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Counting Chains, Antichains and Linear Extensions of Partially Ordered Sets

by

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ABSTRACT

This thesis deals with three distinct yet interrelated problems involving partially ordered sets, namely the counting of antichains, chains and linear extensions. The antichain counting segment of this thesis introduces Dedekind's problem involving the counting of antichains in power sets, relates the problem of counting antichains in posets to that of counting independent sets in graphs and discusses what is known as a "central element" in a poset. The largest portion of this thesis deals with linear extension counting. Results known as correlation inequalities consider the probability that an arbitrary linear extension of a poset possesses certain characteristics. A related problem, the 1/3 – 2/3 Conjecture, is dealt with in detail and some complete proofs are given. The final segment of this thesis discusses chain counting and demonstrates some relationships between the problems of counting chains and of counting linear extensions and also gives ideas on a specific chain counting problem.

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CHAPTER ONE

INTRODUCTION

1.1 THE BASIC PROBLEM

In order to describe the basic problem being dealt with in this thesis, the four specialized terms from the thesis title, namely chains, antichains, linear extensions and partially ordered sets must first be defined.

Let X be a set. A partially ordered set $P = (X,\leq)$ or poset consists of a set X and a binary relation $\leq$ such that for every $x, y, z \in X$

a) $x \leq x$ (reflexive),

b) $x \leq y$ and $y \leq x$ implies $x = y$ (antisymmetric),

c) $x \leq y$ and $y \leq z$ implies $x \leq z$ (transitive).

Often a partially ordered set P is described using a strict order relation "<" so that $x < y$ in P if and only if $x \leq y$ and $x \neq y$. When conditions a) - c) above are stated in terms of "<" we get that P must be irreflexive ($x \nless x$), asymmetric ($x < y$ implies $y \nless x$)) and transitive. It should be noted that unless otherwise stated, all posets in this thesis are assumed to be finite, meaning that the poset has a finite base set X.

A totally ordered set is a partially ordered set $P = (X, \leq)$ such that for every distinct $x, y \in X$ either $x \leq y$ or $y \leq x$ (or both). A subposet $Q = (X',\leq')$ of $P = (X, \leq)$ is a poset such that $X' \subseteq X$ and $\leq'$ is equal to $\leq$ restricted to X'.

Now the three main structures which are being counted in this thesis may be defined. A chain in poset P is a totally ordered subposet of P. A subposet $A = (X',\leq')$ of

P is an antichain if it is totally unordered, in other words for every pair of distinct elements $x, y \in X'$, $x \not\leq' y$ and $y \not\leq' x$. A chain in P is maximal if it is not a subposet of any other chain in P. Similarly, an antichain in P is maximal if it is not a subposet of any other antichain in P. A linear extension of $P = (X, \leq)$, $L = (X, \leq_1)$, is a totally ordered set on X such that $x \leq y$ implies $x \leq_1 y$.

Consider a given poset $P = (X, \leq)$ and let $x, y \in X$. y is said to cover x if $y > x$ and there is no $z \in X$ such that $y > z > x$. Partially ordered sets are represented by Hasse diagrams, which associate each element in X with a point on the plane. For every pair x and y in X such that y covers x, y is placed above x in the plane and a line segment is drawn connecting y with x. For instance, consider a poset P on $X = \{a, b, c, d, e\}$ with $a < b$, $a < c$, $a < d$, $b < e$, $c < e$, $d < e$ and $a < e$. Then the Hasse diagram of P will be as shown in Figure 1.2.1.

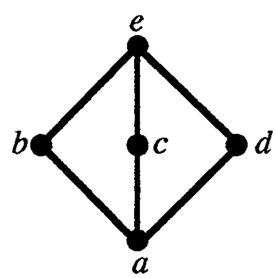


Figure 1.2.1

This thesis deals with counting chains, antichains and linear extensions in partially ordered sets. Now some notation which will be used for this will be specified. Given a poset P, $A(P)$ will be the set of antichains of P and $a(P)$ the number of antichains of P, $C(P)$ will be the set of chains of P and $c(P)$ the number of chains of P, and $E(P)$ will denote the set of linear extensions of P and $e(P)$ the number of such linear extensions. $MC(P)$ will be the set of maximal chains of P with $mc(P)$ the number of maximal chains of P and similarly $MA(P)$ will be the set of maximal antichains of P with $ma(P)$ the number of maximal antichains of P.

The concept of counting these structures will be demonstrated using the poset in Figure 1.2.1 as an example. $A(P) = \{\{\emptyset\}, \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{b,c\}, \{b,d\}, \{c,d\},\{b,c,d\}\}$ so $a(P) = 10$. $C(P) = \{\{\emptyset\}, \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,e\}, \{c,e\}, \{d,e\}, \{a,e\}, \{a,b,e\}, \{a,c,e\}, \{a,d,e\}\}$ so $c(P) = 16$. $E(P) = \{a<b<c<d<e,\ a<c<b<d<e,\ a<b<d<c<e,\ a<c<d<b<e,\ a<d<b<c<e,\ a<d<c<b<e\}$ and thus we have $e(P) = 6$. $MA(P) = \{\{a\}, \{e\}, \{b,c,d\}\}$ so $ma(P) = 3$ and $MC(P) = \{\{a,b,e\}, \{a,c,e\}, \{a,d,e\}\}$ so we also have $mc(P) = 3$.

The purpose of this thesis is to present current results available on the problems of counting chains, antichains and linear extensions in partially ordered sets and to suggest areas which are open to further work. The following section will describe some further notation and definitions required in order to carry out this task.

1.2 NOTATION AND DEFINITIONS

Some further definitions and notation which will be used throughout this thesis will now be given. Definitions for the majority of these terms can be found in [DP]. Consider a given poset $P = (X, \leq)$. If for $x, y \in X$ neither $x \leq y$ nor $y \leq x$ then we say that x and y are unrelated or incomparable and symbolize this as $x \mid y$. If $x \in X$ is unrelated to every other element in X then x is said to be an isolated point. If for $x \in X$ there is no $z \in X$ such that $z > x$ then x is a maximal element of P and if there is no $z \in X$ such that $z < x$ then x is a minimal element of P. A poset is said to be bounded if it has a unique minimal element and a unique maximal element. The height or length of a poset is equal to one less than the number of elements in its longest chain. Similarly, a chain has length equal to one less than its number of elements. The width of a poset is equal to the number of elements in its largest antichain. P is k-thin if every element in X is incomparable with at most k other elements in X. Thus a k-thin poset will be of width $k + 1$ or less.

Two posets can be combined together in various ways to form a new poset. Let $P = (X_1, \leq_1)$ and $Q = (X_2, \leq_2)$ be partially ordered sets where $X_1 \cap X_2 = \emptyset$. The disjoint union of P and Q denoted $P \cup Q$ is the poset $(X, \leq)$ formed by taking $X = X_1 \cup X_2$ and letting $x \leq y$ in $P \cup Q$ for $x, y \in X$ if and only if either

$$\begin{aligned} & x, y \in X_1 \text{ and } x \leq_1 y \\ \text{or } & x, y \in X_2 \text{ and } x \leq_2 y. \end{aligned}$$

The linear sum or ordinal sum of P and Q is denoted $P \oplus Q$ and is the poset $(X, \leq)$ such that $X = X_1 \cup X_2$ and for every $x, y \in X$, $x \leq y$ if and only if either

$$\begin{aligned} & x, y \in X_1 \text{ and } x \leq_1 y \\ \text{or } & x, y \in X_2 \text{ and } x \leq_2 y \\ \text{or } & x \in X_1 \text{ and } y \in X_2. \end{aligned}$$

A poset is called irreducible if it is not the linear sum of two non-empty posets. Let $P_1, P_2, ..., P_n$ be partially ordered sets. The cross product $P_1 \times P_2 \times ... \times P_n$ has as its elements $\{(p_1, p_2, ..., p_n) \mid p_i \text{ is in } P_i \text{ for every } i \in \{1, 2, ..., n\}\}$. The ordering $\leq$ for the cross product is defined by $(x_1, x_2, ..., x_n) \leq (y_1, y_2, ..., y_n)$ if and only if $x_i \leq y_i$ in P_i for every $i \in \{1, 2, ..., n\}$.

Now some terminology relating to a special class of partially ordered sets called lattices will be discussed. Let $P = (X, \leq)$ be a partially ordered set, and let $x, y \in X$. The meet of x and y is the unique maximal element z in X such that $z \leq x$ and $z \leq y$, if such an element exists. Similarly, the join of x and y is the unique minimal element z in X such that $z \geq x$ and $z \geq y$, if such an element exists. The meet of x and y is written $x \wedge y$ and the join is written $x \vee y$. P is a lattice if $x \vee y$ and $x \wedge y$ exist for every $x, y \in X$. A lattice $L = (X, \leq)$ is distributive if the distributive law holds for L, which states that for every a, b, $c \in X$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. Note that the dual distributive law is not required as in a lattice one distributive law holds if and only if the dual law holds (see page 130, [DP]). For a given lattice L, a polynomial is an expression having elements of L as its operands and $\vee$ and $\wedge$ as its operators.

Now a few definitions need to be made from the realm of graph theory. A graph G consists of a set of vertices and a set of edges. The vertices are represented as points on the plane and the edges must have each of their two endpoints incident to a vertex. The degree of a vertex is the number of endpoints of edges incident to it. A graph G is said to be connected if the vertices of G cannot be partitioned into two non-empty classes such that there are no edges from a member of one class to a member of the other. A cycle is a connected graph in which every vertex has degree 2. A subgraph of graph G is a subset V of the vertices of G, along with a subset of the edges of G having both endpoints in V. A tree is a connected graph which contains no cycle as a subgraph. A path is a tree which either has exactly one vertex and no edges, or has two or more vertices exactly two of which have degree one, while all remaining vertices have degree two. An induced subgraph of G is a graph formed by removing some vertices of G and also removing all edges connected to those vertices.

1.3 STRUCTURE OF THE THESIS

As is perhaps apparent from the title, this thesis has three distinct divisions to it. Chapter Two contains the first main division which is the problem of counting antichains.

Several topics relating to this problem will be presented. Firstly, some basic antichain counting results for specific classes of posets will be given. Then the problem from graph theory of counting independent sets in graphs will be given, and a connection between this and the antichain counting problem will be demonstrated which in turn will produce some new antichain counting results. Next a classic problem known as Dedekind's problem which involves counting antichains in power sets will be addressed, and finally, a problem of Colbourne and Rival, and of Rosenthal, which looks at the proportion of antichains possessing a certain characteristic will be examined.

Chapters Three and Four both deal with the problem of counting linear extensions in partially ordered sets. Chapter Three will look at some basic linear extension counting problems and then introduce the topic of correlation inequalities, which are inequalities involving probabilities that a given linear extension has certain properties. It will also give a full proof that the XYZ inequality is a consequence of another inequality. Chapter Four will deal in detail with another well-known problem, the 1/3–2/3 conjecture which also involves probabilities that a given linear extension possesses certain characteristics. Though the 1/3–2/3 conjecture has not been verified to be true for all partially ordered sets, partial results are available and nearly complete proofs will be given showing that the 1/3 – 2/3 conjecture does in fact hold for semiorders and for all but finitely many height-1 partially ordered sets.

Chapter Five will deal with the problem of counting chains, the problem which has been explored the least in current literature of the three problems presented in this thesis. Not only will this chapter give some basic chain counting results, but it will also demonstrate some results which relate the chain counting problem to the linear extension counting problem, thus demonstrating that the three main problems of this thesis are not entirely independent of one another. Finally, this chapter will produce some original results relating to a new open question, "What width restrictions may be placed on a poset, such that for every integer $n \geq 1$ there is a poset P in the restricted class with $c(P) = n$?"

CHAPTER TWO

COUNTING ANTICHAINS

2.1 INTRODUCTION

Much work has been done over the years on the problem of counting antichains in partially ordered sets. This section will survey the antichain counting results available in the current literature by breaking the results up into several categories.

Firstly, antichain counts of various poset classes will be presented. Some of these are direct well-known results such as the number of antichains in a fence, while others involve placing bounds on the maximum and minimum possible number of antichains on a class of posets possessing a certain set of characteristics. Secondly, the problem of counting independent sets in a graph will be examined, and it will be shown that there is a relationship between this problem and the antichain counting problem which allows one to derive further antichain counting results. Thirdly, the century old problem known as Dedekind's problem which involves counting the antichains of a power set will be addressed. Finally a problem of Colbourne and Rival, and of Rosenthal, which considers the proportion of antichains containing an element larger than or equal to a given element x will be looked at.

Before proceeding with the antichain counting results, an interesting relationship will be described. Consider a poset $P = (X,<)$. An order ideal (or down-set or decreasing set) of P is a subset Q of X such that whenever $x \in Q$, $y \in X$ and $y < x$ then $y \in Q$. Similarly an order filter (or up-set or increasing set) of P is a subset Q of X such that whenever $x \in Q$, $y \in X$ and $y > x$ then $y \in Q$. There is a bijection from the order ideals of P to the antichains of P. Consider the set S of maximal elements of any order ideal Q. S will form an antichain in P. Also each antichain in P corresponds to the one order ideal consisting of all elements less than or equal to the elements of that antichain. Similarly there is a bijection from the order filters of P to the antichains of P. Thus any results involving the number of antichains in a poset can have the term "antichain" replaced by "order ideal" or "order filter".

2.2 ANTICHAIN COUNTING RESULTS

There are a number of results in the current literature which give specific counts of the number of antichains in a given poset. One such result is the number of antichains in an n-fence, which is well-known and can be found for instance, in a paper by Beck [Be]. The n-fence P_n is the poset on n vertices $\{1, ..., n\}$ such that for $i \in \{1, ..., n-1\}$, $i < i + 1$ if i is odd and $i > i + 1$ if i is even. P_n is shown in Figure 2.2.1.

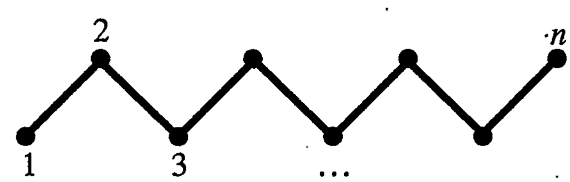


Figure 2.2.1

It is known that $a(P_n) = F_{n+1}$ where F_n is the n'th Fibonacci number which is defined as follows:

$$F_0 = 1, F_1 = 1,$$
$$F_n = F_{n-1} + F_{n-2}.$$

This relationship can be easily seen by viewing $a(P_n)$ as the sum of the number of antichains in P_n containing n and the number not containing n. The number of antichains not containing n will simply be $a(P_{n-1})$. Any antichain containing vertex n cannot contain vertex $n-1$ but can contain any of the other $n-2$ vertices. Thus to get the set of antichains containing n, simply adjoin n to each of the antichains in P_{n-2}, giving a total of $a(P_{n-2})$ antichains. Thus $a(P_n) = a(P_{n-1}) + a(P_{n-2})$. It is easy to see that $a(P_0)= 1$ and $a(P_1)= 2$, and thus the desired result follows.

This relationship between number of antichains (or ideals) in a fence and the Fibonacci numbers has an interesting application. By relating the Fibonacci numbers to a geometric object, new identities amongst the Fibonacci numbers can be found as in a paper by Hopkins and Staton [HS] and another by Beck [Be].

A related problem was examined by Berman and Köhler [BK]. They considered the poset $W_{m,n}$, consisting of the cross product of an n-fence with a chain on m vertices. Such a poset is shown in Figure 2.2.2.

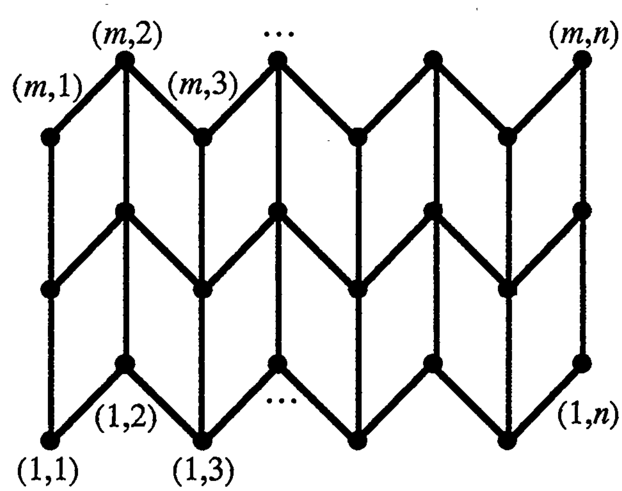


Figure 2.2.2

Berman and Köhler found the number of antichains in such a poset to be recursively defineable as follows:

$$a(W_{m,0}) = a(W_{0,n}) = 1$$

$$a(W_{m,n}) = a(W_{m-1,n}) + \sum_{\substack{i+j=n-1 \\ i \text{ even}}} a(W_{m-1,i}) \cdot a(W_{m,j}) .$$

This result is found by applying Theorem 2.2.1 which will be listed below after giving a few preliminary definitions. Let P be a poset on a set X, and let S be a subset of X. Then $P \backslash S$ will be the subposet of P with underlying set $X \backslash S$. Let x be an element of X. Then define $\text{cone}(x) = \{y \in X \mid y = x \text{ or } y < x \text{ or } y > x\}$. Thus cone($x$) is the subset of elements of X which are related to x.

Theorem 2.2.1. Let x be an element of X. Then

$$a(P) = a(P \backslash x) + a(P \backslash \text{cone}(x)).$$

Proof. The proof is fairly obvious. The number of antichains in P will equal the number of antichains in P not containing x plus the number of antichains containing x. The former is simply $a(P \backslash x)$. Any antichain containing x cannot contain any other element in cone(x), so will only contain elements from $P \backslash \text{cone}(x)$ along with x. There will be $a(P \backslash \text{cone}(x))$ such antichains and the theorem follows. □

By applying Theorem 2.2.1 to any partially ordered set, the number of antichains can be systematically found. Berman and Köhler have implemented this in a computer program to count antichains in any given partially ordered set.

In the same paper, Berman and Köhler [BK] find both recursive and implicit formulas for the number of antichains in posets which are cross-products of other posets. One such result is a recursive count of the number of antichains in the cross product of a poset P with a path on n vertices P_n, which is derived using algebraic methods.

Lemma 2.2.2. Let P be a partially ordered set, P_n be a path on n vertices, and $S(P)$ be the set of order ideals of P. Then

$$a(P \times P_n) = \sum_{J \in S(P)} a(J \times P_{n-1}).$$

Berman and Köhler also generalize this result to Lemma 2.2.3 which follows.

Lemma 2.2.3. Let P be a poset, Z be a poset with a unique minimal element and let P_1 be the one element poset. Then

$$a\big(P \times (P_1 \oplus Z)\big) = \sum_{J \in S(P)} a\,(J \times Z).$$

By substituting P_{n-1} for Z we get Lemma 2.2.2.

In a paper by Stanley [Sy1] the number of antichains in the cross-product of three chains is considered. He derives the following result.

Lemma 2.2.4. Let P_j denote a path on j elements, where j is an integer. Then

$$a(P_k \times P_m \times P_n) = \prod_{j=0}^{k-1} \frac{\binom{n+m+j}{m}}{\binom{m+j}{m}}.$$

Berman and Köhler also consider the cross-product of several chains [BK]. They generate a table of specific values for $a(P_2 \times P_2 \times P_m \times P_n)$ where m ranges from 3 to 5 and n ranges from 1 to 10. They also produce values for $a(P_2 \times P_2 \times P_2 \times P_2 \times P_n)$ where n ranges from 1 to 10.

Unlike the number of antichains in the cross product of two posets, the number of antichains in the linear sum and disjoint sum of two posets can easily be found. These results are given in the following lemma.

Lemma 2.2.5. Let P_1 and P_2 be partially ordered sets. Then

a) $a(P_1 \oplus P_2) = a(P_1) + a(P_2) - 1$;

b) $a(P_1 \cup P_2) = a(P_1) \cdot a(P_2)$.

Faigle, Lovász, Schrader and Turán [FLST] produce recursive equations for the number of antichains in certain classes of posets. Firstly, series-parallel posets will be examined.

Definition 2.2.6. A series parallel poset is defined recursively as follows:

1) A single vertex is series parallel.

2) Let $P_1 = (X_1, <_1)$ and $P_2 = (X_2, <_2)$ be series parallel posets with $X_1 \cap X_2 = \emptyset$. Then $P_1 \oplus P_2$ and $P_1 \cup P_2$ are series parallel posets.

Using the definition above, Lemma 2.2.5 can be implemented as a computer program to count antichains in series-parallel posets.

Faigle et al. [FLST] also produce a recursive relation on the antichains of a certain class of posets as expressed in the following lemma.

Lemma 2.2.7. Let P be a poset and let $\min P$ be the set of minimal elements of P. Suppose there exists $a \in \min P$ such that for every $y \in P \setminus \min P$, $a < y$. Let $P' = P \setminus \{a\}$. Then

$$a(P) = a(P') + 2^{|\min P| - 1}.$$

This follows directly from Lemma 2.2.1.

Another problem related to that of counting antichains is the idea of counting maximal antichains in a partially ordered set. Ziegler [Z] considered the problems of counting antichains and of counting maximal antichains in a partially ordered set of width w on n elements, and was able to place an upper bound on both of these quantities. His results are stated in the following lemma.

Lemma 2.2.8. Let P be a poset on n elements of width w. We then have:

a) P contains at most $a(n,w)$ antichains where

$$a(n,w) = \max_{\substack{c_1+\ldots+c_w=n \\ c_i \geq 1}} \prod_{i=1}^{w} (c_i + 1).$$

P achieves this maximum if and only if it is the disjoint union of w chains of lengths c_1, ..., c_w, where the c_i's are those maximizing the above equation.

b) P contains at most $ma(n,w)$ maximal antichains where

$$ma(n,w) = \max_{\substack{c_1+\ldots+c_w=n \\ c_i \geq 1}} \prod_{i=1}^{w} c_i .$$

P achieves this maximum if (but not only if) it is the disjoint union of w chains of lengths c_1, ..., c_w, where the c_i's are those maximizing the above equation.

This lemma can easily be seen by considering a poset P of width w. Since P has width w, it can be covered by w chains. If any two chains have an element in common , the total number of antichains can be increased by removing enough edges to eliminate that element from one of the chains. Thus any poset maximizing the number of antichains will be the disjoint union of w chains. The number of antichains in such a poset will be $(c_1 + 1)(c_2 + 1)$ ... $(c_i + 1)$ and part a) follows. A similar argument produces the construction in part b). To show that the "but not only if " clause holds in part b), consider the following simple example. Let $n = 3$ and $w = 2$. Then $ma(n, w) = 2$. Let P be the poset on three elements a, b and c with $a < b$ and $a < c$. $ma(P) = 2$, but P is not of the form described in b).

This concludes this section of general antichain counting methods. The problem of counting antichains in a power set on n elements will be dealt with in a separate section as much work has been done on it since the problem was first suggested nearly one hundred years ago. As well, further antichain counting results will be derived using a technique relating antichains to independent sets in a graph.

2.3 ANTICHAINS IN POWER SETS

A well-known antichain counting problem is that of counting the number of antichains in the power set on n elements, $P(n)$. $P(n)$ is the poset having as its element set all subsets of the set $\{1, ..., n\}$ and as its order relation, set inclusion. As an example $P(1)$, $P(2)$ and $P(3)$ are shown in Figure 2.3.1.

