 6.1 For an arbitrarily given point P in R^3 and a non-negative integer n, we define $F_{nP}(Q)$ in R^3-{P} as follows:

$$F_{nP}(Q) = \begin{cases} l_{PQ}^{-1} & (n = 0) \\ \dfrac{1}{n!}\dfrac{\partial^n}{\partial r_P^n} l_{PQ}^{-1} & (n > 0) \end{cases} \tag{6.1.1}$$

where r_P is the radius vector of P with respect to the geocentre, and l_{PQ} is the distance between point P and point Q in R^3-{P}.

6.1.1 Lemmas

Before discussing the properties of $F_{nP}(Q)$, we shall introduce several useful lemmas.

Lemma 6.1 Let r_P, r_Q, ψ_{PQ} be the radius of P and Q and the angle between r_P and r_Q respectively, and

$$h_P(Q) = r_Q \cos\psi_{PQ} - r_P \tag{6.1.2}$$

Then

$$\frac{\partial}{\partial r_P} F_{0P}^2(Q) = 2F_{0P}^4(Q) h_P(Q) \tag{6.1.3}$$

$$\frac{\partial}{\partial r_P}[F_{0P}^2(Q) h_P(Q)] = 2F_{0P}^4(Q) h_P^2(Q) - F_{0P}^2(Q) \tag{6.1.4}$$

Proof. From the definition of $F_{nP}(Q)$, we have

$$\frac{\partial}{\partial r_P} F_{0P}(Q) = (r_P^2 + r_Q^2 - 2r_P^2 r_Q^2 \cos\psi_{PQ})^{-3/2}(r_Q \cos\psi_{PQ} - r_P) = F_{0P}^3(Q) h_P(Q) \tag{6.1.5}$$

$$\frac{\partial}{\partial r_P} h_P(Q) = \frac{\partial}{\partial r_P}(r_Q \cos\psi_{PQ} - r_P) = -1 \tag{6.1.6}$$

Therefore

$$\frac{\partial}{\partial r_P} F_{0P}^2(Q) = 2F_{0P}(Q)\frac{\partial}{\partial r_P} F_{0P}(Q) = 2F_{0P}^4(Q) h_P(Q) \tag{6.1.7}$$

$$\frac{\partial}{\partial r_P}(F_{0P}^2(Q) h_P(Q)) = h_P(Q)\frac{\partial}{\partial r_P} F_{0P}^2(Q) + F_{0P}^2(Q)\frac{\partial}{\partial r_P} h_P(Q)$$

$$= 2F_{0P}^4(Q) h_P^2(Q) - F_{0P}^2(Q) \tag{6.1.8}$$

So lemma 6.1 holds#

Lemma 6.2 Let

$$t_P(Q) = F_{0P}(Q)h_P(Q) \tag{6.1.9}$$

Then

$$-1 \le t_P(Q) \le 1 \tag{6.1.10}$$

Proof. From the definition (6.1.1) and (6.1.2), we have

$$1 - t_P^2(Q) = \frac{r_Q^2(1 - \cos^2 \psi_{PQ})}{l_{PQ}^2} \ge 0 \tag{6.1.11}$$

It follows that (6.1.10) holds #

Lemma 6.3 If V is an infinitely differentiable and harmonic function in an area , then for an arbitrarily given constant vector r, $\frac{\partial}{\partial r}V$ is also an infinitely differentiable and harmonic function in the area .

Proof. We denote the angles between r and the coordinate axes by $\alpha_x, \alpha_y, \alpha_z$, respectively. Then

$$\frac{\partial}{\partial r}V = \cos\alpha_x \frac{\partial}{\partial X}V + \cos\alpha_y \frac{\partial}{\partial Y}V + \cos\alpha_z \frac{\partial}{\partial Z}V \quad . \tag{6.1.12}$$

Since r is a constant vector, $\cos\alpha_x$, $\cos\alpha_y$ and $\cos\alpha_z$ are all constants. $\frac{\partial}{\partial r}V$ is infinitely differentiable because V is infinitely differentiable. And since V is harmonic in , we have that in

$$\bar{A}\frac{\partial}{\partial X}V=\frac{\partial^2}{\partial X^2}\frac{\partial}{\partial X}V+\frac{\partial^2}{\partial Y^2}\frac{\partial}{\partial X}V+\frac{\partial^2}{\partial Z^2}\frac{\partial}{\partial X}V$$
$$=\frac{\partial}{\partial X}\frac{\partial^2}{\partial X^2}V+\frac{\partial}{\partial X}\frac{\partial^2}{\partial Y^2}V+\frac{\partial}{\partial X}\frac{\partial^2}{\partial Z^2}V=\frac{\partial}{\partial X}\bar{A}V=0 \tag{6.1.13}$$

which means $\frac{\partial}{\partial X}V$ is harmonic in . In the same way, we can prove that $\frac{\partial}{\partial Y}V$ and $\frac{\partial}{\partial Z}V$ are also harmonic in . It follows from (6.1.12) that $\frac{\partial}{\partial r}V$ is harmonic in #

6.1.2 Properties of the basic kernel function

Now we shall discuss the properties of $F_{nP}(Q)$.

Property 6.1 For an arbitrarily given point P in R^3, we have

$$F_{nP}(Q)=\frac{2n-1}{n}F_{0P}^2(Q)h_P(Q)F_{n-1P}(Q)-\frac{n-1}{n}F_{0P}^2(Q)F_{n-2P}(Q) \quad (n>0) \tag{6.1.14}$$

$$\frac{\partial}{\partial r_Q}F_{nP}(Q)=-\frac{n+1}{r_Q}[F_{nP}(Q)+r_P F_{n+1P}(Q)] \quad (n\geq 0) \tag{6.1.15}$$

Proof. We use mathematical induction to prove the property.

(a) When n =1, we have

$$F_{1P}(Q)=\frac{\partial}{\partial r_P}(l_{PQ}^{-1})=F_{0P}^3(Q)h_P(Q) \tag{6.1.16}$$

i.e. (6.1.14) holds for n =1. Suppose that (6.1.14) holds for n =k, then by (6.1.3) and (6.1.4),

$$F_{k+1P}(Q) = \frac{1}{k+1}\frac{\partial}{\partial r_P}[\frac{2k-1}{k}F_{0P}^2(Q)h_P(Q)F_{k-1P}(Q) - \frac{k-1}{k}F_{0P}^2(Q)F_{k-2P}(Q)]$$

$$= \frac{2k-1}{k+1}F_{0P}^2(Q)h_P(Q)F_{kP}(Q) + \frac{1}{k+1}\frac{2k-1}{k}[2F_{0P}^4(Q)h_P^2(Q) - F_{0P}^2(Q)]F_{k-1P}(Q)$$

$$-\frac{(k-1)^2}{(k+1)k}F_{0P}^2(Q)F_{k-1P}(Q) - \frac{2(k-1)}{(k+1)k}F_{0P}^4(Q)h_P(Q)F_{k-2P}(Q)$$

$$= \frac{2k+1}{k+1}F_{0P}^2(Q)h_P(Q)F_{kP}(Q) - \frac{k}{k+1}F_{0P}^2(Q)F_{k-1P}(Q) \tag{6.1.17}$$

i.e. (6.1.14) holds for n =k+1 . So by the induction principal, (6.1.14) holds.

