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Teachers’-mathematics-knowledge-building communities

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TEACHERS'-MATHEMATICS-KNOWLEDGE-BUILDING COMMUNITIES

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"Mathematics knowledge for teaching" is by far the most prominent topic of investigation among mathematics education researchers, and there are two main schools of thought. One holds that such teachers' knowledge is mostly formal and explicit – and so specifiable, cataloguable, unpackable, teachable, and testable. The other is that such knowledge is principally tacit, and so the work of defining, developing, and assessing mathematics knowledge for teaching demands much more subtle, participatory, and time-extensive strategies. This report focuses on the latter. I describe the emergence of knowledge-building communities among practicing teachers as they analyze and re-synthesize – that is, substruct – mathematics content for teaching.

El "conocimiento de la matemática para la enseñanza" es el tema de investigación más sobresaliente entre los investigadores de educación matemática y existen dos escuelas de pensamiento principales. Una sostiene que dicho conocimiento de los profesores es mayormente formal y explícito – y por tanto especificable, catalogable, desentrañable, enseñable y verificable. La otra sostiene que dicho conocimiento es fundamentalmente tácito, y por tanto el trabajo de definir, desarrollar, y evaluar conocimiento matemático para el aprendizaje demanda estrategias mucho más sutiles, participativas y de largo aliento. Este informe se centra en esta última. Describo cómo emergen las comunidades constructivas del conocimiento entre profesores cuanto analizan y sintetizan – esto es, sustraen – contenido matemático para la enseñanza.

A BRIEF HISTORY OF RESEARCH INTO TEACHERS' MATHEMATICS KNOWLEDGE

Since the 1970s, investigations of the relationship between teachers' formal knowledge of mathematics and their students' learning have repeatedly reaffirmed an early finding by Begle (1979): there is little correlation between the mathematics courses that teachers take and the performance of their students on standardized tests. This result is, of course, troubling – given that most teacher education programs require candidates to complete several courses in areas of disciplinary specialization.

Why is there a lack of correlation? Widely endorsed answers to this question began to emerge in the 1990s as researchers noted that stock courses in mathematics are focused on completed results that often hide the messiness and complications that led to their production. In contrast, teaching in primary school entails struggling with that messiness and working through the complications as students produce novel mathematical insights. Since teachers must regularly employ associative reasoning as they grapple with the images, analogies, logical connections, and other associations that tend to be buried inside finalized formulations, their knowledge is necessarily different from research mathematicians' knowledge. As Ball and Bass (2003) characterized the difference, whereas it is the research mathematician's task to pack insights into tight formulations (theorems, formulas, etc.), it is

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the teachers' task to *unpack*. Applying Shulman's (1986) notion of "pedagogical content knowledge" to mathematics, the suggestion is that a specific type of professional knowledge is involved here, as Baumert et al. (2010) summed up in a recent review of the research in the area:

> Findings show that [teachers' content knowledge of mathematics] remains *inert* in the classroom unless accompanied by a rich repertoire of mathematical knowledge and skills relating directly to the curriculum, instruction, and student learning. (p. 139, emphasis added)

Despite widespread agreement on this point, there is a decided lack of consensus around the nature of teachers' "inert" mathematical content knowledge and how it might be activated. A large number of formal investigations have focused on explicit manifestations of disciplinary knowledge – that is, the sorts of insights that can be assessed directly through observation, interviews, or written tests. Such explicit knowledge is typically deemed to be teachable.

Other researchers have suggested that the most important knowledge for teaching tends to be enacted and tacit, and so it is neither easily identified nor readily measured. Some of these investigators have emphasized the networks of association that underlie the conceptual fluency of expert teachers. As has been established in other domains (see Ericsson et al., 2006), expert performance is enabled by well-established and automatized (i.e., not necessarily accessible to consciousness) webs of association, from which the expert effortlessly "selects" an interpretation, scenario, or cluster of actions that is likely to best fit unfamiliar situations. Experts are typically unable to explain or justify their choices when asked about them; they simply recognize their interpretations or actions as appropriate. I believe that teachers' content knowledge of mathematics manifests largely in this way – and if this belief is justified, there are significant implications for how mathematical knowledge for teaching is studied, assessed, and developed.