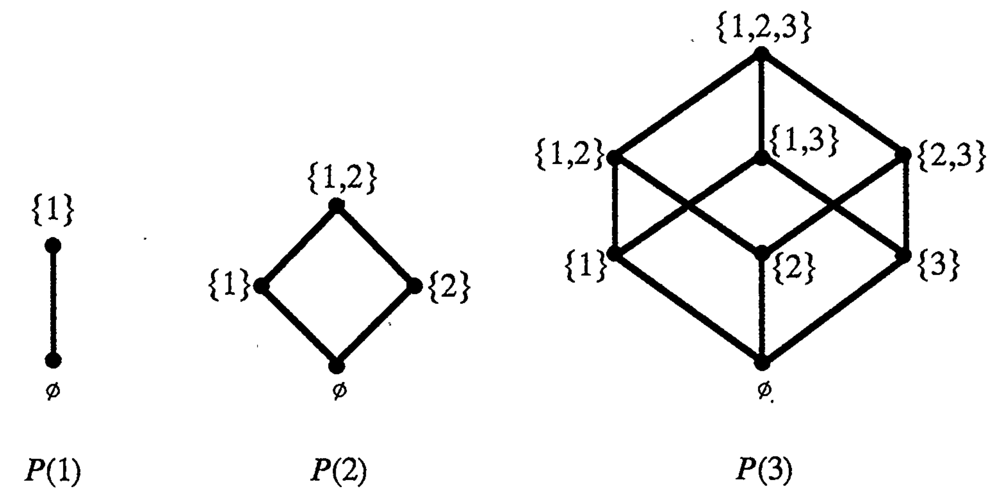


Figure 2.3.1

The problem of evaluationg $a(P(n))$ was first suggested by R. Dedekind in 1897 and hence has become known as "Dedekind's problem", while the value of $a(P(n))$ is called the n'th Dedekind number. Papers discussing this problem are too numerous to mention, however an extensive listing of these can be found in a paper by Kisielewicz [Ki]. Many of these attempt to place bounds on the Dedekind numbers, estimate the Dedekind numbers asymptotically and produce algorithms to find the Dedekind numbers. Currently, only exact values for the first 0 to 8'th Dedekind numbers are known and they are as follows:

$$a(P(0)) = 2$$

$$a(P(1)) = 3$$

$$a(P(2)) = 6$$

$a(P(3)) = 20$

$a(P(4)) = 168$

$a(P(5)) = 7{,}581$

$a(P(6)) = 7{,}828{,}354$

$a(P(7)) = 2{,}414{,}682{,}040{,}998$

$a(P(8)) = 56{,}130{,}437{,}228{,}687{,}557{,}907{,}788.$

The 0 to 4'th Dedekind numbers can easily be calculated by hand, as Dedekind himself did. The fifth was done by Church [C1], who developed an algorithm for calculating these by hand. M. Ward [Wa] found the sixth in 1946 , and Church [C2] found the seventh in 1965 using a computer . The eighth was recently found by Wiedemann [Wd]. He developed a new method for calculating the Dedekind numbers, which required 200 hours of computer time in order to produce $a(P(8))$.

Recently, an algebraic solution was found to Dedekind's problem by Andrzej Kisielewicz using a new approach [Ki]. Let $[x]$ represent the integer portion of x. Kisielewicz's theorem is as follows:

Theorem 2.3.1. For any $n \geq 1$,

$$a(P(n)) = \sum_{k=1}^{2^{2^n}} \prod_{j=1}^{2^n-1} \prod_{i=1}^{j-1} \left(1 - b_i^k b_j^k \prod_{m=0}^{[\log_2 i]} \left(1 - b_m^i + b_m^i b_m^j \right) \right)$$

$$\text{where } b_i^k = \left[\frac{k}{2^i}\right] - 2\left[\frac{k}{2^{i+1}}\right].$$

Kisielewicz proves this by constructing an isomorphic copy of $P(n)$ with a different labelling from which the above equation can be derived. Thus the Dedekind numbers can finally be expressed as a single equation. Unfortunately this does not help in finding further Dedekind numbers as it is no more efficient than the other algorithms which exist for counting antichains in power sets.

Kurepa considered the related problem of counting the number of maximal antichains in a power set [Ku]. In his paper, the number of maximal antichains in a power set on n elements is called the right overturned factorial or the dual factorial of n, written n¡ . Kurepa was able to obtain the values of n¡ for n from 0 to 5 by directly counting the maximal antichains in the specific power sets. They are as follows:

0¡ = 1	3¡ = 7
1¡ = 2	4¡ = 29
2¡ = 3	5¡ = 146.

So far, no algebraic formula has been found to describe the behaviour of n¡ .

A paper by Popadic in 1970 focusses on finding formulas to represent the number of k-element antichains in a power set [P]. Let $a_k(P(n))$ represent the number of k-element antichains in the power set on n elements. Popadic was able to find the following explicit formulas for $a_2(P(n))$ and $a_3(P(n))$.

Lemma 2.3.2. $$a_2(P(n)) = 2^{2n-1} + 2^{n-1} - 3^n.$$

Lemma 2.3.3. Let $I_{nrpq} = 2^n + 2^r + 2^{n-r-p-q} + 2 - 2^{r+p} - 2^{r+q} - 2^{n-r-p} - 2^{n-r-q}$. Then
$$a_3(P(n)) = \frac{1}{6} \sum_{r=0}^{n-2} \sum_{p=1}^{n-r-1} \sum_{q=1}^{n-r-p} \binom{n}{r}\binom{n-r}{p}\binom{n-r-p}{q} I_{rpq}.$$

Lemma 2.3.2 follows easily from a simple combinatorial argument. Lemma 2.3.3 requires a little more work and the basic argument is as follows. Consider two incomparable members of $P(n)$, B and C. Let $r = |B \cap C|$, $p = |B - C|$ and $q = |C - B|$. Then $0 \leq r \leq n-2$, $1 \leq p \leq n-r-1$ and $1 \leq q \leq n-r-p$. The number of elements simultaneously incomparable with B and C is equivalent to the number of 3-element antichains containing B and C which can be shown to equal the equation given by I_{nprq}. The value of I_{nprq} is calculated using basic enumeration techniques. It is then found that the number of incomparable pairs having given p, r and q values is $\binom{n}{r}\binom{n-r}{p}\binom{n-r-p}{q}$. By summing over all possible values of p, r and q and recognizing that each antichain gets counted six times (once for each permutation of the elements of each antichain), Lemma 2.3.3 follows.

Though much work has been done on the problem of counting antichains in power sets, much remains unknown in this area. Values of $a_k(P(n))$ have yet to be found for $k\backslash > 3$ and the Dedekind numbers beyond the eighth are unknown.

2.4 ANTICHAINS AND INDEPENDENT SETS

The problem of counting antichains in partially ordered sets is related to that of counting independent sets in graphs. How these two problems relate will be described after some initial terminology is given.

An independent set in a graph is any induced subgraph of that graph which contains no edges. For example, in a complete graph, the only independent sets will be each of the vertices of that graph and the empty set of vertices. Let $i(G)$ represent the number of independent sets in a given graph, G.

A bipartite graph is any graph whose vertices can be partitioned into two sets such that there are no edges between any two members of the same set.

Let $G(P)$ be the comparability graph of $P = (X,<)$ which is the graph on set X such that for every x and y in X, there is an edge between x and y if and only if either $x<y$ in P

or $y<x$ in P. Whenever P is a poset of height one or less, its comparability graph will be the bipartite graph which looks identical to P. The following lemma describing the relationship between the antichains of P and the independent sets of $G(P)$ can easily be seen to hold.

Lemma 2.4.1 Let $P = (X,<)$ be a partially ordered set and let S be a subset of X. S is an antichain in P if and only if it is an independent set in $G(P)$.

While there are few specific results known about antichain totals for various classes of posets, much more work has been done on the problem of counting independent sets in graphs. Since not every graph is the comparability graph of some poset, it becomes important to be able to distinguish between those graphs which are comparability graphs and those which are not. Then it can be determined whether or not a given independent set counting result can be transformed into an antichain counting result. The following result [GH] due to Gilmore and Hoffman and also to Ghouila-Houri gives a characterization of comparability graphs.

Theorem 2.4.2. A graph G is a comparability graph if it contains no sequences of vertices $v_1, v_2, \ldots, v_n$, with n odd and $n \geq 3$ such that for every i, v_i and v_{i+1} are joined by an edge and v_i and v_{i+2} are not joined by an edge (where addition is modulo n).

This theorem gives a simple means of testing whether or not a given graph is a comparability graph.

The remainder of this section will survey specific classes of graphs for which independent sets have been counted, and then will describe the corresponding classes of posets having these graphs as their comparability graphs. Values for $a(P)$ can then be attached to these derived posets by Lemma 2.4.1.

Firstly, recall the n-fence dealt with in Section 2.2. It has a path of length n as its comparability graph. Rather than directly computing the number of antichains in the n-fence as was previously done, the problem can be thought of as calculating the number of independent sets in the n-path, which of course will be the same as the number of antichains in the n-fence. A related problem is that of counting the number of independent sets of size k in a path on n elements. Kaplansky [Ka] found the value of this to be $\binom{n-k+1}{k}$. This corresponds to the number of antichains of size k in an n-fence.

Now consider a $2n$-crown as in Figure 2.4.1b. This has as its comparability graph, the cycle on $2n$ vertices, as is shown in Figure 2.4.1a.

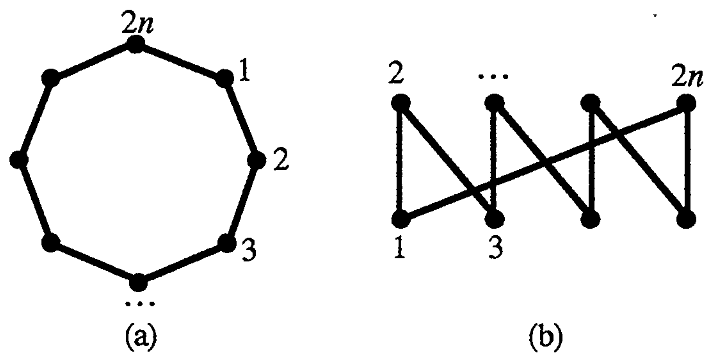


Figure 2.4.1

Prodinger and Tichy [PT] have shown that the number of independent sets in a cycle containing m vertices, $C(m)$, is the m'th Lucas number L_m, for $m \geq 2$. The m'th Lucas number is defined as follows:

$$L_0 = 2;$$

$$L_1 = 1;$$

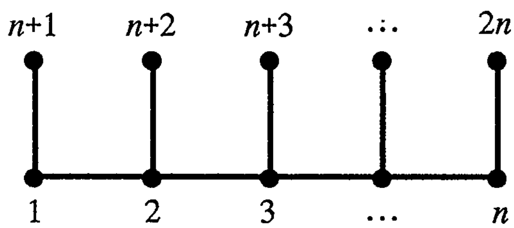
$$L_m = L_{m-1} + L_{m-2}.$$

To see this relationship, label the vertices of the cycle 1, ..., m. The number of independent sets in this cycle will be the number of independent sets in a path of length m minus the number of independent sets in that path containing both vertices 1 and m. The former quantity is the m+1'st Fibonacci number (from Section 2.2) and the latter quantity is the number of independent sets in a path of length m-4 which is the m–3'rd Fibonacci number. Using these ideas, the following is derived:

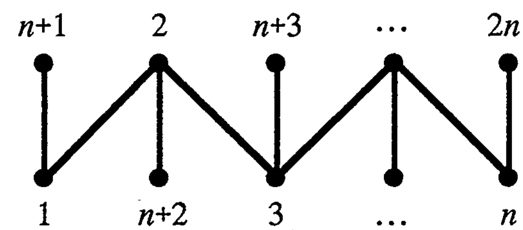
$$\begin{aligned} i(C(m)) &= F_{m+1} - F_{m-3} \\ &= F_m + F_{m-1} - F_{m-4} - F_{m-5} \\ &= i(C(m-1)) + i(C(m-2)). \end{aligned}$$

By verifying that $i(C(m)) = L_m$ for $m = 2$, 3 and 4.the relationship follows. Thus it can be concluded that the number of antichains in the $2n$-crown is equal to the $2n$'th Lucas number.

Another graph considered by Prodinger and Tichy [PT] is that shown in Figure 2.4.2a which will be called R_n. This graph is the comparability graph of poset R'_n shown in Figure 2.4.2b.



(a) R_n



(b) R'_n

Figure 2.4.2

The number of independent sets $i(R_n)$ can be found recursively by adding two new vertices to graph R_n to produce R_{n+1}, and then summing the number of independent sets containing the new vertices and the number not containing the new vertices. Using this method $i(R_n)$ is found to be defined by the following recursion:

$$i(R_1) = 3;$$

$$i(R_2) = 8;$$

$$i(R_{n+1}) = 2i(R_n) + 2i(R_{n-1}).$$

Solving the recursion gives

$$a(R'_n) = i(R_n) = \frac{3 + 2\sqrt{3}}{6}(1 + \sqrt{3})^n + \frac{3 - 2\sqrt{3}}{6}(1 - \sqrt{3})^n.$$

Adding some additional edges to graph R_n produces graph Q_n which is the comparability graph of Q'_n as shown in Figure 2.4.3.

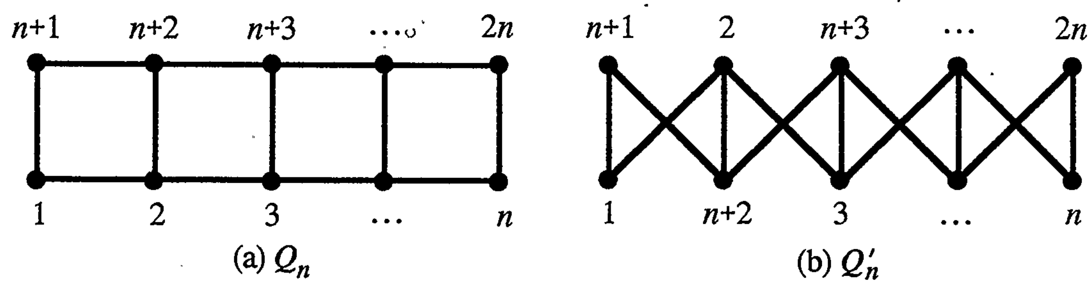


Figure 2.4.3

Using similar methods to those above, Prodinger and Tichy [PT] define $i(Q_n)$ recursively and then solve that recursion to obtain the following result:

$$i(Q_n) = \frac{1}{2}\left[(1 + \sqrt{2})^{n+1} + (1 - \sqrt{2})^{n+1}\right].$$

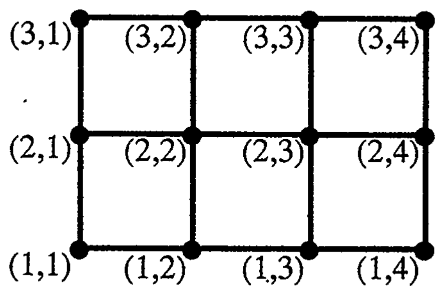
As before, $i(Q_n)$ will equal the number of antichains in Q'_n.

A more difficult problem is that of calculating the number of independent sets in what is known as an $m \times n$ lattice, $L_{m,n}$, not to be confused with the lattices defined in Section 1.2. $L_{m,n}$ is the planar graph resulting from neighbouring points joining to form a rectangular grid. As an example, $L_{3,4}$ is shown in Figure 2.4.4a, and it should be noted that graph Q_n from Figure 2.4.3 is also the $2 \times n$ lattice. As with the graphs dealt with previously in this section, $L_{m,n}$ is the comparability graph of a specific class of partially ordered sets. The partially ordered set $L'_{m,n}$ has vertex set $\{(a,b) \mid 1 \le a \le m, 1 \le b \le n\}$ and order relation $<$ defined as follows:

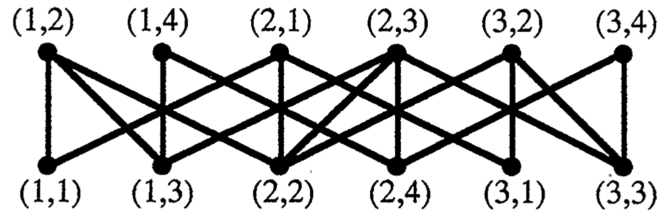
$$\forall\, a, b \text{ such that } 1 \le a \le m \text{ and } 1 \le b \le n-1 \quad \begin{cases} (a, b) < (a, b+1) & \text{if } a+b \text{ even} \\ (a, b) > (a, b+1) & \text{if } a+b \text{ odd} \end{cases}$$

$$\forall\ a, b \text{ such that } 1 \le a \le m\text{-}1 \text{ and } 1 \le b \le n \quad \begin{cases} (a, b) < (a+1, b) & \text{if } a+b \text{ even} \\ (a, b) > (a+1, b) & \text{if } a+b \text{ odd} \end{cases} .$$

$L'_{3,4}$ is shown in Figure 2.4.4b.



(a) $L_{3,4}$



(b) $L'_{3,4}$

Figure 2.4.4

Both Weber [We] and Engel [E] have attempted to determine the number of independent sets in $L_{m,n}$, and so far the exact solution has not been found. Weber was able to place the following bounds on the value of $i(L_{m,n})$, which of course equals $a(L'_{m,n})$.

Theorem 2.4.3. For $mn > 1$, $1.45^{mn} < i(L_{m,n}) < 1.74^{mn}$.

Weber also showed the following:

Lemma 2.4.4. $1.45 \le \lim_{n\to\infty} \left[i(L_{n,n})\right]^{1/n^2} \le 1.554$, and this limit exists.

Engel strengthened Weber's lemma to produce the following improved bounds.

Lemma 2.4.5. $1.503 \le \lim_{n\to\infty} \left[i(L_{n,n})\right]^{1/n^2} \le 1.514$.

Engel also conjectured that $\lim_{n\to\infty} \left[i(L_{n,n})\right]^{1/n^2} = 1.50304808\ldots$.

Work has also been done on finding the number of independent sets in specific classes of trees. Note that any connected cycle-free height-1 poset will have a tree as its comparability graph. Define a complete$t$-ary tree of height $n-1$ as a tree in which exactly one vertex has degree t, every other vertex has degree 1 or $t+1$, and all paths connecting the degree 1 vertices to the degree t vertex contain exactly n vertices. Such a tree will be denoted $T_n(t)$. As an example, $T_3(3)$, the complete 3-ary tree of height 2 is shown in Figure 2.4.5.

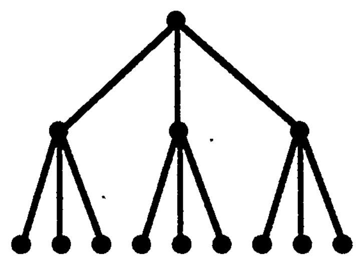


Figure 2.4.5

Let the poset having $T_n(t)$ as its comparability graph be called $P_n(t)$. Kirschenhofer, Prodinger and Tichy [KPT] produced results involving the number of independent sets in $T_n(t)$. These results will be given with the understanding that $i(T_n(t))$ can be replaced by $a(P_n(t))$ to give the number of antichains in $P_n(t)$.

Kirschenhofer et al. first consider the case where $t = 1$. This tree is simply a path, and thus is the comparability graph of a fence, which has previously been considered. If t equals 2, 3, or 4, the following result is produced for $i(T_n(t))$:

Lemma 2.4.6 When $t = 2$, 3 or 4 then

$$i(T_n(t)) \sim D(t) \cdot K(t)^{t^n} \text{ as } n \to \infty,$$

where $D(t)$ and $K(t)$ are constants depending on t such that

$$2^{1/(1-t)} < D(t) < 1 < K(t) < 2^{1/(t-1)}.$$

For $t \geq 5$, Kirschenhofer et al. found the following result for $i(T_n(t))$:

Lemma 2.4.7 When $t \geq 5$ then

$$i(T_{2m}(t)) \sim B(t) \cdot K(t)^{t^{2m}} \quad \text{as } m \to \infty$$
$$\text{and } i(T_{2m+1}(t)) \sim C(t) \cdot K(t)^{t^{2m+1}} \quad \text{as } m \to \infty .$$

where $B(t) > C(t)$ and $B(t)$ and $C(t)$ are constants depending on t with

$$\lim_{t \to \infty} B(t) = \lim_{t \to \infty} C(t) = 1.$$

Finally Kirschenhofer et al. considered binary trees on n vertices. Note that the poset having such a graph as its comparability graph will be a non-cyclical height-1 poset with one vertex of degree 2 and the remaining vertices having degree 3 or 1. Let S_n represent the average value for the number of independent sets in a binary tree on n vertices. Kirschenhofer et al. have shown the following:

Lemma 2.4.8 $S_n \sim (1.12928\ldots)(1.63742\ldots)^n$ as $n \to \infty$.

Füredi [Fu] considered the problem of counting the number of maximal independent sets in a connected graph on n vertices. If G is a graph, let $mi(G)$ represent the number of maximal independent sets of G. By calling an independent set S maximal we mean that no vertices can be added to S to produce another independent set. Now let $m(n)$ be the maximum number of maximal independent sets possible in a connected graph on n vertices. Füredi discovered the following:

Lemma 2.4.9. If $n > 50$ then

$$m(n) = \begin{cases} 2 \cdot 3^{t-1} + 2^{t-1} & \text{for } n = 3t; \\ 3^t + 2^{t-1} & \text{for } n = 3t+1; \\ 4 \cdot 3^{t-1} + 3 \cdot 2^{t-2} & \text{for } n = 3t+2. \end{cases}$$

It should be noted that any poset on n vertices which is connected will have a connected graph on n vertices as its comparability graph. Also note that any set of vertices forms a maximal independent set in a comparability graph if and only if it forms a maximal antichain in the poset having that graph as its comparability graph. Thus it can be concluded that for any connected poset P on n vertices with $n > 50$, the maximum possible number of maximal antichains in P will be less than or equal to $m(n)$. As well, there are posets which achieve this maximum, and their comparability graphs can be found in [Fu].

Griggs, Grinstead and Guichard were able to show that the preceding result holds for all $n \geq 6$ [GGG]. They also added that if $n < 6$, then $m(n) = n$.

Wilf [Wl] considered the same problem restricted to a tree T_n on n vertices and was able to reduce the upper bound on the number of maximal antichains as follows:

Lemma 2.4.10. Let $m(T_n)$ be the maximum number of maximal independent sets that can occur in a connected tree on n vertices. Then

$$m(T_n) = \begin{cases} 2^{n/2-1} + 1 & \text{if } n \text{ even;} \\ 2^{(n-1)/2} & \text{if } n \text{ odd.} \end{cases}$$

This result gives the maximum number of maximal antichains possible in a poset on n elements having a tree as its comparability graph.

One further independent set counting problem was examined by V. Linek [Lk]. He was able to demonstrate the following:

Theorem 2.4.11. For every integer $n \geq 1$ there exists a bipartite graph with exactly n independent sets.

Using Lemma 2.4.1 we get that for every integer $n \geq 1$ there exists a partial order of length at most one with exactly n antichains. This result will be discussed further in the final chapter where a related problem will be addressed.

This completes a survey of results involving the counting of independent sets in graphs. By applying Lemma 2.4.1 to these results, new information about the number of antichains is added to that of the previous section.

2.5 CENTRAL ELEMENTS IN POSETS

One interesting question relating to the idea of counting antichains in partially ordered sets is the following. Let $a(P, \geq x)$ represent the number of antichains of poset P which contain an element $\geq x$, let $a(P, x)$ be the number of antichains in P which contain x, and let $a(P)$ denote the total number of antichains of P as usual. Is there a real number $0 < \lambda \leq 1/2$ such that every poset P has an element x satisfying

$$\lambda \leq \frac{a(P, \geq x)}{a(P)} \leq 1 - \lambda \ ? \tag{2.5.1}$$

An element satisfying (2.5.1) is known as a central element of poset P. Equivalent versions of this question were raised independently by Colbourn and Rival and by Rosenthal. Sands considered this problem [Sa] and was able to prove the following.