(b) It is obvious that (6.1.15) holds for n=0. Suppose it holds for n=k, then

$$\begin{aligned}\frac{\partial}{\partial r_Q}F_{k+1P}(Q) &= \frac{1}{k+1}\frac{\partial}{\partial r_P}\frac{\partial}{\partial r_Q}F_{kP}(Q)\\ &= -\frac{1}{k+1}\frac{\partial}{\partial r_P}\frac{(k+1)}{r_Q}[F_{kP}(Q) + r_P F_{k+1P}(Q)]\\ &= -\frac{1}{r_Q}[(k+1)F_{k+1P}(Q) + F_{k+1P}(Q) + (k+2)r_P F_{k+2P}(Q)]\\ &= -\frac{k+2}{r_Q}[F_{k+1P}(Q) + r_P F_{k+2P}(Q)]\end{aligned} \tag{6.1.18}$$

i.e. (6.1.15) holds for n=k+1. It follows that (6.1.15) holds for n≥0 #

Property 6.2 $$F_{nP}(Q) = P_n(t_P(Q))l_{PQ}^{-(n+1)} \tag{6.1.19}$$

where $P_n(t)$ is the *n*-order Legendre polynomial (Heiskanen and Moritz, 1967).

Proof. Let

$$\overline{P}_{nP}(Q) = F_{nP}(Q) l_{PQ}^{n+1} \tag{6.1.20}$$

then by Property 6.1 and noting that $F_{0P}(Q) = l_{PQ}^{-1}$ we have

$$\overline{P}_{nP}(Q) = \frac{2n-1}{n} t_P(Q) \overline{P}_{n-1P}(Q) + \frac{n-1}{n} \overline{P}_{n-1P}(Q) \tag{6.1.21}$$

and

$$\overline{P}_{0P}(Q) = 1 \tag{6.1.22}$$

Therefore by (1-59) in Heiskanen and Moritz (1967) and (6.1.10), we have

$$\overline{P}_{nP}(Q) = P_n(t_P(Q)) \tag{6.1.23}$$

This means that (6.1.19) holds #

Property 6.3 $F_{nP}(Q)$ is harmonic in R^3-{P}.

Proof. It is easy to see that $F_{0P}(Q) = l_{PQ}^{-1}$ is harmonic and infinitely differentiable in R^3-{P}. By Lemma 6.3, we know that $F_{1P}(Q) = \frac{\partial}{\partial r_P} F_{0P}(Q)$ is harmonic and differentiable and, by extending this, we obtain that for any non-negative integer n, $F_{nP}(Q)$ is harmonic in R^3-{P} #

By Property 6.2, we can obtain the following two properties:

Property 6.4 $F_{nP}(Q)$ is regular, i.e., if $r_Q \to \infty$ or $l_{PQ} \to \infty$, then $F_{nP}(Q) \to 0$.

Property 6.5 With an increase in n, when $r_Q \to \infty$ or $l_{PQ} \to \infty$, the rate of vanishing of $F_{nP}(Q)$ increases.

Property 6.6 For $r_P < r_Q$, we have

$$F_{nP}(Q) = \sum_{k=n}^{\infty} \binom{n}{k} \frac{r_P^{k-n}}{r_Q^{k+1}} P_k(\cos\psi_{PQ}) \tag{6.1.24}$$

$$\frac{\partial}{\partial r_Q} F_{nP}(Q) = -\sum_{k=n}^{\infty} (k+1) \binom{n}{k} \frac{r_P^{k-n}}{r_Q^{k+2}} P_k(\cos\psi_{PQ}) \tag{6.1.25}$$

where ψ_{PQ} is the angle between r_Q and r_P.

Proof. According to Heiskanen and Moritz (1967),

$$\frac{1}{l_{PQ}} = \sum_{k=0}^{\infty} \frac{r_P^k}{r_Q^{k+1}} P_k(\cos\psi_{PQ}) \tag{6.1.26}$$

It then follows from (6.1.1) and (6.1.26) that (6.1.24) and (6.1.25) hold #

The properties 6.4 and 6.5 show that $F_{nP}(Q)$ has some properties of wavelet $\psi_{a,b}(t)$ (see Keller, 1995: $\psi_{a,b}(t) \equiv \psi(\frac{b-t}{a})$, n is similar to a^{-1} and P and Q correspond to b and t respectively; also see Hohlschneider, 1995, Freeden and Schneider, 1998 and Liu *et al.*, 1998 for more information about wavelets and their applications). In fact, from (6.1.19), we can see that $F_{nP}(Q)$ is the product of $P_n(t_P(Q))$ and $l_{PQ}^{-(n+1)}$, where the function $P_n(t_P(Q))$ is a wave function which varies between −1 and 1 and whose frequency increases with an increase in n. The function $l_{PQ}^{-(n+1)}$ can be regarded as the amplitude which vanishes with an increase in l_{PQ}, and the rate of vanishing increases as n increases. Regarding this, we can get further understanding from the following two tables.

We consider a situation in which Q moves only on a curved surface, which might as well be supposed to be the surface of a sphere with radius r_Q, and P is a point which does not belong on the surface, i.e. $r_P \neq r_Q$. Let $d = r_P - r_Q$ and ψ be the angle between r_P and r_Q. Then by the definition of $F_{nP}(Q)$, we know that $F_{nP}(Q)$ relates only to r_Q, n, d and ψ, i.e.

$$F_{nP}(Q) = F(r_Q, n, d, \psi) \quad . \tag{6.1.27}$$

Let

$$R_f = F(r_Q, n, d, \psi)/F(r_Q, n, d, 0) \tag{6.1.28}$$

Then tables 6.1 and 6.2 show the relationships between R_f and n, d, ψ when r_Q=6372 km. From table 6.1, we can see that R_f decreases with an increase in ψ, and the rate of decrease increases with an increase in the distance d of P from the surface on which Q moves. For a pictorial representation, see Figure 6.1. From table 6.2, we can see that R_f decreases with an increase in ψ, and the rate of decrease increases with an increase in n. Figure 6.2 illustrates this behavior.