I use Ma's (1999) notion of "profound understanding of fundamental mathematics" as the backdrop of this discussion. However, while I resonate with her interpretation of the word *profound*, I am less comfortable with the adjective *fundamental*. Specifically, I propose that characterizing teachers' disciplinary expertise as "foundational, primary, and elementary" – terms which suggest a closed set of insights and understandings that might be catalogued and assessed – may be antithetical to the project of researching the complexity of teachers' knowledge. As an alternative, I develop the notion of "profound understanding of *emergent* mathematics" (Davis, 2011; Davis & Renert, 2013). I argue that the knowledge needed by teachers is not simply a clear-cut and well-connected set of basics, but a sophisticated and largely enactive mix of various associations/instantiations of mathematical concepts and an awareness of the complex processes through which mathematics is produced. I use the term *emergent* to flag the adaptive, evolving, coherent-but-never-fixed character of teachers' disciplinary knowledge.

The distinction between fundamental and emergent is not a subtle one. The former is lodged in a web of construction metaphors, and bespeaks fixedness, stability, and constancy. I argue that image is not just inappropriate; it is counterproductive. Teachers' disciplinary knowledge of mathematics is vast, intricate, and evolving. Rather than thinking in terms of a discrete body of foundational knowledge held by individuals, then, it may be more productive to view it as a flexible, vibrant category of living knowledge that is distributed across a body of professionals.

Instead of thinking in terms of a stable collection of facts that must be mastered, then, I prefer to think of teachers' disciplinary knowledge of mathematics as a *learnable disposition*. To develop what I mean by this, I proceed with a discussion of "concept study" – which refers to a collection of strategies developed by teachers of the past 15 years to interrogate and extend their mathematical understandings.

CONCEPT STUDY

A concept study is a participatory, collaborative structure for teachers to engage with one another in the examination and elaboration of mathematical understandings. The phrase *concept study* combines elements of two prominent notions in contemporary mathematics education research: *concept* analysis and lesson *study*.

Concept analysis, which was well represented in mathematics education research from the 1960s to the 1980s, focuses on explicating logical structures and figurative associations that inhere in mathematical concepts. As Usiskin et al. (2003) described it, concept analysis

> … involves tracing the origins and applications of a concept, looking at the different ways in which it appears both within and outside mathematics, and examining the various representations and definitions used to describe it and their consequences. (p. 1)

Usiskin et al. extended their description to include ways of representing ideas to learners, alternative definitions and their implications, histories and evolutions of concepts, applications, and learners' interpretations of what they are learning. Concept study has very much the same sorts of foci. However, unlike concept analysis (which has typically be undertaken by researchers and subsequently published as textbooks and resource materials for teachers), concept study involves classroom practitioners in the work.

That is where and how the focus on concept analysis is blended with the collaborative structures of lesson study:

> Working in a small group, teachers collaborate with one another, meeting to discuss learning goals, to plan an actual classroom lesson (called a 'research lesson'), to observe how it works in practice, and then to revise and report on the results so that other teachers can benefit from it." (Wikipedia, "Lesson Study," retrieved 2013 January 14.)

Unlike lesson studies, concept studies are not concerned with crafting classroom activities or learning tasks. However, like lesson studies, concept studies are oriented toward new pedagogical possibilities through participatory, collective, and ongoing engagements.

Further, and breaking with the popular focus on "unpacking" in contemporary research into teachers' disciplinary knowledge of mathematics, concept studies move well beyond taking concepts apart. While some of the activities that cross-grade groups of teachers undertake do resemble unpacking (e.g., identifying metaphors, analogies, and images that teachers use to render concepts meaningful), the products of such activities serve as gateways to more constructive engagements which can quickly lead to reworkings of the concepts at hand. I refer to this process as *substructing*.

The word *substructing* is derived from the Latin *sub-*, "under, from below" and *struere*, "pile, assembled" (which is the root of *strew* and *construe*, in addition to *structure* and *construct*). To substruct is to build beneath something. In industry, substructing refers to reconstructing a building without demolishing it – and, ideally, without interrupting its use. Likewise, in concept studies, teachers rework mathematical concepts, sometimes radically, while using them almost without interruption in their teaching.

