On Pricing and Hedging Options in Regime-Switching Models with Feedback Effect *

Robert J. Elliott † Tak Kuen Siu ‡ Alexandru Badescu §

Abstract

We study the pricing and hedging of European-style derivative securities in a Markov, regime-switching, model with a feedback effect depending on the economic condition. We adopt a pricing kernel which prices both financial and economic risks explicitly in a dynamically incomplete market and we provide an equilibrium analysis. A martingale representation for a European-style index option's price is established based on the price kernel. The martingale representation is then used to construct the local risk-minimizing strategy explicitly and to characterize the corresponding pricing measure.

Keywords: Pricing and Hedging; Regime-switching; Feedback Effect; Product Price Kernel; Local Risk-Minimization.

JEL Classification: D52, G13, G11.

---

*Robert J. Elliott would like to thank ARC and SSHRC for their continued support. Tak Kuen Siu wishes to thank ARC for its support. Alexandru Badescu would like to thank NSERC for its support.

†Corresponding Author: Robert J. Elliott, School of Mathematical Sciences, University of Adelaide, SA 5005 AUSTRALIA; Haskayne School of Business, University of Calgary, Calgary, Alberta, Canada, T2N 1N4; E-mail: relliott@ucalgary.ca

‡Tak Kuen Siu, Department of Actuarial Studies, Faculty of Business and Economics, Macquarie University, Sydney, NSW 2109, AUSTRALIA; E-mail: ktksiu2005@gmail.com; Tel.: :(+61-2) 9850-8573; Fax: (+61 2) 9850 9481

§Alexandru Badescu, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada, T2N 1N4; E-mail: abadescu@math.ucalgary.ca

§1. Introduction

Since the seminal work of Hamilton (1989), Markov regime-switching models have been an important and popular class of econometric time series models. In the original work of Hamilton (1989), the class of discrete-time Markov-switching autoregressive time series models was proposed. The basic idea of Hamilton's model is that the autoregressive parameters are modulated by a discrete-time, finite-state Markov chain so that one set of parameters is in force at a particular time period according to the state of the chain in that period. The original motivation of introducing this class of models is to provide a simple but realistic way to describe and explain the cyclical behavior of economic data, which may be attributed to business cycles. Numerous empirical studies reveal that the Markov-switching autoregressive time series model can describe and explain the empirical behaviors of many economic and financial data well, especially the long-term behavior of these data. Some examples include Sola and Driffill (1994), Ang and Bekaert (2002), Camacho (2005), Lux and Kaizoji (2007), and others.

Due to their empirical successes, Markovian regime-switching models have found diverse applications in economics and finance. They have been applied to different important areas in finance, including asset allocation, option valuation, risk management and term structure modeling. Recently, motivated by both practical and theoretical interests, option valuation under Markovian regime-switching models has received considerable interest in the literature. Some works in this area include Naik (1993), Guo (2001), Buffington and Elliott (2002), Elliott et al. (2005), and others. From a practical perspective, Markov, regime-switching, models have better empirical performance than their constant-coefficient counterparts. From a theoretical viewpoint, one may need to take into account in option valuation, at least, two sources of risks underlying a Markovian regime-switching model, namely, the diffusion risk and the regime-switching risk. The diffusion risk refers to the source of risk due to fluctuations of market prices or rates and can be considered market or financial risk. The regime-switching risk means the source of risk due to transitions of (macro)-economic conditions and can be thought of as economic risk. With the two sources of risks and two primitive tradeable assets, the market is, in general, incomplete. So there are more than one equivalent martingale measures and no-arbitrage prices. Guo (2001) introduced a set of “fictitious” assets to complete the market described by a Markovian regime-switching Geometric Brownian motion and derives the option valuation result under the completed market. Elliott et al. (2005) employed a well-known tool in actuarial science, namely, the Esscher transform, to pick an equivalent martingale measure for option valuation under a Markovian regime-switching Geometric Brownian motion. In particular, they introduced a regime-switching Esscher transform for option valuation and justified its use by a regime-switching version of the minimal entropy martingale measure. Elliott et al. (2007a) extended the approach of Elliott et al. (2005) to deal with option valuation under a generalized Markovian regime-switching jump-diffusion model, where the jump component is described by a completely random measure. They deal with the pricing of both European-style and American-style options. Elliott et al. (2007b) developed a martingale representation for contingent claims under Markovian regime-switching Geometric Brownian motion based on an equivalent martingale measure chosen by the regime-switching Esscher transform using the concept of stochastic flows under some differentiability conditions. They then use this representation together with a zero-coupon bond to develop a risk-minimizing trading strategy.

The models for valuing options under Markovian regime-switching models developed so far in the literature incorporate only the impact of economic states or conditions on the asset price dynamics. In other words, the causality is from economic conditions to asset price dynamics. The impact of the asset price dynamics on the economic conditions is not modelled. This effect is called a feedback effect. In other words, the causality runs from asset price dynamics to economic conditions. There are theoretical and empirical evidences for supporting this causal relationship, or feedback effect, in different markets, such as the bond, share, currency and energy markets. Fong and See (2003) considered a regime-switching model with feedback effect when modeling the conditional volatility of crude oil futures returns. The key feature of their regime-switching model is that transition probabilities are functions of the basis. They found empirical evidence to support the feedback effect, in the sense that in periods of high volatility, an increase in backwardation is associated with an increase in the probability of switching to, or remaining in, the high-volatility regime. This feedback effect is also consistent with the theory of storage. They also found that a regime-switching model with feedback effect can improve the accuracy of short-term volatility forecasts. Kaminski and Lo (2007) considered a regime-switching model with feedback effect in studying empirically stop-loss rules. They called their model a behavioral regime-switching model, where transition probabilities are state-dependent; that is, the transition probabilities of states of an economy modeled by a Markov chain depend on the lagged return of a risky asset. They found empirical evidence to support the regime-switching model with feedback effect for monthly and annual log-returns for the CRSP Value-Weighted Total Market Index and Ibbotson Associates Long-Term Government Bond Index, from January 1950 to December 2004. Bansal et al. (2010) investigated empirically a bivariate regime-switching with feedback effect in modeling the relationship between bond and share prices, using daily futures-contract returns for the US stock index and ten-year Treasury notes over the crisis-rich 1997-2005 period. They postulated that regime switching transition probabilities depend on lagged implied volatility from equity-index options, or lagged VIX, and found significant empirical evidence that the probability of shifting from the low-stress economic regime to the high-stress economic regime is increasing in lagged VIX and that the probability shifting from the high-stress economic regime to the low-stress regime is decreasing in lagged VIX. All the regime-switching models with feedback effect mentioned above are discrete-time models.

Many existing models used for valuing options, such as the stochastic volatility model, ARCH-type models, the jump-diffusion model, Lévy-based models, etc., can describe the short-term behavior of asset prices and may provide satisfactory results for valuing short-term options. However, for valuing long-term options, it is important to incorporate the impact of economic conditions on asset price dynamics since there could be substantial change in economic conditions over a long period of time. A model which is suitable to price long-term options can be applied to fair valuation of some modern insurance options with embedded options, such as equity-linked life insurance products, which have a long maturity, say 30 to 40 years. Regime-switching models are natural choice to incorporate structural changes in economic conditions in modeling asset price dynamics and pricing options.

In this paper, we study the pricing and hedging of European-style options under a "double" Markovian regime-switching model with feedback effect. We consider a continuous-time economy consisting of two primitive assets, namely, a risk-free asset and a risky asset. The price dynamics of the risky asset are governed by the double Markovian regime-switching model. In such model, the market parameters, including the market interest rate of the bank account, the appreciation rate and the volatility of the risky asset are modulated by an observable, continuous-time, finite-state Markov chain. The states of the chain represent different states of an economy. The Markovian regime-switching models can incorporate the structural changes of the price dynamics of the risky asset due to the changes in the economic conditions. In addition, we provide a novel way to incorporate the feedback effect of the price process of the risky asset on the economic condition by assuming that the rate matrix of the chain is modulated by the price process. This may be considered a continuous-time, abstract, version of the regime-switching models with feedback. As referenced above, there have been some theoretical and empirical evidences which support the regime-switching models with feedback effect. Here we shall provide further justification for such models by investigating their statistical properties and economic features in a general equilibrium framework. The regime-switching model with feedback effect may also provide insights to study the link between asset prices and monetary policies. Indeed, it is known that developments in asset prices in different markets, say equity, housing and foreign exchange markets, have a significant effect on both inflation and real economic activity. For example, large movements in asset prices are associated with prolonged booms and bursts, and extreme adverse changes in asset prices may lead to macroeconomic instability. An understanding on how asset prices affect the economy, (i.e. the feedback effect we refer to here), can provide some insights in developing and analyzing actions and policies adopted by central banks or regulators to minimize the likelihood of macroeconomic instability. For details, interested readers may refer to the Geneva Reports on the World Economy 2 by Cecchetti et al. (2000). Despite the generality of our model, the pricing and hedging of contingent claims presents an interesting and challenging problem due the incompleteness of the market. Here we first specify an equivalent martingale measure, or a price kernel, which prices the two sources of risk, namely, the diffusion risk and the regime-switching risk. By exploiting the methodology similar to that in Colwell et al. (1991) and Colwell and Elliott (1993), we then establish a martingale representation for a European-style index option's price based on this price kernel. Under some suitable differentiability conditions for the coefficients of the price processes we explicitly identify the integrands in the martingale representation of the option's price using the concept of stochastic flows. The martingale representation is used to construct the local risk-minimizing strategy explicitly and the corresponding pricing measure is then obtained.

In a recent paper by Lee and Protter (2008), option pricing and hedging were considered in a jump-diffusion market. However, Lee and Protter do not consider regime switching. Regime switching models have been shown empirically to be a useful tool when modeling financial data. Lee and Protter used a compensated Poisson process while we use the Markov chain to model jump-type randomness. The semimartingale dynamics of a finite state Markov chain are as given in equation (2.1) below. The chain can feed into many components of the processes. For example, the appreciation rate and volatility of the risky asset depend on both the current level of the asset price and the current level of the economic state represented by the Markov chain. However, in the Lee-Protter model, the appreciation rate and volatility of the risky asset depends only on the current level of the risky asset. One may think that the jump component in the jump-diffusion model of Lee and Protter may be related to that in the Markov, regime-switching, jump-diffusion model we considered here by representing the modulating Markov chain in terms of a random measure. However, the jump intensity in the random measure representation of the chain and that in the jump-diffusion process in Lee and Protter may have a different structure. The former depends on the transition intensity of the chain, which is partially determined by the current asset price level. The latter is assumed to be a functional of the price path of the risky asset. Further, we model explicitly the feedback effect of asset prices on the economic conditions, which are empirically and theoretically justified. Feedback effect is also incorporated in the Lee-Protter model. However, Lee and Protter focused on the feedback effect of the current and past asset price behavior on future asset prices. Whereas, we concentrate on modeling the feedback effect of asset prices on the economic conditions, and vice versa, which are empirically and theoretically justified.

Finally, the martingale representation theorem in our paper, which provides the hedging strategy, is new.

This paper is structured as follows: In Section 2 we describe the asset price dynamics of the model. In Section 3 we present a numerical simulation under a discrete time version of the proposed model. In Section 4 we specify an equivalent martingale measure, or a pricing kernel, based on a product of two density processes, one for the Markov, regime-switching, process and one for the Markov chain and provide an economic equilibrium justification for the proposed change of measure. Then, we present the main result for the martingale representation and its derivation in Section 5. We construct the local risk-minimizing strategy for hedging and the corresponding pricing measure in Section 6. The final section summarizes the paper.

§2. The Price Dynamics

We consider a continuous-time economy with two primitive assets, namely, a bank account B and a risky asset S. These assets are tradeable continuously over time in a finite time horizon $[0, T]$, where $T < \infty$. Write $[0, T]$ as $\mathcal{T}$. We describe sources of uncertainties in the economy using a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\mathcal{P}$ is a real-world probability. Define an observable, continuous-time and finite-state Markov chain $\mathbf{X} := \{\mathbf{X}(t) | t \in \mathcal{T}\}$ on $(\Omega, \mathcal{F}, \mathcal{P})$ with a finite state space $\mathcal{X} := \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\} \subset \Re^N$. The states of the chain represent different states of the economy. They may be interpreted as sovereign credit ratings, or proxies of other (macro)-economic indicators, such as the gross domestic product, the retail price index, the growth rate of industrial production, and possibly others. Without loss of generality, we identify the state space of the chain $\mathbf{X}$ with a finite set of unit vectors $\mathcal{E} := \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$, where $\mathbf{e}_i$ is the canonical unit vector in $\Re^N$ and the j^{th} component of $\mathbf{e}_i$ is the Kronecker delta δ_{ij}, for each $i, j = 1, 2, \ldots, N$. The space $\mathcal{E}$ is called the canonical state space of the chain $\mathbf{X}$. Let $\xi := \{\xi(t) | t \in \mathcal{T}\}$ denote a stochastic process on $(\Omega, \mathcal{F}, \mathcal{P})$ with state space $\Re^+$ such that for each $t \in \mathcal{T}$, $\xi(t)$ represents the value of the risky asset at time t. We specify the dynamics of the risky asset later in this section. Let $\mathbf{A}(t, \xi(t))$ denote the generator, or the rate matrix, $[a_{ij}(t, \xi(t))]_{i,j=1,2,\ldots,N}$ of the chain $\mathbf{X}$, where, for $i \neq j$, $a_{ij}(t, \xi(t))$ is the instantaneous intensity of the transition of $\mathbf{X}$ from state i to state j at time t. Then, the rate matrix $\mathbf{A}(t, \xi(t))$ of the chain $\mathbf{X}$ at time t depends on the current value of the asset price $\xi(t)$. This models the feedback effect of the asset price on states described. For each $t \in \mathcal{T}$, $a_{ij}(t, \xi(t)) \geq 0$, for $i \neq j$ and $\sum_{j=1}^{N} a_{ij}(t, \xi(t)) = 0$, so $a_{ii}(t, \xi(t)) \leq 0$. We further assume that $a_{ij}(t, \xi(t)) > 0$, for each $i, j = 1, 2, \ldots, N$ and each $t \in \mathcal{T}$. Then, with the canonical state space $\mathcal{E}$, the semi-martingale representation for $\mathbf{X}$ is given by

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \mathbf{A}(s, \xi(s))\mathbf{X}(s)ds + \mathbf{M}(t) \ . \tag{2.1}$$

Here $\{\mathbf{M}(t)|t \in \mathcal{T}\}$ is an $\Re^N$-valued martingale with respect to the complete enlarged filtration generated by both the processes $\mathbf{X}$ and ξ under the measure $\mathcal{P}$.

The semi-martingale representation for the chain $\mathbf{X}$ with time-dependent family of rate matrices was derived in Elliott et al. (1994). It describes the evolution of the chain $\mathbf{X}$ over time. Note that $\mathbf{M}$ is also a process of finite (locally integrable) variation.

Let $r(t)$ denote the instantaneous market interest rate of the bank account B at time t. The interest rate $r(t)$ is assumed to be modulated by the chain $\mathbf{X}$ as follows:

$$r(t) = \langle \mathbf{r}, \mathbf{X}(t) \rangle \ , \tag{2.2}$$

where $\mathbf{r} := (r_1, r_2, \ldots, r_N)' \in \Re^N$ with $r_i > 0$, for each $i = 1, 2, \ldots, N$. $\mathbf{y}'$ denotes the transpose of a vector or matrix $\mathbf{y}$. The scalar product $\langle \cdot, \cdot \rangle$ in $\Re^N$ selects a component of the vector of interest rates $\mathbf{r}$ that is in force at time t according to the state $\mathbf{X}(t)$ of the economy at time t.

The evolution of the balance of the bank account $B := \{B(t)|t \in \mathcal{T}\}$ over time is then governed by:

$$B(t) = \exp\left(\int_0^t r(u)du \right) \ , \quad B(0) = 1 \ . \tag{2.3}$$

Write, for each $t \in \mathcal{T}$,

$$B(t, T) = B^{-1}(t)B(T) = \exp\left(\int_t^T r(u)du \right) \ .$$

Suppose the price process $\xi := \{\xi(t)|t \in \mathcal{T}\}$ of the risky asset S is governed by the following double Markov, regime-switching, process with feedback effect:

$$\begin{aligned} d\xi(t) &= \xi(t-)\Big((\mu(t,\xi(t),\mathbf{X}(t)) - \langle \boldsymbol{\gamma}(t,\xi(t)), \mathbf{A}(t,\xi(t))\mathbf{X}(t)\rangle)dt \\ &\quad +\sigma(t,\xi(t-),\mathbf{X}(t-))dW(t) + \langle \boldsymbol{\gamma}(t,\xi(t-)), d\mathbf{X}(t)\rangle \Big) . \end{aligned} \tag{2.4}$$

Here $W := \{W(t)|t \in \mathcal{T}\}$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$; $\mu(t,\xi(t),\mathbf{X}(t))$ and $\sigma(t,\xi(t),\mathbf{X}(t))$ are the appreciation rate and the volatility of the risky asset at time t, respectively; $\boldsymbol{\gamma}(t,\xi(t)) := (\gamma_1(t,\xi(t)), \gamma_2(t,\xi(t)), \ldots, \gamma_N(t,\xi(t)))'$, where the difference $\gamma_j(t,\xi(t)) - \gamma_i(t,\xi(t))$ represents the jump amplitude of the asset price when the state of the economy $\mathbf{X}$ transits from state $\mathbf{e}_i$ at time $t-$ to state $\mathbf{e}_j$ at time t. The difference is identical to zero when no transition of the state of the economy occurs at time t. We suppose that $\mu(t,\xi(t),\mathbf{X}(t))$, $\sigma(t,\xi(t),\mathbf{X}(t))$ and $\boldsymbol{\gamma}(t,\xi(t))$ satisfy some technical conditions to ensure that the price process has a unique strong solution. Here the state of the economy $\mathbf{X}$ has "double" effect, hence its name, on the dynamics of the asset price. The transitions of $\mathbf{X}$ cause structural changes in the model parameters, namely, the appreciation rate and the volatility of the risky asset, and jumps in the asset price at the same time. Instead of introducing another random measure as in the Merton jump-diffusion model and its variants, we model the jump component by the chain $\mathbf{X}$ itself. So, jumps in the asset price are directly related to transitions of the state of the economy instead of other exogenous factors, like market events or news, as in the Merton-type jump-diffusion models. The jumps considered here can also be thought of as the shifts in the level of the asset price due to transitions of the state of the economy. They may occur less frequently than those modeled by the Merton-type jump-diffusion models and have long-term impact on the price, which is relevant for modelling the long-term behavior of the risky asset.