Theorem 2.5.1. For every integer $l > 1$, there exists a number $0 < \lambda < 1$ such that for every finite poset P of length l–1 or less, there is an element x such that

$$\lambda \leq \frac{a(P, x)}{a(P)}.$$

Note that Theorem 2.5.1 deals with the number of antichains containing an element x, in a poset of length l, while equation 2.5.1 deals with the number of order ideals containing x, in any poset. These two problems are equivalent when $l = 1$ but not when $l > 1$.

Linial and Saks however, proved the following theorem which conclusively affirms the existence of a λ satisfying (2.5.1) for some x in every poset P [LS].

Theorem 2.5.2. In any finite partially ordered set P, there is an element x in P such that

$$0.17 \cong \frac{3-\log_2 5}{4} < \frac{a(P, \geq x)}{a(P)} < 1 - \frac{3-\log_2 5}{4} \cong 0.83.$$

Shearer (see [LS]) has shown that there are posets for which (2.5.2) fails when λ is taken to be 0.197. Thus 0.17 must be close to the true bound.

Faigle, Lovász, Schrader and Turán [FLST] were able to improve upon Linial and Saks' value of 0.17 for λ when restricting the posets in equation (2.5.1) to series-parallel posets, interval orders and trees . Series parallel posets were defined in Section 2.2. An interval order is a partially ordered set $(X,<)$ such that for every $a, b, c, d \in X$, $a < b$ and $c < d$ implies that either $a < d$ or $c < b$ (or both). In the context of a partially ordered set, a tree is any poset $(X, \leq)$ such that for every $x \in X$, $Y = \{y \in X \mid x \leq y\}$ forms a chain in $(X, \leq)$. Now their theorems can be stated.

Theorem 2.5.3. Let $P = (X,<)$ be a series parallel poset or an interval order. Then there exists an element x in X such that

$$\frac{1}{4} \leq \frac{a(P, \geq x)}{a(P)} \leq \frac{3}{4}.$$

Theorem 2.5.4. Let $P = (X,<)$ be a tree. Then there exists an element x in X such that

$$\frac{1}{3} \leq \frac{a(P, \geq x)}{a(P)} \leq \frac{2}{3}.$$

Faigle et al. have also shown that the bounds in the above two theorems are the best possible. Thus it is not only known that there is a real number λ such that equation (2.5.1) is true for some element x in every poset P, but it is also known that λ can be made larger when we are restricted to certain classes of partially ordered sets.

This section concludes a survey of results involving the counting of antichains in partially ordered sets. Though much work has been done in this area, it can be seen that there remain open questions to be answered and classes of posets to be examined.

CHAPTER THREE

COUNTING LINEAR EXTENSIONS

3.1 INTRODUCTION

In general, the problem of counting linear extensions is considerably more difficult than that of counting antichains or that of counting chains. Consequently there are few linear extension counting results. The linear extension counting problem has actually had more attention in the realm of Computer Science than in Mathematics. Brightwell and Winkler [BrWi] have assessed the difficulty of this problem and determined it to be #P-complete, the exact meaning of which can be found in [J]. It suffices to say that counting linear extensions is generally considered difficult.

Work has been done on finding efficient algorithms to count linear extensions in various classes of posets. Such algorithms exist for counting the number of linear extensions of a poset of width 2 [AC], a tree [A] and a poset of width k [Sr].

There are a few results which involve the direct counting of linear extensions in partially ordered sets. Such results will be handled in the following two sections, the first of which will cover posets of width 2 and the second of which will cover any remaining posets. Section 3.4 will give a full proof based on the FKG inequality of an important result known as the XYZ inequality, a theorem which involves considering the probability that a given relation holds in an arbitrary linear extension of a specific poset. The XYZ inequality is just one result in a group of theorems known as correlation inequalities which will be dealt with in the final section of this chapter.

3.2 POSETS OF WIDTH TWO

Many results have been produced on the problem of counting the number of linear extensions of partially ordered sets of width 2. Some of these posets are shown in the following diagram.

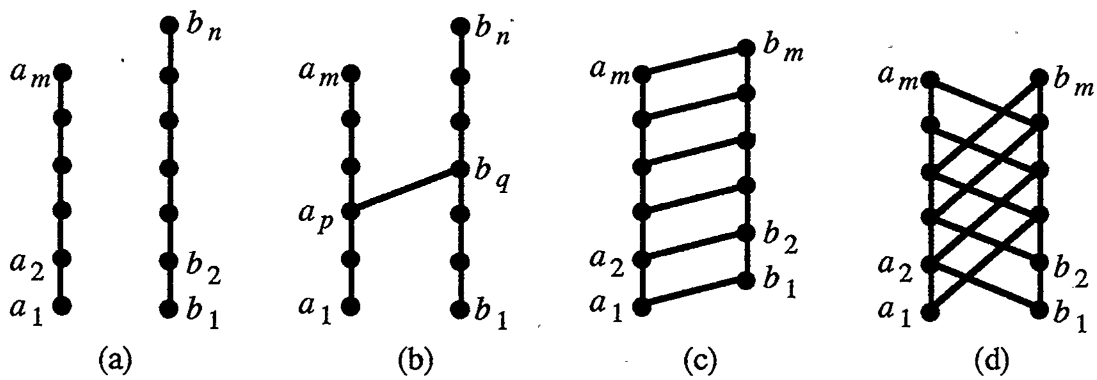


Figure 3.2.1

Consider a poset which consists of two chains, one of length m and the other of length n, as shown in Figure 3.2.1a. Label the elements of the first chain $a_1, ..., a_m$ and the elements of the second chain $b_1, ..., b_n$. It is well known that the number of linear extensions of such a poset will simply be $\binom{m+n}{m}$ since there are a total of m+n positions to fill, and m of these positions must be selected to indicate where the elements of the chain of length m will be placed. As a sorting problem, this poset represents the merging of two sorted sets.

Now consider a poset P that is like the one above, but which has a single constraint of the form $a_p < b_q$ added, where $1 \leq p \leq m$ and $1 \leq q \leq n$, as is shown in Figure 3.2.1b. The following results involving the number of linear extensions of P can be found in Knuth [Kn]. In any linear extension of P, it is obvious that elements $a_{p+1}, ..., a_m$ must lie above a_p and $a_1, ..., a_{p-1}$ must lie below a_p. Also elements $b_q, ..., b_n$ must lie above a_p.

However elements $b_1, ..., b_{q-1}$ could lie either above or below a_p. Let k be any integer such that $0 \le k < q$. Consider the set of linear extensions in which $b_1, ..., b_k$ lie below a_p and $b_{k+1}, ..., b_{q-1}$ lie above a_p. The number of such linear extensions $e(P,k)$ is as follows:

$$e(P,k) = \binom{m-p+n-k}{m-p}\binom{p-1+k}{p-1}.$$

When the sum over all possible values of k of the above quantity is taken, the total number of linear extensions of P is

$$e(P) = \sum_{0 \le k < q} \binom{m-p+n-k}{m-p}\binom{p-1+k}{p-1}.$$

Now, the situation in which many constraints of the the form $a_p < b_q$ and $b_j < a_l$ are added to a poset consisting of two chains will be considered. It is assumed that the added constraints are consistent with one another, meaning that the properties of transitivity and irreflexivity are not violated, and that the transitive closure of the resulting poset is considered. Atkinson and Chang [AC] as well as Mohanty [M] have produced algorithms to count the number of linear extensions in such a poset. As well, there are certain specific examples of this sort of poset for which the total number of linear extensions can be found. Consider the partially ordered set of this type for which both chains are of the same length ($m = n$) and which has added constraints $a_i < b_i$ for all $1 \le i \le m$. This partially ordered set is shown in Figure 3.2.1c. It is known that the number of linear extensions of this poset is the m'th Catalan number C_m which is defined as follows:

$$C_m = \frac{1}{m+1}\binom{2m}{m}.$$

This result can be found in a paper by Atkinson and Chang [AC] and it also follows from the equivalence of this problem to a lattice path counting problem. Consider an $m \times m$ lattice as defined in Section 2.4. It is well-known (see for instance [M]) that the number of paths of $2m$ steps from (0,0) to (m,m) which don't cross below a diagonal line joining

(0,0) to (m,m), is also the Catalan number, C_m. The bijection from the linear extensions of the poset in 3.2.1c to the above described paths is as follows. Given a linear extension $x_1 < x_2 < \ldots < x_{2n}$ of the poset, start at (0,0) on the lattice. Given that the first $i-1$ steps have been taken, let the i'th step be "north" if $x_i \in \{a_1, \ldots, a_m\}$ and "east" if $x_i \in \{b_1, \ldots, b_m\}$. It is easy to check that this produces a bijection.

Finally consider the poset shown in Figure 3.2.1d. As a matter of interest, the complement of the comparability graph of this particular poset is the "zig-zag" graph or $2m$-fence as described in Section 2.2. The number of linear extensions of such a poset on $2m$ elements is the $2m$'th Fibonacci number where the n'th Fibonacci number F_n is defined in Section 2.2. This result was found by Atkinson and Chang [AC].

Now that results involving the number of linear extensions of some types of width-2 posets have been discussed, the problem of counting linear extensions of other posets will be discussed in the following section.

3.3 OTHER POSETS

Posets of width 2 are not the only posets for which attempts have been made to count linear extensions. In this section, results for counting linear extensions of bipartite graphs and power sets will be examined, as well as a number of results involving the general linear extension counting problem.

Stachowiak [Sk1] dealt with the problem of counting linear extensions of bipartite graphs. A bipartite graph can be thought of as a partially ordered set by attaching directions to the edges of the graphs, providing the edges are directed so as not to produce a cycle.

Such an assignment of directions is known as an orientation of the graph. A natural orientation of a bipartite graph is an orientation such that the vertices can be partitioned into two sets X_1 and X_2 so that X_1 and X_2 are antichains and such that if $x \in X_1$, $y \in X_2$ and x and y are related then $x < y$. It is easy to see that every bipartite graph must have at least two natural orientations. Stachowiak provides the following theorem:

Theorem 3.3.1. The number of linear extensions of an orientation of a bipartite graph is less than or equal to the number of linear extensions of a natural orientation of that graph.

This theorem is proved using induction on the number of vertices in the graph. Let G and G' be graphs on the same set of vertices. Then we say $G \subseteq G'$ if the set of edges of G is a subset of the set of edges of G'. The following corollary arises from Theorem 3.3.1.

Corollary 3.3.2. Let P be a height-1 poset. Then P is a natural orientation of $G(P)$, its comparability graph, which will be bipartite. Let Q be a poset on the same set of elements as P. Then a) $G(P) \subseteq G(Q) \Rightarrow e(P) \geq e(Q)$

b) $G(P) = G(Q) \Leftrightarrow e(P) = e(Q)$.

It may be noted that the $\Rightarrow$ direction of b) follows immediately from part a). In another paper, Stachowiak[Sk2] extends this result to posets of arbitrary height.

Theorem 3.3.3. If P and Q are posets on the same set of elements then

a) $G(P) \subseteq G(Q) \Rightarrow e(P) \geq e(Q)$

b) $G(P) = G(Q) \Rightarrow e(P) = e(Q)$.

In order to prove this theorem, the following lemma is required:

Lemma 3.3.4. Let A be an antichain of P. Then

$$\sum_{a \in A} e(P-a) \leq e(P).$$

The lemma follows from an argument using induction on the number of elements in P.

Proof of Theorem 3.3.3. The proof of a) is by induction on the number of elements in P. Assume $G(P) \subseteq G(Q)$. If P has one element the theorem is obvious. Let $n \geq 1$ and assume that the theorem holds for all $n-1$ element posets. Then $G(P-x) \subseteq G(Q-x)$ for every x in P, so $e(P-x) \geq e(Q-x)$ for every x in P. Let I be the set of minimal elements of poset Q. Summing over all x's in I gives

$$\sum_{x \in I} e(P-x) \geq \sum_{x \in I} e(Q-x). \tag{3.3.1}$$

Then it is easy to see the following:

$$\sum_{x \in I} e(Q-x) = e(Q). \tag{3.3.2}$$

I forms an antichain in P so we can apply Lemma 3.3.4 to I to produce the following:

$$e(P) \geq \sum_{x \in I} e(P-x) \tag{3.3.3}$$

Combining equations (3.3.1), (3.3.2) and (3.3.3) gives the desired result. As in Corollary 3.3.3, part b) follows directly from part a). □

Edelman, Hibi and Stanley [EHS] produce a recurrence for the number of linear extensions of a poset. Before their result can be stated, a few preliminary definitions are required. A chain c of a poset P is saturated if there is no $z \in P-c$ such that $x < z < y$ for some $x, y \in c$ and $c \cup \{z\}$ is a chain. Thus in a finite poset, $c = (x_0 < x_1 < \ldots < x_m)$ is saturated if and only if x_i covers x_{i-1} for each $i \in \{1, \ldots, k\}$. Now let $c = (x_0 < x_1 < \ldots < x_m)$ be a saturated chain with $m > 0$. Let P_c be poset P with the elements of c replaced by $x_{0,1}, x_{1,2}, \ldots, x_{m-1,m}$ such that all the following relations as well as those implied by transitivity hold:

a) $x_{0,1} < x_{1,2} < \ldots < x_{m-1,m}$

b) $y < x_{i,i+1}$ if $y \in P\text{–}c$ and $y < x_{i+1}$ in P

c) $y > x_{i,i+1}$ if $y \in P\text{–}c$ and $y > x_i$ in P.

In addition, when $m = 0$, let $P_c = P - x_0$.

Theorem 3.3.5. Let P be a finite poset and let C be a set of saturated chains of P such that every maximal chain of P contains exactly one element of C as a subposet. Then

$$e(P) = \sum_{c \in C} e(P_c).$$

This is proved by constructing a bijection from the linear extensions of P to the union of the linear extensions of the P_c's. The following corollary arises from this theorem:

Corollary 3.3.6. Let P be a finite poset and let A be an antichain of P intersecting every maximal chain (i.e. A is a *cutset* of P). Then

$$e(P) = \sum_{x \in A} e(P\text{–}x).$$

Notice that this corollary gives the case when Lemma 3.3.4 holds with equality. Also note that Corollary 3.3.6 is a generalization of equation (3.3.2).

Sha and Kleitman [SK] considered the problem of finding the number of linear extensions of the power set on n elements, $P(n)$. It is well known that the number of linear extensions of $P(n)$ is equal to the number of maximal chains in the free distributive lattice on n generators. Though a rigorous definition of the free distributive lattice on n generators $L_n = (X, <)$ will not be given, intuitively it is the lattice whose element set X consists of all polynomials arising from the n generators, where no element is less than another unless it is forced to be by the laws of distributive lattices. As an example, the free distributive lattice on 3 generators is shown in Figure 3.3.1.

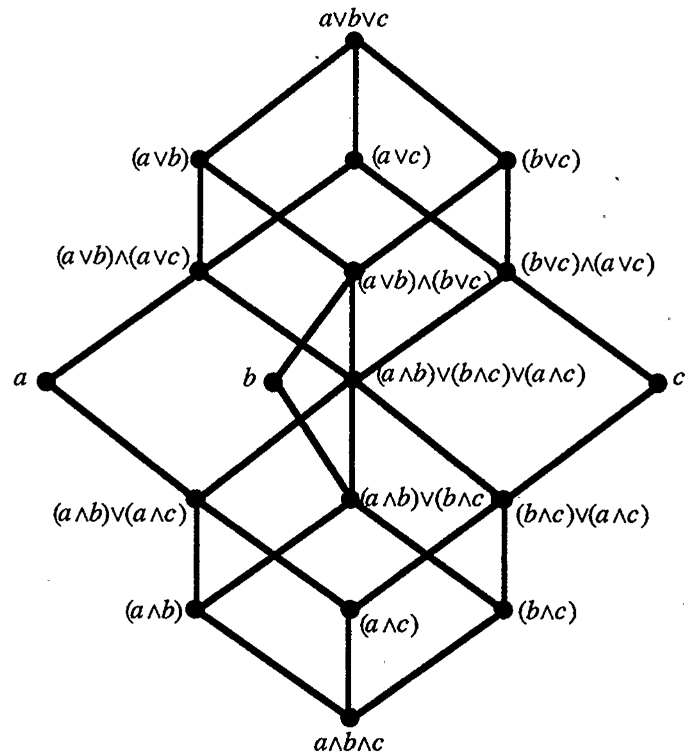


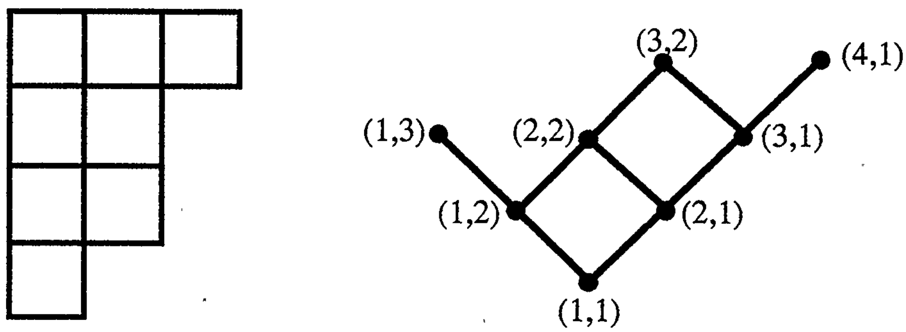
Figure 3.3.1

Sha and Keitman found the following upper bound for the number of linear extensions of $P(n)$.

Theorem 3.3.7.

$$e(P(n)) \le \prod_{k=1}^{n} \binom{n}{k}^{\binom{n}{k}}.$$

Another class of posets for which the total number of linear extensions is known is the class of posets that can be associated with what is called a Young diagram. The ideas stated here can all be found in [Sg]. For a given positive integer n, consider a decreasing sequence of r positive integers $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_r > 0$ such that the sum of the λ_i's is n. Such a sequence describes a Young diagram, which consists of r rows of cells aligned in a grid, such that row i contains λ_i cells for $1 \le i \le r$ and the first column contains the first cell of each row. The Young diagram with $r = 4$, $\lambda_1 = 3$, $\lambda_2 = 2$, $\lambda_3 = 2$ and $\lambda_4 = 1$ is shown in Figure 3.3.2a. Now associate a poset with a given Young diagram. Let X be the set of ordered pairs of the form (p, q) where $1 \le p \le r$, $1 \le q \le \lambda_p$. Now define the order $\le$ on P as follows. Let $(p, q) < (p', q')$ if and only if $p \le p'$ and $q \le q'$. Figure 3.3.2b gives the poset associated with the diagram in Figure 3.3.2a.



a) b)

Figure 3.3.2

From a given Young diagram with n cells we can produce what are called standard Young tableaus by filling the cells with the integers 1, 2, ..., n in such a way that the number in a given cell is less than the number in the cell to its immediate right and the number in the cell below it. Figure 3.3.3 shows several possible Young tableaus resulting from the Young diagram shown in Figure 3.3.2.

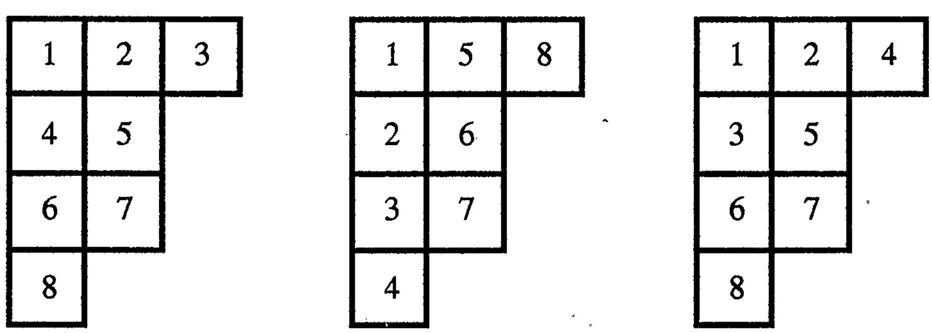


Figure 3.3.3

It is well known that the number of linear extensions of $P(D)$ is equal to the number of standard tableaus arising from diagram D. Thus we are interested in counting the number of standard tableaus resulting from a given diagram. There are numerous results involving counting standard tableaus. One such result, the hook length formula, will be given here.

The hook number of a cell in a diagram is the number of boxes which are either below the cell or to the right of the cell, including the cell itself. Figure 3.3.4 gives the hook lengths of each cell in the Young diagram we have been using as an example.

6	4	1
4	1	
3	1	
1		

Figure 3.3.4

Let $st(D)$ be the number of standard tableaus which can be produced from a given Young diagram D with n cells, and let $d_1, d_2, ..., d_n$ be the hook lengths associated with D. The hook length formula due to Frame, Robinson and Thrall [FRT] is as follows:

Theorem 3.3.8.

$$st(D) = \frac{n!}{d_1 d_2 \ldots d_n}.$$

Because of the relationship described previously, this formula also gives the number of linear extensions in the poset associated with D.

Another well known related problem (found for instance in [Sg]), involves counting the number of linear extensions of rooted trees, where a rooted tree is a poset whose Hasse diagram is a tree containing a single minimal vertex. To each element in such a tree, $T = (X, <)$, can be assigned what is known as an interval number. For a given $x \in X$, let the interval number $in(x)$ be defined as follows:

$$in(x) = |\{y \in X | y \geq x\}|.$$

It is known that the number of linear extensions of such a poset on n elements is

$$\frac{n!}{\prod_{x \in X} in(x)}.$$

Thus some linear extension counting results have been surveyed. In the following two sections, some related problems will be examined.

3.4 THE XYZ INEQUALITY

Related to the problem of counting linear extensions in partially ordered sets is the theorem now known as the XYZ Inequality. This theorem was first conjectured by Ivan Rival and Bill Sands and was eventually proved by L. A. Shepp [Sh2]. Before the problem can be stated, a few preliminary definitions must be given. Let E and F be sets of relations of the form $x_i < x_j$. We will say a permutation satisfies a relation $x_i < x_j$ if x_i comes before x_j in the permutation. Then define $p(E)$ to be the number of permutations of $x_1, ..., x_n$ satisfying all the relations of E divided by the total number of permutations of $x_1, ..., x_n$. Also let $p(E, F)$ be the number of permutations of $x_1, ..., x_n$ satisfying all the relations of both E and F divided by the total number of permutations of $x_1, ..., x_n$. Now define $p(E|F)$ as the number of permutations of $x_1, ..., x_n$ satisfying all the relations of both E and F divided by the total number of permutations of $x_1, ..., x_n$ satisfying all the relations of F. Note that since any partially ordered set is a set of relations of the form $x_i < x_j$, the above definitions apply when E and F are partially ordered sets on the same set of elements. Now the statement of the XYZ Inequality can be made.

Theorem 3.4.1. Let $P = (X,<)$ be a partially ordered set where x, y and z are arbitrary elements of X with $x \mid z$. Then

$$p(x < y \mid P) \leq p(x < y \mid x < z, P),$$

where $p(x < y \mid x < z, P)$ is the proportion of permutations of X satisfying $x < z$ and P which also satisfy $x < y$.

