THE UNIVERSITY OF CALGARY

# TOPICS IN HOMOMORPHISMS IN REPRESENTATION THEORY OF SYMMETRIC GROUPS

by

QIDUAN YANG

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE

DEGREE OF

DOCTOR OF PHILOSOPHY

DEPARTMENT OF

MATHEMATICS AND STATISTICS

CALGARY, ALBERTA

APRIL, 1989

THE UNIVERSITY OF CALGARY

# TOPICS IN HOMOMORPHISMS IN REPRESENTATION THEORY OF SYMMETRIC GROUPS

by

QIDUAN YANG

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE

DEGREE OF

DOCTOR OF PHILOSOPHY

DEPARTMENT OF

MATHEMATICS AND STATISTICS

CALGARY, ALBERTA

APRIL, 1989

ISBN 0-315-50438-2

# THE UNIVERSITY OF CALGARY
# FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Topics in Homomorphisms in Representation Theory of Symmetric Groups", submitted by Qiduan Yang in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

H. K. Farahat.

**Supervisor**, Dr. H.K. Farahat, Department of Mathematics and Statistics

W. K. Nicholson

Dr. W.K. Nicholson, Department of Mathematics and Statistics

Jon Rokne

Dr. J. G. Rokne, Department of Computer Science

Luca Bombelli

Dr. L. Bombelli, Department of Mathematics and Statistics

Gerald Cliff

**External Examiner**, Dr. G. Cliff, University of Alberta

<u>24 April 1989</u>

## ABSTRACT

Incidence matrices associated with a pair of partitions of $n$ are introduced to describe the homomorphisms between permutation modules. A partial solution to the description of the correspondence between two labellings of irreducible modules is given. All partitions $\mu$ such that there are non-zero homomorphisms from $S^{(n-1,1)}$ into $S^{\mu}$ are found. The Specht module $S^{\lambda}$, where $\lambda = (\lambda_1, \lambda_2, 1^r)$ with $r \geq 2$, is proved to have socle length one.

# ACKNOWLEDGEMENT

I would like to express my deepest gratitude to Professor H. K. Farahat for his invaluable guidance and encouragement and, above all, for his kindness and understanding throughout the course of my research.

I wish to thank the University of Calgary for the financial support, without which the work would not have been possible.

# TABLE OF CONTENTS

# CHAPTER 1
# INTRODUCTION

The representation theory of finite groups has its roots in character theory, emerging around the turn of this century as the work of Frobenius, Schur and other authors. In this theory, the symmetric group $\mathfrak{S}_n$ is a simple but important case, simple because its characters and irreducible representations can be found in the rational field, important because every finite group can be embedded in some symmetric group.

It was Alfred Young's achievement to find a natural classification of all irreducible representations of $\mathfrak{S}_n$ in terms of "Young tableaux" over the rational field. W. Specht's alternative approach in this topic showed how to derive representations by considering submodules of a polynomial ring $K[x_1,x_2,\dots,x_n]$, where $K$ is a field, and this method yielded interesting results without referring to the characteristic of the field $K$. We shall follow the approach in [James and Kerber (1981)] by using modules isomorphic to those of Specht.

The modular structure of permutation modules $M^\lambda$ and the Specht modules $S^\lambda$ is still at the center stage of the representation theory of symmetric groups. The homomorphisms between permutation modules and Specht modules provide useful information in many aspects. There are several different ways to describe homomorphisms, for instance, Carter and Lusztig found a $K$–basis for the space of homomorphisms from a Specht module to a permutation module in terms of so–called semistandard tableaux. In this dissertation, we introduce matrices with non–negative integer entries, which we call incidence matrices corresponding to each pair of compositions $\lambda$ and $\mu$ of $n$, and use them to describe homomorphisms from $M^\lambda$ to $M^\mu$. One of the advantages of these incidence matrices is their natural connection with the bilinear forms on $M^\lambda$ and $M^\mu$. Special cases of the incidence matrices are the $\psi$ maps introduced in [James, (1977a)].

The construction of the modular irreducible representations was given by G.D. James in 1976. Each equivalence class of irreducible representations is labelled by a so-called "row $p$-regular" partition of $n$, where $p$ is the characteristic of the ground field $K$. An alternative way of labelling the irreducibles is by using the "column $p$-regular" partitions. An interesting question was raised in [James, (1977b)] : what is the connection between the two labellings? In §3D of this dissertation, we attempt to describe the links between the two labellings by making use of some special features of certain incidence matrices. A partial solution of the above question is obtained and presented in that section.

The complete determination of the homomorphisms between two Specht modules $S^\lambda$ and $S^\mu$ for a pair of distinct partitions $\lambda$ and $\mu$ of $n$ is a difficult open question. G.D. James found all partitions $\mu$ such that $\mathrm{Hom}_{K\mathfrak{S}_n}(S^{(n)},S^\mu)$ is non-zero; and Gwendolen Murphy made a thorough analysis of the space $\mathrm{Hom}_{K\mathfrak{S}_n}(S^{(n-r,r)},S^{(n-k,k)})$. Also, some very useful information concerning this subject can be found in [Carter and Lusztig (1974)] and [Carter and Payne (1980)]. In §3E, all partitions $\mu$ such that $\mathrm{Hom}_{K\mathfrak{S}_n}(S^{(n-1,1)}, S^\mu)$ is non-zero are found and $K$-bases are given in each case, using computation of homomorphisms in terms of incidence matrices.

In Chapter 4, we investigate the socle length of some Specht modules by using endomorphisms of the permutation modules with certain special properties. In the first three sections of Chapter 4, we use our own machinery to reproduce the results concerning the socle lengths of Specht modules associated with hook partitions and two–parts partitions obtained by M. Peel and Gwendolen Murphy in [Peel (1971)] and [Murphy (1982)] respectively. In §4D we extend these result to the Specht modules associated with partitions of the form $(\lambda_1,\lambda_2,1^r)$, with $r \geq 2$ by calculating the $K$–space $\mathrm{Hom}_{K\mathfrak{S}_n}(S^{\lambda *},S^{\lambda})$. This provides another example of the application of incidence matrices.

# CHAPTER 2
# BASIC FACTS

## §2A Modules and Their Duals over a Group Algebra

Let $K$ be a field and $G$ be a finite group. An element $\xi$ in the group algebra $KG$ can be written in the form

$$\xi = \sum_{\sigma \in G} \xi_\sigma \sigma , \qquad \xi_\sigma \in K.$$

The group anti–automorphism

$$\begin{aligned} T: \ G &\longrightarrow G \\ \sigma &\longmapsto \sigma^{-1} \end{aligned}$$

can be extended by linearity to a $K$–algebra anti–automorphism of $KG$ :

$$T: \sum_{\sigma \in G} \xi_\sigma \sigma \longmapsto \sum_{\sigma \in G} \xi_\sigma \sigma^{-1}.$$

We shall write $\xi^* = T(\xi)$ for $\xi \in KG$.

Unless specified, "a $KG$–module" means a left $KG$–module which is also a $K$–space of finite dimension, in most parts of this dissertation. Let $M$ be a $KG$–module. Then $M^* = \mathrm{Hom}_K(M,K)$ becomes a $KG$–module, called the dual of $M$, if we define

$$(\xi f)(m) = f(\xi^* m)$$

for $f \in M^*$, $\xi \in KG$, $m \in M$.

Assume that $M$ is a $KG$–module with a $K$–valued non–degenerate, symmetric bilinear form $< \, , \, >$ satisfying

$$< \xi m_1, \, m_2 > \; = \; < m_1, \, \xi^* m_2 >,$$

for $m_1$, $m_2 \in M$ and $\xi \in KG$. Define

$$\begin{aligned} \theta : \; M &\longrightarrow M^* \\ m &\longmapsto \theta_{\mathrm{m}} \end{aligned}$$

by setting

$$\theta_{\mathrm{m}}(m') \; = \; < m, \, m' >, \qquad m' \in M.$$

Then for $\xi \in KG$, $m, m' \in M$,

$$\theta_{\xi \mathrm{m}}(m') \; = \; < \xi m, \, m' > \; = \; < m, \, \xi^* m' > \; = \; \theta_{\mathrm{m}}(\xi^* m') \; = \; (\xi \theta_{\mathrm{m}})(m').$$

Thus $\theta : \; m \longmapsto \theta_{\mathrm{m}}$ is a $KG$–homomorphism from $M$ to $M^*$. In fact, $\theta$ is a $KG$–isomorphism since $\dim_K M = \dim_K M^*$, and $< \, , \, >$ is non–degenerate:

$$\mathrm{Ker}(\theta) \; = \; \{ \; m \in M \mid \; < m, \, m' > \; = \; 0 \quad (\forall m' \in M) \; \} \; = \; 0.$$

Let $U$ be a $KG$–submodule of $M$, then

$$U^{\perp} \; = \; \{ \; m \in M \mid \; < m, \, u > \; = \; 0 \quad (\forall u \in U) \; \}$$

is a $KG$–submodule of $M$. The following is proved in §1 [James (1978b)]:

**(2.1) LEMMA.** *Let $M$ be a $KG$–module with a non–degenerate, symmetric bilinear form satisfying $< \xi m_1, m_2 > \; = \; < m_1, \xi^{\star} m_2 >$, for $m_1, m_2 \in M$ and $\xi \in KG$. Let $U$, $U_1$ and $U_2$ be $KG$–submodules of $M$ and assume that $U_1 \leq U_2$, then*

(i) *$U^{\perp}$ is a $KG$–submodule of $M$ and $U_2^{\perp} \leq U_1^{\perp}$.*

(ii) $U^{\perp\perp} = U$.

(iii) $\dim_K U + \dim_K U^{\perp} = \dim_K M$.

(iv) *$(U_2/U_1)^* \cong U_1^{\perp}/U_2^{\perp}$, in particular, $U \cong (M/U^{\perp})^*$ as $KG$–modules .*∎

Note. In §1 [James (1978b)], the adjective "non–singular" was used for "non–degenerate", both mean for every non–zero $m$ in $M$, there is some $m'$ in $M$, such that $< m, m' > \; \neq 0$.∎

Let $\xi$ be an element in $KG$. M. Peel proved a lemma concerning the dual of the left ideal $KG\xi$ of $KG$ :

**(2.2) LEMMA.** (Lemma 1. [Peel (1981)]) *Let $K$ be a field, $G$ be a finite group. Then if $\xi \in KG$,*

$$(KG\xi)^* \cong KGT(\xi) = KG\xi^{\star}$$

*as $KG$–modules.*∎

The proof for (2.2) can be found in Peel's original paper. An alternative proof, which was suggested by H. K. Farahat, is given in 4.2 [Yang (1984)].

Next we turn to some general results about $KG$–modules and their dual modules, where $G$ is an arbitrary finite group.

For each $KG$–module $M$, there is a natural $KG$–homomorphism from $M$ onto $M^{**}$:

$$x \longmapsto \hat{x}$$

where $\hat{x}(f) = f(x)$, $x \in M$, $f \in M^{*}$, since

$$(\xi x)\hat{}\ (f) = f(\xi x) = (\xi^{*} f)(x) = \hat{x}(\xi^{*} f) = (\xi\ \hat{x})(f).$$

Let $L$ and $M$ be $KG$–modules. Each $\varphi \in \mathrm{Hom}_{KG}(L,M)$ determines a $KG$–homomorphism

$$\begin{aligned} \varphi^{*} :\ & M^{*} \longrightarrow L^{*} \\ & f \longmapsto f\varphi. \end{aligned}$$

Further, the mapping

$$\varphi \in \mathrm{Hom}_{KG}(L,M) \longmapsto \varphi^{*} \in \mathrm{Hom}_{KG}(M^{*},L^{*})$$

is a $K$–isomorphism, since we can identify $\mathrm{Hom}_{KG}(L^{**},M^{**})$ with $\mathrm{Hom}_{KG}(L,M)$ and the composite mapping

$$\varphi \longmapsto \varphi^{*} \longmapsto \varphi^{**}$$

is essentially the identity mapping on $\mathrm{Hom}_{KG}(L,M)$. Thus we have (c.f. 4.12 [Yang (1984)])

**(2.3) LEMMA.** *Let $L$ and $M$ be be $KG$–modules, then*

$$\mathrm{Hom}_{KG}(L,M) \cong \mathrm{Hom}_{KG}(M^*,L^*)$$

*as $K$–spaces.* ∎

The proof of the following lemma concerning dual modules and short exact sequences can be found in (4.12) [Yang (1984)] :

**(2.4) LEMMA.** *Let $L$, $M$ and $N$ be be $KG$–modules and $L^*$, $M^*$ and $N^*$ be their dual modules. Then*

(i) *The sequence*

$$0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0 \qquad \text{(a)}$$

*is exact if and only if the sequence*

$$0 \longrightarrow N^* \xrightarrow{\psi^*} M^* \xrightarrow{\varphi^*} L^* \longrightarrow 0 \qquad \text{(a}^*\text{)}$$

*is exact.*

(ii) *The sequence* (a) *is split if and only if* (a$^*$) *is split.*

Let $G$ be a finite group and $M$ be a $KG$–module. Let

$$M = M_0 > M_1 > \cdots > M_{r-1} > M_r = 0 \qquad \text{(b)}$$

be a composition series of $M$, i.e. a chain of submodules of $M$, in which

$$\frac{M_{i-1}}{M_i} = J_i$$

is an irreducible $KG$–module, $i = 0, 1, \ldots, r-1$. The composition series (b) of $M$ determines a sequences of irreducible $KG$–modules

$$(J_1, J_2, \ldots, J_r),$$

which is called the composition factor sequence associated to (b). The number $r$ is called the composition length of $M$.

We shall give the proof of the following standard result about the composition factor sequences of a $KG$–module $M$ and its dual $M^*$.

(2.5) **LEMMA.** *Let*

$$M = M_0 > M_1 > \cdots > M_{r-1} > M_r = 0 \qquad \text{(b)}$$

*be a composition series of a KG–module M. Assume that*

$$(J_1, J_2, \ldots, J_r)$$

*is the composition factor sequence of M associated to* (b). *Then there exists a composition series*

$$M^* = N_0 > N_1 > \cdots > N_{r-1} > N_r = 0 \qquad \text{(b}^*\text{)}$$

*such that*

$$(J_r^*, \ldots, J_1^*)$$

*is the composition factor sequence of* $M^*$ *associated to* (b$^*$).

**PROOF.** Use induction on the composition length $r$ of the $KG$–module $M$. The statement is trivial from (2.4) when $r = 1$. Now assume $r \geq 2$ and the statement in (2.5) is true for all $KG$–modules with composition lengths less than or equal to $(r-1)$. The exact sequence

$$0 \longrightarrow J_r \longrightarrow M \longrightarrow M/J_r \longrightarrow 0$$

has its "dual" exact sequence

$$0 \longrightarrow (M/J_r)^* \longrightarrow M^* \longrightarrow J_r^* \longrightarrow 0$$

by (2.4). The $KG$–module $M/J_r$ has a descending chain of $KG$–submodules

$$\frac{M}{J_r} = \frac{M_0}{J_r} > \frac{M_1}{J_r} > \cdots > \frac{M_{r-2}}{J_r} > \frac{M_{r-1}}{J_r} = 0\ . \qquad \text{(c)}$$

But

$$J_i = \frac{M_{i-1}}{M_i} \cong \frac{M_{i-1}/J_r}{M_i/J_r}\ , \qquad i = 0,\ 1,\ \ldots\ ,\ r-2.$$

Therefore (c) is a composition series for $M/J_r$ and the composition factor sequence associated to (c) is $(J_1, \ldots ,J_{r-1})$, up to isomorphism. By induction hypothesis, $(M/J_r)^*$ has a composition series

$$(M/J_r)^* = N_1 > \cdots > N_{r-1} > N_r = 0 \qquad (\text{c}^*)$$

such that the composition factor sequence associated to $(c^*)$ is

$$(J^*_{r-1}, \dots, J^*_1).$$

Clearly

$$M^* = N_0 > N_1 > \cdots > N_{r-1} > N_r = 0 \qquad (b^*)$$

is a composition series of $M^*$, and the composition factor sequence associated to $(b^*)$ is

$$(J^*_r, J^*_{r-1}, \dots, J^*_1). \blacksquare$$

**(2.6) NOTES ON THE RADICAL AND THE SOCLE OF A MODULE $M$** (§5 [Curtis and Reiner (1981)])

Let $A$ be a ring with 1, and $M$ be a (left) $A$–module. The radical of $M$, denoted by rad($M$), is defined as the intersection of all maximal submodules of $M$,

$$\mathrm{rad}(M) = \cap \{ N < M \mid N \text{ is a maximal submodule of } M \}.$$

The socle of $M$, denoted by soc($M$), is the sum of all the irreducible submodules of $M$. The radical of the ring $A$ is the radical of the left $A$–module $A$. In fact, rad($A$) is a two–sided ideal of $A$, and

$$\mathrm{rad}(A) = \{ \mathrm{Ann}_A(S) \mid S \text{ is an irreducible } A\text{–module} \}.$$

Moreover, the radical of the factor ring $A/\mathrm{rad}(A)$ is 0.

A ring $A$ is said to be left artinian if the left ideals of $A$ satisfy the descending chain condition, i.e. for every descending sequence of left ideals of $A$,

$$L_1 \geq L_2 \geq \cdots$$

there exists an integer $k$ such that

$$L_k = L_{k+1} = \cdots .$$

Clearly, if $G$ is a finite group, $K$ is a field, then the group ring $KG$ is left artinian since $KG$ is a finite dimensional $K$-space.

Let $A$ be a left artinian ring, then $A/\mathrm{rad}(A)$ is a semisimple ring. Furthermore, if $M$ is an $A$-module, then

$$\mathrm{rad}(M) = (\mathrm{rad}A)M$$

and $M/\mathrm{rad}M$ is a semisimple $A$-module. In fact, $\mathrm{rad}(M)$ is the smallest submodule of $M$ such that the factor module is semisimple.

The following lemma links up the two semisimple $KG$-modules: $M/\mathrm{rad}M$ and $\mathrm{soc}(M^*)$. The proof can be found on page 57, Chapter 4 [Yang (1984)] :

**(2.7) LEMMA.** *Let $M$ be a $KG$-module, where $G$ is a finite group and $K$ is a field. Then*

$$M/\mathrm{rad}(M) \cong [\mathrm{soc}(M^*)]^*$$

*as $KG$-modules.*∎

## §2B Partitions and Tableaux

We denote by $\mathfrak{S}_n$ the group of permutations of the set $\underline{n} = \{1, 2,\ldots, n\}$, which is called the symmetric group of degree $n$. The alternating group $\mathfrak{A}_n$ is the subgroup of $\mathfrak{S}_n$ consisting of all the even permutations of $\mathfrak{S}_n$. Denote by $K\mathfrak{S}_n$ and $K\mathfrak{A}_n$ the group algebras of $\mathfrak{S}_n$ and $\mathfrak{A}_n$ over a field $K$ respectively.

Let $X$ be a subset of $\underline{n}$. We write

$$\mathfrak{S}[X] = \{ \pi \in \mathfrak{S}_n \mid \pi(i) = i \quad (\forall i \in \underline{n}\backslash X) \}.$$

Clearly $\mathfrak{S}[X]$ is a subgroup of $\mathfrak{S}_n$, and

$$\mathfrak{S}[\underline{n}] = \mathfrak{S}_n, \qquad \mathfrak{S}[\emptyset] = \{1\}.$$

The product $\mathfrak{S}[X]\cdot\mathfrak{S}[Y]$ is again a subgroup of $\mathfrak{S}_n$, if $X \subseteq \underline{n}$, $Y \subseteq \underline{n}$ , and $X \cap Y = \emptyset$. For $X = \{i_1, i_2,\ldots, i_k\} \subseteq \underline{n}$ , we shall write

$$\mathfrak{S}[X] = \mathfrak{S}[i_1, i_2,\ldots, i_k]$$

by omitting the braces $\{ \}$.

Let $\gamma : G \longrightarrow K$ be a $K$-valued function on a finite group $G$. We write for $H \subseteq G$,

$$\gamma(H) = \sum_{\sigma \in H} \gamma(\sigma)\sigma .$$

The following two $K$–valued functions play significant roles in the representation theory of $\mathfrak{S}_n$:

$$\iota(\sigma) = 1, \qquad \forall\sigma \in \mathfrak{S}_n.$$
$$\epsilon(\sigma) = \begin{cases} 1, & \forall\sigma \in \mathfrak{A}_n, \\ -1, & \forall\sigma \in \mathfrak{S}_n \backslash \mathfrak{A}_n. \end{cases}$$

$\iota$ is called the trivial character, while $\epsilon$ is called the alternating character of $\mathfrak{S}_n$.

The left ideals $K\mathfrak{S}_n\iota(\mathfrak{S}_n)$ and $K\mathfrak{S}_n\epsilon(\mathfrak{S}_n)$ are $K$–spaces of dimension 1 (one). Clearly they are irreducible $K\mathfrak{S}_n$–modules.

A composition $\lambda$ of a positive integer $n$ is a sequence $(\lambda_1,\lambda_2,...)$ of non–negative integers such that

$$\sum_{i=1}^{\infty} \lambda_i = n \ ;$$

if, in addition,

$$\lambda_1 \geq \lambda_2 \geq \cdots,$$

then $\lambda$ is called a partition of $n$. Abbreviations such as $(3,0,2,2,0,...) = (3,0,2,2) = (3,0,2^2)$ will usually be adopted.

For a composition $\lambda = (\lambda_1,\lambda_2,...)$ of $n$, define $\lambda' = (\lambda'_1,\lambda'_2,...)$ by setting $\lambda'_i$ equal to the cardinality of

$$\{ \lambda_j \mid \lambda_j \geq i \}.$$

Note that $\lambda'$ is a partition of $n$. For example, if $\lambda = (3,0,2^2)$ and $\mu = (3,2^2)$ then

$$\lambda' = (3^2,1) = \mu'.$$

When $\lambda$ is a partition of $n$, the mapping

$$\lambda \longmapsto \lambda'$$

is a bijection from the set of partitions of $n$ onto itself. $\lambda'$ is called the conjugate partition of $\lambda$ in this case.

A diagram $D$ is a finite subset of $\mathbb{N}\times\mathbb{N}$. For a composition $\lambda$ of $n$, $[\lambda]$ is the diagram

$$\{ (i,j) \mid j = 1, 2,\ldots, \lambda_i, \quad i = 1, 2, \ldots \}$$

A $\lambda$–tableau $t$ is a bijection from $[\lambda]$ onto the set $\underline{n} = \{1, 2,\ldots, n\}$. We shall write $t_{ij}$ for $t(i,j)$, $(i,j) \in [\lambda]$. A $\lambda$–tableau $t$ can be depicted by replacing the nodes $(i,j)$ in $[\lambda]$ by $t_{ij}$. For example, if $\lambda = (3,0,1,2)$, then

$$t = \begin{array}{l} 4\ 2\ 6 \\ \hline \\ 3 \\ 1\ 5 \end{array}$$

is the $\lambda$–tableau such that $t_{11} = 4$, $t_{12} = 2$, ... , etc.

The group $\mathfrak{S}_n$ acts on the set of $\lambda$–tableaux by letter permutations. That is

$$(\pi t)_{ij} = \pi(t_{ij}), \qquad \pi \in \mathfrak{S}_n,\ (i,j) \in [\lambda].$$

For example, if

$$t = \begin{matrix} 2 & 3 & 5 \\ 1 & 4 & \end{matrix},$$

$\pi = (1,2,4) \in \mathfrak{S}_5$, then

$$\pi t = \begin{matrix} 4 & 3 & 5 \\ 2 & 1 & \end{matrix}.$$

We shall use the lower case letters $t$, $x$, $y$, ... to denote tableaux. Let $x$ be a tableau of a composition of $n$ : $\lambda = (\lambda_1,\lambda_2,\ldots)$. For a fixed $i$, the subset of $\underline{n}$ :

$$X_i = \{\ x_{ij} \mid (i,j) \in [\lambda]\ \}$$

is the set of elements in the $i$–th row of (the depiction of) $x$. Similarly for a fixed $j$, the set of elements in the $j$–th column of $x$ is

$$X_j' = \{\ x_{ij} \mid (i,j) \in [\lambda]\ \}$$

The row stabilizer and column stabilizer of the $\lambda$–tableau $x$ are defined as

$$Rx = \prod_i \mathfrak{S}[X_i],$$

$$\underset{\prime}{C}x = \prod_j \mathfrak{S}[X_j'].$$

The elements in $Rx$ ($Cx$) are called row (column) permutations of $x$.

Let $x$ and $y$ be two tableaux of a composition $\lambda$ of $n$. There exists $\pi$ in $\mathfrak{S}_n$, such that $y = \pi x$. It is easy to verify that

$$R(\pi x) = \pi(Rx)\pi^{-1},$$
$$C(\pi x) = \pi(Cx)\pi^{-1}.$$

Two $\lambda$–tableaux $x$ and $y$ are said to be row–equivalent, if $y$ arises from $x$ by a row permutation of $x$ :

$$x \sim y \quad \Leftrightarrow \quad \exists \pi \in Rx,\ y = \pi x.$$

The equivalence class of $x$ is denoted by $\underline{\bar{x}}$ , called the $\lambda$–tabloid of $x$. It is convenient to treat $\underline{\bar{x}}$ as a sequence of subsets of $\underline{n}$ :

$$(X_1,\ X_2,\ \ldots).$$

Our convention is the following

**(2.8) DEFINITION.** *For a $\lambda$–tableau $x$, the $\lambda$–tabloid $\underline{\bar{x}}$ is the column vector*

$$\begin{bmatrix} X_1 \\ X_2 \\ \vdots \end{bmatrix}$$

*whose entries are subsets of $\underline{n}$ :*

$$X_i = \{\ x_{ij} \mid (i,j) \in [\lambda]\ \}, \qquad i = 1,\ 2,\ \ldots\ .$$

When we want to emphasize the composition $\lambda$, the notation

$$\begin{bmatrix} X_1 \\ X_2 \\ \vdots \end{bmatrix}_{\lambda}$$

is adopted.$\blacksquare$

For example, say $\lambda = (3,2^2)$,

$$x = \begin{array}{lll} 4 & 5 & 6 \\ 1 & 7 & \\ 2 & 3 & \end{array}$$

we also follow the notation invented by G.D. James (3.9 [James (1978b)]):

$$\underline{\overline{x}} = \begin{array}{l} \overline{\underline{4\ 5\ 6}} \\ \underline{1\ 7} \\ \underline{2\ 3} \end{array} .$$

There is a natural $\mathfrak{S}_n$–action on the set of $\lambda$–tabloids :

$$\pi\, \underline{\overline{x}} = \pi \begin{bmatrix} X_1 \\ X_2 \\ \vdots \end{bmatrix}_{\lambda} = \begin{bmatrix} \pi X_1 \\ \pi X_2 \\ \vdots \end{bmatrix}_{\lambda} = \underline{\overline{\pi x}} ,$$

where $\pi \in \mathfrak{S}_n$ and $x$ is a $\lambda$–tableau.

Let $x$ be a $\lambda$–tableau and $\{\pi_1, \ldots , \pi_h\}$ be a set of left coset representatives of $Rx$ in $\mathfrak{S}_n$. It is easily seen that

$$\{ \pi_i\, \underline{\overline{x}} \mid i = 1, 2, \ldots , h \}$$

is the set of all $\lambda$–tabloids, and the mapping

$$\pi_i \, \overline{\underline{x}} \longmapsto \pi_i R x$$

is a one–to–one mapping from the set of $\lambda$–tabloids onto the set of left cosets of $Rx$ in $\mathfrak{S}_n$. The $\mathfrak{S}_n$–action on $\lambda$–tabloids agrees with the one which is the left multiplication of $\mathfrak{S}_n$ on the set of the left cosets of $Rx$ in $\mathfrak{S}_n$.

## §2C Specht Modules and Their Duals

Specht modules can be defined in a few equivalent ways. In this section, we shall state the definitions of permutation modules and Specht modules given in [James and Kerber (1981)].

Let $K$ be an arbitrary field. We shall denote the group algebra $K\mathfrak{S}_n$ by $\Gamma$. For a composition $\lambda$ of $n$, consider the $K$–space $M_K^\lambda$ having as basis the set of all $\lambda$–tabloids. The $\mathfrak{S}_n$–action on the $\lambda$–tabloids (c.f. the end of last section)

$$\pi \, \underline{\overline{t}} = \underline{\overline{\pi t}} \,, \qquad \pi \in \mathfrak{S}_n, \; t \text{ is a } \lambda\text{–tableau},$$

extends to an $\mathfrak{S}_n$–action on the $K$–space $M_K^\lambda$ by $K$–linearity and turns $M_K^\lambda$ into a $\Gamma = K\mathfrak{S}_n$ module. Thus $M_K^\lambda$ , which will be abbreviated as $M^\lambda$ if the field $K$ is fixed in the context, is a cyclic $\Gamma$–module, generated over $\Gamma$ by any one $\lambda$–tabloid. We call $M^\lambda$ the permutation module associated with the composition $\lambda$ of $n$.

Let $t_1$ and $t_2$ be $\lambda$–tableaux, such that $t_2 = \pi t_1$, for some $\pi \in \mathfrak{S}_n$. Then in the $\Gamma$–module $M^\lambda$,

$$\epsilon(Ct_2) \, \underline{\overline{t_2}} = \pi \, \epsilon(Ct_1) \, \pi^{-1}(\pi \, \underline{\overline{t_1}} \,) = \pi \, \epsilon(Ct_1) \, \underline{\overline{t_1}} \,.$$

We now define the Specht module $S_K^\lambda = S^\lambda$ associated with a composition $\lambda$ of $n$ to be the cyclic module

$$\Gamma \epsilon(Ct) \, \underline{\overline{t}} \,,$$

where $t$ is an arbitrary $\lambda$–tableau. By the remark above, the definition of $S^\lambda$ does not depend on the choice of $\lambda$–tableau $t$. Clearly $S^\lambda$ is a cyclic $\Gamma$–submodule of $M^\lambda$.

Define a $K$–bilinear form on $M^\lambda$, by setting

$$< \underline{\overline{t_1}}, \underline{\overline{t_2}} > = \begin{cases} 1, & \text{if } \underline{\overline{t_1}} = \underline{\overline{t_2}}, \\ 0 & \text{otherwise,} \end{cases}$$

where $t_1$ and $t_2$ are $\lambda$–tableaux and extending this to $M^\lambda$ by $K$–linearity. We see that $<\ ,\ >$ becomes a non–degenerate, symmetric bilinear from on $M^\lambda$, satisfying

$$< \xi m_1, m_2 > = < m_1, \xi^\star m_2 >, \qquad m_1, m_2 \in M^\lambda, \ \xi \in \Gamma.$$

The crucial result about this bilinear form on $M^\lambda$ is the following

**(2.9) JAMES'S SUBMODULE THEOREM** [4.8 James (1978b)]

*Let $K$ be an arbitrary field. If $\lambda$ is a partition of $n$ and $U \leq M^\lambda$, then either $U \geq S^\lambda$ or $U \leq S^{\lambda\perp}$.* ∎

With the aid of the Submodule Theorem, one can see that if $S^\lambda \nleq S^{\lambda\perp}$, then $S^\lambda \cap S^{\lambda\perp}$ is the unique maximal submodule of $S^\lambda$. This leads to

**(2.10) THEOREM.** (4.9 [James (1978b)]) *Let $\lambda$ be a partition of $n$. Then $S^\lambda/(S^\lambda \cap S^{\lambda\perp})$ is either zero or an irreducible $\Gamma$–module.* ∎

The $\Gamma$–module $S^\lambda/(S^\lambda \cap S^{\lambda\perp})$ is called the James module associated with the partition $\lambda$ of $n$, denoted by $J^\lambda$.

Let $\lambda$ be a partition of $n$. We can write

$$\lambda = (n^{r_n},(n-1)^{r_{n-1}},\ldots,1^{r_1}),$$

where $r_i$ is a non-negative integer, $i = 1, 2,\ldots, n$. For $m \in \mathbb{N}$, we say that $\lambda$ is row $m$-regular, if $r_i < m$ for each $i$ ; otherwise, $\lambda$ is row $m$-singular. Say $\lambda$ is column $m$-regular (singular), if $\lambda'$ is row $m$-regular (singular).

The following theorem, acknowledged as a breakthrough in the representation theory of symmetric groups during the last two decades, was proved by G. James in 1976:

**(2.11) THEOREM.** (Theorem 2 and 6 [James (1976)]) *Let $K$ be a field of characteristic $p$ ($p > 0$), $\lambda$ be a partition of $n$.*

(i) $J^\lambda \neq 0$ *if and only if* $\lambda$ *is row $p$-regular.*

(ii) *The set*

$$\{\ J^\lambda \mid \lambda \text{ is a row } p\text{-regular partition of } n\ \}$$

*is a complete set of inequivalent irreducible $\Gamma$-modules.*∎

It is profitable to find certain left ideals, of the group ring $\Gamma = K\mathfrak{S}_n$, which are isomorphic to the modules $M^\lambda$, $S^\lambda$ and $J^\lambda$ respectively. Recall that if $t$ is a $\lambda$-tableau, where $\lambda$ is a partition of $n$, then

$$\iota(Rt) = \sum_{\sigma\in Rt} \sigma,$$
$$\epsilon(Ct) = \sum_{\sigma\in Ct} \operatorname{sgn}(\sigma)\sigma.$$

Write $\beta_t = \iota(Rt)$, $\alpha_t = \epsilon(Ct)$, then the mapping

$$\begin{aligned} f:\ M^\lambda &\longrightarrow \Gamma\beta_t \\ \xi\ \underline{\overline{t}} &\longmapsto \xi\beta_t \qquad \xi \in \Gamma, \end{aligned}$$

is a $\Gamma$–isomorphism from $M^\lambda$ onto $\Gamma\beta_t$, and the restriction of $f$ to $S^\lambda$, denoted by $\hat{f}$, is a $\Gamma$–isomorphism from $S^\lambda$ onto $\Gamma\alpha_t\beta_t$. We shall call $\Gamma\alpha_t\beta_t$ the Young module associated with the $\lambda$–tableau $t$, denoted by $Y(t)$. If $t_1$ and $t_2$ are $\lambda$–tableaux, $t_2 = \pi t_1$, $\pi \in \mathfrak{S}_n$, then

$$\epsilon(Ct_2)\iota(Rt_2) = \pi\ \epsilon(Ct_1)\pi^{-1}\pi\ \iota(Rt_1)\pi^{-1} = \pi\ \epsilon(Ct_1)\ \iota(Rt_1)\pi^{-1}.$$

The mapping

$$\xi \longmapsto \xi\,\pi^{-1}$$

is a $\Gamma$–isomorphism from $Y(t_1)$ onto $Y(t_2)$.

The fact that the James modules are also isomorphic to some left ideals of $\Gamma = K\mathfrak{S}_n$ was first proved in §2 [Farahat and Peel (1980)].

**(2.12) THEOREM.** *Let $K$ be a field of characteristic $p$ ($p > 0$), and $\lambda$ be a partition of $n$. Assume that $t$ is a $\lambda$–tableau.*

(1) *Either $\Gamma\alpha_t\beta_t\alpha_t$ is zero, or $\Gamma\alpha_t\beta_t$ has the unique maximal submodule*

*and* $\Gamma\alpha_t\beta_t\alpha_t$ *is isomorphic to the corresponding irreducible factor module.*

(2) $\Gamma\alpha_t\beta_t\alpha_t$ *is non-zero if and only if* $\lambda$ *is a row* $p$*-regular partition of* $n$. ∎

Combining (2.9) and (2.12), we have

**(2.13) COROLLARY.** *If* $t$ *is a* $\lambda$*-tableau,* $\lambda$ *is a partition of* $n$*, then*

$$J^{\lambda} \cong \Gamma\alpha_t\beta_t\alpha_t. \ ∎$$

At this stage, we can find two isomorphic copies for the dual of $S^{\lambda}$, when $\lambda$ is a partition of $n$ :

**(2.14) LEMMA.** *Let* $t$ *be a* $\lambda$*-tableau.*

(a) $S^{\lambda *} \cong M^{\lambda}/S^{\lambda\perp}$.

(b) $S^{\lambda *} \cong \Gamma\iota(Rt)\ \epsilon(Ct)$.

**PROOF.** (a) is a straightforward application of (2.1) (iv). For proving (b), recall that

$$T : \sum_{\sigma\in\mathfrak{S}_n} \alpha_{\sigma}\, \sigma \longmapsto \sum_{\sigma\in\mathfrak{S}_n} \alpha_{\sigma}\, \sigma^{-1}$$

is an anti-automorphism of the group algebra $\Gamma = K\mathfrak{S}_n$, and

$$T[\iota(Rt)] = \iota(Rt),$$
$$T[\epsilon(Ct)] = \epsilon(Ct),$$

since $\epsilon(\sigma) = \epsilon(\sigma^{-1})$, $\sigma \in \mathfrak{S}_n$. Note that

$$T[\epsilon(Ct)\ \iota(Rt)] = T[\iota(Rt)]\ T[\epsilon(Ct)] = \iota(Rt)\ \epsilon(Ct).$$

By (2.2)

$$Y(t)^* = [\Gamma\epsilon(Ct)\ \iota(Rt)]^* \cong \Gamma\ T[\epsilon(Ct)\ \iota(Rt)] = \Gamma\iota(Rt)\ \epsilon(Ct).\blacksquare$$

**(2.15) REMARKS.** By the method of the proof in (2.14), one can easily show that

$$\Gamma\beta_t \cong (\Gamma\beta_t)^*,$$
$$\Gamma\alpha_t \cong (\Gamma\alpha_t)^*,$$
$$\Gamma\alpha_t\beta_t\alpha_t \cong (\Gamma\alpha_t\beta_t\alpha_t)^*.$$

Thus the permutation module $M^\lambda$ and the James module $J^\lambda$ are self dual.$\blacksquare$

Let $G$ be a finite group and $K$ be a field. For each linear representation of $G$ :

$$\gamma : \quad G \longrightarrow K,$$

there is a $K$–algebra automorphism

$$\begin{array}{rccc} \Phi_\gamma : & KG & \longrightarrow & KG \\ & \sum\limits_{\sigma\in\mathfrak{S}_n} \alpha_\sigma\, \sigma & \longmapsto & \sum\limits_{\sigma\in\mathfrak{S}_n} \alpha_\sigma\, \gamma(\sigma)\sigma\ . \end{array}$$

Let $M$ and $N$ be $KG$–modules. The tensor product of $M$ and $N$ over $K$, denoted by $M \otimes N$, becomes a $KG$–module such that

$$\sigma(\sum_i m_i \otimes n_i) = \sum_i (\sigma m_i) \otimes (\sigma n_i), \qquad \sigma \in G,\ m_i \in M,\ n_i \in N.$$

The following lemma was first proved in [Peel (1981)].

**(2.16) LEMMA.** *Let $\gamma : G \longrightarrow K$ be a linear representation of a finite group $G$ over a field $K$, such that $\gamma(\sigma) = \gamma(\sigma^{-1})$, $\sigma \in G$. If $\xi \in KG$, then*

$$KG\xi \cong KG\Phi_\gamma(\xi) \otimes KG\gamma(G)$$

*as $KG$–modules.*

**SKETCH OF PROOF.** Define a mapping

$$\begin{aligned} f :\ KG &\longrightarrow KG\Phi_\gamma(\xi) \otimes KG\gamma(G) \\ \eta &\longmapsto \eta\,[\,\Phi_\gamma(\xi) \otimes \gamma(\,G)\,]\,. \end{aligned}$$

Check that $f$ is a $KG$–epimorphism and $\mathrm{Ker}(f) = \ell.\mathrm{Ann}_{KG}(\xi)$, where

$$\ell.\mathrm{Ann}_{KG}(\xi) = \{\ \eta \mid \eta \in KG,\ \eta\xi = 0\ \}. \blacksquare$$

Let $\lambda'$ be the conjugate partition of $\lambda$. Clearly

$$(i,j\,) \in [\lambda'] \quad \Leftrightarrow \quad (\,j,i) \in [\lambda].$$

For a $\lambda$–tableau $t$, let $t'$ be the $\lambda'$–tableau such that

**(2.17)** $t'_{ij} = t_{ji}$, $(i,j\,) \in [\lambda']$.

It can be easily seen that

$$Rt = Ct',$$
$$Ct = Rt'.$$

$t'$ is called the conjugate tableau of $t$. The relationship between the Young modules $Y(t)$ and $Y(t')$ was discovered by M. Peel in 1981 :

**(2.18) PROPOSITION.** [Peel (1981)] *Let $t$ be a $\lambda$-tableau, $t'$ be the conjugate tableau of $t$, where $\lambda$ is a partition of $n$. Then*

$$Y(t') \cong Y(t)^{*} \otimes \Gamma\epsilon(\mathfrak{S}_n).$$

**PROOF.** Recall that

$$\epsilon(\sigma) = \begin{cases} 1, & \text{if } \sigma \in \mathfrak{A}_n, \\ -1, & \text{if } \sigma \in \mathfrak{S}_n \backslash \mathfrak{A}_n, \end{cases}$$

is a linear character of $\mathfrak{S}_n$. Apply (2.16),

$$\begin{aligned} Y(t') &= \Gamma\epsilon(\,Ct')\,\iota(\,Rt') \\ &\cong \Gamma\Phi_{\epsilon}[\epsilon(\,Ct')\,\iota(\,Rt')] \otimes \Gamma\epsilon(\mathfrak{S}_n) \\ &= \Gamma\iota(\,Ct')\epsilon(\,Rt') \otimes \Gamma\epsilon(\mathfrak{S}_n) \\ &= \Gamma\iota(\,Rt)\epsilon(\,Ct) \otimes \Gamma\epsilon(\mathfrak{S}_n). \end{aligned}$$

Thus $Y(t') \cong Y(t)^{*} \otimes \Gamma\epsilon(\mathfrak{S}_n)$ by (2.14) (b) and the fact $Y(t) \cong S^{\lambda}$ stated in the notes following (2.11).∎

**(2.19) REMARKS.** Applying the same method in the proof of (2.18), one obtains

$$\Gamma\iota(Rt) \cong \Gamma\epsilon(Rt) \otimes \Gamma\epsilon(\mathfrak{S}_n),$$
$$\Gamma\epsilon(Ct)\iota(Rt)\epsilon(Ct) \cong \Gamma\iota(Ct)\epsilon(Rt)\iota(Ct) \otimes \Gamma\epsilon(\mathfrak{S}_n)\ .$$

Also, as a corollary of (2.18), we have

$$S^{\lambda'} \cong S^{\lambda *} \otimes \Gamma\epsilon(\mathfrak{S}_n)$$

for each partition $\lambda$ of $n$.∎

At the end of this section, we are to state the result concerning the $K$–dimension and the $K$–bases for the Specht modules.

Let $\lambda$ be a partition of $n$. Say a $\lambda$–tableau $x$ is standard, if

$$x(i,j) < x(i,j+1),$$
$$x(i,j) < x(i+1,j)$$

whenever $(i,j)$, $(i,j+1)$ and $(i+1,j)$ belong to the $\lambda$–diagram $[\lambda]$.

**(2.20) THEOREM.** [3.5. Peel (1975)] *The set*

$$\{\ \epsilon(Cx)\ \underline{\bar{x}}\ |\ x \text{ is a standard } \lambda\text{–tableau}\ \}$$

*forms a $K$–basis for the Specht module $S^{\lambda}$.*∎

# CHAPTER 3
# INCIDENCE MATRICES AND HOMOMORPHISMS

## §3A Homomorphisms

In this section, we shall construct a $K$–basis for the $K$–space $\mathrm{Hom}_\Gamma(M^\lambda, M^\mu)$, where $\lambda$ and $\mu$ are compositions of $n$. A general treatment concerning homomorphisms between permutation modules can be found in §10 [Curtis and Reiner (1981)].

**(3.1) DEFINITION.** Let $\lambda$ and $\mu$ be compositions of $n$. A matrix $M = (m_{ij})$, $(i,j) \in \mathbb{N}\times\mathbb{N}$, is called a $(\lambda,\mu)$–incidence matrix, if $m_{ij}$ is a non–negative integer, such that

$$\sum_i m_{ij} = \lambda_j\,, \qquad \sum_j m_{ij} = \mu_i$$

for all $(i,j)$ in $\mathbb{N}\times\mathbb{N}$. The set of $(\lambda,\mu)$–incidence matrices is denoted by $\mathfrak{M}(\lambda,\mu)$.