Table 6.1 R_f-values for various values of d(in km) and ψ (in degree) when r_Q =6372km, n=2

ψ_o \ d	-0.001	0.001	-1.000	1.000	-200.000	200.000
0.0	1.0	1.0	1.0	1.0	1.0	1.0
1.0E-06	9.637E-01	9.637E-01	9.999E-01	9.999E-01	1.000E+00	1.000E+00
1.0E-05	5.099E-02	5.099E-02	9.999E-01	9.999E-01	1.000E+00	1.000E+00
1.0E-04	-3.505E-04	-3.505E-04	9.996E-01	9.996E-01	9.999E-01	9.999E-01
1.0E-03	-3.634E-07	-3.634E-07	9.637E-01	9.637E-01	9.999E-01	9.999E-01
1.0E-02	-3.635E-10	-3.635E-10	5.100E-02	5.090E-02	9.999E-01	9.999E-01
1.0E-01	-3.635E-13	-3.635E-13	-3.507E-04	-3.507E-04	9.909E-01	9.906E-01
1.0E+00	-3.634E-16	-3.634E-16	-3.635E-07	-3.630E-07	4.341E-01	4.280E-01
1.0E+01	-3.552E-19	-3.552E-19	-3.554E-10	-3.549E-10	-2.829E-03	-2.117E-03

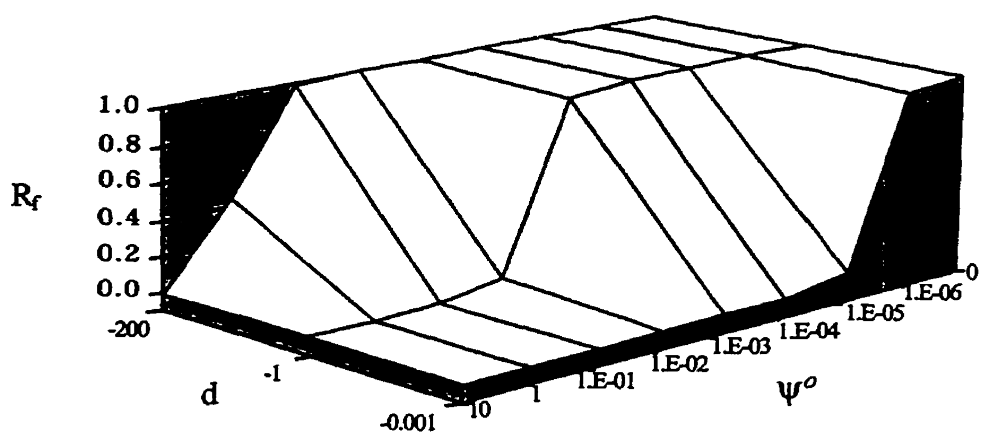

Figure 6.1 The relationship between R_f and d (in km), ψ (degree) when r_Q =6372 km, n=2

Table 6.2 R_f-values for various values of n and ψ (in degree) when r_Q =6372(km) and d=-0.001(km)

ψo \ n	0	1	10	50	100	1000
0.0	1.0	1.0	1.0	1.0	1.0	1.0
1.0E-06	9.938E-01	9.817E-01	6.448E-01	1.810E-02	-7.896E-02	-1.611E-04
1.0E-05	6.686E-01	2.989E-01	-5.211E-04	-1.196E-10	-4.521E-20	0.000E+00
1.0E-04	8.996E-02	7.183E-04	-4.307E-13	7.403E-56	0.000E+00	0.000E+00
1.0E-03	8.991E-03	7.262E-07	-7.608E-24	0.000E+00	0.000E+00	0.000E+00
1.0E-02	8.992E-04	6.565E-10	-7.645E-35	0.000E+00	0.000E+00	0.000E+00
1.0E-01	8.992E-05	-6.329E-12	-7.645E-46	0.000E+00	0.000E+00	0.000E+00
1.0E+00	8.992E-06	-7.048E-13	-7.614E-57	0.000E+00	0.000E+00	0.000E+00
1.0E+01	8.992E-07	-7.056E-14	-4.649E-68	0.000E+00	0.000E+00	0.000E+00

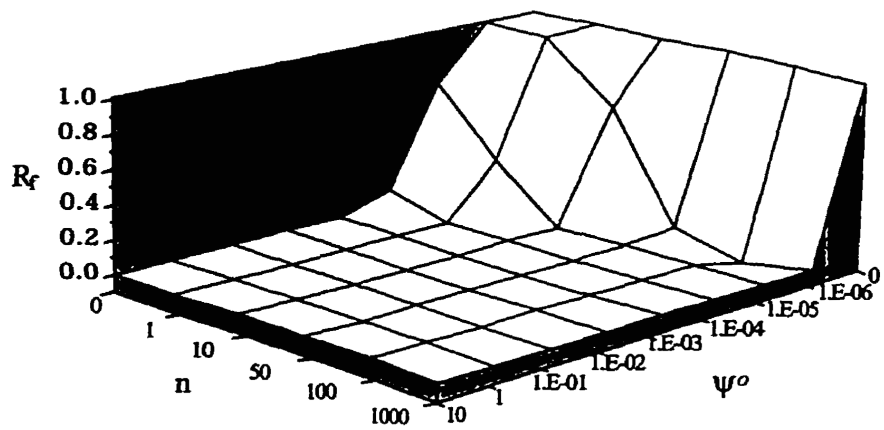

Figure 6.2 The relationship between R_f and n, ψ (degree) when r_Q =6372 (km), d=-0.001 (km)

6.2 Local relationships among the disturbing density, the disturbing potential and the disturbing gravity

In the preceding section, for a non-negative integer n and an arbitrarily given point P in R^3, we defined a function $F_{nP}(Q)$ being harmonic in R^3-{P} and having some properties of wavelet. In this section, we will, by means of $F_{nP}(Q)$, establish the relationships among the disturbing density $\delta\rho$, the disturbing potential T and the disturbing gravity δg (or the gravity anomaly Δg) in the Earth's gravity field.