It should be noted that the preceding equation is equivalent to the following:

$$p(x < y, P)\, p(x < z, P) \leq p(P)\, p(x < y, x < z, P).$$

The proof of this theorem invokes the FKG Inequality of Fortuin, Kastelyn and Ginibre [FKG], which first requires a definition. Given a poset P on a set X, we say a real-valued function f on X is *increasing* if whenever $x < y$ then $f(x) \leq f(y)$. The FKG Inequality is as follows.

Theorem 3.4.2. Let $L = (X,<)$ be a distributive lattice, let f and g be increasing real-valued functions on X and let μ be a real-valued function of X such that for all x and y in X, $\mu(x) \geq 0$ and $\mu(x)\mu(y) \leq \mu(x \wedge y)\mu(x \vee y)$. Then

$$\sum_{x \in X} f(x)g(x)\mu(x) \sum_{y \in X} \mu(y) \geq \sum_{x \in X} f(x)\mu(x) \sum_{y \in X} g(y)\mu(y).$$

We do not prove the FKG inequality, but apply it to derive the XYZ inequality. Before proving Theorem 3.4.1, some preliminary definitions are needed as well as several lemmas. Let S be the set of n-tuples $a = (a_1, ..., a_n)$ where each $a_i \in \{1, 2, ..., N\}$ and N is some integer which will later be allowed to tend to infinity. Define the relation $\leq$ on S by the following:

$x \leq y$ for $x, y \in S$ if and only if $x_1 \geq y_1$, $x_i - x_1 \leq y_i - y_1$ for $i \in \{2, ..., n\}$.

It is easy to verify that $(S, \leq)$ forms a partially ordered set. For instance, if $N = 2$ and $n = 3$, $(S, \leq)$ will be the poset shown in Figure 3.4.1.

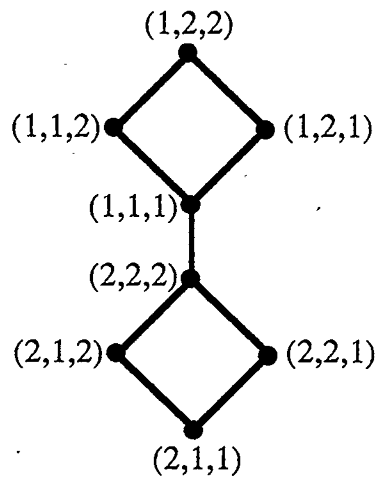


Figure 3.4.1

We refer to the i'th component of $a. \in S$ by a_i

Lemma 3.4.3. For every $a, b \in S$, $a \wedge b$ and $a \vee b$ exist in S, and in fact

a) $(a \wedge b)_i = \min(a_i - a_1, b_i - b_1) + \max(a_1, b_1)$ and

b) $(a \vee b)_i = \max(a_i - a_1, b_i - b_1) + \min(a_1, b_1)$.

Proof. Consider $x_i = \min(a_i - a_1, b_i - b_1) + \max(a_1, b_1)$ and

$y_i = \max(a_i - a_1, b_i - b_1) + \min(a_1, b_1)$.

Let $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$. To verify Lemma 3.4.3, we need to demonstrate the following: i) $x \in S$;

ii) $x \leq a, x \leq b$;

iii) if $z \in S$ and $z \leq a$, $z \leq b$ then $z \leq x$;

and also the following: i) $y \in S$;

ii) $y \geq a, y \geq b$;

iii) if $z \in S$ and $z \geq a$, $z \geq b$ then $z \geq y$.

Only the proof of i) - iii) for x will be shown as the proof for y is similar.

i) To verify $x \in S$, it must be checked that $x_i \in \{1, ..., N\}$ for all $i \in \{2, ..., n\}$.

There are four cases which are as follows:

a) $\min(a_i - a_1, b_i - b_1) = a_i - a_1, \ \max(a_1, b_1) = a_1;$

b) $\min(a_i - a_1, b_i - b_1) = a_i - a_1, \ \max(a_1, b_1) = b_1;$

c) $\min(a_i - a_1, b_i - b_1) = b_i - b_1, \ \max(a_1, b_1) = b_1;$

d) $\min(a_i - a_1, b_i - b_1) = b_i - b_1, \ \max(a_1, b_1) = a_1.$

Now it can be shown that $x_i \in \{1, ..., N\}$ for all $i \in \{2, ..., n\}$.

a) $x_i = a_i - a_1 + a_1$. Then $x_i = a_i \in \{1, ..., N\}$.

b) $x_i = a_i - a_1 + b_1$. But $a_i = a_i - a_1 + a_1 \leq a_i - a_1 + b_1 \leq b_i - b_1 + b_1 = b_i$. Since $a_i, b_i \in \{1, ..., N\}$ then $x_i \in \{1, ..., N\}$.

c) $x_i = b_i - b_1 + b_1$. Then $x_i = b_i \in \{1, ..., N\}$.

d) $x_i = b_i - b_1 + a_1$. But $b_i = b_i - b_1 + b_1 \leq b_i - b_1 + a_1 \leq a_i - a_1 + a_1 = a_i$. Since $a_i, b_i \in \{1, ..., N\}$ then $x_i \in \{1, ..., N\}$.

Thus $x \in S$.

ii) By definition of $\leq$ the following must be shown:

1) $x_1 \geq a_1, x_1 \geq b_1;$

2) $x_i - x_1 \leq a_i - a_1$ for $i \in \{2, ..., n\}$, $x_i - x_1 \leq b_i - b_1$ for $i \in \{2, ..., n\}$.

1) $x_1 = \min(a_1 - a_1, b_1 - b_1) + \max(a_1, b_1) = \max(a_1, b_1)$. Thus $x_1 \geq a_1$ and $x_1 \geq b_1$.

2) Each of the four cases a) – d) from the proof of i) must be considered for each $i \in \{2, ..., n\}$.

a) $x_i - x_1 = a_i - x_1 \leq a_i - a_1 \leq b_i - b_1$.

b) $x_i - x_1 = a_i - a_1 + b_1 - x_1 \leq a_i - a_1 + b_1 - b_1 = a_i - a_1 \leq b_i - b_1$.

c) $x_i - x_1 = b_i - x_1 \leq b_i - b_1 \leq a_i - a_1$.

d) $x_i - x_1 = b_i - b_1 + a_1 - x_1 \leq b_i - b_1 + a_1 - a_1 = b_i - b_1 \leq a_i - a_1$.

Thus 2) is satisfied for all i.

iii) Let $z \in S$ be such that $z \leq a$ and $z \leq b$. We show that $z \leq x$. Since $z \leq a$ and $z \leq b$ then $z_1 \geq a_1$ and $z_1 \geq b_1$ by the definition of $\leq$ on S. Thus $z_1 \geq \max(a_1, b_1) = x_1$. Let $i \in \{2, ..., n\}$. $z \leq a$ and $z \leq b$ also imply that $z_i - z_1 \leq a_i - a_1$ and $z_i - z_1 \leq b_i - b_1$. Thus

$z_i - z_1 \leq \min(a_i - a_1, b_i - b_1) = x_i - \max(a_1, b_1) = x_i - x_1$.

Since $z_1 \geq x_1$ and $z_i - z_1 \leq x_i - x_1$ for all $i \in \{2, ..., n\}$, then $z \leq x$. □

Lemma 3.4.4. $(S, \leq)$ is a distributive lattice.

Proof. $(S, \leq)$ is a lattice by Lemma 3.4.3. To show $(S, \leq)$ is distributive, the distributive law $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

must be verified. To demonstrate this the following preliminary results are required:

$$\min(j, \max(k, l)) = \max(\min(j, k), \min(j, l)) \quad (3.4.1)$$

$$\max(j, \min(k, l)) = \min(\max(j, k), \max(j, l)) \quad (3.4.2)$$

for any numbers j, k, and l. These equations simply say that every chain forms a distributive lattice. Now the distributive law will be verified for $(S, \leq)$.

$$\begin{aligned}(a \wedge (b \vee c))_i &= \min(a_i - a_1, (b \vee c)_i - (b \vee c)_1) + \max(a_1, (b \vee c)_1) \\ &= \min[a_i - a_1, \max(b_i - b_1, c_i - c_1) + \min(b_1, c_1) - \min(b_1, c_1)] + \\ &\quad \max[a_1, \min(b_1, c_1)] \\ &= \min[a_i - a_1, \max(b_i - b_1, c_i - c_1)] + \max[a_1, \min(b_1, c_1)]\end{aligned}$$

Applying (3.4.1) and (3.4.2) to this we obtain the following:

$$\begin{aligned}(a \wedge (b \vee c))_i = &\max[\min(a_i - a_1, b_i - b_1), \min(a_i - a_1, c_i - c_1)] + \\ &\min[\max(a_1, b_1), \max(a_1, c_1)]\end{aligned}$$

$$= \max[\min(a_i - a_1, b_i - b_1) + \max(a_1, b_1) - \max(a_1, b_1),$$
$$\min(a_i - a_1, c_i - c_1) + \max(a_1, c_1) - \max(a_1, c_1)]$$
$$+ \min[\max(a_1, b_1), \max(a_1, c_1)]$$
$$= \max[(a \wedge b)_i - (a \wedge b)_1, (a \wedge c)_i - (a \wedge c)_1,] + \min[(a \wedge b)_1,$$
$$(a \wedge c)_1]$$
$$= ((a \wedge b) \vee (a \wedge c))_i. \qquad \square$$

Now fix poset $P = (X, <)$, and fix the elements of X as $x_4, ..., x_n$ so that x_1 and x_3 are not related. Note that $x_1, x_2, ..., x_n$ need not be a linear extension of P. Next define functions f and g on S as follows:

$$f(a) = \begin{cases} 1 & \text{if } a_1 \leq a_2 \\ 0 & \text{otherwise} \end{cases}$$

$$g(a) = \begin{cases} 1 & \text{if } a_1 \leq a_3 \\ 0 & \text{otherwise} \end{cases}$$

Lemma 3.4.5. f and g are increasing functions.

Proof. Proving f is an increasing function is equivalent to proving that for any $a, b \in S$, if $f(a) = 1$ and $a \leq b$ then $f(b) = 1$. Since $a \leq b$, by definition $a_1 \geq b_1$ and $a_2 - a_1 \leq b_2 - b_1$. Since $f(a) = 1$, by definition of f, $a_1 \leq a_2$. Thus $a_2 - a_1 \geq 0$ so we will have $b_2 - b_1 \geq 0$ which in turn implies that $b_2 \geq b_1$ and thus $f(b) = 1$. A similar proof shows that g is increasing. $\square$

We say that $a \in S$ satisfies the inequalities of poset P if for every $i, j \in \{1, ..., n\}$ such that $x_i < x_j$ in P then $a_i \leq a_j$. Now define function $\mu : S \rightarrow \{0, 1\}$ as follows:

$$\mu(a) = \begin{cases} 1 & \text{if } a \text{ satisfies the inequalities of } P; \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.4.6. $\mu(a)\mu(b) \leq \mu(a \vee b)\mu(a \wedge b)$ for all $a, b \in S$.

Proof. Since $\mu(a)$ can only equal 0 or 1, it is enough to show that if $\mu(a) = \mu(b) = 1$ then $\mu(a \vee b) = \mu(a \wedge b) = 1$. If $\mu(a) = \mu(b) = 1$ then a and b satisfy the inequalities of P. Thus for every i, j such that $x_i < x_j$ is an inequality of P we must have $a_i \le a_j$ and $b_i \le b_j$. Thus

$$\begin{aligned}(a \wedge b)_i &= \min(a_i - a_1, b_i - b_1) + \max(a_1, b_1)\\ &\le \min(a_j - a_1, b_j - b_1) + \max(a_1, b_1) = (a \wedge b)_j.\end{aligned}$$

Since $(a \wedge b)_i \le (a \wedge b)_j$ it is clear that $(a \wedge b)$ satisfies the inequalities of P. Thus $\mu(a \wedge b) = 1$. Similarly it can be shown that $\mu(a \vee b) = 1$. □

Proof of Theorem 3.4.1. By Lemmas 3.4.4, 3.4.5 and 3.4.6, f, g and μ satisfy the initial conditions of the FKG inequality (Theorem 3.4.2). Thus applying this inequality we get

$$\sum_{a \in S} f(a)g(a)\mu(a) \sum_{b \in S} \mu(b) \ge \sum_{a \in S} f(a)\mu(a) \sum_{b \in S} g(b)\mu(b).$$

Dividing both sides by $|S|\cdot|S|$ gives

$$\left(\frac{\sum_{a \in S} f(a)g(a)\mu(a)}{|S|}\right)\left(\frac{\sum_{b \in S} \mu(b)}{|S|}\right) \ge \left(\frac{\sum_{a \in S} f(a)\mu(a)}{|S|}\right)\left(\frac{\sum_{b \in S} g(b)\mu(b)}{|S|}\right). \tag{3.4.3}$$

Now consider the first bracketed quantity in (3.4.3). We show that as $N \to \infty$,

$$\left(\frac{\sum_{a \in S} f(a)g(a)\mu(a)}{|S|}\right) \to p(x_1 < x_2, x_1 < x_3, P).$$

Let $S' = \{a \in S \mid a_i \ne a_j \ \forall\, i, j \in \{1, ..., n\} \text{ with } i \ne j\}$. Thus S' is the set of all elements in S containing no repetitions of its coordinates. First note that $|S| = N^n$ and $|S'| = N! \,/\, (N-n)!$. Thus

$$\frac{|S'|}{|S|} = \frac{N!}{(N-n)!\, N^n} \to 1 \quad \text{as } N \to \infty.$$

Intuitively, as N becomes infinitely large, the proportion of elements of S containing repetitions amongst their coordinates becomes insignificant. The same thing happens when we consider the elements of S for which $x_1 < x_2$, $x_1 < x_3$ and the inequalities of P are satisfied. If we further restrict ourselves to S', the elements without repetitions, and allow N to go to infinity we have

$$\left(\frac{\sum_{a \in S'} f(a)g(a)\mu(a)}{\sum_{a \in S} f(a)g(a)\mu(a)} \right) \to 1.$$

Then as $N \to \infty$, $$\frac{\sum_{a \in S} f(a)g(a)\mu(a)}{|S|} \to \frac{\sum_{a \in S'} f(a)g(a)\mu(a)}{|S'|}.$$

Every $b = \{b_1, b_2, ..., b_n\} \in S'$ corresponds to a permutation of $\{1, ..., n\}$, $a = \{a_1, a_2, ..., a_n\}$ such that $b_i < b_j \Leftrightarrow a_i < a_j$. It can be shown that a fixed number m of elements of S' will correspond to a given permutation of $\{1, ..., n\}$. Also note that $\sum_{a \in S'} f(a)g(a)\mu(a)$ counts the elements of S' for which $x_1 < x_2$, $x_1 < x_3$ and the inequalities of P are satisfied. Hence if we further restrict a to P' where P' is the set of elements of S which are permutations of $\{1, ..., n\}$, we get

$$\sum_{a \in P'} f(a)g(a)\mu(a) = e(P \mid x_1 < x_2, x_1 < x_3),$$

where $e(P \mid x_1 < x_2, x_1 < x_3)$ counts the number of linear extensions of P such that $x_1 < x_2$ and $x_1 < x_3$. Thus

$$\frac{\sum_{a \in S'} f(a)g(a)\mu(a)}{|S'|} = \frac{m \cdot e(P \mid x_1 < x_2, x_1 < x_3)}{m \cdot |P'|} = p(x_1 < x_2, x_1 < x_3, P).$$

Similarly the other three quantities in equation 3.4.3 approach the probabilities in the following equation as N approaches infinity. We get

$$p(x_1 < x_2, x_1 < x_3, P)\, p(P) \geq p(x_1 < x_2, P)\, p(x_1 < x_3, P), \tag{3.4.4}$$

which is the desired result. □

This completes Shepp's proof of the XYZ Inequality. Fishburn [Fi] has done further work on this problem and was able to demonstrate that whenever x_1, x_2 and x_3 are pairwise unrelated, then ≥ can be replaced with > in equation 3.4.4.

3.5 CORRELATION INEQUALITIES

The XYZ Inequality is a relationship which is part of a broader class of theorems, known as correlation inequalities. In order to define correlation in terms of linear extensions of partially ordered sets, a few ideas from probability theory will be given.

Let Q and R be two events. Then we say that Q and R are positively correlated if $p(Q) < p(Q \mid R)$, Q and R are negatively correlated if $p(Q) > p(Q \mid R)$ and Q and R are independent or uncorrelated if $p(Q) = p(Q \mid R)$.

Before proceeding, some definitions of the meaning of probability in terms of linear extensions of partially ordered sets must be given. These definitions can be found in a paper by Brightwell [Br2]. Let Q and R be asymmetric subsets of $X \times X$ where X is some set with $|X| = n$. As in the previous section, the probability $p(Q)$ of subset Q is the proportion of permutations of the n elements of X which satisfy all relations of Q. Now define the conditional probability $p(Q \mid R) = p(Q \cup R) / p(R)$ as the probability that a permutation satisfying all relations in R also satisfies all relations in Q. The above definitions can be applied to posets, since a poset P is an asymmetric set of pairs (x, y) with $(x, y) \in P$ if and only if $x < y$ in P. Thus if Q and R are posets, $p(Q \mid R)$ is the probability that an arbitrary linear extension of R is also a linear extension of Q.

Now the idea of correlation in the context of partially ordered sets will be considered. Let Q and R be posets on a set X and let P be a poset on a set containing X such that $P \cup R$ is asymmetric. Q and R are said to be positively correlated with respect to P (in symbols $Q \uparrow_P R$ using a notation of Brightwell [Br2]) if $p(Q \mid P) \leq p(Q \mid P \cup R)$. Similarly, Q and R are said to be negatively correlated with respect to P ($Q \downarrow_P R$) if $p(Q \mid P) \geq p(Q \mid P \cup R)$. Note that assuming $P \cup Q$ is also asymmetric, we get that $Q \uparrow_P R$ if and only if $R \uparrow_P Q$ and $Q \downarrow_P R$ if and only if $R \downarrow_P Q$. To extend these definitions, say that there is a pair of posets Q, R on set X such that for every poset P on a set containing X with $P \cup R$ asymmetric, then $Q \uparrow_P R$. We say that such a pair is universally positively correlated and write $Q \uparrow R$. Similarly, if there is a pair of posets Q, R on a set X such that for every poset P on a set containing X with $P \cup R$ asymmetric, then $Q \downarrow_P R$, we say that Q and R are universally negatively correlated and write $Q \downarrow R$.

The XYZ Inequality treated in the previous section is an example of a correlation inequality. In this case let Q and R be the posets on $X = \{x_1, x_2, x_3\}$ with strict order relations such that $Q = \{(x_1, x_2)\}$ and $R = \{(x_1, x_3)\}$, and let P be a poset on a set containing X with $x_3 \nless x_1$. Then we can say not only that Q and R are positively correlated with respect to P, but also that Q and R are universally positively correlated ($Q \uparrow R$).

The following well known correlation inequality was proved by Graham, Yao and Yao [GYY].

Theorem 3.5.1. Let A and B be disjoint totally ordered sets. Let $P = A \cup B$ and let Q and R be sets of relations of the form $a_m < b_n$ where $a_m \in A$ and $b_n \in B$. Then

$$p(Q \mid P) \leq p(Q \mid P \cup R).$$

Thus this theorem states that $Q \uparrow_P R$. An example often used to illustrate this theorem is as follows. Consider a tennis tournament between two teams A and B for which each team has a complete ranking of its players. It must be assumed that the tennis players play consistently so that if player x is ranked above player y, then player x will always beat player y. Theorem 3.5.1 says that if some players on team A have already lost to some players on team B, then there is an increased likelihood that a given player on team A will lose to a given player on team B.

Shepp [Sh1] was able to expand upon this result and show that Theorem 3.5.1 also holds when A and B are just disjoint partially ordered sets. Brightwell [Br5] added that if there are $x, y, z, w \in X$ such that $(x, y) \in Q$ and $(z, w) \in R$ and either x and z or y and w are in the same connected component of $G(P)$, the comparability graph of P, then Shepp's result holds strictly. Another extension of Theorem 3.5.1 was found by Graham, Yao and Yao [GYY] and is the following:

Theorem 3.5.2. Let A and B be disjoint totally ordered sets. Let $P = A \cup B$ and let C, Q and R be sets of relations of the form $a_m < b_n$ where $a_m \in A$ and $b_n \in B$. Then

$$p(Q \mid P \cup C) \leq p(Q \mid P \cup C \cup R).$$

Alternate proofs of this were found by Kleitman and Shearer [KS] and also by Shepp [Sh1]. Once again Shepp considered the corresponding problem with A and B disjoint partially ordered sets and found that the theorem does not hold in this case. The simplest counter-example to this is due to the referee of [Sh1]. Let A be the two-element antichain consisting of elements a_1 and a_2 and let B be the two-element antichain consisting of elements b_1 and b_2. Let $C = \{a_2 < b_1\}$, $R = \{a_2 < b_2\}$ and $Q = \{a_1 < b_1\}$. Then $p(Q \mid P \cup C) = 2/3 > 5/8 = p(Q \mid P \cup C \cup R)$.

Brightwell wrote two papers [Br2] and [Br5] dealing with the idea of correlation with respect to another poset. In the first, he was able to classify all posets $P = (X, \leq)$ such that $\{(x, y)\}$ and $\{(z, w)\}$ are correlated with respect to S for every poset S which is an extension of P and has $w \nless z$ in S, where $x, y, z, w \in X$. In the second, he classified all posets $P = (X, <)$ such that $\{(x, y)\}$ and $\{(z, w)\}$ are correlated with respect to S for every poset S on X which is a subposet of P and has $w \nless z$ in S, where $x, y, z, w \in X$.

Winkler [Wn] and Brightwell [Br1] considered the problem of finding conditions for universal correlations amongst posets. Winkler produced a necessary and sufficient condition for posets Q and R on a set X to be universally positively correlated. Brightwell produced another such condition equivalent to $Q \uparrow R$ and was also able to give a condition when Q and R are universally negatively correlated.

It can be seen that much work has been done in the area of correlation. The above is only a summary of results that have been found thus far. To complete this survey of results dealing with the problem of counting linear extensions of posets, the following section will deal in detail with a specific linear extension counting problem, the 1/3 – 2/3 conjecture. Like correlation inequalities, the 1/3 – 2/3 conjecture involves the probability that a given relation will occur in a linear extension of a given poset.

CHAPTER FOUR

THE 1/3 – 2/3 CONJECTURE

4.1 INTRODUCTION

The idea of counting linear extensions gives rise to a well-known unsolved problem, the 1/3 - 2/3 conjecture. Consider a partially ordered set $(X,<)$. The 1/3 – 2/3 conjecture claims that in every poset $(X,<)$ that is not a chain there will be a pair x, $y \in X$ such that x is below y in somewhere between 1/3 and 2/3 of the linear extensions of $(X,<)$. This conjecture is attributed to Fredman in 1976 [Fr].

The motivation behind the 1/3 – 2/3 conjecture is the following question. For every poset, is there a pair (x,y) such that x lies below y in approximately half the linear extensions? The 1/3 – 2/3 conjecture hypothesizes that the answer to this question is yes, when "approximately half" is interpreted as "between 1/3 and 2/3". Formally the conjecture is stated as follows.

Conjecture 4.1.1 Let $P = (X,<)$ be a finite partially ordered set that is not a chain. Then there exist distinct elements x and y in X such that $1/3 \leq p(x<y|P) \leq 2/3$.

It is known that when all posets are considered, there is no λ with $1/3 < \lambda \leq 1/2$ such that Conjecture 4.4.1 with λ and $1-\lambda$ replacing 1/3 and 2/3 can be proved. To see this, consider the poset consisting of a 2-element chain plus an isolated point. When considering any pair of unrelated points x and y in this poset, either $p(x<y|P) = 1/3$ or $p(x<y|P) = 2/3$. Thus the values of 1/3 and 2/3 are the "best" possible.