Let $x$ be a $\lambda$–tableau, and

$$\underline{\bar{x}} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \end{bmatrix}$$

be the corresponding $\lambda$–tabloid, viewed as a column vector with entries subsets of $\underline{n}$ . For a $(\lambda,\mu)$–incidence matrix $M = (m_{ij})$, denote by $(M, \underline{\bar{x}}\,)$ the set of $\mu$–tabloids in the form

$$\begin{bmatrix} X_{11} \cup X_{12} \cup \ldots \\ X_{21} \cup X_{22} \cup \ldots \\ \vdots \end{bmatrix}$$

where for all $(i,j)$,

$$\bigcup_i X_{ij} = X_j\ , \qquad |X_{ij}| = m_{ij}.$$

That is to say, a $\mu$–tabloid $\bar{\underline{y}} = [Y_1,\ Y_2,\ \dots\ ]^T$ belongs to $(M,\ \bar{\underline{x}}\ )$ if and only if for each $i$, $i \geq 1$, $Y_i$ has $m_{i1}$ elements coming from $X_1$, $m_{i2}$ elements coming from $X_2$, ... , etc.

**(3.2) EXAMPLE.** When $\lambda = (3,2)$ and $\mu = (4,1)$, there are two $(\lambda,\mu)$–incidence matrices:

$$M = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}, \qquad N = \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}.$$

Take a (3,2)–tableau $x = \begin{matrix} 1 & 2 & 3 \\ 4 & 5 & \end{matrix}$ , then the first column $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ of $M$ indicates that 1, 2 and 3, the elements in the first row of $x$, all lie in the first row in each of the (4,1)–tabloid in $(M,\ \bar{\underline{x}}\ )$; while 4 and 5, the elements in the second row of $x$, are sent to two different rows in all possible ways to obtain all the (4,1)–tabloids in $(M,\ \bar{\underline{x}}\ )$, according to the second column $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ of $M$. Thus

$$\left(M,\ \begin{array}{l} \overline{1\ 2\ 3} \\ \underline{4\ 5} \end{array}\right) = \left\{\ \begin{array}{l} \overline{1\ 2\ 3\ 4} \\ \underline{5} \end{array}\ ,\ \begin{array}{l} \overline{1\ 2\ 3\ 5} \\ \underline{4} \end{array}\ \right\}.$$

Similarly we have

$$\left(N,\ \begin{array}{l} \overline{1\ 2\ 3} \\ \underline{4\ 5} \end{array}\right) = \left\{\ \begin{array}{l} \overline{1\ 2\ 4\ 5} \\ \underline{3} \end{array}\ ,\ \begin{array}{l} \overline{1\ 3\ 4\ 5} \\ \underline{2} \end{array}\ ,\ \begin{array}{l} \overline{2\ 3\ 4\ 5} \\ \underline{1} \end{array}\ \right\}.\ \blacksquare$$

The set $(M,\ \bar{\underline{x}}\ )$ is in fact an $Rx$–orbit of $\mu$–tabloids. Some useful facts concerning the $(\lambda,\mu)$–incidence matrices are listed in the following

**(3.3) OBSERVATION.** *Let $x$ be a $\lambda$–tableau.*

(i) *Each $\mu$–tabloid falls into a set $(M, \overline{\underline{x}})$ for some $(\lambda,\mu)$–incidence matrix $M$.*

(ii) *The set $(M, \overline{\underline{x}})$ is an $Rx$–orbit of $\mu$–tabloids, where $Rx$ is the row stabilizer of $x$, for each $M$ in $\mathfrak{M}(\lambda,\mu)$.*

(iii) *For $\pi \in \mathfrak{S}_n$ and $M \in \mathfrak{M}(\lambda,\mu)$, we always have*

$$(M, \pi\overline{\underline{x}}) = \pi(M, \overline{\underline{x}}).$$

(iv) *$(M, \overline{\underline{x}})$ and $(N, \overline{\underline{x}})$ are distinct $Rx$–orbits if $M \neq N$.*

**PROOF.** (i) Let

$$\overline{\underline{y}} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \end{bmatrix}$$

be an arbitrary $\mu$–tabloid. Put $X_{ij} = Y_i \cap X_j$, for all $i$ and $j$, then

$$Y_i = \bigcup_j X_{ij}, \qquad X_j = \bigcup_i X_{ij}.$$

Define a matrix $M = (m_{ij})$ via $m_{ij} = |X_{ij}|$, for all $(i,j)$ in $\mathbb{N}\times\mathbb{N}$. Then

$$\overline{\underline{y}} \in (M, \overline{\underline{x}}).$$

(ii) If $\overline{\underline{y}}$, $\overline{\underline{z}}$ are in $(M, \overline{\underline{x}})$, then

$$|Y_i \cap X_j| = m_{ij} = |Z_i \cap X_j|$$

for all $i$ and $j$. Thus there exists $\sigma$ in $Rx$, such that $\sigma Y_i = Z_i$ for each $i$, hence $\sigma\overline{\underline{y}} = \overline{\underline{z}}$. Conversely, if $\sigma \in Rx$, it is clear that

$$|\sigma Y_i \cap X_j| = |Y_i \cap X_j| = m_{ij},$$

therefore $\sigma\underline{\bar{y}} \in (M, \underline{\bar{x}})$, whenever $\sigma \in Rx$.

(iii) Take $\underline{\bar{y}} \in (M, \underline{\bar{x}})$, then

$$\underline{\bar{y}} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \end{bmatrix},$$

where $|Y_i \cap X_j| = m_{ij}$ for all $i$ and $j$. Thus

$$\pi\underline{\bar{y}} = \begin{bmatrix} \pi Y_1 \\ \pi Y_2 \\ \vdots \end{bmatrix},$$

where $\pi \in \mathfrak{S}_n$, and

$$(\pi Y_i) \cap (\pi X_j) = \pi(Y_i \cap X_j)$$

has cardinality $m_{ij}$ for all $i$ and $j$. Thus $\pi\underline{\bar{y}} \in (M, \pi\underline{\bar{x}})$. This shows that $\pi(M, \underline{\bar{x}}) \subseteq (M, \pi\underline{\bar{x}})$. The inclusion in the other direction can be proved by the same argument. The proof of (iv) is omitted.∎

**(3.4) PROPOSITION.** *Let $P$ be a $(\lambda,\mu)$–incidence matrix. There is a $\Gamma$–homomorphism*

$$\varphi_P\colon\ M^\lambda \longrightarrow M^\mu,$$

*such that for any $\lambda$–tableau $x$*

$$\varphi_P(\underline{\bar{x}}) = \Sigma\{\underline{\bar{y}} \mid \underline{\bar{y}} \in (P, \underline{\bar{x}})\}.$$

**PROOF.** First we fix an arbitrary $\lambda$–tableau $z$. According to (3.3) (ii) above, there exists a $\Gamma$–homomorphism

$$\varphi_P\colon\ M^\lambda \longrightarrow M^\mu$$

such that

$$\varphi_P( \bar{z} ) = \Sigma \{ \bar{y} \mid \bar{y} \in (P, \bar{z} ) \}.$$

Let $x$ be a $\lambda$–tableau. Then we can find $\pi \in \mathfrak{S}_n$, such that $x = \pi z$. Thus by (3.3)(iii),

$$\begin{aligned}\varphi_P( \bar{x} ) = \varphi_P( \pi\bar{z} ) = \pi\varphi_P( \bar{z} ) &= \pi\Sigma \{ \bar{y} \mid \bar{y} \in (P, \bar{z} ) \} \\ &= \Sigma \{ \bar{y} \mid \bar{y} \in \pi(P, \bar{z} ) \} \\ &= \Sigma \{ \bar{y} \mid \bar{y} \in (P, \bar{x} ) \}.\blacksquare\end{aligned}$$

**(3.5) PROPOSITION.** *Let $\lambda$ and $\mu$ be compositions of $n$. Then the set*

$$\{ \varphi_P \mid P \in \mathfrak{M}(\lambda,\mu) \}$$

*forms a $K$–basis for the $K$–space* $\mathrm{Hom}_\Gamma(M^\lambda,M^\mu)$.

**PROOF.** The above set is linearly independent over $K$ because of (3.3)(iv). Let $\varphi$: $M^\lambda \longrightarrow M^\mu$ be a $\Gamma$–homomorphism, $x$ be a $\lambda$–tableau. Write

$$\varphi( \bar{x} ) = \Sigma\ \alpha( \bar{y} )\bar{y}, \qquad \alpha( \bar{y} ) \in K,$$

where the sum is taken over all $\mu$–tabloids $\bar{y}$. We shall show that if $\bar{y}$ and $\bar{z}$ are $\mu$–tabloids and $\bar{z} = \sigma\bar{y}$, $\sigma \in Rx$, then $\alpha( \bar{y} ) = \alpha( \bar{z} )$. We have

$$\begin{aligned}\alpha( \bar{z} ) = < \varphi( \bar{x} ), \bar{z} > &= < \varphi( \bar{x} ), \sigma\bar{y} > = < \sigma^{-1}\varphi( \bar{x} ), \bar{y} > \\ &= < \varphi(\sigma^{-1}\bar{x} ), \bar{y} > = < \varphi( \bar{x} ), \bar{y} > = \alpha( \bar{y} ).\end{aligned}$$

Thus $\bar{y}$ and $\bar{z}$ have their coefficients equal in $\varphi( \bar{x} )$, whenever they are in the same $Rx$–orbit. This proves $\varphi$ is a $K$–linear combination of $\varphi_P$, $P \in \mathfrak{M}(\lambda,\mu)$. ∎

**(3.6) REMARKS.**

(i) In practice, there is no harm in writing a $(\lambda,\mu)$–incidence matrix as a row– and column– finite matrix according to the number of non–zero parts of $\lambda$ and $\mu$. Also, it is convenient to identify a $(\lambda,\mu)$–incidence matrix $P$ and the $\Gamma$–homomorphism $\varphi_P$ given in (3.4) above without much risk of ambiguity. Thus, we shall say that $\mathfrak{M}(\lambda,\mu)$ is a $K$–basis for $\mathrm{Hom}_\Gamma(M^\lambda, M^\mu)$, and for $P$ in $\mathfrak{M}(\lambda,\mu)$, we write

$$P(\ \underline{\bar{x}}\ ) = \Sigma\ \{\ \underline{\bar{z}}\ |\ \underline{\bar{z}} \in (P,\ \underline{\bar{x}}\ )\ \}.$$

(ii) Let $x$ and $y$ be $\lambda$– and $\mu$–tableaux respectively. There is a one–to–one correspondence between the set of $Rx$–$Ry$ double coset representatives in $\mathfrak{S}_n$ and the set $\mathfrak{M}(\lambda,\mu)$. For $\pi \in \mathfrak{S}_n$, define a matrix $P^\pi = (p_{ij})$ in the following manner:

$$p_{ij} = |\pi Y_i \cap X_j|, \qquad (i.j) \in \mathbb{N}\times\mathbb{N}.$$

If $\pi$, $\pi' \in \mathfrak{S}_n$, then $P^\pi = P^{\pi'}$ if and only if for all $i$ and $j$ ,

$$|\pi Y_i \cap X_j\ | = |\pi' Y_i \cap X_j\ |,$$

if and only if $\pi' y = \sigma\pi y$ for some $\sigma \in Rx$, if and only if

$$\pi' = \sigma\pi\tau, \qquad \sigma \in Rx,\ \tau \in Ry. ∎$$

In §9 [James (1977a)], the Specht module $S^\lambda$, where $\lambda$ is a partition of

$n$, is characterized as the intersection of some particular $\Gamma$–homomorphisms from $M^\lambda$ to other permutation modules. Those homomorphisms, which are called $\psi$–maps by James, will play very important roles in our discussion. We shall give the definition of those maps in our language, which is equivalent to the one given by James in §8 [James (1977a)].

Let $\lambda = (\lambda_1,\lambda_2,\ldots)$ be a composition of $n$. Note that the $(\lambda,\lambda)$–incidence matrix

$$\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix}$$

represents the identity mapping of the $\Gamma$–module $M^\lambda$. Let $k$ and $l$ be two distinct positive integers. If $0 < w \leq \lambda_k$ , denote by $(k \xrightarrow{w} l)$ the $(\lambda,\mu)$–incidence matrix

$$\left[\begin{array}{ccc|ccccc|ccc}
\lambda_1 & & & & & & & & & & \\
& \ddots & & & & & & & & & \\
& & \lambda_{k-1} & & & & & & & & \\
\hline
& & & \lambda_k - w & & \cdots & & 0 & & & \\
& & & & \lambda_{k+1} & & & & & & \\
& & & & & \ddots & & & & & \\
& & & & & & \lambda_{l-1} & & & & \\
& & & w & & \cdots & & \lambda_l & & & \\
\hline
& & & & & & & & \lambda_{l+1} & & \\
& & & & & & & & & \ddots & \\
\end{array}\right]$$

in case $k < l$, and with the similar convention if $k > l$.

By (3.4), we have immediately if $x$ is a $\lambda$–tableau and $\underline{\bar{x}} = [X_1,X_2,\ldots]^{\mathrm{T}}$ is the corresponding tabloid,

$$(3.7)\ (k \xrightarrow{w} l)\ \underline{\bar{x}} = \sum_{\substack{W \subseteq X_k \\ |W| = w}} [X_1,\dots,X_{k-1},\ X_k \setminus W,\ X_{k+1},\dots,X_{l-1},\ X_l \cup W,\ X_{l+1},\dots]^T .$$

For example, if $\lambda = (3,3,2)$ and $x$ is the $\lambda$–tableau

$$\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & \end{array}$$

then

$$(2 \xrightarrow{2} 1)\ \overline{\begin{array}{l} \underline{1\ 2\ 3} \\ \underline{4\ 5\ 6} \\ \underline{7\ 8} \end{array}} = \overline{\begin{array}{l} \underline{1\ 2\ 3\ 5\ 6} \\ \underline{4} \\ \underline{7\ 8} \end{array}} + \overline{\begin{array}{l} \underline{1\ 2\ 3\ 4\ 6} \\ \underline{5} \\ \underline{7\ 8} \end{array}} + \overline{\begin{array}{l} \underline{1\ 2\ 3\ 4\ 5} \\ \underline{6} \\ \underline{7\ 8} \end{array}} .$$

**(3.8) LEMMA.** *If* $w > \lambda_k - \lambda_l$ , $\lambda_k \geq w > 0$, *then for any* $\lambda$*–tableau* $x$ ,

$$(k \xrightarrow{w} l)\ \epsilon(Cx)\ \underline{\bar{x}} = 0,$$

**PROOF.** Let $\underline{\bar{y}}$ be an arbitrary tabloid in the set

$$\left((k \xrightarrow{w} l), \underline{\bar{x}}\right).$$

Then $\underline{\bar{y}}$ is of the form

$$[\ X_1,\dots,X_{k-1},\ X_k \setminus W,\ X_{k+1},\dots,X_{l-1},\ X_l \cup W,\ X_{l+1}\ ,\dots]^T$$

where $W$ is some subset of $X_k$ with cardinality $w$. The $l$–th row of $\underline{\bar{y}}$ has $\lambda_l + w > \lambda_k$ elements, all from $X_k \cup X_l$ . The elements of the set $X_k \cup X_l$ appear in $\max\{\lambda_k, \lambda_l\}$ columns of $x$. It follows that two elements in the $l$–th row of $\underline{\bar{y}}$ , say $a$ and $b$, appear in the same column of $x$. Thus

$$[1 - (a,b)]\ \underline{\bar{y}} = 0.$$

Hence $\epsilon(Cx)$ annihilates $\overline{\underline{y}}$, since $\{1,(a,b)\}$ is a subgroup of $Cx$. As $\overline{\underline{y}}$ is arbitrary, we have

$$0 = \epsilon(Cx)(k \xrightarrow{w} l)(\ \overline{\underline{x}}\ ) = (k \xrightarrow{w} l)\epsilon(Cx)(\ \overline{\underline{x}}\ ). \blacksquare$$

If $\lambda = (\lambda_1,\lambda_2,\ldots)$ is a partition of $n$, then $\lambda_k - \lambda_l < w$ for all $k > l$ and $w > 0$. Therefore, one can easily deduce that

$$S^\lambda \subseteq \bigcap_{k>l} \bigcap_{0<w\leq\lambda_k} \mathrm{Ker}(k \xrightarrow{w} l).$$

One of James' results is the following

**(3.9) THEOREM.** (17.18 [James (1978b)]) *Let* $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_h)$ *be a partition of* $n$ *with* $\lambda_h > 0$, *then*

$$S^\lambda = \bigcap_{i=1}^{h-1} \bigcap_{0<w\leq\lambda_k} \mathrm{Ker}(i+1 \xrightarrow{w} i). \quad \blacksquare$$

This powerful theorem has many applications. For instance, if $\varphi \in \mathrm{Hom}_\Gamma(M^\lambda,M^\mu)$, where $\mu$ is a partition of $n$, the $\mathrm{Im}\varphi \subseteq S^\mu$ if and only if

$$(i+1 \xrightarrow{w} i)\varphi = 0, \qquad i = 1, 2, \ldots, 0 < w \leq \mu_{i+1}.$$

To determine the $K$–dimension of $\mathrm{Hom}_\Gamma(M^\lambda,S^\mu)$, we must calculate in general $(k \xrightarrow{w} l)P$, where $P$ is a $(\lambda,\mu)$–incidence matrix. We shall derive a formula, found by G.D. James in his proof of 24.6 [James (1978b)], by using our own machinery. Many of our later discussions and calculations are based on that formula.

Consider compositions $\lambda$ and $\mu$ of $n$. Assume that $0 < w \leq \mu_k$. Let $\nu$ be the composition defined as

$$\nu_i = \begin{cases} \mu_i & \text{if } i \neq k\ ,\ i \neq l\ ; \\ \mu_k - w, & \text{if } i = k\ ; \\ \mu_l + w, & \text{if } i = l\ . \end{cases}$$

For each $P \in \mathfrak{M}(\lambda,\mu)$, there is a diagram:

$$\begin{tikzcd} M^\lambda \arrow[r, "P"] \arrow[dr, "(k \xrightarrow{w} l)P"'] & M^\mu \arrow[d, "(k \xrightarrow{w} l)"] \\ & M^\nu \end{tikzcd}$$

Thus $(k \xrightarrow{w} l)P$ is a $\Gamma$–homomorphism from $M^\lambda$ to $M^\nu$, and we can write

$$(k \xrightarrow{w} l)P = \sum_{Q \,\in\, \mathfrak{M}(\lambda,\mu)} \alpha(Q)Q, \qquad \alpha(Q) \in K.$$

Since the mapping $(k \xrightarrow{w} l)$ keeps all the rows of a $\mu$–tabloid unchanged except the rows $k$ and $l$, we may study a two–parts composition $\mu = (\mu_1,\mu_2)$ first, and then extend the result easily to the general case.

Let $\mu = (\mu_1,\mu_2)$ be a two–parts composition of $n$, and

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \end{bmatrix}$$

be a $(\lambda,\mu)$–incidence matrix. Our goal is to find $\alpha(Q)$ in the expression

$$(2 \xrightarrow{w} 1)P = \sum_{Q \in \mathfrak{M}(\lambda,\mu)} \alpha(Q)\ Q\ .$$

Let $x$ be a $\lambda$–tableau. If $\overline{y}$ is a $\mu$–tabloid in $(P, \overline{x})$, then

$$(2 \xrightarrow{w} 1) \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \end{bmatrix} = \sum_{\substack{W \subseteq Y \\ |W| = w}} \begin{bmatrix} Y_1 \cup W \\ Y_2 \setminus W \end{bmatrix}.$$

Look at one of the resulting $\nu$–tabloid

$$\begin{bmatrix} Y_1 \cup W \\ Y_2 \setminus W \end{bmatrix}$$

where $W$ is a subset of $Y_2$ with cardinality $w$. Write

$$w_j = |\ W \cap X_j\ |, \qquad j = 1,\ 2,\ \ldots$$

Then $(w_1,w_2,\ldots)$ is a composition of the integer $w$, satisfying $0 \leq w_j \leq p_{2j}$ , for all $j$ . Thus the $\nu$–tabloid

$$\begin{bmatrix} Y_1 \cup W \\ Y_2 \setminus W \end{bmatrix}$$

belongs to the $Rx$–orbit $(Q, \overline{x})$, where

$$Q = \begin{bmatrix} p_{11} + w_1 & \cdots & p_{1j} + w_j & \cdots \\ p_{21} - w_1 & \cdots & p_{2j} - w_j & \cdots \end{bmatrix} .$$

Thus $(2 \xrightarrow{w} 1)P$ is a $\mathbb{Z}$–linear combination of the $(\lambda,\nu)$–incidence matrices of the form $Q$ above, where $(w_1,w_2, \ldots)$ is a composition of $w$ satisfying $w_j \le p_{2j}$ for all $j$. To see the (integer) coefficient of $Q$ above in the expression of $(2 \xrightarrow{w} 1)P$, we notice that in the $R\underline{x}$–orbit $(P, \bar{\underline{x}})$, there are exactly

$$\prod_j \begin{bmatrix} p_{1j}+w_j \\ w_j \end{bmatrix}$$

$\mu$–tabloids yielding the $\nu$–tabloid

$$\begin{bmatrix} Y_1 \cup W \\ Y_2 \setminus W \end{bmatrix}$$

under the action of $(2 \xrightarrow{w} 1)$. Therefore, we have

$$(2 \xrightarrow{w} 1)P = \sum \prod_j \begin{bmatrix} p_{1j}+w_j \\ w_j \end{bmatrix} \begin{bmatrix} p_{11}+w_1 & \cdots & p_{1j}+w_j & \cdots \\ p_{21}-w_1 & \cdots & p_{2j}-w_j & \cdots \end{bmatrix}$$

where the sum is over all compositions $(w_1,w_2,\ldots)$ of the positive integer $w$, satisfying $0 \le w_j \le p_{2j}$.

In general, if $\lambda$ and $\mu$ are compositions of $n$, $k$ and $l$ are positive integers, $0 < w \le \mu_k$, $P = (p_{ij})$ is a $(\lambda,\mu)$–incidence matrix, then

(3.10) FORMULA.

$$(k \xrightarrow{w} l)P = \sum \prod_j \begin{bmatrix} p_{lj}+w_j \\ w_j \end{bmatrix} \begin{bmatrix} & \vdots & & & \\ p_{l1} & + w_1 & p_{l2} & + w_2 & \cdots \\ & \vdots & & & \\ p_{k1} & - w_1 & p_{k2} & - w_2 & \cdots \\ & \vdots & & & \end{bmatrix}$$

*where the sum is over all compositions* $(w_1, w_2, \ldots)$ *of* $w$ *satisfying* $0 < w_j \leq p_{kj}$ *for all* $j$.

In particular, if $\lambda$ is a composition of $n$, $0 < w \leq \lambda_k$, $u + v = w$, we have the following diagram:

$$
\begin{array}{ccc}
M^\lambda & \xrightarrow{(k \xrightarrow{u} l)} & M^\mu \\
& \searrow & \downarrow (k \xrightarrow{v} l) \\
& & M^\nu
\end{array}
$$

where

$$
\begin{aligned}
\mu_i &= \nu_i = \lambda_i, \qquad \text{if } i \neq k,\ i \neq l; \\
\mu_k &= \lambda_k - u; \\
\mu_l &= \lambda_l + u; \\
\nu_k &= \lambda_k - w; \\
\nu_l &= \lambda_l + w.
\end{aligned}
$$

Noticing that $(k \xrightarrow{u} l)$ is the $(\lambda,\mu)$–incidence matrix

$$
\begin{bmatrix}
\lambda_1 & & & & & & & \\
& \ddots & & & & & & \\
& & \lambda_{l-1} & & & & & \\
& & & \lambda_l & \cdots & u & & \\
& & & \vdots & \lambda_{l+1} & \vdots & & \\
& & & & \ddots & & & \\
& & & 0 & \cdots & \lambda_k - u & & \\
& & & & & & \lambda_{k+1} & \\
& & & & & & & \ddots
\end{bmatrix}
$$

we can apply (3.10) above to obtain

**(3.11) COROLLARY.** $(k \xrightarrow{v} l)\,(k \xrightarrow{u} l) = \begin{bmatrix} u+v \\ v \end{bmatrix} (k \xrightarrow{u+v} l)$.∎

A reasonable question is whether any of the kernels in (3.9) can be ommitted. A discussion of which kernels are redundant when the field has characteristic 2 or zero is given in §3 [James (1976)] and 12.1 [James (1977a)]. We shall study the general case by making use of the formula in (3.11) above.

If char($K$) = 0, the binomial coefficient $\binom{u+v}{v}$ is never zero. Thus by (3.11),

$$\mathrm{Ker}(k \xrightarrow{u+v} l) \supseteq \mathrm{Ker}(k \xrightarrow{u} l),$$
$$\mathrm{Ker}(k \xrightarrow{w} l) \supseteq \mathrm{Ker}(k \xrightarrow{u} l), \qquad w \geq u.$$

we have at once

**(3.12) COROLLARY.** *When* char($K$) = 0, if $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_h)$ *is a partition of* $n$ *with* $\lambda_h > 0$

$$S^\lambda = \bigcap_{i=1}^{h-1} \mathrm{Ker}(i+1 \xrightarrow{1} i).∎$$

Now consider a field $K$ of characteristic $p$, where $p$ is a positive prime. Let

$$a = a_0 + a_1 p + \dots + a_r p^r, \qquad 0 \leq a_i < p\ ,$$
$$b = b_0 + b_1 p + \dots + b_r p^r, \qquad 0 \leq b_i < p\ .$$

The congruence $(1 + x)^p \equiv 1 + x^p \pmod{p}$ easily gives the following

**(3.13) LEMMA.** $\begin{bmatrix} a \\ b \end{bmatrix} \equiv \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \cdots \begin{bmatrix} a_r \\ b_r \end{bmatrix} \pmod{p}.$

*In particular, $p$ divides $\begin{bmatrix} a \\ b \end{bmatrix}$ if and only if $a_i < b_i$ for some $i$.* ∎

Consider the intersection

$$\bigcap_{0<w\le\lambda_k} \operatorname{Ker}(k \xrightarrow{w} l).$$

Write

$$\lambda_k = a_0 + a_1 p + \dots + a_r p^r, \quad 0 \le a_i < p\ ,\ a_r > 0;$$
$$w = b_0 + b_1 p + \dots + b_r p^r, \quad 0 \le b_i < p.$$

Assume that $b_s > 0$, $b_j = 0$ if $0 \le j < s$. Then

$$\begin{aligned} \begin{bmatrix} w \\ p^s \end{bmatrix} &\equiv \begin{bmatrix} b_s \\ 1 \end{bmatrix} \begin{bmatrix} b_{s+1} \\ 0 \end{bmatrix} \cdots \begin{bmatrix} b_r \\ 0 \end{bmatrix} \pmod{p}. \\ &\equiv \begin{bmatrix} b_s \\ 1 \end{bmatrix} \pmod{p} \\ &\equiv b_s \pmod{p} \end{aligned}$$

which is nonzero modulo $p$ . Therefore for each $w$, $0 < w \le \lambda_k$, there exist some $s$, $0 \le s \le r$, such that

$$\mathrm{Ker}(k \xrightarrow{w} l) \supseteq \mathrm{Ker}(k \xrightarrow{p^s} l).$$

This proves that

(3.14)
$$\bigcap_{0<w\leq\lambda_k} \mathrm{Ker}(k \xrightarrow{w} l) = \bigcap_{s=0}^{r} \mathrm{Ker}(k \xrightarrow{p^s} l) \blacksquare$$

As a summary, we have

**(3.15) PROPOSITION.** *Let* $\lambda = (\lambda_1,\lambda_2, \dots ,\lambda_h)$ *be a partition of* $n$ *with* $\lambda_h > 0$.

(i) If $\mathrm{char}(K) = 0$, then

$$S^\lambda = \bigcap_{i=1}^{h-1} \mathrm{Ker}(i+1 \xrightarrow{1} i).$$

(ii) *If* $\mathrm{char}(K) = p > 0$,

$$S^\lambda = \bigcap_{i=1}^{h-1} \bigcap_{j} \mathrm{Ker}(i+1 \xrightarrow{j} i)$$

*where the second intersection is taken over all* $j \leq \lambda_{i+1}$ *, such that* $j$ *is a power of* $p$.$\blacksquare$

## §3B Adjoint Maps

Recall that for each composition $\lambda$ of $n$, a non–degenerate, symmetric $K$–bilinear form is defined on the permutation module $M^\lambda$ via

$$< \overline{\underline{x}} \, , \, \overline{\underline{y}} > \; = \begin{cases} 1, & \text{if } \overline{\underline{x}} = \overline{\underline{y}} \, , \\ 0, & \text{otherwise,} \end{cases}$$

where $\overline{\underline{x}}$ and $\overline{\underline{y}}$ are $\lambda$–tabloids, and the submodule of $M^\lambda$ which is orthogonal to $S^\lambda$ is given by

$$S^{\lambda\perp} = \{ \; m \in M^\lambda \;\; | \; < \mathrm{m}, \, \mathrm{m}' > \; = \; 0, \; (\forall \mathrm{m}' \in S^\lambda) \; \}$$

(c.f. §2C). In §3 [James (1977b)], $S^{\lambda\perp}$ is characterized as the sum of images of $\Gamma$–homomorphisms from some permutation modules into $M^\lambda$, where $\Gamma$ = $K\mathfrak{S}_n$. In order to study the $K$–space

$$\mathrm{Hom}_\Gamma(M^\lambda/S^{\lambda\perp}, M^\lambda),$$

we shall use the concept of adjoint maps, which will provide a shorter proof of Theorem 2 in [James (1977b)], and make a comparison between the $K$–spaces $\mathrm{Hom}_\Gamma(M^\lambda, S^\lambda)$ and $\mathrm{Hom}_\Gamma(M^\lambda/S^{\lambda\perp}, M^\lambda)$.

In general, let us consider $KG$–modules $M_1$ and $M_2$ (here we allow $M_i$ to be a finite or infinite dimensional $K$–space), where $K$ is an arbitrary field and $G$ is a finite group. Assume that a non–degenerate, symmetric $K$–bilinear form $< \; , \; >_i$ is defined on $M_i$, which is $G$–invariant, i.e. satisfying

$$< \xi m,\ m' >_i \ = \ < m,\ \xi^\star m' >_i$$

for $m,\ m' \in M_i$, $\xi = \sum_{\sigma\in G} \alpha_\sigma \sigma$, $\xi^\star = \sum_{\sigma\in G} \alpha_\sigma \sigma^{-1}$ in $KG$. Let $\phi$: $M_1 \longrightarrow M_2$ be a $K$–homomorphism. A $K$–homomorphism $\psi$: $M_2 \longrightarrow M_1$ is called an adjoint of $\phi$, if the following equality holds for all $m_i$ in $M_i$, $i = 1, 2$:

$$< \phi(m_1),\ m_2 >_2 \ = \ < m_1,\ \psi(m_2) >_1.$$

We shall state the following basic facts on the adjoints without proofs:

(3.16) FACTS.

(i) *If* $\phi$: $M_1 \longrightarrow M_2$ *has an adjoint, then it is unique. The adjoint of* $\phi$ *is denoted by* $\phi^A$.

(ii) *If both* $M_1$ *and* $M_2$ *are finite dimensional* $K$*–spaces, then each* $K$*–homomorphism* $\phi$: $M_1 \longrightarrow M_2$ *has its adjoint.*

(iii) *If* $M_3$ *is also a* $KG$*–module equipped with a non–degenerate,* $G$*–invariant, symmetric* $K$*–bilinear form* $< \ , \ >_3$, *and*

$$\phi:\ M_1 \longrightarrow M_2\ , \qquad \psi:\ M_2 \longrightarrow M_3$$

*are* $K$*–homomorphisms, such that* $\phi^A$, $\psi^A$ *and* $(\psi\phi)^A$ *exist, then*

$$(\psi\phi)^A = \phi^A\psi^A. \blacksquare$$

We are interested in the case $\phi$: $M_1 \longrightarrow M_2$ is a $KG$–homomorphism.

**(3.17) LEMMA.** *If $\phi$: $M_1 \longrightarrow M_2$ is a $KG$–homomorphism and $\phi^A$ exists, then $\phi^A$ : $M_2 \longrightarrow M_1$ is also a $KG$–homomorphism.*

**PROOF.** It suffices to show that for all $m_1 \in M_1$, $m_2 \in M_2$, $\xi \in KG$,

$$< m_1,\ \phi^A(\xi m_2) >_1 = < m_1,\ \xi\phi^A(m_2) >_1$$

But

$$\begin{aligned} < m_1,\ \phi^A(\xi m_2) >_1 &= < \phi(m_1),\ \xi m_2 >_2 \\ &= < \xi^\star\phi(m_1),\ m_2 >_2 \\ &= < \phi(\xi^\star m_1),\ m_2 >_2 \\ &= < \xi^\star m_1,\ \phi^A(m_2) >_1 \\ &= < m_1,\ \xi\phi^A(m_2) >_1. \blacksquare \end{aligned}$$

Before working on the family of permutation modules

$$\{\ M^\lambda \mid \lambda \text{ is a composition of } n\ \}$$

we shall try to unify the $K$–bilinear forms on $M^\lambda$'s by defining a $\Gamma$–module with each $M^\lambda$ being a direct summand, where $\Gamma = K\mathfrak{S}_n$. Fix an integer $n$. Let $M_n$ be the $K$–space with the following set as a $K$–basis:

$$\bigcup_\lambda \{\ \bar{\underline{x}} \mid \bar{\underline{x}} \text{ is a } \lambda\text{–tabloid }\},$$

the union is taken over all compositions of $n$. The $\mathfrak{S}_n$–action on the above set is defined via

$$\sigma \begin{bmatrix} X_1 \\ X_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \sigma X_1 \\ \sigma X_2 \\ \vdots \end{bmatrix}, \qquad \sigma \in \mathfrak{S}_n,$$

which makes $M_n$ a $\Gamma$–module by $K$–linearity. It is clear that $M_n$ is the internal direct sum of all the permutation modules $M^\lambda$, where $\lambda$ is a composition of $n$ :

$$M_n = \oplus \{ M^\lambda \mid \lambda \text{ is a composition of } n \}.$$

Define a $K$–bilinear form on $M_n$, by setting

$$< \underline{\bar{x}} , \underline{\bar{y}} > = \begin{cases} 1, & \text{if } \underline{\bar{x}} = \underline{\bar{y}} , \\ 0, & \text{otherwise.} \end{cases}$$

and extending this to $M_n$ by $K$–linearity. It can be easily shown that $< , >$ above is $\mathfrak{S}_n$–invariant, non–degenerate, symmetric. Also, we can see that

$$< M^\lambda, M^\mu > = 0, \qquad \text{whenever } \lambda \neq \mu.$$

The bilinear form on $M^\lambda$ defined in §2C is the restriction of the one above to the $\Gamma$–submodule $M^\lambda$ of $M_n$ .

Let $\lambda$ and $\mu$ be compositions of $n$. We have seen in §3A that the sets of incidence matrices $\mathfrak{M}(\lambda,\mu)$ and $\mathfrak{M}(\mu,\lambda)$ are $K$–bases for $\mathrm{Hom}_\Gamma(M^\lambda,M^\mu)$ and $\mathrm{Hom}_\Gamma(M^\mu,M^\lambda)$ respectively. The following lemma reveals that the natural bijection

$$P \in \mathfrak{M}(\lambda,\mu) \longmapsto P^{\mathrm{T}} \in \mathfrak{M}(\mu,\lambda)$$

is identical to the map $P \longmapsto P^{\mathrm{A}}$:

**(3.18) LEMMA.** *Each* $P \in \mathfrak{M}(\lambda,\mu)$ *has its adjoint, and* $P^{\mathrm{A}} = P^{\mathrm{T}} \in \mathfrak{M}(\mu,\lambda)$.

**PROOF.** It is enough to show that

$$< P(\, \bar{\underline{x}} \,),\ \bar{\underline{y}} > \; = \; < \bar{\underline{x}},\ P^{\mathrm{T}}(\, \bar{\underline{y}} \,) >,$$

for arbitrary $\lambda$–tabloids $\bar{\underline{x}} = [\ X_1,\ X_2,\ \dots\ ]^{\mathrm{T}}$, $\mu$–tabloids $\bar{\underline{y}} = [\ Y_1,\ Y_2,\ \dots\ ]^{\mathrm{T}}$. Recall that

$$P(\, \bar{\underline{x}} \,) = \Sigma \left\{ \bar{\underline{z}} = [\ Z_1,\ Z_2,\ \dots\ ]^{\mathrm{T}} \ \middle|\ |\ Z_i \cap X_j\ | = p_{ij} \quad (\forall i,j) \right\} .$$

Thus

$$< P(\, \bar{\underline{x}} \,),\ \bar{\underline{y}} > \; = \begin{cases} 1, & \text{if } |\ Y_i \cap X_j\ | = p_{ij}\ , \text{ for all } i,\ j, \\ 0, & \text{otherwise.} \end{cases}$$

Meanwhile,

$$< \bar{\underline{x}},\ P^{\mathrm{T}}(\, \bar{\underline{y}} \,) > \; = \begin{cases} 1, & \text{if } |\ X_i \cap Y_j\ | = p_{ji}\ , \text{ for all } i,\ j, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} 1, & \text{if } |\ Y_i \cap X_j\ | = p_{ij}\ , \text{ for all } i,\ j, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \; < P(\, \bar{\underline{x}} \,),\ \bar{\underline{y}} >. \blacksquare$$

**(3.19) COROLLARY.** *The adjoint of*

$$\sum_{P \in \mathfrak{M}(\lambda,\ \mu)} \alpha(P)P :\ M^{\lambda} \longrightarrow M^{\mu}, \qquad \alpha(P) \in K,$$

*is* $\displaystyle\sum_{P \in \mathfrak{M}(\lambda,\mu)} \alpha(P)P^{\mathrm{T}}:\ M^{\mu} \longrightarrow M^{\lambda}.$ ∎

Let $i,j$ and $w$ be positive integers, with $j > i$. The $K$–homomorphism

$$(j \xrightarrow{w} i) :\ M_n \longrightarrow M_n$$

is defined in the following manner:

$$(j \xrightarrow{w} i)(\,\underline{\bar{x}}\,) = 0, \qquad \text{if } |\,X_j| < w\ ;$$

$$(j \xrightarrow{w} i)(\,\underline{\bar{x}}\,) = \sum_{\substack{|W|=w \\ W \subseteq X_j}} \Big[\ X_1,\ \dots\ ,\ X_{i-1},\ X_i \cup W,\ \dots\ ,\ X_j \cup W,\ X_{j+1},\ \dots\ \Big]^{\mathrm{T}}$$

if $|\,X_j| \geq w$. It can be easily verified that $(j \xrightarrow{w} i) :\ M_n \longrightarrow M_n$ is a $\Gamma$–homomorphism (comparing with the definition of $(k \xrightarrow{w} l)$ in §3A). Similarly, one can defined $(i \xrightarrow{w} j)$ when $j > i$. Let $\lambda$ be a composition of $n$, and assume that $\lambda_j \geq w$. The restriction of $(j \xrightarrow{w} i)$ to $M^{\lambda}$, still denoted by $(j \xrightarrow{w} i)$, is a $\Gamma$–homomorphism from $M^{\lambda}$ to $M^{\mu}$, where $\mu$ is the composition

$$(\lambda_1,\ \dots,\ \lambda_{i-1},\ \lambda_i+w,\ \lambda_{i+1},\dots,\ \lambda_{j-1},\ \lambda_j-w,\ \lambda_{j+1},\dots).$$

The $\Gamma$–homomorphism $(j \xrightarrow{w} i) :\ M^{\lambda} \longrightarrow M^{\mu}$ is an element in $\mathfrak{M}(\lambda,\mu)$ (c.f.

the matrix illustration of $(k \xrightarrow{w} l)$ in §3A). We have at once

$$(j \xrightarrow{w} i)^{\mathrm{T}} = (i \xrightarrow{w} j) : \; M^{\mu} \longrightarrow M^{\lambda}.$$

Thus according to the viewpoint in (3.18), we obtain

**(3.20) PROPOSITION.** *The* $\Gamma$*-homomorphisms*

$$(j \xrightarrow{w} i) : \; M^{\lambda} \longrightarrow M^{\mu},$$
$$(i \xrightarrow{w} j) : \; M^{\mu} \longrightarrow M^{\lambda}$$

*are adjoints of each other.*∎

**(3.21) REMARKS.** As $\Gamma$–endomorphisms of $M_n$, the relation

$$(j \xrightarrow{w} i)^{\mathrm{A}} = (i \xrightarrow{w} j)$$

still holds, since it is true on every direct summand $M^{\lambda}$ of $M_n$.∎

The following lemma is a restatement of Theorem 2 in [James (1977b)], but our proof is shorter due to the adoption of adjoint maps.

**(3.22) LEMMA.** *The sequence*

$$M^{\mu} \xrightarrow{(i \xrightarrow{w} j)} M^{\lambda} \xrightarrow{(j \xrightarrow{w} i)} M^{\mu}$$

*has the property*

$$\mathrm{Im}(i \xrightarrow{w} j) = [\mathrm{Ker}(j \xrightarrow{w} i)]^{\perp}.$$

**PROOF.** We have to show that $m \in \mathrm{Ker}(j \xrightarrow{w} i)$ if and only if

$$< m, m' > = 0, \quad \forall m' \in \mathrm{Im}(i \xrightarrow{w} j).$$

Note that

$$
\begin{aligned}
& m \in \mathrm{Ker}(j \xrightarrow{w} i) \\
\Leftrightarrow\quad & (j \xrightarrow{w} i)m = 0 \\
\Leftrightarrow\quad & < (j \xrightarrow{w} i)m,\ m'' > = 0 \qquad (\forall m'' \in M^{\mu}) \\
\Leftrightarrow\quad & < m,\ (i \xrightarrow{w} j)m'' > = 0 \qquad (\forall m'' \in M^{\mu}) \\
\Leftrightarrow\quad & < m,\ m' > = 0, \qquad (\forall m' \in \mathrm{Im}(i \xrightarrow{w} j)). \blacksquare
\end{aligned}
$$

By (3.9), if $\lambda$ is a partition of $n$,

$$
\begin{aligned}
S^{\lambda} &= \bigcap_{j > i} \ \bigcap_{w \geq 1} \mathrm{Ker}(j \xrightarrow{w} i) \\
&= \bigcap_{i \geq 1} \ \bigcap_{w \geq 1} \mathrm{Ker}(i{+}1 \xrightarrow{w} i)
\end{aligned}
$$

Therefore, combining the result in (3.22), we deduce that

**(3.23) PROPOSITION.** (Corollary 2 [James (1977b)]) *If $\lambda$ is a partition of $n$,*

$$
\begin{aligned}
S^{\lambda\perp} &= \sum_{j > i} \ \sum_{w \geq 1} \mathrm{Im}(i \xrightarrow{w} j) \\
&= \sum_{i \geq 1} \ \sum_{w \geq 1} \mathrm{Im}(i \xrightarrow{w} i{+}1). \blacksquare
\end{aligned}
$$

Consider the subspace of $\mathrm{End}_{\Gamma}(M^{\lambda})$:

$$
\mathrm{H}_{\lambda} = \{ \phi \in \mathrm{End}_{\Gamma}(M^{\lambda}) \mid \mathrm{Ker}(\phi) \supseteq S^{\lambda\perp} \},
$$

where $\lambda$ is a partition of $n$. It is clear that

$$\mathrm{H}_{\lambda} \cong \mathrm{Hom}_{\Gamma}(M^{\lambda}/S^{\lambda\perp},\ M^{\lambda})$$

as $K$-spaces. The K-space $\mathrm{H}_{\lambda}$ and $\mathrm{Hom}_{\Gamma}(M^{\lambda},S^{\lambda})$ are linked up in the following

**(3.24) PROPOSITION.** $\phi \in \mathrm{H}_{\lambda}$ *if and only if* $\phi^{\mathrm{A}}$, *the adjoint of* $\phi$, *belongs to* $\mathrm{Hom}_{\Gamma}(M^{\lambda},S^{\lambda})$. *In other words, the mapping*

$$\phi \in \mathrm{H}_{\lambda} \longmapsto \phi^{\mathrm{A}} \in \mathrm{Hom}_{\Gamma}(M^{\lambda},S^{\lambda})$$

*is a* $K$*-isomorphism.*

**PROOF.** Let $\phi$ be a $\Gamma$-endomorphism of $M^{\lambda}$. Then $\phi \in \mathrm{H}_{\lambda}$ if and only if

$$\phi\ (i \xrightarrow{w} i{+}1) = 0, \qquad i \geq 1,\ w \geq 1,$$

by (3.23); if and only if

$$[\phi(i \xrightarrow{w} i{+}1)]^{\mathrm{A}} = 0, \quad i \geq 1,\ w \geq 1$$

if and only if

$$(i{+}1 \xrightarrow{w} i)\phi^{\mathrm{A}} = 0,\ldots i \geq 1,\ w \geq 1$$

by (3.16)(iii); if and only if $\phi^{\mathrm{A}} \in \mathrm{Hom}_{\Gamma}(M^{\lambda},S^{\lambda})$ by (3.9).∎

## §3C Bases for $\mathrm{Hom}_\Gamma(S^\lambda, M^\mu)$

A theorem concerning $K$–bases and $K$–dimension of $\mathrm{Hom}_\Gamma(S^\lambda, M^\mu)$ was first proved in §3 [Carter and Lusztig (1974)], and the proof was modified in §13 [James (1978b)]. In this section, we shall describe the ideas of the above authors briefly and state the main results about $\mathrm{Hom}_\Gamma(S^\lambda, M^\mu)$ needed in our later discussions.