6.2.1 Local relationship between the disturbing potential and the disturbing gravity on a leveling surface

The disturbing potential T satisfies the following relation (Guan and Ning, 1981):

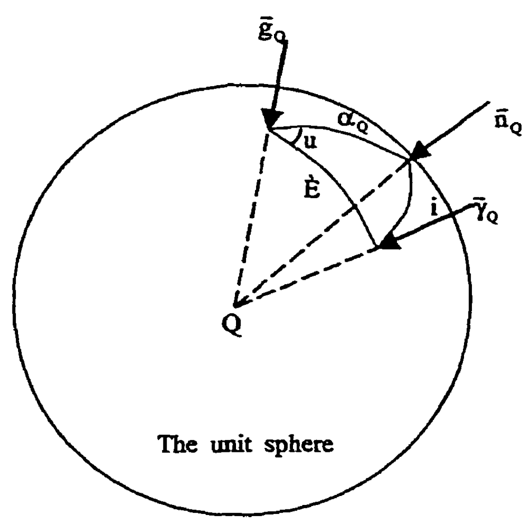

Figure 6.3 The relation among $\vec{g}_Q$, $\vec{\gamma}_Q$ and $\vec{n}_Q$

$$\frac{\partial}{\partial \bar{n}_Q} T(Q) \approx -[\delta g(Q) - \gamma_Q \Theta \cos u \tan \alpha_Q] \cos \alpha_Q \tag{6.2.1}$$

where $\bar{n}_Q$ is an arbitrarily given vector at Q, α_Q is the angle between $\bar{n}_Q$ and the gravity vector $\bar{g}_Q$ at Q, Θ is the total deflection of the vertical and u is an angle between the plane $(\bar{g}_Q, \bar{n}_Q)$ and the plane $(\bar{g}_Q, \bar{\gamma}_Q)$ (see Figure 6.3).

When n_Q is the interior normal vector of a surface σ, which is a level surface (real or normal) or a spherical surface with its centre at the geocentre, α_Q will be very small. So we can neglect the effect of the deflections of the vertical in (6.2.1) and obtain that

$$\frac{\partial}{\partial \bar{n}_Q} T(Q) = -\delta g(Q) \qquad \text{(Q is on } \sigma) \tag{6.2.2}$$

After proper reductions, we can represent the boundary surface by a level surface σ_E (for example the geoid or the reference ellipsoid). Thus the disturbing potential T is harmonic outside σ_E. Now we choose a level surface σ completely surrounding σ_E and take a point P inside σ. Then since the Earth and P are all inside σ, both $F_{nP}(Q)$ and T are harmonic and regular outside σ. It follows from Green's second formula (Heiskanen and Moritz, 1967) that

Model 6.1a:

$$\int_\sigma T(Q) \frac{\partial}{\partial \bar{n}_Q} F_{nP}(Q) dQ + \int_\sigma \delta g(Q) F_{nP}(Q) dQ = 0 \tag{6.2.3}$$

Furthermore, from the fundamental equation of physical geodesy, we have

$$-\frac{\partial}{\partial \bar{n}_Q} T(Q) = \delta g(Q) = \Delta g(Q) - \frac{1}{\gamma_Q} \frac{\partial \gamma_Q}{\partial \bar{h}_Q} T(Q). \tag{6.2.4}$$

So Model 6.1a can be rewritten as

Model 6.1b: $$\int_\sigma T(Q)[\frac{\partial}{\partial \bar{n}_Q} F_{nP}(Q) - \frac{1}{\gamma_Q}\frac{\partial \gamma_Q}{\partial \bar{h}_Q} F_{nP}(Q)]dQ + \int_\sigma \Delta g(Q) F_{nP}(Q)dQ = 0 \qquad (6.2.5)$$

Model 6.1 shows the relationship among the information on some frequency of the disturbing potential T and the disturbing gravity δg or gravity anomaly Δg on a level surface σ completely surrounding the Earth. From Property 6.2, we know that the kernel functions in the model decrease when Q goes away from P, and with the increase in n or the decrease in the distance between P and the surface σ the kernel functions have higher frequencies and their rates of decrease increase. This means that with the increase in n or the decrease in the distance between P and the surface σ, the integrals in the model can be evaluated in a smaller neighborhood of $\bar{P}$ (the nearest point on σ from P) and the data (input and output) should contain higher frequencies. So when n increases or the distance between P and the curved surface σ decreases, the data's information is projected in the neighborhood of $\bar{P}$. Since P is an arbitrarily given point inside σ, by selecting P we can project the local information of the disturbing potential T and the disturbing gravity δg (or gravity anomaly Δg) in the neighborhood of any point on σ. A more detailed discussion will be given in the next section.

6.2.2 Local relationship among the disturbing density, the disturbing potential and the disturbing gravity

Now we consider the relationships among the disturbing density $\delta\rho$, the disturbing potential T and the disturbing gravity δg.

Let τ be the space inside σ_E. By Poisson's equation, we have

$$\Delta T(Q)=\begin{cases}-4\pi G\delta\rho(Q), & Q \text{ is in } \tau\\ 0, & Q \text{ is outside } \tau\end{cases} \tag{6.2.6}$$

where G is the gravitation constant. For a level surface σ completely surrounding σ_E and a point P outside σ, we know from Property 6.3 that

$$\Delta F_{nP}(Q)=0 \qquad \text{(Q is inside } \sigma) \tag{6.2.7}$$

So by Green's second formula (Heiskanen and Moritz, 1967), equation (6.2.2) and noting that σ surrounds τ, we obtain

Model 6.2: $$\int_\sigma T(Q)\frac{\partial}{\partial \vec{n}_Q}F_{nP}(Q)dQ+\int_\sigma \delta g(Q)F_{nP}(Q)dQ-4\pi G\int_\tau \delta\rho(Q)F_{nP}(Q)dQ=0 \tag{6.2.8}$$

In Model 6.2, the impact of the disturbing density of the interior point Q decreases with the increase in the depth of Q, and the rate of the decrease increases with increasing n. So when n is bigger, the model shows the relationship between the high frequency information of the disturbing density $\delta\rho$ in shallow layers of the Earth and the disturbing gravity and disturbing potential on a surface surrounding the Earth.

6.2.3 Local relationship between the disturbing potential and the disturbing gravity on different layers

Finally, we give the relationship between the disturbing potential T and the disturbing gravity δg on different layers.

Let σ_1 and σ_2 be the level surfaces satisfying $\tau\subset\sigma_1\subset\sigma_2$, and P be a point outside σ_2. Then from (6.2.8) we can easily obtain

Model 6.3: $$\int_{\sigma_1} T(Q)\frac{\partial}{\partial \bar{n}_Q}F_{nP}(Q)dQ + \int_{\sigma_1} \delta g(Q)F_{nP}(Q)dQ =$$

$$= \int_{\sigma_2} T(Q')\frac{\partial}{\partial \bar{n}_{Q'}}F_{nP}(Q')dQ' + \int_{\sigma_2} \delta g(Q')F_{nP}(Q')dQ' \quad . \qquad (6.2.9)$$

Model 6.3 shows the relationship among the information at some frequency of the disturbing potential T and the disturbing gravity δg on different layers. Since P is close to σ_2, the frequency of the information on σ_2 is higher than that on σ_1. But when n increases, the frequencies of the information on both layers will be higher. So when determining the data on σ_1 from the data on σ_2, we can get the higher resolution of the data on σ_1 by increasing the resolution of the data on σ_2 without a change in the distance between the two layers. This is very important for processing satellite gravity data.