At this point, one comment on notation will be made. When the poset, P, is fixed, then we will write $p(x<y)$ instead of $p(x<y|P)$.

The reason that there has been so much interest in the 1/3 – 2/3 conjecture is that it has direct applications to the problem of finding time efficient sorting algorithms for computer programs. Consider a set of partially sorted data which is to be completely sorted. This forms a poset P of data items. The sorting is done by making comparisons between unrelated items in the poset to determine their ordering. Let n be the number of comparisons that must be made to sort such a poset, and find the relationship between the number of comparisons needed and the number of linear extensions of P. Consider the worst case situation. This occurs if whenever a comparison is made and it is found that say, $x<y$, then x is below y in more than half of the linear extensions. In this situation, the best that can happen is that for every comparison between 2 elements x and y, $p(x<y) = p(y<x) = 1/2$. Thus we will have

$$2^n = e(P).$$

In general, the worst case says

$$2^n \geq e(P)$$
$$\Rightarrow \; n \geq \log_2 e(P).$$

The lower bound on n is known as the information theoretic bound. It should be noted that if the 1/3 – 2/3 conjecture can be proved, then we will have

$$\left(\frac{3}{2}\right)^n \leq e(P)$$
$$\Rightarrow \; n \leq \log_{3/2} e(P) \cong 1.7 \log_2 e(P).$$

To date, though much progress has been made on the 1/3 – 2/3 conjecture, it still remains an open problem. Kahn and Saks [KS] succeeded in showing that if the values 1/3 and 2/3 in Conjecture 4.1.1 are replaced with 3/11 and 8/11, the conjecture can be proved for all finite posets. Using this 3/11 – 8/11 result, the best upper limit that can be placed on the number of comparisons of elements required to sort P is as follows.

$$\left(\frac{11}{8}\right)^n \le e(P)$$

$$\Rightarrow \; n \le \log_{11/8} e(P) \cong 2.2 \log_2 e(P).$$

Other researchers have had success at considering specific classes of finite posets and proving the 1/3 – 2/3 conjecture for those classes (see [Ll], [BrW], [Sr], [GHP], [Br4] and [TGF]). With some classes of posets, a result much closer to 1/2 can be achieved. As well, Brightwell [Br4] has extended the definition of $p(x<y)$ to include a certain class of infinite posets and has been able to show that in the infinite case, there are counter-examples to the 1/3 – 2/3 conjecture. These ideas will be examined in more detail in the following sections.

4.2 THE 3/11 – 8/11 THEOREM

One of the first major breakthroughs in the efforts to prove the 1/3 – 2/3 conjecture is the following result shown by Kahn and Saks in 1984 [KS].

Theorem 4.2.1. Every finite partially ordered set $(X,<)$ which is not totally ordered contains a pair $x, y \in X$ such that $3/11 < p(x<y) < 8/11$.

The key to Kahn's and Saks' proof of this theorem is in considering the "average height" of an element over all linear extensions of the poset. In order to define the average height of an element x in X, two other definitions must first be made. Let $P = (X, <)$ be a partially ordered set, and as before let $E(P)$ be the set of linear extensions of P and let $e(P) = |E(P)|$. For a given $L \in E(P)$ define a function f from X to the natural numbers by $f(x)$

$= k$ if x is at height k in L, that is, if there are exactly $k - 1$ elements of X below x in L. Now let $p(f(x) = k)$ represent the proportion of linear extensions of P in which x is at height k. Mathematically, this can be stated as

$$p(f(x)=k) = \frac{e(P \mid f(x)=k)}{e(P)}.$$

Finally, the average height $h(x)$ of an element x in a poset P of size n is defined as follows:

$$h(x) = \sum_{k=1}^{n} kp(f(x)=k).$$

This definition allows the critical theorem in Kahn's and Saks' proof to be stated.

Theorem 4.2.2. Any pair of elements x and y in X satisfying $|h(y)-h(x)| < 1$ also satisfies $3/11 < p(x<y) < 8/11$.

A complete proof of Theorem 4.2.2 will not be given here, though the main ideas Kahn and Saks used to prove this theorem will be briefly outlined. First they let $e_k(x<y)$ be the number of linear extensions in which $f(x)-f(y)=k$. Then they proved a series of lemmas which describes various relationships amongst the $e_k(x<y)$'s and leads to the following lemma.

Lemma 4.2.3. Let $\{a_i\}$ and $\{b_i\}$ where $i \geq 1$ be sequences of non-negative real numbers which satisfy the following:

(1) $a_1 = b_1$;

(2) $a_i = 0 \Rightarrow a_{i+1} = 0,\ b_i = 0 \Rightarrow b_{i+1} = 0$, if $i > 1$;

(3) $\sum_{i \geq 1} a_i - \sum_{i \geq 1} b_i = 1$;

(4) $a_2 + b_2 \leq a_1 + b_1$;

(5) $a_i \leq a_{i+1} + a_{i-1}$;

(6) $a_i \geq a_{i+1}\, a_{i-1}$;

(7) $\quad \sum_{i\geq 1} ia_i + \sum_{i\geq 1} ib_i < 1.$

Then $\sum_{i\geq 1} b_i > 3/11$.

Kahn and Saks then let x, y be a pair of incomparable elements satisfying $|h(y) - h(x)| < 1$ and let a_k and b_k be defined by the following equations:

$$a_k = \frac{e_k(x<y)}{e(P)} \quad \text{for } k\geq 1,$$

$$b_k = \frac{e_{-k}(x<y)}{e(P)} \quad \text{for } k\geq 1.$$

It can be shown that $\{a_k\}$ and $\{b_k\}$ satisfy (1) – (7). Now $p(x<y) = \Sigma_{k\geq 1}\, b_i > 3/11$ by Lemma 4.2.2. Similarly it can be shown that $p(y < x) > 3/11$ so we get $3/11 < p(x<y) < 8/11$.

Then it remains to show that every partially ordered set that is not a chain contains a pair (x, y) such that $|h(x) - h(y)| < 1$, since this in turn will imply $3/11 < p(x<y) < 8/11$. Let $\{1, 2, ..., n\}$ represent the elements of X where $|X| = n$. Then $1 \leq h(j) \leq n$ where $j \in \{1, ..., n\}$. In the worst case these elements will be evenly spaced one unit apart across the interval $[1, n]$, in which case $(X,<)$ must be a chain. Otherwise there must be two elements i and j that are closer together than one unit. Then $|h(i) - h(j)| < 1$ as required.

Thus it has been shown that for every finite poset $(X,<)$ that is not a chain, there are distinct elements x and y in X such that $3/11 < p(x<y) < 8/11$.

4.3 CASES WHERE THE 1/3 – 2/3 CONJECTURE HAS BEEN PROVED

Since thus far no proof has been found to improve upon the 3/11 – 8/11 bound for all finite posets, efforts have been concentrated on trying to prove the 1/3 – 2/3 conjecture for special classes of partially ordered sets. A number of such results have emerged, thus strengthening the belief that the 1/3 – 2/3 conjecture holds for all finite posets. These results will be examined in this section.

The earliest special case for which the 1/3 – 2/3 conjecture was shown to be valid was for the case of a partial order which can be covered by two chains (or equivalently, a partial order of width two) [Ll]. Let $P=(X,<)$ be a poset consisting of two chains $A = (a_1 > a_2 > \ldots > a_m)$ and $B = (b_1 > b_2 > \ldots > b_n)$ along with some relations between the elements of A and B. We sketch the proof of Linial to show that there exist distinct elements x and y in X such that $1/3 \leq p(x<y) \leq 2/3$.

Before the main theorem can be proved, a few definitions, assumptions and a lemma must be introduced. Assume that the 1/3 – 2/3 conjecture fails for P. Without loss of generality it can be assumed that a_1 and b_1 are unrelated since otherwise one of these would be a maximal element and would occur in the top position in each linear extension and thus could be disregarded. Also without loss of generality it can be assumed that $p(a_1>b_1) \leq 1/3$. Now define a sequence $\{q_1, \ldots, q_n\}$ as follows.

$$q_1 = p(a_1 > b_1),$$
$$q_i = p(b_{i-1} > a_1 > b_i) \quad \forall i \text{ such that } 2 \leq i \leq n,$$
$$q_{n+1} = p(b_n > a_1).$$

Lemma 4.3.1. The set $\{q_1, \ldots, q_{n+1}\}$ satisfies the following:

a) $\sum_{i=1}^{n+1} q_i = 1,$

b) $1/3 \geq q_1 \geq ... \geq q_{n+1} \geq 0.$

Proof. a) Clearly every linear extension of P is accounted for in exactly one of the q_i's, so the sum of all these probabilities must be 1.

b) It needs to be shown that $q_i \geq q_{i+1}$ for every $i \in \{1, ..., n\}$. Consider a linear extension satisfying $b_i > a_1 > b_{i+1}$, an event which occurs with probability q_{i+1}. Since a_1 is the maximal element in A, there can be no elements between b_i and a_1 in this linear extension, and a_1 and b_i must be unrelated. Thus a_1 and b_i can be interchanged to produce another linear extension of P. This linear extension will satisfy $b_{i-1} > a_1 > b_i$, an event occurring with probability q_i. The interchange of a_1 and b_i forms an injection from the linear extensions satisfying the event having probability q_{i+1} to those satisfying the event having probability q_i. Therefore $q_i \geq q_{i+1}$. □

Theorem 4.3.2. If P is a partially ordered set which can be covered by two chains, then there exist elements x and y in X such that $1/3 \leq p(x<y) \leq 2/3$.

Proof. Let r be an integer such that:

$$\sum_{i=1}^{r-1} q_i \leq \frac{1}{2} \leq \sum_{i=1}^{r} q_i.$$

Then

$$p(a_1 > b_{r-1}) = \sum_{i=1}^{r-1} q_i \leq \frac{1}{2}.$$

This implies that $p(a_1 < b_{r-1}) < 1/3$ by the initial assumption, which in turn implies that $p(a_1 > b_{r-1}) > 2/3$. Similarly,

$$p(a_1 > b_r) = \sum_{i=1}^{r} q_i \geq \frac{1}{2}.$$

Again by the initial assumption, $p(a_1<b_r) > 2/3$. Thus we get that $p(b_r<a_1<b_{r-1}) > 1/3$. Thus $q_r > 1/3$ which contradicts the assumption that $1/3 \geq q_r$, so the 1/3 – 2/3 conjecture must hold for P. □

This simple proof for posets of width two was followed up quite recently with a much more complex proof by Brightwell and Wright that the 1/3 – 2/3 conjecture holds for all k–thin posets with $k \leq 5$. [BrW]. By assuming that $(X,<)$ is a 5-thin poset for which the 1/3 – 2/3 conjecture fails, they were able to limit themselves without loss of generality to 5-thin posets possessing some additional specific characteristics. They then showed that if a finite list of posets can be found such that every 5-thin poset that need be considered bears a certain relation to one of the posets on the finite list, then every 5-thin poset must satisfy the 1/3 – 2/3 conjecture. By using a computer, Brightwell and Wright were able to produce a finite list of posets possessing the desired characteristics, and thus prove the 1/3 – 2/3 conjecture for 5-thin posets.

Series parallel posets are another group of partial orders for which the 1/3 – 2/3 conjecture has been verified. Recall that series parallel posets were defined in Section 2.5. Steiner [Sr] is responsible for the proof of the 1/3 – 2/3 conjecture for these posets, the idea of which is as follows. If one goes back early enough in the building process of a series parallel poset P, a subposet P_1 consisting only of two chains can be found. This will occur just after the first parallel composition is performed. According to the previous proof for posets that can be covered by two chains, P_1 has a pair x,y such that $1/3 \leq p(x<y) \leq 2/3$. After performing a series composition with some P_2 to form a poset P', it is found that $p(x<y)$ in P_1 is equal to $p(x<y)$ in P'. Similarly, a parallel composition of P_1 with some P_2 will preserve $p(x<y)$. Thus every such poset P will have a pair (x, y) satisfying the 1/3 – 2/3 conjecture, unless there are no parallel compositions performed in the formation of P, in which case P must be a chain.

One further case for which the 1/3 – 2/3 conjecture has been proved is the class of posets having a non-trivial automorphism [GHP]. Let $P=(X,<)$ be a poset having a non-trivial automorphism α and assume that the 1/3 – 2/3 conjecture fails for P. Let P' be the poset produced when α is applied to P. Thus P and P' differ only in their labellings. Since α is an automorphism, $p(x<y) = p(\alpha(x)<\alpha(y))$ for every $x, y \in X$. Let $x<_0y$ if and only if $p(x<y)>2/3$. It can be shown than $<_0$ is a total order on X, and thus is a linear extension of P. Then $<_0$ must also be a linear extension of P', and P' and P must be identical since otherwise there will exist $x, y \in X$ such that $x<_0y$ but $\alpha(y) <_0 \alpha(x)$. This contradicts the assumption that α is a non-trivial automorphism, so no poset P exists such that P contains a non-trivial automorphism and the 1/3 – 2/3 conjecture fails. This particular class of posets will be examined again in Section 4.6.

Recently it has also been shown that the 1/3 – 2/3 conjecture holds for semiorders [Br4], and for height-1 posets [TGF]. These two proofs will be given in detail in the next two sections. The existence of all these classes of posets for which the 1/3 – 2/3 conjecture is satisfied further strengthens the belief that the 1/3 – 2/3 conjecture must hold for all finite posets.

4.4 THE 1/3 – 2/3 CONJECTURE FOR SEMIORDERS

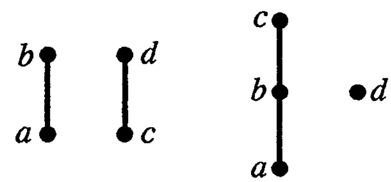
Brightwell addressed the problem of proving the 1/3 – 2/3 conjecture for semiorders in both the finite and infinite cases. He was able to show that the 1/3 – 2/3 conjecture holds for all finite semiorders and certain classes of infinite semiorders [Br4]. He also produced a list of infinite semiorders for which the 1/3 – 2/3 conjecture fails, and conjectured that all such semiorders are linear sums of those on his list. In this section, the proof for the finite case will be given, while the infinite case will be dealt with in section 4.7.

A semiorder is defined to be a poset $(X,<)$ such that for every $a, b, c, d \in X$,

a) if $a<b$ and $c<d$ then either $a<d$ or $c<b$ (or both);

b) if $a<b$ and $b<c$ then either $a<d$ or $d<c$ (or both).

Requirement a) implies that $(X,<)$ doesn't contain the poset in Figure 4.4.1a as a subposet, and b) implies that $(X,<)$ doesn't contain the poset in Figure 4.4.1b as a subposet.



a) b)

Figure 4.4.1

Let $(X,<)$ be a finite poset and let x, y be incomparable elements in X. It is said that $z \in X$ is *good for* $x<y$ if either: a) $x<z \mid y$ or b) $x \mid z<y$. In other words, an element z is good for a pair $x<y$ if it tends to encourage x to lie below y in a linear extension of the partially ordered set.

Lemma 4.4.1. A finite partial order is a semiorder if and only if for every $x, y \in X$ there are never $z, w \in X$ such that z is good for $x<y$ and w is good for $y<x$.

Proof. To prove the forward direction, assume that $(X,<)$ is a semiorder and that there is an element z good for $x<y$ and an element w good for $y<x$. Since there are two possible situations when an element is good for a pair, we end up with four possible cases. When $x<z \mid y$ and $y<w \mid x$ or when $x \mid z<y$ and $y \mid w<x$; x, y, z and w form a subposet consisting of two 2-element chains, as in Figure 4.4.1a. Similarly, when $x<z \mid y$ and $y \mid w<x$ or when

$x \mid z<y$ and $y<w \mid x$; x, y, z and w form a subposet consisting of a three element chain plus an isolated point as in Figure 4.4.1b. Thus each case produces a contradiction to the assumption that we are dealing with a semiorder. In the opposite direction, if $(X, <)$ is not a semiorder then it contains a subposet which consists of either two 2-element chains or a 3-element chain plus an isolated point. Each of these posets contains a set of elements x, y, z and w such that w is good for $y<x$ and z is good for $x<y$. □

From this point on, it will be assumed that $(X,<)$ is a finite poset and that $(X,<)$ does not satisfy the 1/3 – 2/3 conjecture. An order $<_0$ is defined on X by letting $x<_0y$ if $p(x<y)>1/2$. It is claimed that $<_0$ defines a total order on $(X,<)$. This order can be easily seen to be irreflexive and total. To see that $<_0$ is transitive, recall that the 1/3 – 2/3 conjecture is assumed to fail for $(X,<)$. If x, y, $z \in X$ such that $p(x<y)>1/2$ and $p(y<z)>1/2$ then it must be true that $p(x<y)>2/3$ and $p(y<z)>2/3$. This implies that $p(x<y<z)>1/3$ which in turn implies that $p(x<z)>1/3$ and thus $p(x<z)>1/2$, as required for transitivity.

Given this definition for $<_0$ the following theorem will be proved.

Theorem 4.4.2. Let $(X,<)$ be a finite poset which does not satisfy the 1/3 – 2/3 conjecture. Then for every $x, y \in X$ such that $x<_0y$ and $x \mid y$ in $(X,<)$, either:

a) There are at least two elements of X good for $x<y$, or

b) There is an element z of X good for $x<y$ such that $x<_0z<_0y$.

Proof. Fix $x, y \in X$ such that $x <_0 y$ and $x \mid y$. Let Λ be the set of all linear extensions of $(X,<)$. Given a linear extension $\lambda \in \Lambda$, let $<_\lambda$ represent the ordering of λ. Now partition the elements of Λ into the following classes:

$\Lambda_1 = \{\lambda \in \Lambda \mid y<_\lambda x\}$;

$\Lambda_2 = \{\lambda \in \Lambda \mid x<_\lambda y$ and λ with x and y interchanged is also in $\Lambda\}$;

$\Lambda_3 = \{\lambda \in \Lambda \mid x <_\lambda y$ and λ with x and y interchanged is not in $\Lambda\}$.

It can be easily seen that Λ_3 can be rewritten as follows:

$\Lambda_3 = \{\lambda \in \Lambda \mid x <_\lambda y$ and there exists $z \in X$ such that $x <_\lambda z <_\lambda y$ and either $x < z$ or $z < y\}$.

Note that any z which satisfies the requirement in Λ_3 will also be good for $x<y$. Now let $p(\Lambda_i)$ represent the probability that a given linear extension is an element of Λ_i. Then $p(\Lambda_2) \leq p(\Lambda_1)$ since every element in Λ_2 corresponds to a unique element in Λ_1. However, $p(\Lambda_1) = p(y<x) < 1/3$ so $p(\Lambda_2)$ must also be less than 1/3. Since the sum of $p(\Lambda_1)$, $p(\Lambda_2)$, and $p(\Lambda_3)$ is 1, then $p(\Lambda_3) > 1/3$ which in turn implies that Λ_3 is non-empty and thus that there is at least one $z \in X$ that is good for $x<y$. Now there are two possibilities: either there is exactly one element z of X good for $x<y$ or there are at least two elements of X good for $x<y$. The latter case gives situation a). The former case implies that for every $\lambda \in \Lambda_3$ then $x <_\lambda z <_\lambda y$. This implies that $p(x<z<y) > 1/3$, so $p(x<z)$ and $p(z<y)$ must be both greater than 1/3 which in turn implies they are both greater than 1/2 by the assumption that the 1/3 – 2/3 conjecture fails. Thus $x <_0 z <_0 y$ as in situation b). □

The following lemma is proved by showing that when the poset in Theorem 4.4.2 is restricted to a finite semiorder, only situation a) can hold.

Lemma 4.4.3. If $(X,<)$ is a finite semiorder that does not satisfy the 1/3 – 2/3 conjecture, then for every $x, y \in X$ such that $x <_0 y$ and $x \mid y$ in $(X,<)$, there are at least two elements of X good for $x<y$.

Proof. Assume that situation b) in Theorem 4.4.2 holds. In other words, assume that there are elements x, y and z of X such that $x \mid y$, z is good for $x<y$, and $x <_0 z <_0 y$. Then either $x < z \mid y$ or $x \mid z < y$. In the former case, x is good for $y<z$. However, Theorem 4.4.2 along with the assumption that $z <_0 y$ implies that there is a $w \in X$ good for $z<y$ which

contradicts Lemma 4.4.1. Similarly in the latter case, y is good for $z<x$ but $x<_0z$ implies that there exists a w good for $x<z$. Again there is a contradiction to Lemma 4.4.1. Thus situation b) cannot hold for a finite semiorder, and the lemma follows directly. □

Now a few generalizations need to be made and an additional two lemmas shown before proving that every finite non-chain semiorder $(X,<)$ satisfies the 1/3 – 2/3 conjecture.

Let $(X,<)$ be a finite semiorder that is not a chain. Without loss of generality several assumptions can be made about the structure of $(X,<)$. Firstly, it may be assumed that $(X,<)$ is irreducible, since if the 1/3 – 2/3 conjecture fails for a poset then it fails for each of that poset's irreducible parts. To see this, consider a poset P for which the 1/3 - 2/3 conjecture fails and which is the linear sum of two posets, P' and P''. Then every unrelated pair of elements x and y in P (and thus every pair in P') has either $p(x<y) < 1/3$ or $p(x<y) > 2/3$ in P. Since P is the linear sum of P' and P'', any linear extension of P is simply a linear extension of P' with a linear extension of P'' adjoined. Thus for every unrelated pair $x, y \in P'$, $p(x<y)$ in P is equal to $p(x<y)$ in P', so the 1/3 – 2/3 conjecture must also fail for P'. It can be concluded that only irreducible semiorders need be considered.

Secondly, it may be assumed that X has at least 2 elements since $(X,<)$ is not a chain.

Thirdly the assumption can be made that $(X,<)$ has at least two minimal and two maximal elements. If $(X,<)$ has only one maximal element z, then it is the linear sum of z and $(X, <) \setminus z$, and thus is not irreducible, contradicting our first generalization. Similarly, $(X, <)$ must have more than one minimal element.

Now the elements can be ordered as $\{x_1, ..., x_n\}$ where $x_i <_0 x_j$ whenever $i<j$. The following lemma is needed in order to prove the main theorem.

Lemma 4.4.4. An element of X can only be good for at most two pairs of the form $x_i < x_{i+1}$ where $1 \leq i \leq n-1$.

Proof. Let $x_k \in X$ where $k \in \{1, ..., n\}$. We first show by contradiction that x_k can only be good for at most one pair $x_i < x_{i+1}$ where $i<k$. Assume that x_k is good for $x_i < x_{i+1}$ and $x_j < x_{j+1}$ where $i<j<k$. Then $x_i < x_k | x_{i+1}$ and $x_j < x_k | x_{j+1}$ since $<_0$ is a linear extension of $(X,<)$. It must be true that $x_j | x_{i+1}$ because $x_j < x_{i+1}$ implies that $j<i+1$ which contradicts $i<j$, and $x_{i+1} < x_j$ implies that $x_{i+1} < x_k$ which contradicts $x_{i+1} | x_k$. Since $x_j < x_k | x_{i+1}$ and $x_j | x_{i+1}$, then x_k is good for $x_j < x_{i+1}$. However, $x_{i+1} <_0 x_j$ since $i<j$ which implies there is an element of X good for $x_{i+1} < x_j$ by Theorem 4.4.2, which contradicts Lemma 4.4.1. Therefore, x_k can be good for at most one pair $x_i < x_{i+1}$ where $i<k$. Similarly, it can be shown that x_k is only good for at most one pair $x_i < x_{i+1}$ with $i > k$. Thus x can only be good for at most two pairs of form $x_i < x_{i+1}$. □

Note also that if x_k in Lemma 4.4.4 is a maximal element then it can't be good for a pair $x_i < x_{i+1}$ with $i>k$. To see this, assume that x_k is maximal and good for some $x_i < x_{i+1}$ with $i>k$. Then either $x_i | x_k < x_{i+1}$ which implies x_k is not maximal or $x_i < x_k | x_{i+1}$ which implies $i<k$. Since both of these implications are contradictions, it must be true that x_k can't be good for a pair $x_i < x_{i+1}$ where $i>k$. Similarly it can be shown that if x_k is a minimal element, then it can't be good for a pair $x_i < x_{i+1}$ with $i<k$.