The price process considered here also incorporates the feedback effect on the economic state. More specifically, the state of an economy affects the evolution of the asset price, and at the same time, the state of an economy is also a consequence of the evolution of the asset price. We may think of the share index price as a public signal available for investors. The market agents react to this public signal and incorporate their reactions in their investment behavior. The change of the investment behavior of the market agent will affect the economic state again. The idea is not unlike the concept of reflexibility advocated by George Soros. The feedback effect is often overlooked in some of the existing literature, but is of importance when a risky asset is considered.

In the sequel, we describe the flows of the solutions of the dynamics of the risky asset and their derivative with respect to the initial condition. Let $\xi_{s,t}(y)$ denote the solution of the price process with initial condition y at time s; that is, $\xi_{s,s}(y) = y$. Write $F := \{\mathcal{F}(t) | t \in \mathcal{T}\}$ for the right-continuous complete filtration generated by the price process ξ so that $\mathcal{F}(t)$ represents the information set generated by the price process ξ up to and including time t. Let

$$
\begin{aligned}
f(u, \xi(u), \mathbf{X}(u)) &:= \mu(u, \xi(u), \mathbf{X}(u))\xi(u) \ , \\
g(u, \xi(u), \mathbf{X}(u)) &:= \sigma(u, \xi(u), \mathbf{X}(u))\xi(u) \ , \\
\mathbf{h}(u, \xi(u)) &:= \boldsymbol{\gamma}(u, \xi(u))\xi(u) \ .
\end{aligned}
$$

Then,

$$
\begin{aligned}
\xi_{s,t}(y) &= y + \int_s^t f(u, \xi(u), \mathbf{X}(u))du + \int_s^t g(u, \xi(u-), \mathbf{X}(u-))dW(u) \\
&\quad + \int_s^t \langle \mathbf{h}(u, \xi(u-)), d\mathbf{M}(u) \rangle \ . \qquad (2.5)
\end{aligned}
$$

Now, we suppose that μ, σ and $\boldsymbol{\gamma}$ are measurable functions that are three times differentiable in ξ, and which, together with their derivatives, are bounded, for each $t \in \mathcal{T}$ and $\mathbf{x} \in \mathcal{E}$. We further assume that f, g and $\mathbf{h}$ satisfy suitable integrability conditions. By adopting arguments similar to those of Kunita (1978, 1982), Bismut (1981) and Elliott and Kohlmann (1989), there exists a flow of diffeomorphisms $y \to \xi_{s,t}(y)$ associated with the price process almost surely. Write $D_{s,t}(y) := \frac{\partial \xi_{s,t}(y)}{\partial y}$. Then, $D_{s,t}(y)$ satisfies the following linearized equation:

$$
\begin{aligned}
dD_{s,t}(y) &= f_\xi(t, \xi_{s,t}(y), \mathbf{X}(t))D_{s,t}(y)dt + g_\xi(t, \xi_{s,t-}(y), \mathbf{X}(t-))D_{s,t-}(y)dW(t) \\
&\quad + D_{s,t-}(y) \langle \mathbf{h}_\xi(t, \xi(t-)), d\mathbf{M}(t) \rangle \ , \\
D_{s,s}(y) &= 1 \ . \qquad (2.6)
\end{aligned}
$$

Here f_ξ, g_ξ and $\mathbf{h}_\xi$ are the derivatives of the functions f, g and $\mathbf{h}$ with respect to ξ, respectively. Equation (2.6) can be obtained by following the techniques presented in Kunita (1978, 1982), Bismut (1981) and Elliott and Kohlmann (1989).

§3. A simulation study

In this section we present a simulation study to illustrate the impact of the feedback effect on the behavior of asset prices. We consider a discrete time version of the model from (2.4). For any $k = 1...T$, we assume the asset price dynamic is given by:

$$\xi_k \;=\; \xi_{k-1}\Big(1 + \langle \mu, X_{k-1}\rangle - \langle \boldsymbol{\gamma}, \mathbf{A}(\xi_{k-1})\mathbf{X}_{k-1}\rangle + \langle \sigma, X_{k-1}\rangle\, W_k + \langle \boldsymbol{\gamma}, \mathbf{X}_k - \mathbf{X}_{k-1}\rangle\Big)\ .$$

Here W_k is a sequence of independent standard Brownian Motions and X_k is discrete-time, 2-state, Markov chain with the transition matrix $A(\xi_k)$. We assume the model parameters do not depend on the asset price, the feedback effect being incorporated only in the transition matrix. We let $\mu = (0.004, 0.0035)'$ and we consider the two regimes are characterized by a low (state 1) and a high (state 2) volatility, with $\sigma = (0.01, 0.03)'$. The ratio of the two volatilities represents a reasonable choice for asset prices (see for example Turner *et al.* (1989)). The jump parameter is $\boldsymbol{\gamma} = (0.005, 0.001)'$, so whenever there is a regime shift from state 1 to state 2 the jump amplitude is -0.004, whereas there is a jump of 0.004 when the economy switches from the high volatility state to the lower one. The feedback effect is represented using a threshold level for the asset price. Thus, whenever the share price ξ_k drops below of a predetermined level C, then $A(\xi_k) = A_2$, while $A(\xi_k) = A_1$ for the values of the index greater than C. The transition probabilities of A_1 are, $a^1_{11} = 0.9$ and $a^1_{22} = 0.8$, therefore whenever $\xi_k > C$, both regimes are highly persistent. When the stock price drops below C, we let the two regimes to be less persistent with $a^2_{11} = 0.8$ and $a^2_{22} = 0.65$ for A_2. The reason behind choosing these values is that various empirical studies (for example Ang and Bekaert (2002) and Bansal and Zhou (2002)) found that when transition probabilities are time dependent, the expected probability of remaining in a high volatility state is lower.

To assess the impact of the proposed feedback effect we simulate sample paths according to the above model based on the transition matrix $A(\xi_k)$ and we compare our results with the same model constructed using a Markov chain based on A_1. For a pe-

Figure 1: Expected stock price for feedback vs. no feedback models

riod consisting of 2,500 observations (which roughly corresponds to a 10 year period) we compute the expected asset price for both models at any time point. The expectations are computed using a Monte-Carlo simulation based on 10,000 sample paths. We let $\xi_0 = 1000$ and $C = 900$. The results are illustrated in the graph below:

From Figure 1 we note that the expected value of the asset price when there is no feedback effect is in general lower than its counterpart with the difference being more pronounced as time increases. This finding can be explained by the choice of A_2, since whenever the price drops below 900, despite the low volatility state is not as persistent as in A_1, the probability of switching from a high volatility period is relatively high. For example, the differences between the expected asset prices after one year is approximately 0.5, after two years is around 6.6, while at the end of the period becomes 40. Although these differences might not look very large, they may have a significant impact on pricing long term insurance products such as participating life insurance policies and equity-linked securities. However, the behavior of the feedback model versus the no feedback one is strongly determined by the choices of the parameters. One could expect significant changes once we let the other parameters to depend on the asset price, so a more detailed empirical analysis should be performed. Our purpose here was to investigate the feedback effect within a very simple setting.

We now take a closer look at only one sample path simulated from both models over a period of two years. This time the simulations are done using the same random seed generator. We use the same threshold limit as in the first numerical experiment. From Figure 2 we note that the two processes move together only until the asset price drops first time below the threshold level which takes place at day 73. At this moment the transition matrix for the feedback model changes to A_2. However, the two process continue to move together until the first difference in regime switching occurs do to the new transition probabilities. The feedback process recovers more quickly from the higher volatility state (when its value is below 900) and it resembles a shifted version of the no feedback counterpart (when its value is greater than 900). There are 102 regime switches with feedback effect and only 80 changes in regimes without feedback effect. The value of the model with feedback has been below the threshold level for 184 days. An intuitive interpretation for the price movements illustrated in Figure 2 can be made relative to some of the typical outcomes of a downward market trend. For example, when the feedback effect is incorporated, the asset price picks up after a relative short period of time, and this behavior is, in general, associated with a market "correction", which may be attributed to the presence of the feedback effect. It appears that in the presence of the feedback effect, the momentum of the asset price continues after a short period of small reversal. It seems that an asset price bubble may remain for a longer period of time in a market with the feedback effect than in one without the effect. On the other hand, a substantial drop in the asset price for a longer period of time in the no feedback case characterizes a "bear" market.

Another important issue regarding the statistical properties of our proposed model is its order of integration. Although, data generated from the stock price dynamic according to (2.4) might not be stationary, its log returns can generate a stationary data set. There has been a growing amount of literature on deciding whether a set of time series data is

Figure 2: Simulated Asset Price for feedback vs. no feedback

stationary or not. In general there are two main types of statistical tests to assess this issue: the unit root tests such as Dickey-Fuller (DF) test and Philips-Perron (PP) test, which use the presence of unit roots as the null hypothesis, and the stationarity versus non-stationarity tests with the reverse null hypothesis, such as the KPSS test. A known problem with the two unit root tests is that in general they cannot distinguish very well non stationary time series from highly persistent ones, so other more efficient tests should be employed.

We tested the stationarity for data generated according to our feedback model based on 5,000 observations. We repeated this experiment several times for our parameter set using the KPPS and the modified efficient PP tests and our findings suggests that there is no strong evidence against stationarity assumption. However, a more detailed statistical analysis should be of interest to asses the quality of our model to generates price movements observed in markets.

§4. An Equivalent Martingale Measure

We first specify an equivalent martingale measure, or price kernel, which can price the two sources of risk. This equivalent martingale measure is defined as a product of two density processes, one for a measure change for the diffusion component and one for a measure change for the chain. Then we discuss how the proposed change of measure can be used to obtain equilibrium valuation relationships.

Firstly, we describe the Girsanov density for a measure change of the diffusion component. Let $F^{\mathbf{X}} := \{\mathcal{F}^{\mathbf{X}}(t) | t \in \mathcal{T}\}$ denote the right-continuous complete filtration generated by the chain $\mathbf{X}$. For each $t \in \mathcal{T}$, $\mathcal{G}(t) := \mathcal{F}(t) \vee \mathcal{F}^{\mathbf{X}}(t)$, which represents the enlarged σ-field generated by $\mathcal{F}(t)$ and $\mathcal{F}^{\mathbf{X}}(t)$. Write $G := \{\mathcal{G}(t) | t \in \mathcal{T}\}$ for the corresponding complete enlarged filtration.

For each $s, t \in \mathcal{T}$ with $s \leq t$, we define a Girsanov density:

$$\Lambda^1_{s,t}(y) \ := \ 1 + \int_s^t \Lambda^1_{s,u-}(y) \theta_1(u, \xi_{s,u-}(y), \mathbf{X}(u-)) dW(u) \ . \tag{4.1}$$

The term $\theta_1(t, \xi, \mathbf{x})$ is used to compensate for the diffusion risk and may be interpreted as "a market price of risk" for the diffusion component.

Here we suppose that $\theta_1(t, \xi, \mathbf{x})$ is such that

1. $\{\Lambda^1_{0,t}(y_0) | t \in \mathcal{T}\}$ is a square-integrable $(G, \mathcal{P})$-martingale, where y_0 is the initial condition for the solution flow $\{\xi_{0,t}(y_0) | t \in \mathcal{T}\}$ of the price process at time zero;
2. for each $(t, \xi, \mathbf{x}) \in \mathcal{T} \times \Re^+ \times \mathcal{E}$, $\theta_1(t, \xi, \mathbf{x}) \in \Re$;
3. for each $(t, \mathbf{x}) \in \mathcal{T} \times \mathcal{E}$, $\theta_1(t, \cdot, \mathbf{x})$ has continuous first derivatives in ξ.

We are now ready to define a density process for a measure change for the Markov chain. We employ the parametrization of the product density processes of Elliott and Siu (2010) to provide a parsimonious way for the measure change for the chain. More specifically, we suppose that all instantaneous intensities of the transitions of the chain are scaled or rotated by the same amount when changing the probability measures.

Firstly, we describe some notation from Dufour and Elliott (1999) which will be useful here. For any feedback rate matrix $\mathbf{\Pi}(t, \xi(t))$, let

$$\begin{aligned} & \boldsymbol{\pi}(t, \xi(t)) \\ := \; & (\pi_{11}(t, \xi(t)), \ldots, \pi_{ii}(t, \xi(t)), \ldots, \pi_{NN}(t, \xi(t)))' \in \Re^N \ , \end{aligned}$$

and

$$\mathbf{\Pi}_0(t, \xi(t)) := \mathbf{\Pi}(t, \xi(t)) - \mathbf{diag}(\boldsymbol{\pi}(t, \xi(t))) \ ,$$

where $\mathbf{diag}(\mathbf{y})$ is a diagonal matrix with the diagonal elements given by the vector $\mathbf{y}$.

Let $\theta_2 : \mathcal{T} \times \Re^+ \times \mathcal{E} \to \Re^+$ be a function such that $\forall (t, \mathbf{x}) \in \mathcal{T} \times \mathcal{E}$, $\theta_2(t, \cdot, \mathbf{x})$ has a continuous first derivative in ξ. The term θ_2 is used to compensate for the regime-switching risk described by the chain $\mathbf{X}$ and may be interpreted as a (nonlinear) version of "a market price of risk" of the chain $\mathbf{X}$. Define, for each $t \in \mathcal{T}$, a feedback generator, or rate matrix, $\mathbf{A}^{\theta_2}(t, \xi(t)) := [a^{\theta_2}_{ij}(t, \xi(t))]_{i,j=1,2,\ldots,N}$ of the chain $\mathbf{X}$ such that for each $i, j = 1, 2, \ldots, N$,

$$a^{\theta_2}_{ij}(t, \xi(t)) := \theta_2(t, \xi(t), \mathbf{X}(t)) a_{ij}(t, \xi(t)) \ , \quad t \in \mathcal{T} \ .$$

Define, for each $t \in \mathcal{T}$, $\mathbf{D}^{\theta_2}(t, \xi(t))$ as the matrix $[a^{\theta_2}_{ij}(t, \xi(t))/a_{ij}(t, \xi(t))]$. Recall that $a_{ij}(t, \xi(t)) > 0$, for each $t \in \mathcal{T}$, so $\mathbf{D}^{\theta_2}(t, \xi(t))$ is well-defined. Let $\mathbf{1} := (1, 1, \ldots, 1)' \in \Re^N$ and $\mathbf{I}$ denote the $(N \times N)$-identity matrix.

Consider a vector of counting processes $\mathbf{N} := \{\mathbf{N}(t) | t \in \mathcal{T}\}$ defined by:

$$\mathbf{N}(t) := \int_0^t (\mathbf{I} - \mathbf{diag}(\mathbf{X}(u-))) d\mathbf{X}(u) \ , \quad t \in \mathcal{T} \ , \tag{4.2}$$

where its component $N_i(t)$ counts the number of times the chain $\mathbf{X}$ jumps to state $\mathbf{e}_i$ in the time interval $[0, t]$, for each $i = 1, 2, \ldots, N$.

The predictable compensator or dual predictable projection of $\mathbf{N}$ can be determined by a modified version of a result from Dufour and Elliott (1999).

Lemma 4.1: Let $\mathbf{a}(t,\xi(t)) := (a_{11}(t,\xi(t)), a_{22}(t,\xi(t)), \cdots, a_{NN}(t,\xi(t)))' \in \Re^N$. Write $\mathbf{diag}[\mathbf{a}(t,\xi(t))]$ for the diagonal matrix with diagonal elements given by the components of $\mathbf{a}(t,\xi(t))$. Define $\mathbf{A}_0(t,\xi(t)) := \mathbf{A}(t,\xi(t)) - \mathbf{diag}[\mathbf{a}(t,\xi(t))]$ and

$$\tilde{\mathbf{N}}(t) := \mathbf{N}(t) - \int_0^t \mathbf{A}_0(u,\xi(u))\mathbf{X}(u)du \ , \quad t \in \mathcal{T} \ . \tag{4.3}$$

Then $\tilde{\mathbf{N}} := \{\tilde{\mathbf{N}}(t) | t \in \mathcal{T}\}$ is a $(G, \mathcal{P})$-martingale.

Proof: This result follows from Remark 2.1 in Dufour and Elliott (1999) with the constant rate matrix replaced by the feedback rate matrix $\mathbf{A}(t,\xi(t))$.