We have seen in §3A that $\mathfrak{M}(\lambda,\mu)$ forms a $K$–basis for $\mathrm{Hom}_\Gamma(M^\lambda, M^\mu)$ since we can characterize an $Rx$–orbit of $\mu$–tabloids by a $(\lambda,\mu)$–incidence matrix, where $x$ is a $\lambda$–tableau. The following sequence of definitions is essentially an alternative way of describing an $Rx$–orbit of $\mu$–tabloids.

Let $\lambda$ and $\mu$ be compositions of $n$. A function $T : [\lambda] \longrightarrow \mathbb{N}$, where $[\lambda]$ is the diagram of $\lambda$, is called a $(\lambda,\mu)$–tableau, if the cardinality of

$$\{ \ (k,l) \in [\lambda] \mid T(k,l) = i \ \}$$

is equal to $\mu_i$ for each $i$. ($T$ was called a $\lambda$–tableau of type $\mu$ by Carter, Lusztig and James in the references above.) For example, if $\lambda = (4,1)$, $\mu = (3,2)$, we define

$$T(1,1) = T(1,2) = 2,$$
$$T(1,3) = T(1,4) = T(2,1) = 1,$$

then $T$ is a $((4,1),(3,2))$–tableau, depicted as

$$\begin{array}{llll} 2 & 2 & 1 & 1 \\ 1 & & & \end{array} .$$

The set of all $(\lambda,\mu)$–tableaux is denoted by $\mathfrak{T}(\lambda,\mu)$.

A $\lambda$–tableau $x$ induces an $\mathfrak{S}_n$–action on the diagram $[\lambda]$ via

$$\sigma\cdot(i,j) = x^{-1}\sigma\, x(i,j), \qquad (i,j) \in [\lambda],\ \sigma \in \mathfrak{S}_n.$$

That is to say, $(\sigma\cdot)$: $[\lambda] \longrightarrow [\lambda]$ is the bijection which makes the following diagram commute

$$\begin{tikzcd} {[\lambda]} \arrow[r, "(\sigma\cdot)"] \arrow[d, "x"'] & {[\lambda]} \\ \underline{n} \arrow[r, "\sigma"'] & \underline{n} \arrow[u, "x^{-1}"'] \end{tikzcd}$$

An $\mathfrak{S}_n$–action on the set $\mathfrak{T}(\lambda,\mu)$ can be defined in the following manner:

$$\sigma T = T(\sigma^{-1}\cdot).$$

It is easy to check that the above action on $\mathfrak{T}(\lambda,\mu)$ is eventually the place permutation of $(\lambda,\mu)$–tableaux. For example, if

$$x = \begin{matrix} 1 & 2 & 3 & 4 \\ 5 & & & \end{matrix}$$

and $T$ is the $((4,1),(3,2))$–tableau $\begin{matrix} 2 & 2 & 1 & 1 \\ 1 & & & \end{matrix}$, then

$$(23)T = \begin{matrix} 2 & 1 & 2 & 1 \\ 1 & & & \end{matrix}$$
$$(123)T = \begin{matrix} 1 & 2 & 2 & 1 \\ 1 & & & \end{matrix}.$$

We say that $T$ and $T_1$ in $\mathfrak{T}(\lambda,\mu)$ are row equivalent, if

$$\pi T = T_1, \qquad \text{for some } \pi \in Rx.$$

In the above example, the row equivalence class of $T = \begin{smallmatrix} 2 & 2 & 1 & 1 \\ 1 \end{smallmatrix}$ is

$$\left\{ \begin{smallmatrix} 2 & 2 & 1 & 1 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 2 & 1 & 2 & 1 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 2 & 1 & 1 & 2 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 1 & 2 & 1 & 2 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 1 & 1 & 2 & 2 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 1 & 2 & 2 & 1 \\ 1 \end{smallmatrix} \right\}$$

A one–to–one correspondence between $\mathfrak{T}(\lambda,\mu)$ and the set of $\mu$–tabloids can be established as follows: fix a $\lambda$–tableau $x$, for each $T \in \mathfrak{T}(\lambda,\mu)$, let $\gamma_x(T)$ be the $\mu$–tabloid satisfying that $x_{ij}$ lies in the $T(i,j)$–th row of $\gamma_x(T)$. It is not difficult to check that $\gamma_x$ is a bijection. We shall prove that if $T \in \mathfrak{T}(\lambda,\mu)$, then

$$\{\ \gamma_x(T_1)\ |\ T_1 \text{ is row equivalent to } T\ \}$$

is an $Rx$–orbit of $\mu$–tabloids, stated in the following lemma.

**(3.25) LEMMA.** *Fix a $\lambda$–tableau $x$, let $T$, $T_1 \in \mathfrak{T}(\lambda,\mu)$, $P \in \mathfrak{M}(\lambda,\mu)$. Then*

(i) *$\gamma_x(T)$ belongs to the $Rx$–orbit $(P, \bar{\underline{x}}\ )$ if and only if the number of $i$'s in the $j$–th row of $T$ is equal to $p_{ij}$ for all $i$ and $j$.*

(ii) *$T_1$ is row equivalent to $T$ if and only if $\gamma_x(T_1)$ and $\gamma_x(T)$ belong to the same $Rx$–orbit.*

**PROOF.** It suffices to show (i). Let

$$\gamma_x(T) = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \end{bmatrix}.$$

Recall that $x_{kl}$ lies in the $T(k,l)$–th row of $\gamma_x(T)$. Thus

$$\begin{aligned} &\text{the number of } i\text{'s in the } j\text{–th row of } T \\ = {} &\text{the cardinality of } \{\, x_{jl} \mid T(j,l) = i \,\} \\ = {} &|\, Y_i \cap X_j| = p_{ij}\,, \qquad \text{for all } i \text{ and } j. \blacksquare \end{aligned}$$

**(3.26) DEFINITIONS.**

(i) A $(\lambda,\mu)$–tableau $T$ is said to be semistandard (reverse semistandard) if the numbers in the depiction of $T$ are non–decreasing (non–increasing) along the rows of $T$ and strictly increasing ( strictly decreasing) down the columns of $T$.

(ii) A $\Gamma$–homomorphism $P \in \mathfrak{M}(\lambda,\mu)$ is said to be semistandard (reverse semistandard), if for some $\lambda$–tableau $x$, $(P, \bar{\underline{x}})$ contains a $\mu$–tabloid $\gamma_x(T)$, where $T$ is semistandard (reverse semistandard). This condition is independent of the choice of $x$. Note also that $T$ is uniquely determined by $\gamma_x(T)$.

**(3.27) EXAMPLE.** Let $\lambda = (2,2)$, $\mu = (2,1^2)$, and $x$ be a $(2,2)$–tableau. There are four $(\lambda,\mu)$–incidence matrices ($Rx$–orbits):

$$\left( \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \bar{\underline{x}} \right) = \left\{ \gamma_x\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, \gamma_x\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} \right\},$$

$$\left( \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \bar{\underline{x}} \right) = \left\{ \gamma_x\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}, \gamma_x\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \gamma_x\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}, \gamma_x\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \right\},$$

$$\left( \begin{bmatrix} 0 & 2 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}, \bar{\underline{x}} \right) = \left\{ \gamma_x\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}, \gamma_x\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \right\},$$

$$\left( \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \bar{\underline{x}} \right) = \left\{ \gamma_x\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}, \gamma_x\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}, \gamma_x\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}, \gamma_x\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \right\}.$$

The $\Gamma$–homomorphism $\begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$ is semistandard, while $\begin{bmatrix} 0 & 2 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$ is reverse semistandard.∎

**(3.28) THEOREM.** (§3 [Carter and Lusztig (1974)], and §13 [James (1978b)] *Let $\lambda$ and $\mu$ be partitions of $n$, $\hat{P}$ be the restriction of $P \in \mathfrak{M}(\lambda,\mu)$ to $S^\lambda$. Unless $char(K) = 2$ and $\lambda$ is row 2–singular,*

$$\{\ \hat{P} \mid P \in \mathfrak{M}(\lambda,\mu),\ P \text{ is semistandard}\ \}$$

*forms a $K$–basis of* $\mathrm{Hom}_\Gamma(S^\lambda,M^\mu)$. *The set*

$$\{\ \hat{P} \mid P \in \mathfrak{M}(\lambda,\mu),\ P \text{ is reverse semistandard}\ \}$$

*also forms a $K$–basis of* $\mathrm{Hom}_\Gamma(S^\lambda,M^\mu)$.∎

It is sometimes handy if we have a criterion for semistandard (reverse semistandard) homomorphisms according to the features of the incidence matrices themselves. The following facts concerning reverse semistandard homomorphisms will be used in the later sections.

**(3.29) LEMMA.** *Let $\lambda$ and $\mu$ be partitions of $n$, and let $N = (n_{ij})$ be a $(\lambda,\mu)$– incidence matrix. Then $N : M^\lambda \longrightarrow M^\mu$ is a reverse semistandard homomorphism if and only if*

$$\sum_{i=k}^{\infty} (n_{i+1,j} - n_{i,j+1}) \geq 0, \qquad k = 1, 2, \dots, j = 1, 2, \dots .$$

**PROOF.** We shall prove that the above condition is necessary, the converse part can be proved by the similar argument.

Assume that $N$: $M^\lambda \longrightarrow M^\mu$ is reverse semistandard. Let $x$ be a $\lambda$–tableau and $T$ be the reverse semistandard $(\lambda,\mu)$–tableau such that $\gamma_x(T)$ belongs to $(N, \bar{\underline{x}})$. The numbers along each row of $T$ are non–increasing, and the number of $i$'s in the $j$–th row of $T$ is $n_{ij}$ (c.f. 3.25 and 3.26). The $(j+1)$–th row of $T$ is thus of the form

$$\underbrace{l \ \dots \ l}_{n_{l,j+1}} \ \underbrace{(l+1) \ \dots \ (l+1)}_{n_{l-1,j+1}} \ \dots \ \underbrace{2 \ \dots \ 2}_{n_{2,j+1}} \ \underbrace{1 \ \dots \ 1}_{n_{1,j+1}}$$

For each entry 1 (one) in the $(j+1)$–th row of $T$, there must be an entry in the same column and the $j$–th row, which has the value greater than 1 (one). Therefore, necessarily, the number of entries with values greater 1 in the $j$–th row of $T$ must be at least the number of all entries in the $(j+1)$–th row of $T$. That is

$$\sum_{i=1}^{\infty} n_{i+1,j} \geq \sum_{i=1}^{\infty} n_{i,j+1}$$

i.e.

$$\sum_{i=1}^{\infty} (n_{i+1,j} - n_{i,j+1}) \geq 0.$$

Similarly, for each entry 2 in the $(j+1)$–th row of $T$, there must be an entry with its value greater than 2 in the same column and the $j$–th row of $T$. This forces

$$\sum_{i=2}^{\infty} (n_{i+1,j} - n_{i,j+1}) \geq 0.$$

Repeating this process, we have

$$\sum_{i=k}^{\infty} (n_{i+1,j} - n_{i,j+1}) \geq 0, \qquad k = 1, 2, \dots , j = 1, 2, \dots .$$

The converse part of the proof can be done by the same argument.∎

## §3D On the Labelling of the Irreducibles

When the ground field $K$ has its characteristic $p$, $p > 0$, the irreducible $\Gamma$–modules can be labeled by row $p$–regular partitions of $n$, where $\Gamma$ is the group algebra $K\mathfrak{S}_n$, since each Specht module $S^\lambda$, with $\lambda$ row $p$–regular, has the unique maximal submodule $S^\lambda \cap S^{\lambda\perp}$, and the set of factor modules

$$\{ J^\lambda = S^\lambda/(S^\lambda \cap S^{\lambda\perp}) \mid \lambda \text{ is a row } p\text{–regular partition of } n \}$$

forms a complete set of inequivalent irreducible $\Gamma$–modules (see 2.11). By working with $S^\mu$, where $\mu$ is a column $p$–regular partition of $n$, we can find another labelling of irreducible $\Gamma$– modules, based on the following facts:

**(3.30) LEMMA.** *Every Specht module $S^\mu$, with $\mu$ a column $p$–regular partition of $n$, has the unique irreducible submodule $L^\mu$, and the set*

$$\{ L^\mu \mid \mu \textit{ is a column } p\textit{–regular partition of } n \}$$

*forms a complete set of inequivalent irreducible $\Gamma$–modules.*

**SKETCH OF THE PROOF.** Note that a partition $\lambda$ of $n$ is row $p$–regular if and only if its conjugate partition $\lambda'$ is column $p$–regular. A Specht module $S^\lambda$ has the unique maximal submodule if and only if $(S^\lambda)^*$, the dual of $S^\lambda$, has the unique irreducible submodule by (2.7); if and only if $S^{\lambda'}$ has the unique irreducible submodule, because of (2.18):

$$S^{\lambda'} \cong (S^\lambda)^* \otimes \Gamma\epsilon(\mathfrak{S}_n).$$

It is easy to see that

$$L^{\lambda'} \cong J^{\lambda} \otimes \Gamma\epsilon(\mathfrak{S}_n),$$

and thus the mapping $J^{\lambda} \longmapsto L^{\lambda'}$ is a bijection from the set

$$\{\ J^{\lambda}\ |\ \lambda \text{ is a row } p\text{–regular partition of } n\ \}$$

onto the set

$$\{\ L^{\mu}\ |\ \mu \text{ a column } p\text{–regular partition of } n\ \}. \blacksquare$$

A difficult question was raised at the end of [James (1977c)]: what is the connection between the two labellings? In the proof of the above lemma, we have seen that tensoring with $\Gamma\ \epsilon(\mathfrak{S}_n)$ yields a bijection (up to isomorphism) from the set of inequivalent irreducibles onto itself :

$$J^{\lambda} \longmapsto J^{\lambda} \otimes \Gamma\epsilon(\mathfrak{S}_n) \cong L^{\lambda'},$$

for each row $p$–regular partition of $n$. A much harder problem is the following:

**(3.31) PROBLEM.** *For each row $p$–regular partition $\lambda$ of $n$, search for a column $p$–regular partition of $n$, denoted by $\Phi_p(\lambda)$, such that $J^{\lambda}$ is isomorphic to the unique irreducible submodule $L^{\Phi_p(\lambda)}$ of $S^{\Phi_p(\lambda)}$.* $\blacksquare$

**(3.32) NOTE.** When $p = 2$, $\epsilon(\mathfrak{S}_n) = \iota(\mathfrak{S}_n)$. Thus

$$L^{\lambda'} \cong J^{\lambda} \otimes \Gamma\epsilon(\mathfrak{S}_n) = J^{\lambda} \otimes \Gamma\iota(\mathfrak{S}_n) \cong J^{\lambda},$$

and the mapping $\Phi_2$ is the trivial one:

$$\Phi_2(\lambda) = \lambda', \qquad \text{for all 2-regular } \lambda.$$

If $p > n$, for each partition $\lambda$ of $n$ the Specht module $S^\lambda$ is irreducible over $\Gamma$, hence $\Phi_p(\lambda) = \lambda$.∎

We shall give a partial solution for Problem 3.31 by showing that if $\lambda$ is a row $p$-regular partition satisfying some stronger conditions, there is an easy combinatorial way to describe $\Phi_p(\lambda)$.

Let $\lambda$ and $\mu$ be partitions of $n$. Write $\mu = (\mu_1, \mu_1, \ldots, \mu_h)$, $\mu_h > 0$. According to (3.9), a $\Gamma$-homomorphism $\phi$: $M^\lambda \longrightarrow M^\mu$ has its image contained in $S^\mu$ if and only if

$$(i+1 \xrightarrow{w} i)\phi = 0, \qquad i = 1, 2, \ldots, h-1; \quad 0 < w \leq \mu_{i+1}.$$

In some ideal cases, we can find an element $N$ in the basis $\mathfrak{M}(\lambda,\mu)$ of $\mathrm{Hom}_\Gamma(M^\lambda, M^\mu)$, satisfying

$$(i+1 \xrightarrow{w} i)N = 0, \qquad i = 1, 2, \ldots, h-1; \quad 0 < w \leq \mu_{i+1}.$$

Let $a$ and $b$ be non-negative integers. Write

$$b = b_0 + b_1 p + \ldots + b_r p^r, \qquad 0 \leq b_i < p, \ b_r > 0.$$

The integer $p^{r+1}$ is the smallest power of $p$ larger than $b$. Denote $(r+1)$ by

$\ell_p(b)$. When $b = 0$, define $\ell_p(b) = 1$. The following result can be deduced from 22.5 [James (1978 )] .

**(3.33) LEMMA.** $\binom{a+w}{w} \equiv 0 (\mathrm{mod}\ p)$ *for all* $w$, $0 < w \leq b$, *if and only if*

$$a \equiv -1 (\mathrm{mod}\ p^{\ell_p(b)}). \blacksquare$$

The following proposition, first proved by G.D. James, is our main tool in this section:

**(3.34) PROPOSITION.** (24.6 (i) [James (1978b)] *Let* $\lambda$ *and* $\mu$ *be partitions of* $n$. *If* $N = (n_{ij}) \in \mathfrak{M}(\lambda,\mu)$ *satisfies*

$$n_{ij} \equiv -1 (\mathrm{mod}\ p^{\ell_p(n_{i+1,j})}). \qquad i = 1, 2, \dots ,$$

*then* $\mathrm{Im}(N) \subseteq S^{\mu}$.

**PROOF.** Recall the formula in (3.10), for each $i$, $w$ :

$$(i{+}1 \xrightarrow{w} i)N = \sum \prod_j \left[ \begin{matrix} n_{ij}+w_j \\ w_j \end{matrix} \right] N_{(w_1,w_2,\dots)} ,$$

where the sum is taken over all compositions $(w_1,w_2,\dots)$ of $w$ satisfying

$$0 \leq w_j \leq n_{i+1,j}, \qquad \text{for all } j ,$$

and $N_{(w_1,w_2,\dots)}$ is the incidence matrix having the rows $i$ and $(i{+}1)$ of the form

$$\begin{bmatrix} n_{i1} + w_1 & n_{i2} + w_2 & \dots & n_{ij} + w_j & \dots \end{bmatrix}$$
$$\begin{bmatrix} n_{i+1,1} - w_1 & n_{i+1,2} - w_2 & \dots & n_{i+1,j} - w_j & \dots \end{bmatrix}$$

and the other rows identical to the corresponding rows of $N$. The conditions

$$n_{ij} \equiv -1(\text{mod } p^{\ell_p(n_{i+1,j})}). \qquad i = 1, 2, \dots ,$$

assures that

$$\prod_j \begin{bmatrix} n_{ij}+w_j \\ w_j \end{bmatrix} \equiv 0(\text{mod } p)$$

for all $i$, all compositions $(w_1,w_2,\dots)$ of $w$ satisfying

$$0 \le w_j \le n_{i+1,j}, \text{ for all } j ,$$

by (3.33) above. Therefore

$$(i{+}1 \xrightarrow{w} i)N = 0, \qquad \text{for all } i,\ w,$$

and it follows that $\mathrm{Im}(N) \subseteq S^{\mu}$ by (3.9).∎

G.D. James obtained the following result concerning $\mathrm{Hom}_{\Gamma}(S^{(n)}, S^{\mu})$ by applying the proposition above:

**(3.35) COROLLARY.** (24.4. [James (1978b)]) *Let $\mu = (\mu_1,\mu_2,\dots,\mu_h)$ be a partition of $n$. The trivial module $J^{(n)} = S^{(n)}$ is isomorphic to a submodule of $S^{\mu}$ if and only if*

$$\mu_i \equiv -1(\mathrm{mod}\ p^{\ell_p(\mu_{i+1})}). \qquad i = 1, 2, \ldots, h-1. \blacksquare$$

(3.34) has many other applications. For each partition $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_h)$ of $n$, with $\lambda_h > 0$, we can construct a partition $\theta_p(\lambda)$ and a $(\lambda,\theta_p(\lambda))$ incidence matrix $N^{(p,\lambda)}$, such that

$$\mathrm{Im}(\ N^{(p,\lambda)}) \subseteq S^{\theta_p(\lambda)}.$$

**(3.36) CONSTRUCTION.** *Let $\lambda = (\lambda_1,\lambda_2,\ldots)$ be a partition of $n$. Write for each $j$,*

$$\lambda_j = s_j(p-1) + r_j, \qquad 0 \le r_j < p-1.$$

*Let $N^{(p,\lambda)}$ be the integral matrix whose $j$–th column is*

$$\begin{bmatrix} p-1 \\ \vdots \\ p-1 \\ r_j \\ 0 \\ \vdots \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} s_j \\ \\ \\ \\ \end{matrix}.$$

*The row sums of $N^{(p,\lambda)}$ determines a partition of $n$, denoted by $\theta_p(\lambda)$. Clearly $N^{(p,\lambda)} \in \mathfrak{M}(\lambda,\theta_p(\lambda))$.* $\blacksquare$

According to this construction and (3.34), we have at once

**(3.37)** $\mathrm{Im}(\ N^{(p,\lambda)}) \subseteq S^{\theta_p(\lambda)}$.

When $\lambda$ is a partition satisfying certain conditions, more information about the irreducible submodules of $S^{\theta_p(\lambda)}$ can be found through the $\Gamma$–homomorphism $N^{(p,\lambda)}$. We shall prove the following combinatorial lemma:

**(3.38) LEMMA.** *Let* $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_h)$ *be a partition of* $n$ *satisfying*

$$\lambda_h > 0,$$
$$\lambda_j - \lambda_{j+1} \geq p-1, \qquad j = 1, 2, \ldots, h-1.$$

*Then*

(i) *The* $\Gamma$*–homomorphism* $N^{(p,\lambda)}: M^{\lambda} \to M^{\theta_p(\lambda)}$ *is reverse semistandard.*

(ii) $\theta_p(\lambda)$ *is a column* $p$*–regular partition of* $n$.

**PROOF.** (i) Write $N^{(p,\lambda)} = (n_{ij})$ and

$$\lambda_j = s_j(p-1) + r_j, \qquad 0 \leq r_j < p-1,\ j = 1, 2, \ldots, h.$$

For each $j$, $1 \leq j \leq h-1$, we have

$$\lambda_j - \lambda_{j+1} = (s_j - s_{j+1})(p-1) + (r_j - r_{j+1}) \geq p-1.$$

It is clear that $|r_j - r_{j+1}| < p-1$. If

$$0 \leq r_j - r_{j+1} < p-1,$$

one must have

$$(s_j - s_{j+1})(p-1) \geq (p-1) - (r_j - r_{j+1}) > 0.$$

Thus $s_j - s_{j+1} \geq 1$, it follows that the $(s_{j+1}+2,j)$–entry in $N^{(p,\lambda)}$ is either

$(p{-}1)$ or $r_j$, i.e. the columns $j$ and $(j{+}1)$ are of the form

$$\begin{array}{l} 1 \\ \vdots \\ s_{j+1} \\ \\ s_{j+1}+2 \\ \\ \end{array} \begin{bmatrix} p-1 & p-1 \\ \vdots & \vdots \\ p-1 & p-1 \\ p-1 & r_{j+1} \\ r_j & 0 \\ \vdots & \vdots \end{bmatrix}, \qquad \text{if } s_j - s_{j+1} = 1;$$

or

$$\begin{array}{l} 1 \\ \vdots \\ s_{j+1} \\ \\ s_{j+1}+2 \\ \\ s_j+1 \\ \\ \\ \end{array} \begin{bmatrix} p-1 & p-1 \\ \vdots & \vdots \\ p-1 & p-1 \\ p-1 & r_{j+1} \\ p-1 & 0 \\ \vdots & \vdots \\ r_j & 0 \\ 0 & 0 \\ \vdots & \vdots \end{bmatrix}, \qquad \text{if } s_j - s_{j+1} \geq 2.$$

If $r_j - r_{j+1} < 0$, then

$$(s_j - s_{j+1})(p-1) \geq (p-1) + (\ r_{j+1} - r_j\ ).$$

Hence $s_j - s_{j+1} \geq 2$, and the $j$-th and the $(j{+}1)$-th column of $N^{(p,\lambda)}$ are of the form shown above. In either cases,

$$\begin{aligned} n_{i+1,j} - n_{i,j+1} &= 0, \qquad 1 \leq i \leq s_{j+1}, \\ n_{k+2,j} - n_{k+1,j+1} &\geq 0, \qquad \text{where } k = s_{j+1}. \end{aligned}$$

Therefore the conditions in (3.29) are satisfied:

$$\sum_{i=k}^{\infty} (n_{i+1,j} - n_{i,j+1}) \geq 0, \qquad k = 1, 2, \ldots .$$

Since $j$ is arbitrary, we have proved that $N^{(p,\lambda)}$ is reverse semistandard.

(ii) The first column of the matrix $N^{(p,\lambda)}$ determines the lengths of the first $(p - 1)$ columns in the diagram of $\theta_p(\lambda)$. Since the first column of $N^{(p,\lambda)}$ is

$$\begin{bmatrix} p-1 \\ \vdots \\ p-1 \\ r_1 \\ 0 \\ \vdots \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} p \\ \vdots \\ p \end{matrix}}\right\} s_1 \\ \\ \\ \\ \end{matrix}$$

the first $r_1$ columns in the diagram of $\theta_p(\lambda)$ have their lengths equal to $s_1 + 1$, and the next $(p - 1 - r_1)$ columns have lengths equal to $s_1$. According to the discussion in the proof of (i), either

$$s_1 - s_2 = 1 \quad \text{or} \quad s_1 - s_2 \geq 2.$$

When $s_1 - s_2 = 1$, $r_1 - r_2 \geq 0$, since $\lambda_1 - \lambda_2 \geq p - 1$. There are

$$(p - 1 - r_1) + r_2$$

columns with lengths $s_1$ in the diagram of $\theta_p(\lambda)$. But

$$(p - 1 - r_1) + r_2 = p - 1 - (r_1 - r_2) \leq p - 1.$$

When $s_1 - s_2 \geq 2$, the columns $p$ to $2(p-1)$ in the diagram of $\theta_p(\lambda)$ have their lengths less than $s_1$. Repeating this process, we can prove that no more than $(p-1)$ columns in the diagram of $\theta_p(\lambda)$ have the same lengths, hence $\theta_p(\lambda)$ is column $p$–regular.∎

The Γ–homomorphism $N^{(p,\lambda)}$ has some interesting properties when $\lambda$ is a partition described in (3.38):

**(3.39) PROPOSITION.** *Let* $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_h)$ *be a partition of* $n$ *satisfying*

$$\lambda_h > 0,$$
$$\lambda_j - \lambda_{j+1} \geq p-1, \qquad j = 1,\ 2,\ \ldots,\ h\text{–}1.$$

*Then*

(i) $S^\lambda$ *is not contained in* $\mathrm{Ker}[N^{(p,\lambda)}]$.

(ii) $S^{\lambda\perp} = \mathrm{Ker}[N^{(p,\lambda)}]$, *hence* $(S^\lambda)^*$ *is isomorphic to a submodule of* $S^{\theta_p(\lambda)}$.

**PROOF.** (i) By (3.38)(i), $N^{(p,\lambda)}\colon M^\lambda \to M^{\theta_p(\lambda)}$ is a reverse semistandard Γ–homomorphism. Thus $N^{(p,\lambda)}(S^\lambda) \neq 0$, since the restriction of $N^{(p,\lambda)}$ to $S^\lambda$ is an element in a $K$–basis of $\mathrm{Hom}_\Gamma(S^\lambda, M^{\theta_p(\lambda)})$, according to (3.28).

(ii) We first show that $S^{\lambda\perp} \subseteq \mathrm{Ker}[N^{(p,\lambda)}]$. By (3.23),

$$S^{\lambda\perp} = \sum_{i\geq 1}\sum_{w\geq 1} \mathrm{Im}(i \xrightarrow{w} i+1).$$

Thus it is enough to show that

$$N^{(p,\lambda)}(i \xrightarrow{w} i+1) = 0, \qquad 1 \leq i \leq h-1,\ w \geq 1.$$

Let $N^{\mathrm{T}}$ be the transpose of $N^{(p,\lambda)}$. By (3.16)(iii), it is enough to show that

$$[\, N^{(p,\lambda)}(i \xrightarrow{w} i+1)]^{\mathrm{A}} = 0,$$

i.e.

$$(i+1 \xrightarrow{w} i)N^{\mathrm{T}} = 0, \qquad 1 \leq i \leq h-1,\ w \geq 1,$$

by (3.18) and (3.20). Notice that in the matrix $N^{(p,\lambda)}$, the last non–zero entry in the $j$–th column, $r_{\mathrm{j}}$, is also the last non–zero entry in its row, by the proof of (3.38) (i). Thus the matrix $N^{\mathrm{T}}$ satisfies the conditions in (3.34), which imply that

$$(i+1 \xrightarrow{w} i)N^{\mathrm{T}} = 0, \qquad 1 \leq i \leq h-1,\ w \geq 1.$$

Thus $S^{\lambda\perp} \subseteq \mathrm{Ker}[\, N^{(p,\lambda)}]$. The submodule $\mathrm{Ker}[\, N^{(p,\lambda)}]$ of $M^{\lambda}$ does not contain $S^{\lambda}$ by (i), hence it is contained in $S^{\lambda\perp}$ by the Submodule Theorem (2.8). Therefore $S^{\lambda\perp} = \mathrm{Ker}[\, N^{(p,\lambda)}]$. To verify the last statement, we simply apply (2.14):

$$(S^{\lambda})^{*} \cong M^{\lambda}/S^{\lambda\perp},$$

and note that there is a monomorphism from $M^{\lambda}/S^{\lambda\perp}$ into $S^{\mathbb{0}_{\mathrm{p}}(\lambda)}$ induced by $N^{(p,\lambda)}$.∎

It is worth naming those partitions described in (3.39) at this stage:

**(3.40) DEFINITION.** *A partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)$ *is said to be strongly row* $p$*–regular, if* $\lambda_h > 0$, *and*

$$\lambda_j - \lambda_{j+1} \geq p - 1, \quad j = 1, 2, \ldots, h\text{–}1.$$

*A partition* $\mu$ *is said to be strongly column* $p$*–regular, if its conjugate* $\mu'$ *is strongly row* $p$*–regular.*∎

**(3.41) THEOREM.** *Let* $\lambda$ *be a strongly row* $p$*–regular partition of* $n$. *Let* $\theta_p(\lambda)$ *be the partition constructed in* (3.36). *Then* $\theta_p(\lambda)$ *is column* $p$*–regular, and the unique irreducible submodule of* $S^{\theta_p(\lambda)}$ *is isomorphic to* $J^\lambda$.

**PROOF.** Only the last statement need to be verified. By (3.39) above, $(S^\lambda)^*$ is isomorphic to a submodule of $S^{\theta_p(\lambda)}$. But $(S^\lambda)^*$ has its unique irreducible submodule isomorphic to $J^\lambda$, by (2.7). Thus as $\theta_p(\lambda)$ is column $p$–regular, $J^\lambda$ is isomorphic to the unique irreducible submodule of $S^{\theta_p(\lambda)}$.∎

An alternative proof of the last statement above is the following (see also, 24.6, 24.7 in [James (1978b)]):

**(3.42) PROPOSITION.** *If* $\lambda$ *is a strongly row* $p$*–regular partition of* $n$, *then* $\hat{N}^{(p,\lambda)}$, *the restriction of* $N^{(p,\lambda)}$ *to* $S^\lambda$, *is a non–zero* $\Gamma$*–homomorphism from* $S^\lambda$ *to* $S^{\theta_p(\lambda)}$. *Furthermore, there exists a* $\Gamma$*–homomorphism*

$$\psi : \; J^\lambda \longrightarrow S^{\theta_p(\lambda)},$$

*which makes the following diagram commute:*

$$\begin{tikzcd}
S^{\lambda} \arrow[d, "\pi"'] \arrow[dr, "\hat{N}^{\theta_p(\lambda)}"] & \\
J^{\lambda} = \dfrac{S^{\lambda}}{S^{\lambda} \cap S^{\lambda\perp}} \arrow[r, "\psi"] & S^{\theta_p(\lambda)}
\end{tikzcd}$$

*where $\pi$ is the coset map.*

**PROOF.** $\mathrm{Im}[N^{(p,\lambda)}] \subseteq S^{\theta_p(\lambda)}$ by (3.37). $\hat{N}^{(p,\lambda)}: S^{\lambda} \longrightarrow S^{\theta_p(\lambda)}$ is non–zero since $N^{(p,\lambda)}$ is reverse semistandard, hence $N^{(p,\lambda)}(S^{\lambda}) \neq 0$, as seen in (3.39). By (3.39) (ii),

$$S^{\lambda\perp} = \mathrm{Ker}[N^{(p,\lambda)}],$$

thus

$$\mathrm{Ker}[\,\hat{N}^{(p,\lambda)}] = S^{\lambda} \cap S^{\lambda\perp},$$

and

$$\mathrm{Im}[\,\hat{N}^{(p,\lambda)}] \cong J^{\lambda} = \frac{S^{\lambda}}{S^{\lambda} \cap S^{\lambda\perp}}. \blacksquare$$

**(3.43) REMARKS.** For a positive prime $p$, $\theta_p$ is a map from the set of partitions of $n$ to itself. When $p = 2$, $\theta_2$ sends each partition to its conjugate, hence $\theta_2$ is a bijection. When $p > 2$, $\theta_p$ is neither one–to–one nor onto. For example, when $p = 3$, $n = 9$,

$$\theta_3(7,2) = (4,2^2,1) = \theta_3(7,1^2),$$

while the partitions $(1^9)$, $(2,1^7)$, etc., are not in the image set of $\Theta_3$. Recall that in Problem (3.31), we look for a column $p$–regular partition $\Phi_p(\lambda)$, when $\lambda$ is row $p$–regular.

According to our Theorem (3.41), we have $\Phi_p(\lambda) = \Theta_p(\lambda)$, when $\lambda$ is strongly row $p$–regular. Now we reverse the problem as follows: if a strongly column $p$–regular partition $\mu$ is given, can we find a row $p$–regular partition $\lambda$, such that

$$\Phi_p(\lambda) = \mu \ ?$$

The answer is yes and the proof is based on Peel's result (2.18) and Theorem (3.41) above.▮

We state the following lemma without proof:

**(3.44) LEMMA.** *Let $M$ and $N$ be $\Gamma$–modules. The mapping*

$$\varphi \in \mathrm{Hom}_\Gamma(M,N) \longmapsto \varphi' \in \mathrm{Hom}_\Gamma[M \otimes \Gamma\epsilon(\mathfrak{S}_n),\ N \otimes \Gamma\epsilon(\mathfrak{S}_n)]$$

*where $\varphi'$ is defined by*

$$\varphi'[m \otimes \epsilon(\mathfrak{S}_n)] = \varphi(m) \otimes \epsilon(\mathfrak{S}_n), \qquad m \in M,$$

*is a $K$–isomorphism. Furthermore, $\varphi$ is onto (one–to–one) if and only if $\varphi'$ is onto (one–to–one).*▮

Let $\lambda$ be a row $p$–regular partition of $n$, $\nu$ be a column $p$–regular partition of $n$, such that $\nu = \Phi_p(\lambda)$. That is to say, there are $\Gamma$–homomorphisms in the following sequence

$$S^\lambda \xrightarrow{\pi} J^\lambda \xrightarrow{\psi} S^\nu,$$

such that $\pi$ is onto and $\psi$ is one–to–one. In the dual sequence, by (2.4),

$$(S^\nu)^* \xrightarrow{\psi^*} (J^\lambda)^* \xrightarrow{\pi^*} (S^\lambda)^*,$$

we have $\psi^*$ onto and $\pi^*$ one–to–one. Applying (3.44) above, one obtains

$$(S^\nu)^* \otimes \Gamma\epsilon(\mathfrak{S}_n) \xrightarrow{(\psi^*)'} (J^\lambda)^* \otimes \Gamma\epsilon(\mathfrak{S}_n) \xrightarrow{(\pi^*)'} (S^\lambda)^* \otimes \Gamma\epsilon(\mathfrak{S}_n),$$

in which $(\psi^*)'$ is onto and $(\pi^*)'$ is one–to–one. Notice that $J^\lambda \cong (J^\lambda)^*$ by (2.19), and by (2.18) and (2.19) :

$$(J^\lambda)^* \otimes \Gamma\epsilon(\mathfrak{S}_n) \cong J^\lambda \otimes \Gamma\epsilon(\mathfrak{S}_n),$$
$$(S^\nu)^* \otimes \Gamma\epsilon(\mathfrak{S}_n) \cong S^{\nu'},$$
$$(S^\lambda)^* \otimes \Gamma\epsilon(\mathfrak{S}_n) \cong S^{\lambda'}.$$

Therefore we have a sequence

$$S^{\nu'} \xrightarrow{\pi_1} J^\lambda \otimes \Gamma\epsilon(\mathfrak{S}_n) \xrightarrow{\psi_1} S^{\lambda'},$$

in which $\pi_1$ is onto and $\psi_1$ is one–to–one, $\nu'$ is row $p$–regular, $\lambda'$ is column $p$–regular. It also means

$$J^{\nu'} \cong J^{\lambda} \otimes \Gamma\epsilon(\mathfrak{S}_n)$$

is isomorphic to the unique irreducible submodule of $S^{\lambda'}$. That is to say,

**(3.45) PROPOSITION.** *Let $\lambda$ ($\nu$) be row (column) $p$-regular partition of $n$. Then $\nu = \Phi_p(\lambda)$ if and only if $\lambda' = \Phi_p(\nu')$.* ∎

For example, if $p = 3$, $n = 9$, noticing that $(8,1)' = (2,1^7)$, we have

$$\Phi_3(8,1) = (3,2^3),$$
$$\Phi_3(4^2,1) = (2,1^7).$$

The above observation is useful when we look for a row $p$-regular partition $\lambda$, such that $\Phi_p(\lambda) = \mu$, where $\mu$ is a given strongly column $p$-regular partition. Since $\mu'$ is strongly row $p$-regular, we have

$$\Phi_p(\mu') = \Theta_p(\mu')$$

by (3.41). Thus we can find $\Phi_p^{-1}(\mu)$ through the following algorithm:

$$\mu \longmapsto \mu' \longmapsto \Theta_p(\mu') \longmapsto \Theta_p(\mu')' = \Phi_p^{-1}(\mu)$$

for each strongly column $p$-regular partition $\mu$ of $n$.

When $\lambda$ is a row and column $p$-regular partition of $n$ and the Specht module $S^{\lambda}$ is irreducible over $\Gamma = K\mathfrak{S}_n$, the answer to (3.31) concerning $\lambda$ is certainly $\Phi_p(\lambda) = \lambda$. We now summarize our results in this section which provide partial answers to Problem (3.31):

**(3.46) THEOREM.** (1) *Let $\lambda$ be a strongly row $p$-regular partition of $n$. Then* $\Phi_p(\lambda) = \Theta_p(\lambda)$.

(2) *If $\mu$ is a strongly column $p$-regular partition of $n$,* $\Phi_p^{-1}(\mu) = \Theta_p(\mu')'$.

(3) *If $\lambda$ is a row and column $p$-regular partition of $n$ and the Specht module $S^\lambda$ is irreducible over $\Gamma = K\mathfrak{S}_n$, then* $\Phi_p(\lambda) = \lambda$ .

**(3.47) EXAMPLE.** Take $n = 6$. All partitions of 6 are listed below in pairs of conjugation:

$$(6),\ (1^6)$$
$$(5,1),\ (2,1^4)$$
$$(4,2),\ (2^2,1^2)$$
$$(4,1^2),\ (3,1^3)$$
$$(3^2),\ (2^3)$$
$$(3,2,1),\ (3,2,1)$$

(a) When char($K$) $= 2$, the mapping $\Phi_2$ sends every 2-regular partition to its conjugate, as we have seen in (3.32).

(b) When char($K$) $= p > 6$, every partition of 6 is row and column $p$-regular, and the mapping $\Phi_p$ is the identity mapping on the set of partitions of 6, by (3.32).

(c) When char($K$) $= 3$, there are seven row 3-regular partitions: (6), (5,1), (4,2), $(3^2)$, $(4,1^2)$, (3,2,1), $(2^2,1^2)$. Among them, (6), (5,1) and (4,2) are strongly row 3-regular, hence their conjugates $(1^6)$, $(2,1^4)$ and $(2^2,1^2)$ are strongly column 3-regular. The following table lists all strongly 3-regular partitions of 6, the matrices $N^{(p,\lambda)}$, and the partitions $\Theta_3(\lambda)$ by the construction of (3.36):

| $\lambda$ | $N^{(3,\lambda)}$ | $\Theta_3(\lambda)$ |
|---|---|---|
| $(6)$ | $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ | $(2^3)$ |
| $(5,1)$ | $\begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}$ | $(3,2,1)$ |
| $(4,2)$ | $\begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix}$ | $(4,2)$ |
| $(3^2)$ | $\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$ | $(4,2)$ |
| $(4,1^2)$ | $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix}$ | $(4,2)$ |
| $(3,2,1)$ | $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ | $(5,1)$ |
| $(2^2,1^2)$ | $\begin{bmatrix} 2 & 2 & 1 & 1 \end{bmatrix}$ | $(6)$ |

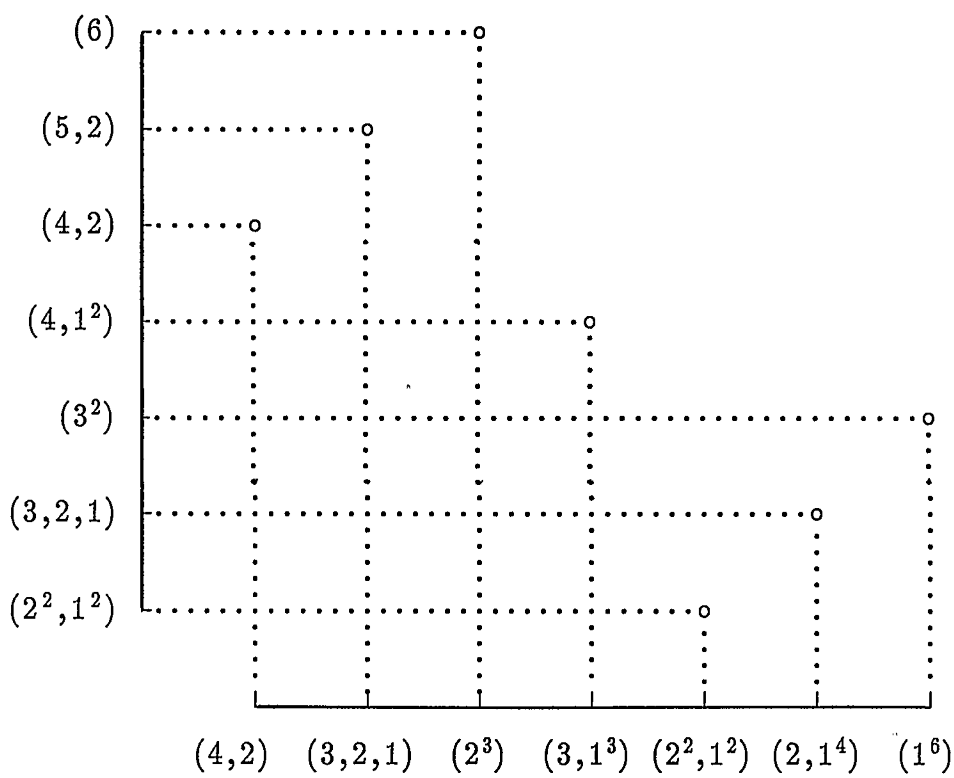
By applying (3.46)(2), we have

$$\Phi_3^{-1}(1^6) = (2^3)' = (3^2),$$
$$\Phi_3^{-1}(2,1^4) = (3,2,1)' = (3,2,1),$$
$$\Phi_3^{-1}(2^2,1^2) = (4,2)' = (2^2,1^2).$$

In the set of row 3–regular partitions of 6, $(4,1^2)$ is the only one whose $\Phi_3$–image can not be found by using (3.45). But $(3,1^3)$ is the only leftover in the set of column 3–regular partitions, thus we are lucky to see that

$$\Phi_3(4,1^2) = (3,1^3).$$

In the following diagram, we list all 3–regular partitions of 6 along the vertical line, in the dictionary order. Their conjugates are listed along the horizontal line, from right to the left. The circles with coordinates $(\Phi_3(\lambda),\lambda)$ illustrates the correspondence $\Phi_3$.