6.3 'Multi-resolution' representation of the single-layer density of the disturbing potential

In section 1.4.2, we discussed three indirect parameter methods for representing the disturbing potential: the spherical harmonic representation; Bjerhammar's representation and the 'fictitious' single layer density representation. These methods were proposed for simplifying the representation of the disturbing potential. Another significance of these models is that their model parameters, consisting of a set of spherical harmonic coefficients or a function distributed on a spherical surface, can be determined from all kinds of gravity data. As we showed in section 1.5.4, however, these methods are hard to be employed in the processing of high-resolution gravity data since in these models the local relationships between the model parameters and the high-resolution gravity data are very weak.

In this section, we will generalize the 'fictitious' single layer density representation of the disturbing potential to establish a new model in which the local character of the model parameters is considered.

6.3.1 Establishment of the model

The 'fictitious' single layer density representation of the disturbing potential is given as follows (Hsu and Zhu, 1984):

$$T(P) = \frac{1}{4\pi}\int_{S_B}\frac{1}{l_{PQ}}\rho_Q^{*}dQ \tag{6.3.1}$$

where S_B is the surface of the Bjerhammar sphere with radius R_B and ρ^* is the "model parameters" which will be determined from the gravity data.

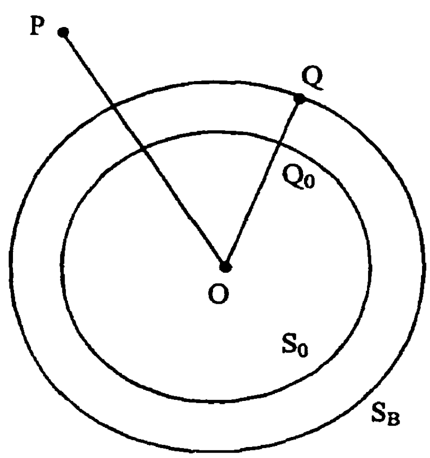

Figure 6.4 The sphere S_0 and the Bjerhammar sphere S_B

For $d_0>0$, we draw a sphere S_0 centred at the centre O of S_B and with radius $R_0=R_B-d_0$ (see figure 6.4)

Replacing the geocentre O_E by Q_0 in (6.1.26), we obtain

$$\frac{1}{l_{PQ}} = \sum_{n=0}^{\infty} \frac{d_0^n}{l_{PQ_0}^{n+1}} P_n(\cos\angle PQ_0Q) \tag{6.3.2}$$

Therefore from (6.1.19), we have

$$\frac{1}{l_{PQ}} = \sum_{n=0}^{\infty} d_0^n F_{nQ_0}(P) \tag{6.3.3}$$

It follows from (6.3.1) that

$$T(P) = \sum_{n=0}^{\infty} \frac{1}{4\pi} \int_{S_0} F_{nQ_0}(P) \rho_n^{\bullet}(Q_0) dQ_0 \tag{6.3.4}$$

where

$$\rho_n^{\bullet}(Q_0) = \rho_Q^{\bullet} d_0^n \frac{R_B^2}{R_0^2} \tag{6.3.5}$$

Equation (6.3.4) is the model we want.

6.3.2 Further discussion on the model

Obviously model (6.3.4) is a generalization of the 'fictitious' single layer density representation of the disturbing potential (6.3.1). This can be seen by taking $d_0=0$.

The model (6.3.4) expresses the disturbing potential T as a summation of an infinite series. From the properties of $F_{nQ_0}(P)$, for each n, the term T_n in the series

$$T_n(P) \equiv \frac{1}{4\pi}\int_{S_0} F_{nQ_0}(P)\rho_n^*(Q_0)dQ_0 \tag{6.3.6}$$

represents some frequency information of the disturbing potential and the frequency goes from low to high when n goes from 0 to ∞. In this sense, (6.3.4) can be regarded as a generalization of the spherical harmonic expansion (1.4.19) of the disturbing potential. Actually, when taking d_0=R_B in (6.3.2), we can obtain (1.4.19) and T_n corresponds to the Laplace spherical harmonics Y_n of the disturbing potential.

It is obvious from the following equation (6.3.7) that the parameters $\{C_{nm}, S_{nm}\}_{m=0}^{n}$ in Y_n have no local relationships with the disturbing potential T.

$$\begin{pmatrix} C_{nm} \\ S_{nm} \end{pmatrix} = k_n \int T(\theta,\lambda)P_{nm}(\cos\theta)\begin{pmatrix} \cos m\lambda \\ \sin m\lambda \end{pmatrix} d\theta d\lambda \tag{6.3.7}$$

Although the parameters ρ_n^* can not be expressed by means of a simple closed formula like (6.3.7) of the disturbing potential T, there exist local relationships between ρ_n^* and T. In the following, we will discuss how to obtain the relationships.

First of all, for equation (6.3.1), we use the method discussed in section 1.5.4 to obtain a solution of the parameters ρ^*. Since the kernel function in the integral equation (6.3.1) decreases slowly, the resolution of the resulting ρ^* will be low. From these ρ^*, we can obtain ρ_0^* by means of (6.3.5).

Then we rewrite equation (6.3.4) as

$$T^1(P) \equiv T(P) - T_0(P) = \sum_{k=1}^{\infty} \frac{1}{4\pi} \int_{S_0} F_{nQ_0}(P)\rho_n^*(Q_0)dQ_0$$
$$= \frac{1}{4\pi} \int_{S_0} K_1(P,Q_0)\rho_1^*(Q_0)dQ_0 \tag{6.3.8}$$

where $T_0(P)$ can be obtain from ρ_0^* by means of (6.3.6) and the kernel function K_1 is given by

$$K_1(P,Q_0) = \frac{1}{d_0}\left(\frac{1}{l_{PQ}} - F_{0Q_0}(P)\right) = F_{1Q_0}(P) + \sum_{n=2}^{\infty} d_0^{n-1} F_{nQ_0}(P) \tag{6.3.9}$$

Obviously, the kernel function K_1 decreases faster than the kernel function in (6.3.1) when Q_0 moves away from P. Therefore, (6.3.8) shows a stronger local relationship between ρ_1^* and T^1, and from this integral equation, we can obtain ρ_1^* with a resolution higher than ρ_0^*.

Finally, after n steps, we can obtain

$$T^n(P) \equiv T(P) - \sum_{k=0}^{n-1} T_k(P) = \frac{1}{4\pi} \int_{S_0} K_n(P,Q_0)\rho_n^*(Q_0)dQ_0 \tag{6.3.10}$$

where

$$K_n(P,Q_0) = F_{nQ_0}(P) + \sum_{k=n+1}^{\infty} d_0^{k-n} F_{kQ_0}(P) \tag{6.3.11}$$

For a big enough n, the relation (6.3.10) shows a strong local relationship between ρ_n^* and T^n and from this integral equation, we can obtain ρ_1^* with a high resolution.