□

Let $d_{ij}^{\theta_2}(t,\xi(t)) := a_{ij}^{\theta_2}(t,\xi(t))/a_{ij}(t,\xi(t))$. We adopt the convention that $\frac{0}{0} = 0$. So, $d_{ij}^{\theta_2}(t,\xi(t))$ is taken to be zero when $a_{ij}(t,\xi(t)) = 0$. Write

$$\mathbf{d}^{\theta_2}(t,\xi(t)) := (d_{11}^{\theta_2}(t,\xi(t)), d_{22}^{\theta_2}(t,\xi(t)), \ldots, d_{NN}^{\theta_2}(t,\xi(t)))' \ .$$

Define

$$\mathbf{D}_0^{\theta_2}(t,\xi(t)) := \mathbf{D}^{\theta_2}(t,\xi(t)) - \mathbf{diag}[\mathbf{d}^{\theta_2}(t,\xi(t))] \ , \quad \theta_2 = \theta_2(t,\xi(t),\mathbf{X}(t)) \ .$$

Consider a density process $\{\Lambda_{s,t}^2(y) | t, s \in \mathcal{T}, t \geq s\}$ for a measure change for the Markov chain defined by:

$$\begin{aligned}\Lambda_{s,t}^2(y) &= 1 + \int_s^t \Lambda_{s,u-}^2(y)[\mathbf{D}_0^{\theta_2}(u,\xi(u-))\mathbf{X}(u-) - \mathbf{1}]' \\ &\times (d\mathbf{N}(u) - \mathbf{A}_0(u,\xi(u))\mathbf{X}(u)du) \ .\end{aligned} \tag{4.4}$$

We assume that $\{\Lambda_{0,t}^2(y_0) | t \in \mathcal{T}\}$ is a $(G, \mathcal{P})$-martingale.

Now, we define a product density process for a measure change for both the diffusion process and the Markov chain by setting:

$$\Lambda_{s,t}(y) := \Lambda_{s,t}^1(y) \cdot \Lambda_{s,t}^2(y) \ .$$

Applying Itô's product rule to $\Lambda_{s,t}(y)$ gives:

$$\begin{aligned}\Lambda_{s,t}(y) &= 1+\int_s^t \Lambda_{s,u-}(y)\theta_1(u,\xi_{s,u-}(y),\mathbf{X}(u-))dW(u)\\ &\quad +\int_s^t \Lambda_{s,u-}(y)[\mathbf{D}_0^{\theta_2}(u,\xi(u-))\mathbf{X}(u-)-\mathbf{1}]'\\ &\quad \times(d\mathbf{N}(u)-\mathbf{A}_0(u,\xi(u))\mathbf{X}(u)du)\ ,\end{aligned} \tag{4.5}$$

so, $\{\Lambda_{0,t}(y_0)|t\in\mathcal{T}\}$ is a $(G,\mathcal{P})$-(local)-martingale.

We assume that $\{\Lambda_{0,t}(y_0)|t\in\mathcal{T}\}$ is a $(G,\mathcal{P})$-martingale.

Write $\boldsymbol{\theta}$ for (θ_1,θ_2), where $\theta_1=\theta_1(t,\xi(t),\mathbf{X}(t))$ and $\theta_2=\theta_2(t,\xi(t),\mathbf{X}(t))$. Then, we define a new probability measure $\mathcal{P}^{\boldsymbol{\theta}}$ equivalent to $\mathcal{P}$ on $\mathcal{G}(T)$ by setting

$$\frac{d\mathcal{P}^{\boldsymbol{\theta}}}{d\mathcal{P}}=\Lambda_{0,T}(y_0)\ . \tag{4.6}$$

To determine an equivalent martingale measure for valuation, we need to determine θ_1 and θ_2 for compensating for the diffusion risk and the regime-switching, respectively.

By the Girsanov theorem, under the new measure $\mathcal{P}^{\boldsymbol{\theta}}$,

$$W^{\boldsymbol{\theta}}(t):=W(t)-\int_0^t\theta_1(u,\xi(u),\mathbf{X}(u))du\ ,\quad t\in\mathcal{T}\ ,$$

is a standard Brownian motion with respect to the enlarged filtration G.

Let

$$\mathbf{a}^{\theta_2}(t,\xi(t)):=(a_{11}^{\theta_2}(t,\xi(t)),a_{22}^{\theta_2}(t,\xi(t)),\cdots,a_{NN}^{\theta_2}(t,\xi(t)))'\in\Re^N\ ,\quad t\in\mathcal{T}\ .$$

Write $\mathbf{diag}[\mathbf{a}^{\theta_2}(t,\xi(t))]$ for the diagonal matrix with diagonal elements given by the components of $\mathbf{a}^{\theta_2}(t,\xi(t))$. Define $\mathbf{A}_0^{\theta_2}(t,\xi(t)):=\mathbf{A}^{\theta_2}(t,\xi(t))-\mathbf{diag}[\mathbf{a}^{\theta_2}(t,\xi(t))]$ and

$$\tilde{\mathbf{N}}^{\boldsymbol{\theta}}(t):=\mathbf{N}(t)-\int_0^t\mathbf{A}_0^{\theta_2}(u,\xi(u))\mathbf{X}(u)du\ ,\quad t\in\mathcal{T}\ , \tag{4.7}$$

where $\theta_2 = \theta_2(t, \xi(t), \mathbf{X}(t))$.

Then, we have the following result.

Lemma 4.2: $\tilde{\mathbf{N}}^{\boldsymbol{\theta}} := \{\tilde{\mathbf{N}}^{\boldsymbol{\theta}}(t) | t \in \mathcal{T}\}$ is a $(G, \mathcal{P}^{\boldsymbol{\theta}})$-martingale.

Proof: This result follows if $\{\Lambda_{0,t}(y_0)\tilde{\mathbf{N}}^{\boldsymbol{\theta}}(t) | t \in \mathcal{T}\}$ is a $(G, \mathcal{P})$-martingale. Applying Itô's product rule on $\Lambda^1_{0,t}(y_0)\Lambda^2_{0,t}(y_0)\tilde{\mathbf{N}}^{\boldsymbol{\theta}}(t)$,

$$\begin{aligned} d(\Lambda^1_{0,t}(y_0)\Lambda^2_{0,t}(y_0)\tilde{\mathbf{N}}^{\boldsymbol{\theta}}(t)) &= \Lambda^1_{0,t-}(y_0)d(\Lambda^2_{0,t}(y_0)\tilde{\mathbf{N}}^{\boldsymbol{\theta}}(t)) + \Lambda^2_{0,t}(y_0)\tilde{\mathbf{N}}^{\boldsymbol{\theta}}(t)d\Lambda^1_{0,t}(y_0) \\ &\quad + d < \Lambda^1_{0,t}(y_0), \Lambda^2_{0,t}(y_0)\tilde{\mathbf{N}}^{\boldsymbol{\theta}}(t) > \ . \end{aligned}$$

where $< \Lambda^1_{0,t}(y_0), \Lambda^2_{0,t}(y_0)\tilde{\mathbf{N}}^{\boldsymbol{\theta}}(t) >$ is the quadratic variation of $\Lambda^1_{0,t}(y_0)$ and $\Lambda^2_{0,t}(y_0)\tilde{\mathbf{N}}^{\boldsymbol{\theta}}(t)$.

Since $\mathbf{N}$ is a jump process and W is a standard Brownian motion, $< \mathbf{N}(t), W(t) >= 0$, for each $t \in \mathcal{T}$. Note that $\Lambda^1_{0,t}(y_0)$ is a stochastic integral with respect to W and that $\Lambda^2_{0,u}(y_0)\tilde{\mathbf{N}}^{\boldsymbol{\theta}}(u)$ is the sum of stochastic integrals with respect to $\mathbf{N}$. So,

$$< \Lambda^1_{0,t}(y_0), \Lambda^2_{0,t}(y_0)\tilde{\mathbf{N}}^{\boldsymbol{\theta}}(t) >= 0 \ , \quad t \in \mathcal{T} \ ,$$

and, hence

$$d(\Lambda^1_{0,t}(y_0)\Lambda^2_{0,t}(y_0)\tilde{\mathbf{N}}^{\boldsymbol{\theta}}(t)) = \Lambda^1_{0,t-}(y_0)d(\Lambda^2_{0,t}(y_0)\tilde{\mathbf{N}}^{\boldsymbol{\theta}}(t)) + \Lambda^2_{0,t}(y_0)\tilde{\mathbf{N}}^{\boldsymbol{\theta}}(t)d\Lambda^1_{0,t}(y_0) \ .$$

Therefore, $\{\Lambda_{0,t}(y_0)\tilde{\mathbf{N}}^{\boldsymbol{\theta}}(t) | t \in \mathcal{T}\}$ is a $(G, \mathcal{P})$-martingale if and only if $\{\Lambda^2_{0,t}(y_0)\tilde{\mathbf{N}}^{\boldsymbol{\theta}}(t) | t \in \mathcal{T}\}$ is a $(G, \mathcal{P})$-martingale. The rest of the proof is adapted to the first part of the proof of Lemma 2.3 in Dufour and Elliott (1999). Again, applying Itô's product rule on $\Lambda^2_{0,t}(y_0)\tilde{\mathbf{N}}^{\boldsymbol{\theta}}(t)$,

$$\begin{aligned} & \Lambda^2_{0,t}(y_0)\tilde{\mathbf{N}}^{\boldsymbol{\theta}}(t) \\ = \ & \int_0^t \Lambda^2_{0,u-}(y_0)d\mathbf{N}(u) - \int_0^t \Lambda^2_{0,u-}(y_0)\mathbf{A}^{\theta_2}_0(u, \xi(u))\mathbf{X}(u)du \\ & + \int_0^t \tilde{\mathbf{N}}^{\boldsymbol{\theta}}(u-)d\Lambda^2_{0,t}(y_0) + \int_0^t \Lambda^2_{0,u}(y_0)\mathbf{diag}[\mathbf{A}_0(u, \xi(u))\mathbf{X}(u)] \\ & \times(\mathbf{D}^{\theta_2}_0(u, \xi(u))\mathbf{X}(u) - \mathbf{1})du \\ & + \int_0^t \Lambda^2_{0,u-}(y_0)\mathbf{diag}[d\mathbf{N}(u) - \mathbf{A}_0(u, \xi(u))\mathbf{X}(u)du] \\ & \times(\mathbf{D}^{\theta_2}_0(u, \xi(u-))\mathbf{X}(u-) - \mathbf{1}) \ . \end{aligned} \tag{4.8}$$

By noticing that

$$\begin{aligned}
& \mathbf{diag}[\mathbf{A}_0(u,\xi(u))\mathbf{X}(u)](\mathbf{D}_0^{\theta_2}(u,\xi(u))\mathbf{X}(u)-\mathbf{1}) \\
= \; & \mathbf{A}_0^{\theta_2}(u,\xi(u))\mathbf{X}(u)-\mathbf{A}_0(u,\xi(u))\mathbf{X}(u) \ ,
\end{aligned}$$

we have

$$\begin{aligned}
& \Lambda^2_{0,t}(y_0)\tilde{\mathbf{N}}^{\boldsymbol{\theta}}(t) \\
= \; & \int_0^t \Lambda^2_{0,u-}(y_0)(d\mathbf{N}(u)-\mathbf{A}_0(u,\xi(u))\mathbf{X}(u)du)+\int_0^t \tilde{\mathbf{N}}^{\boldsymbol{\theta}}(u-)d\Lambda^2_{0,u}(y_0) \\
& +\int_0^t \Lambda^2_{0,u-}(y_0)\mathbf{diag}[d\mathbf{N}(u)-\mathbf{A}_0(u,\xi(u))\mathbf{X}(u)du] \\
& \times(\mathbf{D}_0^{\theta_2}(u,\xi(u-))\mathbf{X}(u-)-\mathbf{1}) \ , \qquad (4.9)
\end{aligned}$$

and so, $\{\Lambda^2_{0,t}(y_0)\tilde{\mathbf{N}}^{\boldsymbol{\theta}}(t)|t\geq 0\}$ is a $(G,\mathcal{P})$-martingale.

□

The following result specifies the statistical properties of the Markov chain $\mathbf{X}$ under the new measure $\mathcal{P}^{\boldsymbol{\theta}}$. This result is similar to that of Lemma 2.3 in Dufour and Elliott (1999).

Proposition 4.3: $\mathbf{X}$ is a Markov chain with a family of generators $\mathbf{A}^{\theta_2}(t,\xi(t))$, $t\in\mathcal{T}$, under $\mathcal{P}^{\boldsymbol{\theta}}$.

Proof: From Lemma 3.2, $\tilde{\mathbf{N}}^{\boldsymbol{\theta}}$ is a $(G,\mathcal{P}^{\boldsymbol{\theta}})$-martingale. Using Remark 2.1 in Dufour and Elliott (1999) with the constant rate matrix replaced by the feedback rate matrix $\mathbf{A}(t,\xi(t))$,

$$\begin{aligned}
\mathbf{X}(t) &= \mathbf{X}(0)+\int_0^t(\mathbf{I}-\mathbf{X}(u-)\mathbf{1}')d\mathbf{N}(u) \\
&= \mathbf{X}(0)+\int_0^t(\mathbf{I}-\mathbf{X}(u)\mathbf{1}')\mathbf{A}_0^{\theta_2}(u,\xi(u))\mathbf{X}(u)du \\
&\quad +\int_0^t(\mathbf{I}-\mathbf{X}(u-)\mathbf{1}')(d\mathbf{N}(u)-\mathbf{A}_0^{\theta_2}(u,\xi(u))\mathbf{X}(u)du) \\
&= \mathbf{X}(0)+\int_0^t \mathbf{A}^{\theta_2}(u,\xi(u))\mathbf{X}(u)du+\mathbf{M}^{\boldsymbol{\theta}}(t) \ ,
\end{aligned}$$

$$(4.10)$$

where $\mathbf{M}^{\boldsymbol{\theta}}(t) := \int_0^t (\mathbf{I} - \mathbf{X}(u-)\mathbf{1}')(d\mathbf{N}(u) - \mathbf{A}_0^{\theta_2}(u, \xi(u))\mathbf{X}(u)du)$, $t \in \mathcal{T}$, and $\{\mathbf{M}^{\boldsymbol{\theta}}(t) | t \in \mathcal{T}\}$ is a $(G, \mathcal{P}^{\boldsymbol{\theta}})$, $\Re^N$-valued, martingale.

Hence, the result follows.

□

For each $t \in \mathcal{T}$, we define the discounted price of the risky asset $\tilde{\xi}(t) := (B(0,t))^{-1}\xi(t)$ at time t. Under the new measure $\mathcal{P}^{\boldsymbol{\theta}}$, the discounted price process $\tilde{\xi}$ of the risky asset has the following dynamics:

$$\begin{aligned} d\tilde{\xi}(t) &= \sigma(t, \tilde{\xi}(t-)B(0,t-), \mathbf{X}(t-))\tilde{\xi}(t-)(dW(t) - \theta_1(t, \tilde{\xi}(t)B(0,t), \mathbf{X}(t))dt) \\ &+ \tilde{\xi}(t-)\Big[\mu(t, \tilde{\xi}(t-)B(0,t-), \mathbf{X}(t-)) - r(t-) + \sigma(t, \tilde{\xi}(t-)B(0,t-), \mathbf{X}(t-)) \\ &\times \theta_1(t, \tilde{\xi}(t-)B(0,t-), \mathbf{X}(t-)) + (\theta_2(t, \tilde{\xi}(t-)B(0,t-), \mathbf{X}(t-)) - 1) \\ &\times \Big\langle \boldsymbol{\gamma}(t, \tilde{\xi}(t-)B(0,t-)), \mathbf{A}(t, \tilde{\xi}(t-)B(0,t-))\mathbf{X}(t) \Big\rangle \Big] dt \\ &+ \tilde{\xi}(t-) \Big\langle \boldsymbol{\gamma}(t, \tilde{\xi}(t-)B(0,t-)), d\mathbf{M}^{\boldsymbol{\theta}}(t) \Big\rangle \ . \end{aligned} \tag{4.11}$$

Hence the discounted price process of the risky asset $\tilde{\xi}$ is a $(G, \mathcal{P}^{\boldsymbol{\theta}})$-(local)-martingale if and only if, for each $t \in \mathcal{T}$,

$$\begin{aligned} &\mu(t, \xi(t), \mathbf{X}(t)) - r(t) + \sigma(t, \xi(t), \mathbf{X}(t))\theta_1(t, \xi(t), \mathbf{X}(t)) \\ &+ (\theta_2(t, \tilde{\xi}(t-)B(0,t-), \mathbf{X}(t-)) - 1) \\ &\times \Big\langle \boldsymbol{\gamma}(t, \tilde{\xi}(t-)B(0,t-)), \mathbf{A}(t, \tilde{\xi}(t-)B(0,t-))\mathbf{X}(t) \Big\rangle = 0 \ , \quad \mathcal{P}\text{-a.s.} \end{aligned} \tag{4.12}$$

Since there are infinitely many choices of (θ_1, θ_2) satisfying the above equation, there are infinitely many equivalent martingale measures under which the discounted price process $\tilde{\xi}$ is a (local)-martingale.

Consider a function $c(\cdot) : \Re^+ \to \Re^+$ such that $c(\cdot)$ is twice differentiable with bounded derivatives and $B^{-1}(t,T)c(\xi_{0,T}(y_0))$ is integrable with respect to $\mathcal{P}^{\boldsymbol{\theta}}$. Consider a $(G, \mathcal{P}^{\boldsymbol{\theta}})$-martingale defined by

$$\hat{V}(t) := E^{\boldsymbol{\theta}}[B^{-1}(t,T)c(\xi_{0,T}(y_0)) | \mathcal{G}(t)] \ . \tag{4.13}$$

Here $E^{\boldsymbol{\theta}}[\cdot]$ denotes expectation under $\mathcal{P}^{\boldsymbol{\theta}}$.