(d) When char($K$) = 5, all the partitions of 6 are row 5–regular except $(1^6)$, but (6) and (5,1) are the only strongly row 5–regular ones. By applying (3.46), we have

$$\Phi_5(6) = (4,2),$$
$$\Phi_5(5,1) = (5,1),$$
$$\Phi_5(2^2,1^2) = (1^6),$$

$$\Phi_5(2,1^4) = (2,1^4).$$

We may apply (3) in Theorem (3.46) to get more information about the correspondence $\Phi_5$ . The detail is omitted here. In general there might be more leftovers after we apply $\Theta_p$ to strongly row $p$–regular partitions and $\Phi_p^{-1}$ to strongly column $p$–regular partitions. A complete description of the correspondence $\Phi_p$ is still an open question.∎

## §3E The Homomorphisms from $S^{(n-1,1)}$ to Other Specht Modules

The determination of the $K$–dimension of $\mathrm{Hom}_{\Gamma}(S^{\lambda}, S^{\mu})$ for a pair of partitions of $n$, where $\Gamma = K\mathfrak{S}_n$, is a difficult question. When $\mathrm{char}(K) = 0$, or $\mathrm{char}(K) = p$, $p > n$,

$$\dim_K \mathrm{Hom}_{\Gamma}(S^{\lambda}, S^{\mu}) = \delta_{\lambda\mu} = \begin{cases} 1, & \text{if } \lambda = \mu, \\ 0, & \text{if } \lambda \neq \mu, \end{cases}$$

since $\{ S^{\lambda} \mid \lambda \text{ is a partition of } n \}$ forms a complete set of inequivalent irreducible $\Gamma$–modules. In the case $\mathrm{char}(K) = p < n$, some partial results have been obtained by a few authors. G. James solved the problem when $\lambda$ is the partition $(n)$ and $\mu$ is arbitrary in 24.4 [James (1978b)] (see 3.35 in §3D), while another special case, when both $\lambda$ and $\mu$ are two–parts partitions of $n$, was studied in [Gwendolen Murphy (1982)].

We set up our goal in this section as the following:

**(3.48) PROBLEM.** *Let $K$ be a field of characteristic $p > 0$.*

(a) *Find all partitions $\mu$ of $n$ such that*

$$\mathrm{Hom}_{\Gamma}( S^{(n-1,1)}, S^{\mu}) \neq 0.$$

*When $\mathrm{Hom}_{\Gamma}( S^{(n-1,1)}, S^{\mu}) \neq 0$, exhibit a $K$–basis for this $K$–space thus find the $K$–dimension.*

(b) *Find all partitions $\mu$ of $n$, $n \geq 3$, such that the irreducible module*

$J^{(n-1,1)}$ *is isomorphic to a submodule of* $S^{\mu}$.∎

For a pair of partitions $\lambda$ and $\mu$ of $n$, we may list all the semistandard (reverse semistandard) $\Gamma$–homomorphisms from $M^{\lambda}$ to $M^{\mu}$, and then test whether some $K$–linear combinations of them send $S^{\lambda}$ into the kernel intersection in (3.9).

**(3.49) NOTATIONS AND NOTES.**

(i) For a partition $\mu = (\mu_1,\mu_2,\dots,\mu_h)$ of $n$, $\mu_h > 0$, there are $h$ $((n-1,1),\mu)$–incidence matrices, each of which has zero in all but one place in its second column. Denote by $P_i$ the $((n-1,1),\mu)$–incidence matrix whose second column has 1 (one) in the $i$–th row, i.e.

$$\mathfrak{M}((n-1,1),\mu) = \left\{ P_i = \begin{bmatrix} \mu_1 & 0 \\ \vdots & \vdots \\ \mu_{i-1} & 0 \\ \mu_i - 1 & 1 \\ \mu_{i+1} & 0 \\ \vdots & \vdots \\ \mu_h & 0 \end{bmatrix} \;\middle|\; i = 1, 2, \dots, h. \right\}.$$

It can be easily checked that $P_1, \dots, P_{h-1}$ are reverse semistandard, by the criteria in (3.29).

(ii) If $\varphi : M^{(n-1,1)} \longrightarrow M^{\mu}$ is a $\Gamma$–homomorphism, we shall use $\hat{\varphi}$ to denote the restriction of $\varphi$ to $S^{(n-1,1)}$. Thus according to (3.28):

$$\{ \hat{P}_i \mid i = 1, 2, \dots, h-1 \}$$

is a $K$–basis for the $K$–space $\mathrm{Hom}_{\Gamma}(S^{(n-1,1)}, M^{\mu})$. In this section we choose those reverse semistandard homomorphisms to work with because they seem to have a better performance in our very last section, although we believe that the results of this section are independent of the choice of the basis.

(iii) Let $x$ be the $(n-1,1)$–tableau

$$\begin{matrix} 1 & 3 & \cdots & n \\ 2 & & & \end{matrix} .$$

Our later analysis is based on the positions of the elements 1 and 2 in some tabloids. Let $\nu$ be a composition of n. The sum of all $\nu$–tabloids with 1 (one) in the $i$–row and 2 in the $j$–th row is written as

$$\sum \left\{ \overline{\underline{y_\nu}} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \end{bmatrix}_\nu \;\middle|\; 1 \in Y_i,\ 2 \in Y_j \right\}$$

or abbreviated by

$$\sum \left\{ \overline{\underline{y_\nu}} \;\middle|\; 1 \in Y_i,\ 2 \in Y_j \right\} = \sum_{\substack{1 \in Y_i \\ 2 \in Y_j}} \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \end{bmatrix}_\nu .$$

For example, if $P_j$ is the $((n-1,1),\mu)$–incidence matrix defined in (i) then we can write

$$P_j(\,\overline{\underline{x}}\,) = \sum_{i=1}^{h} \sum \left\{ \overline{\underline{y_\mu}} \;\middle|\; 1 \in Y_i,\ 2 \in Y_j \right\} .$$

(iv) In the notation of (iii), we have

$$(1,2) \sum \left\{ \overline{\underline{y_\nu}} \;\middle|\; 1 \in Y_i,\ 2 \in Y_j \right\} = \sum \left\{ \overline{\underline{y_\nu}} \;\middle|\; 1 \in Y_j,\ 2 \in Y_i \right\}$$

where (1,2) is the transposition in $\mathfrak{S}_n$. It is clear that

$$[1 - (1,2)] \sum \left\{ \overline{\underline{y_\nu}} \;\middle|\; 1 \in Y_i,\ 2 \in Y_i \right\} = 0,$$

for all $i$ and any composition $\nu$ of $n$.∎

Assume that

$$\varphi = \sum_{i=1}^{h-1} z_i P_i \ :\ M^{(n-1,1)} \longrightarrow M^{\mu}$$

is a $\Gamma$–homomorphism, $z_i \in K$. Let $x$ be the $(n-1,1)$–tableau $\begin{matrix} 1 & 3 & \cdots & n \\ 2 \end{matrix}$. From (3.9) and (3.28), the restriction $\hat{\varphi}$ of $\varphi$ to $S^{(n-1,1)}$ is a $\Gamma$–homomorphism from $S^{(n-1,1)}$ to $S^{\mu}$ if and only if $\varphi\{[1-(12)]\ \overline{\underline{x}}\ \} \in S^{\mu}$, if and only if

$$(3.50) \quad [1 - (1,2)]\ (i{+}1 \xrightarrow{w} i) \sum_{i=1}^{h-1} z_i P_i(\ \overline{\underline{x}}\ ) = 0, \quad 1 \le i \le h{-}1,\ 1 \le w \le \mu_{i+1}\ ;$$

if and only if

$$(3.51) \quad [1 - (1,2)]\ (k \xrightarrow{w} l) \sum_{i=1}^{h-1} z_i P_i(\ \overline{\underline{x}}\ ) = 0, \quad k > l \ge 1,\ 1 \le w \le \mu_k\ .$$

Both (3.50) and (3.51) yield systems of linear equations in unknowns $\{z_i\}_{i=1}^{h-1}$ over $K$. When we want to find the necessary conditions on $\varphi$, (3.51) provides a faster algorithm, although some of the linear equations given by (3.51) are clearly consequences of those given by (3.50). The following arises from (3.10):

**(3.52) FORMULAE.**

(i) For $k = 2, 3, \ldots, h$, $w = 1, 2, \ldots, \mu_k$,

$$(k \xrightarrow{w} 1)P_1 = (k \xrightarrow{w} 1)\begin{bmatrix} \mu_1-1 & 1 \\ \mu_2 & 0 \\ \vdots & \vdots \\ \mu_k & 0 \\ \vdots & \vdots \\ \mu_h & 0 \end{bmatrix} = \begin{bmatrix} \mu_1-1+w \\ w \end{bmatrix}\begin{bmatrix} \mu_1-1+w & 1 \\ \mu_2 & 0 \\ \vdots & \vdots \\ \mu_k-w & 0 \\ \vdots & \vdots \\ \mu_h & 0 \end{bmatrix}.$$

(ii) For $k = 2, \ldots, h-1$,

$$(k \xrightarrow{w} 1)P_k = (k \xrightarrow{w} 1)\begin{bmatrix} \mu_1 & 0 \\ \vdots & \vdots \\ \mu_{k-1} & 0 \\ \mu_k - 1 & 1 \\ \mu_{k+1} & 0 \\ \vdots & \vdots \\ \mu_h & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \mu_1 + w \\ w \end{bmatrix} \begin{bmatrix} \mu_1+w & 0 \\ \vdots & \vdots \\ \mu_{k-1} & 0 \\ \mu_k-1-w & 1 \\ \mu_{k+1} & 0 \\ \vdots & \vdots \\ \mu_h & 0 \end{bmatrix} + \begin{bmatrix} \mu_1 + w-1 \\ w-1 \end{bmatrix} \begin{bmatrix} \mu_1+w-1 & 1 \\ \vdots & \vdots \\ \mu_{k-1} & 0 \\ \mu_k-w & 0 \\ \mu_{k+1} & 0 \\ \vdots & \vdots \\ \mu_h & 0 \end{bmatrix},$$

if $1 \leq w < \mu_k$; and

$$(k \xrightarrow{\mu_k} 1)P_k = (k \xrightarrow{\mu_k} 1) \begin{bmatrix} \mu_1 & 0 \\ \vdots & \vdots \\ \mu_{k-1} & 0 \\ \mu_k - 1 & 1 \\ \mu_{k+1} & 0 \\ \vdots & \vdots \\ \mu_h & 0 \end{bmatrix} = \begin{bmatrix} \mu_1 + \mu_k - 1 \\ \mu_k - 1 \end{bmatrix} \begin{bmatrix} \mu_1+\mu_k-1 & 1 \\ \vdots & \vdots \\ \mu_{k-1} & 0 \\ 0 & 0 \\ \mu_{k+1} & 0 \\ \vdots & \vdots \\ \mu_h & 0 \end{bmatrix}.$$

(iii) For $k = 2, \dots, h-1$, $2 \leq i \leq h-1$ and $i \neq k$,

$$(k \xrightarrow{w} 1)P_i = (k \xrightarrow{w} 1) \begin{array}{c} \\ \\ k \\ \\ i \\ \\ \\ \end{array} \begin{bmatrix} \mu_1 & 0 \\ \vdots & \vdots \\ \mu_k & 0 \\ \vdots & \vdots \\ \mu_i-1 & 1 \\ \vdots & \vdots \\ \mu_h & 0 \end{bmatrix} = \begin{bmatrix} \mu_1 + w \\ w \end{bmatrix} \begin{bmatrix} \mu_1+w & 0 \\ \vdots & \vdots \\ \mu_k-w & 0 \\ \vdots & \vdots \\ \mu_i-1 & 1 \\ \vdots & \vdots \\ \mu_h & 0 \end{bmatrix},$$

for all $w$, $1 \leq w \leq \mu_k$.

**(3.53) LEMMA.** (a) *Let* $\varphi = \sum_{i=1}^{h} z_i P_i : M^{(n-1,1)} \longrightarrow M^{\mu}$ *be a*

$\Gamma$*–homomorphism satisfying* (3.51). *Then*

$$z_k + \mu_1 z_1 = 0, \quad k = 2, \dots, h-1.$$

Hence $\varphi$ *must be of the form*

$$z_1(P_1 - \mu_1 P_2 - \dots - \mu_1 P_{h-1}), \qquad z_1 \in K.$$

(b) *The* $K$*–dimension of* $\mathrm{Hom}_\Gamma(S^{(n-1,1)}, S^{\mu})$ *is either zero or one.* $\mathrm{Hom}_\Gamma(S^{(n-1,1)}, S^{\mu})$ *has dimension one if and only if*

$$P_1 - \mu_1(P_2 + \cdots + P_{h-1})$$

*satisfies* (3.50) *or* (3.51).

**PROOF.** To prove (a), we take $w = 1$ in Formulae (3.52) above. Let $\nu$ be the composition of $n$

$$(\mu_1+1, \mu_2, \dots, \mu_k-1, \dots, \mu_h).$$

We have by (i) in (3.52)

$$(k \xrightarrow{1} 1)P_1 = \begin{bmatrix} \mu_1 \\ 1 \end{bmatrix} \begin{bmatrix} \mu_1 & 1 \\ \mu_2 & 0 \\ \vdots & \vdots \\ \mu_k-1 & 0 \\ \vdots & \vdots \\ \mu_h & 0 \end{bmatrix}.$$

Thus in $(k \xrightarrow{1} 1)P_1(\,\underline{\bar{x}}\,)$, the sum of $\nu$–tabloids

$$\sum\left\{ \underline{\overline{y_\nu}} \;\middle|\; 1 \in Y_h,\ 2 \in Y_1 \right\}$$

has coefficient $\mu_1$. Meanwhile, for $2 \le k \le h-1$, by (3.52)(ii),

$$(k \xrightarrow{1} 1)P_k = = \begin{bmatrix}\mu_1+1\\1\end{bmatrix} \begin{bmatrix} \mu_1+1 & 0 \\ \vdots & \vdots \\ \mu_{k-1} & 0 \\ \mu_k-2 & 1 \\ \mu_{k+1} & 0 \\ \vdots & \vdots \\ \mu_h & 0 \end{bmatrix} + \begin{bmatrix} \mu_1 & 1 \\ \vdots & \vdots \\ \mu_{k-1} & 0 \\ \mu_k-1 & 0 \\ \mu_{k+1} & 0 \\ \vdots & \vdots \\ \mu_h & 0 \end{bmatrix}, \text{ if } \mu_k \ge 2;$$

$$(k \xrightarrow{1} 1)P_k = \begin{bmatrix} \mu_1 & 1 \\ \vdots & \vdots \\ \mu_{k-1} & 0 \\ 0 & 0 \\ \mu_{k+1} & 0 \\ \vdots & \vdots \\ \mu_h & 0 \end{bmatrix}, \text{ if } \mu_k = 1.$$

In both cases, the sum of $\nu$–tabloids

$$\sum\left\{ \underline{\overline{y_\nu}} \;\middle|\; 1 \in Y_h,\ 2 \in Y_1 \right\}$$

in $(k \xrightarrow{1} 1)P_k( \underline{\overline{x}} )$ has coefficient 1 (one). If $i \ne k$, $2 \le i \le h-1$, by (3.52)(iii),

$$(k \xrightarrow{1} 1)P_i = \begin{bmatrix} \mu_1 + 1 \\ 1 \end{bmatrix} \begin{bmatrix} \mu_1+1 & 0 \\ \vdots & \vdots \\ \mu_k-1 & 0 \\ \vdots & \vdots \\ \mu_i-1 & 1 \\ \vdots & \vdots \\ \mu_h & 0 \end{bmatrix}.$$

Therefore $\sum\left\{ \overline{\underline{y_\nu}} \;\middle|\; 1 \in Y_h,\ 2 \in Y_1 \right\}$ does not occur in $(k \xrightarrow{1} 1)P_i(\ \overline{\underline{x}}\ )$, if $i \neq k$, $2 \leq i \leq h-1$. Also, we notice that

$$\sum\left\{ \overline{\underline{y_\nu}} \;\middle|\; 1 \in Y_1,\ 2 \in Y_h \right\}$$

has coefficient zero in $(k \xrightarrow{1} 1)P_j$ , $1 \leq j \leq h-1$. Therefore in the expression of

$$(k \xrightarrow{1} 1) \sum_{i=1}^{h-1} z_i P_i(\ \overline{\underline{x}}\ )$$

the sum of $\nu$–tabloids $\sum\left\{ \overline{\underline{y_\nu}} \;\middle|\; 1 \in Y_h,\ 2 \in Y_1 \right\}$ has coefficient

$$z_k + \mu_1 z_1\ ,$$

while $\sum\left\{ \overline{\underline{y_\nu}} \;\middle|\; 1 \in Y_1,\ 2 \in Y_h \right\}$ has coefficient zero. It follows that in the expression

$$[1 - (1,2)]\ (k \xrightarrow{1} 1) \sum_{i=1}^{h-1} z_i P_i(\ \overline{\underline{x}}\ ),$$

the sum

$$\sum\left\{ \underline{\overline{y_\nu}} \;\middle|\; 1 \in Y_h,\ 2 \in Y_1 \right\} - \sum\left\{ \underline{\overline{y_\nu}} \;\middle|\; 1 \in Y_1,\ 2 \in Y_h \right\}$$

has coefficient $(z_k + \mu_1 z_1)$. Since $\varphi = \sum_{i=1}^{h-1} z_i P_i$ satisfies (3.51), we must have

$$z_k + \mu_1 z_1 = 0, \qquad k = 2, \ldots, h-1.$$

(b) is an easy corollary of (a).∎

When $\mu = (\mu_1, \mu_2)$, a two–parts partition, $\{P_1\}$ is a $K$–basis for $\mathrm{Hom}_\Gamma(S^{(n-1,1)}, M^\mu)$. Thus the K–space $\mathrm{Hom}_\Gamma(S^{(n-1,1)}, S^\mu)$ is non–zero if and only if

$$P_1(S^{(n-1,1)}) \subseteq S^\mu$$

This is the special case of 2.12 in [Gwendolen Murphy (1982)]. We shall state and sketch the proof of the following

**(3.54) PROPOSITION.** *If $\mu = (\mu_1, \mu_2)$, $\mu_1 \geq \mu_2 > 1$, is a two–parts partition of $n$, then the $K$–dimension of* $\mathrm{Hom}_\Gamma(S^{(n-1,1)}, S^\mu)$ *is either zero or one.* $\mathrm{Hom}_\Gamma(S^{(n-1,1)}, S^\mu)$ *has dimension one if and only if*

$$\mu_1 \equiv 0 \pmod{p^{\ell_p(\mu_2-1)}}.$$

Note. The definition of $\ell_p(b)$, $b$ a positive integer, is given in §3D, prior to (3.33).

**PROOF OF (3.54).** Take $k = 2$ in Formulae (3.52)(i):

$$(2 \xrightarrow{w} 1)P_1 = (2 \xrightarrow{w} 1) \begin{bmatrix} \mu_1 - 1 & 1 \\ \mu_2 & 0 \end{bmatrix} = \begin{bmatrix} \mu_1 - 1 + w \\ w \end{bmatrix} \begin{bmatrix} \mu_1 - 1 + w & 1 \\ \mu_2 - w & 0 \end{bmatrix}$$

where $w = 1, 2, \ldots, \mu_2$. For $w = 1, 2, \ldots, \mu_2 - 1$, let $\nu(w)$ be the partition of $n : (\mu_1 + w, \mu_2 - w)$, then

$$(2 \xrightarrow{w} 1)P_1(\, \underline{\bar{x}}\, )$$

$$= \begin{bmatrix} \mu_1 - 1 + w \\ w \end{bmatrix} \left[ \sum_{\substack{1 \in Y_1 \\ 2 \in Y_1}} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}_{\nu(w)} + \sum_{\substack{1 \in Y_2 \\ 2 \in Y_1}} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}_{\nu(w)} \right].$$

By applying (3.49)(iv), we have

$$[1 - (1,2)]\ (2 \xrightarrow{w} 1)P_1(\, \underline{\bar{x}}\, )$$

$$= \begin{bmatrix} \mu_1 - 1 + w \\ w \end{bmatrix} \left[ \sum_{\substack{1 \in Y_2 \\ 2 \in Y_1}} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}_{\nu(w)} - \sum_{\substack{1 \in Y_1 \\ 2 \in Y_2}} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}_{\nu(w)} \right].$$

Therefore, $P_1(\, S^{(n-1,1)}) \subseteq S^{\mu}$ only if

$$\begin{bmatrix} \mu_1 - 1 + w \\ w \end{bmatrix} \equiv 0 \pmod{p}, \qquad \text{for } w = 1, 2, \ldots, \mu_2 - 1.$$

By (3.33), this is equivalent to,

$$\mu_1 - 1 \equiv -1 \ (\mathrm{mod}\ p^{\ell_p(\mu_2-1)}),$$

i.e.

$$\mu_1 \equiv 0 \ (\mathrm{mod}\ p^{\ell_p(\mu_2-1)}).$$

Conversely if $\mu_1 \equiv 0(\mathrm{mod}\ p^{\ell_p(\mu_2-1)})$, then

$$[1 - (1,2)]\ (2 \xrightarrow{w} 1)P_1(\ \underline{\bar{x}}\ ) = 0, \qquad \text{for } w = 1,\ 2,\ \dots\ ,\ \mu_2-1.$$

It is clear that $[1 - (1,2)]\ (2 \xrightarrow{\mu_2} 1)P_1(\ \underline{\bar{x}}\ ) = 0$, since 1 and 2 lie in the same row of each tabloid occuring in $(2 \xrightarrow{\mu_2} 1)P_1(\ \underline{\bar{x}}\ )$.∎

Now we assume that $\mu = (\mu_1,\ \mu_2,\ \dots\ ,\ \mu_h)$, $\mu_h > 0$, $h \geq 3$. Consider the following cases:

**CASE A.** $p$ divides $\mu_1$.

**CASE B.** $p$ does not divide $\mu_1$.

In CASE A, $\mu_1 \equiv 0\ (\mathrm{mod}\ p)$, $\mathrm{Hom}_\Gamma(\ S^{(n-1,1)},\ S^{\mu})$ is non–zero if and only if $P_1(\ S^{(n-1,1)}) \subseteq S^{\mu}$, by (3.53)(iii). We have the the following assertion by making use of Formulae (3.52):

**(3.55) PROPOSITION.** *Let $\mu = (\mu_1,\ \dots\ ,\ \mu_h)$, such that $\mu_h > 0$, $h \geq 3$. Then $P_1(\ S^{(n-1,1)}) \subseteq S^{\mu}$ if and only if*

$$\mu_1 \equiv 0\ (\mathrm{mod}\ p^{\ell_p(\mu_2)})$$
$$\mu_i \equiv -1\ (\mathrm{mod}\ p^{\ell_p(\mu_{i+1})}), \qquad i = 2,\ \dots,\ h-1.$$

**PROOF.** First assume that $P_1(S^{(n-1,1)}) \subseteq S^{\mu}$, i.e.

$$[1-(1,2)]\ (i+1 \xrightarrow{w} i)\ P_1(\ \underline{\bar{x}}\ ) = 0, \quad 1 \leq i \leq h-1,\ 1 \leq w \leq \mu_{i+1}.$$

Take $k = 2$ in (3.52),

$$(2 \xrightarrow{w} 1)P_1 = \begin{bmatrix} \mu_1-1+w \\ w \end{bmatrix} \begin{bmatrix} \mu_1-1+w & 1 \\ \mu_2-w & 0 \\ \mu_3 & 0 \\ \vdots & \vdots \\ \mu_h & 0 \end{bmatrix},\ 1 \leq w \leq \mu_2.$$

Since $h \geq 3$, the sum

$$\sum\left\{ \underline{\overline{y_{\nu(2,w)}}}\ \middle|\ 1 \in Y_h,\ 2 \in Y_1 \right\}$$

where $\nu(2,w)$ is the composition $(\mu_1+w,\ \mu_2-w,\ \mu_3,\ \dots,\ \mu_h)$, has coefficient $\begin{bmatrix} \mu_1-1+w \\ w \end{bmatrix}$ in the expression of $(2 \xrightarrow{w} 1)P_1(\ \underline{\bar{x}}\ )$, $1 \leq w \leq \mu_2$. Thus the sum

$$\sum\left\{ \underline{\overline{y_{\nu(2,w)}}}\ \middle|\ 1 \in Y_h,\ 2 \in Y_1 \right\} - \sum\left\{ \underline{\overline{y_{\nu(2,w)}}}\ \middle|\ 1 \in Y_1,\ 2 \in Y_h \right\}$$

has coefficient $\begin{bmatrix} \mu_1-1+w \\ w \end{bmatrix}$ in

$$[1-(1,2)]\ (2 \xrightarrow{w} 1)P_1(\ \underline{\bar{x}}\ ) = 0, \qquad \text{for } w = 1,\ 2,\ \dots,\ \mu_2.$$

Therefore

$$\begin{bmatrix} \mu_1-1+w \\ w \end{bmatrix} \equiv 0 \pmod{p}, \qquad \text{for } w = 1,\ 2,\ \dots,\mu_2;$$

i.e.

$$\mu_1 \equiv 0 \ (\mathrm{mod}\ p^{\ell_p(\mu_2)}).$$

Furthermore, for each $i$, $2 \leq i \leq h-1$,

$$(i+1 \xrightarrow{w} i)P_1 = \begin{bmatrix} \mu_i+w \\ w \end{bmatrix} \begin{bmatrix} \mu_1-1 & 1 \\ \mu_2 & 0 \\ \vdots & \vdots \\ \mu_i+w & 0 \\ \mu_{i+1}-w & 0 \\ \vdots & \vdots \\ \mu_h & 0 \end{bmatrix}, \qquad 1 \leq w \leq \mu_{i+1}.$$

By counting the coefficients of

$$\sum\left\{ \overline{\underline{y_{\nu(i,w)}}} \;\middle|\; 1 \in Y_i,\ 2 \in Y_1 \right\}$$

where $\nu_{(i,w)}$ is the composition

$$(\mu_1, \ \dots\ , \mu_i+w, \mu_{i+1}-w, \ \dots\ , \mu_h),$$

in $(i+1 \xrightarrow{w} i)P_1(\ \bar{\underline{x}}\ )$, we must have $\begin{bmatrix} \mu_i+w \\ w \end{bmatrix} \equiv 0 \ (\mathrm{mod}\ p)$, $w = 1,2,\dots,\mu_{i+1}$.
Thus

$$\mu_i \equiv -1(\mathrm{mod}\ p^{\ell_p(\mu_{i+1})}), \qquad i = 2,\ \dots,\ h-1.$$

The converse part of the proof is clear from the above discussion.∎

It remains to study the case $p$ does not divide $\mu_1$. We first assume that $\mu = (\mu_1, \mu_2, \mu_3)$, a 3–parts partition of $n$. Then $\mathrm{Hom}_\Gamma(\ S^{(n-1,1)},\ S^{\mu})$ is non–zero if and only if

$$(P_1 - \mu_1 P_2)(\, S^{(n-1,1)}) \subseteq S^{\mu},$$

by (3.53). Put $k = 3$, $w = 1$ in (3.52):

$$(3 \xrightarrow{1} 1)P_1 = \begin{bmatrix} \mu_1 \\ 1 \end{bmatrix} \begin{bmatrix} \mu_1 & 1 \\ \mu_2 & 0 \\ \mu_3-1 & 0 \end{bmatrix}$$

$$(3 \xrightarrow{1} 1)P_2 = \begin{bmatrix} \mu_1+1 \\ 1 \end{bmatrix} \begin{bmatrix} \mu_1+1 & 0 \\ \mu_2-1 & 1 \\ \mu_3-1 & 0 \end{bmatrix}$$

Thus the sum

$$\sum \left\{ \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}_{\nu} \middle| \; 1 \in Y_2,\ 2 \in Y_1 \right\},$$

where $\nu = (\mu_1+1,\ \mu_2,\ \mu_3-1)$, has coefficient $\mu_1$ in $(3 \xrightarrow{1} 1)P_1(\ \underline{\bar{x}}\ )$, and zero in $(3 \xrightarrow{1} 1)P_2(\ \underline{\bar{x}}\ )$, while

$$\sum \left\{ \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}_{\nu} \middle| \; 1 \in Y_1,\ 2 \in Y_2 \right\}$$

has coefficient $(\mu_1+1)$ in $(3 \xrightarrow{1} 1)P_2(\ \underline{\bar{x}}\ )$, zero in $(3 \xrightarrow{1} 1)P_1(\ \underline{\bar{x}}\ )$. Therefore the sum

$$\sum \left\{ \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}_{\nu} \middle| \; 1 \in Y_2,\ 2 \in Y_1 \right\} - \sum \left\{ \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}_{\nu} \middle| \; 1 \in Y_1,\ 2 \in Y_2 \right\}$$

has coefficient $\mu_1 - (-\mu_1)(\mu_1+1)$ in

$$[1 - (1,2)](3 \xrightarrow{1} 1)(P_1 - \mu_1 P_2)(\,\underline{\bar{x}}\,).$$

It forces that

$$\mu_1 + \mu_1(\mu_1+1) \equiv 0 (\mathrm{mod}\ p),$$

i.e.

$$\mu_1(\mu_1+2) \equiv 0 (\mathrm{mod}\ p).$$

Because $p$ does not divide $\mu_1$ by assumption, it follows that

**(3.56)** $(\mu_1+2) \equiv 0 (\mathrm{mod}\ p)$.

Hence we have

**(3.57) PROPOSITION.** *If $\mu = (\mu_1, \mu_2, \mu_3)$ is a partition of $n$, $\mu_1$ is an odd integer, then*

$$\mathrm{Hom}_\Gamma( S^{(n-1,1)}, S^{\mu}) = 0$$

*when $\Gamma = K\mathfrak{S}_n$, $n \geq 3$, $char(K) = 2$.* ∎

Assume now char$(K) = p \geq 3$. By taking $k = 2$ in (3.52):

$$(2 \xrightarrow{w} 1)P_1 = \begin{bmatrix} \mu_1-1+w \\ w \end{bmatrix} \begin{bmatrix} \mu_1-1+w & 1 \\ \mu_2-w & 0 \\ \mu_3 & 0 \end{bmatrix}, \qquad 1 \le w \le \mu_2;$$

$$(2 \xrightarrow{w} 1)P_2 = \begin{bmatrix} \mu_1+w \\ w \end{bmatrix} \begin{bmatrix} \mu_1+w & 0 \\ \mu_2-w-1 & 1 \\ \mu_3 & 0 \end{bmatrix} + \begin{bmatrix} \mu_1+w-1 \\ w-1 \end{bmatrix} \begin{bmatrix} \mu_1+w-1 & 1 \\ \mu_2-w & 0 \\ \mu_3 & 0 \end{bmatrix},$$

if $1 \le w \le \mu_2-1$;

$$(2 \xrightarrow{\mu_2} 1)P_2 = \begin{bmatrix} \mu_1+\mu_2-1 \\ \mu_2-1 \end{bmatrix} \begin{bmatrix} \mu_1+\mu_2-1 & 1 \\ 0 & 0 \\ \mu_3 & 0 \end{bmatrix}.$$

Write $\nu(w) = (\mu_1+w,\ \mu_2-w,\ \mu_3)$. For $w = 1, 2, \dots, \mu_2-1$, the coefficient of

$$\sum \left\{ \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}_{\nu(w)} \middle|\ 1 \in Y_3,\ 2 \in Y_2 \right\}$$

in $(2 \xrightarrow{w} 1)P_2(\,\underline{\underline{x}}\,)$ is $\begin{bmatrix} \mu_1+w \\ w \end{bmatrix}$. Thus

$$\sum \left\{ \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}_{\nu(w)} \middle|\ 1 \in Y_3,\ 2 \in Y_2 \right\} - \sum \left\{ \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}_{\nu(w)} \middle|\ 1 \in Y_2,\ 2 \in Y_3 \right\}$$

has coefficient $-\mu_1\begin{bmatrix} \mu_1+w \\ w \end{bmatrix}$ in

$$[1 - (1,2)](2 \xrightarrow{w} 1)(P_1 - \mu_1 P_2)(\,\underline{\underline{x}}\,).$$

This forces

$$\begin{bmatrix}\mu_1+w\\ w\end{bmatrix} \equiv 0(\text{mod } p), \qquad \text{for } w = 1, 2, \ldots ,\mu_2-1;$$

i.e.

$$\mu_1 \equiv -1 \ (\text{mod } p^{\ell_p(\mu_2-1)}).$$

Combining this with (3.56):

$$\mu_1+2 \equiv 0 \ (\text{mod } p),$$

we must have $\mu_2 - 1 = 0$, hence $\mu_2 = \mu_3 = 1$. We now state the following :

**(3.58) PROPOSITION.** *Let $\mu = (\mu_1, \mu_2, \mu_3)$ be a 3–parts partition of $n$, such that $\mu_1$ is not be divisible by $p$. When $p \geq 3$,* $\text{Hom}_\Gamma(S^{(n-1,1)}, S^\mu)$ *is non–zero if and only if*

$$\mu_1 \equiv -2 \ (\text{mod } p)$$
$$\mu_2 = \mu_3 = 1.$$

*Under the above conditions, the restriction of $(P_1 + 2P_2)$ to $S^{(n-1,1)}$ is a $K$–basis for* $\text{Hom}_\Gamma(S^{(n-1,1)}, S^\mu)$.

**PROOF.** We have shown that the conditions

$$\mu_1 \equiv -2 \ (\text{mod } p)$$
$$\mu_2 = \mu_3 = 1.$$

are necessary for $\mathrm{Hom}_\Gamma( S^{(n-1,1)}, S^\mu)$ being non–zero in the note above. To show that they are sufficient, it is enough to check that

$$[1 - (1,2)](2 \xrightarrow{1} 1)(P_1 + 2P_2)( \underline{\bar{x}} ) = 0,$$
$$[1 - (1,2)](3 \xrightarrow{1} 2)(P_1 + 2P_2)( \underline{\bar{x}} ) = 0.$$

We have

$$(2 \xrightarrow{1} 1)(P_1 + 2P_2) = (2 \xrightarrow{1} 1) \left\{ \begin{bmatrix} \mu_1-1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} + 2 \begin{bmatrix} \mu_1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

$$= \mu_1 \begin{bmatrix} \mu_1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} + 2 \begin{bmatrix} \mu_1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$= (\mu_1+2) \begin{bmatrix} \mu_1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} = 0,$$

since $(\mu_1+2) \equiv 0 \pmod{p}$; and

$$(3 \xrightarrow{1} 2)(P_1 + 2P_2)( \underline{\bar{x}} )$$

$$= \left\{ 2 \begin{bmatrix} \mu_1-1 & 1 \\ 2 & 0 \\ 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} \mu_1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \right\} ( \underline{\bar{x}} )$$

$$= 2 \sum_{\substack{i=1,2 \\ j=1,2}} \sum \left\{ \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}_{\nu} \middle| \; 1 \in Y_i, \; 2 \in Y_j \right\},$$

therefore

$$[1 - (1,2)](3 \xrightarrow{1} 2)(P_1 + 2P_2)( \underline{x} ) = 0. \blacksquare$$

The remaining case is $\mu = ( \mu_1, \ldots, \mu_h)$, with $\mu_h > 0$, such that $h > 3$ and $p$ does not divide $\mu_1$. We intend to show that $\mathrm{Hom}_\Gamma( S^{(n-1,1)}, S^{\mu}) = 0$ whenever $\mu$ is a partition of the above type. Suppose

$$\mathrm{Hom}_\Gamma( S^{(n-1,1)}, S^{\mu}) \neq 0,$$

then according to (3.53), the restriction of

$$\varphi = P_1 - \mu_1(P_2 + \cdots + P_h)$$

to $S^{(n-1,1)}$ is a non–zero $\Gamma$–homomorphism from $S^{(n-1,1)}$ to $S^{\mu}$. Necessarily,

$$[1 - (1,2)](2 \xrightarrow{w} 1)\varphi( \underline{x} ) = 0, \qquad 1 \leq w \leq \mu_2.$$

Recalling (iii) in Formulae (3.52), for each $i$, $2 < i \leq h-1$,

$$(2 \xrightarrow{w} 1)P_{\mathrm{i}} = \begin{bmatrix}\mu_1+w\\ w\end{bmatrix} \begin{array}{c} \\ \\ \\ \\ \mathrm{i} \\ \\ \\ \end{array} \begin{bmatrix} \mu_1+w & 0 \\ \mu_2-w & 0 \\ \mu_3 & 0 \\ \vdots & \vdots \\ \mu_{\mathrm{i}}-1 & 1 \\ \vdots & \vdots \\ \mu_{\mathrm{h}} & 0 \end{bmatrix}, \qquad 1 \le w \le \mu_2.$$

Thus $\sum\left\{ \underline{\overline{y_\nu}} \;\middle|\; 1 \in Y_{\mathrm{h}},\ 2 \in Y_{\mathrm{i}} \right\}$, where $\nu = (\mu_1+w,\ \mu_2-w,\ \mu_3, \ldots,\ \mu_{\mathrm{h}})$, has coefficient $\begin{bmatrix}\mu_1+w\\ w\end{bmatrix}$ in $(2 \xrightarrow{w} 1)P_{\mathrm{i}}(\ \underline{\bar{x}}\ )$. We notice that if $j \ne i$, $1 \le j \le h-1$, neither $\sum\left\{ \underline{\overline{y_\nu}} \;\middle|\; 1 \in Y_{\mathrm{h}},\ 2 \in Y_{\mathrm{i}} \right\}$ nor $\sum\left\{ \underline{\overline{y_\nu}} \;\middle|\; 1 \in Y_{\mathrm{i}},\ 2 \in Y_{\mathrm{h}} \right\}$ occurs in $(2 \xrightarrow{w} 1)P_{\mathrm{j}}(\ \underline{\bar{x}}\ )$. It follows that

$$\sum\left\{ \underline{\overline{y_\nu}} \;\middle|\; 1 \in Y_{\mathrm{h}},\ 2 \in Y_{\mathrm{i}} \right\} - \sum\left\{ \underline{\overline{y_\nu}} \;\middle|\; 1 \in Y_{\mathrm{i}},\ 2 \in Y_{\mathrm{h}} \right\}$$

has coefficient $-\mu_1\begin{bmatrix}\mu_1+w\\ w\end{bmatrix}$ in

$$[1-(1,2)](2 \xrightarrow{w} 1)\varphi(\ \underline{\bar{x}}\ ),$$

since $p$ does not divide $\mu_1$, that forces

$$\begin{bmatrix}\mu_1+w\\ w\end{bmatrix} \equiv 0 \pmod{p}, \qquad 1 \le w \le \mu_2.$$

Therefore

$$\mu_1 \equiv -1 \pmod{p^{\ell_{\mathrm{p}}(\mu_2)}}.$$

Since $\mu_2 \ge 1$, the above condition implies

$$\mu_1 \equiv -1 (\mathrm{mod}\ p).$$

Hence

$$\varphi = P_1 - \mu_1(P_2 + \cdots + P_h) = P_1 + P_2 + \cdots + P_h.$$

On the other hand, $\varphi$ must satisfy

$$[1 - (1,2)](h \xrightarrow{1} h-1)\varphi(\underline{\bar{x}}) = 0.$$

We have

$$(h \xrightarrow{1} h-1)P_1 = \left[\begin{smallmatrix}\mu_{h-1}+1\\ 1\end{smallmatrix}\right] \cdot \begin{bmatrix} \mu_1-1 & 1 \\ \mu_2 & 0 \\ \vdots & \vdots \\ \mu_{h-1}+1 & 0 \\ \mu_h-1 & 0 \end{bmatrix},$$

$$(h \xrightarrow{1} h-1)P_{h-1} = (h \xrightarrow{1} h-1)\begin{bmatrix} \mu_1 & 0 \\ \vdots & \vdots \\ \mu_{h-2} & 0 \\ \mu_{h-1}-1 & 1 \\ \mu_h & 0 \end{bmatrix} = \left[\begin{smallmatrix}\mu_{h-1}\\ 1\end{smallmatrix}\right]\begin{bmatrix} \mu_1 & 0 \\ \vdots & \vdots \\ \mu_{h-2} & 0 \\ \mu_{h-1} & 1 \\ \mu_h-1 & 0 \end{bmatrix}.$$

Let $\nu(h)$ be the composition $(\mu_1, \ldots, \mu_{h-1}+1, \mu_h-1)$. The sum of $\nu(h)$–tabloids

$$\sum\left\{ \overline{\underline{y_{\nu(h)}}} \;\middle|\; 1 \in Y_{h-1},\ 2 \in Y_1 \right\}$$

has coefficient $(\mu_{h-1}+1)$ in $(h \xrightarrow{1} h-1)P_1(\underline{\bar{x}})$, zero in $(h \xrightarrow{1} h-1)P_j(\underline{\bar{x}})$, for

$j = 2, 3, \ldots, h{-}1$; while

$$\sum\left\{ \overline{\underline{y_{\nu(h)}}} \;\middle|\; 1 \in Y_1,\ 2 \in Y_{h\text{-}1} \right\}$$

has coefficient $\mu_{h\text{-}1}$ in $(h \xrightarrow{1} h{-}1)P_{h\text{-}1}(\ \underline{\bar{x}}\ )$, zero in $(h \xrightarrow{1} h{-}1)P_j(\ \underline{\bar{x}}\ )$, when $j = 1, 2, \ldots, h{-}2$. Therefore,

$$\sum\left\{ \overline{\underline{y_{\nu(h)}}} \;\middle|\; 1 \in Y_{h\text{-}1},\ 2 \in Y_1 \right\} - \sum\left\{ \overline{\underline{y_{\nu(h)}}} \;\middle|\; 1 \in Y_1,\ 2 \in Y_{h\text{-}1} \right\}$$

has coefficient

$$(\mu_{h\text{-}1}{+}1) - \mu_{h\text{-}1} = 1.$$

in $[1 - (1,2)](h \xrightarrow{1} h{-}1)\varphi(\ \underline{\bar{x}}\ )$. This is a contradiction, since we must have

$$[1 - (1,2)](h \xrightarrow{1} h{-}1)\varphi(\ \underline{\bar{x}}\ ) = 0,$$

according to our analysis above. Thus we have proved

**(3.59) PROPOSITION.** *Let $\mu = (\ \mu_1, \ldots, \mu_h)$ be a partition of $n$, such that $\mu_h > 0$, $h > 3$, $p$ does not divide $\mu_1$. Then*

$$\mathrm{Hom}_\Gamma(\ S^{(n-1,1)}, S^{\mu}) = 0. \blacksquare$$

As a summary, we have the following

**(3.60) THEOREM.** *Let $K$ be a field of charcteristic $p$, $p > 0$. The*

*$K$–dimension of* $\mathrm{Hom}_\Gamma( S^{(n-1,1)}, S^{\mu})$ *is either one or zero. The $K$–space* $\mathrm{Hom}_\Gamma( S^{(n-1,1)}, S^{\mu})$ *has dimension one if and only if $\mu$ is a partition of $n$ belonging to one of the following categories:*

(a) $\mu_1 \equiv 0 \pmod{p^{\ell_p(\mu_2)}}$, $\mu_i \equiv -1 \pmod{p^{\ell_p(\mu_{i+1})}}$, $i \geq 2$. *In this case,* $\tilde{P}_1$ *is a $K$–basis for* $\mathrm{Hom}_\Gamma( S^{(n-1,1)}, S^{\mu})$.