Lemma 4.4.5. If $i<j$ where $x_i | x_j$, then there is no element good for $x_j < x_i$.

Proof. Let $i<j$. Then $x_i <_0 x_j$. Since $x_i | x_j$, by Lemma 4.4.3 there is an element z of X good for $x_i < x_j$ so by Lemma 4.4.1 there is no element w of X good for $x_j < x_i$. □

Lemma 4.4.6. x_i and x_{i+1} are incomparable for every $i \in \{1, ..., n-1\}$.

Proof. It is not possible that $x_{i+1} < x_i$ because that would imply $x_{i+1} <_0 x_i$, contradicting the choice of the sequence $\{x_i\}$. Assume now that $x_i < x_{i+1}$. We show that this implies that $(X,<)$ is not irreducible. Let A be the set of maximal elements of $\{x_{i+1}, ..., x_n\}$ with respect to the order relation $<$, and let set B be the set of minimal elements of $\{x_{i+1}, ..., x_n\}$, also with respect to $<$. Let $x_a \in A$ and $x_b \in B$, and note that $a \leq i$ and $i+1 \leq b$. It will be shown that $x_a < x_b$ by considering four cases.

Case i) Assume that $a=i$ and $b \neq i+1$. Then $x_b \mid x_{i+1}$ since both are minimal elements of $\{x_{i+1}, ..., x_n\}$. Next we show that $x_i < x_b$. If $x_b < x_i$ then $x_b < x_i < x_{i+1}$ which contradicts $x_b \mid x_{i+1}$. If $x_b \mid x_i$, then x_i is good for $x_b < x_{i+1}$. However, $i+1 < b$ so Lemma 4.4.5 is contradicted. Thus it must be true that $x_i < x_b$. Since $i=a$, then $x_a < x_b$.

Case ii) Assume that $a \neq i$ and $b=i+1$. The argument is similar to that for case i) so will not be repeated here.

Case iii) Assume that $a=i$ and $b=i+1$. By the assumption that $x_i < x_{i+1}$, we have $x_a < x_b$.

Case iv) Assume that $a \neq i$ and $b \neq i+1$. We will show that $x_a < x_b$. Note that $x_a \mid x_i$ and $x_b \mid x_{i+1}$. $x_b < x_a$ cannot occur since $a < b$ implies $x_a <_0 x_b$ and $<_0$ is a linear extension of $<$. Now assume that $x_a \mid x_b$. By case ii) $x_a < x_{i+1}$, so x_a must be good for $x_b < x_{i+1}$. However $i+1 < b$, so Lemma 4.4.5 is again contradicted. Thus x_a and x_b must be related. Therefore $x_a < x_b$.

The above four cases show that $x_a < x_b$ is always true. Thus every maximal element in $\{x_1, ..., x_i\}$ is less than every minimal element in $\{x_{i+1}, ..., x_n\}$, which implies that

$(X,<)$ is the linear sum of $\{x_1, ..., x_i\}$ and $\{x_{i+1}, ..., x_n\}$ and produces a contradiction to the assumption that $(X,<)$ is irreducible. □

Finally the main theorem can be stated and proved.

Theorem 4.4.7. Let $(X,<)$ be a semiorder that is not a chain. Then there exist incomparable elements x and y in X such that $1/3 \leq p(x<y) \leq 2/3$.

Proof. The proof of this theorem is by a simple combinatorial argument. There are $n-1$ pairs of the form (x_i, x_{i+1}) in the set $\{x_1, ..., x_n\}$. Since each pair is unrelated by the previous lemma, there must be at least 2 elements good for each such pair (x_i, x_{i+1}) by Lemma 4.4.3. Thus there must be at least $2n-2$ instances in which an element is good for a pair. By Lemma 4.4.4 each element in X can be good in at most 2 instances, with the exception of maximal and minimal elements which are at most good in one less instance than the nonmaximal nonminimal elements. Since there are at least 2 maximal and 2 minimal elements in X, the n elements of X are good in at most $2n-4$ instances. Evidently such a poset cannot exist, so one can conclude that there is no finite non-chain semiorder for which the $1/3 - 2/3$ conjecture fails. □

It should be noted that Brightwell proved Theorem 4.4.2 and Lemma 4.4.3 for the larger class of finitely generated, thin (possibley infinite) posets. As we are concerned with the finite case, the proofs presented are simplifications of his approach.

4.5 THE 1/3 – 2/3 CONJECTURE FOR HEIGHT-1 POSETS

Trotter, Gehrlein, and Fishburn [TGF] proved that the 1/3 - 2/3 conjecture holds for all partially ordered sets of height one. The proof breaks into several parts since the height–1 posets must be partitioned into four different classes which each must be handled separately. In this section, the proof for only one of the classes will be given. However, this particular case deals with all but finitely many of the height-1 posets, so is the case of most interest. A brief description of the idea behind the other three cases will be given at the end of this section.

Firstly, a few basic definitions need to be made. A height-1 poset is a partially ordered set whose longest chain has length 1. The vertices of a height-1 poset $(X,<)$ can be partitioned into three sets. Let X_0 be the set of all nonmaximal minimal points in X, and let X_1 be the set of all nonminimal maximal points in X. The remaining points are the isolated points, and these fall into the third set. Let $n_0 = |X_0|$, $n_1 = |X_1|$ and $n = |X|$.

At this point the main theorem will be stated, and then certain restrictions can be made without loss of generality on the properties of $(X,<)$.

Theorem 4.5.1. Let $(X,<)$ be a height-1 partially ordered set that is not a chain. Then there are elements x and y in X such that $1/3 \leq p(x<y) \leq 2/3$.

Now assume that $(X,<)$ is a partially ordered set for which the $1/3 - 2/3$ conjecture fails. Without loss of generality, three things can be said about $(X,<)$. Firstly, it can be assumed that $(X,<)$ has no isolated points. Assume for a contradiction that $P = (X,<)$ has some isolated point z. Then for every pair x, y in X and hence for every x, y in $X–z$ either $p(x<y|P) < 1/3$ or $p(x<y|P) > 2/3$. Since z is an isolated point, it occurs in each position equally often in the set of linear extensions of $(X,<)$ and so $p(x<y|P) = p(x<y|P–z)$. Thus for every pair x, y in $X–z$, either $P(x<y|P–z) < 1/3$ or $P(x<y|P–z) > 2/3$ and so the $1/3 - 2/3$ conjecture fails for $P–z$ also. Thus only posets containing no isolated points need be considered. Note that this implies that $n = n_0 + n_1$.

It can also be assumed that $n \geq 3$. The only posets with fewer than three vertices are the one and two element chains and the poset consisting of two isolated points which by the previous paragraph need not be considered. Since chains are excluded from Theorem 4.5.1, one can assume without loss of generality that $n \geq 3$.

Thirdly, assume that $n_1 \geq n_0$. If $(X,<)$ is a poset for which the 1/3 – 2/3 conjecture fails, then the dual of $(X,<)$ ($(X,<)$ turned upside down), will not satisfy the 1/3 – 2/3 conjecture either since $p(x<y)$ in P will equal $p(x>y)$ in the dual of P.

Now several further definitions can be made. Let $\mathcal{P}$ represent the set of all height-1 posets such that $n \geq 3$, $n = n_0 + n_1$ and $n_1 \geq n_0$. Given a poset $P \in \mathcal{P}$ and a linear extension $L \in E(P)$, consider the function f on X_1 which gives the "height" of $x \in X_1$ in L as defined in Section 4.2. Then let $t_x = p(f(x) = n)$, which is the probability that x is in the maximal position of a given linear extension. Note that $p(f(x) = 1) = 0$ since $x \in X_1$ implies x is above at least one element in P, and thus can never occur in the minimal position in a linear extension of P. This implies that the "average height" of x as defined in Section 4.2 can be rewritten as follows:

$$h(x) = \sum_{k=2}^{n} kp(f(x)=k).$$

$h(x)$ represents the average height of x taken over all the linear extensions of P. Finally, let N be the following set of pairs:

$$N = \{(8,8),\ (7,7),\ \ldots,\ (2,2)\} \cup \{(7,6),\ (6,5),\ \ldots,\ (2,1)\}.$$

The following theorem will be proved in this section:

Theorem 4.5.2. Let $P \in \mathcal{P}$ such that $(n_1, n_0) \notin N$. Then there exist distinct elements x and y in X_1 such that $1/3 \leq p(x<y) \leq 2/3$.

The desired result will be reached by verifying a series of relationships which will be grouped together as a sequence of lemmas and theorems.

Lemma 4.5.3. a) $t_x = p(f(x)=n) \geq p(f(x)=n-1) \geq \ldots \geq p(f(x) = 2)$;

b) $p(f(x)=n) + p(f(x)=n-1) + \ldots + p(f(x)=2) = 1$;

c) $t_x \geq \frac{1}{n-1}$;

d) $h(x) \geq \frac{n}{2} + 1$;

e) $h(x) \leq n + \frac{1}{2} - \frac{1}{2t_x}$.

Proof. a) To show this relationship it needs to be shown that $p(f(x) = k) \geq p(f(x) = k-1)$ for every $k \in \{3, \ldots, n\}$. Let L be a linear extension of P in which x is at height $k-1$. Let y be the element directly above x in L. Elements x and y can be interchanged to produce another linear extension of P (if not, then it must be true that $y>x$ which contradicts the fact that $x \in X_1$), which will have x at height k. This produces an injective mapping from the elements with $f(x) = k-1$ to the elements with $f(x) = k$ which implies that $E(P \mid f(x)=k) \geq E(P \mid f(x) = k-1)$ and finally that $p(f(x)=k) \geq p(f(x)=k-1)$.

b) The left side of this equation sums the probabilities that a given $x \in X_1$ is at each of the possible heights, and must equal 1 since x is at some height in each linear extension.

c) Using a) and b) the following is obtained:

$$\begin{aligned}(n-1)t_x &= (n-1)p(f(x)=n)\\ &= p(f(x)=n) + p(f(x)=n) + \ldots + p(f(x)=n)\\ &\geq p(f(x)=n) + p(f(x)=n-1) + \ldots + p(f(x)=2)\\ &=1.\end{aligned}$$

Thus c) holds as required.

d) $$h(x) = \sum_{k=2}^{n} kp(f(x)=k)$$
$$\geq \sum_{k=2}^{n} k\left(\frac{1}{n-1}\right).$$

This last inequality is true since given the constraints imposed by a) and b) on the $p(f(x)=k)$'s, $h(x)$ will be minimized when the $p(f(x)=k)$'s are all equal to one another. Since there are $n-1$ of them and they have a sum of 1, this will happen when $p(f(x)=k) = 1/(n-1)$ for all $k \in \{2, ..., n\}$. Performing the summation, the following is arrived at:

$$h(x) \geq \left(\frac{n(n+1)}{2} - 1\right)\left(\frac{1}{n-1}\right)$$
$$= \frac{n}{2} + 1.$$

e) Let $q = \lfloor 1/t_x \rfloor$. and let a be the remainder when 1 is divided by t_x. Then $qt_x + a = 1$ where $0 \leq a < t_x$. In this section the aim is to find an upper bound on $h(x)$. As before, $h(x) = \sum_{k=2}^{n} kp(f(x)=k)$. To maximize $h(x)$ subject to a) and b), it is necessary to distribute as much of the probability as possible to the $p(f(x)=k)$'s with k large and as little as possible to those with k small. Since $p(f(x)=n) = t_x$, maximization of $h(x)$ will occur when $p(f(x)=n-1) = t_x$, $p(f(x)=n-2) = t_x$, ..., until q t_x's have been distributed. The next $p(f(x)=k)$ will equal the remainder a, and the remaining $p(f(x)=k)$'s will equal 0. Thus the following is produced:

$$h(x) = np(f(x)=n) + (n-1)p(f(x)=n-1) + ... + 2p(f(x)=2)$$
$$\leq nt_x + (n-1)t_x + ... + (n-(q-1))t_x + (n-q)a$$
$$= [n + (n-1) + (n-2) + ... + (n-(q-1))]t_x + (n-q)(1-qt_x)$$
$$= n + q\left(\frac{qt_x}{2} + \frac{t_x}{2} - 1\right)$$
$$\leq n + \frac{1}{t_x}\left(\frac{\left(\frac{1}{t_x}\right)t_x}{2} + \frac{t_x}{2} - 1\right)$$
$$= n + \frac{1}{2} - \frac{1}{2t_x}.$$

Thus the desired inequality in e) is arrived at. □

Several more definitions must be made before proceeding. Let x and y be distinct elements of X_1. Let $B = p(x<y)$. Let $b = p(f(y)-f(x)=1)$. Note that $b = p(f(x)-f(y)=1)$ also since there is a bijection from the linear extensions in which y is immediately above x to the linear extensions in which y is immediately below x. Let $a_k = p(f(x)-f(y)=k)$ and let $b_k = p(f(y)-f(x)=k)$ for all $k\geq 1$. Note that the four following relations follow almost directly from these definitions:

$$B = p(x<y) = \sum_{k=1}^{n-1} b_k; \tag{4.5.1}$$

$$1 - B = p(y<x) = \sum_{k=1}^{n-1} a_k; \tag{4.5.2}$$

$$b = a_1 = b_1; \tag{4.5.3}$$

$$h(x) - h(y) = \sum_{k=1}^{n-1} k(a_k - b_k). \tag{4.5.4}$$

Given the above set of definitions, another series of relationships can be proved, which will again be grouped together as a lemma.

Lemma 4.5.4.

a) $b = a_1 \geq a_2 \geq \ldots \geq a_{n-1};$

b) $\displaystyle\sum_{i=1}^{k} b_i \geq \sum_{i=1}^{k} b\left(1-\frac{b}{B}\right)^{i-1} \quad \forall\, k \in \{1, \ldots, n-1\};$

c) $\displaystyle\sum_{i=1}^{\infty} ib\left(1 - \frac{b}{B}\right)^{i-1} = \frac{B^2}{b};$

d) $\displaystyle h(x) - h(y) > \frac{1-2B-B^2}{2b}.$

Proof. a) In order to prove this inequality, it needs to be shown that $p(f(x)-f(y)=k) \geq p(f(x)-f(y)=k+1)$ for every $k\in\{1, \ldots, n-2\}$. Consider a linear extension in which $f(x)-f(y)=k+1$. Interchanging y with the element immediately above it, say z, produces another linear extension of $(X,<)$ (if not, then $z>y$ which contradicts $y \in X_1$) with $f(x)-f(y)=k$. Thus there is an injection from the linear extensions with $f(x)-f(y)=k+1$ to the linear extensions with $f(x)-f(y)=k$, and so $p(f(x)-f(y)=k) \geq p(f(x)-f(y)=k+1)$ as required.

b) The proof of this inequality comes from a paper by Kahn and Saks [KS]. It relies upon the following relation which will not be verified here but which is derived using a method of Stanley's [Sy3]:

$$b_i^2 \geq b_{i+1} b_{i-1} \quad \text{for } i \geq 2.$$

As an aside, any sequence $\{b_n\}$ of positive real numbers satisfying the above inequality is called *log concave*. This inequality will be used in the following form:

$$\frac{b_i}{b_{i-1}} \geq \frac{b_{i+1}}{b_i} \quad \text{for } i \geq 2. \tag{4.5.5}$$

Now for simplicity, let $b'_i = b(1-b/B)^{i-1}$. Verifying inequality b) then amounts to showing that

$$\sum_{i=1}^{k} b_i \geq \sum_{i=1}^{k} b_i' \quad \forall\, k \in \{1, \ldots, n-1\}. \tag{4.5.6}$$

(4.5.6) says that the sequence $\{b_i\}$ *majorizes* sequence b'_i. By using the formula for the infinite sum of a geometric series, the following can be verified:

$$\sum_{i=1}^{\infty} b'_i = B.$$

Now let h be the least integer such that $b_h < b'_h$. Note that $h>1$ since $b_1 = b'_1 = b$. Then $b_{h-1} \geq b'_{h-1}$ and the following sequence is derived:

$$\frac{b_h}{b_{h-1}} \leq \frac{b'_h}{b'_{h-1}} = 1 - \frac{b}{B}. \tag{4.5.7}$$

Combining (4.5.5) and (4.5.7) we obtain

$$\frac{b_i}{b_{i-1}} \leq 1 - \frac{b}{B} \quad \text{for all } i \geq h.$$

If $j < h$, then $b_j \geq b'_j$. If $j \geq h$,

$$\begin{aligned} b_j &= b_h \cdot \left(\frac{b_{h+1}}{b_h}\right) \cdot \left(\frac{b_{h+2}}{b_{h+1}}\right) \cdot \ \dots \cdot \left(\frac{b_j}{b_{j-1}}\right) \\ &\leq b_h \cdot \left(1 - \frac{b}{B}\right) \cdot \left(1 - \frac{b}{B}\right) \cdot \ \dots \cdot \left(1 - \frac{b}{B}\right) \\ &< b'_h \cdot \left(1 - \frac{b}{B}\right)^{j-h} \\ &= b \cdot \left(1 - \frac{b}{B}\right)^{h-1} \cdot \left(1 - \frac{b}{B}\right)^{j-h} \\ &= b'_j. \end{aligned}$$

Now to show (4.5.6), consider the following two cases.

Case i) Let $k<h$. Then it follows directly that

$b_1 + b_2 + \dots + b_k \geq b'_1 + b'_2 + \dots + b'_k$ as required.

Case ii) Let $k \geq h$.

$$\begin{aligned} b_1 + b_2 + \dots + b_k &= B - (b_{k+1} + \dots + b_{n-1}) \\ &\geq B - (b'_{k+1} + b'_{k+2} + \dots + b'_{n-1}) \\ &\geq B - (b'_{k+1} + b'_{k+2} + \dots + b'_{n-1} + \dots) \\ &= b'_1 + b'_2 + \dots + b'_k. \end{aligned}$$

Thus (4.5.6) has been verified, so b) holds as required.

c) It is well known that

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}.$$

Taking the derivative of both sides gives

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}. \tag{4.5.8}$$

Letting $x = 1-b/B$, substituting into (4.5.8) and simplifying produces identity c) as required.

d) From (4.5.4) we get

$$h(x) - h(y) = \sum_{k=1}^{n-1} ka_k - kb_k.$$

This implies

$$h(x) - h(y) \geq \min\left(\sum_{k=1}^{n-1} ka_k\right) - \max\left(\sum_{k=1}^{n-1} kb_k\right),$$

where the minimum is taken over all sequences of $n-1$ real numbers $\{a_i\}$ having a sum of $1-B$ with $b = a_1 \geq a_2 \geq ... \geq a_{n-1} \geq 0$, and the maximum is taken over all sequences of $n-1$ real numbers $\{b_i\}$ having a sum of B and satisfying

$$\sum_{i=1}^{k} b_i \geq \sum_{i=1}^{k} b\left(1-\frac{b}{B}\right)^{i-1} \quad \forall\, k \in \{1, \ldots, n-1\}.$$

Now it needs to be determined for which values of a_k the first sum is minimized, and for which values of b_k the second sum is maximized. Note that

$$\sum_{k=1}^{n-1} ka_k = a_1 + 2a_2 + \ldots + (n-1)a_{n-1}. \tag{4.5.9}$$

Given the restrictions on the a_i's, it can be seen that the above sum is minimized when a_1 is made as large as possible, then a_2 as large as possible, etc., until $1-B$ is exhausted. It is known that $a_1 = b$, so the largest that a_2 can be is b, and the largest a_3 can be is b, until $1-B$ is used up. Let $r = \lfloor (1\text{-}B)/b \rfloor$, the integer portion of the quotient when $1-B$ is divided by b. Then let z be the remainder when $1-B$ is divided by b. Thus $br + z = 1 - B$. Note that from this, the following are obtained:

$$r > \frac{1-B}{b} - 1; \tag{4.5.10}$$

$$\frac{br}{2} \leq \frac{1-B}{2}. \tag{4.5.11}$$

In conclusion, sum (4.5.9) will be minimized when the first $a_1, ..., a_r$ are equal to b, $a_{r+1} = z$, and all other a_i's equal 0.

Now consider the second sum

$$\sum_{k=1}^{n-1} kb_k = b_1 + 2b_2 + \dots +(n-1)b_{n-1}.$$

The b_i's have the fixed sum B, so this sum is maximized when b_{n-1} is made as large as possible, then b_{n-2} is as large as possible, etc., until B is exhausted. This is equivalent to minimizing b_1, then b_2, etc., which is the same as minimizing the partial sums b_1, $b_1 + b_2$, $b_1 + b_2 + b_3$, ..., $b_1 + b_2 + \dots + b_{n-1}$. However, from b) it is known that the partial sums are minimized when $b_i = b(1-b/B)^{i-1}$, and thus this will also maximize $\sum_{k=1}^{n-1} kb_k$.

Combining these ideas gives

$$\begin{aligned}
h(x) - h(y) \quad &\geq \sum_{k=1}^{r} kb + (r+1)z - \sum_{k=1}^{n-1} kb\left(1-\frac{b}{B}\right)^{k-1} \\
&> \frac{br(r+1)}{2} + (r+1)(1-B-rb) - \frac{B^2}{b} \qquad \text{(using (4.5.8))} \\
&= r\left(1 - B - \frac{rb}{2} - \frac{b}{2}\right) + 1 - B - \frac{B^2}{b} \\
&> \left(\frac{1-B}{b} - 1\right)\cdot\left(1 - B - \frac{b}{2} - \left(\frac{1-B}{2}\right)\right) + 1 - B - \frac{B^2}{b} \\
&\qquad \text{(from (4.5.10) and (4.5.11)).} \\
&= \frac{(1-B)^2}{2b} + \frac{b}{2} - \frac{B^2}{b} \\
&> \frac{(1-B)^2}{2b} - \frac{B^2}{b} \\
&= \frac{1-2B-B^2}{2b}.
\end{aligned}$$

Thus the required identity has been verified, and so the final part of Lemma 4.5.4 is complete. □

Lemma 4.5.5. Let $x, y \in X_1$. If $h(x) \geq h(y)$ and $b[h(x) - h(y)] \leq 1/9$, then $1/3 < p(x<y) < 2/3$.

Proof. The proof of this lemma is divided into two sections. First it will be shown that if $h(x) \geq h(y)$, then $p(x<y) < 2/3$.