Suppose ξ is a Markov process with respect to the enlarged filtration G under $\mathcal{P}^{\boldsymbol{\theta}}$. Then,

$$\begin{aligned}
\hat{V}(t) &= E^{\boldsymbol{\theta}}[B^{-1}(t,T)c(\xi_{0,T}(y_0))|\mathcal{G}(t)] \\
&= E^{\boldsymbol{\theta}}[B^{-1}(t,T)c(\xi_{0,T}(y_0))|\xi_{0,t}(y_0)=y, \mathbf{X}(t)=\mathbf{x}] \\
&= E^{\boldsymbol{\theta}}[B^{-1}(t,T)c(\xi_{t,T}(y))|\mathbf{X}(t)=\mathbf{x}] \\
&:= V(t,y,\mathbf{x}) \qquad (4.14)
\end{aligned}$$

So, a no-arbitrage price at time t of a contingent claim of the form $c(\xi_{0,T}(y_0))$, which is the payoff of the claim at maturity $T > t$, with respect to an equivalent martingale measure $\mathcal{P}^{\boldsymbol{\theta}}$ is given by:

$$\tilde{V}(t,y,\mathbf{x}) = B(t,T)V(t,y,\mathbf{x}) \ . \qquad (4.15)$$

Hence, $V(t,y,\mathbf{x})$ can be interpreted as a no-arbitrage discounted price of the contingent claim with respect to an equivalent martingale measure $\mathcal{P}^{\boldsymbol{\theta}}$.

In the remainder of this section we investigate under what assumptions the martingale measure defined in (4.6) is consistent with equilibrium arguments. There are various assumptions on the economy and the representative agent one can use to obtain general equilibrium prices. For example, Kou (2002) proposes an equilibrium pricing formula for a jump diffusion asset process when there is an outside endowment process but the asset pays no dividends. In this setting, the pricing measure is constructed using a specified jump diffusion dynamic for the endowment process. Similar approaches were considered by Camara *et al.* (2009) and Camara and Wang (2010), amongst others. Another approach for equilibrium pricing in a jump-diffusion economy was considered by Naik and Lee (1990). Their approach is based on a pure-exchange economy where the asset pays continuous dividends, but there is no outside endowment. The equilibrium asset price is then derived based on a constat relative risk aversion utility and a special jump diffusion dynamic for the underlying dividend model. In this paper we consider a similar setting to the one in Kou (2002), although our pricing measure could also be related to equilibrium arguments under the Naik and Lee (1990) framework.

We assume the representative agent in the economy wants to maximize his expected utility of consumption, $\max_c E[\int_0^\infty u(c(t),t)dt]$. Here $c(t)$ represents the level of consumption at time t, and $u(\cdot,\cdot)$ is differentiable, strictly concave and strictly increasing with respect to the first argument. We denote by $e(t)$ the endowment process with initial value e_0. Then in equilibrium the asset price at any time t should satisfy the well-known Euler equation:

$$\xi(t) = E\Big[\frac{u_c(e(T),T)}{u_c(e(t),t)}\xi(T)\Big|\mathcal{G}_t\Big]$$

Here, u_c represent the partial derivative of u with respect to consumption. Note that in equilibrium $e(t) = c(t)$. There are different specifications one can use for the utility function. As in Kou (2002) one can specify the joint dynamics of the endowment and asset process and derive the risk neutral process of the asset price using the above Euler equation. However, in this paper we follow a different approach by using the pricing kernel defined in (4.5) and deriving dynamics for the endowment process which are consistent with the rational expectation equilibrium price from the Euler equation for two special forms of utilities.

Proposition 4.4: Suppose the price process is governed by the double Markov, regime-switching, process from (2.4):

(i) If the utility function is of power form, $u(c(t),t) = e^{-\rho t}(c(t))^\Upsilon/\Upsilon,\ \ \Upsilon < 1$, where ρ is the subjective rate of discount and Υ is the risk aversion parameter, then the Euler equation holds if the endowment process satisfies the SDE:

$$\begin{aligned} de(t) &= e(t-)\alpha_1(t,\xi(t),\mathbf{X}(t))dt + e(t-)\alpha_2(t,\xi(t),\mathbf{X}(t))dW(t) \\ &\quad + e(t-)\boldsymbol{\beta}'(t,\xi(t-),\mathbf{X}(t-))d\tilde{\mathbf{N}}(t)\ , \\ e(0) &= 1\ , \end{aligned}$$

where

$$\begin{aligned} \alpha_1(t,\xi(t),\mathbf{X}(t)) &= (1-\Upsilon)^{-1}\Big(r(t) - \rho + \frac{(\Upsilon-2)\theta_1^2(t,\xi(t),\mathbf{X}(t))}{\Upsilon-1} \\ &\quad -(\Upsilon-1)\boldsymbol{\beta}'(t,\xi(t),\mathbf{X}(t))\mathbf{A}_0(t,\xi(t))\mathbf{X}(t) \end{aligned}$$

$$
\begin{aligned}
& \quad + \sum_{i=1}^{N} \left\langle \tilde{\mathbf{D}}_0^{\theta_2}(t, \xi(t)), \mathbf{e}_i \right\rangle \langle \mathbf{e}_i, \mathbf{A}_0(t, \xi(t)) \mathbf{X}(t) \rangle \Big) , \\
\alpha_2(t, \xi(t), \mathbf{X}(t)) &= \frac{\theta_1(t, \xi(t), \mathbf{X}(t))}{\Upsilon - 1} , \\
\beta_i(t, \xi(t), \mathbf{X}(t)) &= \left\langle \tilde{\mathbf{D}}_0^{\theta_2}(t, \xi(t)) - \mathbf{1}, \mathbf{e}_i \right\rangle^{1/(\Upsilon - 1)} - 1 ,
\end{aligned}
$$

and

$$
\boldsymbol{\beta}(t, \xi(t), \mathbf{X}(t)) := (\beta_1(t, \xi(t), \mathbf{X}(t)), \beta_2(t, \xi(t), \mathbf{X}(t)), \cdots, \beta_N(t, \xi(t), \mathbf{X}(t)))' \in \Re^N .
$$

(ii) If the utility function is of exponential form, $u(c(t), t) = e^{-\rho t}(1 - e^{-\Upsilon c(t)})/\Upsilon$, $\Upsilon > 0$, then the Euler equation holds if the endowment process $\{\tilde{e}(t) | t \in \mathcal{T}\}$ satisfies the SDE:

$$
\begin{aligned}
d\tilde{e}(t) &= \tilde{\alpha}_1(t, \xi(t), \mathbf{X}(t)) dt + \tilde{\alpha}_2(t, \xi(t), \mathbf{X}(t)) dW(t) + \boldsymbol{\zeta}'(t, \xi(t-), \mathbf{X}(t-)) d\tilde{\mathbf{N}}(t) , \\
\tilde{e}(0) &= 0 ,
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\alpha}_1(t, \xi(t), \mathbf{X}(t)) &= \Upsilon^{-1} \Big(r(t) - \rho + \Upsilon \boldsymbol{\zeta}'(t) \mathbf{A}_0(t, \xi(t)) \mathbf{X}(t) + \frac{1}{2} \theta_1^2(t, \xi(t), \mathbf{X}(t)) \\
& \quad + \sum_{i=1}^{N} \left\langle \tilde{\mathbf{D}}_0^{\theta_2}(t, \xi(t)) - \mathbf{1}, \mathbf{e}_i \right\rangle \langle \mathbf{e}_i, \mathbf{A}_0(t, \xi(t)) \mathbf{X}(t) \rangle \Big) , \\
\tilde{\alpha}_2(t, \xi(t), \mathbf{X}(t)) &= \frac{\theta_1(t, \xi(t), \mathbf{X}(t))}{\Upsilon} , \\
\zeta_i(t, \xi(t), \mathbf{X}(t)) &= - \frac{\ln \left\langle \tilde{\mathbf{D}}_0^{\theta_2}(t, \xi(t)), \mathbf{e}_i \right\rangle}{\Upsilon} .
\end{aligned}
$$

and

$$
\boldsymbol{\zeta}(t, \xi(t), \mathbf{X}(t)) = (\zeta_1(t, \xi(t), \mathbf{X}(t)), \zeta_2(t, \xi(t), \mathbf{X}(t)), \cdots, \zeta_N(t, \xi(t), \mathbf{X}(t)))' \in \Re^N .
$$

Proof:

(i) For the power utility the Euler equation becomes:

$$
\xi(t) = E \Big[e^{-\rho(T-t)} \Big(\frac{e(T)}{e(t)} \Big)^{\Upsilon - 1} \xi(T) \Big| \mathcal{G}(t) \Big] .
$$

Thus, the measure change from (4.5) is consistent with equilibrium pricing if the following condition is satisfied:

$$\Lambda(t) = B(t)e^{-\rho t}(e(t))^{\Upsilon-1} \ .$$

Here we write $\Lambda(t)$ for $\Lambda_{0,t}(y)$ and $B(t)$ for $B(0,t)$ to simplify the notations.

We now derive the stochastic differential equation governing the evolution of the endowment process $\{e(t)|t \in \mathcal{T}\}$ over time. Suppose $\{e(t)|t \in \mathcal{T}\}$ has the following form:

$$\begin{aligned} de(t) &= e(t-)\alpha_1(t, \xi(t), \mathbf{X}(t))dt + e(t-)\alpha_2(t, \xi(t), \mathbf{X}(t))dW(t) \\ &\quad + e(t-)\boldsymbol{\beta}'(t, \xi(t), \mathbf{X}(t))d\tilde{\mathbf{N}}(t) \ , \\ e(0) &= 1 \ . \end{aligned}$$

To simplify the notation, write $\alpha_1(t)$, $\alpha_2(t)$, $\boldsymbol{\beta}(t)$ and $\beta_i(t)$, $(i = 1, 2, \cdots, N)$, for $\alpha_1(t, \xi(t), \mathbf{X}(t))$, $\alpha_2(t, \xi(t), \mathbf{X}(t))$, $\boldsymbol{\beta}(t, \xi(t), \mathbf{X}(t))$ and $\beta_i(t, \xi(t), \mathbf{X}(t))$, respectively. Also write $\theta_1(t)$ and $\mathbf{A}_0(t)$ for $\theta_1(t, \xi(t), \mathbf{X}(t))$ and $\mathbf{A}_0(t, \xi(t))$, respectively.

Applying Itô's differentiation rule to $B(t)e^{-\rho t}(e(t))^{\Upsilon-1}$ gives:

$$\begin{aligned} & B(t)e^{-\rho t}(e(t))^{\Upsilon-1} \\ = & \ 1 + \int_0^t B(u)e^{-\rho u}(e(u))^{\Upsilon-1}(r(u) - \rho)du + \int_0^t B(u)e^{-\rho u}(\Upsilon - 1)e^{\Upsilon-2}(u)de(u) \\ & + \frac{1}{2}\int_0^t B(u)e^{-\rho u}(\Upsilon - 1)(\Upsilon - 2)e^{\Upsilon-1}(u)\alpha_2^2(u)du \\ & + \sum_{0<u\le t} B(u)e^{-\rho u}\Big(e^{\Upsilon-1}(u) - e^{\Upsilon-1}(u-) - (\Upsilon - 1)e^{\Upsilon-1}(u-)\boldsymbol{\beta}'(u)\Delta\mathbf{N}(u)\Big) \\ = & \ 1 + \int_0^t B(u)e^{-\rho u}(e(u))^{\Upsilon-1}[r(u) - \rho + \alpha_1(u)(\Upsilon - 1)]du \\ & + \int_0^t B(u)e^{-\rho u}e^{\Upsilon-1}(u)(\Upsilon - 1)\alpha_2(u)dW(u) \\ & + \frac{1}{2}\int_0^t B(u)e^{-\rho u}(\Upsilon - 1)(\Upsilon - 2)e^{\Upsilon-1}(u)\alpha_2^2(u)du \\ & - \int_0^t B(u)e^{-\rho u}(\Upsilon - 1)e^{\Upsilon-1}(u)\boldsymbol{\beta}'(u)\mathbf{A}_0(u)\mathbf{X}(u)du \\ & + \sum_{0<u\le t} B(u)e^{-\rho u}\Big(e^{\Upsilon-1}(u) - e^{\Upsilon-1}(u-)\Big) \ . \end{aligned}$$

We now evaluate the term:

$$\sum_{0<u\leq t} B(u)e^{-\rho u}\Big(e^{\Upsilon-1}(u)-e^{\Upsilon-1}(u-)\Big)\ .$$

Note that $\Delta\mathbf{N}(t) := (\Delta N^1(t), \Delta N^2(t), \cdots, \Delta N^N(t))$, where $\Delta N^i(t)$ can be either 0 or 1 and $\Delta N^i(t)\Delta N^j(t) = 0$, if $i \neq j$. Consequently,

$$\begin{aligned}
& \sum_{0<u\leq t} B(u)e^{-\rho u}\Big(e^{\Upsilon-1}(u)-e^{\Upsilon-1}(u-)\Big) \\
= & \sum_{0<u\leq t} B(u)e^{-\rho u}e^{\Upsilon-1}(u-)\Big[(1+\boldsymbol{\beta}'(u)\Delta\mathbf{N}(u))^{\Upsilon-1}-1\Big] \\
= & \sum_{i=1}^{N}\int_0^t B(u)e^{-\rho u}e^{\Upsilon-1}(u-)\Big[(1+\beta_i(u))^{\Upsilon-1}-1\Big]\langle\mathbf{e}_i, d\mathbf{N}(u)\rangle\ .
\end{aligned}$$

Then

$$\begin{aligned}
& B(t)e^{-\rho t}(e(t))^{\Upsilon-1} \\
= & 1+\int_0^t B(u)e^{-\rho u}(e(u))^{\Upsilon-1}[r(u)-\rho+\alpha_1(u)(\Upsilon-1)]du \\
& +\int_0^t B(u)e^{-\rho u}e^{\Upsilon-1}(u)(\Upsilon-1)\alpha_2(u)dW(u) \\
& +\frac{1}{2}\int_0^t B(u)e^{-\rho u}(\Upsilon-1)(\Upsilon-2)e^{\Upsilon-1}(u)\alpha_2^2(u)du \\
& -\int_0^t B(u)e^{-\rho u}(\Upsilon-1)e^{\Upsilon-1}(u)\boldsymbol{\beta}'(u)\mathbf{A}_0(u)\mathbf{X}(u)du \\
& +\sum_{i=1}^{N}\int_0^t B(u)e^{-\rho u}e^{\Upsilon-1}(u-)\Big[(1+\beta_i(u))^{\Upsilon-1}-1\Big]\langle\mathbf{e}_i, d\mathbf{N}(u)\rangle \\
= & 1+\int_0^t B(u)e^{-\rho u}(e(u))^{\Upsilon-1}[r(u)-\rho+\alpha_1(u)(\Upsilon-1)]du \\
& +\int_0^t B(u)e^{-\rho u}e^{\Upsilon-1}(u)(\Upsilon-1)\alpha_2(u)dW(u) \\
& +\frac{1}{2}\int_0^t B(u)e^{-\rho u}(\Upsilon-1)(\Upsilon-2)e^{\Upsilon-1}(u)\alpha_2^2(u)du \\
& -\int_0^t B(u)e^{-\rho u}(\Upsilon-1)e^{\Upsilon-1}(u)\boldsymbol{\beta}'(u)\mathbf{A}_0(u)\mathbf{X}(u)du \\
& +\sum_{i=1}^{N}\int_0^t B(u)e^{-\rho u}e^{\Upsilon-1}(u-)\Big[(1+\beta_i(u))^{\Upsilon-1}-1\Big]\langle\mathbf{e}_i, \mathbf{A}_0(u)\mathbf{X}(u)\rangle\, du
\end{aligned}$$

$$+\sum_{i=1}^{N}\int_0^t B(u)e^{-\rho u}e^{\Upsilon-1}(u-)\Big[(1+\beta_i(u))^{\Upsilon-1}-1\Big]\left\langle \mathbf{e}_i, d\tilde{\mathbf{N}}(u)\right\rangle \ .$$

Note that

$$\begin{aligned}
\Lambda(t) &= 1+\int_0^t \Lambda(u)\theta_1(u)dW(u)+\int_0^t \Lambda(u)[\tilde{\mathbf{D}}_0^{\theta_2}(t,\xi(t))-\mathbf{1}]'d\tilde{\mathbf{N}}(u) \\
&= 1+\int_0^t \Lambda(u)\theta_1(u)dW(u)+\sum_{i=1}^{N}\int_0^t \Lambda(u)\left\langle \tilde{\mathbf{D}}_0^{\theta_2}(t,\xi(t))-\mathbf{1},\mathbf{e}_i\right\rangle\left\langle \mathbf{e}_i, d\tilde{\mathbf{N}}(u)\right\rangle \ .
\end{aligned}$$

By the unique decomposition of a special semimartingale and the assumption that $\Lambda(t)=B(t)e^{-\rho t}(e(t))^{\Upsilon-1}$, comparing the coefficients we obtain the following explicit expressions for $\alpha_1(t)$, $\alpha_2(t)$ and $\beta_i(t)$, $i=1,2,\cdots,N$:

$$\begin{aligned}
\alpha_1(t) &= (1-\Upsilon)^{-1}\Big(r(t)-\rho+\frac{(\Upsilon-2)\theta_1^2(t)}{\Upsilon-1}-(\Upsilon-1)\boldsymbol{\beta}'(t)\mathbf{A}_0(t)\mathbf{X}(t) \\
&\quad +\sum_{i=1}^{N}\left\langle \tilde{\mathbf{D}}_0^{\theta_2}(t,\xi(t)),\mathbf{e}_i\right\rangle\langle \mathbf{e}_i,\mathbf{A}_0(t)\mathbf{X}(t)\rangle\Big) \ , \\
\alpha_2(t) &= \frac{\theta_1(t)}{\Upsilon-1} \ , \\
\beta_i(t) &= \left\langle \tilde{\mathbf{D}}_0^{\theta_2}(t,\xi(t))-\mathbf{1},\mathbf{e}_i\right\rangle^{1/(\Upsilon-1)}-1 \ .
\end{aligned}$$

(ii) Similarly, we can derive the dynamics for the endowment process $\{e(t)|t\in\mathcal{T}\}$ satisfying the following condition:

$$\Lambda(t)=B(t)e^{-\rho t-\Upsilon e(t)} \ .$$

Again we suppose that $\{e(t)|t\in\mathcal{T}\}$ has the following form:

$$\begin{aligned}
de(t) &= \tilde{\alpha}_1(t,\xi(t),\mathbf{X}(t))dt+\tilde{\alpha}_2(t,\xi(t),\mathbf{X}(t))dW(t) \\
&\quad +\boldsymbol{\zeta}'(t,\xi(t),\mathbf{X}(t))d\tilde{\mathbf{N}}(t) \ , \\
e(0) &= 1 \ .
\end{aligned}$$

To simplify the notation, we write $\tilde{\alpha}_1(t)$, $\tilde{\alpha}_2(t)$, $\boldsymbol{\zeta}(t)$ and $\zeta_i(t)$, $(i=1,2,\cdots,N)$, for $\tilde{\alpha}_1(t,\xi(t),\mathbf{X}(t))$, $\tilde{\alpha}_2(t,\xi(t),\mathbf{X}(t))$, $\boldsymbol{\zeta}(t,\xi(t),\mathbf{X}(t))$ and $\zeta_i(t,\xi(t),\mathbf{X}(t))$, respectively.