(b) $\mu = (\mu_1, \mu_2)$, $\mu_2 > 1$, $\mu_1 \equiv 0 \pmod{p^{\ell_p(\mu_2-1)}}$. *In this case,* $\tilde{P}_1$ *is a $K$–basis for* $\mathrm{Hom}_\Gamma( S^{(n-1,1)}, S^{\mu})$.

(c) $\mu = (\mu_1, 1^2)$, $\mu_1 \equiv -2 \pmod{p}$, $p \geq 3$. *In this case,* $\tilde{P}_1 + 2\tilde{P}_2$ *is a $K$–basis for* $\mathrm{Hom}_\Gamma( S^{(n-1,1)}, S^{\mu})$.

(d) $\mu = (n-1,1)$ *itself.*∎

**(3.61) COROLLARY.** *The James module* $J^{(n-1,1)}$ *is isomorphic to a submodule of* $S^{\mu}$*, where* $\mu \neq (n-1,1)$*, if and only if $\mu$ is a partition in one of the categories* (a), (b) *and* (c) *in* (3.60).

**PROOF.** Assume that $J^{(n-1,1)}$ is isomorphic to a submodule of $S^{\mu}$, then

$$S^{(n-1,1)} \xrightarrow{\pi} J^{(n-1,1)} \xrightarrow{\theta} S^{\mu}$$

gives a non–zero $\Gamma$–homomorphism $\theta\pi$ from $S^{(n-1,1)}$ to $S^{\mu}$, where $\pi$ is the coset map and $\theta$ is the monomorphism. Thus $\mathrm{Hom}_\Gamma( S^{(n-1,1)}, S^{\mu}) \neq 0$. Now apply (3.60).

Conversely, assume that $\mathrm{Hom}_\Gamma( S^{(n-1,1)}, S^\mu) \neq 0$. We need the structure theorem of the Γ–module $S^{(n-1,1)}$, proved by H.K. Farahat in 1961:

**THEOREM.** (5.2 [Farahat (1962)]) *Let $K$ be a field with characteristic $p$, $p > 0$. $S^{(n-1,1)}$ is an irreducible $\Gamma$–module if and only if $p$ does not divide $n$. When $p$ divides $n$, $S^{(n-1,1)}$ is reducible, and there exists an exact sequence*

$$0 \to J^{(n)} \to S^{(n-1,1)} \to J^{(n-1,1)} \to 0.$$

According to the above result, we have only to verify that $J^{(n)}$ is not isomorphic to a submodule of $S^\mu$ if $\mathrm{Hom}_\Gamma( S^{(n-1,1)}, S^\mu)$ is non–zero. (3.35) in §3D gives the criterion of $\mu$, for $J^{(n)}$ being isomorphic to a submodule of $S^\mu$. Since none of the partitions in categories (a), (b) and (c) satisfies the conditions in (3.35):

$$\mu_i \equiv -1(\mathrm{mod}\ p^{\ell_p(\mu_{i+1})}), \qquad i \geq 1,$$

we can conclude that $J^{(n-1,1)}$ is a submodule of $S^\mu$.∎

# CHAPTER 4
# SOCLE LENGTH OF SOME SPECHT MODULES

## §4A The Problem of Calculating $\mathrm{Hom}_\Gamma(M^\lambda,S^\lambda)$

Let $\lambda$ be a partition of $n$. Write $\Gamma = K\mathfrak{S}_n$. If $J$ is an irreducible submodule of $S^\lambda$, then

$$M^\lambda/J^\perp \cong J^* \cong J$$

by (2.1)(iv) and (2.19). Thus

$$M^\lambda \xrightarrow{\pi} M^\lambda/J^\perp \cong J \xrightarrow{\theta} S^\lambda$$

gives a non–zero element in $\mathrm{Hom}_\Gamma(M^\lambda,S^\lambda)$, where $\pi$ is the coset map and $\theta$ is the inclusion map. Assume that the socle of $S^\lambda$ (c.f. 2.6) is the direct sum of irreducible submodules $J_1$, ..., $J_k$ of $S^\lambda$:

$$\mathrm{soc}(S^\lambda) = J_1 \oplus \cdots \oplus J_k ,$$

and $\varphi_i$ is the non–zero homomorphism constructed above corresponding to $J_i$, $1 \leq i \leq k$. Then it is not difficult to show that $\varphi_1$, ..., $\varphi_k$ are linearly independent elements in $\mathrm{Hom}_\Gamma(M^\lambda,S^\lambda)$. Hence we have

**(4.1) LEMMA.** *The socle length of $S^\lambda$ does not exceed the $K$–dimension of* $\mathrm{Hom}_\Gamma(M^\lambda,S^\lambda)$.∎

We notice that the $K$–dimension of $\mathrm{Hom}_\Gamma(S^\lambda,M^\lambda)$ is equal to 1 (one) in most of the cases, according to (3.28).

**(4.2) LEMMA.** *Unless* $\mathrm{char}(K) = 2$ *and* $\lambda$ *is a row 2–singular partition, the* $K$*–space* $\mathrm{Hom}_\Gamma(S^\lambda, M^\lambda)$ *is isomorphic to* $K$.

**PROOF.** By (3.28), unless $\mathrm{char}(K) = 2$ and $\lambda$ is a row 2–singular, the $K$–dimension of $\mathrm{Hom}_\Gamma(S^\lambda, M^\lambda)$ is equal to the number of semistandard $(\lambda,\lambda)$ tableaux. But there is only one semistandard $(\lambda,\lambda)$–tableau for each partition $\lambda$ of $n$.∎

When $\mathrm{char}(K) = 0$, the $K$–algebra $\Gamma = K\mathfrak{S}_n$ is semisimple, thus $\mathrm{Hom}_\Gamma(M^\lambda, S^\lambda)$ has also $K$–dimension one. In the case $\mathrm{char}(K) = p > 0$, we only know that

$$\dim_K \mathrm{Hom}_\Gamma(M^\lambda, S^\lambda) \geq 1.$$

Since the dimension of $\mathrm{Hom}_\Gamma(M^\lambda, S^\lambda)$ gives an upper bound of the socle length of $S^\lambda$, by (4.1), we shall discuss this problem in general and calculate the $K$–dimension of $\mathrm{Hom}_\Gamma(M^\lambda, S^\lambda)$ when $\lambda$ is a "hook" partition of $n$ and $\mathrm{char}(K) \neq 2$ in this section.

Let $\varphi$: $M^\lambda \to M^\lambda$ be a $\Gamma$–endomorphism, where $\lambda$ is a partition of $n$. Then $\mathrm{Im}(\varphi) \leq S^\lambda$ if and only if

$$(k \xrightarrow{w} \ell)\varphi = 0, \qquad k > \ell \geq 1,\ w > 0,$$

by (3.9). Let $\mathfrak{M}(\lambda,\lambda)$ be the set of $(\lambda,\lambda)$–incidence matrices, viewed as a $K$–basis of $\mathrm{End}_\Gamma(M^\lambda)$ (see 3.5 and 3.6). We can express $\varphi$ as

$$\varphi = \sum_{P \in \mathfrak{M}(\lambda,\lambda)} z_P P, \qquad z_P \in K.$$

The conditions

$$\text{(4.3a)} \quad (k \xrightarrow{w} \ell) \sum_{P \in \mathfrak{M}(\lambda,\lambda)} z_P P = 0, \qquad k > \ell \geq 1,\ w > 0.$$

yield a system of homogeneous linear equations in the set of unknowns

$$\{ z_P \mid P \in \mathfrak{M}(\lambda,\lambda) \}$$

over $K$. In fact,

(4.3b) **OBSERVATION.** *The $K$–dimension of* $\mathrm{Hom}_\Gamma(M^\lambda,S^\lambda)$ *is equal to the dimension of the solution space determined by* (4.3a).∎

Solving such a linear system is certainly a tedious task, but not altogether impossible. We shall try to classify the set $\mathfrak{M}(\lambda,\lambda)$, hoping that some of the "large" linear equations will break into "smaller" linear equations which are easier to work with.

Let $\lambda$ be a partition of $n$, such that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_h > 1,$$
$$\lambda_{h+1} = \cdots = \lambda_{h+r} = 1,$$
$$\lambda_k = 0, \text{ if } k > h+r.$$

**(4.4) DEFINITION.** The block of a $(\lambda,\lambda)$–incidence matrix $M$ consisting the first $h$ rows of $M$ is called the hat of $M$. Say that $M$ and $N$ in $\mathfrak{M}(\lambda,\lambda)$ are hat–equivalent, if the hats of $M$ and $N$ are the same. This is an equivalence relation on the set $\mathfrak{M}(\lambda,\lambda)$. An equivalence class in $\mathfrak{M}(\lambda,\lambda)$ is called a hat–class.∎

Let $M$ be a $(\lambda,\lambda)$–incidence matrix, viewed as an $(h+r)\times(h+r)$ matrix. If $h < i \leq h+r$, the $i$–th row of $M$ is $E_{\mathrm{k}}$ for some $k$, where $E_{\mathrm{k}}$ is the $k$–th basic row vector with $(h+r)$ components:

$$E_{\mathrm{k}} = (0, \ldots , 0, \overset{\mathrm{k}}{1}, 0, \ldots , 0).$$

Thus every $M$ in $\mathfrak{M}(\lambda,\lambda)$ defines a function

$$m : \{ h+1, \ldots , h+r \} \rightarrow \{ 1, 2, \ldots , h+r \},$$

such that $E_{m(i)}$ is the $i$–th row vector of $M$, $i = h+1, \ldots , h+\mathrm{r}$. If we denote the hat of $M$ by $H_M$ , then $M$ can be written as

$$M = \begin{bmatrix} H_M \\ E_{m(\,h+1)} \\ \vdots \\ E_{m(\,h+r)} \end{bmatrix} .$$

**(4.5) LEMMA.** *For each $i$, $h < i < h+r$,*

$$(i{+}1 \xrightarrow{1} i) \sum_{M \in \mathfrak{M}(\lambda\,,\,\lambda)} z_M M = 0, \qquad z_M \in K,$$

*implies that*

$$(i{+}1 \xrightarrow{1} i) \sum_{M \in \mathfrak{H}} z_M M = 0, \qquad z_M \in K,$$

*for each hat–class $\mathfrak{H}$ in $\mathfrak{M}(\lambda,\lambda)$.*

**PROOF.** $(i{+}1 \xrightarrow{1} i)\ M$ can be calculated by applying the formula in (3.10), for $i = h{+}1, \dots, h{+}r{-}1$ :

$$(i{+}1 \xrightarrow{1} i)\ M = (i{+}1 \xrightarrow{1} i) \begin{bmatrix} H_M \\ E_{m(\,h+1)} \\ \vdots \\ E_{m(\,h+r)} \end{bmatrix}$$

$$= k_{i,M}\ M_{i,1}$$

where

$$M_{i,1} = \begin{bmatrix} H_M \\ E_{m(\,h+1\,)} \\ \vdots \\ E_{m(h+i)} + E_{m(h+i+1)} \\ 0 \\ \vdots \\ \mathrm{E}_{m(\,h+r\,)} \end{bmatrix}$$

and $k_{i,M}$ is 1 if $m(h{+}i) = m(h{+}i{+}1)$, or 2 otherwise, according to (3.10). Thus if $M$ and $N$ are in $\mathfrak{M}(\lambda,\lambda)$, $H_M \neq H_N$ , then $M_{i,1} \neq N_{i,1}$ in

$$(i+1 \xrightarrow{1} i) \sum_{M \in \mathfrak{M}(\lambda\,,\,\lambda)} z_M M = 0.$$

Therefore

$$(i+1 \xrightarrow{1} i) \sum_{M \in \mathfrak{H}} z_M M = 0$$

for each hat–class $\mathfrak{H}$ in $\mathfrak{M}(\lambda,\lambda)$.∎

The above lemma suggests a thorough investigation on the equations

$$(i+1 \xrightarrow{1} i) \sum_{M \in \mathfrak{H}} z_M M = 0$$

for a hat–class $\mathfrak{H}$ and for each $i$, $i = h+1, \ldots, h+r-1$. The following simple observation is the base of our further discussion:

**(4.6a) LEMMA.** *If $M$ and $M'$ are $(\lambda,\lambda)$–incidence matrices in a hat–class $\mathfrak{H}$, then $M'$ arises from $M$ by permuting the rows $h+1, \ldots, h+\mathrm{r}$ of $M$.*

**PROOF.** Here we treat $M$ ($M'$) as an integral matrix (not a $\Gamma$–homomorphism), hence we can perform matrix operations on $M$ ($M'$). The partition $\lambda$ of $n$ can be viewed as a row vector

$$[\lambda_1, \ldots \lambda_h, \lambda_{h+1}, \ldots, \lambda_{h+r}].$$

Write

$$M = \begin{bmatrix} H_M \\ E_{m(h+1)} \\ \vdots \\ E_{m(h+r)} \end{bmatrix}.$$

We have, by the definition of $(\lambda,\lambda)$–incidence matrices,

$$\begin{aligned} \lambda &= [\lambda_1, \dots \lambda_h, \lambda_{h+1}, \dots, \lambda_{h+r}] \\ &= [1, 1, \dots, 1]M \\ &= [1, 1, \dots, 1]H_M + E_{m(h+1)} + \dots + E_{m(h+r)}. \end{aligned}$$

If $M'$ is also a $(\lambda,\lambda)$–matrix with the same hat as $M$, it follows that

$$E_{m(h+1)} + \dots + E_{m(h+r)} = E_{m'(h+1)} + \dots + E_{m'(h+r)},$$

where $m'$ is the function defined by $M'$. The basic row vectors $E_k$ thus occur with the same frequency on both sides of this. And thus the lemma follows.∎

Let $G_r$ be the group of all the permutations on the set $\{1, 2, \dots, h+r\}$, such that

$$\pi(i) = i, \qquad 1 \leq i \leq h .$$

Define a $G_r$–action on the hat–class $\mathfrak{H}$ in $\mathfrak{M}(\lambda,\lambda)$ in the following manner: if $\pi \in G_r$, $M \in \mathfrak{H}$, then $\pi * M$ is the incidence matrix

$$\begin{bmatrix} H_M \\ E_{m\pi^{-1}(h+1)} \\ \vdots \\ E_{m\pi^{-1}(h+r)} \end{bmatrix}.$$

It is easy to check that

$$(\pi_1\pi_2)*M = \pi_1*(\pi_2*M),$$

$\pi_1$, $\pi_2 \in G_r$, $M \in \mathfrak{H}$. We can restate (4.6a) as follows

**(4.6b)** *If $M$ and $M'$ are $(\lambda,\lambda)$–incidence matrices in some hat–class $\mathfrak{H}$ of $\mathfrak{M}(\lambda,\lambda)$, then*

$$M' = \pi*M$$

for some $\pi$ in $G_r$.∎

For example, if $\lambda = (2,1^3)$ and

$$M = \begin{bmatrix} 2&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix} \in \mathfrak{M}(\lambda,\lambda),$$

then

$$(2,3)*M = \begin{bmatrix} 2&0&0&0 \\ 0&0&1&0 \\ 0&1&0&0 \\ 0&0&0&1 \end{bmatrix}, \qquad (2,3,4)*M = \begin{bmatrix} 2&0&0&0 \\ 0&0&0&1 \\ 0&1&0&0 \\ 0&0&1&0 \end{bmatrix}.$$

It is worth noting that the group $G_r$ is generated by the transpositions

$$(h+1,h+2),\ (h+2,h+3),\ \dots\ ,\ (h+r-1,h+r).$$

**(4.7) LEMMA.** *If $M$ and $N$ are $(\lambda,\lambda)$-incidence matrices in some hat-class* $\mathfrak{H}$, such that $N = (i,i+1)*M$ *for some $i$, $h < i < h+r$, then*

$$(i+1 \xrightarrow{1} i) \sum_{P \in \mathfrak{H}} z_P P = 0$$

*implies*

$$z_N + z_M = 0.$$

**PROOF.** Write

$$(i+1 \xrightarrow{1} i)\ M = (i+1 \xrightarrow{1} i) \begin{bmatrix} H_M \\ E_{m(h+1)} \\ \vdots \\ E_{m(h+r)} \end{bmatrix} = k_{i,M}\ M_{i,1}$$

where

$$M_{i,1} = \begin{bmatrix} H_M \\ E_{m(h+1)} \\ \vdots \\ E_{m(h+i)} + E_{m(h+i+1)} \\ 0 \\ \vdots \\ E_{m(h+r)} \end{bmatrix},$$

$$k_{i,M} = \begin{cases} 2, & \text{if } m(h+r) = m(h+r+1), \\ 1, & \text{if } m(h+r) \neq m(h+r+1). \end{cases}$$

It is clear that if $M'$ belongs to the same hat-class in which $M$ lies, then

$$(i+1 \xrightarrow{1} i)\ M' = (i+1 \xrightarrow{1} i)\ M$$

if and only if

$$E_{m'(h+j)} = E_{m(h+j)}\,, \qquad j \neq i,\ j \neq i+1,$$
$$\{\, E_{m'(h+i)}, E_{m'(h+i+1)} \,\} = \{\, E_{m(h+i)}, E_{m(h+i+1)} \,\}\ ;$$

if and only if

$$M' = (i,i+1)*M, \quad \text{or} \quad M' = M.$$

Assume now that

$$(i+1 \xrightarrow{1} i) \sum_{P \in \mathfrak{H}} z_P P = 0$$

for some hat–class $\mathfrak{H}$ of $\mathfrak{M}(\lambda,\lambda)$. It follows that

$$\sum_{P \in \mathfrak{H}} z_P\, k_{i,P} P_{i,1} = 0.$$

If $M \in \mathfrak{H}$ and $k_{i,M} = 2$, then $M = (i,i+1)*M$, and

$$0 = 2z_M = z_M + z_{(i,i+1)*M}$$

from the comments above. If $k_{i,M} = 1$, $M \neq (i,i+1)*M$, then

$$[z_M + z_{(i,\,i+1)*M}]M_{i,1} = 0,$$

it follows that

$$z_M + z_{(i,\,i+1)*M} = 0.$$

Thus (4.7) holds.∎

(4.8) **COROLLARY.** *Let $M$ and $N$ be two arbitrary $(\lambda,\lambda)$–incidence matrices in a hat–class $\mathfrak{H}$. The conditions*

$$(i+1 \xrightarrow{1} i) \sum_{P \in \mathfrak{H}} z_P P = 0, \qquad h < i < h+r$$

*imply that either $z_N = z_M$ or $z_N = -z_M$.*

**PROOF.** There exists a set of transpositions $\tau_1, \tau_2, \dots, \tau_u$ in the set of generators of the group $G_r$:

$$\{ (i,i+1) \mid i = h+1, \dots, h+r-1 \}$$

such that

$$N = \tau_u * \cdots * \tau_1 * M.$$

Now apply (4.7) above repeatedly.∎

(4.9) **COROLLARY.** *Assume that* char$(K) \neq 2$. *If $M$ is a $(\lambda,\lambda)$–incidence matrix of the form*

$$\begin{bmatrix} H_M \\ E_{m(h+1)} \\ \vdots \\ E_{m(h+r)} \end{bmatrix},$$

*such that the function $m$: $\{h+1,\dots,h+r\} \rightarrow \{1,2,\dots,h+r\}$ is not one–to–one, then the conditions*

$$(i+1 \xrightarrow{1} i) \sum_{P \in \mathfrak{M}(\lambda,\lambda)} z_P P = 0, \qquad h < i < h+r$$

*imply* $z_M = 0$.

**PROOF.** Let $\mathfrak{H}$ be the hat–class in $\mathfrak{M}(\lambda,\lambda)$ to which $M$ belongs. By (4.5),

$$(i+1 \xrightarrow{1} i) \sum_{P \in \mathfrak{M}(\lambda,\lambda)} z_P P = 0, \qquad h < i < h+r$$

imply

$$(i+1 \xrightarrow{1} i) \sum_{P \in \mathfrak{H}} z_P P = 0, \qquad h < i < h+r.$$

According to (4.8), it is enough to show that $z_Q = 0$, for some $Q$ in $\mathfrak{H}$. Note that if $Q \in \mathfrak{H}$, then $H_M = H_Q$,

$$Q = \begin{bmatrix} H_M \\ E_{q(h+1)} \\ \vdots \\ E_{q(h+r)} \end{bmatrix}.$$

The function $q\colon \{h+1, \dots, h+r\} \to \{1, 2, \dots, h+r\}$ determined by $Q$ is not one–to–one, since $q = m\sigma$ for some $\sigma$ in $G_r$. Take $Q$ in $\mathfrak{H}$, such that

$$q(i) = q(i+1), \qquad \text{for some } i, \ h+1 \le i \le h+r,$$

then $Q = (i,i+1)*Q$ since $E_{q(i)} = E_{q(i+1)}$, and

$$(i+1 \xrightarrow{1} i) \sum_{P \in \mathfrak{H}} z_P P = 0$$

implies $2z_Q = 0$ by (4.7). Therefore $z_Q = 0$ since $\mathrm{char}(K) \neq 2$.∎

Now we are ready to study the $K$–space $\mathrm{Hom}_\Gamma(M^\lambda, S^\lambda)$, where

$$\lambda = (n-r, 1^r), \qquad 1 \leq r \leq n-2,$$

a hook partition of $n$. Our goal is to show the following

**(4.10) PROPOSITION.** *Assume that* $\mathrm{char}(K) \neq 2$, *and* $\lambda$ *is the partition*

$$(n-r, 1^r), \qquad 1 \leq r \leq n-2.$$

*Then* $\dim_K \mathrm{Hom}_\Gamma(M^\lambda, S^\lambda) \leq 1$.∎

The proof consists of a sequence of notes ending on page 126.

**(4.11) NOTES.** Let $\lambda = (n-r, 1^r)$, $1 \leq r \leq n-2$.

(i) Two $(\lambda,\lambda)$–incidence matrices $M$ and $N$ belong to the same hat–class if and only if their first rows are identical.

(ii) If $M \in \mathfrak{H}$, where $\mathfrak{H}$ is some hat–class in $\mathfrak{M}(\lambda,\lambda)$, we shall write

$$H_M = [\ m_{11}, \ \dots\ , \ m_{1,r+1}],$$

$$M = \begin{bmatrix} H_M \\ E_{m(2)} \\ \vdots \\ E_{m(h+r)} \end{bmatrix}.$$

The function $m\colon \{2,\dots,r+1\} \longrightarrow \{1,2,\dots,r+1\}$ is not one–to–one if and only if for some $u$ and $v$, $2 \le u < v \le r+1$,

$$E_{m(u)} = E_{m(v)}.$$

It is clear that in this case we must have

$$E_{m(u)} = E_{m(v)} = [1,\ 0,\ \dots\ ,\ 0] = E_1.$$

Thus $m$ is not one–to–one if and only if $m_{11} \le n-r-2$. By (4.9), the conditions

$$(i+1 \xrightarrow{1} i) \sum_{P \in \mathfrak{M}(\lambda,\lambda)} z_P P = 0, \qquad 2 < i \le r\ ,$$

imply that

$$z_M = 0, \qquad \text{whenever } m_{11} \le n-r-2,$$

if $\mathrm{char}(K) \neq 2$.

(iii) Thus we only have to take account of the following hat–classes in the calculation of $\mathrm{Hom}_\Gamma(M^\lambda,S^\lambda)$ :

$$\mathfrak{H}(1) = \{ M \in \mathfrak{M}(\lambda,\lambda) \mid m_{11} = n-r \}$$
$$\mathfrak{H}(q) = \{ M \in \mathfrak{M}(\lambda,\lambda) \mid m_{11} = n-r-1,\ m_{1q} = 1 \}$$

for $q = 2, 3, \dots, r+1$.

(iv) From (4.8), we know that if $M, N \in \mathfrak{H}(i)$, for some $i$, $1 \leq i \leq r+1$, then either $z_N = z_M$ *or* $z_N = -z_M$. Our next task is to discover the links between $\mathfrak{H}(i)$ and $\mathfrak{H}(j)$ when $i \neq j$. We shall concentrate on the conditions

$$(2 \xrightarrow{1} 1) \sum_{P \in \mathfrak{M}(\lambda,\lambda)} z_P P = 0. \blacksquare$$

Let us start with the case $r \geq 2$ and consider

$$\mathfrak{H}(2) = \{ M \in \mathfrak{M}(\lambda,\lambda) \mid m_{11} = n-r-1,\ m_{12} = 1 \}$$
$$\mathfrak{H}(3) = \{ M \in \mathfrak{M}(\lambda,\lambda) \mid m_{11} = n-r-1,\ m_{13} = 1 \}$$

Take

$$M = \left[\begin{array}{ccc|c} n-r-1 & 1 & 0 & \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & \\ \hline & 0 & & I_{r-2} \end{array}\right] \in \mathfrak{H}(2),$$

$$N = \left[\begin{array}{ccc|c} n-r-1 & 0 & 1 & \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & \\ \hline & 0 & & I_{r-2} \end{array}\right] \in \mathfrak{H}(3).$$

Apply (3.10),

$$(2 \xrightarrow{1} 1)M = M_{1,1} = (2 \xrightarrow{1} 1)N,$$

where

$$M_{1,1} = \left[\begin{array}{ccc|c} n-r-1 & 1 & 1 & \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \\ \hline & 0 & & I_{r-2} \end{array}\right].$$

We want to show that if

$$P \in \bigcup_{i=1}^{r+1} \mathfrak{H}(i), \qquad P \neq M,\ P \neq N,$$

then

$$P_{1,1} \neq M_{1,1},$$

where $P_{1,1}$ arises from $(2 \xrightarrow{1} 1)P = k_{1,P}P_{1,1}$, $k_{1,P} \in \mathbb{Z}$. Consider the following cases:

(i) $P \in \mathfrak{H}(1)$. Then $p_{11} = n-r$, $p_{21} = 0$. Hence the (1,1)–entry of $P_{1,1}$ is n–$r$. It is clear that $P_{1,1} \neq M_{1,1}$.

(ii) $P \in \mathfrak{H}(2)$, $P \neq M$. There exists an integer $\ell$, $2 < \ell \leq r+1$, such that the $\ell$–th row of $M$ of $P$ is different from the $\ell$–th row of $M$. Therefore the $\ell$–th row of $P_{1,1}$ is different from the $\ell$–th row of $M_{1,1}$.

(iii) $P \in \mathfrak{H}(3)$, $P \neq N$. Use the same argument in (ii).

(iv) $P \in \mathfrak{H}(q)$, $q > 3$. Then $p_{1q} = 1$ and $p_{2q} = 0$. The $(1,q)$–entry of $P_{1,1}$ is then $p_{1q} + p_{2q} = 1 + 0 = 1$. Therefore $P_{1,1} \neq M_{1,1}$ since the

$(1,q)$–entry of $M_{1,1}$ is 0 when $q > 3$.

Hence

$$(2 \xrightarrow{1} 1) \sum \Big\{ z_P P \ \Big|\ P \in \bigcup_{i=1}^{r+1} \mathfrak{H}(i) \Big\} = 0$$

implies

$$(2 \xrightarrow{1} 1)\ [z_M M + z_N N\,] = (z_M + z_N)\ M_{1,1} = 0,$$

which forces

$$z_M + z_N = 0.$$

The above discussion shows that if $M$, $N \in \mathfrak{H}(2) \cup \mathfrak{H}(3)$, then either $z_N = z_M$ *or* $z_N = -\,z_M$, by combining the result in (4.8). In general, we have

**(4.12) LEMMA.** *Assume* $\lambda = (n-r,1^r)$, $r \geq 2$. *Let* $M$ *and* $N$ *be elements in*

$$\mathfrak{H}(2) \cup \mathfrak{H}(3) \cup \cdots \cup \mathfrak{H}(r+1),$$

*then*

$$(i+1 \xrightarrow{1} i) \sum_{P \in \mathfrak{M}(\lambda,\lambda)} z_P P = 0. \qquad 1 \leq i \leq r,$$

imply that $z_N = z_M$ *or* $z_N = -\,z_M$.

**PROOF.** The similar argument can be applied to

$$\mathfrak{H}(2) \cup \mathfrak{H}(4),\ \ldots\ ,\ \mathfrak{H}(2) \cup \mathfrak{H}(r+1).\blacksquare$$

Finally, we study the links between $\mathfrak{H}(1)$ and $\mathfrak{H}(q)$, $q > 1$. Take

$$P = \left[\begin{array}{cc|c} n-r & 0 & \\ 0 & 1 & 0 \\ \hline \multicolumn{2}{c|}{0} & I_{r-1} \end{array}\right] \in \mathfrak{H}(1),$$

$$Q = \left[\begin{array}{cc|c} n-r-1 & 1 & \\ 1 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & I_{r-1} \end{array}\right] \in \mathfrak{H}(2).$$

We intend to show that if $\lambda = (n-r,1^r)$, $r \geq 2$, then

**(4.13) OBSERVATION.** *The condition*

$$(2 \xrightarrow{1} 1)\sum\left\{ z_M M \;\middle|\; M \in \bigcup_{i=1}^{r+1} \mathfrak{H}(i) \right\} = 0$$

*implies that*

$$z_P + (n-r)z_Q = 0.$$

**PROOF.** By applying (3.10) again, we have

$$(2 \xrightarrow{1} 1)P = P_{1,1},$$

$$(2 \xrightarrow{1} 1)Q = (n-r)P_{1,1},$$

where

$$P_{1,1} = \left[\begin{array}{cc|c} n-r & 1 & \\ 0 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & I_{r-1} \end{array}\right].$$

We must show that if

$$M \in \bigcup_{i=1}^{r+1} \mathfrak{H}(i), \quad M \neq P,\ M \neq Q,$$

then $M_{1,1} \neq P_{1,1}$, where $M_{1,1}$ arises from $(2 \xrightarrow{1} 1)M = kM_{1,1}$, $k \in \mathbb{Z}$. Consider the following cases:

(i) $M \in \mathfrak{H}(1)$, $M \neq P$. Notice that in the incidence matrix $P$,

$$E_{p(i)} = E_i\ , \qquad i = 2, \dots, r+1.$$

If $M \neq P$, There exists an integer $\ell$, $2 < \ell \leq r+1$, such that

$$E_{m(\ell)} \neq E_\ell = E_{p(\ell)}.$$

But $E_{m(\ell)}$ and $E_{p(\ell)}$ are the $\ell$–th rows of $M_{1,1}$ and $P_{1,1}$ respectively. Thus

$$M_{1,1} \neq P_{1,1}.$$

(ii) $M \in \mathfrak{H}(2)$, $M \neq Q$. The argument is similar to (i).

(iii) $M \in \mathfrak{H}(v)$, $v \geq 3$. Then $m_{1v} = 1$, and thus the $(1,v)$–entry of $M_{1,1}$ is one, but the $(1,v)$–entry of $P_{1,1}$ is zero, which means $M_{1,1} \neq P_{1,1}$.

Therefore the condition

$$(2 \xrightarrow{1} 1)\sum\left\{ z_M M \;\middle|\; M \in \bigcup_{i=1}^{r+1} \mathfrak{H}(i) \right\} = 0$$

implies

$$(2 \xrightarrow{1} 1)\ [z_P P + z_Q Q] = 0,$$

i.e.

$$[z_P + (n-r)z_Q]\ P_{1,1} = 0.$$

Thus

$$z_P + (n-r)z_Q = 0$$

as claimed.∎

When $\lambda = (n-1,1)$, there are only two $(\lambda,\lambda)$–incidence matrices

$$P = \begin{bmatrix} n-1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad Q = \begin{bmatrix} n-2 & 1 \\ 1 & 0 \end{bmatrix},$$

it is easy to check that $(2 \xrightarrow{1} 1)(z_P P + z_Q Q) = 0$ implies that

$$z_P + (n-1)z_Q = 0.$$

We summarize the above investigations in the following :

(4.14) NOTES. *Assume that* $\mathrm{char}(K) \neq 2$ *and* $\lambda = (n-r,1^r)$, $1 \le r \le n-2$. *Let* $\{ z_M \mid M \in \mathfrak{M}(\lambda,\lambda) \}$ *be a solution to*

$$(i+1 \xrightarrow{1} i) \sum_{P \in \mathfrak{M}(\lambda,\lambda)} z_P P = 0. \qquad 1 \le i \le r.$$

*Then*

(1) $z_M = 0$ *whenever* $M \notin \bigcup_{i=1}^{r+1} \mathfrak{H}(i)$.

(2) *If* $z_{M} = t$, $t \in K$, *for some* $M \in \mathfrak{H}(2)$, *then* $z_{N} \in Kt$, *for each* $N$ *in* $\bigcup_{i=1}^{r+1} \mathfrak{H}(i)$.∎

The notes above gives a proof for Proposition (4.10).

(4.15) **EXAMPLE.** Take $\lambda = (3,1^2)$. The following table gives the $(\lambda,\lambda)$–incidence matrices and the results of $(i{+}1 \longrightarrow i)M$ by applying (3.10) :

| $M \in \mathfrak{M}(\lambda,\lambda)$ | $(3 \xrightarrow{1} 2)M$ | $(2 \xrightarrow{1} 1)M$ |
|---|---|---|
| $M_1 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 3 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |
| $M_2 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 3 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ |
| $M_3 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ | $3\begin{bmatrix} 3 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |
| $M_4 = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ | $3\begin{bmatrix} 3 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ |
| $M_5 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ |
| $M_6 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ |
| $M_7 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ | $2\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ |

Note that $M_7 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is the only element in $\mathfrak{M}(\lambda,\lambda)$ with two rows identical below the hat. We can see from the column $(3 \xrightarrow{1} 2)M$ in the table that

$$(3 \xrightarrow{1} 2)M_7 = 2\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

hence $\sum_{i=1}^{7} z_i(3 \xrightarrow{1} 2)M_i = 0$ yields

$$2z_7 = 0,$$

i.e. $z_7 = 0$, when $\mathrm{char}(K) \neq 2$.

The other six matrices split into three hat–classes:

$$\mathfrak{H}(1) = \{ M_1, M_2 \},$$
$$\mathfrak{H}(2) = \{ M_3, M_5 \},$$
$$\mathfrak{H}(3) = \{ M_4, M_6 \}.$$

Again from the column $(3 \xrightarrow{1} 2)M$ in the table, we deduce that (refer to 4.8):

$$z_1 + z_2 = 0,$$
$$z_3 + z_5 = 0,$$
$$z_4 + z_6 = 0.$$

the condition $\sum_{i=1}^{6} z_i(2 \xrightarrow{1} 1)M_i = 0$ yields three equations (Refer to 4.13):

$$z_1 + 3z_3 = 0,$$

which links up $\mathfrak{H}(1)$ and $\mathfrak{H}(2)$;

$$z_2 + 3z_4 = 0,$$

which links up $\mathfrak{H}(1)$ and $\mathfrak{H}(3)$; and the equation

$$z_5 + z_6 = 0,$$

linking up $\mathfrak{H}(2)$ and $\mathfrak{H}(3)$. Put $z_5 = t$, then

$$\begin{aligned} z_4 &= t, \\ z_3 &= z_6 = -t, \\ z_1 &= 3t, \\ z_2 &= -3t. \end{aligned}$$

It is easy to check that

$$\begin{aligned} \varphi_0 &= 3M_1 - 3M_2 + M_4 - M_3 + M_5 - M_6 \\ &= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 3\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{aligned}$$

is a $\Gamma$–homomorphism from $M^\lambda$ to $S^\lambda$, and spans the $K$–space $\mathrm{Hom}_\Gamma(M^\lambda,S^\lambda)$, where $\Gamma = K\mathfrak{S}_5$, $\mathrm{char}(K) \neq 2$.∎

In (4.1), we have seen that

$$\dim_K \mathrm{Hom}_\Gamma(M^\lambda,S^\lambda) \geq \text{the socle length of } S^\lambda > 0.$$

Thus (4.10) in fact provides the proof of the following

(4.10a) **THEOREM.** *Assume that* $\mathrm{char}(K) \neq 2$ *and*

$$\lambda = (n-r,1^r),\ 1 \leq r \leq n-2.$$

*Then* $\mathrm{Hom}_\Gamma(M^\lambda,S^\lambda)$ *has* $K$*-dimension* 1 (*one*), *and* $S^\lambda$ *has a unique irreducible submodule.*$\blacksquare$

## §4B The Two–parts Partitions : Special Cases

When $r$ is an integer, $1 \leq r \leq n/2$, $\lambda = (n-r,r)$ is called a two–parts partition. Although a great effort has been made by many authors at clearing up the modular structure of the Specht modules $S^{\mu}$, $\mu$ is a partition of $n$, it seems that most of the Specht modules are still left in mystery, except those of hook partitions and two–parts partitions. The homomorphisms from $S^{(n-r,r)}$ to $S^{(n-k,k)}$ were studied by Gwendolen Murphy, who proved that $S^{(n-r,r)}$ has a unique irreducible submodule, hence its socle length is equal to one (see [Murphy, (1982)]). G.D. James found a way of determining the decomposition numbers of $S^{(n-r,r)}$ in §24 [James, (1978b)]. According to his result, each composition factor of $S^{(n-r,r)}$ has multiplicity exactly one. Combining these two facts we can deduce that the $K$–space $\mathrm{Hom}_{K\mathfrak{S}_n}(S^{\lambda^*},S^{\lambda})$ has dimension one, when $\lambda = (n-r,r)$. In the following two sections, we attempt to prove the fact $\dim_K \mathrm{Hom}_{K\mathfrak{S}_n}(S^{\lambda^*},S^{\lambda}) = 1$, $\lambda = (n-r,r)$, by direct computations with incidence matrices corresponding to endomorphisms of $M^{(n-r,r)}$. Our interest in $\mathrm{Hom}_{K\mathfrak{S}_n}(S^{\lambda^*},S^{\lambda})$ originated in a separate study of the Specht modules restricted to the alternating groups.

Let $\lambda = (\lambda_1,\lambda_2)$ be a two–parts partition of $n$, i.e.

$$\lambda_1 + \lambda_2 = \mathrm{n},$$
$$0 < \lambda_2 \leq \lambda_1$$

The set $\mathfrak{M}(\lambda,\lambda)$ consists of the following incidence matrices:

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \begin{bmatrix} \lambda_1-1 & 1 \\ 1 & \lambda_2-1 \end{bmatrix}, \ \ldots, \begin{bmatrix} \lambda_1-\lambda_2 & \lambda_2 \\ \lambda_2 & 0 \end{bmatrix}.$$

It is clear that each $P = (p_{ij})$ in $\mathfrak{M}(\lambda,\lambda)$ is uniquely determined by $p_{12}$ or $p_{11}$. We shall denote the incidence matrix

$$\begin{bmatrix} \lambda_1-k & k \\ k & \lambda_2-k \end{bmatrix}$$

by $P_k$, $k = 0,1,\ldots,\lambda_2$. We notice that each $P_k$ in $\mathfrak{M}(\lambda,\lambda)$ is a symmetric matrix, hence in the points of view in §3B, we have

**(4.16) LEMMA.** *Let $\lambda = (\lambda_1,\lambda_2)$ be a two-parts partition of $n$.*

(i) *Each $\Gamma$-endomorphism $\varphi$ of $M^\lambda$ is self-adjoint, i.e. $\varphi^A = \varphi$.*

(ii) *If $\varphi$ is a $\Gamma$-endomorphism of $M^\lambda$, then* $\mathrm{Im}(\varphi) \leq S^\lambda$ *if and only if* $\mathrm{Ker}(\varphi) \geq S^{\lambda\perp}$.

**PROOF.** Write $\varphi = \sum_k z_k P_k$, $z_k \in K$. Then

$$\varphi^A = \sum_k z_k P_k^A = \sum_k z_k P_k^T = \varphi,$$

according to (3.18). Recall that $\mathrm{Im}(\varphi) \leq S^\lambda$ if and only if

$$(2 \xrightarrow{w} 1)\varphi = 0, \quad w = 1, 2, \ldots, \lambda_2,$$

by (3.9); if and only if for $w = 1, 2, \ldots, \lambda_2$,

$$\varphi^A(2 \xrightarrow{w} 1)^A = 0;$$

if and only if

$$\varphi(1 \xrightarrow{w} 2) = 0, \quad w = 1, 2, \ldots, \lambda_2,$$

since $\varphi^{A} = \varphi$ and $(2 \xrightarrow{w} 1)^{A} = (1 \xrightarrow{w} 2)$ by (3.20); if and only if

$$\varphi(S^{\lambda\perp}) = 0$$

by (3.23).∎

The lemma above implies that

(4.17) COROLLARY. *If $\lambda$ is a two-parts partition of $n$, then the $K$-dimensions of* $\mathrm{Hom}_{\Gamma}(M^{\lambda},S^{\lambda})$ *and* $\mathrm{Hom}_{\Gamma}(M^{\lambda}/S^{\lambda\perp},S^{\lambda})$ *are equal.*∎

Let $\varphi = \sum_{k} z_k P_k$ be a $\Gamma$-endomorphism of $M^{\lambda}$, $\lambda = (\lambda_1,\lambda_2)$. For an integer $w$, $0 < w \leq \lambda_2$, the condition

(4.18) $(2 \xrightarrow{w} 1)\sum_{k} z_k P_k = 0$

yields a homogeneous linear system in the set of unknowns $\{z_k \mid k = 0,\dots,\lambda_2\}$ over $K$. Applying Formula (3.10), we have

(4.19)
$$(2 \xrightarrow{w} 1)\begin{bmatrix}\lambda_1-k & k\\ k & \lambda_2-k\end{bmatrix} = \sum_{\substack{w_1+w_2=w\\ 0\leq w_1\leq k}} \begin{bmatrix}\lambda_1-k+w_1\\ w_1\end{bmatrix}\begin{bmatrix}k+w_2\\ w_2\end{bmatrix}\begin{bmatrix}\lambda_1-k+w_1 & k+w_2\\ k-w_1 & \lambda_2-k-w_2\end{bmatrix}$$

which is a $\mathbb{Z}$-linear combination of $(\lambda,\mu(w))$-incidence matrices $Q_i$, for all $i$, $0 \leq i \leq \lambda_2-w$, where

$$\mu(w) = (\lambda_1+w,\ \lambda_2-w),$$

$$Q_i = \begin{bmatrix} \lambda_1-i & w+i \\ i & \lambda_2-(w+i) \end{bmatrix}, \qquad 0 \le i \le \lambda_2-w.$$

Let $\overset{w}{c}_{ij}$ be the (integer) coefficient of $z_j$ in the equation concerning $Q_i$ given by (4.18) above. Then the homogeneous system on $z_j$, $j = 0,1,\ldots,\lambda_2$ has its coefficient matrix $C^{(w)}$ of the size $(\lambda_2-w+1)$ by $(\lambda_2+1)$.

(4.20) OBSERVATIONS.