Unlike "unpacking," the term "substructing" is both reductive and productive. It is reductive in that it starts by re-collecting and re-membering experiential, linguistic, and other elements that infuse the meanings of concepts. It is productive as it compels new integrative structures and novel interpretations, which in turn become the raw materials for further substructing.

For the purposes of illustration, I will now move to a brief report of recent substructing work conducted by a group of 11 practicing teachers. Their focus was the concept of *function*.

A BRIEF ACCOUNT

The teachers involved in the concept study reported below were participants in a master's program at the University of Calgary. All grade levels are represented in the group, and the episode described here occurred at the start of the second (and final) year of their program.

Prior to their joint substructing of the concept of function, group members had been exposed to concept studies of "multiplication" and "zero." They thus began this study with some familiarity of different strategies and emphases that might be used to substruct concepts.

The structures of concept study illustrated below (and described in more detail in the subsequent section) have a 10-year history. They arose initially out of casual encounters of groups of teachers who shared an interest in better understanding specific concepts and topics in mathematics. Based on that work, five emphases were distilled from the strategies that teachers invented: meanings, landscapes, entailments, blends, and pedagogical problem solving. After relating some of the details of this illustrative episode, I will offer more explicit descriptions of these emphases.

Emphasis 1: Meanings

The group's concept study of functions did not start out smoothly. Because the concept is covered explicitly only in high school, most of the primary- and middle-school teachers in the group were at first a little at sea when faced with the task of identifying different associations for functions. This emphasis was thus dominated by the high school teachers, as they mentioned "black box," "input/output," "vertical line test," and so on (a more complete list is presented in the first column of Figure 2).

Emphasis 2: Landscapes

Things became even more sluggish as the group attempted to organize the list of associations into a coherent mapping to highlight connections and gaps. Given that there is no explicit mention of function prior to high school, the group struggled to find connections between early, middle, and senior grades. Nevertheless, participants persevered – enabled in large part by the insistence of one lower-grades teacher that higher-grades counterparts explain in detail every one of their realizations so that connections to foci at the elementary level might be identified. It took hours, but eventually the group created a landscape that revealed a flow across three major topics of study: pattern, equation, function. The group elected to illustrate the insight as a nested, emergent flow (see fig. 1).

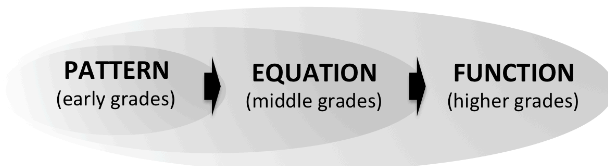

Figure 1: A depiction of the nested emergence of the function concept.

It is fair to say that group members were somewhat disappointed at this result, with several voicing the sentiment, "We already knew that."

If a function is defined in terms of …	… then the type of data is …	… then it is represented as …	… the in elementary school it looks like …	… then in middle school it looks like …	… then in high school it looks like …	… then the representation type is …	Conceptual Limitation(s)
vertical line test	discrete/ **continuous**	graph	n/a	n/a	graph	enactive and **iconic**	rejects some functions when $x = f(y)$; includes others that are not function
black box	discrete	two related sets of data	missing values in an equation	missing equation problems	two sets of data	Iconic	"magical mystery"
input/output	discrete	ordered pairs	sequenced practice (e.g., skip counting)	using linear equations to find missing values; ordered pairs	ordered pairs	iconic and **symbolic**	piecemeal production; missing the whole
independent/ dependent relationship	discrete/ **continuous**	rule	pattern rules	linear equations; graphing	rule	symbolic	unidirectional; may suggest they're not reversible
1-to-1 relationship/ correspondence	discrete/ **continuous**	anything in this column	counting based on 1-to-1 correspondence	ordered pairs; T-chart	anything in this column	enactive, iconic, and/or symbolic	implication of uniqueness – i.e., for every x there is a unique y
manipulating/ responding variables	discrete/ **continuous**	physical event ➡ collect data ➡ graph ➡ interpret	experimental data – growth chart	data; graphing; interpolating/ extrapolating tasks	event ➡ data ➡ graph ➡ interpolate/extrapolate	enactive ➡ iconic ➡ symbolic	contrived, controlled, contextual
equation with 2 or more variables	discrete/ **continuous**	equations (implicit: $y = \ldots$)	multiple representation problems	equations (implicit: $y = \ldots$)	equations (implicit: $y = \ldots$)	symbolic	Some equations with 2 or more variables may not be functions (e.g., $y = \pm\sqrt{x}$)
indexed pattern	discrete	sequence	growing patterns	sequences	sequences	enactive, iconic, and/or **symbolic**	implication of uniqueness; for every i, there is a unique n; dependence on initial position, or starting point; requires a starting point; domain is always natural numbers
mapping	discrete	arrow diagram	classification	transformational arrows	arrow diagram	**iconic** and symbolic	implied container schema
creating a table of values	discrete	horizontal or vertical T-chart	T-chart	T-chart	T-chart	iconic and **symbolic**	orientation of data; idea of discreteness; possible for left column to be random; implication of uniqueness (all of the limitations?)
$f(x)$	discrete/ **continuous**	equations (explicit: $f(x) = \ldots$)	n/a	equations (implicit: $y = \ldots$)	equations (explicit: $f(x) = \ldots$)	symbolic	notation implies multiplication; coordinates are $(x, f(x))$ they don't get that $f(x)$ is y