Applying Itô's differentiation rule to $B(t)e^{-\rho t-\Upsilon e(t)}$ gives:

$$
\begin{aligned}
& B(t)e^{-\rho t-\Upsilon e(t)} \\
= \; & 1+\int_0^t B(u)e^{-\rho u-\Upsilon e(u)}(r(u)-\rho)du-\int_0^t \Upsilon B(u)e^{-\rho u-\Upsilon e(u)}de(u) \\
& +\frac{1}{2}\int_0^t B(u)\Upsilon^2 e^{-\rho u-\Upsilon e(u)}\tilde{\alpha}_2^2(u)du \\
& +\sum_{0<u\le t} B(u)e^{-\rho u}\Big(e^{-\Upsilon e(u)}-e^{-\Upsilon e(u-)}+\Upsilon e^{-\Upsilon e(u-)}\boldsymbol{\zeta}'(u-)\Delta\mathbf{N}(u)\Big) \\
= \; & 1+\int_0^t B(u)e^{-\rho u-\Upsilon e(u)}(r(u)-\rho)du-\int_0^t \Upsilon B(u)e^{-\rho u-\Upsilon e(u)}\tilde{\alpha}_1(u)du \\
& -\int_0^t \Upsilon B(u)e^{-\rho u-\Upsilon e(u)}\tilde{\alpha}_2(u)dW(u)+\int_0^t \Upsilon B(u)e^{-\rho u-\Upsilon e(u)}\boldsymbol{\zeta}'(u)\mathbf{A}_0(u)\mathbf{X}(u)du \\
& +\frac{1}{2}\int_0^t B(u)\Upsilon^2 e^{-\rho u-\Upsilon e(u)}\tilde{\alpha}_2^2(u)du \\
& +\sum_{0<u\le t} B(u)e^{-\rho u}\Big(e^{-\Upsilon e(u)}-e^{-\Upsilon e(u-)}\Big)\;.
\end{aligned}
$$

Note that

$$
\begin{aligned}
& \sum_{0<u\le t} B(u)e^{-\rho u}\Big(e^{-\Upsilon e(u)}-e^{-\Upsilon e(u-)}\Big) \\
= \; & \sum_{0<u\le t} B(u)e^{-\rho u-\Upsilon e(u-)}\Big(e^{-\Upsilon \boldsymbol{\zeta}'(u)\Delta\mathbf{N}(u)}-1\Big) \\
= \; & \sum_{i=1}^N \int_0^t B(u)e^{-\rho u-\Upsilon e(u-)}(e^{-\Upsilon\zeta_i(u)}-1)\langle \mathbf{e}_i, d\mathbf{N}(u)\rangle \;.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
& B(t)e^{-\rho t-\Upsilon e(t)} \\
= \; & 1+\int_0^t B(u)e^{-\rho u-\Upsilon e(u)}(r(u)-\rho)du-\int_0^t \Upsilon B(u)e^{-\rho u-\Upsilon e(u)}\tilde{\alpha}_1(u)du \\
& -\int_0^t \Upsilon B(u)e^{-\rho u-\Upsilon e(u)}\tilde{\alpha}_2(u)dW(u)+\int_0^t \Upsilon B(u)e^{-\rho u-\Upsilon e(u)}\boldsymbol{\zeta}'(u)\mathbf{A}_0(u)\mathbf{X}(u)du \\
& +\frac{1}{2}\int_0^t B(u)\Upsilon^2 e^{-\rho u-\Upsilon e(u)}\tilde{\alpha}_2^2(u)du \\
& +\sum_{i=1}^N \int_0^t B(u)e^{-\rho u-\Upsilon e(u-)}(e^{-\Upsilon\zeta_i(u)}-1)\langle \mathbf{e}_i, \mathbf{A}_0(u)\mathbf{X}(u)\rangle du
\end{aligned}
$$

$$+\sum_{i=1}^{N}\int_0^t B(u)e^{-\rho u-\Upsilon e(u-)}(e^{-\Upsilon\zeta_i(u)}-1)\left\langle \mathbf{e}_i, d\tilde{\mathbf{N}}(u)\right\rangle \ .$$

Again, note that

$$\Lambda(t)=1+\int_0^t \Lambda(u)\theta_1(u)dW(u)+\sum_{i=1}^{N}\int_0^t \Lambda(u)\left\langle \tilde{\mathbf{D}}_0^{\theta_2}(t,\xi(t))-\mathbf{1},\mathbf{e}_i\right\rangle\left\langle \mathbf{e}_i, d\tilde{\mathbf{N}}(u)\right\rangle \ .$$

So, by the unique decomposition of a special semimartingale, comparing the coefficients we obtain the following explicit formulas for $\tilde{\alpha}_1(t)$, $\tilde{\alpha}_2(t)$, $\boldsymbol{\zeta}(t)$ and $\zeta_i(t)$, $i=1,2,\cdots,N$:

$$\begin{aligned}
\tilde{\alpha}_1(t) &= \Upsilon^{-1}\Big(r(t)-\rho+\Upsilon\boldsymbol{\zeta}'(t)\mathbf{A}_0(t)\mathbf{X}(t)+\frac{1}{2}\theta_1^2(t) \\
&\quad +\sum_{i=1}^{N}\left\langle \tilde{\mathbf{D}}_0^{\theta_2}(t,\xi(t))-\mathbf{1},\mathbf{e}_i\right\rangle\langle \mathbf{e}_i,\mathbf{A}_0(t)\mathbf{X}(t)\rangle\Big)\ , \\
\tilde{\alpha}_2(t) &= \frac{\theta_1(t)}{\Upsilon}\ , \\
\zeta_i(t) &= -\frac{\ln\left\langle \tilde{\mathbf{D}}_0^{\theta_2}(t,\xi(t)),\mathbf{e}_i\right\rangle}{\Upsilon}\ .
\end{aligned}$$

The results from Proposition 4.4 indicate that under both utility specifications the endowment process follows a similar type of dynamics as the asset price process ξ_t. In particular, both processes are driven by the same Brownian motion with different diffusion coefficients, or volatility coefficients. However, the jump dynamics are different; the jump component for the price process is modelled by the modulating Markov chain while that for the endowment process is represented by a vector-valued martingale counting process $\tilde{\mathbf{N}}(t)$. There are some interesting open problems about the relationships between these two processes one may further investigate. For example, since we obtain explicit formulae for the coefficients of the endowment process as functions of the coefficients of the price process, we can examine possible comparisons between the drift, diffusion coefficient and jump amplitude for the two processes. However, it seems difficult, if not impossible, to derive inequality constraints for the above quantities in our framework unless certain stringent assumptions for the model parameters are provided. For example, in the power utility case, the coefficients $\alpha_1(t,\xi(t),\mathbf{X}(t))$, $\alpha_1(t,\xi(t),\mathbf{X}(t))$ and $\boldsymbol{\beta}(t,\xi(t),\mathbf{X}(t))$ of the endowment process depend not only on some quantities in the price process, but also on the two prices of risk $\theta_1(t, \xi(t), \mathbf{X}(t))$ and $\theta_2(t, \xi(t), \mathbf{X}(t))$. Since the market price of risk and the regime switching risk must satisfy the martingale condition (4.12) we focus on the special case where the latter quantity is given. Consequently, we consider the case where the regime switching risk is not priced. This corresponds to the case where $\theta_2(t, \xi(t), \mathbf{X}(t)) = 1$. Then, from (4.12), it follows that $\theta_1(t, \xi(t), \mathbf{X}(t)) = (r(t) - \mu(t, \xi(t), \mathbf{X}(t)))/\sigma(t, \xi(t), \mathbf{X}(t))$. If we further assume that the drift and volatility of ξ_t do not have the feedback effect, and given that the Markov chain is in state $\mathbf{e}_j$, the price process is more volatile than the endowment one if the following relation holds:

$$\frac{(r(t) - \mu(t, \mathbf{e}_i))}{\sigma^2(t, \mathbf{e}_i)(\Upsilon - 1)} < 1 \ .$$

Similarly, other conditions can be constructed for comparing the coefficients of the two processes. However, the assumption that $\theta_2(t, \xi(t), \mathbf{X}(t)) = 1$ may be rather restrictive, and one of the main goal is to build a regime switching model, where the regime switching risk can be priced and model parameters depend on the current price level. Nevertheless, the impacts of our proposed price processes on economic fundamentals is an interesting topic to be further investigated.

§5. The Martingale Representation

This section gives the martingale representation for the discounted price V of the claim. It is a key result for constructing the local risk-minimizing hedging strategy and characterizing the corresponding martingale pricing measure in the next section.

First, by the Bayes' rule and the Markov property,

$$V(t, y, \mathbf{x}) = E[B^{-1}(t, T)\Lambda_{t,T}(y)c(\xi_{t,T}(y))|\mathbf{X}(t) = \mathbf{x}] \ . \tag{5.1}$$

Let

$$\begin{aligned} \tilde{\mathbf{D}}^{\theta_2}(u, \xi(u)) \ &:= \ \mathbf{D}^{\theta_2}(u, \xi(u))\mathbf{X}(u) \in \Re^N \ , \\ \tilde{\mathbf{A}}^{\theta_2}(u, \xi(u)) \ &:= \ \mathbf{A}^{\theta_2}(u, \xi(u))\mathbf{X}(u) \in \Re^N \ . \end{aligned}$$

Then,

$$\begin{aligned}\Lambda_{s,t}(y) &= 1+\int_s^t \Lambda_{s,u-}(y)\theta_1(u,\xi_{s,u-}(y),\mathbf{X}(u-))dW(u) \\ &\quad +\int_s^t \Lambda_{s,u-}(y)[\tilde{\mathbf{D}}_0^{\theta_2}(u,\xi_{s,u-}(y))-\mathbf{1}]' \\ &\quad \times(d\mathbf{N}(u)-\tilde{\mathbf{A}}_0(u,\xi_{s,u}(y))du) \qquad (5.2)\end{aligned}$$

By exploiting the unique decomposition of special semi-martingales and the Itô differentiation rule, we obtain the following martingale representation result.

Proposition 5.1: Define the processes

$$\kappa^{\theta_2}(u,\xi(u)) := \frac{(\tilde{\mathbf{D}}_0^{\theta_2}(u,\xi(u))-\mathbf{1})'\frac{\partial \tilde{\mathbf{D}}_0^{\theta_2}}{\partial \xi}(u,\xi(u))}{1+(\tilde{\mathbf{D}}_0^{\theta_2}(u,\xi(u))-\mathbf{1})'\mathbf{1}} \in \Re \ , \qquad (5.3)$$

$$\begin{aligned}& L_{s,t}(y) \\ :=\ & \int_s^t \frac{\partial \theta_1}{\partial \xi}(u,\xi_{s,u-}(y),\mathbf{X}(u-))D_{s,u-}(y)(dW(u)-\theta_1(u,\xi_{s,u}(y),\mathbf{X}(u))du) \\ & +\int_s^t D_{s,u-}(y)\Big(\frac{\partial \tilde{\mathbf{D}}_0^{\theta_2}}{\partial \xi}(u,\xi_{s,u-}(y))\Big)'(d\mathbf{N}(u)-\tilde{\mathbf{A}}_0(u,\xi_{s,u}(y))du) \\ & -\int_s^t \tilde{\mathbf{D}}_0^{\theta_2}(u,\xi_{s,u}(y))\frac{\partial \tilde{\mathbf{A}}_0}{\partial \xi}(u,\xi_{s,u}(y))D_{s,u}(y)du \\ & -\int_s^t D_{s,u-}(y)\kappa^{\theta_2}(u,\xi_{s,u-}(y))\mathbf{1}'d\mathbf{N}(u) \ . \qquad (5.4)\end{aligned}$$

Let $\mathbf{X}(0) := \mathbf{x}_0$, which represents the initial value of the chain $\mathbf{X}$ at time zero. Then, the $(G, \mathcal{P}^\theta)$-martingale $V(t,\xi_{0,t}(y_0),\mathbf{x})$ has the following representation:

$$\begin{aligned}& V(t,\xi_{0,t}(y_0),\mathbf{x}) \\ =\ & V(0,y_0,\mathbf{x}_0)+\int_0^t \phi^c(u,\xi(u-),\mathbf{X}(u-))(dW(u)-\theta_1(u,\xi(u),\mathbf{X}(u))du) \\ & +\int_0^t \Big\langle \boldsymbol{\phi}^d(u,\xi(u-),\mathbf{X}(u-)), d\mathbf{M}^{\boldsymbol{\theta}}(u)\Big\rangle \ . \qquad (5.5)\end{aligned}$$

Here

$$
\begin{aligned}
& \phi^c(u,y,\mathbf{x}) \\
= \;& E^{\boldsymbol{\theta}}\Big[B^{-1}(u,T)\Big(L_{u,T}c(\xi_{u,T}(y)) + \frac{\partial c}{\partial \xi}(\xi_{u,T}(y))D_{u,T}(y)\Big)|\mathbf{X}(u)=\mathbf{x}\Big]g(u,y,\mathbf{x}) \ ,
\end{aligned}
\tag{5.6}
$$

and

$$
\boldsymbol{\phi}^d(u,y_-,\mathbf{x}_-,\mathbf{x}) = [V(t,y_-(1+\langle\boldsymbol{\gamma},\mathbf{x}-\mathbf{x}_-\rangle),\mathbf{x}) - V(t,y_-,\mathbf{x}_-)]\mathbf{x} + \mathbf{V}(u,y_-) \ , \tag{5.7}
$$

where $y := \xi_{0,u}(y_0)$, $y_- := \xi_{0,u-}(y_0)$, $\mathbf{x} := \mathbf{X}(u)$ and $\mathbf{x}_- := \mathbf{X}(u-)$; $\mathbf{V}(u,y_-) := (V(u,y_-,\mathbf{e}_1),V(u,y_-,\mathbf{e}_2),\ldots,V(u,y_-,\mathbf{e}_N))' \in \Re^N$; $g(u,y,\mathbf{x}) := \sigma(u,y,\mathbf{x})y$ (see Page 9).