Assume $h(y) \geq h(x)$. This implies that $h(x) - h(y) \leq 0$, so by Lemma 4.5.4d, we have

$$\frac{1-2B-B^2}{2b} < 0$$
$$\Rightarrow B^2 + 2B - 1 > 0$$
$$\Rightarrow [B + (\sqrt{2} + 1)] \cdot [B - (\sqrt{2} - 1)] > 0$$
$$\Rightarrow B > \sqrt{2} - 1$$
$$\Rightarrow p(x<y) > \sqrt{2} - 1.$$

Thus $h(y) \geq h(x)$ implies that $p(x<y) > \sqrt{2} - 1$. Interchanging x and y gives:

$$h(x) \geq h(y) \Rightarrow p(y<x) > \sqrt{2} - 1$$
$$\Rightarrow 1 - B > \sqrt{2} - 1$$
$$\Rightarrow B < 2 - \sqrt{2} < \frac{2}{3}$$
$$\Rightarrow p(x<y) < \frac{2}{3}.$$

This completes the first part of the lemma. Now for the second part, it will be shown that if $b[h(x) - h(y)] \leq 1/9$ then $p(x<y) > 1/3$. Assume $b[h(x) - h(y)] \leq 1/9$. Multiplying both sides of Lemma 4.5.4d by b gives

$$b[h(x) - h(y)] > \frac{1-2B-B^2}{2}.$$

Combining the two equations gives:

$$\frac{1}{9} > \frac{1-2B-B^2}{2}$$
$$\Rightarrow B^2 + 2B - \frac{7}{9} > 0$$
$$\Rightarrow \left(B + \frac{7}{3}\right)\left(B - \frac{1}{3}\right) > 0$$
$$\Rightarrow B > \frac{1}{3}$$
$$\Rightarrow p(x<y) > \frac{1}{3}.$$

Thus the second part of the lemma has been verified, and combining the two portions gives the desired result. □

A few further definitions must be made. Let $m = n_1$, for convenience. Also let $X_1 = \{1, 2, ..., m\}$ and recalling Lemma 4.5.3c suppose without loss of generality

$$\frac{1}{n-1} \leq t_1 \leq t_2 \leq \ldots \leq t_m.$$

Let k be a fixed element in $\{2, 3, ..., m\}$.

Lemma 4.5.6. For any k there exist distinct integers $i, j \leq k$ such that $h(i) \geq h(j)$ and

$$h(i) - h(j) \leq \frac{(n-1)t_k - 1}{2t_k(k-1)}.$$

Proof. First it will be shown that:

$$h(1), h(2), \ldots, h(k) \in \left[\frac{n}{2} + 1, n + \frac{1}{2} - \frac{1}{2t_k}\right].$$

The lower bound on the $h(i)$'s follows directly from Lemma 4.5.3d. The upper bound follows from Lemma 4.5.3e and the fact that $t_i \leq t_k$ for all i such that $1 \leq i \leq k$.

The $h(1), ..., h(k)$ are all in the above interval, so the question becomes, how close together must at least two of the $h(i)$'s be? In the worst case, the $h(i)$'s will be spaced evenly across the interval, so the distance between any two adjoining $h(i)$'s will be the length of the interval divided by the number of subintervals, which is:

$$\frac{\left(n + \frac{1}{2} - \frac{1}{2t_k}\right) - \left(\frac{n}{2} + 1\right)}{k-1}.$$

Then in any case, there must be some adjacent pair $h(i), h(j)$ with $h(i) \geq h(j)$ such that

$$\begin{aligned} h(i) - h(j) &\leq \frac{\left(n + \frac{1}{2} - \frac{1}{2t_k}\right) - \left(\frac{n}{2} + 1\right)}{k-1} \\ &= \frac{(n-1)t_k - 1}{2t_k(k-1)}. \quad \square \end{aligned}$$

This lemma can now be strengthened to produce the following result.

Lemma 4.5.7. There exist distinct elements x and y in X_1 such that $h(x) \geq h(y)$ and

$$b[h(x) - h(y)] \le \frac{n-m-1}{m\,(m-1)}.$$

Proof. First it will be shown that for the values of i and j from Lemma 4.5.6, and for any k with $1 \le k \le m$, $b \le t_k$. By the initial definition, $b = p(f(i)-f(j) = 1)$. It will be shown that $p(f(i)-f(j) = 1) \le p(f(i)=n)$. Consider a linear extension in which $f(i)-f(j) = 1$. Note that i is immediately above j in this extension. It can be easily seen that another linear extension will be formed if i is moved to the top position, and this new linear extension will have $f(i) = n$. In this manner an injection is formed from the extensions with $f(i)-f(j) = 1$ to the extensions with $f(i) = n$. Thus $b \le t_i$ as required and $t_i \le t_k$ implies that $b \le t_k$.

Now consider the value of k which minimizes the quantity $((n - 1)t_k - 1) / (2t_k(k-1))$. By Lemma 4.5.6 there exist integers $i, j \le k$ such that $h(i) - h(j)$ is less than or equal to this quantity and such that $h(i) \ge h(j)$. In general it can be said that there exist x, $y \in X_1$ such that $h(x) \ge h(y)$ and

$$h(x) - h(y) \le \min_{2\le k\le m}\left(\frac{(n-1)t_k - 1}{2t_k(k-1)}\right).$$

Using the fact that $b \le t_k$, the previous inequality implies

$$b[h(x) - h(y)] \le \min_{2\le k\le m}\left(\frac{(n-1)t_k - 1}{2(k-1)}\right).$$

Let

$$Z = \min_{2\le k\le m}\left(\frac{(n-1)t_k - 1}{2(k-1)}\right).$$

Now the aim is to find the maximum possible value of Z over all combinations of the t_i's. Since the t_i's have a fixed sum of 1, Z must be a maximum when $((n-1)t_k-1)/(2(k-1))$ is the same for all values of k. If there were a k for which the corresponding quantity was smaller than the others, then increasing t_k and decreasing the other t_i's would consequently increase Z. Thus for Z to be a maximum

$$Z = \frac{(n-1)t_k - 1}{2(k-1)} \quad \forall\ k \in \{2, 3, \ldots, m\}.$$

This can be rewritten as

$$2(k-1)Z = (n-1)t_k - 1 \quad \forall\, k \in \{2, 3, \ldots, m\}$$

Summing over all values of k gives

$$\sum_{k=2}^{m} 2(k-1)Z = \sum_{k=2}^{m} ((n-1)t_k - 1)$$

$$\Rightarrow\ Z\,(m-1)\,m = (n-1)(1-t_1) - (m-1)$$

$$\Rightarrow\ Z = \frac{(n-1)(1-t_1)-(m-1)}{(m-1)\,m}$$

$$\leq \frac{(n-1)\left(1-\frac{1}{n-1}\right)-(m-1)}{(m-1)\,m}$$

$$= \frac{n-m-1}{m\,(m-1)}.$$

Thus it can be concluded that there exists an x and y in X_1 such that $h(x) \geq h(y)$ and $b[h(x) - h(y)] \leq \frac{n-m-1}{m\,(m-1)}$. □

Proof of Theorem 4.5.2. Finally the main theorem can be demonstrated. Combining Lemmas 4.5.5 and 4.5.7 we get that if

$$\frac{n-m-1}{m\,(m-1)} \leq \frac{1}{9} \tag{4.5.12}$$

then there exist $x, y \in X_1$ such that $1/3 < p(x<y) < 2/3$. All that is needed to show is that for every pair $(n_1, n_0) \notin N$ (4.5.12) holds, where $m = n_1$ and $n = n_0 + n_1$.

A brief proof will show that (4.5.12) holds whenever $m \geq 9$. Let $m \geq 9$. Recall that $m \geq n_0$ and $m + n_0 = n$. Thus $n \leq 2m$. Then as desired,

$$\frac{n-m-1}{m\,(m-1)} \leq \frac{2m-m-1}{m\,(m-1)} = \frac{1}{m} \leq \frac{1}{9}.$$

The cases remaining are those for which $m < 9$ and (m, n_0) is not in N. There are finitely many of these and (4.5.12) can be verified individually for each one. □

Thus Theorem 4.5.2, which covers all but finitely many of the posets included in Theorem 4.5.1, has been proved. At this point a brief idea of how the remaining cases can be proved will be given.

The second part of the proof covers the cases where $(n_1, n_0) = \{(8, 8), (7, 7), (7, 6), (6, 5)\}$. By going through part 1 more carefully, an upper bound that is smaller than

$$\frac{n - m - 1}{m\,(m - 1)}$$

can be found on $b[h(x) - h(y)]$ in Lemma 4.5.7. It then remains to verify that this bound is less than or equal to 1/9 for the desired (n_0, n_1) pairs.

The third part of this proof covers the cases where $(n_1, n_0) = \{(4, 4), (3, 3), (2, 2), (5, 4), (4, 3), (3, 2), (2, 1)\}$. First let V_m be the poset on $2m$ vertices with m minimal points $\{l_1, ..., l_m\}$, and m maximal points $\{u_1, ..., u_m\}$, such that $l_i < u_j$ if and only if $i \leq j$. Let V_m^+ be V_m plus one isolated point. Define $\delta(P)$ as follows:

$$\delta(P) = \max_{x,\, y} \min \{p(x<y),\, p(y<x)\}.$$

Showing that the 1/3 – 2/3 conjecture holds for a poset then, is equivalent to showing that $\delta(P) \geq 1/3$. Now define δ_n and $\delta(m)$ as follows:

$$\delta_n = \min \{\delta(P) \colon P \text{ is an } n\text{-point height-1 poset}\};$$

$$\delta(m) = \min \{\delta(P) \colon \text{width of } P = m \text{ and } P \text{ has height } 1\}.$$

Conjecture 4.5.8. $\delta(m+1) = \delta_{2m} = \delta_{2m+1} = \delta(V_m) = \delta(V_m^+)$.

Trotter, Gehrlein and Fishburn have verified that this conjecture holds for the cases where $m = 2$, 3 and 4. They have also verified that V_2, V_3, and V_4 all have $\delta(P) \geq 1/3$. These two facts confirm that the 1/3 – 2/3 conjecture holds for the desired subset of N.

The fourth part of the proof covers the two remaining cases, (6,6) and (5,5), and is based on the following lemma.

Lemma 4.5.9. Let $P \in \mathcal{P}$.

a) If $x, y \in X_1$ and $p(x<y) \leq 1/3$, then x must cover at least two points in X_0 that y doesn't cover.

b) If $x, y \in X_0$ and $p(x<y) \leq 1/3$, then y must be covered by at least two points in X_1 that x isn't covered by.

The proof of this lemma employs methods similar to many of the proofs in Section 4.4 on semiorders. Note that conditions a) and b) each imply that there are at least two elements good for $y<x$, where "good" is as defined in Section 4.4.

To prove cases (6,6) and (5,5) one begins to build a height one poset subject to the restrictions imposed by Lemma 4.5.9. It is found that in order to have four vertices in X_1, at least 6 vertices are required in X_0, in order to satisfy Lemma 4.5.9. This automatically demonstrates that no such poset with $(n_1, n_0) = (5, 5)$ can exist. By continuing the building process, it is found that there are exactly five possibilities for a (6, 6) poset satisfying Lemma 4.5.9. It can be shown that for each of these five posets, we can alter Lemma 4.5.3d to be $h(x) \geq (n + 2.8)/2$. Recall that in its original form, Lemma 4.5.3d was $h(x) \geq (n + 2)/2$. Going through part one again with this tighter lower bound on $h(x)$, reduces the upper bound of $b[h(x)-h(y)]$ sufficiently to admit the case when $(n_1, n_0) =$ (6, 6).

4.6 CASES PRODUCING BETTER THAN THE 1/3 – 2/3 CONJECTURE

There are special classes of posets for which it has been shown that the 1/3 – 2/3 conjecture can be improved upon. One such case is that of a finite cycle-free ordered set containing a non-trivial automorphism of P. Note that a cycle-free poset is any poset not haveing the poaset $\{a<b,\ b<c,\ a<d,\ d<c,\ a<c\}$ as a subposet. As was shown in Section 4.3, any poset containing a non-trivial automorphism satisfies the 1/3 – 2/3 conjecture, and in fact, Ganter, Häfner and Poguntke conjecture that this bound can be improved upon [GHP]. When this class of posets is further restricted to those which contain no cycles, it is found that there is a pair $x, y \in X$ such that $p(x<y)=1/2$. The formal statement of the theorem is as follows:

Theorem 4.6.1. If $P=(X,<)$ is a finite cycle-free ordered set and α is a non-trivial automorphism of P, then $p(x<\alpha(x))=1/2$ for any $x \in X$ with $\alpha(x) \neq x$.

Proof. Only a general description of this proof will be given. Consider the covering graph of P, CP. CP is the undirected graph on X with an edge between y and z in X if and only if y is a lower or upper cover of z in P. Now partition X into equivalence classes so that y is related to z if and only if $y=z$ or there is a path in CP between y and z containing no fixed points of α. Let $[x]$ represent the equivalence class containing $x \in X$. Now one can show there is a bijection between $E(P \mid x<\alpha(x))$ and $E(P \mid \alpha(x)<x)$, the set of linear extensions having x below $\alpha(x)$, and the set of linear extensions having $\alpha(x)$ below x. For $\lambda \in E(P \mid x<\alpha(x))$, let $f_\lambda(x)$ be the "height " of x in λ as in Section 4.2 and define $\Phi(\lambda)$ as follows:

$$\Phi(\lambda)(y)=\begin{cases} f_\lambda(\alpha(y)) & \text{if } y \in [x]; \\ f_\lambda(\alpha^{-1}(y)) & \text{if } y \in [\alpha(x)]; \\ f_\lambda(y) & \text{otherwise.} \end{cases}$$

It is then demonstrated that Φ is a bijection which implies $E(P \mid x<\alpha(x)) = E(P \mid \alpha(x)<x)$ and thus $p(x<\alpha(x))=1/2$ as desired. □

This proof in fact demonstrates the following statement which is stronger than Theorem 4.6.1:

Theorem 4.6.2. If $P=(X,<)$ is a finite partially ordered set with a non-trivial automorphism α, and if $x \in X$ with $\alpha(x) \neq x$ is such that there is a fixed point of α on every path connecting x and $\alpha(x)$ in CP, then $p(x<\alpha(x))=1/2$.

The relationship between Theorems 4.6.1 and 4.6.2 can be easily seen. If P is a finite cycle-free partially ordered set, then every $x \in X$ such that $x \neq \alpha(x)$ will have a fixed point of α on the path (if it exists) connecting x and $\alpha(x)$.

Another case where the 1/3–2/3 conjecture can be strengthened is the case of a poset containing a large antichain. Kahn and Saks [KS] conjectured that if a poset $(X,<)$ contains a large antichain, then there will be elements x and y in X such that $p(x<y)$ is close to 1/2. Komlos has proved that this conjecture is true whenever $(X,<)$ is a poset containing a large number of minimal elements, which includes all large bipartite graphs [Ko]. Formally, Komlos' result can be stated as follows.

Theorem 4.6.3. For every $\varepsilon > 0$ there is a function $M(n)$ such that if $(X,<)$ is any n element height-1 poset with width at least $M(n)$ minimal elements, then there are minimal elements x, y in X such that $1/2 - \varepsilon < p(x<y) < 1/2 + \varepsilon$.

Using the definition of $\delta(P)$ from Section 4.5, Theorem 4.6.3 can be rephrased as

$$\lim_{m \to \infty} \min\{\delta(P) \mid P \text{ has height one and width } m\} = \frac{1}{2}.$$

Komlos' proof of this theorem uses Ramsey theory to prove that two elements x and y can always be selected from a large collection of random variables such that $p(x<y)$ is approximately 1/2. When this idea is applied to the poset problem, Theorem 4.6.3 follows directly.

There is at least one further situation in which the 1/3 – 2/3 conjecture can be strengthened. Recall from Section 4.5, the definition of V_m, a certain class of height-1 posets. Trotter, Gehrlein and Fishburn conjecture the following [TGF]:

$$\lim_{m \to \infty} V_m = \frac{1}{2}.$$

The authors have only thus far been able to find the values of V_2, V_3 and V_4 and verify that $V_2 < V_3 < V_4 < 1/2$, so this conjecture remains an open problem.

4.7 THE 1/3 – 2/3 CONJECTURE FOR INFINITE POSETS

In two separate papers [Br3] and [Br4], Brightwell has considered the 1/3 – 2/3 conjecture for a certain class of infinite posets. Since an infinite poset can have an infinite number of linear extensions, $p(x<y)$ cannot be determined using its previous definition, so a new definition is needed.

Consider a poset $P=(X,<)$. Define Y a subset of X to be convex if whenever x, $y \in Y$, $z \in X$ and $x<z<y$ then $z \in Y$. Suppose $(X_n)_1^\infty$ is an increasing sequence of finite convex subsets of X whose union is X, and which each contain both x and y. Then

$$p(x<y|P) = \lim_{n \to \infty} p(x<y|(P|_{X_n}))$$

if that limit exists for every choice of (X_n), and all such limits agree.

Brightwell shows that when P is a thin poset (each element in P is incomparable with at most k elements for some finite k), then this limit exists and is independent of the choice of (X_n). He then proves the following theorem:

Theorem 4.7.1. For every infinite thin poset $(X,<)$ that is not a chain, there are elements x, y of X such that $3/11 \le p(x<y) \le 8/11$.

It should be noted that in the infinite case the proof does not produce strict inequalities whereas with the finite case the inequalities can be shown to be strict. Counter-examples of infinite posets for which the 1/3 – 2/3 conjecture does not hold have been found. Interestingly, these posets are all semiorders which can be derived using a theorem of Brightwell. Firstly define a semiorder $(X,<)$ to be *2-separated* if for every $x, y \in X$ with $x \mid y$ there are either two elements good for $x < y$ or two elements good for $y < x$, where "good" is as defined in Section 4.4. Brightwell's theorem follows.

Theorem 4.7.2. Every irreducible 2-separated semiorder of width k is isomorphic to a poset $(X,<)$ with $X=\{x_1, x_2, x_3, ...\}$, and $x_i<x_j$ if and only if $j \ge i+r_j$, where each r_j is either k or $k-1$, with at least one $r_i=k$, and $r_i=r_{i+k-1}$ for all i. As well, every poset of the previous form is an irreducible 2-separated semiorder of width k.

Three of these posets are shown in Figure 4.7.1. A locally finite poset is a poset $(X, <)$ such that for every $x, y \in X$, there are finitely many $z \in X$ such that $x < z < y$. Brightwell conjectures that all locally finite, thin posets for which the 1/3 – 2/3 conjecture fails are linear sums of these three posets along with the one-element poset.

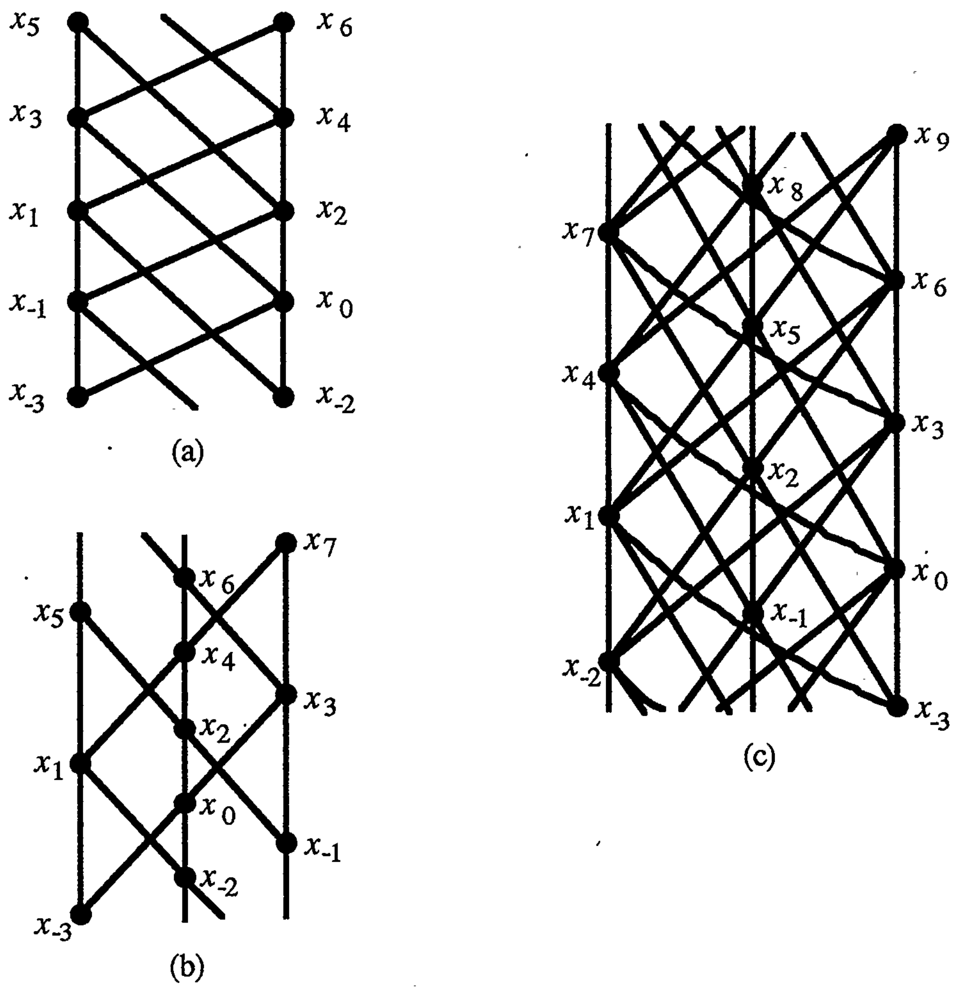


Figure 4.7.1

Figure 4.7.1 (a) has k=2 and $\{r_i\}$ = {.., 2, 2, 2, ...}, (b) has k=3 and $\{r_i\}$ = {.., 3, 2, 3, 2, 3, 2, ...} and (c) has k=3 and $\{r_i\}$ = { .., 3, 3, 3, ...}. All of these posets satisfy $3/11 \leq \delta(P) \leq 1/3$ and specifically, a) has $\delta(P) \approx 0.2764$ and b) has $\delta(P) \approx 0.3106$.

The question which now arises is, what is the bound on $p(x<y)$ for the most central pair of an infinite thin poset? As of yet, it is not known whether the 3/11 – 8/11 bound can be improved upon.

There are many open questions surrounding the $1/3 - 2/3$ conjecture. In the finite case it has yet to be proved whether or not the conjecture holds for all posets. In the infinite case, the $1/3 - 2/3$ conjecture is known to fail, yet the greatest λ with $0 < \lambda < 1/3$ such that λ, $1-\lambda$ can replace $1/3$, $2/3$ in the conjecture has not been found.

CHAPTER FIVE

COUNTING CHAINS

5.1 INTRODUCTION

Of the three basic counting problems being addressed in this thesis, that of counting chains seems to have received the least amount of attention. Results are not abundant in the current literature. Perhaps the most work has been done on the problem of counting chains in power sets, an obvious extension of Dedekind's problem on counting antichains in power sets. This problem will be dealt with in Section 5.3. Other specific chain counting results will be dealt with in Section 5.2, and the Section 5.3 will examine some relationships between the problem of counting chains and that of counting linear extensions. The final section will give some results on a new chain counting problem.

5.2 CHAIN COUNTING RESULTS

Some work has been done on directly counting chains in certain classes of posets, and these results will be dealt with in this section.

One simple result is the number of chains in a poset P that is itself a chain on n elements. It can easily be seen that P contains 2^n chains, since every subset of the n elements of P is itself a chain.

Kurepa [Ku] counts the number of maximal chains in a special class of trees. Define T_n to be the tree formed from $\emptyset$ (the empty sequence) and all sequences of the form $a = (a_1, a_2, ..., a_j)$ where $1 \le j \le n$ and $0 \le a_i \le i-1$ for all i, $1 \le i \le j$, and where $a < b$ if and only if sequence a is an initial part of sequence b. T_3 is shown as an example in Figure 5.2.1.