(i) Let $\overset{w}{c}_{ij}$ be the integer coefficient of

$$Q_i = \begin{bmatrix} \lambda_1-i & w+i \\ i & \lambda_2-(w+i) \end{bmatrix}$$

in

$$(2 \xrightarrow{w} 1)P_j = (2 \xrightarrow{w} 1) \begin{bmatrix} \lambda_1-j & j \\ j & \lambda_2-j \end{bmatrix}.$$

From (4.19), it is easily seen that

$$\overset{w}{c}_{ij} = 0 \qquad \text{if } i > j \text{ or } j > w+i.$$

(ii) If we agree with the convention

$$\begin{bmatrix} a \\ b \end{bmatrix} = 0 \qquad \text{if } b < 0,$$

the general formula can be written as

$$\overset{w}{c}_{ij} = \begin{bmatrix} \lambda_1-i \\ j-i \end{bmatrix} \begin{bmatrix} w+i \\ w-(j-i) \end{bmatrix}, \quad 0 \le j \le \lambda_2,\ 0 \le i \le \lambda_2-w,\ 1 \le w \le \lambda_2 .$$

(iii) From (ii) above the set of non–zero entries of the integral matrix $C^{(w)}$ has the shape of a parallelogram :

$$\begin{bmatrix} * & \cdots & * & & 0 \\ & * & \cdots & * & \\ & & \ddots & & \ddots \\ 0 & & * & \cdots & * \end{bmatrix}$$

in which the $(i+1)$–th row is as follows :

$$\begin{array}{ccccccccccc} z_0 & \cdots & z_{i-1} & z_i & & z_{i+1} & & \cdots & z_{i+w} & z_{i+w+1} & z_{\lambda_2} \\ \hline 0 & \cdots & 0 & \begin{bmatrix}\lambda_1-i\\0\end{bmatrix}\begin{bmatrix}w+i\\w\end{bmatrix} & & \begin{bmatrix}\lambda_1-i\\1\end{bmatrix}\begin{bmatrix}w+i\\w-1\end{bmatrix} & & \cdots & \begin{bmatrix}\lambda_1-i\\w\end{bmatrix}\begin{bmatrix}w+i\\0\end{bmatrix} & 0 \cdots & 0 \end{array}$$

In particular, the first and the last (from left to the right) non–zero entries in the $(i+1)$–th row of $C^{(w)}$ are

$$c^{w}_{ii} = \begin{bmatrix} w+i \\ w \end{bmatrix}, \quad , c^{w}_{i,i+w} = \begin{bmatrix} \lambda_1-i \\ w \end{bmatrix} .$$

(4.21) EXAMPLE. Take $\lambda = (12,4)$, a partition of 16. There are 5 $(\lambda,\lambda)$–incidence matrices, labelled as

$$P_0 = \begin{bmatrix}12 & 0\\ 0 & 4\end{bmatrix},\ P_1 = \begin{bmatrix}11 & 1\\ 1 & 3\end{bmatrix},\ P_2 = \begin{bmatrix}10 & 2\\ 2 & 2\end{bmatrix},\ P_3 = \begin{bmatrix}9 & 3\\ 3 & 1\end{bmatrix},\ P_4 = \begin{bmatrix}8 & 4\\ 4 & 0\end{bmatrix}.$$

The condition

$$(2 \xrightarrow{1} 1) \sum_{k=0}^{4} z_k P_k = 0$$

yields a homogeneous linear system in the unknowns $z_k$, $k = 0,1,2,3,4$, with coefficient matrix

$$C^{(1)} = \begin{bmatrix} \binom{12}{0}\binom{1}{1} & \binom{12}{1}\binom{1}{0} & & & \\ & \binom{11}{0}\binom{2}{1} & \binom{11}{1}\binom{2}{0} & & \\ & & \binom{10}{0}\binom{3}{1} & \binom{10}{1}\binom{3}{0} & \\ & & & \binom{9}{0}\binom{4}{1} & \binom{9}{1}\binom{4}{0} \end{bmatrix}.$$

When $K$ is the field of rational numbers $\mathbb{Q}$, it is clear that the solution space has dimension one, since the first four columns of $C^{(1)}$ form a submatrix which has non–zero determinant.

(4.22) **OBSERVATION.** *If $K$ has characteristic zero, the $K$–space $\mathrm{Hom}_\Gamma(M^\lambda,S^\lambda)$ has dimension one, where $\lambda$ is a two–parts partition of $n$ .*

**PROOF.** $S^\lambda = \mathrm{Ker}(2 \xrightarrow{1} 1)$ when $\mathrm{char}(K) = 0$, by (3.12). Thus the K–dimension of $\mathrm{Hom}_\Gamma(M^\lambda,S^\lambda)$ is equal to the dimension of the solution space on unknowns $\{ z_k \}_{k=0}^{\lambda_2}$ yielded by

$$(2 \xrightarrow{1} 1) \sum_k z_k P_k = 0.$$

The coefficient matrix $C^{(1)}$ has rank $\lambda_2$, according to the analysis in (4.20). Therefore

$$\dim_K \mathrm{Hom}_\Gamma(M^\lambda,S^\lambda) = 1. \blacksquare$$

The above result is true for all partition $\lambda$ when $\mathbb{Q}$ is the ground field, in fact $\mathrm{Hom}_\Gamma(M^\lambda,S^\lambda)$ is isomorphic to $\mathrm{Hom}_\Gamma(S^\lambda,M^\lambda)$ as $\mathbb{Q}$–spaces since $\Gamma = \mathbb{Q}\mathfrak{S}_n$ is a semisimple ring. When char($K$) is a positive prime $p$, the matrix $\mathcal{C}^{(1)}$ has rank less than or equal to $\lambda_2$ over $K$. Denote by $\mathcal{C}^{(w)}(\mathrm{mod}\ p)$ the matrix obtained from $\mathcal{C}^{(w)}$ by taking every entry modulo $p$, where $p = \mathrm{char}(K)$. For instance, in Example (4.21),

$$\mathcal{C}^{(1)}(\mathrm{mod}\ 2) = \begin{bmatrix} 1 & 0 & & & \\ & 0 & 1 & & \\ & & 1 & 0 & \\ & & & 0 & 1 \end{bmatrix}$$

which has rank 3 over $\mathbb{Z}_2$ ; and

$$\mathcal{C}^{(1)}(\mathrm{mod}\ 3) = \begin{bmatrix} 1 & 0 & & & \\ & 2 & 2 & & \\ & & 0 & 1 & \\ & & & 1 & 0 \end{bmatrix}$$

which has also rank 3 over $\mathbb{Z}_3$.

Let $K$ be a field of characteristic $p$ where $p$ is a positive prime. If $\lambda = (\lambda_1,\lambda_2)$ is a two–parts partition of $n$, then

$$S^\lambda = \bigcap_{s\geq 0} \mathrm{Ker}(2 \xrightarrow{p^s} 1)$$

by (3.14) and (3.15). If

$$\lambda_2 = b_r p^r + b_{r-1}p^{r-1} + \cdots + b_1 p + b_0, \qquad 0 \leq b_i \leq p-1,\ b_r > 0,$$

we can work on the linear system with coefficient matrix

$$C = \begin{bmatrix} C^{(1)} \\ C^{(p)} \\ \vdots \\ C^{(p^r)} \end{bmatrix} \qquad \text{or} \qquad C(\text{mod } p) = \begin{bmatrix} C^{(1)}\ (\text{mod } p) \\ C^{(p)}\ (\text{mod } p) \\ \vdots \\ C^{(p^r)}(\text{mod } p) \end{bmatrix}.$$

In fact,

**(4.23) LEMMA.** *The $K$–dimension of* $\mathrm{Hom}_\Gamma(M^\lambda, S^\lambda)$ *is equal to*

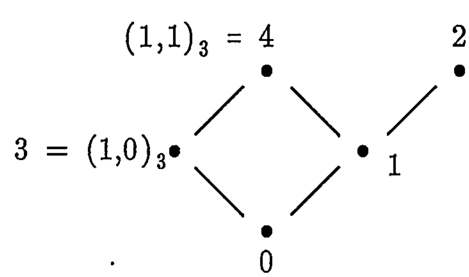
$$(\lambda_2+1) - \mathit{rank}[C(\text{mod } p)]. \ \blacksquare$$

Let $\mathbb{N}_0$ be the set of non–negative integers. We shall write

$$a = (a_s, a_{s-1}, \ldots, a_1, a_0)_p$$

if $a \in \mathbb{N}_0$ and $a = a_s p^s + a_{s-1}p^{s-1} + \cdots + a_1 p + a_0$, $0 \le a_i < p$. For non–negative integers $a$ and $b$, we say that $a \underset{p}{\succeq} b$ if $a_k \ge b_k$, $k \ge 0$. This defines a partial order on the set $\mathbb{N}_0$.

If $a$, $b \in \mathbb{N}_0$ and $a \le b$, we write $[a,b]$ to denote the set of consecutive integers $\{a, a+1, \ldots, b\}$. For example, $[0,4] = \{0,1,2,3,4\}$ and the diagram of the poset $([0,4], \underset{3}{\succeq})$ is as follows :



Note that $2 = (2)_3$ and $4 = (1,1)_3$ are the two maximal elements in this set under the order "$\underset{3}{\succ}$".

In general, there might be more than one maximal elements in the poset $(S, \underset{p}{\succ})$, for a subset $S$ of $\mathbb{N}_0$. We shall soon see that in the homogeneous system on $z_0, z_1, \ldots, z_{\lambda_2}$ with coefficient matrix

$$C = \begin{bmatrix} C^{(1)} \\ C^{(p)} \\ \vdots \\ C^{(p^r)} \end{bmatrix},$$

the values of $z_m$'s, where $m$ is one of the maximal elements in the poset $([0,\lambda_2], \underset{p}{\succ})$, play important roles in the solutions.

**(4.24) LEMMA.** *Assume that* $0 \leq a < b \leq \lambda_2$, *and* $b \underset{p}{\succ} j$ *for all* $j$ *in* $([a,b], \underset{p}{\succ})$. *If* $[\zeta_0, \zeta_1, \ldots, \zeta_{\lambda_2}]^T$ *is a solution to the system*

$$C \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_{\lambda_2} \end{bmatrix} = 0,$$

*then* $\zeta_j \in K\zeta_b$ *for all* $j$ *in* $[a,b]$.

**PROOF.** It suffices to show the following

(4.25) *There exists a submatrix of* $C$, *consisting of* $(b-a)$ *rows of* $C$, *which is* *of the form*

$$
\begin{array}{cccc}
z_0 \cdots z_{a-1} & z_a \; z_{a+1} \cdots z_{b-1} \; z_b & z_{b+1} \cdots z_{\lambda_2} \\
\left[\begin{array}{c|cccc|c}
 & t_a & * & * \cdots & * & \\
0 & & t_{a+1} & * \cdots & * & 0 \\
 & & & \ddots \quad \ddots & \vdots & \\
 & 0 & & t_{b-1} & * &
\end{array}\right]
\end{array}
$$

*satisfying* $t_i \not\equiv 0 \pmod p$, $i = a, a+1, \ldots, b-1$.

Write $b = (b_r, b_{r-1}, \ldots, b_1, b_0)_p$, $b_r > 0$. If $i = (i_r, i_{r-1}, \ldots, i_1, i_0)_p$ is an integer in the set $[a,b]$, then

$$b_k \geq i_k, \qquad k = 0, 1, \ldots, r,$$

since $b \underset{p}{\succ} i$ by assumption. If $a \leq i < b$, there exists $s$, $0 \leq s \leq r$, such that

$$0 \leq i_s < b_s \leq p-1,$$
$$i_k = b_k, \ 0 \leq k < s.$$

Consider the block $C^{(p^s)}$ of $C$. By (4.20) (iii), the first non-zero integer entry in the $(i+1)$-th row of $C^{(p^s)}$ is

$$c_{ii}^{p^s} = \begin{bmatrix} i+p^s \\ p^s \end{bmatrix} .$$

By (3.13),

$$c_{ii}^{p^s} \equiv \begin{bmatrix} i_s+1 \\ 1 \end{bmatrix} \pmod p \not\equiv 0 \pmod p$$

since $0 \le i_s < p-1$. Furthermore, if $j > b$, then $j > i+p^s$, since $i_s < b_s$. Thus by (4.20) (i),

$$c_{ij}^{p^s} = 0, \qquad \text{whenever } j > b.$$

Therefore the submatrix claimed in (4.25) does exist and (4.24) is proved.∎

(4.26) **NOTES ON THE PARTIAL ORDER " $\underset{p}{\succ}$ "**

(1) Let $p$ be a positive prime and $d = (d_r,\ldots,d_1,d_0)_p > 0$, $0 \le d_i \le p-1$, $(\forall i)$. Examine the sequence $d_0, d_1, \ldots, d_r$. Let $k$ be the first index such that $k \ge 1$, $d_k \ne 0$, and the set $\{d_0, d_1, \ldots, d_{k-1}\}$ contains a digit different from $(p-1)$, if such a $k$ exists. Then define

$$f(d) = (d_r,\ldots,d_{k+1},d_k-1,p-1,\ldots,p-1)_p \ .$$

This defines $f(d)$ for all $d$ not of the form $d = (d_r,p-1,\ldots,p-1)_p$ for $r \ge 1$, or $0 < d \le p-1$. We shall state and prove the following facts.

(2) If $f(d)$ is defined for some $d$, then $f(d)$ is a maximal element in the poset $([0,d], \underset{p}{\succ})$.

PROOF. It is enough to show that if $j \in ([0,d], \underset{p}{\succ})$ and $j \underset{p}{\succ} f(d)$, then $j = f(d)$. Obviously $f(d) \le j \le d$. Write $j = (j_r,\ldots,j_1,j_0)_p$, then $j_r = d_r,\ldots,$ $j_{k+1} = d_{k+1}$, and $j_{k-1} = \cdots = j_0 = p-1$. $j_k$ is either $d_k$ or $d_k-1$. Notice that one of $d_{k-1}, d_{k-2}, \ldots, d_0$ is less than $p-1$, according to the construction of $f(d)$. Suppose $j_k = d_k$, then

$$j = (d_r,\dots,d_{k+1},d_k,p-1,\dots,p-1)_p > d,$$

a contradiction. Therefore $j_k = d_k-1$. It follows that $j = f(d)$.

(3) If $f(d)$ is defined for some $d$, then $d \underset{p}{\succ} i$ for all $i$ in $[f(d)+1,d]$.

PROOF. Assume that $f(d) < i \leq d$, $i = (i_r,\dots,i_1,i_0)_p$, $0 \leq i_s \leq p-1$, for all $s$. We must have $i_r = d_r, \dots, i_{k+1} = d_{k+1}$. Noticing that

$$i \geq f(d)+1 = (d_r,\dots,d_{k+1},d_k,0,\dots,0)_p,$$

we can see that $i_k = d_k$, hence $i = (d_r,\dots,d_{k+1},d_k,i_{k-1},\dots,i_0)_p$. Suppose that

$$d_{k-1} = \cdots = d_{k-1} = 0, \quad d_{k-h-1} \neq 0.$$

Then

$$i_{k-1} = \cdots = i_{k-1} = 0, \quad i_{k-h-1} \leq d_{k-h-1},$$

and by construction of $f(d)$ $d_{k-h-2} = \cdots = d_0 = p-1$. It follows that $d \underset{p}{\succ} i$ for all $i$ in $[f(d)+1,d]$.

(4) If $f(d)$ is defined for some $d$, then $f(d)$ is the largest maximal element in $([0,d-1], \underset{p}{\succ})$.

PROOF. This is a corollary of (3).

(5) For $d = (d_r,\dots,d_1,d_0)_p > 0$, $0 \leq d_i \leq p-1$, $(\forall i)$, $d_r > 0$, we can construct a sequence in the set $[0,d]$ as follows. Take $d$ as the first term. If $f(d)$ is not defined, stop. Otherwise, take $f(d)$ to be the second term. Applying the process in (1) to $f(d)$, we either stop or construct

$$f^2(d) = f(f(d)).$$

Repeat this process, to obtain

$$d, f(d), \ \dots \ , \ f^s(d),$$

such that $f[f^s(d)]$ is not defined. This is the descending sequence of all the maximal elements in the poset $([0,d], \underset{p}{\succ})$, by (2) and (3) above. In particular, the smallest maximal element in $([0,d], \underset{p}{\succ})$ is $d$ itself when $f(d)$ is not defined, and is $(d_r-1,p-1,\dots,p-1)_p = d_rp^r-1$ otherwise.∎

In the poset $([0,\lambda_2], \underset{p}{\succ})$, let $\lambda_2 = m(0) > \cdots > m(s)$ be the descending sequence of all the maximal elements constructed as in (4.26)(5), then we have immediately from (4.24) and (4.26)(2) and (3):

**(4.27) COROLLARY.** *If* $[\zeta_0,\zeta_1,\dots,\zeta_{\lambda_2}]^T$ *is a solution to the system*

$$C \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_{\lambda_2} \end{bmatrix} = 0,$$

*then* $\zeta_j \in K\zeta_{m(k)}$, $m(k+1) < j \leq m(k)$, $k = 0, 1, \dots, s-1$ ; *and*

$$\zeta_j \in K\zeta_{m(s)} \ , \qquad \textit{for all } j \textit{ in } [0,m(s)].∎$$

In (4.25), we carefully choose a submatrix whose $(a+1)$–th, $(a+2)$–th,...,$b$–th columns form an upper triangular matrix. A parallel discussion leads to a lower triangular one.

**(4.28) LEMMA.** *Assume that* $\lambda_1-\lambda_2 \leq a < b \leq \lambda_1$, and $b \underset{p}{\succ} j$ *for all* $j$ *in* $([a,b], \underset{p}{\succ})$. *If* $[\zeta_0,\zeta_1,\dots,\zeta_{\lambda_2}]^T$ *is a solution to the system*

$$C \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_{\lambda_2} \end{bmatrix} = 0,$$

*then* $\zeta_{\lambda_1 - k} \in K\zeta_{\lambda_1 - b}$ *for all* $k$ *in* $[a,b]$.

**PROOF.** If $k < b$ and $k = (k_r, k_{r-1}, \ldots, k_1, k_0)_p \underset{p}{\precneqq} b = (b_r, b_{r-1}, \ldots, b_1, b_0)_p$, $b_r > 0$, there exists $s$, $0 \leq s \leq r$, such that

$$0 \leq k_s < b_s \leq p-1,$$
$$k_j = b_j, \quad 0 \leq j < s.$$

In the $(i+1)$–th row of $C^{(p^s)}$, the last (from left to the right) non–zero entry is

$$c^{p^s}_{i,\, i+p^s} = \begin{bmatrix} \lambda_1 - i \\ p^s \end{bmatrix},$$

by (4.20) (iii). Put $i = \lambda_1 - k - p^s$, then

$$c^{p^s}_{i,\, i+p^s} = c^{p^s}_{i,\, \lambda_1 - k} = \begin{bmatrix} k+p^s \\ p^s \end{bmatrix} \equiv \begin{bmatrix} k_s+1 \\ 1 \end{bmatrix} \pmod p \not\equiv 0 \pmod p.$$

Thus we can choose a submatrix of $C$, consisting of $(b-a)$ rows of $C$, in the form

(4.29)

$$\begin{array}{ccccccc} z_{\lambda_1-\lambda_1} \cdots & z_{\lambda_1-b-1} & z_{\lambda_1-b} & z_{\lambda_1-b+1} \cdots & z_{\lambda_1-a} & \cdots & z_{\lambda_2} \end{array}$$
$$\left[\begin{array}{c|ccc|c} & * & t_{b-1} & 0 & \\ 0 & \vdots & \ddots & \ddots & 0 \\ & * & \cdots\; * & t_a & \end{array}\right]$$

satisfying $t_j \not\equiv 0 \pmod p$, $j = a, a+1, \dots, b-1$. ∎

**(4.30) EXAMPLES AND REMARKS.**

(1) We shall complete the computation started in (4.21) dealing with $\lambda = (12,4)$. By applying (4.20), we have

$$C^{(1)} = \begin{bmatrix} \binom{12}{0}\binom{1}{1} & \binom{12}{1}\binom{1}{0} & & & \\ & \binom{11}{0}\binom{2}{1} & \binom{11}{1}\binom{2}{0} & & \\ & & \binom{10}{0}\binom{3}{1} & \binom{10}{1}\binom{3}{0} & \\ & & & \binom{9}{0}\binom{4}{1} & \binom{9}{1}\binom{4}{0} \end{bmatrix} = \begin{bmatrix} 1 & 12 & & & \\ & 2 & 11 & & \\ & & 3 & 10 & \\ & & & 4 & 9 \end{bmatrix}$$

$$C^{(2)} = \begin{bmatrix} \binom{12}{0}\binom{2}{2} & \binom{12}{1}\binom{2}{1} & \binom{12}{2}\binom{2}{0} & & \\ & \binom{11}{0}\binom{3}{2} & \binom{11}{1}\binom{3}{1} & \binom{11}{2}\binom{3}{0} & \\ & & \binom{10}{0}\binom{4}{2} & \binom{10}{1}\binom{4}{1} & \binom{10}{2}\binom{4}{0} \end{bmatrix} = \begin{bmatrix} 1 & 24 & 66 & & \\ & 3 & 33 & 55 & \\ & & 6 & 40 & 45 \end{bmatrix}$$

$$C^{(3)} = \begin{bmatrix} \binom{12}{0}\binom{3}{3} & \binom{12}{1}\binom{3}{2} & \binom{12}{2}\binom{3}{1} & \binom{12}{3}\binom{3}{0} & \\ & \binom{11}{0}\binom{4}{3} & \binom{11}{1}\binom{4}{2} & \binom{11}{2}\binom{4}{1} & \binom{11}{3}\binom{4}{0} \end{bmatrix} = \begin{bmatrix} 1 & 36 & 198 & 220 & \\ & 4 & 66 & 220 & 165 \end{bmatrix}$$

$$C^{(4)} = \begin{bmatrix} \binom{12}{0}\binom{4}{4} & \binom{12}{1}\binom{4}{3} & \binom{12}{2}\binom{4}{2} & \binom{12}{3}\binom{4}{1} & \binom{12}{4}\binom{4}{0} \end{bmatrix} = \begin{bmatrix} 1 & 48 & 396 & 880 & 495 \end{bmatrix}$$

(2) Assume that $\mathrm{char}(K) = 2$. The diagram of the poset $([0,4], \underset{2}{\succ})$ is as follows :

$$\begin{tikzcd}
 & 3 = (1,1)_2 \arrow[d, dash] & \\
2 = (1,0)_2 \arrow[ur, dash] \arrow[dr, dash] & 1 \arrow[d, dash] & 4 = (1,0,0)_2 \\
 & 0 \arrow[ur, dash] &
\end{tikzcd}$$

$3 = (1,1)_2$ is the smallest maximal element in the poset above, while $4 = (1,0,0)_2$ is another maximal element. The $K$–dimension of

$$\mathrm{Hom}_{\Gamma}(M^{(12,4)}, S^{(12,4)}), \quad \text{where } \Gamma = K\mathfrak{S}_{16},$$

is now equal to the solution space of the system

$$C \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = 0, \qquad \text{where } C = \begin{bmatrix} C^{(1)} \\ C^{(2)} \\ C^{(4)} \end{bmatrix} .$$

Using (1), we have

$$C = \left[\begin{array}{ccccc} 1 & 12 & & & \\ & 2 & 11 & & \\ & & 3 & 10 & \\ & & & 4 & 9 \\ \hline 1 & 24 & 66 & & \\ & 3 & 33 & 55 & \\ & & 6 & 40 & 45 \\ \hline 1 & 48 & 396 & 880 & 495 \end{array}\right] = \left[\begin{array}{ccccc} 1 & 0 & & & \\ & 0 & 1 & & \\ & & 1 & 0 & \\ & & & 1 & 1 \\ \hline 1 & 0 & 0 & & \\ & 1 & 1 & 1 & \\ & & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 1 \end{array}\right] \pmod 2.$$

In the integral matrix $C$, the first, sixth and the third rows form form a submatrix

$$\begin{bmatrix} 1 & 12 & & & \\ & 3 & 33 & 55 & \\ & & 3 & 10 & 0 \end{bmatrix} = \begin{array}{c} \begin{array}{ccccc} z_0 & z_1 & z_2 & z_3 & z_4 \end{array} \\ \begin{bmatrix} 1 & 0 & & & \\ & 1 & 1 & 1 & \\ & & 1 & 0 & 0 \end{bmatrix} \end{array} \pmod 2.$$

Put $z_3 = t$, $t \in K$, then

$$z_0 = z_2 = 0, \quad z_1 = t.$$

On the other hand, the poset $([8,12], \underset{2}{\succ})$ has two maximal elements

$$11 = (1,0,1,1)_2 \quad \text{and} \quad 12 = (1,1,0,0)_2.$$

We can find a submatrix of $C$ according to (4.28) :

$$\begin{bmatrix} 0 & 0 & 1 & & \\ & 1 & 1 & 1 & \\ & & & 0 & 1 \end{bmatrix}$$

which gives $z_4 = z_2 = 0$, $z_1 = z_3$. Combining these results, we have the general solution

$$[\zeta_0,\zeta_1,\zeta_2,\zeta_3,\zeta_4]^{\mathrm{T}} = t\,[0,1,0,1,0]^{\mathrm{T}}, \quad t \in K.$$

(3) Assume $\mathrm{char}(K) = 3$. There are two maximal elements in $([0,4], \underset{3}{\succ})$: $2 = (2)_3$ and $4 = (1,1)_3$. We are now working on the linear system

$$C \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = 0,$$

where

$$C = \begin{bmatrix} C^{(1)} \\ C^{(3)} \end{bmatrix} = \left[\begin{array}{ccccc} 1 & 12 & & & \\ & 2 & 11 & & \\ & & 3 & 10 & \\ & & & 4 & 9 \\ \hline 1 & 36 & 198 & 220 & \\ & 4 & 66 & 220 & 165 \end{array}\right] \equiv \left[\begin{array}{ccccc} 1 & 0 & & & \\ & 2 & 2 & & \\ & & 0 & 1 & \\ & & & 1 & 0 \\ \hline 1 & 0 & 0 & 1 & \\ & 1 & 0 & 1 & 0 \end{array}\right] \pmod 3.$$

The first two rows of $C$ constitute the submatrix corresponding to $([0,2],\underset{3}{\succ})$. We have $z_0 = 0$, $z_1 = -z_2$. The poset $([8,12],\underset{3}{\succ})$ has maximal elements

$$8 = (2,2)_3, \quad 11 = (1,0,2)_3, \quad 12 = (1,1,0)_3.$$

The second and the third rows make up the submatrix corresponding to $([9,11], \underset{3}{\succ})$, which gives $z_3 = 0$, $z_1 = -z_2$. This example show that the information from (4.25) and (4.28) is not adequate in determining the dimension of the solution space. In fact, we need the last row of $C$:

$$z_1 + z_3 = 0.$$

Thus $z_1 = -z_3 = 0$, and $z_2 = -z_1 = 0$. Setting

$$z_4 = t, \qquad t \in K,$$

we obtain the general solution

$$[\zeta_0,\zeta_1,\zeta_2,\zeta_3,\zeta_4]^{\mathrm{T}} = t\,[0,0,0,0,1]^{\mathrm{T}}, \quad t \in K. \blacksquare$$

In the remaining part of this section, we shall prove that $\mathrm{Hom}_\Gamma(M^\lambda,S^\lambda)$ has dimension one if $\lambda$ is a two–parts partition with some special features, by making use of (4.25) and (4.28), and leave the proof for a general two–parts partition to the subsequent section by using mathematical induction.

(4.31) **LEMMA.** *Let* $\lambda = (\lambda_1,\lambda_2)$ *be a two–parts partition of* $n$. *If*

$$\lambda_2 \underset{p}{\succ} i, \qquad \text{for all } i, \quad 0 \leq i \leq \lambda_2,$$

*then*

$$\dim_K \operatorname{Hom}_\Gamma(M^\lambda, S^\lambda) = 1.$$

**PROOF.** According to (4.24),

$$\dim_K \operatorname{Hom}_\Gamma(M^\lambda, S^\lambda) \leq 1.$$

Recall (4.1),

$$\dim_K \operatorname{Hom}_\Gamma(M^\lambda, S^\lambda) > 0.$$

Therefore $\dim_K \operatorname{Hom}_\Gamma(M^\lambda, S^\lambda) = 1.$ ∎

(4.32) **LEMMA.** *The $K$-dimension of* $\operatorname{Hom}_\Gamma(M^\lambda, S^\lambda)$ *is equal to one, if $\lambda$ is the partition $(b,b)$ and $\Gamma = K\mathfrak{S}_{2b}$.*

**PROOF.** Write $b = (b_r, b_{r-1}, \ldots, b_1, b_0)_p$, $0 \leq b_i \leq p-1$, $(\forall i)$, $b_r > 0$. If $b < p$ or $b \equiv -1 \pmod{p^r}$, we can apply (4.31). Now assume that $b > p$ and $b \not\equiv -1 \pmod{p^r}$. We shall prove that the dimension of the solution space to

$$C \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_b \end{bmatrix} = 0, \qquad \text{where } C = \begin{bmatrix} C^{(1)} \\ C^{(p)} \\ \vdots \\ C^{(p^r)} \end{bmatrix},$$

does not exceed one.

By (4.26)(5), $m = b_r p^r - 1$ is the smallest maximal element in the poset $([0,b], \underset{p}{\succ})$. Thus, if $[\zeta_0, \zeta_1, \ldots, \zeta_b]^T$ is a solution to the above system, then

$$\zeta_j \in K\zeta_m, \qquad j \in [0,\ m],$$

by applying (4.27). Meanwhile, (4.28) gives

$$\zeta_{b-k} \in K\zeta_{b-m}, \qquad k \in [0,\ m].$$

Note that

$$\begin{aligned} 2m &= 2(b_r p^r - 1) \\ &= (b_r p^r - 1) + (b_r - 1)p^r + (p-1)p^{r-1} + \cdots + (p-1)p + (p-1) \\ &= (b_r, p-1, \ldots, p-1)_p - 1 + (b_r - 1)p^r \\ &\geq (b_r, b_{r-1}, \ldots, b_1, b_0)_p = b. \end{aligned}$$

Therefore

$$b - m \leq m,$$

and it follows from the discussion above

$$\zeta_{b-m} \in K\zeta_m.$$

Hence

$$\zeta_j \in K\zeta_m \qquad \text{for all } j \text{ in } [0,b].$$

This proves that

$$\dim_K \mathrm{Hom}_\Gamma(M^\lambda, S^\lambda) \leq 1.$$

Thus

$$\dim_K \mathrm{Hom}_\Gamma(M^\lambda, S^\lambda) = 1$$

by (4.1).∎

## §4C Two–parts Partitions : General Case

In this section we shall complete the proof of the following :

(4.33) **THEOREM.** *Let $\lambda = (\lambda_1,\lambda_2)$ be a two–parts partition of $n$ and $K$ is an arbitrary field. Then*

$$\dim_K \mathrm{Hom}_{K\mathfrak{S}_n}(M^\lambda, S^\lambda) = 1. \blacksquare$$

In the previous section, we have seen in (4.30) that if $\lambda_2 \underset{p}{\succ} i$ for all $i$, $0 \le i \le \lambda_2$, then the conclusion in (4.33) above holds. In particular, if $\lambda$ is the partition $(n-1,1)$, $\dim_K \mathrm{Hom}_{K\mathfrak{S}_n}(M^\lambda, S^\lambda) = 1$ for all $n$. This suggests the mathematical induction for proving (4.33), based on the following

(4.34) **LEMMA.** *If $2 \le r \le n/2$, then*

$$\dim_K \mathrm{Hom}_{K\mathfrak{S}_n}(M^{(n-r,r)}, S^{(n-r,r)}) \le \dim_K \mathrm{Hom}_{K\mathfrak{S}_{n-1}}(M^{(n-r,r-1)}, S^{(n-r,r-1)}). \blacksquare$$

The method applied in the proof of the lemma above is quite different from the argument used in §4B. We need to take some notes on the invariants and standard tableaux.

Let $\lambda$ be a partition of $n$ and $x$ be a $\lambda$–tableau. Recall that $Rx$ is the row group of $x$. Define

$$S^\lambda_x = \{\ u \in S^\lambda \mid \pi u = u \ \text{ for all } \pi \text{ in } Rx\ \},$$

then $S_x^\lambda$ is a $K$–subspace of $S^\lambda$. Note that $M^\lambda$ is a cyclic $K\mathfrak{S}_n$–module generated by $\underline{\bar{x}}$ , where $\underline{\bar{x}}$ is the corresponding $\lambda$–tabloid. For each $u$ in $S_x^\lambda$ , there is a $K\mathfrak{S}_n$–homomorphism $\varphi_u$: $M^\lambda \longrightarrow S^\lambda$ satisfying $\varphi_u( \underline{\bar{x}} ) = u$ . Conversely, if $\psi$ : $M^\lambda \longrightarrow S^\lambda$ is a $K\mathfrak{S}_n$–homomorphism, then it is easy to verify that

$$\psi = \varphi_u, \qquad \text{where } u = \psi( \underline{\bar{x}} ).$$

Thus

(4.35) *The map* $u \in S_x^\lambda \longmapsto \varphi_u \in \mathrm{Hom}_{K\mathfrak{S}_n}(M^\lambda, S^\lambda)$ *is a* $K$*–isomorphism.*

Let $\mathfrak{S}$ be the group of all permutations on the set $\mathbb{N} = \{1,2,\ldots\}$. The symmetric group $\mathfrak{S}_n$ on the set $\underline{n}$ can be viewed as the subgroup of $\mathfrak{S}$ :

$$\{ \pi \in \mathfrak{S} \mid \pi(k) = k, \quad k > n \}.$$

In this point of view, $\mathfrak{S}_m$ is a subgroup of $\mathfrak{S}_n$ whenever $m \leq n$. We shall write $\Gamma_n$ for the group algebra $K\mathfrak{S}_n$ in this section. Thus every $\Gamma_n$–module is also a $\Gamma_{n-1}$–module in the natural manner. In the following we shall have a classification of standard $(n-r,r)$–tableaux (ref. 2.20) in order to analyze the $\Gamma_{n-1}$–structure of $M^{(n-r,r)}$ and $S^{(n-r,r)}$. A general treatment on the restrictions of Specht modules can be found in §15 [Peel (1969)].

## (4.36) NOTES ON STANDARD TABLEAUX

(1) Recall that a $\lambda$–tableau is a bijection $x : [\lambda] \longrightarrow \underline{n}$ , where $[\lambda]$ is the diagram of the partition $\lambda$. Let $\rho$ : $\mathbb{N}\times\mathbb{N} \longrightarrow \mathbb{N}$ be the function

$$\rho(i,j) = i, \qquad (i,j) \in \mathbb{N}\times\mathbb{N}.$$

Write

$$\mathfrak{T}^{\lambda} = \{ x \mid x \text{ is a standard } \lambda\text{-tableau} \},$$
$$\mathfrak{T}^{\lambda}_{i} = \{ x \mid x \in \mathfrak{T}^{\lambda},\ \rho x^{-1}(n) = i \}, \quad i = 1,2,\ldots,$$

then

$$\mathfrak{T}^{(n-r,r)} = \mathfrak{T}^{(n-r,r)}_{1} \cup \mathfrak{T}^{(n-r,r)}_{2}.$$

(2) Assume $r < n/2$, hence $r < n-r$. Let

$$y : [(n-r-1,r)] \longrightarrow \underline{n-1}$$

be a standard $(n-r-1,r)$-tableau. define a $\lambda$-tableau $f_0(y) : [(n-r-1,r)] \longrightarrow \underline{n}$ via

$$f_0(y)(i,j) = y_{ij} \text{ if } (i,j) \in [(n-r-1,r)] ;$$
$$f_0(y)(1,n-r) = n.$$

Then $f_0 : \mathfrak{T}^{(n-r-1,r)} \longrightarrow \mathfrak{T}^{(n-r,r)}_{1}$ is a bijection.

(3) Let $x : [(n-r,r)] \longrightarrow \underline{n}$ be a $\lambda$-tableau in $\mathfrak{T}^{(n-r,r)}_{2}$, hence $x_{2r} = n$. Define $g_0(x)$ to be the restriction of $x$ to $[(n-r,r-1)]$. Then

$$g_0 : \mathfrak{T}^{(n-r,r)}_{2} \longrightarrow \mathfrak{T}^{(n-r,r-1)}$$

is a bijection.∎

## (4.37) NOTES ON THE $\Gamma_{n-1}$-MODULE STRUCTURE OF $M^{(n-r,r)}$

(1) The $\Gamma_n$–module $M^{(n-r,r)}$ has a $K$–basis (ref. §2C)

$$\{ \underline{\bar{x}} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \;\Big|\; X_1 \cup X_2 = \underline{n}\ ,\quad X_1 \cap X_2 = \emptyset,\ |X_1| = n-r \}.$$

Let

$$M_1 = \sum_{X_1 \ni n} K\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$M_2 = \sum_{X_2 \ni n} K\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} .$$

Then $M_1$ and $M_2$ are $\Gamma_{n-1}$–submodules of $M^{(n-r,r)}$. Furthermore, $M_1$ is isomorphic to $M^{(n-r-1,r)}$, $M_2$ is isomorphic to $M^{(n-r,r-1)}$ as $\Gamma_{n-1}$–modules.

(2) $M^{(n-r,r)}$ is the internal direct sum of $M_1$ and $M_2$ over $\Gamma_{n-1}$.

(3) For a $(n-r-1,r)$–tabloid $\underline{\bar{y}} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$ , define

$$f:\ \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}_{(n-r-1,r)} \longmapsto \begin{bmatrix} Y_1 \cup \{n\} \\ Y_2 \end{bmatrix}_{(n-r,r)},$$

then $f$ extends to a $\Gamma_{n-1}$–homomorphism from $M^{(n-r-1,r)}$ to $M^{(n-r,r)}$.

(4) For a $(n-r,r)$–tabloid $\underline{\bar{x}} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ , define

$$g\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{cases} \begin{bmatrix} X_1 \\ X_2 \backslash \{n\} \end{bmatrix}, & \text{if } n \in X_2, \\ 0, & \text{otherwise.} \end{cases}$$

Then $g$ extends to a $\Gamma_{n-1}$–homomorphism from $M^{(n-r,r)}$ to $M^{(n-r,r-1)}$.

(5) The short $\Gamma_{n-1}$–sequence

$$0 \longrightarrow M^{(n-r-1,r)} \xrightarrow{f} M^{(n-r,r)} \xrightarrow{g} M^{(n-r,r-1)} \longrightarrow 0$$

is split exact.∎

**(4.38) LEMMA.** *Assume that* $2 \leq r < n/2$.

(i) $f(S^{(n-r-1,r)}) \leq S^{(n-r,r)}$.

(ii) $g(S^{(n-r,r)}) \leq S^{(n-r,r-1)}$.

(iii) *The short* $\Gamma_{n-1}$*–sequence*

$$0 \longrightarrow S^{(n-r-1,r)} \xrightarrow{\hat{f}} S^{(n-r,r)} \xrightarrow{\hat{g}} S^{(n-r,r-1)} \longrightarrow 0$$

*is exact, where* $\hat{f}$ *and* $\hat{g}$ *are corresponding restrictions of* $f$ *and* $g$ *respectively.*

**PROOF.** (i) Let

$$y = \begin{matrix} y_{11} & \cdots & y_{1r} & y_{1,r+1} & \cdots & y_{1,n-r-1} \\ y_{21} & \cdots & y_{2r} & & & \end{matrix}$$

be an $(n-r-1,r)$–tableau. $S^{(n-r-1,r)}$ is generated over $\Gamma_{n-1}$ by $\epsilon(Cy)\ \overline{y}$, where $\epsilon(Cy)$ can be written as

$$\prod_{j=1}^{r} [1 - (y_{1j}, y_{2j})] .$$

Let $z$ be the $(n-r,r)$–tableau

$$\begin{array}{llllll} y_{11} & \cdots & y_{1r} & y_{1,r+1} \cdots & y_{1,n-r-1} & n \\ y_{21} & \cdots & y_{2r} & & & \end{array}$$

Then the column groups of $y$ and $z$ are identical, hence $\epsilon(Cz) = \epsilon(Cy)$. Note that $\underline{\bar{z}} = f(\ \underline{\bar{y}}\ )$ by (4.37) (3), and $\epsilon(Cy) \in \Gamma_{n-1}$,

$$f[\epsilon(Cy)\ \underline{\bar{y}}\ ] = \epsilon(Cy)\ f(\ \underline{\bar{y}}\ ) = \epsilon(Cz)\ \underline{\bar{z}}\ ,$$

which is an element of $S^{(n-r,r)}$. This proves (i).

(ii) $S^{(n-r,r)}$ has its standard basis :

$$\{\ \epsilon(Cx)\ \underline{\bar{x}}\ |\ x \in \mathfrak{T}^{(n-r,r)}\ \}.$$

If $x \in \mathfrak{T}_1^{(n-r,r)}$, then $x_{1,n-r} = n$. Since $n-r > r$ by assumption,

$$1 \leq x_{1j} < x_{2j} < n, \qquad 1 \leq j \leq r,$$

thus

$$\epsilon(Cx) = \prod_{j=1}^{r} [1 - (x_{1j}, x_{2j})] \in \Gamma_{n-1}.$$

Therefore

$$g[\epsilon(Cx)\ \underline{\bar{x}}\ ] = \epsilon(Cx) g(\ \underline{\bar{x}}\ ) = 0$$

by (4.37) (4).

If $x \in \mathfrak{T}_2^{(n-r,r)}$, then $x_{2r} = n$, consider the $(n-r,r-1)$-tableau

$$t = g_0(x) = \begin{array}{lllll} x_{11} & \cdots & x_{1,r-1} & \cdots & x_{1,n-r} \\ x_{21} & \cdots & x_{2,r-1} & & \end{array} .$$

We have

$$\epsilon(Cx) = \prod_{j=1}^{r} [1 - (x_{1j}, x_{2j})] = \epsilon(Ct)[1 - (x_{1r}, n)].$$

Therefore

$$g[\epsilon(Cx)\ \underline{\overline{x}}\ ] = \epsilon(Ct)\ g\ \{[1 - (x_{1r}, n)]\ \underline{\overline{x}}\ \}$$

$$= \epsilon(Ct)\ g\left[\underline{\overline{x}} - \underline{\overline{\begin{array}{lllllll} x_{11} & \cdots & x_{1,r-1} & n & \cdots & x_{1,n-r} \\ x_{21} & \cdots & x_{2,r-1} & x_{1r} & & \end{array}}}\right]$$

$$= \epsilon(Ct)\ g\ (\ \underline{\overline{x}}\ ) = \epsilon(Ct)\ g\ (\ \underline{\overline{t}}\ )$$

by (4.37) (4).

(iii) The restriction $\hat{f}$ of $f$ to $S^{(n-r-1,r)}$ is one–to–one, since $f$ is. In fact, if $y$ is a standard $(n-r-1,r)$–tableau, then $\epsilon(Cy)\ \underline{\overline{y}}$ is a member in the standard basis of $S^{(n-r-1,r)}$, and we have seen in the proof of (i) that

$$f\ (\epsilon(Cy)\ \underline{\overline{y}}\ ) = \epsilon(Cz)\ \underline{\overline{z}}\ ,$$

where $z = f_0(y)$ (ref. 4.36). Thus

$$f\ (S^{(n-r-1,r)}) = \sum_{z \in \mathfrak{T}_1^{(n-r,r)}} K\ \epsilon(Cz)\ \underline{\overline{z}}\ .$$

By the proof of (ii), if $x \in \mathfrak{T}_2^{(n-r,r)}$,

$$g[\epsilon(Cx)\ \overline{\underline{x}}\ ] = \epsilon(Ct)\ \overline{\underline{t}}\ ,$$

where $t = g_0(x)$. Since $g_0 : \mathfrak{T}_2^{(n-r,r)} \longrightarrow \mathfrak{T}^{(n-r,r-1)}$ is a bijection,

$$\hat{g} : S^{(n-r,r)} \longrightarrow S^{(n-r,r-1)}$$

is onto. It is also clear from the proof of (ii) that

$$\mathrm{Ker}(\hat{g}) = \sum_{z \in \mathfrak{T}_1^{(n-r,r)}} K\,\epsilon(Cz)\ \overline{\underline{z}} = \mathrm{Im}(\hat{f}). \blacksquare$$

Recall that if $x \in \mathfrak{T}_2^{(n-r,r)}$, the row group of $g_0(x)$ can be viewed as a subgroup of $Rx$ :

$$\{\ \pi \in Rx \mid \pi(n) = n\ \}.$$

Thus if $u \in S_x^{(n-r,r)}$ and $\pi$ is in the row group of $g_0(x)$,

$$\pi g(u) = g(\pi u) = g(u).$$

This proves

**(4.39) LEMMA.** $g(\,S_x^{(n-r,r)}\,) \subseteq S_{g_0(x)}^{(n-r,r-1)}$ . $\blacksquare$

We shall analyze the $K$–space $f(\,S^{(n-r-1,r)}) \cap S_x^{(n-r,r)})$, in the $\Gamma_{n-1}$–exact sequence

$$0 \longrightarrow S^{(n-r-1,r)} \xrightarrow{\hat{f}} S^{(n-r,r)} \xrightarrow{\hat{g}} S^{(n-r,r-1)} \longrightarrow 0.$$

If we can prove that $f(\,S^{(n-r-1,r)}) \cap S_x^{(n-r,r)} = 0$, then the restriction of $g$ to $S^{(n-r-1,r)}$ is a one–to–one $K$–linear mapping into $S_{g_0(x)}^{(n-r,r-1)}$, and (4.34) will follow.