Proof: First, applying the Itô differentiation on $V(t,\xi(t),\mathbf{X}(t))$ and using the unique decomposition of special semi-martingales gives:

$$
\begin{aligned}
& V(t,\xi(t),\mathbf{X}(t)) \\
= \;& V(0,y_0,\mathbf{x}_0) + \int_0^t \frac{\partial V}{\partial u}du + \int_0^t \frac{\partial V}{\partial y}[\mu(u,\xi(u),\mathbf{X}(u)) - \langle\boldsymbol{\gamma}(u,\xi(u)),\mathbf{A}(u,\xi(u))\mathbf{X}(u)\rangle]\xi(u)du \\
& + \int_0^t \frac{\partial V}{\partial y}\sigma(u,\xi(u-),\mathbf{X}(u-))\xi(u-)dW(u) + \frac{1}{2}\int_0^t \frac{\partial^2 V}{\partial y^2}\sigma^2(u,\xi(u),\mathbf{X}(u))\xi^2(u)du \\
& + \int_0^t \frac{\partial V}{\partial y}\xi(u-)\left\langle\boldsymbol{\gamma}(u,\xi(u-)),d\mathbf{X}(u)\right\rangle \\
& + \sum_{0<u\le t}\Big\{\Big[V(u,\xi(u-)(1+\langle\boldsymbol{\gamma}(u,\xi(u)),\mathbf{X}(u)-\mathbf{X}(u-)\rangle),\mathbf{X}(u)) - V(u,\xi(u-),\mathbf{X}(u-))\Big] \\
& \times\langle\mathbf{X}(u),d\mathbf{X}(u)\rangle - \frac{\partial V}{\partial y}\xi(u-)\left\langle\boldsymbol{\gamma}(u,\xi(u-)),\mathbf{X}(u)-\mathbf{X}(u-)\right\rangle\Big\} + \int_0^t\left\langle\frac{\partial V}{\partial \mathbf{x}},d\mathbf{X}(u)\right\rangle \\
& + \sum_{0<u\le t}\Big[V(u,\xi(u),\mathbf{X}(u)) - V(u,\xi(u),\mathbf{X}(u-)) - \left\langle\frac{\partial V}{\partial \mathbf{x}},\Delta\mathbf{X}(u)\right\rangle\Big] \\
= \;& V(0,y_0,\mathbf{x}_0) + \int_0^t \frac{\partial V}{\partial u}du + \int_0^t \frac{\partial V}{\partial y}[\mu(u,\xi(u),\mathbf{X}(u)) - \langle\boldsymbol{\gamma}(u,\xi(u)),\mathbf{A}(u,\xi(u))\mathbf{X}(u)\rangle]\xi(u)du \\
& + \int_0^t \frac{\partial V}{\partial y}\sigma(u,\xi(u-),\mathbf{X}(u-))\xi(u-)dW(u) + \frac{1}{2}\int_0^t \frac{\partial^2 V}{\partial y^2}\sigma^2(u,\xi(u),\mathbf{X}(u))\xi^2(u)du \\
& + \int_0^t\Big[V(u,\xi(u-)(1+\langle\boldsymbol{\gamma}(u,\xi(u)),\mathbf{X}(u)-\mathbf{X}(u-)\rangle),\mathbf{X}(u)) - V(u,\xi(u-),\mathbf{X}(u-))\Big]
\end{aligned}
$$

$$
\begin{aligned}
& \times \langle \mathbf{X}(u), d\mathbf{X}(u)\rangle + \int_0^t \langle \mathbf{V}(u, \xi(u)), d\mathbf{X}(u)\rangle \\
= \; & V(0, y_0, \mathbf{x}_0) + \int_0^t \frac{\partial V}{\partial y}\sigma(u, \xi(u-), \mathbf{X}(u))\xi(u-)(dW(u) - \theta_1(u, \xi(u), \mathbf{X}(u))du) \\
& + \int_0^t [V(u, \xi(u-)(1 + \langle \boldsymbol{\gamma}(u, \xi(u)), \mathbf{X}(u) - \mathbf{X}(u-)\rangle), \mathbf{X}(u)) - V(u, \xi(u-), \mathbf{X}(u-))] \\
& \times \left\langle \mathbf{X}(u), d\mathbf{M}^{\boldsymbol{\theta}}(u)\right\rangle + \int_0^t \left\langle \mathbf{V}(u, \xi(u)), d\mathbf{M}^{\boldsymbol{\theta}}(u)\right\rangle \\
& + \int_0^t \Big[\frac{\partial V}{\partial u} + \frac{\partial V}{\partial y}(\mu(u, \xi(u), \mathbf{X}(u)) - \langle \boldsymbol{\gamma}(u, \xi(u)), \mathbf{A}(u, \xi(u))\mathbf{X}(u)\rangle \\
& + \sigma(u, \xi(u), \mathbf{X}(u))\theta_1(u, \xi(u), \mathbf{X}(u)))\xi(u) + \frac{1}{2}\frac{\partial^2 V}{\partial y^2}\sigma^2(u, \xi(u), \mathbf{X}(u))\xi^2(u) \\
& + [V(u, \xi(u-)(1 + \langle \boldsymbol{\gamma}(u, \xi(u)), \mathbf{X}(u) - \mathbf{X}(u-)\rangle), \mathbf{X}(u)) - V(u, \xi(u-), \mathbf{X}(u-))] \\
& \times \left\langle \mathbf{X}(u), \mathbf{A}^{\theta_2}(u, \xi(u))\mathbf{X}(u)\right\rangle + \left\langle \mathbf{V}(u, \xi(u)), \mathbf{A}^{\theta_2}(u, \xi(u))\mathbf{X}(u)\right\rangle \Big] du \; . \qquad (5.8)
\end{aligned}
$$

Note that the final du integral is a continuous (local)-martingale of finite variation under the new measure $\mathcal{P}^{\boldsymbol{\theta}}$. Since $V(t, \xi(t), \mathbf{X}(t))$ is a $(G, \mathcal{P}^{\boldsymbol{\theta}})$-martingale, the du integral above must be identical to zero almost surely. This gives:

$$
\begin{aligned}
& V(t, \xi(t), \mathbf{X}(t)) \\
= \; & V(0, y_0, \mathbf{x}_0) + \int_0^t \frac{\partial V}{\partial y}\mu(u, \xi(u-), \mathbf{X}(u-))\xi(u-)(dW(u) - \theta_1(u, \xi(u), \mathbf{X}(u))du) \\
& + \int_0^t [V(u, \xi(u-)(1 + \langle \boldsymbol{\gamma}(u, \xi(u)), \mathbf{X}(u) - \mathbf{X}(u-)\rangle), \mathbf{X}(u)) - V(u, \xi(u-), \mathbf{X}(u-))] \\
& \times \left\langle \mathbf{X}(u), d\mathbf{M}^{\boldsymbol{\theta}}(u)\right\rangle + \int_0^t \left\langle \mathbf{V}(u, \xi(u-)), d\mathbf{M}^{\boldsymbol{\theta}}(u)\right\rangle \; .
\end{aligned} \tag{5.9}
$$

Hence,

$$
\phi^c(u, y, \mathbf{x}) = \frac{\partial V}{\partial y} g(u, y, \mathbf{x}) \; , \tag{5.10}
$$

and

$$
\boldsymbol{\phi}^d(u, y_-, \mathbf{x}_-, \mathbf{x}) = [V(t, y_-(1 + \langle \boldsymbol{\gamma}, \mathbf{x} - \mathbf{x}_-\rangle), \mathbf{x}) - V(t, y_-, \mathbf{x}_-)]\mathbf{x} + \mathbf{V}(u, y_-) \; . \tag{5.11}
$$

Now,

$$
\frac{\partial V}{\partial y}(t, y, \mathbf{x})
$$

$$= E\Big[B^{-1}(t,T)\Big(\frac{\partial \Lambda_{t,T}}{\partial y}(y)c(\xi_{t,T}(y)) + \Lambda_{t,T}(y)\frac{\partial c}{\partial \xi}(\xi_{t,T}(y))D_{t,T}(y)\Big)|\mathbf{X}(u)=\mathbf{x}\Big] \ . \tag{5.12}$$

Note that

$$\begin{aligned}
\frac{\partial \Lambda_{t,T}(y)}{\partial y} &= \int_t^T \frac{\partial \Lambda_{t,u-}(y)}{\partial y}\theta_1(u,\xi_{t,u-}(y),\mathbf{X}(u-))dW(u) \\
&+\int_t^T \Lambda_{t,u-}(y)\frac{\partial \theta_1}{\partial \xi}(u,\xi_{t,u-}(y),\mathbf{X}(u-))D_{t,u-}(y)dW(u) \\
&+\int_t^T \Big\{\frac{\partial \Lambda_{t,u-}(y)}{\partial y}[\tilde{\mathbf{D}}_0^{\theta_2}(u,\xi_{t,u-}(y))-\mathbf{1}]' \\
&+\Lambda_{t,u-}(y)\Big(\frac{\partial \tilde{\mathbf{D}}_0^{\theta_2}}{\partial \xi}(u,\xi_{t,u-}(y))D_{t,u-}(y)\Big)'\Big\} \\
&\times(d\mathbf{N}(u)-\tilde{\mathbf{A}}_0(u,\xi_{t,u}(y))du) \\
&-\int_t^T \Lambda_{t,u}(y)[\tilde{\mathbf{D}}_0^{\theta_2}(u,\xi_{t,u}(y))-\mathbf{1}]' \\
&\times\frac{\partial \tilde{\mathbf{A}}_0}{\partial \xi}(u,\xi_{t,u}(y))D_{t,u}(y)du \ .
\end{aligned} \tag{5.13}$$

Applying the Itô product rule on $\Lambda_{t,T}(y)L_{t,T}(y)$ gives

$$\begin{aligned}
&\Lambda_{t,T}(y)L_{t,T}(y) \\
=& \int_t^T L_{t,u-}(y)\Lambda_{t,u-}(y)\theta_1(u,\xi_{t,u-}(y),\mathbf{X}(u-))dW(u) \\
&+\int_t^T \Lambda_{t,T}(y)\frac{\partial \theta_1}{\partial \xi}(u,\xi_{t,u-}(y),\mathbf{X}(u-)) \\
&\times D_{t,u-}(y)(dW(u)-\theta_1(u,\xi_{t,u}(y),\mathbf{X}(u))du) \\
&+\int_t^T \Lambda_{t,u}(y)\theta_1(u,\xi_{t,u}(y),\mathbf{X}(u))\frac{\partial \theta_1}{\partial \xi}(u,\xi_{t,u}(y),\mathbf{X}(u))D_{t,u}(y)du \\
&+\int_t^T L_{t,u-}(y)\Lambda_{t,u-}(y)(\tilde{\mathbf{D}}_0^{\theta_2}(u,\xi_{t,u-}(y))-\mathbf{1})' \\
&\times(d\mathbf{N}(u)-\tilde{\mathbf{A}}_0(u,\xi_{t,u}(y))) \\
&+\int_t^T \Lambda_{t,u-}(y)\frac{\partial \tilde{\mathbf{D}}_0^{\theta_2}}{\partial \xi}(u,\xi_{t,u-}(y))D_{t,u-}(y) \\
&\times(d\mathbf{N}(u)-\tilde{\mathbf{A}}_0(u,\xi_{t,u}(y))du) \\
&-\sum_{t<u\leq T}\Lambda_{t,u-}(y)D_{t,u-}(y)\kappa^{\theta_2}(u,\xi_{t,u-}(y))\Delta\mathbf{N}'(u)\Delta\mathbf{N}(u)
\end{aligned}$$

$$
\begin{aligned}
& - \int_t^T \Lambda_{t,u}(y)(\tilde{\mathbf{D}}_0^{\theta_2}(u, \xi_{t,u}(y)) - \mathbf{1})' \\
& \times \frac{\partial \tilde{\mathbf{A}}_0}{\partial \xi}(u, \xi_{t,u}(y)) D_{t,u}(y) du \\
& + \sum_{t<u\leq T} D_{t,u-}(y) \Lambda_{t,u-}(y)(\tilde{\mathbf{D}}_0^{\theta_2}(u, \xi_{t,u}(y)) - \mathbf{1})' \\
& \times \frac{\partial \tilde{\mathbf{A}}_0}{\partial \xi}(u, \xi_{t,u}(y)) \Delta \mathbf{N}'(u) \Delta \mathbf{N}(u) \\
& - \sum_{t<u\leq T} \Lambda_{t,u-} D_{t,u-}(y) \kappa^{\theta_2}(u, \xi_{t,u-}(y)) \\
& \times (\tilde{\mathbf{D}}_0^{\theta_2}(u, \xi_{t,u-}(y)) - \mathbf{1})' \mathbf{1} \Delta \mathbf{N}'(u) \Delta \mathbf{N}(u) \\
= & \int_t^T L_{t,u-}(y) \Lambda_{t,u-}(y) \theta_1(u, \xi_{t,u-}, \mathbf{X}(u-)) dW(u) \\
& + \int_t^T \Lambda_{t,u-}(y) \frac{\partial \theta_1}{\partial \xi}(u, \xi_{t,u-}, \mathbf{X}(u-)) D_{t,u-}(y) dW(u) \\
& + \int_t^T \Big\{ L_{t,u-}(y) \Lambda_{t,u-}(y) [\tilde{\mathbf{D}}_0^{\theta_2}(u, \xi_{t,u-}(y)) - \mathbf{1}]' \\
& + \Lambda_{t,u-}(y) \Big(\frac{\partial \tilde{\mathbf{D}}_0^{\theta_2}}{\partial \xi}(u, \xi_{t,u-}(y)) D_{t,u-}(y) \Big)' \Big\} \\
& \times (d\mathbf{N}(u) - \tilde{\mathbf{A}}_0(u, \xi_{t,u}(y)) du) \\
& - \int_t^T \Lambda_{t,u}(y) [\tilde{\mathbf{D}}_0^{\theta_2}(u, \xi_{t,u}(y)) - \mathbf{1}]' \\
& \times \frac{\partial \tilde{\mathbf{A}}_0}{\partial \xi}(u, \xi_{t,u}(y)) D_{t,u}(y) du \ . \qquad (5.14)
\end{aligned}
$$

So, $\Lambda_{t,T}(y) L_{t,T}(y)$ satisfies the same stochastic differential equation. So, by the uniqueness of the solutions,

$$
\frac{\partial \Lambda_{t,T}(y)}{\partial y} = \Lambda_{t,T}(y) L_{t,T}(y) \ . \qquad (5.15)
$$

Hence, by the Bayes' rule,

$$
\begin{aligned}
& \phi^c(t, y, \mathbf{x}) \\
= & \frac{\partial V}{\partial y}(t, y, \mathbf{x}) g(t, y, \mathbf{x}) \\
= & E^{\boldsymbol{\theta}} \Big[B^{-1}(t, T) \Big(L_{t,T}(y) c(\xi_{t,T}(y)) + \frac{\partial c}{\partial \xi}(\xi_{t,T}(y)) D_{t,T}(y) \Big) | \mathbf{X}(t) = \mathbf{x} \Big] g(t, y, \mathbf{x}) \ . \qquad (5.16)
\end{aligned}
$$

$\square$

From the proof of Proposition 5.1,

$$\frac{\partial V}{\partial t}+\frac{\partial V}{\partial y}\xi(t)\Big(\mu(t,\xi(t),\mathbf{X}(t))-\langle\boldsymbol{\gamma}(t,\xi(t)),\mathbf{A}(t,\xi(t))\mathbf{X}(t)\rangle$$
$$+\sigma(t,\xi(t),\mathbf{X}(t))\theta_1(t,\xi(t),\mathbf{X}(t))\Big)+\frac{1}{2}\frac{\partial^2 V}{\partial y^2}\xi^2(t)\sigma^2(t,\xi(t),\mathbf{X}(t))$$
$$+\Big(V(t,\xi(t-)(1+\langle\boldsymbol{\gamma}(t,\xi(t)),\mathbf{X}(t)-\mathbf{X}(t-)\rangle),\mathbf{X}(t))-V(t,\xi(t-),\mathbf{X}(t-))\Big)$$
$$\times\big\langle\mathbf{X}(t),\mathbf{A}^{\theta_2}(t,\xi(t))\mathbf{X}(t)\big\rangle+\big\langle\mathbf{V}(t,\xi(t)),\mathbf{A}^{\theta_2}(t,\xi(t))\mathbf{X}(t)\big\rangle=0\ , \tag{5.17}$$

with terminal condition $V(T,y,\mathbf{x})=B(T,T)c(y)$.

Let $\tilde{V}_i:=\tilde{V}(t,y,\mathbf{e}_i)$, for each $i=1,2,\ldots,N$. Write

$$\tilde{\mathbf{V}}(t,y):=(\tilde{V}_1,\tilde{V}_2,\ldots,\tilde{V}_N)^T\ .$$

Recall that $\tilde{V}(t,y,\mathbf{x})=B(t,T)V(t,y,\mathbf{x})$, which represents a price of the contingent claim. Since $\mathcal{P}^{\boldsymbol{\theta}}$ is an equivalent martingale measure, from the martingale condition, the following Markov, regime-switching, partial differential equation for $\tilde{V}(t,y)$ is obtained:

$$\frac{\partial\tilde{V}}{\partial t}+\frac{\partial\tilde{V}}{\partial y}y\Big(r(t)-\theta_2(t,y,\mathbf{x})\left\langle\boldsymbol{\gamma}(t,y),\mathbf{A}(t,y)\mathbf{x}\right\rangle\Big)+\frac{1}{2}\frac{\partial^2\tilde{V}}{\partial y^2}y^2\sigma^2(t,y,\mathbf{x})$$
$$+\Big(V(t,y_-(1+\langle\boldsymbol{\gamma}(t,y_-),\mathbf{x}-\mathbf{x}_-\rangle),\mathbf{x})-V(t,y_-,\mathbf{x}_-)\Big)\left\langle\mathbf{x},\mathbf{A}^{\theta_2}(t,y)\mathbf{x}\right\rangle$$
$$+\Big\langle\tilde{\mathbf{V}}(t,y),\mathbf{A}^{\theta_2}(t,y)\mathbf{x}\Big\rangle=0\ , \tag{5.18}$$

with terminal condition $\tilde{V}(T,y,\mathbf{x})=c(y)$.