Figure 2: An entailments chart for the concept of *function*.

Having the advantage of watching from the side, I offered that perhaps they had known it on some level … but that knowledge was likely implicit, as it did not "present" itself in any way when the task for first undertaken. Rendering explicit, as I witnessed, was hard work.

Emphasis 3: Entailments

They also examined how particular instantiations might be troublesome at the high school level, which is represented in the final column of the above chart which scans some of the conceptual limitations of varied realizations. In the group context, filling in this column afforded an opportunity to make sense of what a function is by making it clear what a function isn't.

Emphasis 4: Blends

Following that discussion, the suggestion was made that each person should write out their own answer to the question, "What is a function?" on a sheet of paper and then post their responses on the wall. When that was done, the group proceeded to cluster responses according to core theme, and three emerged: function as RELATIONSHIP, PATH, and VALUE.

Three subgroups then formed around these themes, and each took on the task of pulling together a cluster of thematically similar interpretations into a consolidated description of function. The three summary descriptions were then contrasted, and the group undertook another consolidation effort, soon arriving at:

A function is a relationship[1] between two sets of values, in which each value in the source set corresponds to at most one value in the target set.

There are three important details to notice about this emergent realization. First, notice the way in which teachers still acknowledged the role of distinct meanings in their footnote, but displaced the particularity of each meaning from the more distilled definition. That separation-and-connection signals an important difference between the disciplinary knowledge of mathematics teachers and the disciplinary knowledge of mathematicians. Teachers must be more attentive to the nuances of varied *meanings*, while research mathematician are more oriented to the formulation of logically sound and more encompassing *definitions*.

Second, unlike much of mathematics instruction, this definition was derived from meanings – not the other way around. Participants arrived it by consolidating what they knew, not by attempting to make sense of someone else's formulation.

It's also important to note that this emergent definition was not assembled for dissemination. It was a product of the group, intended for the group. And meaningful to the group. As one lower-grades teacher commented, "I didn't give a rip about functions when we started. Now not only do I know what they are, I actually care about them."

[1] Articulated as a pattern, equation, mapping rule, ordered pairings, axes-associating curve, transformation process, etc.

Emphasis 5: Pedagogical Problem Solving

The balance of the concept study time was given to pedagogical problem solving, as the teachers grappled with question their own students had posed. These questions spanned patterning, algebra, and functions across the grades. Only a few of the dozen-or-so problems actually focused on functions, but that didn't seem to matter. The group was clearly disposed to substructing whatever concepts seemed to be involved in the problems raised.

SOME EMPHASES IN A CONCEPT STUDY

As mentioned in the preceding section, the various strategies and emphases that we use in concept studies were invented by teachers themselves as they grappled with the complex natures of various mathematical concepts. Although I offer discrete descriptions of each, they shouldn't be construed as separate or sequential. I see them as aspects of inquiry that are always already present (alongside other interpretive strategies that might not yet have been noticed or made explicit). The word *emphasis* (rather than, e.g., *steps*) was chosen to signal the simultaneity of different elements of concept study. Indeed, if space permitted a more fulsome account of the event, it would be clear that as the activity unfolded, members of the cohort revised and refined strategies in a recursively elaborative manner.