From the proof of (4.38) :

$$f(\,S^{(n-r-1,r)}) = \sum_{z \,\in\, \mathfrak{T}_1^{(n-r,r)}} K\,\epsilon(Cz)\ \underline{\bar{z}}\ .$$

A finer classification of the set $\mathfrak{T}_1^{(n-r,r)}$ is needed at this stage. Let

$$\mathfrak{T}_{11} = \{\ x \in \mathfrak{T}^{(n-r,r)}\ \mid\ \rho x^{-1}(n) = 1,\ \rho x^{-1}(n-1) = 1\ \},$$
$$\mathfrak{T}_{12} = \{\ x \in \mathfrak{T}^{(n-r,r)}\ \mid\ \rho x^{-1}(n) = 1,\ \rho x^{-1}(n-1) = 2\ \}.$$

Informally speaking, $\mathfrak{T}_{11}$ is the set of standard $(n-r,r)$–tableaux such that both $n$ and $(n-1)$ lie in the first row, while $\mathfrak{T}_{12}$ consists of all standard $(n-r,r)$–tableaux with $n$ in the first row and $(n-1)$ in the second row. It is clear that

$$\mathfrak{T}_1^{(n-r,r)} = \mathfrak{T}_{11} \cup \mathfrak{T}_{12}.$$

Assume that $t$ is a standard $(n-r,r)$–tableau in $\mathfrak{T}_{11}$. Then

$$t_{1j} < t_{2j} < n-1, \qquad 1 \leq j \leq r,$$

hence

$$\epsilon(Ct) = \prod_{j=1}^{r} [1 - (t_{1j}, t_{2j})] \in \Gamma_{n-2} = K\mathfrak{S}_{n-2},$$

and $\epsilon(Ct)$ commutes with the transposition $(n-1,n)$. Therefore

$$(n-1,n)\ \epsilon(Ct)\ \underline{\bar{t}} = \epsilon(Ct)\ (n-1,n)(\ \underline{\bar{t}}\ ) = \epsilon(Ct)\ \underline{\bar{t}}\ .$$

If $t \in \mathfrak{T}_{12}$, say

$$t = \begin{array}{lllll} t_{11} & \cdots & t_{1,r} & t_{1,r+1} \cdots & t_{1,n-r} \\ t_{21} & \cdots & t_{2,r} & & \end{array},$$

with $t_{1,n-r} = n$, $t_{2r} = n-1$. Then

$$\epsilon(Ct) = \prod_{j=1}^{r} [1 - (t_{1j}, t_{2j})]$$

$$= \left\{ \prod_{j=1}^{r-1} [1 - (t_{1j}, t_{2j})] \right\} [1 - (t_{1r}, n-1)]\ .$$

It is clear that $\prod_{j=1}^{r-1} [1 - (t_{1j}, t_{2j})]$ commutes with $(n-1,n)$, and

$$(n-1,n) \cdot [1 - (t_{1r}, n-1)] \cdot (n-1,n) = 1 - (t_{1r}, n),$$

hence

$$(n-1,n)\ \epsilon(Ct)\ \underline{\bar{t}}$$

$$= \prod_{j=1}^{r-1} [1 - (t_{1j},t_{2j})]\cdot(n-1,n)\cdot[1 - (t_{1r},n-1)](\ \bar{\underline{t}}\ )$$

$$= \prod_{j=1}^{r-1} [1 - (t_{1j},t_{2j})]\cdot(n-1,n)\cdot[1 - (t_{1r},n-1)]\cdot(n-1,n)^2(\ \bar{\underline{t}}\ )$$

$$= \prod_{j=1}^{r-1} [1 - (t_{1j},t_{2j})]\cdot[1 - (t_{1r},n)]\cdot(n-1,n)(\ \bar{\underline{t}}\ )$$

$$= \epsilon(Ct^*)\ \overline{\underline{t^*}}\ ,$$

where

$$t^* = \begin{matrix} t_{11} & \cdots & t_{1,r-1} & t_{1,r} \cdots & n-1 \\ t_{21} & \cdots & t_{2,r-1} & n & \end{matrix} \qquad ,$$

which is a standard $(n-r,r)$–tableau in $\mathfrak{T}_2^{(n-r,r)}$ (c.f. 4.36 (i)). We claim the following based on the discussion above :

(4.40) **OBSERVATION.** $f(\,S^{(n-r-1,r)}) \cap S_x^{(n-r,r)} \subset \sum_{z \in \mathfrak{T}_{11}} K\ \epsilon(Cz)\ \bar{\underline{z}}$ .

**PROOF.** Let $u$ be an element in $f(\,S^{(n-r-1,r)}) \cap S_x^{(n-r,r)}$. Write

$$u = \sum_{z \in \mathfrak{T}_1} \alpha_z\ \epsilon(Cz)\ \bar{\underline{z}}\ , \qquad \alpha_z \in K.$$

Then necessarily $(n-1,n)u = u$, hence by the notes above

$$u = \sum_{z \in \mathfrak{T}_{11}} \alpha_z \, \epsilon(Cz) \, \underline{\bar{z}} + \sum_{t \in \mathfrak{T}_{12}} \alpha_t \, \epsilon(Ct) \, \underline{\bar{t}}$$

$$= (n-1,n)u$$

$$= \sum_{z \in \mathfrak{T}_{11}} \alpha_z \, \epsilon(Cz) \, \underline{\bar{z}} + \sum_{t \in \mathfrak{T}_{12}} \alpha_t \, \epsilon(Ct^*) \, \underline{\overline{t^*}}$$

it follows that $\alpha_t = 0$, $t \in \mathfrak{T}_{12}$.▮

If $n-r = r+1$, then $\mathfrak{T}_{11} = \emptyset$, $f(S^{(n-r-1,r)}) \cap S_x^{(n-r,r)} = 0$ by (4.40). Hence (4.34) holds in this case. Assume that $n-r > r+1$, i.e.

$$n-r \geq r+2,$$

then $\mathfrak{T}_{11} \neq \emptyset$ and $\mathfrak{T}_{11} = \mathfrak{T}_{111} \cup \mathfrak{T}_{112}$, where

$$\mathfrak{T}_{111} = \{ x \in \mathfrak{T}_{11} \mid \rho x^{-1}(n-2) = 1 \}$$
$$\mathfrak{T}_{112} = \{ x \in \mathfrak{T}_{11} \mid \rho x^{-1}(n-2) = 2 \}.$$

Let $u$ be an element in

$$S_x^{(n-r,r)} \cap \sum_{z \in \mathfrak{T}_{11}} K \, \epsilon(Cz) \, \underline{\bar{z}} \, .$$

Then by comparing the both sides of $(n-2,n-1)u = u$, the similar argument as in (4.40) leads us to

$$f(S^{(n-r-1,r)}) \cap S_x^{(n-r,r)} \subset \sum_{z \in \mathfrak{T}_{111}} K \, \epsilon(Cz) \, \underline{\bar{z}} \, .$$

Repeating this process, we find that it remains to consider the following case :

(4.41) $n-r \geq 2r$.

In this case,

$$f(\, S^{(n-r-1,r)}\,) \cap S_x^{(n-r,r)} \subset \sum_{z \in \mathfrak{T}_*} K\ \epsilon(Cz)\ \bar{\underline{z}}\ ,$$

where

$$\mathfrak{T}_* = \{\ z \mid z \text{ is a standard } (n-r,r)\text{–tableau, } \rho z^{-1}(k) = 1, \\ k = n-r+1,\ \dots\ ,\ n-1,\ n\ \}$$

Informally speaking a standard $(n-r,r)$–tableau $z$ is in $\mathfrak{T}_*$ if and only if the numbers $n-r+1, \dots, n-1, n$ are all lying in the first row of $z$.

Note that

$$U = \sum_{z \in \mathfrak{T}_*} K\ \epsilon(Cz)\ \bar{\underline{z}}$$

is a $\Gamma_{n-r}$–submodule of $S^{(n-r,r)}$, isomorphic to the Specht module $S^{(n-2r,r)}$ over $\Gamma_{n-r}$. Take the $(n-r,r)$–tableau $x$ to be

$$x = \begin{matrix} 1 & 2 & \cdots & r & r+1 & \cdots & n-r \\ n-r+1 & n-r+2 & \cdots & n & & & \end{matrix}\ ,$$

then the stabilizer group of the first row of $x$ is the symmetric group $\mathfrak{S}_{n-r}$ on the set $\{1,2,\dots,n-r\}$. If

$$u \in U \cap S_x^{\lambda}\ ,$$

then necessarily

$$\pi u = u \qquad \text{for all } \pi \text{ in } \mathfrak{S}_{n-r}.$$

Suppose $u \neq 0$, then $Ku$ is a $\Gamma_{n-r}$–submodule of

$$U = \sum_{z \in \mathfrak{T}_*} K\, \epsilon(Cz)\, \bar{\underline{z}}\ .$$

Note that $Ku$ is isomorphic to the trivial module over $\Gamma_{n-r}$, while $U$ is isomorphic to the Specht module $S^{(n-2r,r)}$ over $\Gamma_{n-r}$. By appying (3.35) to the $\Gamma_{n-r}$–module $S^{(n-2r,r)}$, we learn that $S^{(n-2r,r)}$ has a submodule isomorphic to the trivial $\Gamma_{n-r}$–module if and only if

$$n - 2r \equiv -1 \pmod{p^{\ell}},$$

where $\ell = \ell_p(r)$ (c.f. the note prior to 3.33). Therefore, we have proved

(4.42) **OBSERVATION.** *Assume that* $2 \leq r < n/2$. *Unless*

$$n - 2r \equiv -1 \pmod{p^{\ell}}, \qquad \ell = \ell_p(r),$$

$f(S^{(n-r-1,r)}) \cap S_x^{(n-r,r)} = 0$, *hence*

$$\dim_K S_x^{(n-r,r)} \leq \dim_K S_{g_0(x)}^{(n-r,r-1)}\ . \blacksquare$$

If $\lambda = (n-r,r)$, $2 \leq r < n/2$, $n - 2r \equiv -1 \pmod{p^{\ell}}$, where $\ell = \ell_p(r)$, we can prove that the Specht module $S^{(n-r,r)}$ is irreducible over $\Gamma_n$. The general criterion for a Specht module being irreducible was conjectured by R. W. Carter (ref. 1.2 [James (1978a)]) and was proved in [James (1978a)] and [James and Murphy (1979)]. Instead of quoting this deep result, we adopt a sufficient condition found by James in 24.9 [James (1978b)] :

**(4.43) LEMMA.** *Let* $\lambda = (\lambda_1,\lambda_2)$ *be a two–parts partition of n. Assume that* $\mathrm{char}(K) = p$ *and*

$$\lambda_1 - \lambda_2 \equiv -1 \pmod{p^\ell},$$

*where* $\ell = \ell_p(\lambda_2)$, *then the Specht module* $S^\lambda$ *is irreducible over* $\Gamma_n = K\mathfrak{S}_n$.

**PROOF.** (c.f. the proof of 24.1 in [James (1978b)])

Consider the $\Gamma_n$–endomorphism of $M^\lambda$ :

$$Q = \begin{bmatrix} \lambda_1 - \lambda_2 & \lambda_2 \\ \lambda_2 & 0 \end{bmatrix},$$

By (3.29), $Q$ is reverse semistandard homomorphism, hence $Q\,(\,S^\lambda) \neq 0$ (c.f. 3.28). By the Submodule Theorem (2.8), $\mathrm{Ker}(\,Q\,) \leq S^{\lambda\perp}$. The condition

$$\lambda_1 - \lambda_2 \equiv -1 \pmod{p^\ell}$$

assures that

$$(2 \xrightarrow{w} 1)Q = 0, \qquad 1 \leq w \leq \lambda_2,$$

by (3.33) and (3.34). Thus $\mathrm{Im}(\,Q\,) \leq S^\lambda$. Also notice that $Q = Q^{\mathrm{T}}$,

$$Q\,(1 \xrightarrow{w} 2) = [(2 \xrightarrow{w} 1)Q^{\mathrm{T}}]^{\mathrm{T}} = [(2 \xrightarrow{w} 1)Q\,]^{\mathrm{T}} = 0$$

by (3.16) (iii) and (3.18). It follows from (3.23) that $S^{\lambda\perp} \leq \mathrm{Ker}(\,Q\,)$, hence

$$S^{\lambda\perp} = \mathrm{Ker}(\,Q\,).$$

$Q$ thus induces a $\Gamma_n$–isomorphism from $M^\lambda / S^{\lambda\perp}$ onto $S^\lambda$ since

$$M^\lambda / S^{\lambda\perp} \cong (\,S^\lambda)^*$$

and $\dim_K S^\lambda = \dim_K (S^\lambda)^*$. Therefore $(\,S^\lambda)^* \cong S^\lambda$. Let $U$ be an irreducible submodule of $S^\lambda$, then $S^\lambda$ has a submodule $V$, such that

$$S^\lambda / V \cong U,$$

by (2.4) and (2.19), therefore

$$S^\lambda \longrightarrow S^\lambda / V \longrightarrow U \leq S^\lambda$$

is a non–zero homomorphism from $S^\lambda$ to itself. By assumption, $\lambda_1 > \lambda_2$, $\lambda$ is row $p$–regular for any prime $p$. But (4.2) gives

$$\dim_K \mathrm{Hom}_{\Gamma_n}(S^\lambda, M^\lambda) = 1.$$

This forces that $S^\lambda = U$ is irreducible.∎

(4.44) **LEMMA.** *Assume that* $S^\mu$ *is irreducible over* $\Gamma_n$, *where* $\mu$ *is a partition of* $n$. *Unless* $\mathrm{char}(K) = 2$ *and* $\mu$ *is row* $2$–*singular,*

$$\dim_K \mathrm{Hom}_{\Gamma_n}(M^{\mu}, S^{\mu}) = 1.$$

**PROOF.** According to (2.3) and (2.19),

$$\mathrm{Hom}_{\Gamma_n}(M^{\mu}, S^{\mu}) \cong \mathrm{Hom}_{\Gamma_n}(S^{\mu *}, M^{\mu *}) \cong \mathrm{Hom}_{\Gamma_n}(S^{\mu}, M^{\mu})$$

as $K$-spaces. Now apply (4.2).∎

(4.45) **COROLLARY.** *Let* $\lambda = (\lambda_1, \lambda_2)$ *be a two-parts partition of* $n$. *Assume that* $\mathrm{char}(K) = p$ *and*

$$\lambda_1 - \lambda_2 \equiv -1 \pmod{p^{\ell}},$$

*where* $\ell = \ell_p(\lambda_2)$. *Then* $\dim_K \mathrm{Hom}_{\Gamma_n}(M^{\lambda}, S^{\lambda}) = 1$.∎

When $\lambda_1 = n-r$, $\lambda_2 = r$, the condition

$$n - 2r \equiv -1 \pmod{p^{\ell}}, \qquad \ell = \ell_p(r)$$

is equivalent to

$$\lambda_1 - \lambda_2 \equiv -1 \pmod{p^{\ell}}, \qquad \ell = \ell_p(\lambda_2).$$

Therefore, combining (4.42) and (4.45), we have proved Lemma (4.34), presented at the beginning of this section.

(4.46) **THE PROOF OF THEOREM (4.33).** Let $\lambda = (\lambda_1, \lambda_2)$ be a two-parts partition. If $\lambda_1 = \lambda_2$, (4.33) follows from (4.32). Assume $\lambda_1 > \lambda_2$. Use induction on $\lambda_2$. When $\lambda_2 = 1$, (4.33) holds because of (4.31). Assume that

(4.33) holds for some $S^{(\lambda_1,\lambda_2-1)}$, $\lambda_2 > 1$. That is to say,

$$\dim_K S_y^{(\lambda_1,\lambda_2-1)} = \dim_K \mathrm{Hom}_{\Gamma_{n-1}}(M^{(\lambda_1,\lambda_2-1)},S^{(\lambda_1,\lambda_2-1)}) = 1$$

for any $(\lambda_1,\lambda_2-1)$–tableau $y$. Apply Lemma (4.34), when $\lambda_1 = n-r$, $\lambda_2 = r$,

$$\dim_K \mathrm{Hom}_{\Gamma_n}(M^{(\lambda_1,\lambda_2)},S^{(\lambda_1,\lambda_2)}) \leq \dim_K \mathrm{Hom}_{\Gamma_{n-1}}(M^{(\lambda_1,\lambda_2-1)},S^{(\lambda_1,\lambda_2-1)}) = 1.$$

But $\dim_K \mathrm{Hom}_{\Gamma_n}(M^\lambda,S^\lambda) \geq 1$ by (4.1). Therefore (4.33) holds.∎

By using (4.1), we have immediatedly

**(4.47) COROLLARY.** *The Specht module $S^\lambda$ has unique irreducible submodule over $K\mathfrak{S}_n$ for arbitrary field $K$, when $\lambda$ is a two–parts partition of $n$.*∎

## §4D The Calculations of $\mathrm{Hom}_{\Gamma}(S^{\lambda^*},S^{\lambda})$

Let $\lambda$ be a partition of $n$. Consider the $K$-subspace of $\mathrm{End}_{\Gamma}(M^{\lambda})$, where $\Gamma = K\mathfrak{S}_n$ :

$$D_{\lambda} = \{ \varphi \in \mathrm{End}_{\Gamma}(M^{\lambda}) \mid \mathrm{Im}(\varphi) \subseteq S^{\lambda}, \mathrm{Ker}(\varphi) \supseteq S^{\lambda\perp} \}.$$

Write $\pi$: $M^{\lambda} \longrightarrow M^{\lambda}/S^{\lambda\perp}$ as the coset map. Each $\varphi \in D_{\lambda}$ induces a $\Gamma$-homomorphism $\psi$ from $M^{\lambda}/S^{\lambda\perp} \cong (S^{\lambda})^*$ into $S^{\lambda}$ :

$$\begin{tikzcd} M^{\lambda} \arrow[r, "\varphi"] \arrow[d, "\pi"'] & S^{\lambda} \subseteq M^{\lambda} \\ M^{\lambda}/S^{\lambda\perp} \arrow[ur, "\psi"'] & \end{tikzcd}$$

which makes the above diagram commute. On the other hand, if

$$\psi : \ M^{\lambda}/S^{\lambda\perp} \longrightarrow S^{\lambda}$$

is a $\Gamma$-homomorphism, then it is clear that $\psi\pi \in D_{\lambda}$. Therefore $D_{\lambda}$ is isomorphic to $\mathrm{Hom}_{\Gamma}(S^{\lambda^*},S^{\lambda})$ as $K$-spaces. $D_{\lambda}$ can also be viewed as a $K$-subspace of $\mathrm{Hom}_{\Gamma}(M^{\lambda},S^{\lambda})$. In (4.1), we have seen that the socle length of $S^{\lambda}$ does not exceed the $K$-dimension of $\mathrm{Hom}_{\Gamma}(M^{\lambda},S^{\lambda})$. In fact, the similar argument leads to

**(4.48)** $1 \leq$ **socle length of** $S^{\lambda} \leq \dim_K(D_{\lambda}) \leq \dim_K \mathrm{Hom}_{\Gamma}(M^{\lambda},S^{\lambda})$.∎

Let $\varphi$ be an element in $\mathrm{End}_{\Gamma}(M^{\lambda})$. Take $\mathfrak{M}(\lambda,\lambda)$ as the $K$-basis of

$\mathrm{End}_\Gamma(M^\lambda)$. Then

$$\varphi = \sum_{P \in \mathfrak{M}(\lambda,\lambda)} z_P P, \qquad z_P \in K,$$

is in $D_\lambda$ if and only if, by (3.9) and (3.23),

(4.49) (i) $$(\ell \xrightarrow{w} m)\varphi = (\ell \xrightarrow{w} m) \sum_{P \in \mathfrak{M}(\lambda,\lambda)} z_P P = 0,$$

(ii) $$\varphi\,(m \xrightarrow{w} \ell) = \sum_{P \in \mathfrak{M}(\lambda,\lambda)} z_P P\,(m \xrightarrow{w} \ell) = 0,$$

for all $\ell > m \geq 1$, $w > 0$.∎

(4.50) REMARKS.

(i) In §4A, we have proved that if $\mathrm{char}(K) \neq 2$, $\lambda = (n-r,1^r)$, then $\mathrm{Hom}_\Gamma(M^\lambda,S^\lambda)$ has $K$–dimension 1 (one). Thus (4.48) above implies that

$$1 = \text{socle length of } S^\lambda = \dim_K(D_\lambda) = \dim_K \mathrm{Hom}_\Gamma(M^\lambda,S^\lambda).$$

(ii) In §4B and §4C, we have proved that for any $\mathrm{char}(K) = p$, $\dim_K \mathrm{Hom}_\Gamma(M^\lambda,S^\lambda) = 1$ when $\lambda = (\lambda_1,\lambda_2)$ is a two–parts partition of $n$. By (4.16) and (4.17),

$$D_\lambda = \mathrm{Hom}_\Gamma(M^\lambda,S^\lambda),$$

it follows that $\dim_K(D_\lambda) = 1$ when $\lambda = (\lambda_1,\lambda_2)$.

(iii) In this section, we shall study the $K$–space $D_\lambda$, when $\lambda$ is the partition $(\lambda_1,\lambda_2,1^r)$, by combining the results and techniques in the previous three sections. Since the case $\lambda_2 = 1$ and the case $r = 0$ have been discussed before, we shall always assume that $\lambda_2 \geq 2$ and $r \geq 1$. In the first half of this section, we shall be carefully classifying the $(\lambda,\lambda)$–incidence matrices in order to analyze the solution space of the linear system yielded by (4.49) in the unknowns $\{ z_P \mid P \in \mathfrak{M}(\lambda,\lambda) \}$.

(4.51) NOTES ON THE CLASSIFICATION OF $(\lambda_1,\lambda_2,1^r)$–INCIDENCE MATRICES

(i) Each $(\lambda,\lambda)$–incidence matrix $Q$, where $\lambda = (\lambda_1,\lambda_2,1^r)$, can be partitioned into blocks in the following manner:

$$Q = \begin{array}{cc} \begin{array}{cc} \scriptstyle 2 & \scriptstyle r \end{array} & \\ \begin{bmatrix} A & B \\ C & D \end{bmatrix} & \begin{array}{c} \scriptstyle 2 \\ \scriptstyle r \end{array} \end{array} .$$

In the point of view of §4A (c.f. the note following 4.4), the block $[\ A,B\ ]$ is called the hat of $Q$. Each row of $[\ C,D\ ]$ is of the form $E_k$ for some $k$, $1 \leq k \leq r+2$, where $E_k$ is the $k$–th basic row vector with $(r+2)$–components. Each $Q$ in $\mathfrak{M}(\lambda,\lambda)$ defines a function:

$$q : [3,r+2] \longrightarrow [1,r+2],$$

such that $E_{q(i)}$ is the $i$–th row of $Q$, $3 \leq i \leq r+2$. Thus, we can write

$$[\ C,D\ ] = \begin{bmatrix} E_{q(3)} \\ \vdots \\ E_{q(r+2)} \end{bmatrix} .$$

Similarly $Q$ also defines a function

$$q' : [3,r+2] \longrightarrow [1,r+2],$$

such that $E^{\mathrm{T}}_{q'(j)}$ is the $j$–th column of $Q$, $3 \leq j \leq r+2$. This is to say,

$$\begin{bmatrix} B \\ D \end{bmatrix} = \begin{bmatrix} E^{\mathrm{T}}_{q'(1)} & \cdots & E^{\mathrm{T}}_{q'(r+1)} \end{bmatrix} .$$

(ii) We shall concentrate on those $Q$'s in $\mathfrak{M}(\lambda,\lambda)$ satisfying that both functions $q$ and $q'$ are one–to–one. The cardinality of the set

$$\{\ i \in [3,r+2]\ \mid\ q(i) \leq 2\ \}$$

is equal to the number of non–zero entries in block $C$ of $Q$, which can be 0, 1 or 2.

(iii) If $C = 0$ in $Q$, then

$$q(i) \geq 2, \qquad 3 \leq i \leq r+2.$$

and $q(i) \neq q(j)$ if $i \neq j$, since $q$ is one–to–one. That implies that the $r{\times}r$ matrix $D$ can be obtained by permuting the rows of the $r{\times}r$ identity matrix

$I_r$. Hence $B$ must also be zero matrix. The block $A$ can then be written as

$$\begin{bmatrix} \lambda_1-k & k \\ k & \lambda_2-k \end{bmatrix}, \qquad 0 \leq k \leq \lambda_2,$$

which is a $[(\lambda_1,\lambda_2), (\lambda_1,\lambda_2)]$–incidence matrix.

(iv) We claim that if

$$Q = \begin{array}{cc} \begin{array}{cc} 2 & r \end{array} & \\ \begin{bmatrix} A & B \\ C & D \end{bmatrix} & \begin{array}{c} 2 \\ r \end{array} \end{array}$$

is a $(\lambda,\lambda)$–incidence matrix such that both functions $q$ and $q'$ are one–to–one, then the block $C$ has exactly one non–zero entry if and only if $B$ does.

From the definition of $(\lambda,\lambda)$–incidence matrices, each row of $[\ C,D\ ]$ has its entries summed up to 1 (one). If $C$ has exactly one non–zero entry (which is 1), $D$ must have $(r-1)$ non–zero entries located in different columns, since $q$ is one–to–one. Hence $D$ has exactly one zero column. This forces that $B$ has exactly one non–zero entry. The converse part can be shown by the same argument.

(v) The notes (iii) and (iv) above imply that if $r \geq 2$, $C$ has exactly two non–zero entries if and only if $B$ does. It is clear that the two non–zero entries of $C$ $(B)$ are in different columns (rows), since $q$ and $q'$ are one–to–one. The block $A$ in this case is a $(\lambda\dagger,\lambda\dagger)$–incidence matrix:

$$\begin{bmatrix} \lambda_1-1-k & k \\ k & \lambda_2-1-k \end{bmatrix}, \qquad 0 \le k \le \lambda_2-1,$$

here $\lambda\dagger$ is the partition $(\lambda_1-1,\lambda_2-1)$ of $n-2$.

(vi) Return to the case $C$ (hence $B$) has exactly one non–zero entry. Let $Q$ be such a $(\lambda,\lambda)$–incidence matrix, satisfying

$$q(i) \le 2, \; q(k) > 2 \text{ if } k \ne i;$$
$$q'(j) \le 2, \; q'(k) > 2 \text{ if } k \ne j.$$

There are four possibilities concerning the values of $q(i)$ and $q'(j)$ :

If $q(i) = q'(j) = 1$, then $A$ must be of the form

$$\begin{bmatrix} \lambda_1-1-k & k \\ k & \lambda_2-k \end{bmatrix}, \qquad 0 \le k \le \min\{ \lambda_1-1,\lambda_2\}.$$

If $q(i) = 1$, $q'(j) = 2$, then $A$ must be of the form

$$\begin{bmatrix} \lambda_1-k & k \\ k-1 & \lambda_2-k \end{bmatrix}, \qquad 1 \le k \le \lambda_2.$$

If $q(i) = 2$, $q'(j) = 1$, then $A$ must be of the form

$$\begin{bmatrix} \lambda_1-k & k-1 \\ k & \lambda_2-k \end{bmatrix}, \qquad 1 \le k \le \lambda_2.$$

If $q(i) = q'(j) = 2$, then $A$ must be of the form

$$\begin{bmatrix} \lambda_1-k & k \\ k & \lambda_2-1-k \end{bmatrix}, \qquad 0 \leq k \leq \lambda_2-1. \blacksquare$$

(4.52) **NOTES AND DEFINITIONS ON $(\ell \xrightarrow{w} m)P$**

(i) This might be the right time to introduce some new tools before diving into the calculations in front of us. Let's turn back to a partition $\mu = (\mu_1, \ldots, \mu_h)$ of $n$. Assume that $1 \leq \ell, m \leq h$, $\ell \neq m$, $0 < w \leq \mu_\ell$. Let $\nu$ be the composition of $n$, such that

$$\nu_\ell = \mu_\ell - w,$$
$$\nu_m = \mu_m + w,$$
$$\nu_j = \mu_j, \qquad \text{if } j \neq \ell,\ j \neq m.$$

Let $P$ be a $(\mu,\mu)$–incidence matrix. $(\ell \xrightarrow{w} m)P$ is then a $\Gamma$–homomorphism from $M^\mu$ to $M^\nu$:

$$\begin{array}{ccc} M^\mu & \xrightarrow{P} & M^\mu \\ & {\scriptstyle (\ell \xrightarrow{w} m)P} \searrow & \downarrow {\scriptstyle (\ell \xrightarrow{w} m)} \\ & & M^\nu \end{array}$$

(ii) $(\ell \xrightarrow{w} m)P$, $P \in \mathfrak{M}(\mu,\mu)$, is a $\mathbb{Z}$–linear combination of elements in $\mathfrak{M}(\mu,\nu)$ by (3.10), say

$$(\ell \xrightarrow{w} m)P = \sum_{Q \in \mathfrak{M}(\mu,\nu)} \alpha(Q)Q\ , \qquad \alpha(Q) \in \mathbb{Z}.$$

Define the support of $(\ell \xrightarrow{w} m)P$ to be the subset of $\mathfrak{M}(\mu,\nu)$ in which each element has non–zero coefficient in the expression of $(\ell \xrightarrow{w} m)P$, denoted by

$$\operatorname{supp}\{(\ell \xrightarrow{w} m)P\} = \{\ Q \in \mathfrak{M}(\mu,\nu) \mid 0 \neq \alpha(Q) \in \mathbb{Z} \text{ in } (\ell \xrightarrow{w} m)P\ \}.$$

(iii) Let $Q$ be a $(\mu,\nu)$–incidence matrix. $(m \xrightarrow{w} \ell)$ is a $\Gamma$–homomorphism from $M^{\nu}$ to $M^{\mu}$, hence $(m \xrightarrow{w} \ell)Q$ is a $\Gamma$–endomorphism of $M^{\mu}$:

$$\begin{tikzcd} M^{\mu} & M^{\mu} \arrow[l, "(m \xrightarrow{w} \ell)Q"'] \arrow[d, "Q"] \\ & M^{\nu} \arrow[ul, "(m \xrightarrow{w} \ell)"] \end{tikzcd}$$

Write according to (3.10):

$$(m \xrightarrow{w} \ell)Q = \sum_{R \,\in\, \mathfrak{M}(\mu,\,\mu)} \beta(R)R, \qquad \beta(R) \in \mathbb{Z}.$$

Similar to (ii) above, we can define

$$\operatorname{supp}\{(m \xrightarrow{w} \ell)Q\} = \{\ R \in \mathfrak{M}(\mu,\mu) \mid 0 \neq \beta(R) \in \mathbb{Z} \text{ in } (m \xrightarrow{w} \ell)Q\ \}.$$

(iv) For $P \in \mathfrak{M}(\mu,\mu)$, $Q \in \mathfrak{M}(\mu,\nu)$, we have the following simple but significant fact: $P$ belongs to $\operatorname{supp}\{(m \xrightarrow{w} \ell)Q\}$ if and only if $Q$ belongs to $\operatorname{supp}\{(\ell \xrightarrow{w} m)P\}$.

(v) Consider the condition (c.f. 4.49 (i)):

$$(\ell \xrightarrow{w} m) \sum_{P \in \mathfrak{M}(\lambda,\lambda)} z_P P = 0.$$

For each $Q \in \mathfrak{M}(\mu,\nu)$, one can find supp$\{(m \overset{w}{\dashv} \ell)Q\}$ by computing $(m \xrightarrow{w} \ell)Q$ according to (3.10). For each $P$ in supp$\{(m \overset{w}{\dashv} \ell)Q\}$, assume that the integer coefficient of $Q$ in $(\ell \xrightarrow{w} m)P$ is $\gamma(Q,P)$, then the coefficient of $Q$ in

$$(\ell \xrightarrow{w} m) \sum_{P \in \mathfrak{M}(\lambda,\lambda)} z_P P = 0$$

is

$$\sum \left\{ \gamma(Q,P)z_P \;\middle|\; P \in \mathrm{supp}\{(m \overset{w}{\dashv} \ell)Q\} \right\}.$$

Thus, we have an equation on $\{ z_P \mid P \in \mathfrak{M}(\mu,\mu) \}$ regarding $Q$ in $\mathfrak{M}(\mu,\nu)$:

$$\sum \left\{ \gamma(Q,P)z_P \;\middle|\; P \in \mathrm{supp}\{(m \overset{w}{\dashv} \ell)Q\} \right\} = 0.$$

In fact, every linear equation on $z_P$'s yielded by (4.49) (i) arises in this manner.

(vi) In our later calculation, we are sometimes concentrating on $z_M$ for some particular $M$ in $\mathfrak{M}(\mu,\mu)$. The following algorithm will be used for finding linear equations involving $z_M$ yielded by

$$(\ell \xrightarrow{w} m) \sum_{P \in \mathfrak{M}(\lambda,\lambda)} z_P P = 0.$$

(4.53) ALGORITHM.

(a) *Find* supp$[(\ell \xrightarrow{w} m)M]$, *for* $M$ *in* $\mathfrak{M}(\mu,\mu)$.

(b) *For* $Q$ *in* supp$[(\ell \xrightarrow{w} m)M]$, *find* supp$\{(m \overset{w}{\to} \ell)Q\}$.

(c) *For each* $P \in$ supp$\{(m \overset{w}{\to} \ell)Q\}$, *calculate the (integer) coefficient* $\gamma(Q,P)$ *of* $Q$ *in* $(\ell \xrightarrow{w} m)P$.

(d) *Write down the equation*

$$\sum\Big\{\ \gamma(Q,P)z_{\mathrm{P}}\ \mid\ P \in \mathrm{supp}\{(m \overset{w}{\to} \ell)Q\}\ \Big\} = 0. \blacksquare$$

(4.54) **EXAMPLE.** Let $\mu = (5,2,1)$, $\nu = (6,1,1)$, partitions of 8. Take

$$M = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \in \mathfrak{M}(\mu,\mu).$$

Consider the equations yielded by

$$(2 \xrightarrow{1} 1) \sum_{P \in \mathfrak{M}(\lambda,\lambda)} z_{\mathrm{P}} P = 0.$$

(a) supp$\{(2 \xrightarrow{1} 1)M\} = \{\ Q,\ Q'\ \}$, where

$$Q = \begin{bmatrix} 4 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad Q' = \begin{bmatrix} 4 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

(b) supp$\{(1 \xrightarrow{1} 2)Q\} = \{\ M,\ N,\ P\ \}$, where

$$N = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

(c) By applying (3.10), we have the coefficient of $Q$ in $(2 \xrightarrow{1} 1)M$, $(2 \xrightarrow{1} 1)N$, $(2 \xrightarrow{1} 1)P$ are

$$\gamma(Q,M) = 1, \quad \gamma(Q,N) = 4, \quad \gamma(Q,P) = 1.$$

(d) Thus the equation regarding $Q$ is

$$z_M + 4z_N + z_P = 0.$$

Similarly we can find the equation regarding $Q'$. The detail is ommitted here.∎

(4.55) NOTES ON $P(m \xrightarrow{w} \ell)$

Let $\mu$ and $\nu$ be compositions of $n$ as in (4.52). The linear equations yielded by

$$\sum_{P \in \mathfrak{M}(\lambda,\lambda)} z_P P(m \xrightarrow{w} \ell) = 0$$

are closely related to those yielded by

$$(\ell \xrightarrow{w} m) \sum_{P \in \mathfrak{M}(\lambda,\lambda)} z_P P = 0.$$

Notice that

$$\sum_{P \in \mathfrak{M}(\lambda,\lambda)} z_P P(m \xrightarrow{w} \ell) = 0$$

is equivalent to

$$\sum_{P \in \mathfrak{M}(\lambda,\, \lambda)} z_P[P\,(m \xrightarrow{w} \ell)]^A = 0.$$

By (3.16) (iii) and (3.19),

$$\sum_{P \in \mathfrak{M}(\lambda,\lambda)} z_P(\ell \xrightarrow{w} m)P^T = 0.$$

Let $Q$ be an arbitrary $(\mu,\nu)$–incidence matrix, $\gamma(Q,P^T)$ be the (integer) coefficient of $Q$ in $(\ell \xrightarrow{w} m)P^T$, then (4.53) gives

$$\sum\left\{\ \gamma(Q,P^T)z_P \ \mid\ P^T \in \operatorname{supp}\{(m \overset{w}{\to} \ell)Q\}\ \right\} = 0.$$

Note that the following equation

$$\sum\left\{\ \gamma(Q,P)z_P \ \mid\ P \in \operatorname{supp}\{(m \overset{w}{\to} \ell)Q\}\ \right\} = 0$$

is yielded by

$$(\ell \xrightarrow{w} m) \sum_{P \in \mathfrak{M}(\lambda,\lambda)} z_P P = 0.$$

Therefore, there is a one–to–one correspondence between the two homogeneous systems, given by (i) and (ii) in (4.49), summarized as the following:

**(4.56) OBSERVATION.** *The condition* (4.49) (i) *yields an equation*

$$\sum_{P \in \mathfrak{M}(\mu,\mu)} \gamma(P) z_P = 0, \qquad \gamma(P) \in K,$$

*if and only if the condition* (4.49) (ii) *yields*

$$\sum_{P \in \mathfrak{M}(\mu,\mu)} \gamma(P) z_Q = 0, \qquad \gamma(P) \in K,$$

*where* $Q = P^T$.∎

Let us now turn back to the partition $\lambda = (\lambda_1,\lambda_2,1^r)$, $\lambda_2 \geq 2$, $r \geq 1$. Assume that $(z_P)_{P\in\mathfrak{M}(\lambda,\lambda)}$ is a solution to (4.49) (i) and (ii). We intend to give an estimation of the dimension of the solution space by revealing the interrelation of the $z_P$'s.

(4.57) **OBSERVATION.** *Let $Q$ be a $(\lambda,\lambda)$–incidence matrix, $q$ and $q'$ be the functions from* [3,r+2,] to [1,r+2] *determined by $Q$ as in* (4.51) (i).

(i) *If* $q$ : [3,r+2,] $\longrightarrow$ [1,r+2] *is not one–to–one, then* (4.49) (i) *yields an equation*

$$2z_Q = 0.$$

(ii) *If* $q'$: [3,r+2,] $\longrightarrow$ [1,r+2] *is not one–to–one, then* (4.49) (i) *yields an equation*

$$2z_Q = 0.$$

**PROOF.** (i) Although this follows from the proof of (4.9) in §4A, we shall sketch the proof by making use of the machinery in (4.53). Write

$$Q = \begin{array}{cc} & \\ 2 & r & \\ \left[\begin{matrix} A & B \\ C & D \end{matrix}\right] & \begin{matrix} 2 \\ r \end{matrix} \end{array}, \qquad \text{where } [\ C,\ D\ ] = \begin{bmatrix} E_{q(3)} \\ \vdots \\ E_{q(r+2)} \end{bmatrix}.$$

There exist $\ell$ and $m$, $3 \leq m < \ell \leq r+2$, satisfying $q(\ell) = q(m)$, since $q$ is not one–to–one by asumption. Apply Algorithm (4.53) to $Q$ :

(a) $\operatorname{supp}\{(\ell \xrightarrow{1} m)Q\} = \{\ Q^*\}$, where

$$Q^* = \begin{array}{cc} 2 \quad r & \\ \left[\begin{matrix} A & B \\ C^* & D^* \end{matrix}\right] & \begin{matrix} 2 \\ r \end{matrix} \end{array}, \qquad \text{where } [\ C^*,\ D^*] = \begin{array}{c} \\ \\ m \\ \\ \ell \\ \\ \\ \end{array} \begin{bmatrix} E_{q(3)} \\ \vdots \\ 2E_{q(m)} \\ \vdots \\ 0 \\ \vdots \\ E_{q(r+2)} \end{bmatrix}.$$

(b) $\operatorname{supp}\{(m \xrightarrow{1} \ell)Q^*\} = \{\ Q\ \}$.

(c) $(\ell \xrightarrow{1} m)Q = 2Q^*$.

(d) The equation regarding $Q^*$ in

$$(\ell \xrightarrow{w} m) \sum_{P \in \mathfrak{M}(\lambda,\lambda)} z_P P = 0$$

is $2z_Q = 0$.

(ii) is a corollary of (4.56) and (i) above.∎

**(4.58) OBSERVATIONS.** *Let $Q$ be a $(\lambda,\lambda)$–incidence matrix such that both functions $q$ and $q'$ are one–to–one. Then*

$$z_Q \in K z_{Q_0} ,$$

*for some $Q_0$, which is one of the following $(\lambda,\lambda)$–incidence matrices listed below:*

(1) $$L(k) = \left[\begin{array}{cc|c} \lambda_1-k & k & \\ k & \lambda_2-k & 0 \\ \hline \multicolumn{2}{c|}{0} & I_r \end{array}\right], \qquad k = 0,\ 1,\ \dots,\ \lambda_2.$$

(2) $$M(k) = \left[\begin{array}{cc|c|c} \lambda_1-k & k & 0 & \\ k & \lambda_2-1-k & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 & I_{r-1} \end{array}\right], \qquad k = 0,\ 1,\ \dots,\ \lambda_2-1.$$

(3) $$N(k) = \left[\begin{array}{cc|c|c} \lambda_1-k & k-1 & 1 & \\ k & \lambda_2-k & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 & I_{r-1} \end{array}\right], \qquad k = 1,\ \dots,\ \lambda_2.$$

$$(4)\quad P(k) = \left[\begin{array}{cc|c|c} \lambda_1-k & k & 0 & \\ k-1 & \lambda_2-k & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 & I_{r-1} \end{array}\right], \qquad k = 1, \dots, \lambda_2.$$

$$(5)\quad Q(k) = \left[\begin{array}{cc|c|c} \lambda_1-1-k & k & 1 & \\ k & \lambda_2-k & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 & I_{r-1} \end{array}\right], \qquad 0 \le k \le \min\{\lambda_1-1,\lambda_2\}.$$

$$(6)\quad R(k) = \left[\begin{array}{cc|cc|c} \lambda_1-1-k & k & 1 & 0 & 0 \\ k & \lambda_2-1-k & 0 & 1 & \\ \hline 1 & 0 & & & \\ 0 & 1 & \multicolumn{2}{c|}{0} & 0 \\ \hline \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{0} & I_{r-2} \end{array}\right], \qquad k = 0, 1, \dots \lambda_2-1.$$

Note. $R(k)$ occurs only if $r \geq 2$.

**PROOF.** In this proof, we shall write $z[Q]$ for the unknown $z_Q$, $Q \in \mathfrak{M}(\lambda,\lambda)$ ( for the sake of typing). Write (c.f. the notes 4.51)

$$Q = \begin{array}{c} \left[\begin{array}{cc} A & B \\ C & D \end{array}\right] \end{array} \begin{array}{l} 2 \\ r \end{array}.$$

According to (4.51), there are three possibilities for the block $C$:

(a) $C = 0$.

(b) $C$ (hence $B$) has exactly one non–zero entry.

(c) $C$ (hence $B$) has two non–zero entries in different rows (columns).

In the case (b) above, We have four subcases as shown in (4.51) (vi). We shall prove the following:

If $q(i) = q'(j) = 1$, for some $i$ and $j$ in [3,r+2], $q(l) > 2$, $q'(l) > 2$ whenever $l \neq i$, $l \neq j$, then $z[Q] \in Kz[Q(k)]$ for some $k$, $0 \leq k \leq \min\{\lambda_1-1,\lambda_2\}$.