So, if $\mathbf{x}=\mathbf{e}_i$ $(i=1,2,\ldots,N)$,

$$\begin{aligned}
r(t) &= r_i\ ,\\
\sigma(t,y,\mathbf{x}) &= \sigma(t,y,\mathbf{e}_i)\ ,\\
\theta_2(t,y,\mathbf{x}) &= \theta_2(t,y,\mathbf{e}_i)\ ,\\
V(t,y_-(1+\langle\boldsymbol{\gamma}(t,y_-),\mathbf{x}-\mathbf{x}_-\rangle),\mathbf{x}) &= V(t,y_-(1+\langle\boldsymbol{\gamma}(t,y_-),\mathbf{e}_i-\mathbf{x}_-\rangle),\mathbf{e}_i)\ ,
\end{aligned}$$

and, hence $\tilde{\mathbf{V}}(t,y)$ satisfies the following system of coupled partial differential equations:

$$\begin{aligned}
&\frac{\partial \tilde{V}}{\partial t} + \frac{\partial \tilde{V}}{\partial y} y\Big(r(t) - \theta_2(t,y,\mathbf{e}_i)\,\langle \boldsymbol{\gamma}(t,y), \mathbf{A}(t,y)\mathbf{e}_i\rangle\Big) + \frac{1}{2}\frac{\partial^2 \tilde{V}}{\partial y^2} y^2 \sigma^2(t,y,\mathbf{e}_i)\\
&+\Big(V(t, y_-(1+\langle \boldsymbol{\gamma}(t,y_-), \mathbf{e}_i - \mathbf{x}_-\rangle), \mathbf{e}_i) - V(t,y_-,\mathbf{x}_-)\Big)\,\langle \mathbf{e}_i, \mathbf{A}^{\theta_2}(t,y)\mathbf{e}_i\rangle\\
&+\Big\langle \tilde{\mathbf{V}}(t,y), \mathbf{A}^{\theta_2}(t,y)\mathbf{e}_i\Big\rangle = 0\ ,
\end{aligned} \tag{5.19}$$

with terminal condition $\tilde{V}(T,y,\mathbf{e}_i) = c(y)$, for each $i = 1, 2, \ldots, N$.

§6. Local Risk-Minimization and the p-Measure

In this section, we employ the martingale representation for the discounted price V of the claim to construct the local risk-minimization hedging strategy and to characterize the corresponding martingale measure for option valuation. The work of Föllmer and Sondermann (1986) gave the risk-minimizing approach for hedging contingent claims when the price process is a martingale. Föllmer and Sondermann (1986) showed that if the price process is a martingale, a unique risk-minimizing trading strategy exists, and, the necessary and sufficient condition for a trading strategy to be risk-minimizing is that the cost process corresponding to the strategy is orthogonal to the price process. When the price process is a semi-martingale, Schweizer (1990) extended the risk-minimizing approach and introduced the local risk-minimizing approach for hedging contingent claims by solving a projection problem in the space of square-integrable semi-martingales, rather than the one in the space of square-integrable martingales as considered in Föllmer and Sondermann (1986). The local risk-minimization approach provides a more flexible way to hedge contingent claims than the risk-minimization one. Colwell and Elliott (1993) employed the local risk-minimization approach to develop a hedging strategy for a contingent claim when the price process is governed by a jump-diffusion process. Here we deal with the double Markov, regime-switching, model. The results developed here provide a theoretically sound method to price and hedge a contingent claim under the double Markov, regime-switching, model.

Firstly, we briefly introduce some concepts related to the (local) risk-minimization approach for hedging contingent claims. For detailed discussions, we refer to Föllmer and Sondermann (1986), Schweizer (1990) and Colwell and Elliott (1993).

For each $t \in \mathcal{T}$, let $\phi(t)$ and $\psi(t)$ denote the amount of the discounted asset price and the amount of the risk-free asset whose price is taken as a constant, respectively, held at time t. Write $\phi := \{\phi(t)|t \in \mathcal{T}\}$ and $\psi := \{\psi(t)|t \in \mathcal{T}\}$. Let Φ denote a pair of processes (ϕ, ψ). We call Φ a trading strategy and assume that it satisfies the following conditions.

i. ϕ is G-predictable,

ii. ψ is G-adapted,

iii. $E[\int_0^T \phi^2(t) d < \tilde{\xi}(t), \tilde{\xi}(t) > + (\int_0^T |\phi(t)\psi(t)|dt)^2] < \infty$, where $d < \tilde{\xi}(t), \tilde{\xi}(t) >$ is the quadratic variation of $\tilde{\xi}(t)$;

iv. for each $t \in \mathcal{T}$, let $V(t, \Phi) := \phi(t)\tilde{\xi}(t) + \psi(t)$. Then, the process $V(\Phi) := \{V(t, \Phi)|t \in \mathcal{T}\}$ has right-continuous sample paths, and, for each $t \in \mathcal{T}$, $V(t, \Phi) \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$, where $\mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$ is the space of square-integrable random variables on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$.

Note that $V(t, \Phi)$ represents the discounted value of the portfolio associated with the portfolio process Φ at time t and that $V(\Phi)$ is the corresponding discounted value process. Then, the discounted cumulative cost process $C(\Phi) := \{C(t, \Phi)|t \in \mathcal{T}\}$ for a trading strategy Φ is defined by setting

$$C(t, \Phi) := V(t, \Phi) - \int_0^t \phi(u) d\tilde{\xi}(u) \ , \quad t \in \mathcal{T} \ , \tag{6.1}$$

where $\int_0^t \phi(t) d\tilde{\xi}(t)$ represents the discounted gain process up to time t.

A trading strategy Φ is said to be self-financing if $C(t, \Phi) = C(0, \Phi)$, for each $t \in \mathcal{T}$, where $C(0, \Phi)$ is an initial cost and is a constant. The notion of a self-financing strategy can be generalized to a mean self-financing strategy. More specifically, a trading strategy Φ is said to be mean self-financing if its discounted cumulative cost process $C(\Phi)$ is a $(G, \mathcal{P})$-martingale. Note that due the the presence of the regime-switching effect, $C(\Phi)$ should be a martingale with respect to the enlarged filtration G instead of the one generated by the price process ξ only.

When a trading strategy Φ is self-financing and $V(T, \Phi) = B^{-1}(0, T)H$, $\mathcal{P}$-a.s., for some $H \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$, Φ is a risk-free hedge portfolio for H and H is an attainable claim. In this case, the price of H is $C(0, \Phi)$. In general, the cost process $C(\Phi)$ is random and H is not attainable. In this case, we need to quantify the residual risk that cannot be hedged by considering the following risk process with respect to the enlarged information set $G(t)$:

$$R(t, \Phi) := E[(C(T, \Phi) - C(t, \Phi))^2 | G(t)] \ , \quad t \in \mathcal{T} \ . \tag{6.2}$$

When the discounted price process $\tilde{\xi}$ is a semi-martingale, Schweizer (1990) extended the risk process and introduced a kind of relative local risk, namely, the R-quotient using the concept of a small perturbation of a trading strategy. The notion of a local risk-minimizing trading strategy is then introduced using the R-quotient. For detail, interested readers may refer to Schweizer (1990).

According to Schweizer (1990), a local risk-minimizing trading strategy Φ which generates the discounted claim $B^{-1}(0, T)c(\xi_{0,T}(y_0))$ must satisfy the following conditions.

1. $V(T, \Phi) = B^{-1}(0, T)c(\xi_{0,T}(y_0))$;
2. $V(t, \Phi) = V(0, \Phi) + \int_0^t \phi(t-) d\tilde{\xi}(t) + \Gamma(t)$, $t \in \mathcal{T}$;
3. $\Gamma := \{\Gamma(t) | t \in \mathcal{T}\}$ is a $(G, \mathcal{P})$-martingale, and Γ is orthogonal to the martingale part of the discounted price process $\tilde{\xi}$ under $\mathcal{P}$.

In addition, we also require the following conditions.

4. $\theta_1(t, \xi(t), \mathbf{X}(t))$ and $\theta_2(t, \xi(t), \mathbf{X}(t))$ are such that $\tilde{\xi}$ is a $(G, \mathcal{P}^{\boldsymbol{\theta}})$-martingale;
5. $V(\Phi)$ is a $(G, \mathcal{P}^{\boldsymbol{\theta}})$-martingale.

Suppose $\Phi := (\phi, \psi)$ is a local risk-minimizing strategy. Then, from Conditions (1) and (5),

$$\begin{aligned} V(t, \Phi) &= E^{\boldsymbol{\theta}}[V(T, \Phi)|G(t)] \\ &= E^{\boldsymbol{\theta}}[B^{-1}(t, T)c(\xi_{0,T}(y_0))|\xi_{0,t}(y_0) = y, \mathbf{X}(t) = \mathbf{x}] \\ &= V(t, y, \mathbf{x}) \ . \end{aligned} \tag{6.3}$$

From Proposition 5.1,

$$\begin{aligned} & V(t, \xi_{0,t}(y_0), \mathbf{x}) \\ = \ & V(0, y_0, \mathbf{x}_0) + \int_0^t \phi^c(u, \xi(u-), \mathbf{X}(u-))(dW(u) - \theta_1(u, \xi(u), \mathbf{X}(u))du) \\ & + \int_0^t \left\langle \boldsymbol{\phi}^d(u, \xi(u-), \mathbf{X}(u-)), d\mathbf{M}^{\boldsymbol{\theta}}(u) \right\rangle \ . \end{aligned} \tag{6.4}$$

Following the terminology in Colwell and Elliott (1993), we call an equivalent probability measure $\mathcal{P}^{\boldsymbol{\theta}}$ a p-measure for the claim $c(\xi_{0,T}(y_0))$ if there exists a local risk-minimizing strategy Φ such that Φ and $\mathcal{P}^{\boldsymbol{\theta}}$ satisfy Conditions (1) - (5). In the sequel, we are going to find the p-measure from Conditions (1) - (5) and to determine the local risk-minimizing trading strategy. Note that the p-measure may not be unique.

First, from Condition (5), $V(t, \Phi) = V(t, \xi(t), \mathbf{X}(t))$. So, from Condition (2) and Equation (6.4),

$$\begin{aligned} \Gamma(t) &= \int_0^t \phi^c(u, \xi(u-), \mathbf{X}(u-))(dW(u) - \theta_1(u, \xi(u), \mathbf{X}(u))du) \\ &\quad + \int_0^t \left\langle \boldsymbol{\phi}^d(u, \xi(u-), \mathbf{X}(u-)), d\mathbf{M}^{\boldsymbol{\theta}}(u) \right\rangle - \int_0^t \phi(u-)d\tilde{\xi}(u) \\ &= \int_0^t \phi^c(u, \xi(u-), \mathbf{X}(u-))(dW(u) - \theta_1(u, \xi(u), \mathbf{X}(u))du) \\ &\quad + \int_0^t \langle \boldsymbol{\phi}^d(u, \xi(u-), \mathbf{X}(u-)), d\mathbf{M}(u) \rangle + \int_0^t (1 - \theta_2(u, \xi(u), \mathbf{X}(u))) \\ &\quad \times \langle \boldsymbol{\phi}^d(u, \xi(u), \mathbf{X}(u-)), \mathbf{A}(u, \xi(u))\mathbf{X}(u) \rangle \, du - \int_0^t \phi(u-)d\tilde{\xi}(u) \ . \end{aligned} \tag{6.5}$$

So, Γ is a martingale under $\mathcal{P}$ if and only if the drift term is identical to zero; that is,

$$\phi(t)B^{-1}(0, t)\xi(t)(r(t) - \mu(t, \xi(t), \mathbf{X}(t))) - \phi^c(t, \xi(t), \mathbf{X}(t))\theta_1(t, \xi(t), \mathbf{X}(t))$$

$$+(1-\theta_2(t,\xi(t),\mathbf{X}(t))\left\langle \boldsymbol{\phi}^d(t,\xi(t),\mathbf{X}(t)),\mathbf{A}(t,\xi(t))\mathbf{X}(t)\right\rangle = 0\ ,$$
$$t \in \mathcal{T}\ ,\quad \mathcal{P}\text{-a.s.} \tag{6.6}$$

Again, from Condition (5), Γ and the martingale part of $\tilde{\xi}$ are orthogonal under $\mathcal{P}$. Let $M^{\tilde{\xi}} := \{M^{\tilde{\xi}}(t)|t \in \mathcal{T}\}$ denote the martingale part of the discounted price process $\tilde{\xi}$ under $\mathcal{P}$ defined by setting

$$M^{\tilde{\xi}}(t) \ := \ \int_0^t \tilde{\xi}(u-)\sigma(u,\xi(u-),\mathbf{X}(u-))dW(u) + \int_0^t \tilde{\xi}(u-)\left\langle \boldsymbol{\gamma}(u,\xi(u-)),d\mathbf{M}(u)\right\rangle\ . \tag{6.7}$$

Then, the product process $\Gamma M^{\tilde{\xi}}$ is a $(G,\mathcal{P})$-martingale.

Let $[\mathbf{M},\mathbf{M}](t)$ denote the quadratic variation of the martingale $\mathbf{M}$ and $<\mathbf{M},\mathbf{M}>(t)$ the unique predictable process such that $[\mathbf{M},\mathbf{M}]-<\mathbf{M},\mathbf{M}>$ is a martingale under $\mathcal{P}$. Then, it can be shown that for each $t \in \mathcal{T}$,

$$\begin{aligned}
& <\mathbf{M},\mathbf{M}>(t) \\
= & \int_0^t \Big(\mathbf{diag}(\mathbf{A}(u,\xi(u))\mathbf{X}(u)) - \mathbf{diag}(\mathbf{X}(u))\mathbf{A}'(u,\xi(u)) - \mathbf{A}(u,\xi(u))\cdot\mathbf{diag}(\mathbf{X}(u))\Big)du\ ,
\end{aligned}$$

see Elliott et al. (1994) for detail.

Write, for each $t \in \mathcal{T}$,

$$\begin{aligned}
& \mathbf{K}(t,\xi(t),\mathbf{X}(t)) \\
:= \ & \mathbf{diag}(\mathbf{A}(t,\xi(t))\mathbf{X}(t)) - \mathbf{diag}(\mathbf{X}(t))\mathbf{A}'(t,\xi(t)) - \mathbf{A}(t,\xi(t))\cdot\mathbf{diag}(\mathbf{X}(t))\ .
\end{aligned}$$

Applying Itô's product rule on $\Gamma M^{\tilde{\xi}}$ gives

$$\begin{aligned}
& \Gamma(t)M^{\tilde{\xi}}(t) \\
= \ & \int_0^t M^{\tilde{\xi}}(u)\phi^c(u,\xi(u-),\mathbf{X}(u-))(dW(u)-\theta_1(u,\xi(u),\mathbf{X}(u))) \\
& + \int_0^t M^{\tilde{\xi}}(u-)\left\langle \boldsymbol{\phi}^d(u,\xi(u-)),d\mathbf{M}(u)\right\rangle + \int_0^t M^{\tilde{\xi}}(u)(1-\theta_2(u,\xi(u),\mathbf{X}(u))) \\
& \times \left\langle \boldsymbol{\phi}^d(u,\xi(u),\mathbf{X}(u)),\mathbf{A}(u,\xi(u))\mathbf{X}(u)\right\rangle - \int_0^t M^{\tilde{\xi}}(u-)\phi(u-)dM^{\tilde{\xi}}(u)
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \Gamma(u-)dM^{\tilde{\xi}}(u)+\int_0^t \tilde{\xi}(u)\sigma(u,\xi(u),\mathbf{X}(u))\phi^c(u,\xi(u),\mathbf{X}(u))du \\
&-\int_0^t \phi(u)\tilde{\xi}^2(u)\sigma^2(u,\xi(u),\mathbf{X}(u))du \\
&+\int_0^t \tilde{\xi}(u)\boldsymbol{\gamma}'(u,\xi(u))(d<\mathbf{M},\mathbf{M}>(u))\boldsymbol{\phi}^d(u,\xi(u)) \\
&-\int_0^t \tilde{\xi}^2(u)\phi(u)\boldsymbol{\gamma}'(u,\xi(u))(d<\mathbf{M},\mathbf{M}>(u))\boldsymbol{\gamma}(u,\xi(u)) \\
= &\int_0^t M^{\tilde{\xi}}(u)\phi^c(u,\xi(u-),\mathbf{X}(u-))(dW(u)-\theta_1(u,\xi(u),\mathbf{X}(u))) \\
&+\int_0^t M^{\tilde{\xi}}(u-)\left\langle \boldsymbol{\phi}^d(u,\xi(u-)),d\mathbf{M}(u)\right\rangle+\int_0^t M^{\tilde{\xi}}(u)(1-\theta_2(u,\xi(u),\mathbf{X}(u))) \\
&\times\left\langle \boldsymbol{\phi}^d(u,\xi(u),\mathbf{X}(u)),\mathbf{A}(u,\xi(u))\mathbf{X}(u)\right\rangle-\int_0^t M^{\tilde{\xi}}(u-)\phi(u-)dM^{\tilde{\xi}}(u) \\
&+\int_0^t \Gamma(u-)dM^{\tilde{\xi}}(u)+\int_0^t \tilde{\xi}(u)\sigma(u,\xi(u),\mathbf{X}(u))\phi^c(u,\xi(u),\mathbf{X}(u))du \\
&-\int_0^t \phi(u)\tilde{\xi}^2(u)\sigma^2(u,\xi(u),\mathbf{X}(u))du \\
&+\int_0^t \tilde{\xi}(u)\boldsymbol{\gamma}'(u,\xi(u))\mathbf{K}(u,\xi(u),\mathbf{X}(u))\boldsymbol{\phi}^d(u,\xi(u)) \\
&-\int_0^t \tilde{\xi}^2(u)\phi(u)\boldsymbol{\gamma}'(u,\xi(u))\mathbf{K}(u,\xi(u),\mathbf{X}(u))\boldsymbol{\gamma}(u,\xi(u))\ .
\end{aligned}
$$