Thus, step by step, we obtain the model parameters $\{\rho_k^*\}_{k=0}^{n}$ and with increase in k, the resolution of ρ_k^* increases. So we call the model (6.3.4) the 'Multi-resolution' representation of the single-layer density of the disturbing potential T.

6.4 Detailed discussion on Model 6.1 and its practical evaluation

Under spherical approximation, the value of the geoidal height (or the disturbing potential) at a single point on the geoid can be given by Stokes's formula

$$N(P) = \frac{T(P)}{4\pi\gamma_P} = \frac{R}{4\pi\gamma_P}\int_\sigma \Delta g(Q)S(\psi_{PQ})dQ. \tag{6.4.1}$$

Since the Stokes function $S(\psi_{PQ})$ can be expressed as

$$S(\psi_{PQ}) = \sum_{n=2}^{\infty}\frac{2k+1}{k-1}P_k(\cos\psi_{PQ}), \tag{6.4.2}$$

we have

$$T(P) = \sum_{k=2}^{\infty}\frac{R}{k-1}\Delta g_k(P), \tag{6.4.3}$$

where the surface harmonics

$$\Delta g_k(P) = \frac{2k+1}{4\pi}\int_\sigma \Delta g(Q)P_k(\cos\psi_{PQ})d\sigma \tag{6.4.4}$$

represent global information of the gravity anomaly at specific frequencies. (6.4.3) means that T(P) contains global gravity information on all frequencies (except for n<2). Stokes's formula (6.4.1) is the relation between the geoidal height at a single point on the geoid and the gravity anomalies on the entire geoid. It can be said that Stokes's formula is the rigorous formula for computing the geoidal height from the globally and continually distributed gravity anomalies in spherical approximation. However, the numerical evaluation of such a formula is generally hindered for two reasons: one is the lack of adequate global coverage of gravity anomalies; the other is that the gravity data are given only at discrete points.

In order to give a supplement to Stokes's formula, Paul (1991) established a model as follows:

$$\int_\sigma T(Q)F_2(\psi_{PQ})d\sigma = \int_\sigma \Delta g(Q)F_1(\psi_{PQ})d\sigma \tag{6.4.5}$$

with

$$F_1(\psi_{PQ}) = \frac{Q(x, \mathring{a}, v)}{Q_0(\mathring{a}, v)} \tag{6.4.6}$$

$$F_2(\psi_{PQ}) = \frac{1}{2Q_0(\varepsilon, v)}\sum_{k=2}^{\infty}(2k+1)(k-1)Q_k(\varepsilon, v)P_k(\cos\psi_{PQ}) \tag{6.4.7}$$

where $\varepsilon>0$, $v>0.5$, $x=\cos\psi_{PQ}$,

$$Q(x,\mathring{a}, v) = \frac{\varepsilon^{2v-1}}{[(x-1)^2+\varepsilon^2]^v}, \tag{6.4.8}$$

$$Q_0(\mathring{a}, v) = \int_{-1}^{1} Q(x,\mathring{a}, v)dx, \tag{6.4.9}$$

and $Q_k(\varepsilon, v)$ (k>0) are given by the following recurrence relations:

$$(2-2v)Q_1(\varepsilon,v)-(2-2v)Q_0(\varepsilon,v)=\varepsilon-\varepsilon^{2v-1}/(\varepsilon^2+4)^{v-1} \qquad (6.4.10a)$$

$$(2-\frac{3}{4}v)Q_2(\varepsilon,v)+(2v-4)Q_1(\varepsilon,v)+(2-\frac{2}{3}v+\varepsilon^2)Q_0(\varepsilon,v)=\varepsilon+\varepsilon^{2v-1}/(\varepsilon^2+4)^{v-1} \qquad (6.4.10b)$$

$$\begin{aligned}(\frac{12}{5}-\frac{6}{5}v)Q_3(\varepsilon,v)+(2v-6)Q_2(\varepsilon,v)+(\frac{28}{5}-\frac{4}{5}v+3\varepsilon^2)Q_1(\varepsilon,v)-2Q_0(\varepsilon,v)&\\ =\varepsilon-\varepsilon^{2v-1}/(\varepsilon^2+4)^{v-1}&\end{aligned} \qquad (6.4.10c)$$

$$Q_k(\varepsilon,v)=a_0[\sum_{j=1}^{4}a_jQ_{k-j}(\varepsilon,v)] \qquad (k>3) \qquad (6.4.10d)$$

$$a_0=\frac{(2k-1)}{k(k-2v+1)}, \quad a_1=2(k-v),$$

$$a_2=-\frac{(k-1)(k-2v+1)}{2k-1}-\frac{(k-2)(k+2v-4)}{2k-5}-(2k-3)(1+\varepsilon^2)$$

$$a_3=2(k+v-3), \quad a_4=-\frac{(k-3)(k+2v-4)}{2k-5}$$

The kernel functions $F_1(\psi_{PQ})$ and $F_2(\psi_{PQ})$ in model (6.4-5) were shown to decay sharply when ψ_{PQ} goes from 0 to π.

Model (6.4.5) was called a local relationship between disturbing potential (geoidal height) and gravity anomaly. It provides the possibility of reducing the integration area of the gravity anomaly in (6.4.1) to a local area of the geoid while the integral will simultaneously be made equal to an integral transform of the local geoidal height. However, the kernel function $F_2(\psi_{PQ})$ is given by the summation of an infinite series (6.4.7), which is hard to be expressed by an analytical formula. In addition, the relationship is established only under spherical approximation.

Model 6.1b established in the preceding section has the same characteristic as Paul's model in some aspects. Since the kernel functions in both models vanish sharply when the moving point moves away from the computation point, they both show the local relationship between the disturbing potential (the geoidal height) and the gravity anomaly. However, Model 6.1b holds not only under spherical approximation, but it can hold for a level surface, and under the spherical approximation the kernel functions in Model 6.1b can be easily computed from (6.1.14) and (6.1.15) and the following equation (6.4.12).