For a $(\lambda,\lambda)$–incidence matrix $Q$ described above, its block $A$ is a $(\mu,\mu)$–incidence matrix, where $\mu = (\lambda_1-1,\ \lambda_2)$. Thus $A$ must be of the form

$$\begin{bmatrix} \lambda_1-1-k & k \\ k & \lambda_2-k \end{bmatrix}$$

for some $k$, $0 \leq k \leq \min\{\lambda_1-1,\lambda_2\}$. Applying (4.8) to $Q$, we have

$$z[Q] = z[M], \qquad \text{or} \qquad z[Q] = -z[M]$$

for some

$$M = \begin{bmatrix} A & B \\ C_1 & D_1 \end{bmatrix}, \qquad \text{where } C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}.$$

Note that

$$M^{\mathrm{T}} = \begin{bmatrix} A^{\mathrm{T}} & C_1^{\mathrm{T}} \\ B_1^{\mathrm{T}} & D_1^{\mathrm{T}} \end{bmatrix}.$$

Applying (4.8) to $M^{\mathrm{T}}$,

$$z[M^{\mathrm{T}}] = z[N^{\mathrm{T}}], \qquad \text{or} \qquad z[M^{\mathrm{T}}] = -z[N^{\mathrm{T}}]$$

for some

$$N^{\mathrm{T}} = \begin{bmatrix} A^{\mathrm{T}} & C_1^{\mathrm{T}} \\ B_2 & D_2 \end{bmatrix}$$

where

$$B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, \qquad D_2 = \begin{array}{c} \begin{array}{cc} \scriptstyle 1 & \scriptstyle r-1 \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & I_{r-1} \end{bmatrix} \end{array} \begin{array}{l} \scriptstyle 1 \\ \scriptstyle r-1 \end{array} .$$

From (4.56) the equation $z[M^{\mathrm{T}}] = z[N^{\mathrm{T}}]$ (or $z[M^{\mathrm{T}}] = -z[N^{\mathrm{T}}]$ ) has its "dual" equation $z[M] = z[N]$ (or $z[M] = -z[N]$ ). Therefore we have

$$z[Q] = z[N] \qquad \text{or} \qquad z[Q] = -z[N].$$

Notice that

$$N = \left[\begin{array}{cc|c|c} \lambda_1-1-k & k & 1 & \\ k & \lambda_2-k & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 & I_{r-1} \end{array}\right] = Q(k),$$

for some $k$, $0 \le k \le \min\{\lambda_1-1, \lambda_2\}$. Thus we arrive at

$$z[Q] \in Kz[Q(k)].$$

The similar discussions will cover the other cases. The detail is omitted here. ∎

(4.59) OBSERVATION.

(1) $z_{L(k)} + kz_{N(k)} + (\lambda_1 - k)z_{Q(k)} = 0,$ *for each k,* $1 \le k \le \lambda_2$ .

(2) $z_{L(0)} + \lambda_1 z_{Q(0)} = 0.$

**PROOF.** We apply Algorithm (4.53) to the following cases.

If $k = 0$, $L(0) = \left[\begin{array}{ccc|c} \lambda_1 & 0 & 0 & \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 1 & \\ \hline & 0 & & I_{r-1} \end{array}\right]$, and

(a) $\text{supp}\{\ (3 \xrightarrow{1} 1)L(0)\ \} = \left\{\ L_0 = \left[\begin{array}{ccc|c} \lambda_1 & 0 & 1 & \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & \\ \hline & 0 & & I_{r-1} \end{array}\right]\ \right\}.$

(b) $\text{supp}\{\ (1 \xrightarrow{1} 3)L_0\ \} = \left\{\ L(0),\ \left[\begin{array}{ccc|c} \lambda_1 - 1 & 0 & 1 & \\ 0 & \lambda_2 & 0 & 0 \\ 1 & 0 & 0 & \\ \hline & 0 & & I_{r-1} \end{array}\right]\ \right\} = \{L(0),\ Q(0)\}.$

(c) $(3 \xrightarrow{1} 1)L(0) = L_0,\qquad (3 \xrightarrow{1} 1)Q(0) = \lambda_1 L_0.$

(d) There arises an equation

$$z_{L(0)} + \lambda_1 z_{Q(0)} = 0.$$

If $0 < k \le \lambda_2 < \lambda_1$, or $0 < k < \lambda_2 = \lambda_1$,

(a) $\text{supp}\{\ (3 \xrightarrow{1} 1)L(k)\ \} = \left\{\ L_k = \left[\begin{array}{cc|c|c} \lambda_1 - k & k & 1 & \\ k & \lambda_2 - k & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 & I_{r-1} \end{array}\right]\ \right\}.$

(b) $\mathrm{supp}\{\ (1 \xrightarrow{1} 3)L_k\ \} = \{\ L(k),\ N(k),\ Q(k)\ \}$.

(c) The coefficient of $L_k$ in $(3 \xrightarrow{1} 1)L(k)$, $(3 \xrightarrow{1} 1)N(k)$ and $(3 \xrightarrow{1} 1)Q(k)$ are 1, $k$ and $(\lambda_1 - k)$ respectively.

(d) There arises an equation

$$z_{L(k)} + kz_{N(k)} + (\lambda_1 - k)z_{Q(k)} = 0.$$

If $\lambda_2 = \lambda_1$ and $k = \lambda_2$,

(a) $$\mathrm{supp}\{\ (3 \xrightarrow{1} 1)L(\lambda_2)\ \} = \left\{ L_{\lambda_2} = \left[\begin{array}{ccc|c} 0 & \lambda_2 & 1 & \\ \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \\ \hline & 0 & & I_{r-1} \end{array}\right] \right\}.$$

(b) $\mathrm{supp}\{\ (1 \xrightarrow{1} 3)L_{\lambda_2}\ \} = \{\ L(\lambda_2),\ N(\lambda_2)\ \}$.

(c) $(3 \xrightarrow{1} 1)L(\lambda_2) = L_{\lambda_2}$, $\quad (3 \xrightarrow{1} 1)N(\lambda_2) = \lambda_1 L_{\lambda_2}$.

(d) There arises an equation, which is a special case of (1) :

$$z_{L(k)} + kz_{N(k)} = 0, \qquad \text{where } k = \lambda_2. \blacksquare$$

Note. (2) can be viewed as the special case of (1) if we agree with the convention $z_{N(0)} = 0$.

(4.60) **OBSERVATION.** For each $k$, $0 \le k \le \lambda_2 - 1$,

$$z_{M(k)} + (\lambda_1 - k)z_{N(k+1)} + kz_{N(k)} = 0.$$

**PROOF.** We can apply (4.53) to the following cases:

(i) $k = 0$.

(ii) $0 < k \leq \lambda_2-1$.

We shall work out the detail for case (ii), and case (i) can be done by the same argument.

(a) $\mathrm{supp}\{(2 \xrightarrow{1} 1)M(k)\}$ contains

$$M' = \left[\begin{array}{cc|c|c} \lambda_1-k & k & 1 & \\ k & \lambda_2-1-k & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 & I_{r-1} \end{array}\right].$$

(b) $\mathrm{supp}\{(1 \xrightarrow{1} 2)M'\} = \{M(k),\ N(k+1),\ N(k)\}$.

(c) $(2 \xrightarrow{1} 1)M(k) = M'$. The coefficient of $M'$ in $(2 \xrightarrow{1} 1)N(k+1)$ and $(2 \xrightarrow{1} 1)N(k)$ are $(\lambda_1-k)$ and $k$ respectively.

(d) There arises an equation

$$z_{M(k)} + (\lambda_1-k)z_{N(k+1)} + kz_{N(k)} = 0.$$

In case (i), we have the equation

$$z_{M(0)} + \lambda_1 z_{N(1)} = 0,$$

viewed as the special case of the equation obtained in (d) when $k = 0$.∎

(4.61) OBSERVATION.

(1) $z_{P(k)} + (\lambda_1-k)z_{Q(k)} + kz_{Q(k-1)} = 0$, *for each* $k$, $1 \leq k \leq \lambda_2$ *if* $\lambda_1 > \lambda_2$; *and for each* $k$, $1 \leq k < \lambda_2$ *if* $\lambda_1 = \lambda_2$.

(2) $z_{P(k)} + kz_{Q(k-1)} = 0$ if $\lambda_1 = \lambda_2 = k$.

**PROOF.** In order to prove (1) we apply (4.53):

(a) $$P' = \left[\begin{array}{cc|c|c} \lambda_1-k & k & 1 & \\ k-1 & \lambda_2-k & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 & I_{r-1} \end{array}\right] \in \operatorname{supp}\{(2 \xrightarrow{1} 1)P(k)\}\ ,$$

(b) $\operatorname{supp}\{(1 \xrightarrow{1} 2)P'\} = \{P(k),\ Q(k),\ Q(k-1)\}$.

where

$$Q(k-1) = \left[\begin{array}{cc|c|c} \lambda_1-k & k-1 & 1 & \\ k-1 & \lambda_2-k+1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 & I_{r-1} \end{array}\right].$$

(c) The coefficient of $P'$ in $(2 \xrightarrow{1} 1)P(k)$, $(2 \xrightarrow{1} 1)Q(k)$, $(2 \xrightarrow{1} 1)Q(k-1)$ are 1, $(\lambda_1-k)$ and $k$ respectively. Thus

(d) $$z_{P(k)} + (\lambda_1-k)z_{Q(k)} + kz_{Q(k-1)} = 0.$$

When $k = \lambda_2 = \lambda_1$, we obtain equation (2) by the similar method:

$$z_{P(\lambda_2)} + \lambda_1 z_{Q(\lambda_2-1)} = 0. \blacksquare$$

Note. (2) can be viewed as the special case of (1) in (4.61) if we agree to define

$$z_{Q(\lambda_2)} = 0 \qquad \text{when } \lambda_2 = \lambda_1.$$

(4.62) **OBSERVATION.** *For each $k$, $1 \leq k \leq \lambda_2$,*

$$z_{N(k)} + (\lambda_1-k)z_{Q(k)} + kz_{Q(k-1)} = 0.$$

**PROOF.** This is a corollary of (4.56) and (4.61) above. Notice that

$$N(k) = P(k)^{\mathrm{T}}, \quad Q(k) = Q(k)^{\mathrm{T}}.$$

From (4.61), we have

$$z_{P(k)} + (\lambda_1-k)z_{Q(k)} + kz_{Q(k-1)} = 0,$$

and (4.56) claims that

$$z_{N(k)} + (\lambda_1-k)z_{Q(k)} + kz_{Q(k-1)} = 0, \qquad 1 \leq k \leq \lambda_2. \blacksquare$$

(4.63) **OBSERVATION.** *Assume that* $\mathrm{char}(K) \neq 2$. *For each $k$, $0 \leq k < \lambda_2$,*

$$z_{Q(k)} + (\lambda_2-k)z_{R(k)} = 0.$$

**PROOF.** By computing

$$(4 \xrightarrow{1} 2)Q(k) = (4 \xrightarrow{1} 2)\left[\begin{array}{cc|c|c} \lambda_1-1-k & k & 1 & \\ k & \lambda_2-k & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 & I_{r-1} \end{array}\right] = Q'$$

where

$$Q' = \left[\begin{array}{cc|cc|c} \lambda_1-1-k & k & 1 & 0 & 0 \\ k & \lambda_2-k & 0 & 1 & \\ \hline 1 & 0 & & 0 & 0 \\ 0 & 0 & & & \\ \hline & 0 & & 0 & I_{r-2} \end{array}\right],$$

we have $\mathrm{supp}\{(4 \xrightarrow{1} 2)Q(k)\} = \{\ Q'\}$. When $0 < k < \lambda_2$,

$$\mathrm{supp}\{(2 \xrightarrow{1} 4)Q'\} = \{Q(k),\ R(k),\ V(k)\},$$

where

$$V(k) = \left[\begin{array}{cc|cc|c} \lambda_1-1-k & k & 1 & 0 & 0 \\ k-1 & \lambda_2-k & 0 & 1 & \\ \hline 1 & 0 & & 0 & 0 \\ 1 & 0 & & & \\ \hline & 0 & & 0 & I_{r-2} \end{array}\right].$$

The coefficients of $Q'$ in $(4 \xrightarrow{1} 2)Q(k)$, $(4 \xrightarrow{1} 2)R(k)$ and $(4 \xrightarrow{1} 2)V(k)$ are 1, $(\lambda_2-k)$ and $k$ respectively. Therefore

$$z_{Q(k)} + (\lambda_2-k)z_{R(k)} + kz_{V(k)} = 0.$$

By (4.57) (i), $2z_{V(k)} = 0$, because there are two rows identical in $V(k)$. Thus $z_{V(k)} = 0$, since $\mathrm{char}(K) \neq 2$. We then have for $0 < k < \lambda_2$,

$$z_{Q(k)} + (\lambda_2-k)z_{R(k)} = 0.$$

When $k = 0$, $\mathrm{supp}\{(2 \xrightarrow{1} 4)Q'\} = \{Q(0),\ R(0)\}$, and the coefficients of

$Q'$ in $(4 \xrightarrow{1} 2)Q(k)$ and $(4 \xrightarrow{1} 2)R(k)$ are 1 and $\lambda_2$ respectively. There arises

$$z_{Q(0)} + \lambda_2 z_{R(0)} = 0.$$

Therefore the equation

$$z_{Q(k)} + (\lambda_2 - k) z_{R(k)} = 0.$$

is valid for $k = 0, 1, \ldots, \lambda_2-1.$ ∎

(4.64) **REMARKS.** The above observation shows that every $z_{Q(k)}$ can be solved in terms of $z_{R(k)}$, for $k = 0, 1, \ldots, \lambda_2-1$. If $\lambda_1 = \lambda_2$,

$$Q(0),\ Q(1),\ \ldots\ ,\ Q(\lambda_2-1)$$

are all the $(\lambda,\lambda)$-incidence matrices of type (v) in (4.58). When $\lambda_1 > \lambda_2$, the incidence matrix

$$Q(\lambda_2) = \left[\begin{array}{cc|c|c} \lambda_1-\lambda_2-1 & \lambda_2 & 1 & \\ \lambda_2 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 & I_{r-1} \end{array}\right]$$

is not covered by (4.63). But it is not hard to show that if $\mathrm{char}(K) \neq 2$, $z_{Q(\lambda_2)} = 0$, by the same type of calculations in (4.63). Note that

$$(4 \xrightarrow{1} 2)Q(\lambda_2) = \left[\begin{array}{cc|cc|c} \lambda_1-\lambda_2-1 & \lambda_2 & 1 & 0 & \\ \lambda_2 & 0 & 0 & 1 & 0 \\ \hline 1 & 0 & & & \\ 0 & 0 & \multicolumn{2}{c|}{0} & 0 \\ \hline \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{0} & I_{r-2} \end{array}\right] = Q^*.$$

The support of $(2 \xrightarrow{1} 4)Q^*$ contains two elements: $Q(\lambda_2)$ and

$$V(\lambda_2) = \left[\begin{array}{cc|cc|c} \lambda_1-\lambda_2-1 & \lambda_2 & 1 & 0 & \\ \lambda_2-1 & 0 & 0 & 1 & 0 \\ \hline 1 & 0 & & & \\ 1 & 0 & \multicolumn{2}{c|}{0} & 0 \\ \hline \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{0} & I_{r-2} \end{array}\right].$$

Thus, by calculating the coefficient of $Q^*$, we have

$$z_{Q(\lambda_2)} + \lambda_2 z_{V(\lambda_2)} = 0.$$

Hence $z_{Q(\lambda_2)} = 0$, since $\mathrm{char}(K) \neq 2$ and $z_{V(\lambda_2)} = 0.$ ▮

At this stage it is wise to make a general survey of the situation after obtaining the observations from (4.59) to (4.64). We order the unknowns $z_Q$'s, where $Q$ is a $(\lambda,\lambda)$–incidence matrix listed in (4.59) from (i) to (vi), in the following manner:

$$z_{L(0)}, \cdots , z_{L(\lambda_2)};$$

$$z_{M(0)}, \ldots, z_{M(\lambda_2-1)};$$

$$z_{N(1)}, \ldots, z_{N(\lambda_2)};$$

$$z_{P(1)}, \ldots, z_{P(\lambda_2)};$$

$$z_{Q(0)}, \ldots, z_{Q(m)}, \qquad m = \min\{\lambda_1-1,\lambda_2\};$$

$$z_{R(0)}, \ldots, z_{R(\lambda_2-1)}.$$

We also order the equations obtained in those observations as

(4.59) $z_{L(k)} + kz_{N(k)} + (\lambda_1-k)z_{Q(k)} = 0, \qquad k = 0, \ldots, \lambda_2.$

(4.60) $z_{M(k)} + kz_{N(k)} + (\lambda_1-k)z_{N(k+1)} = 0, \qquad k = 0, \ldots, \lambda_2-1.$

(4.62) $z_{N(k)} + kz_{Q(k-1)} + (\lambda_1-k)z_{Q(k)} = 0, \qquad k = 1, \ldots, \lambda_2.$

(4.61) $z_{P(k)} + kz_{Q(k-1)} + (\lambda_1-k)z_{Q(k)} = 0, \qquad k = 1, \ldots, \lambda_2.$

(4.63) $z_{Q(k)} + (\lambda_2-k)z_{R(k)} = 0, \qquad k = 0, \ldots, \min\{\lambda_1-1,\lambda_2\}.$
(4.64)

The coefficient matrix of the equations above is of the form

(4.65)

$$
\begin{array}{cccccc}
z_{L(k)} & z_{M(k)} & z_{N(k)} & z_{P(k)} & z_{Q(k)} & z_{R(k)} \\
\hline
\begin{smallmatrix}1&&\\&\ddots&\\&&1\end{smallmatrix} & 0 & * & 0 & * & 0 \\
 & \begin{smallmatrix}1&&\\&\ddots&\\&&1\end{smallmatrix} & * & 0 & 0 & 0 \\
 & & \begin{smallmatrix}1&&\\&\ddots&\\&&1\end{smallmatrix} & 0 & * & 0 \\
 & & & \begin{smallmatrix}1&&\\&\ddots&\\&&1\end{smallmatrix} & * & 0 \\
 & & & & \begin{smallmatrix}1&&\\&\ddots&\\&&1\end{smallmatrix} & *
\end{array}
$$

This suggests the further analysis on the equations yielded by (4.49), in which $R(k)$, $k = 1, 2, \ldots, \lambda_2-1$, are involved.

Each $R(k)$ (c.f. (vi) in 4.58) can be written as

$$
\begin{bmatrix} R_k & I_2 & 0 \\ I_2 & 0 & 0 \\ 0 & 0 & I_{r-2} \end{bmatrix}, \qquad \text{where } R_k = \begin{bmatrix} \lambda_1-1-k & k \\ k & \lambda_2-1-k \end{bmatrix},
$$

for $0 \leq k \leq \lambda_2-1$, here $\lambda^\dagger$ is the partition $(\lambda_1-1,\lambda_2-1)$ of $n-2$. The "embedding" $R_k \longmapsto R(k)$ is a one-to-one mapping from $\mathfrak{M}(\lambda^\dagger,\lambda^\dagger)$ onto the set $\{ R(k) \mid 0 \leq k \leq \lambda_2-1 \}$. We shall compare the equation system yielded by

$$(2 \xrightarrow{w} 1) \sum_k z_{R_k} R_k = 0$$

with the one yielded by

$$(2 \xrightarrow{w} 1) \sum_k z_{R(k)} R(k) = 0.$$

It is conceivable that our results concerning two–parts partitions in §4B and §4C are coming into play.

(4.66) NOTES.

(i) Let us start with a 2×2 matrix with non–negative entries:

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

viewed as an incidence matrix in $\mathfrak{M}(\mu,\nu)$ for suitable compositions $\mu$ and $\nu$. The matrix

$$\begin{bmatrix} P & I_2 & 0 \\ I_2 & 0 & 0 \\ 0 & 0 & I_{r-2} \end{bmatrix}$$

is abbreviated in this note by

$$[\![ P, I_2 ]\!] = \left[\!\left[ \begin{matrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{matrix} \right]\!\right],$$

its 2×4 block at the upper left corner, since the rows from 3 to (r+2), as well as the columns from 5 to (r+2), remain unchanged under the left action of $(2 \xrightarrow{w} 1)$ and $(1 \xrightarrow{w} 2)$.

(ii) For each $w$, $0 < w \leq c+d$,

$$(2 \xrightarrow{w} 1)P = \sum \begin{bmatrix} a+w_1 \\ w_1 \end{bmatrix} \begin{bmatrix} b+w_2 \\ w_2 \end{bmatrix} \begin{bmatrix} a+w_1 & b+w_2 \\ c-w_1 & d-w_2 \end{bmatrix},$$

the sum is taken over all ordered pairs $(w_1,w_2)$, such that

$$0 \leq w_1 \leq c, \quad 0 \leq w_2 \leq d, \quad w_1 + w_2 = w,$$

by Formula (3.10). Therefore, every element in the set $\mathrm{supp}\{(2 \xrightarrow{w} 1)P\}$ is uniquely determined by some ordered pair $(w_1,w_2)$ satisfying the above conditions. Similarly

$$(2 \xrightarrow{w} 1)[\![ P, I_2 ]\!] = \sum \begin{bmatrix} a+w_1 \\ w_1 \end{bmatrix} \begin{bmatrix} b+w_2 \\ w_2 \end{bmatrix} \left[\!\!\left[ \begin{matrix} a+w_1 & b+w_2 & 1 & w_3 \\ c-w_1 & d-w_2 & 0 & 1-w_3 \end{matrix} \right]\!\!\right]$$

where the sum is taken over all triples $(w_1, w_2, w_3)$, satisfying

$$0 \leq w_1 \leq c, \quad 0 \leq w_2 \leq d, \quad 0 \leq w_3 \leq 1, \quad w_1 + w_2 + w_3 = w.$$

It is clear that each element in $\mathrm{supp}\{ (2 \xrightarrow{w} 1)[\![ P, I_2 ]\!] \}$ is uniquely determined by some triple $(w_1, w_2, w_3)$ described above. There is an injection from the set $\mathrm{supp}\{(2 \xrightarrow{w} 1)P\}$ into the set $\mathrm{supp}\{ (2 \xrightarrow{w} 1)[\![ P, I_2 ]\!] \}$ :

$$\begin{array}{ccc} \begin{bmatrix} a+w_1 & b+w_2 \\ c-w_1 & d-w_2 \end{bmatrix} & & [\![ \begin{matrix} a+w_1 & b+w_2 & 1 & 0 \\ c-w_1 & d-w_2 & 0 & 1 \end{matrix} ]\!] \\ \downarrow & & \uparrow \\ (w_1, w_2) & \longrightarrow & (w_1,\ w_2,\ 0) \end{array}$$

Or we can simply write this injection as

$$Q \in \operatorname{supp}\{(2 \xrightarrow{w} 1)P\} \longmapsto [\![\ Q,\ I_2\ ]\!] \in \operatorname{supp}\{\ (2 \xrightarrow{w} 1)[\![\ P,\ I_2\ ]\!]\ \}.$$

(iii) Let

$$Q = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

be a 2×2 matrix with non–negative entries. For each $w$, $0 < w \le e+f$, there is a one–to–one correspondence between the set $\operatorname{supp}\{\ (1 \xrightarrow{w} 2)[\![\ Q,\ I_2\ ]\!]\ \}$ and the set of triples $(w_1,\ w_2,\ w_3)$, satisfying

$$0 \le w_1 \le e, \quad 0 \le w_2 \le f, \quad 0 \le w_3 \le 1, \quad w_1 + w_2 + w_3 = w.$$

The correspondence is given by

$$[\![ \begin{matrix} e-w_1 & f-w_2 & 1-w_3 & 0 \\ g+w_1 & h+w_2 & w_3 & 1 \end{matrix} ]\!] \longmapsto (w_1,\ w_2,\ w_3).$$

Similar to (ii) there is an injection from the set $\operatorname{supp}\{(1 \xrightarrow{w} 2)Q\}$ into the set $\operatorname{supp}\{\ (1 \xrightarrow{w} 2)[\![\ Q,\ I_2\ ]\!]\ \}$ :

$$\begin{tikzcd}
\begin{bmatrix} e-w_1 & f-w_2 \\ g+w_1 & h+w_2 \end{bmatrix} \arrow[d] & \left[\!\!\begin{bmatrix} e-w_1 & f-w_2 & 1 & 0 \\ g+w_1 & h+w_2 & 0 & 1 \end{bmatrix}\!\!\right] \\
(w_1,\ w_2) \arrow[r] & (w_1,\ w_2,\ 0) \arrow[u]
\end{tikzcd}$$

(iv) Now we turn back to the partition $\lambda = (\lambda_1, \lambda_2, 1^r)$ of $n$, $\lambda_2 \geq 2$, $r \geq 2$, and the partition $\lambda^\dagger = (\lambda_1-1, \lambda_2-1)$. For an integer $w$, $0 < w \leq \lambda_2-1$, take an incidence matrix

$$Q = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

such that

$$e + g = \lambda_1 - 1, \quad f + h = \lambda_2 - 1,$$
$$e + f = \lambda_1 - 1 + w, \quad g + h = \lambda_2 - 1 - w.$$

In $\mathrm{supp}\{(1 \xrightarrow{w} 2)Q\}$, the $(\lambda^\dagger,\lambda^\dagger)$–incidence matrix determined by the ordered pair $(w_1, w_2)$, where $0 \leq w_1 \leq c$, $0 \leq w_2 \leq d$, $w_1 + w_2 = w$, is

$$\begin{bmatrix} e-w_1 & f-w_2 \\ g+w_1 & h+w_2 \end{bmatrix}.$$

The (integer) coefficient of $Q$ in $(2 \xrightarrow{w} 1)\begin{bmatrix} e-w_1 & f-w_2 \\ g+w_1 & h+w_2 \end{bmatrix}$ is $\binom{e}{w_1}\binom{f}{w_2}$.
The coefficient of $Q$ in

$$(2 \xrightarrow{w} 1) \sum_{P \in \mathfrak{M}(\lambda^\dagger, \lambda^\dagger)} z_P P = 0$$

is zero, which yields a linear equation on $z_P$'s. If $P \in \mathrm{supp}\{(1 \xrightarrow{w} 2)Q\}$ and $P$ is determined by $(w_1, w_2)$, we shall write

$$z_P = \zeta(w_1, w_2).$$

Thus

$$\sum_{\substack{w_1+w_2=w \\ 0\le w_1\le e \\ 0\le w_2\le f}} \begin{bmatrix} e \\ w_1 \end{bmatrix} \begin{bmatrix} f \\ w_2 \end{bmatrix} \zeta(w_1, w_2) = 0.$$

Also, we know that every linear equation on $z_P$'s yielded by

$$(2 \xrightarrow{w} 1) \sum_P z_P P = 0$$

arises in this manner.

(v) Consider the set supp$\{ (1 \xrightarrow{w} 2)[\![ Q, I_2 ]\!] \}$ in $\mathfrak{M}(\lambda,\lambda)$. The incidence matrix in supp$\{ (1 \xrightarrow{w} 2)[\![ Q, I_2 ]\!] \}$ determined by the triple $(w_1,w_2,w_3)$ is

$$\left[\!\!\left[ \begin{matrix} e-w_1 & f-w_2 & 1-w_3 & 0 \\ g+w_1 & h+w_2 & w_3 & 1 \end{matrix} \right]\!\!\right],$$

where $0 \le w_1 \le e$, $0 \le w_2 \le f$, $0 \le w_3 \le 1$, $w_1 + w_2 + w_3 = w$. The coefficient of $[\![ Q, I_2 ]\!]$ in

$$(2 \xrightarrow{w} 1)\left[\!\!\left[ \begin{matrix} e-w_1 & f-w_2 & 1-w_3 & 0 \\ g+w_1 & h+w_2 & w_3 & 1 \end{matrix} \right]\!\!\right]$$

is $\begin{bmatrix} e \\ w_1 \end{bmatrix} \begin{bmatrix} f \\ w_2 \end{bmatrix}$. If $M \in$ supp$\{ (1 \xrightarrow{w} 2)[\![ Q, I_2 ]\!] \}$, determined by the triple $(w_1,w_2,w_3)$, we write

$$z_M = \zeta(w_1, w_2, w_3).$$

Then the coefficient of $[\![ Q, I_2 ]\!]$ in

$$(2 \xrightarrow{w} 1) \sum_{N \in \mathfrak{M}(\lambda,\lambda)} z_N N = 0$$

is

$$0 = \sum_{(w_1, w_2, w_3)} \begin{bmatrix} e \\ w_1 \end{bmatrix} \begin{bmatrix} f \\ w_2 \end{bmatrix} \zeta(w_1, w_2, w_3)$$

$$= \sum_{\substack{w_1+w_2=w \\ 0 \le w_1 \le e \\ 0 \le w_2 \le f}} \begin{bmatrix} e \\ w_1 \end{bmatrix} \begin{bmatrix} f \\ w_2 \end{bmatrix} \zeta(w_1, w_2, 0) + \sum_{\substack{w_1+w_2=w-1 \\ 0 \le w_1 \le e \\ 0 \le w_2 \le f}} \begin{bmatrix} e \\ w_1 \end{bmatrix} \begin{bmatrix} f \\ w_2 \end{bmatrix} \zeta(w_1, w_2, 1).$$

(vi) If $N \in \operatorname{supp}\{ (1 \xrightarrow{w} 2)[\![ Q, I_2 ]\!] \}$ and $N$ arises from a triple $(w_1, w_2, 1)$, then

$$N = \left[\begin{array}{cc|cc|c} e-w_1 & f-w_2 & 0 & 0 & 0 \\ g+w_1 & h+w_2 & 1 & 1 & \\ \hline 1 & 0 & & & \\ 0 & 1 & \multicolumn{2}{c|}{0} & 0 \\ \hline \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{0} & I_{r-2} \end{array}\right].$$

The third and the fourth columns of $N$ are identical. By (4.57), $2z_N = 0$. Thus, if $\operatorname{char}(K) \neq 2$,

$$\zeta(w_1, w_2, 1) = z_N = 0.$$

Hence the equations obtained in (iv) and (v) are

$$\sum_{(w_1,w_2)} \begin{bmatrix} e \\ w_1 \end{bmatrix} \begin{bmatrix} f \\ w_2 \end{bmatrix} \zeta(w_1,\ w_2) = 0,$$

$$\sum_{(w_1,w_2)} \begin{bmatrix} e \\ w_1 \end{bmatrix} \begin{bmatrix} f \\ w_2 \end{bmatrix} \zeta(w_1,\ w_2,\ 0) = 0,$$

where $(w_1,w_2)$ runs over all ordered pairs such that $w_1 + w_2 = w.\ 0 \leq w_1 \leq e$, $0 \leq w_2 \leq f.$ ▮

The notes above provide the proof of the following lemma:

(4.67) LEMMA. *Assume that* char$(K) \neq 2$,. *For each linear equation*

$$\sum_{k=0}^{\lambda_2 - 1} \alpha_k z_{R_k} = 0$$

*yielded by*

$$\sum_{k} (2 \xrightarrow{w} 1) z_{R_k} R_k = 0,$$

*there exists an equation, yielded by*

$$(\ell \xrightarrow{v} m) \sum_{M \in \mathfrak{M}(\lambda,\lambda)} z_M M = 0, \qquad \ell > m,\ v > 0,$$

*which is of the form*

$$\sum_{k=0}^{\lambda_2 - 1} \alpha_k z_{R(k)} = 0. ▮$$

**(4.68) THEOREM.** *Assume that* $\mathrm{char}(K) \neq 2$. *If* $\lambda = (\lambda_1, \lambda_2, 1^r)$ *is a partition of* $n$, $\lambda_2 \geq 2$, $r \geq 2$. *Then*

$$\dim_K \mathrm{Hom}_\Gamma(S^{\lambda^*}, S^\lambda) = 1.$$

**PROOF.** By (4.48), it is enough to show that the solution space of (4.49) (i) and (ii) has dimension less than or equal to 1 (one).

By (4.33) in §4C, we know that

$$\dim_K \mathrm{Hom}_\Gamma(M^{\lambda\dagger}, S^{\lambda\dagger}) = 1, \qquad \lambda\dagger = (\lambda_1 - 1,\ \lambda_2 - 1).$$

That is, the solution space of

$$\sum_{k=0}^{\lambda_2 - 1} (2 \xrightarrow{w} 1) z_{R_k} R_k = 0, \qquad w > 0,$$

has dimension 1 (one). Assume that $T$ is the coefficient matrix of the unknowns $\{ z_{R_k} \mid k = 0, \dots, \lambda_2 - 1 \}$. i.e.

$$T \begin{bmatrix} z_{R_0} \\ \vdots \\ z_{R_{\lambda_2 - 1}} \end{bmatrix} = 0.$$

Then the lemma (4.67) above gives

$$T \begin{bmatrix} z_{\mathrm{R}(0)} \\ \vdots \\ z_{\mathrm{R}(\lambda_2 - 1)} \end{bmatrix} = 0.$$

Combining this system with the one in (4.65), we conclude that the solution space of (4.49) (i) and (ii) has dimension less or equal to 1 (one).∎

(4.69) **REMARKS.** Surprizingly, it is hard to give a satisfactory estimation of the $K$–dimension of $\mathrm{Hom}_{\Gamma}(S^{\lambda^*}, S^{\lambda})$, when $\lambda = (\lambda_1, \lambda_2, 1)$, $\lambda_2 \geq 2$. For time being, we only know that (4.68) fails to hold for $\lambda = (5,2,1)$, when char($K$) is 3.

(1) One can check that the trivial $K\mathfrak{S}_8$–module $S^{(8)}$ is isomorphic to a submodule of $S^{(5,2,1)}$, by applying (3.35) in §3D.

(2) The Specht module $S^{(5,3)}$ is irreducible over $K\mathfrak{S}_8$, by (4.45) in §4C.

(3) The $K\mathfrak{S}_8$–homomorphism

$$\theta = \begin{bmatrix} 2 & 3 \\ 2 & 0 \\ 1 & 0 \end{bmatrix} : \; M^{(5,3)} \longrightarrow M^{(5,2,1)}$$

has the property $\theta(S^{(5,3)}) \neq 0$, since $\theta$ is reverse semistandard (c.f. 3.28 and 3.29 in §3C); furthermore, $\mathrm{Im}(\theta) \subseteq S^{(5,2,1)}$, since over the field $K = \mathbb{Z}_3$,

$$(3 \xrightarrow{1} 2)\theta = 0,$$

$$(2 \xrightarrow{w} 1)\theta = 0, \qquad 0 < w \leq 2$$

(c.f. 3.9 in §3A).

(4) From (ii) and (iii) above, we know that $S^{(5,2,1)}$ has at least two non–isomorphic irreducible submodules, isomorphic to $S^{(8)}$ and $S^{(5,3)} = J^{(5,3)}$ respectively.

(5) By direct calculations on the system

$$\sum_{P \in \mathfrak{M}(\lambda,\lambda)} (\ell \xrightarrow{w} m)\, z_{\mathrm{P}} P = 0, \qquad \ell > m,\ w > 0,$$

when $\mathrm{char}(K) = 3$, we can find that

$$\dim_K \mathrm{Hom}_{K\mathfrak{S}_8}(M^{\lambda}, S^{\lambda}) = 2.$$

In fact the homomorphisms

$$P = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and

$$\psi = \sum_{\substack{Q \in \mathfrak{M}(\lambda,\lambda) \\ Q \neq P}} Q$$

form a $K$–basis for $\mathrm{Hom}_{K\mathfrak{S}_8}(M^{\lambda}, S^{\lambda})$. It can also be verified that

$$P(S^{\lambda\perp}) = 0, \qquad \psi(S^{\lambda\perp}) = 0.$$

Thus the K–dimension of $\mathrm{Hom}_{K\mathfrak{S}_8}(S^{\lambda^*}, S^{\lambda})$ is equal to two.

(iv) From the facts above, we conclude that $S^{(5,2,1)}$ has socle length 2.∎

(4.70) **COROLLARY.** *When* char(K) $\neq 2$, *if* $\lambda = (\lambda_1,\lambda_2,1^r)$ *is a partition of* $n$, $r \geq 2$, *the Specht module* $S^\lambda$ *has unique irreducible submodule.*∎

# LIST OF SYMBOLS

| Symbol | Meaning | Definition on p. |
|---|---|---|
| **Abbreviations** | | |
| Ann | annihilator | |
| char | characteristic | |
| dim | dimension | |
| End | endomorphism ring | |
| Ker | kernel | |
| rad | Jacobson Radical | |
| soc | socle | |
| **Number System** | | |
| $\mathbb{N}$ | the set of positive integers | |
| $\mathbb{N}_0$ | the set of non–negative integers | |
| $\mathbb{Z}$ | the ring of integers | |
| $\mathbb{Q}$ | the field of rationals | |
| $a \mid b$ | $a$ divides $b$ | |
| $(i_r,\ldots,i_1,i_0)_p$ | $= i_r p^r + \cdots + i_1 p + i_0, \quad 0 \leq i_k \leq p-1.$ | 63, 137 |
| $\ell_p(i)$ | $= r+1, \;$ if $i = (i_r,\cdots,i_1,i_0)_p, \; i_r \neq 0.$ | 63 |
| $\underset{p}{\succeq}$ | a partial order on $\mathbb{N}_0$ | 137 |
| **Set Theory and Combinatorics** | | |
| $\emptyset$ | empty set | |
| $\lvert A\rvert$ | cardinality of set $A$ = number of elements in $A$ | |
| $\forall x$ | for all $x$ | |
| $\underline{n}$ | the set $\{1, 2, \ldots , n\}$ | 13 |
| $\lambda, \mu, \nu$ | compositions (partitions)of $n$ | 14 |
| $\lambda'$ | the conjugate of a partition $\lambda$ | 14 |
| $[\lambda]$ | $\lambda$–diagram | 15 |
| $[a,b]$ | the set of integers $\{a,a+1,\ldots,b\}$ if $b \geq a$ | 13 |
| $t, x, y$ | tableaux | 16 |

| | | |
|---|---|---|
| $\underline{x} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \end{bmatrix}_\lambda$ | $\lambda$–tabloid of the tableau $x$ | 17 |
| $t'$ | conjugate tableau of $t$ | 27 |
| $M,\ N,\ P,$ | incidence matrices | 29 |
| $\mathfrak{M}(\lambda,\mu)$ | set of $(\lambda,\mu)$–incidence matrices | 29 |
| $(M,\ \underline{x})$ | $Rx$–orbit of tabloids determined by $M$ | 29 |
| $(k \xrightarrow{w} m)$ | certain homomorphism | 35 |
| $\Theta_p$ | mapping on set of partitions | 66 |
| $\Phi_p$ | correspondence between two labellings of irreducibles | 62 |
| $\mathfrak{T}(\lambda,\mu)$ | set of $(\lambda,\mu)$–tableaux | 55 |
| $\gamma_x(T)$ | a $\mu$–tabloid determined by $(\lambda,\mu)$–tableau $T$ and $\lambda$–tableau $x$ | 56 |
| $\mathfrak{T}^\lambda$ | set of standard $\lambda$–tableaux | 152 |
| $\mathfrak{T}^\lambda_i$ | | 152 |
| $\mathfrak{T}^\lambda_{ij}$ | certain subsets of $\mathfrak{T}^\lambda$ | 158 |
| $\mathfrak{T}_*$ | | 162 |
| $\begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}$ | 2×4 block at the upper left corner of some incidence matrix | 196 |

## Group Theory and Group Algebra

| | | |
|---|---|---|
| $G$ | finite group | |
| $\mathfrak{S}_n$ | symmetric group on the set $\underline{n}$ | 13 |
| $\mathfrak{A}_n$ | alternating group on $\underline{n}$ | 13 |
| $\mathfrak{S}[B]$ | $\{\pi \in \mathfrak{S}_n \mid \pi(k) = k, \text{ if } k \notin B\},\ B \subseteq \underline{n}$ | 13 |
| $\gamma(H)$ | certain element in $KG$ determined by $\gamma$ and $H \subseteq G$ | 13 |
| $\epsilon$ | alternating character | 14 |
| $\iota$ | trivial character | 14 |
| $Rt$ | row stabilizer of a tableau $t$ | 16 |
| $Ct$ | column stabilizer of a tableau $t$ | 16 |
| $\alpha_t$ | $= \epsilon(Ct)$ | 23 |

| | | |
|---|---|---|
| $\beta_t$ | $= \iota(Rt)$ | 23 |
| $(a,b)$ | transposition of the numbers $a$ and $b$ | |

**Module Theory and Linear Algebra**

| | | |
|---|---|---|
| $K$ | a field | |
| $K\mathfrak{S}_n$ | | 13 |
| $\Gamma$ | group algebra of $\mathfrak{S}_n$ over a field $K$ | 20 |
| $\Gamma_n$ | | 151 |
| $\mathrm{Hom}_\Gamma(M,N)$ | $K$–spaces of $\Gamma$–homomorphisms from $M$ to $N$ | |
| $\mathrm{End}_\Gamma(M)$ | $= \mathrm{Hom}_\Gamma(M,M)$ | |
| $N \leq M$ | module inclusion | |
| $\underset{i}{\oplus} M_i$ | | |
| $M_1 \oplus M_2$ | internal direct sum of modules | |
| $M \otimes N$ | tensor product of modules $M$ and $N$ over $K$ | |
| $<,>$ | $K$–bilinear form | 5, 21 |
| $M^*$ | dual module of a module $M$ | 4 |
| $U^\perp$ | $= \{m \in M \mid <m,u> = 0, \forall u \in U\}$, for $U \leq M$ | 5 |
| $M^\lambda$ | permutation module associated with $\lambda$ | 20 |
| $S^\lambda$ | Specht module associated with $\lambda$ | 20 |
| $J^\lambda$ | James module associated with $\lambda$ | 21 |
| $Y(t)$ | Young module associated with tableau $t$ | 23 |
| $\varphi^A$ | adjoint of homomorphism $\varphi$ | 46 |
| $\mathrm{rad}(M)$ | radical of module $M$ | 11 |
| $\mathrm{rad}(A)$ | Jacobson radical of a ring $A$ | 11 |
| $\Phi_\gamma$ | a $K$–algebra isomorphism of $KG$ defined by a linear character $\gamma$ of $G$ | 25 |
| $\mathrm{H}_\lambda$, $\mathrm{D}_\lambda$ | certain subspaces of $\mathrm{End}_\Gamma(M^\lambda)$ | 52, 168 |
| $E_k$ | $k$–th basic row vector | 109 |
| $\mathfrak{H}$ | hat–class | 109 |
| $\mathfrak{H}(i)$ | some hat class | 120 |
| $C^{(w)}$ | a block in the coefficient matrix of some linear system | 133 |

# INDEX

# REFERENCES


Carter, R.W., and Lusztig, G., "On the modular representations of the general linear and symmetric groups", Math. Z. 136(1974), 193 – 242.

Carter, R.W., and Payne, M.T.J., "On homomorphisms between Weyl modules and Specht modules", Math. Proc. Cambridge Philos. Soc. 87(1980), no 3, 419 – 425.

Curtis, C.W., and Reiner, I., "Methods of representation theory, Vol 1", John Wiley and Sons, (1981).

Farahat, H.K., "On the natural representation of the symmetric groups", Proc. of Glasgow Math. Assoc. Vol. 5, Part 3(1962), 121 – 136.

Farahat, H.K., and Peel, M.H., "On the representation theory of the symmetric groups", J. Algebra 67 (1980) 280 – 304.

James, G.D., "The irreducible representations of the symmetric groups", Bull. London Math, Soc. 8 (1976), 229 – 232.

James, G.D., "A characteristic–free approach to the representation theory of $\mathfrak{S}_n$", J. Algebra 46 (1977a), 430 – 450.

James, G.D., "The module orthogonal to the Specht module", J. Algebra 46 (1977b), 451 – 456.

James, G.D., "Some counterexamples in the theory of Specht modules", J. Algebra 46 (1977c) 457 – 461.

James, G.D., "On a conjecture of Carter concerning irreducible Specht modules", Math. Proc. Cambridge Philos. Soc. 83 (1978a) 11 – 17.

James, G.D., "The representation theory of the symmetric groups", Lecture Notes in Math., Vol. 682, Springer–Verlag (1978b).

James, G.D., and Murphy, G.E., "The determinant of the Gram matrix for a Specht module", J. Algebra 59(1979), 222 – 235.

James, G.D., and Kerber, A., "The representation theory of the symmetric groups", Encyclopedia of Mathematics and its applications, Vol. 16 (1981), Addison–Wesley.

Murphy, Gwndolen, "Submodule structure of some Specht modules", J. Algebra 74, 524 – 534, (1982).

Peel, M.H., "Modular representations of the symmetric groups", Ph.D. Thesis, Dept of Pure Math, The University of Sheffield, (1969).

Peel, M.H., "Hook representations of the symmetric groups", Glasgow Math. Journal 12 (1971), 136 – 149.

Peel, M.H., "Specht modules and the symmetric groups", J. of Algebra 36(1975), 88 – 97.

Peel, M.H., "Duality of left ideals in a group ring", J. Algebra 71 (1981) 524 – 526.

Yang, Q., "Modular representations of the alternating groups", MSc. Thesis, the University of Calgary, (1984).