So, $\Gamma M^{\tilde{\xi}}$ is a $(G,\mathcal{P})$-martingale if and only if

$$
\begin{aligned}
&\sigma(t,\xi(t),\mathbf{X}(t))[\phi^c(t,\xi(t),\mathbf{X}(t))-\phi(t)B^{-1}(0,t)\xi(t)\sigma(t,\xi(t),\mathbf{X}(t))] \\
&+B^{-1}(0,t)\xi(t)\boldsymbol{\gamma}'(t,\xi(t))\mathbf{K}(t,\xi(t),\mathbf{X}(t))(\boldsymbol{\phi}^d(t,\xi(t),\mathbf{X}(t))-\phi(t)B^{-1}(0,t)\xi(t)\boldsymbol{\gamma}(t,\xi(t))) \\
&=0\ ,\quad t\in\mathcal{T}\ ,\quad \mathcal{P}\text{-a.s.}
\end{aligned}
\tag{6.8}
$$

From Condition (4), $\tilde{\xi}$ is a $(G,\mathcal{P}^{\boldsymbol{\theta}})$-martingale. So, from Equation (4.12),

$$
\begin{aligned}
&\mu(t,\xi(t),\mathbf{X}(t))-r(t)+\sigma(t,\xi(t),\mathbf{X}(t))\theta_1(t,\xi(t),\mathbf{X}(t)) \\
&+(\theta_2(t,\xi(t),\mathbf{X}(t))-1)\left\langle \boldsymbol{\gamma}(t,\xi(t)),\mathbf{A}(t,\xi(t))\mathbf{X}(t)\right\rangle \\
&=0\ ,\quad t\in\mathcal{T}\ ,\quad \mathcal{P}\text{-a.s.}
\end{aligned}
\tag{6.9}
$$

From (6.8), we obtain the following closed-form expression for the hedge ratio ϕ:

$$\begin{aligned}&\phi(t,\xi(t),\mathbf{X}(t))\\ =&\ \frac{\sigma(t,\xi(t),\mathbf{X}(t))\phi^{c}(t,\xi(t),\mathbf{X}(t))+\xi(t)B^{-1}(0,t)\boldsymbol{\gamma}'(t,\xi(t))\mathbf{K}(t,\xi(t),\mathbf{X}(t))\boldsymbol{\phi}^{d}(t,\xi(t),\mathbf{X}(t))}{B^{-1}(0,t)\xi(t)(\sigma^{2}(t,\xi(t),\mathbf{X}(t))+B^{-1}(0,t)\xi(t)\boldsymbol{\gamma}'(t,\xi(t))\mathbf{K}(t,\xi(t),\mathbf{X}(t))\boldsymbol{\gamma}(t,\xi(t)))}\ ,\end{aligned} \tag{6.10}$$

and

$$\psi(t,\xi(t),\mathbf{X}(t))=V(t,\Phi)-\phi(t,\xi(t),\mathbf{X}(t))\tilde{\xi}(t)\ . \tag{6.11}$$

Guo (2001) introduced a set of change-of-state (COS) contracts to price the regime-switching risk. A COS contract can be regarded as an insurance contract to which it compensates its holder for any losses that incur when the next change in the state of an economy occurs. The key idea is to introduce a new security to complete the market. However, COS contracts are "fictitious" contracts and are not traded in practice. Using the local-minimization trading strategy one may adjust the hedge ratio (ϕ,ψ) to hedge risk due to transitions of states of the economy.

Note that the hedge ratio ϕ is related to the measure change via its connection to ϕ^{c} and $\boldsymbol{\phi}^{d}$, which are given in (5.6) and (5.7), respectively. Also, the effects of switching regimes and the measure change of the Markov chain are incorporated in the integrands ϕ^{c} and $\boldsymbol{\phi}^{d}$, and, hence in the hedge ratio ϕ. So, the hedge ratio obtained here is different from the one arising from Colwell and Elliott (1993), where the latter does not taken into account the effects of switching regimes.

From (6.6) and (6.9),

$$\begin{aligned}&\theta_{2}(t,\xi(t),\mathbf{X}(t))\\ =&\ 1+\frac{(r(t)-\mu(t,\xi(t),\mathbf{X}(t)))(\phi^{c}(t,\xi(t),\mathbf{X}(t))-\sigma(t,\xi(t),\mathbf{X}(t))\phi(t)B^{-1}(0,t)\xi(t))}{\langle\phi^{c}(t,\xi(t),\mathbf{X}(t))\boldsymbol{\gamma}(t,\xi(t))-\sigma(t,\xi(t),\mathbf{X}(t))\boldsymbol{\phi}^{d}(t,\xi(t),\mathbf{X}(t)),\mathbf{A}(t,\xi(t))\mathbf{X}(t)\rangle}\ ,\end{aligned} \tag{6.12}$$

where $\phi(t):=\phi(t,\xi(t),\mathbf{X}(t))$ is given by (6.10).

Note that θ_1 can be regarded as a traditional market price of risk together with the feedback effect with regime-switching risk.

From (6.9),

$$
\begin{aligned}
& \theta_1(t,\xi(t),\mathbf{X}(t)) \\
= \; & \frac{r(t)-\mu(t,\xi(t),\mathbf{X}(t))-(\theta_2(t,\xi(t),\mathbf{X}(t))-1)\left\langle \boldsymbol{\gamma}(t,\xi(t)),\mathbf{A}(t,\xi(t))\mathbf{X}(t)\right\rangle}{\sigma(t,\xi(t),\mathbf{X}(t))} \\
= \; & \sigma^{-1}(t,\xi(t),\mathbf{X}(t))[(r(t)-\mu(t,\xi(t),\mathbf{X}(t))) \\
& \times \left\langle \sigma(t,\xi(t),\mathbf{X}(t))\phi(t)B^{-1}(0,t)\xi(t)\boldsymbol{\gamma}(t),\mathbf{A}(t,\xi(t))\mathbf{X}(t)\right\rangle \\
& - \left\langle \sigma(t,\xi(t),\mathbf{X}(t))\boldsymbol{\phi}^d(t,\xi(t),\mathbf{X}(t)),\mathbf{A}(t,\xi(t))\mathbf{X}(t)\right\rangle] \;,
\end{aligned}
\tag{6.13}
$$

where $\phi(t) := \phi(t,\xi(t),\mathbf{X}(t))$ is given by (6.10).

So, the price kernel is given by:

$$
\begin{aligned}
& \Lambda_{0,t}(y_0) \\
= \; & 1+\int_0^t \Big(\frac{r(u)-\mu(u,\xi(u-),\mathbf{X}(u-))}{\sigma(u,\xi(u-),\mathbf{X}(u-))}\Big)\Lambda_{0,u-}(y_0)dW(u) \\
& -\int_0^t \frac{(\theta_2(u,\xi(u-),\mathbf{X}(u-))-1)\left\langle \boldsymbol{\gamma}(u,\xi(u-)),\mathbf{A}(u,\xi(u-))\mathbf{X}(u-)\right\rangle}{\sigma(u,\xi(u-),\mathbf{X}(u-))}\Lambda_{0,u-}(y_0)dW(u) \\
& +\int_0^t \Lambda_{0,u-}(y_0)[\tilde{\mathbf{D}}_0^{\theta_2}(u,\xi_{0,u-}(y_0))-\mathbf{1}]'(d\mathbf{N}(u)-\tilde{\mathbf{A}}_0(u,\xi_{0,u}(y_0))du) \;,
\end{aligned}
\tag{6.14}
$$

where $\theta_2(u,\xi(u),\mathbf{X}(u))$ is given by (6.12).

When $\theta_2(t,\xi(t),\mathbf{X}(t))=1$, $\lambda\otimes\mathcal{P}$-a.e. on $\mathcal{T}\times\Omega$, where λ is the Lebesgue measure on $\mathcal{T}$, there is no measure change for the chain, and, hence the regime-switching risk is not priced. In this case,

$$
\theta_1(t,\xi(t),\mathbf{X}(t)) = \frac{r(t)-\mu(t,\xi(t),\mathbf{X}(t))}{\sigma(t,\xi(t),\mathbf{X}(t))} \;, \tag{6.15}
$$

so the price kernel becomes:

$$
\Lambda_{0,t}(y_0) = 1+\int_0^t \Big(\frac{r(u)-\mu(u,\xi(u-),\mathbf{X}(u-))}{\sigma(u,\xi(u-),\mathbf{X}(u-))}\Big)\Lambda_{0,u-}(y_0)dW(u) \;. \tag{6.16}
$$

Therefore, in this particular case, the measure $\mathcal{P}^{\boldsymbol{\theta}}$ coincides with the risk-neutral measure in Guo (2001) and the risk-neutral regime-switching Esscher transform in Elliott et al. (2005).

We end this section by discussing the difference in the martingale representation and the hedging result developed here from those in the standard Black-Scholes-Merton model. In the standard Black-Scholes-Merton model, the market is complete and perfect hedging is possible. In this case, the martingale representation of a contingent claim is given by a stochastic integral with respect to the Brownian motion, which is the only source of randomness in the standard Black-Scholes-Merton model. This martingale presentation can be regarded as a special case of our martingale representation when the Markov chain only has one state and the jump term is switched off, or absent. Further, in the standard Black-Scholes-Merton model, the hedge ratio is determined by the integrand of the martingale representation, which is just the delta of the claim. This hedge ratio gives a perfect hedge result. In the Markov, regime-switching, jump-diffusion model, there are two fundamental sources of risk, namely, the diffusion risk and the regime-switching risk, where, as explained above, the diffusion risk is due to Brownian motion and the regime-switching risk is due to the Markov chain. The jump risk in the asset price is then also determined by the regime-switching risk. The martingale representation we developed here takes into account explicitly the two fundamental sources of risk. In particular, the stochastic integrals with respect to the Brownian motion and the martingale associated with Markov chain take into account explicitly the diffusion risk and the regime-switching risk, respectively. However, unlike the standard Black-Scholes-Merton model, we have two fundamental sources of randomness and two primitive securities. Consequently, the market in our model is incomplete, and so, there is more than one pricing kernel and perfect hedging is not possible for new non-reductant contingent claims. In particular, we have two market prices of risk here, one for the diffusion risk and another one for the regime-switching risk. The martingale condition for precluding arbitrage opportunities is not sufficient to determine the two market prices of risk, and hence, to determine a pricing kernel which is able to price explicitly both the diffusion risk, (or the regime-dependent risk), and the regime-switching risk. So additional conditions or restrictions are required. Here we employ the local risk-minimization approach to determine a pricing kernel and a hedging strategy for the contingent claim. The rationale of the local risk-minimization approach is to determine a pricing kernel and a hedging strategy so as to minimize the quadratic cost of hedging, which is used as a measure of the residual risk of hedging. Unlike the delta in the standard Black-Scholes-Merton model, the hedging strategy determined by the local risk-minimization approach is a partial, or incomplete, hedging strategy.

§6. Conclusion

We developed a model for pricing and hedging European-style index options under a double Markov regime-switching model with feedback effect. We supposed that the transitions of the state of the economy modeled by the chain cause structural changes in the model parameters and jumps in the asset price at the same time. We provided a novel way to incorporate the feedback effect of the price process on the economic condition by allowing the rate matrix of the driving Markov chain of the price process to be modulated by the price process. We investigated the statistical properties of the model and studied the impact of the feedback effect in a simulation study based on a discrete version of our model. We determined an equivalent martingale measure in the incomplete market setting by a product of two density processes so as to price two sources of risk, namely, the diffusion risk and the regime-switching risk. We further provided an economic equilibrium justification for the proposed change of measure. By exploiting the methodology similar to Colwell et al. (1991) and Colwell and Elliott (1993), we established a martingale representation for a European-style index option's price on the price kernel. A local risk-minimizing strategy and a pricing measure were then constructed using the martingale representation. In the particular case when the regime-switching risk is not priced, the pricing measure selected here coincides with that chosen in the existing literature.

Acknowledgements:

The authors would like to thank the editor and the two referees for their helpful comments and suggestions. Robert J. Elliott and Tak Kuen Siu would like to thank Australian Research Council, (ARC), for their support. Alexandru Badescu wishes to thank NSERC for its continuing support.

References:

1. Ang, A., Bekaert, G. 2002. Short rate nonlinearities and regime switches. Journal of Economic Dynamics and Control 26(7-8), 1243-1274.

2. Ang, A., Bekaert, G. 2002. Regime switches in interest rates. Journal of Business and Economic Statistics 20(2), 163-182.

3. Bansal, N., Connolly, R.A., Stivers, C., 2010. Regime-switching in stock index and treasury futures returns and measure of stock stress. The Journal of Future Markets 30(8), 753-779.

4. Bansal, R. and Zhou, H. 2002. Term structure of interest rates with regime shifts. Journal of Finance 57(4), 1997-2043.

5. Bismut, J.M., 1991. Martingales, the Malliavin calculus and hypoellipticity under general Hörmander's conditions. Zeitschrift f́ur Wahrscheinlichkeitstheorie 56, 469-505.

6. Buffington, J., Elliott, R.J., 2002. American options with regime switching. International Journal of Theoretical and Applied Finance 5, 497-514.

7. Camacho, M. 2005. Markov-switching stochastic trends and economic fluctuations. Journal of Economic Dynamics and Control 29(1-2), 135-158.

8. Camara, A., Krehbiel, T.L., Li, W., 2009. Displaced jump-diffusion option valuation. Journal of Derivatives 17(2), 41-58.

9. Camara, A., Wang, Y.H., 2010. A new simple square root option pricing model. Journal of Future Markets 30(11), 1007-1105.

10. Cecchetti, S.G., Genberg, H., Lipsky, J. and Wadhwani, S., 2000. Asset Prices and Central Bank Policy. Geneva Reports on the World Economy 2. International Center for Monetary and Banking Studies (ICMB). Information Press: Oxford, UK.

11. Colwell, B.D., Elliott, R.J., Kopp, P.E., 1991. Martingale representation and hedging policies. Stochastic Process and its Applications 38, 335-345.

12. Colwell, B.D., Elliott, R.J., 1993. Discontinuous asset prices and nonattainable contingent claims. Mathematical Finance 3, 295-308.

13. Dufour, F., Elliott, R.J., 1999. Filtering with discrete state observations. Applied Mathematics and Optimization 40, 259-272.

14. Elliott, R.J., Kohlmann, M., 1989. Integration by parts, homogoensous chaos expansions and smooth densities. Annals of Probability 17, 194-207.

15. Elliott, R.J., Aggoun, L., Moore J.B., 1994. Hidden Markov Models: Estimation and Control. Springer-Verlag, Berlin-Heidelberg-New York.

16. Elliott, R.J., Chan, L.L., Siu, T.K., 2005. Option pricing and Esscher transform under regime switching. Annals of Finance 1(4), 423-432.

17. Elliott, R.J., Siu, T.K., Chan, L.L., Lau J.W., 2007a. Pricing options under a generalized Markov-modulated jump diffusion model. Stochastic Analysis and Applications 25(4), 821-843.

18. Elliott, R.J., Siu, T.K., Yang, H., 2007b. Martingale representation for contingent claims with regime switching. Communications on Stochastic Analysis 1(2), 279-292.

19. Elliott, R.J., Siu, T.K., 2010. On risk minimizing portfolios under a Markovian regime-switching Black-Scholes economy. Annals of Operations Research 176(1), 271-291.

20. Föllmer, H., Sondermann, D., 1986. Hedging of contingent claims under incomplete information, in: Hildenbrand, W., Mas-Colell, A., Debreu, G. (Eds.), Contributions to Mathematical Economics in Honor of Gèrard Debreu, North Holland, Amsterdam, pp. 205-223.

21. Fong, W.M. and See, K.H., 2003. Basis variations and regime shifts in the oil futures market. The European Journal of Finance 9, 499-513.

22. Guo, X., 2001. Information and option pricing. Quantitative Finance 1, 38-44.

23. Hamilton, J.D., 1989. A new approach to the economic analysis of nonstationary time series and the business cycle. Econometrica 57(2), 357-384.

24. Kou, S.G., 2002. A jump-diffusion model for option pricing. Management Science 48(8), 1086-1101.

25. Kaminski, K., Lo, A.W. 2007. When do stop-loss rules stop losses?. Preprint. MIT Sloan School of Management and National Bureau of Economic Research (NBER).

26. Kunita, H., 1978. Stochastic Partial Differential Equations Connected with Nonlinear Filtering. Lecture Notes in Mathematics 972. Springer-Verlag, Berlin-Heidelberg-New York.

27. Kunita, H., 1982. Stochastic Differential Equations and Stochastic Flows of Diffeomorphisms. Lecture Notes in Mathematics 1097. Springer-Verlag, Berlin-Heidelberg-New York.

28. Lux, T., Kaizoji, T., 2007. Forecasting volatility and volume in the Tokyo stock market: long memory, fractality and regime switching. Journal of Economic Dynamics and Control 31(6), 1808-1843.

29. Naik, V., 1993. Option valuation and hedging strategies with jumps in the volatility of asset returns. Journal of Finance 48(2), 1969-1984.

30. Naik, V., Lee, M., 1990. General equilibrium pricing of options on the market portfolio with discontinuous returns. The Review of Financial Studies 3(4), 493-521.

31. Schweizer, M., 1990. Risk-minimality and orthogonality of martingales. Stochastics and Stochastics Reports 30, 123-131.

32. Sola, M., Driffill, J., 1994. Testing the term structure of interest rates using a stationary vector autoregression with regime switching. Journal of Economic Dynamics and Control 18(3-4), 601-628.

33. Turner, C.M., Startz, R., Neison, C.R., 1989. A Markov model of heteroskedasticity, risk, and learning in the stock market. Journal of Financial Economics 25, 3-22.