To obtain a further understanding of the local character of Model 6.1b, we will discuss a special case: First, we use a spherical surface σ_0 with the mean radius of the Earth R instead of σ in Model 6.1b, and obtain

$$\int_\sigma T(Q)\overline{F}_{nP}(Q)d\sigma = \int_\sigma \Delta g(Q)F_{nP}(Q)d\sigma \tag{6.4.11}$$

where (note (6.1.15))

$$\overline{F}_{nP}(Q) = \frac{\partial}{\partial r_Q}F_{nP}(Q) - \frac{2}{R}F_{nP}(Q) = \frac{n-1}{R}F_{nP}(Q) + (n+1)\frac{r_P}{R}F_{n+1P}(Q) \tag{6.4.12}$$

Then if we choose P to be the centre of σ_0, we have

$$r_P = 0 \text{ and } F_{nP}(Q) = \frac{1}{R^{n+1}}P_n(\cos\psi_{PQ})$$

It follows from (6.4.11) and (6.4.12) that

$$\int_\sigma T(Q)P_n(\cos\psi_{PQ})d\sigma = \frac{R}{n-1}\int_\sigma \Delta g(Q)P_n(\cos\psi_{PQ})d\sigma \tag{6.4.13}$$

The above formula is a well-known formula which shows the relationship between the information of T and Δg at some frequency. Since the kernel function $P_n(\cos\psi_{PQ})$ varies between -1 and 1 when ψ_{PQ} goes from 0 to π, the integrals in (6.4.13) have to be evaluated globally. This means that the relationship showed in (6.4.13) is a global relationship.

As a generalization of (6.4.13), Model 6.1 (where P is not at the centre of σ) has its advantages. Unlike the case in (6.4.13), the distance between the moving point Q and the fixed point P increases when ψ_{PQ} goes from 0 to π. It follows that the kernel functions in Model 6.1 vanish as ψ_{PQ} goes from 0 to π. Thus, in the integrals in Model 6.1, the information on the area that is closer to P is bulged more than others. So in this respect, Model 6.1 has local character that is different from (6.4.13).

Furthermore, from Property 6.6, (6.4.11) can be rewritten as

$$L_{nP}(\Delta g) = \int_{\sigma} \Delta g(Q) F_{nP}(Q) d\sigma = 4\pi \sum_{k=n}^{\infty} \frac{k-1}{2k+1} \binom{n}{k} \frac{r_P^{k-n}}{R^k} T_k(\theta, \lambda) \tag{6.4.14}$$

where $T_k(\theta,\lambda)$ is the k-degree Laplace surface harmonic of the disturbing potential T. This means that, with the increase in n or the decrease in d (=R-r_P), the high degree harmonic coefficients are amplified in the model and, for a given n, the harmonic potential coefficients of degree less than n are not contained in the model.

Model 6.1b establishes the relations between the gravity anomaly Δg and the disturbing potential T on the surface σ. Can we then get T from Δg by this model for a given n? Rigorously speaking, the answer is no. This is because (6.2.5) is an integral equation of the first kind that is improperly posed. This can be further explained in view of spectral analysis from (6.4.14): first, $L_{nP}(\Delta g)$ lacks the first n-1 degree harmonic potential coefficients; second, $L_{nP}(\Delta g)$ is less sensitive than T to the harmonic potential coefficients of very large degree because

$$\frac{k-1}{2k+1}\binom{n}{k}\frac{r_P^{k-n}}{R^k} \to 0 \qquad \text{(when } k \to \infty\text{)} \tag{6.4.15}$$

Thanks to the advent of satellite geodesy, the low degree harmonic coefficients have already been obtained with very high accuracy, which is expected to increase further with the planned dedicated gravity satellite missions. We can take these coefficients into account in computing the normal gravity field so that the disturbing potential T does not contain these coefficients:

$$T(\theta,\lambda) = \sum_{k=n_0}^{\infty} T_k(\theta,\lambda)$$

where n_0 satisfies that the first n_0-1 harmonic coefficients have already been known and subtracted. In fact, in the Stokes formula, the zero and first degree coefficients are excluded.

For the second problem, we can suppose that the harmonic potential coefficients of very large degree are zero because they are very small compared to the lower degree coefficients. Although Stokes's formula, the rigorous solution of the Robin boundary-value problem on a spherical surface, expresses the disturbing potential T over all frequencies, it is impossible in practical applications to get the disturbing potential T over all frequencies because of the fact that the input data (gravity anomalies Δg) are only given at discrete points. What we can get is T with finite frequency extension or finite resolution.

In harmonic spectral analysis, T can be approximated by a set of finite spherical harmonic coefficients $\{C_{nm}, S_{nm}\}_{n=n_0}^{n_{max}}$:

$$T(\theta,\lambda) = \sum_{k=n_0}^{n_{max}} T_k(\theta,\lambda)$$

So (6.4.14) is approximated by

$$L_{nP}(\Delta g) = \int_\sigma \Delta g(Q) F_{nP}(Q) d\sigma \approx 4\pi \sum_{k=n}^{n_{max}} \frac{k-1}{2k+1} \binom{n}{k} \frac{r_P^{k-n}}{R^k} T_k(\theta,\lambda) \tag{6.4.16}$$

If $n \le n_0$, (6.4.16) contains a finite set of spherical harmonic coefficients $\{C_{nm}, S_{nm}\}_{n=n_0}^{n_{max}}$ as the unknown parameters. Thus we can get a unique solution T from (6.4.16) by properly choosing P (the closer $\frac{r_P}{R}$ is to 1, the more amplified the high degree coefficients are).

In spatial analysis, we divide the area into many blocks according to the resolution of the gravity data and suppose that T is constant in each block. Then the unknown function T becomes a vector $\{T_i\}$ with finite dimension and can be estimated by the least squares technique from the known gravity data. In more detail, we divide the surface σ into a far-area σ_{far} and a near-area σ_{near}. Furthermore, we divide σ_{near} into a set of grid elements $\{\sigma_i\}$ by meridians and parallels. The size of the grid elements is chosen according to the resolution of the data. Thus Model 6.1b becomes

$$\sum_i \Delta g_i A_{iP}^n + \sum_i T_i B_{iP}^n = f_P^n \tag{6.4.17}$$

where Δg_i and T_i are, respectively, the mean values of Δg and T on σ_i; and

$$A_{iP}^{n} = F_{nP}(Q_i)\Delta\sigma_i \tag{6.4.18a}$$

$$B_{iP}^{n} = [\frac{\partial}{\partial \bar{n}_{Q_i}} F_{nP}(Q_i)]\Delta\sigma_i - \frac{1}{\gamma_{Q_i}}[\frac{\partial}{\partial \bar{h}_{Q_i}}\gamma(Q_i)]F_{nP}(Q_i)\Delta\sigma_i \tag{6.4.18b}$$

$$f_P^{n} = -\int_{\sigma_{far}} \Delta g(Q)F_{nP}(Q)dQ - \int_{\sigma_{far}} T(Q)\{\frac{\partial}{\partial \bar{n}_Q}F_{nP}(Q) - \frac{1}{\gamma_Q}[\frac{\partial}{\partial \bar{h}_Q}\gamma(Q)]F_{nP}(Q)\}dQ \tag{6.4.18c}$$