Figure 3 is intended as a visual metaphor of these points. As I move into elaborated descriptions of the emphases, it is important to re-emphasize that I in no way see them as acting in isolation or as following a particular order. Rather, in every case, it is a matter of relative emphasis.

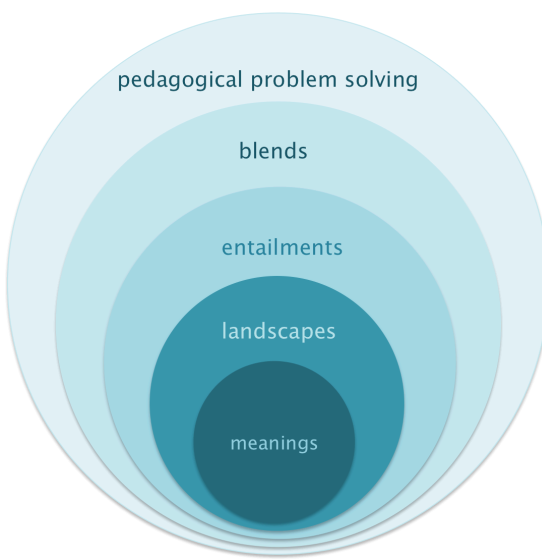

Figure 3: The nested emphases of concept study.

Emphasis 1: Meanings

The term *meanings* is used to collect all manner of associations that a learner might draw on and connect in efforts to make sense of a mathematical construct. Among many possible elements, meanings might draw on:

- Formal definitions (e.g., a function is a formal mapping of one set onto another)
- Metaphors (e.g., a function is a black box)
- Images (e.g., a function can be illustrated as a graph)
- Applications and other examples (e.g., quadratics, trigonometry)

This list can be extended, depending on the type of concept under examination. (For example, other categories of meaning include gestures and algorithms.) To be clear, the assertion and assumption here is not that any particular meaning is right, wrong, superior, inferior, adequate, or insufficient. It is that personal understanding of a mathematical concept is an emergent form, arising in the weaves of such experiential and conceptual elements.

The process of collectively identifying meanings is neither linear nor obvious. Each knower holds and utilizes a personal set of meanings. Some of these are common to all participants, while others are idiosyncratic or shared by only a few. Moreover, meanings are not fixed. They evolve through the process of learning. Not only do they become more numerous, some earlier ones are discarded or expanded when new interpretations arise. Well-rehearsed meanings (e.g., "a function is a relationship") can be so well practices that they may eclipse other interpretive possibilities.

To circumvent the tendency to go directly to well-rehearsed meanings, concept study sessions are typically framed as invitations to explore how a concept such as function is introduced, taken up, applied, and/or elaborated at different grade levels. This strategy also helps to sidestep the temptation to leap straight for the practiced definition.

Emphasis 2: Landscapes

There are dramatic differences of conceptual worth among varied meanings. Some can reach across most contexts in which a learner might encounter a concept; others are situation-specific or even perhaps learner-specific. This realization compelled a "landscapes" strategy, to organize and contrast assembled lists of meanings. Briefly, a landscape is a macro-level view, whereas a meaning is a micro-level view, of a concept.

Most often, landscapes are created in grid-like formats by identifying two dimensions. Different criteria can be used to organize information, and "grade level" has been by far the most common dimension among teacher groups and across concepts. Others might include type of meaning (i.e., formal definition, metaphor, etc.), branches of mathematics linked to the meaning (e.g., arithmetic, algebra, geometry), processes versus objects, and so on.

With such ranges of possibility, landscapes tend to be very detailed and complex (see Davis & Renert, 2013, for other examples). The elegant simplicity in the above example (fig. 2) is a distinct exception, but still illustrates a key point: the purpose of a landscape is to afford senses of connection and trajectory – not in the least so that all participants, across grade levels, can see clearly how they contribute to the emergence of a concept.

Emphasis 3: Entailments

As already mentioned, every meaning of a concept carries a set of implications. The intention of this emphasis is to examine the entailments of different realizations to related concepts (e.g., how is the "function as input/output" meaning manifested in the early years? the middle years? etc.). In the process of exploring entailments, participants are forced to consider the concept afresh and not only in well-rehearsed, practiced ways.