where Q_i is a point in σ_i; $\Delta\sigma_i$ is the area of σ_i; and f_P^n can be evaluated from a global gravity field model and the error of doing so is very small because the values of the kernel functions in σ_{far} are very small relative to those in the near-area. From (6.4.17), we can estimate $\{T_i\}$ from $\{\Delta g_i\}$ by the least squares technique. By properly choosing n ($n \le n_0$) and P, we can make A_{iP}^n or B_{iP}^n decrease rapidly when the distance between P and σ_i increases. From table 1 and table 2, we know that for a fixed n, A_{iP}^n and B_{iP}^n decrease when the distance between P and σ_i increases and the rate of decrease is slow when the size of the grid elements is small, but for grid elements with a small size, we can increase n or decrease the distance between P and σ so that the rate of decrease of A_{iP}^n and B_{iP}^n is still rapid. Thus, the coefficient matrix of the normal equations is a sparse and very strongly diagonal-dominant. This guarantees the stability of the least square solution of (6.4.17) (see Keller, 1995 and Fei, 1994). So from Model 6.1b, we can obtain T from Δg with certain resolution in a local area by properly choosing n and the distance between P and σ, and the resulting solution will be stable.

Like Paul (1991), we can call Model 6.1 the local relationship model between the gravity data and the disturbing potential data. The local relationship has two meanings. One is that we can evaluate with high accuracy the integrals in the model by using mainly the high-accuracy and high-resolution data in a local area. The other is that we can get a stable solution with the required resolution when we invert the integrals because of the rapidly decreasing kernel function of the integrals in the model.

6.5 Chapter summary

In the preceding sections, we established three models showing the relationships among the disturbing density, the disturbing potential and the disturbing gravity (gravity disturbance or anomaly) and a 'Multi-resolution' representation of the single-layer density of the disturbing potential. These models have the following characteristics:

1. The basic kernel function $F_{nP}(Q)$ (6.1.1) has some properties of wavelet and can be evaluated from the recurrence formula (6.1.14) or directly from (6.1.19).

2. The models 6.1, 6.2 and 6.3 show the relationships among the information at some frequency of the disturbing density inside the Earth, the disturbing potential and the disturbing gravity outside the Earth, and with increase in n, the frequency of the information increases and the local character of the information is projected.

3. The multi-resolution representation of the single-layer density of the disturbing potential expresses the disturbing potential T as a summation of an infinite series. For each n, the term T_n in the series represents some frequency information of the disturbing potential and the parameters in T_n have some local relationships with the disturbing potential. When n goes from 0 to ∞, the frequency goes from low to high while the local character goes from weak to strong.

4. These models all involve integral equations of the first kind, which are improperly posed. However, when the disturbing potential T in the models is replaced by its discretized form $\{T_i\}$, we can solve $\{T_i\}$ from the models by means of the least squares technique. Since the kernel functions in the integrals decrease rapidly by properly choosing n, the coefficient matrices of the resulting normal equations can be very sparse and very strongly diagonal-dominant, thus the method will be very efficient.

7 Conclusions and Recommendations

In this thesis, we developed some refinements of the solutions of the geodetic boundary value problems. The following are the conclusions and recommendations of the thesis.

7.1 Summary and conclusions

1. We supplemented the Runge theorems so that the derivatives of the disturbing potential are involved. The new theorems, which state that the disturbing potential and its derivatives can be approximated simultaneously by a function harmonic outside an inner point of the Earth and its corresponding derivatives, are more suitable for providing theoretical guarantee to the approximate theories in physical geodesy.

2. We derived new ellipsoidal correction formulas to Stokes's formula and the inverse Stokes/Hotine formulas. By adding these corrections to the corresponding formulas, the system errors decrease from $O(e^2)$ to $O(e^4)$. Compared to the other relevant spherical formulas, the new formulas are very effective since they are simple closed formulas and the input data are those already obtained via the spherical formulas. A numerical test for the ellipsoidal correction to Stokes's formula in the US showed that the contribution of the ellipsoidal correction ranges from –31 cm to –1 cm and a global geoid model with a resolution of 1 degree is sufficient for the computation if the required accuracy is of the order of 1cm.

3. We investigated the second geodetic boundary value problem based on ground gravity disturbances:

- We analyzed the significance of the second geodetic boundary value problem. The conclusion of the analysis is that in the era of GPS, the second geodetic boundary value problem is most important for the determination of a high accuracy geoid model and the external gravity field, especially for the purpose of replacing the conventional leveling by GPS observations.

- We generalized Hotine's formula to the outside space so that we can evaluate the exterior disturbing potential from the ground gravity disturbances, and obtained an ellipsoidal Hotine formula which is expressed as the spherical Hotine formula plus an ellipsoidal correction term.

- We obtained three approximate solutions to the second geodetic boundary value problem. Two of them were obtained from applying Hotine's formula or the ellipsoidal Hotine formula to solve the second geodetic boundary value problem by means of the Helmert condensation reduction and the analytical continuation method. The third is an integral equation solution obtained from directly solving the second geodetic boundary value problem. Among the three solutions, the Helmet condensation reduction solution has the simplest formula but needs the most assumptions, the analytical continuation solution is in the middle and the integral equation solution is the most complicated but needs the least assumptions.

4. We established four models showing the local characters of the disturbing potential and other gravity parameters. Three of them show the relationships among the disturbing density, the surface disturbing potential and the surface disturbing gravity. The fourth model gives the 'multi-resolution' single-layer density representation of the disturbing potential. The important character of these models is that their kernel functions decrease fast, which guarantees that the integrals in the models can be evaluated with high accuracy by using mainly the high-accuracy and high-resolution data in a local area, and stable solutions with high resolution can be obtained when inverting the integrals. A brief analysis indicated that these local relationship models are useful in the processing of high-resolution gravity data.

7.2 Recommendations

1. For the determination of a geoid model with 1-cm accuracy, the ellipsoidal correction computed from a global geoid model with 1^0 resolution or better should be added to the spherical geoid model obtained via Stokes's formula.

2. For the accurate estimation of the effect of the Earth's flattening on the inverse Hotine/Stokes formulas, further numerical tests are needed.

3. To make the solution of the second geodetic boundary value problem realizable, it is necessary to measure the positions of the gravity data points via GPS. It is recommended to produce an instrument that integrates the gravimeter and the GPS receiver so that the position and the gravity value of the observation point can be measured simultaneously. As for the selection of the three solutions discussed in this thesis, it is recommended from a theoretical analysis to use the integral equation solution in areas with complicated topography, the analytical continuation solution in areas where the inclination angles and the elevations of the topography are very small, and Helmert's condensation reduction solution in areas where the mass density of the topography is known and the gravity disturbance δg is linearly related with the elevation h. This recommendation should be verified by future numerical tests.

4. A numerical test on the models given in chapter 6 will be given in a future investigation to show how significant these models are for practical work.

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