The principal device used within this emphasis is a grid that was invented by a group of teachers who were struggling to understand why 1 is a prime number (Davis, 2008). An "entailments chart" comprises a list of realizations in the first column, with as many additional columns as desired to record the usually uninterrogated implications of those meanings.

Across concept studies, a major consequence of examining entailments has been surprise and confusion over the inconsistencies and occasional contradictions that arise among difference meanings. The final column of the chart in figure 3 represents an explicit attempt to capture some of these tensions and disconnects. And it served as the focus of a protracted conversation on whether a few meanings in particular should be introduced at all.

Emphasis 4: Blends

The three emphases described so far are focused mainly on making fine-grained distinctions among realizations and their entailments. Not surprisingly, many of the participating teachers voiced some frustrations as the shared work unfolded. "Function," after all, is a mathematically coherent concept, not an assemblage of images and implications.

The blends emphasis is about seeking out (or crafting) meta-level coherences by exploring the deep connections among identified meanings and/or assembling those meanings into a more encompassing interpretation – which, of course, might introduce emergent meanings in the never-ending cycle of sense making.

This emphasis draws on research into conceptual blends (e.g., Fauconnier & Turner, 1998) which analyzes how metarepresentations arise when learners blend meanings into more encompassing, further reaching constructs. Subprocesses of blending include inventing and designing new representations, comparing and critiquing them, applying and explaining them, and learning new representations. In the first concept studies that I was involved in, blends tended to arise spontaneously (Davis, 2008; Davis & Simmt, 2006). Those experiences prompted the desire to be more systematic, in the hope that teachers would leave concept studies with powerful, coherent understandings of the concepts they were substructing.

There's really no universal strategy for undertaking a blend. However, vital elements include categorization and distillation of meaning (i.e., looking for the BIG ideas – such as the three principal interpretations of function identified by the teachers). Collective process is also critical. Plain and simply, teachers need to be prepared to disagree and to argue their points and to work through tensions in the interests of their students.

Emphasis 5: Pedagogical Problem Solving

The emphasis of pedagogical problem solving returns teachers to the original emphasis of teaching. It revolves around the questions and quandaries that both capture students' imaginations and stop them in their tracks. Familiar examples include:

- Is ∞ a number?
- What does it mean to divide by zero?
- What's the difference between *undefined, indeterminate,* and *infinite*?

To the experience mathematics knower, these questions might seem trivial, with established and unambiguous responses. Such is not usually the case for novices, as most experienced teachers will attest. As part of a concept study, the intention is to provide a space for participants to bring their diverse areas of conceptual expertise to bear on the sorts of questions that can both captivate and stall groups of learners. They are also intended as opportunities to explore the deep and extensive "root systems" of mathematical concepts.

To explain, the three questions noted above turn out to me intimately linked. As detailed elsewhere (Davis & Renert, 2013), fulsome responses to any of these questions will inevitably compel examination of the others. Moreover, matters of number, operation, and function must be engaged in order to appreciate the responses.

WHY SHOULD WE CARE?

In addition to its vastness, its complexity and its distributed character, a quality of teachers' disciplinary knowledge of mathematics that is made particularly manifest through concept study is its volatility. Plain and simply, on both individual and collective levels, it is a moving form.

It is for this reason that I prefer to think of this category of expertise in terms of learnable disposition in addition to a domain of knowledge. I thus close with a few comments on how I understand the relationship between *mathematics* and *mathematics for teaching.* The simultaneous foci of concept study are (1) the "explicititification" of current mathematics knowledge and (2) the creation of new possibilities for mathematics teaching that are tooted in more nuanced understandings and elaborations of extant mathematics. The goal is not to create new formal mathematics – a task that would require very different validation criteria. The essential questions for us do not revolve around the ontological status of mathematical concepts or around teachers' production of new mathematics.

As a research community, mathematics educators are still far from making definitive claims about the relationships between teachers' profound understandings of mathematics and their students' mathematical understandings. My suspicion is that efforts to address this vexing quandary will require more fine-grained analysis than large-scale assessments, in large part because many of the most important aspects of teachers' knowledge are simply unavailable for explicit and immediate assessment. They are tacit and can only emerge through participation in collective explorations, such as concept studies.

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