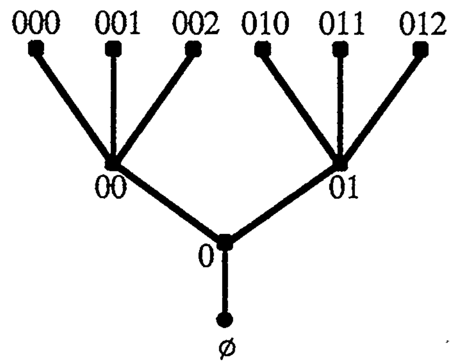


Figure 5.2.1

It can easily be seen that T_n will have $n!$ maximal chains.

Ziegler [Z] considered the problem of placing an upper bound on the number of chains and maximal chains in a length $l-1$ poset on n elements. Recall that in Section 2.2 results produced by Ziegler are given for the same problem applied to antichains. The following lemma bears a strong resemblance to Lemma 2.2.8.

Lemma 5.2.2. Let P be a poset on n elements with length $l-1$.

a) Then P contains at most $c(n,l)$ chains where

$$c(n,l) = \max_{\substack{c_1+\ldots+c_l=n \\ c_i \ge 1}} \prod_{i=1}^{l} (c_i + 1).$$

P achieves this maximum if and only if it is the ordinal sum of l antichains on $c_1, ..., c_l$ elements, where the c_i's are those which maximize the above equation.

b) P contains at most $mc(n,l)$ maximal chains where

$$mc(n,l) = \max_{\substack{c_1+\ldots+c_l=n \\ c_i \geq 1}} \prod_{i=1}^{l} c_i$$

P achieves this maximum if (but not only if) it is the ordinal sum of l antichains on $c_1, .., c_l$ elements, where the c_i's are those which maximize the above equation.

This lemma can be seen intuitively by considering a poset which is the ordinal sum of l antichains. Remove an edge between any two elements, say a and b. In the original poset a and b form a chain, but in the new poset a and b do not form a chain. Since the removal of an edge cannot create any new chains, it is apparent that the new poset will have fewer chains than the original poset. Add to this the observation that all posets can be created by starting with some poset that is the ordinal sum of antichains and removing edges, then part a) follows easily. Part b) is not as obvious since removing an edge between two related elements does not necessarily decrease the number of maximal chains, so Ziegler uses an inductive proof to demonstrate this.

Ziegler also evaluates the maximum value of $mc(n,l)$ for a given n over all l and produces the following result:

$$\max_{0<l\leq n} mc(n,l) = \begin{cases} 3^{n/3} & \text{for } n \equiv 0 \bmod 3; \\ 4 \cdot 3^{(n-4)/3} & \text{for } n \equiv 1 \bmod 3; \\ 2 \cdot 3^{(n-2)/3} & \text{for } n \equiv 2 \bmod 3. \end{cases} \tag{5.2.1}$$

To show this, start with the equation given for $mc(n, l)$ in Lemma 5.2.2b. Note that for a maximum we can't have any $c_i \geq 4$, since $2\,(c_i - 2) \geq 4$ whenever $c_i \geq 4$. Similarly, we won't have $c_i = 1$, since $c_j+1 > c_j \cdot 1$. Thus all c_i's must equal either 2 or 3. However, we won't have three or more 2's since $2 + 2 + 2 = 3 + 3$ but $2^3 < 3^2$. Thus equation (5.2.1) follows.

A paper of Greene's [Gn] deals with the problem of counting chains in a poset of shuffles. Let x and y be words of length m and n from some alphabet A where all $m + n$ letters are distinct. Write $u \leq v$ if u is a subword of v, write $l(u)$ for the set of letters found in word u, and let $v|u$ denote the subword found by restricting v to the letters in u. The vertices of a poset of shuffles consist of all words w such that $l(w) \subseteq l(x) \cup l(y)$, $w|x \leq x$ and $w|y \leq y$. In other words, w can only contain letters from x and y, and the letters from x and y must appear in the same order as they do in x and y. Now define the relation $\leq_O$ setting $u \leq_O v$ if and only if $u|x \geq v|x$, $u|y \leq v|y$ and $u|v = v|u$. Thus $u \leq_O v$ if and only if v can be obtained from u by deleting some (or no) letters from x and inserting some (or no) letters from y. As an example let x be the 2-letter word "AB" and y be the 1-letter word "c". The poset of shuffles on x and y is shown in Figure 5.2.2.

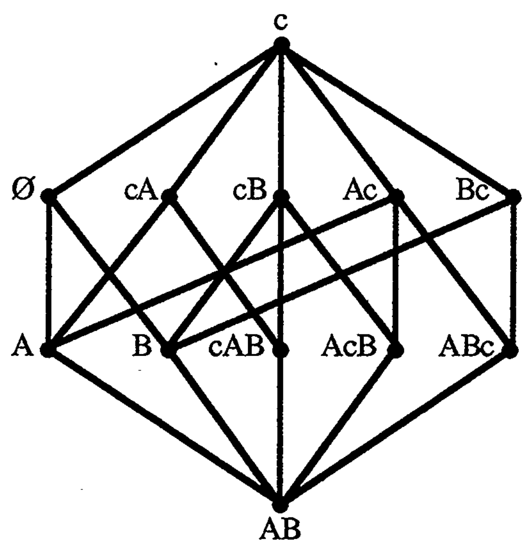


Figure 5.2.2

It is easy to see that whenever x is a 2-letter word and y is a 1-letter word the above poset will be formed. Let $mc(m,n)$ be the number of maximal chains in a poset of shuffles on an m-letter word and an n-letter word. Greene finds the following:

$$mc(m,n) = (m+n)! \sum_{j \geq 0} \binom{m}{j}\binom{n}{j}\left(\frac{1}{2}\right)^j. \tag{5.2.2}$$

Greene proves this and other properties of posets of shuffles using Jacobi polynomials.

5.3 COUNTING CHAINS IN POWER SETS

In a recent article, Nelsen and Schmidt [NS] deal with the problem of counting chains in a power set on n elements, $P(n)$. Firstly they consider the number of chains of given length k in $P(n)$, which will be denoted $c_k(P(n))$, and develop the following recursion formula:

$$c_k(P(n+1)) = k \cdot c_{k-1}(P(n)) + (k+2) \cdot c_k(P(n)). \qquad (5.3.1)$$

Let $X_n = \{1, 2, ..., n\}$.This result is found by intersecting every element of each chain C of length k in $P(n+1)$ with the set X_n to produce a chain C' in $P(n)$. Any such C' will either contain k different vertices, or will have 2 identical vertices and thus will have $k-1$ different vertices. A chain C producing the former type of C' will be called non-degenerate, while a chain producing the latter type will be called degenerate. Thus the number of chains of length k in $P(n+1)$ will equal the sum of the number of non-degenerate chains and degenerate chains of length k in $P(n+1)$. Now let D: $N_0 \subset N_1 \subset \ldots \subset N_k$ be a chain of length k in $P(n)$. $k+2$ non-degenerate chains of length k in $P(n+1)$ can be formed from this by counting the chain D itself and the $k+1$ chains of the form $N_0 \subset N_1 \subset \ldots \subset N_{i-1} \subset N_i \cup \{n+1\} \subset \ldots \subset N_k \cup \{n+1\}$, where $i \in \{0, 1, ..., k\}$. It can be shown that every non-degenerate chain in $P(n+1)$ will be produced exactly once in this manner. Now let D: $N_0 \subset N_1 \subset \ldots \subset N_{k-1}$ be a chain of length $k-1$ in $P(n)$. k degenerate chains of length k in $P(n+1)$ are formed from this by considering chains of the form $N_0 \subset N_1 \subset \ldots \subset N_i \subset N_i \cup \{n+1\} \subset \ldots \subset N_{k-1} \cup \{n+1\}$, where $i \in \{0, 1, ..., k-1\}$. As before, it can be shown that every degenerate chain in $P(n+1)$ will be produced exactly once, and (5.3.1) follows.

Nelsen and Schmidt also produce the following non-recursive equation for the number of chains of length k in a power set:

$$c_k(P(n)) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k+2-j)^n. \tag{5.3.2}$$

Nelsen and Schmidt describe two different ways of deriving this from equation (5.3.1). One menthd uses a formal power series, and the other uses Stirling Numbers.

Finally by taking the sum of (5.3.2) over all possible values of k, and then simplifying using a relationship involving Stirling numbers, Nelsen and Schmidt develop the following equation for the number of chains in $P(n)$:

$$c(P(n)) = 2\sum_{j=2}^{\infty} j^n 2^{-j}, \quad n \geq 1. \tag{5.3.3}$$

Although at first glance, this formula seems to produce the most direct method of calculating $c(P(n))$, in fact using the recursion given in (5.3.1) and then summing over all possible values of k gives the fastest way of calculating $c(P(n))$. Using this method, Nelsen and Schmidt create a table of values listing the number of chains in a power set on 0 to 10 elements.

The problem of counting maximal chains in a power set on n elements is much simpler than that of counting all chains. The number of maximal chains in $P(n)$ is just $n!$, as mentioned in a paper by Kurepa [Ku]. This result follows easily by recognizing that there is a bijection from the permutations of n elements to the maximal chains in a power set on n elements. For instance, the permutation 1, 3, 2 of the numbers 1, 2 and 3, corresponds to the chain $\{\emptyset\} \subset \{1\} \subset \{1,3\} \subset \{1,3,2\}$ in $P(3)$. It is well-known that the number of permutations of an n element set is $n!$, and the result follows.

Griggs, Stahl and Trotter [GST] produced a paper in which they discuss the problem of counting the number of pairwise unrelated chains in a power set. Given two chains C_1 and C_2, C_1 and C_2 are pairwise unrelated if and only if for every $x \in C_1$ and $y \in C_2$, then neither $x < y$ nor $y < x$. The notation $uc_k(P(n))$ will be used to denote the maximum possible number of pairwise unrelated chains of length k in a power set on n elements. The result of Griggs et al. is as follows:

$$uc_k(P(n)) = \binom{n-k}{\lfloor (n-k)/2 \rfloor}. \tag{5.3.4}$$

The construction used to produce a set of $uc_k(P(n))$ unrelated chains is also described by Griggs et al. The chains are designated by $D(i, 0) \subset D(i, 1) \subset ... \subset D(i, k)$, where the $D(i, j)$'s represent the elements in the chain. The set $D(i, 0)$ is a subset of size $\lfloor (n-k)/2 \rfloor$ of $\{k+1, k+2, ..., n\}$. Then the set $D(i, j) = D(i, 0) \cup \{1, 2, ..., j\}$. It is apparent there will be $\binom{n-k}{\lfloor (n-k)/2 \rfloor}$ possible $D(i, 0)$'s and thus the same number of chains, and it can be verified that chains formed in this manner will be unrelated to one another.

The preceding construction demonstrates that the value of $uc_k(P(n))$ must be greater than or equal to $\binom{n-k}{\lfloor (n-k)/2 \rfloor}$ Using a known inequality, Griggs et al. show that $uc_k(P(n))$ must be less that or equal to $\binom{n-k}{\lfloor (n-k)/2 \rfloor}$ and thus equation (5.3.4) follows.

From these results it can be seen that the problem of counting chains in power sets is much more straight-forward than that of counting antichains in power sets. Both recursive and closed formulas exist which describe the number of chains in a power set on n elements.

5.4 RELATING CHAINS AND LINEAR EXTENSIONS

There are a number of recent results which relate the number of chains to the number of linear extensions of a partially ordered set. The first such result involves a certain class of linear extensions known as greedy linear extensions. Let $l = x_1 < x_2 < \dots < x_n$ be a linear extension of a poset P on n elements. l is called greedy if whenever there is a minimal element x in $P-\{x_1, x_2, \dots, x_i\}$ satisfying $x > x_i$, then $x_{i+1} > x_i$. Construct a linear extension by successively picking elements of P to represent $x_1, \dots, x_n$. l is greedy if for each i, x_{i+1} is a minimal element of $P \backslash \{x_1, .., x_i\}$ that is greater in P than x_i. Now let $g(P)$ represent the number of greedy linear extensions. As usual, $mc(P)$ will denote the number of maximal chains in P, and $e(P)$ will be the number of linear extensions of P.

Simion [Si] produces the following result relating $e(P)$, $g(P)$ and $mc(P)$.

Lemma 5.4.1. For any poset P, $mc(P) \leq g(P) \leq e(P)$.

The second inequality is obvious, and to see the first, let $x_1 < x_2 < \dots < x_k$ be a maximal chain in P. Let $F(x_i)$ be the order filter with x_i as its only minimal element. Form a greedy linear extension of P by concatenating greedy linear extensions of $(P-F(x_1), F(x_1)-F(x_2), \dots, F(x_{k-1})-F(x_k), F(x_k))$. It can be shown that distinct maximal chains map to distinct greedy linear extensions and thus the first inequality is verified.

The cases when equality holds in Lemma 5.4.1 are described in the following lemma:

Lemma 5.4.2. Let P be any poset. Then

a) $g(P) = e(P)$ if and only if $P = w_1 \oplus w_2 \oplus \ldots \oplus w_l$, where the w_i's are antichains.

b) $mc(P) = e(P)$ if and only if $P = w_1 \oplus w_2 \oplus \ldots \oplus w_l$, where the w_i's are antichains on either 1 or 2 elements.

c) let P_0 be poset P with a minimum element adjoined. $mc(P) = g(P)$ if and only if P_0 has the property that for a and b in P_0,a covers b implies $F(b) - F(a)$ is a chain.

Proof. a) If all extensions of poset P of height l–1 are greedy, then for $1 \le k \le l$ we must have every element at height k covering every element at height k–1. This produces the class of posets given in a). To see the reverse direction, recognize that every linear extension of such a poset must be greedy.

b) Since $mc(P) = e(P)$, by Lemma 5.4.1, $g(P) = e(P)$. Thus we may only have posets of the form described in a). Note that in such a poset

$$mc(P) = \prod_{i=1}^{l} w_i \quad \text{and} \quad e(P) = \prod_{i=1}^{l} w_i!.$$

Since $mc(P) = e(P)$ we must have $w_i = w_i!$ for i=1 to l. Thus w_i equals one or two for each $i \in \{1, 2, \ldots, l\}$. In the reverse direction, since each w_i equals either 1 or 2 and since $1! = 1$ and $2! = 2$, $mc(P)$ and $e(P)$ are equal.

c) Consider the mapping from maximal chains in P to grredy extensions of P described in Lemma 5.4.1. If there is a pair a, b in P_0 such that a covers b but $F(b) - F(a)$ is not a chain, then each maximal chain containing both a and b will map to more than one linear extension. Thus we will have $mc(P) < g(P)$ which is a contradiction. To see the reverse direction, note that when P_0 has the property that for a and b in P_0, a covers b implies $F(b) - F(a)$ is a chain, then the mappping given in Lemma 5.4.1 will be a bijection. □

Finally using an inductive argument, Simion proves the following theorem about the relationship between $e(P)$ and $mc(P)$.

Theorem 5.4.3. Let P be a finite poset such that $e(P) > mc(P)$. Then

$$e(P)/mc(P) \geq 3/2.$$

Ivan Rival [R] has done work on relating the number of antichains to the number of linear extensions. Before his results can be described, a few preliminary definitions are needed.

Definition 5.4.4. An element x in a lattice $L = (X,<)$ is join-irreducible if

a) x is not the minimum element in L

b) $x = a \vee b$ implies $x = a$ or $x = b$ for all $a, b \in X$.

It should be noted that in a finite lattice L, an element is join-irreducible if and only if it covers exactly one element in X. The partially ordered set of all join-irreducible elements of L will be denoted $J(L)$. Now Rival's theorem can be stated.

Theorem 5.4.4. Let L be a finite lattice. Then $mc(L) \leq e(J(L))$ with equality if and only if L is distributive.

To prove the inequality, Rival sets up a mapping from the linear extensions of $J(L)$ to the maximal chains in L. Assume that L has n join-irreducible elements. Consider a given linear extension of $J(L)$, $l = (x_1, x_2, ..., x_n)$ where $x_i < x_j$ in L implies $i < j$. Now let $c_0 =$ "0", the unique minimal element of L, let $c_1 = x_1$ and let $c_i = x_m \vee c_{i-1}$ where $m = \min\{j \mid x_j \vee c_{i-1} \text{ covers } c_{i-1}\}$. $\{c_0 < c_1 < ...\}$ will form a maximal chain in L. Rival shows that this mapping is onto which means that every maximal chain is produced by some linear extension, and then verifies that no extension can produce two different maximal chains. Stanley [Sy2] had previously demonstrated that if a finite lattice L is distributive then $mc(L) = e(J(L))$ and Rival adds that finite distributive lattices are the only lattices for which equality holds.

One special case of this result was previously discussed in Chapter 3. It is the idea that the number of linear extensions of a power set on n elements is equal to the number of maximal chains in the free distributive lattice on n generators. Since the poset of join irreducibles of the free distributive lattice on n elements is simply the power set on n elements, this result follows from Theorem 5.4.4.

This completes the summary of existing results dealing with the problem of counting chains in partially ordered sets. The following section will deal with a related chain counting problem, for which some new results can be found.

5.5 A CHAIN COUNTING PROBLEM

This section explores some results relating to a specific chain counting problem. In Section 2.4, Theorem 2.4.12 (due to Linek [Lk]) states that for every positive integer there is a partially ordered set of at most height one containing exactly n antichains. The truth of this statement leads the author of this thesis to make the following conjecture, which stems from a question of Bill Sands.

Conjecture 5.5.1. For every integer $n \geq 1$ there exists a partially ordered set of width at most two containing exactly n chains.

This conjecture arises from the question, "What restrictions may be placed on a family of posets such that for every integer $n \geq 1$, there is a partially ordered set in that family with exactly n chains?" Note that it is natural to restrict the width since if width is left unrestricted, we find that the poset which is an antichain on $n-1$ elements has exactly n chains.

It is interesting to note that Linek's theorem follows immediately from Conjecture 5.5.1. To demonstrate this, a few definitions must first be made. Let $P = (X,<)$ be a partially ordered set. Let Λ be a collection of linear orderings of X. We say that P is *realized by* Λ (and Λ *realizes* P) if for every $x, y \in X$,

a) $x < y$ in P if and only if $x < y$ in every $\lambda \in \Lambda$

and b) $x \mid y$ in P if and only if $x < y$ in some $\lambda_1 \in \Lambda$ and $y < x$ in some $\lambda_2 \in \Lambda$.

An alternate way of defining this is to consider the poset P and the elements of Λ as sets of ordered pairs, so that $(x, y) \in P$ if and only if $x < y$ in P for $x, y \in X$. Then it can be said that P is realized by Λ if $P = \cap\Lambda$. The *dimension* of P is defined as the smallest number m for which there is a set of m linear orderings of X which realize P. P is *reversible* if there exists a poset Q on X such that for all distinct $x, y \in X$, $x < y$ in P or $x > y$ in P if and only if $x \mid y$ in Q. Such a poset Q is called a *conjugate* partial order of P.

Now two lemmas are needed, the first of which is due to Dushnik and Miller [DM].

Lemma 5.5.2. A poset P is reversible if and only if the dimension of P is less than or equal to two.

Note that the reverse direction of this implication follows easily. If P has dimension 2, let λ_1 and λ_2 be a pair of linear extensions realizing P. Then reverse λ_1 to form λ_1', and the poset realized by λ_1' and λ_2 will be a conjugate partial order to P. If a poset P on n elements only has dimension 1, its conjugate will be the antichain on n elements.

The following lemma is due to Hiraguchi [H].

Lemma 5.5.3. The dimension of a poset is less than or equal to its width.

Combining Lemmas 5.5.2 and 5.5.3 gives that every width 2 poset is reversible. Now the following result can be demonstrated.

Lemma 5.5.4. Let $P = (X, <)$ be a partially ordered set of width 2, and let Q be a conjugate partial order of P. Then $c(P) = a(Q)$.

Proof. Let C be a chain in P. Then for $x, y \in C$ either $x < y$ or $y < x$ in P which implies $x \mid y$ in Q. Thus C must be an antichain in Q. Conversely, let A be an antichain in Q. Then for $x, y \in A$, $x \mid y$ in Q, so either $x < y$ or $y < x$ in P. Thus A is a chain in P. Since every chain in P forms an antichain in Q and every antichain in Q forms a chain in P, $c(P) = a(Q)$.

□

Note that if Q is a conjugate of P, then P will be a conjugate of Q. Since the largest antichain in P will contain two elements, the longest chain in Q will contain two elements, and thus the conjugate poset Q must have height 1. Thus if for a given integer $n \geq 1$ we can find a width two poset P such that $c(P) = n$, then $a(Q) = n$ where Q is a conjugate of P of height 1. Then if Conjecture 5.5.1 is true, we get that Theorem 2.4.12 must be true also.

If Conjecture 5.5.1 can be proved true, it will also provide an affirmative answer to the following open question of Linek [Lk]: "Does there exist $m > 1$ such that for any $n \geq 1$ there is a partial order of length 1 with n antichains and dimension at most m?" This follows by noting that a conjugate Q of a poset P of dimension two or less will also have dimension two or less. Thus if for an integer n, we can find a width two poset with $c(P) = n$, then taking a conjugate Q of P will give a height one poset with $a(Q) = n$ that also has dimension two or less. This implies that Linek's question can be answered affirmatively with $m = 2$.

The first thing to note in attempting to prove Conjecture 5.5.1 is the following:

Lemma 5.5.5. If for every prime integer $p \geq 1$ there exists a poset P of width two or less such that $c(P) = p$, then for every integer $n \geq 1$, there exists a poset P of width two or less such that $c(P) = n$.

Proof. It is easy to see that the linear sum of two posets P and Q has $c(P \oplus Q) = c(P) \cdot c(Q)$. Assume for every prime number p there is a poset containing exactly p chains. Then consider a composite number n. This can be written as a product of primes $p_1, ..., p_k$, so a poset can be constructed by linearly summing those posets of width at most two having $p_1, ..., p_k$ chains. Such a poset will also have width at most two. □

A weaker version of this lemma is useful in further discussions. We can say that if for every odd integer $q \geq 1$ and for $q = 2$ there exists a poset P such that $c(P) = q$, then for every integer $n \geq 1$, there exists a poset P such that $c(P) = n$. Note that the single element poset has two chains, so we only need find posets for each odd integer $q \geq 1$.

Rather than dealing immediately with Conjecture 5.5.1, one answer will be given to the original question which asks, "what restrictions can be placed on a family of posets in such a way that all possible values of n occur?"

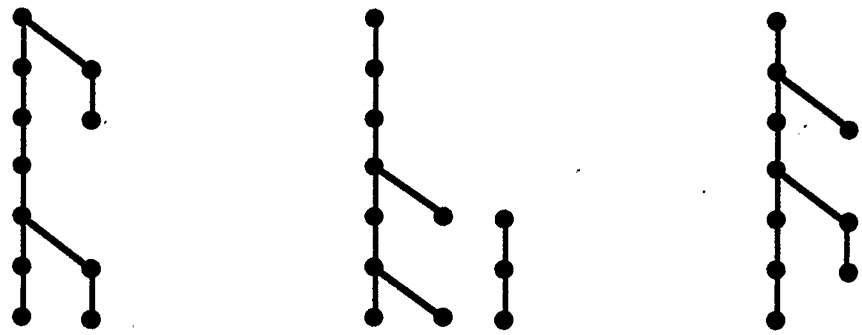
Let n be an integer and let $f(n)$ be the number of clusters of ones in n when it is expressed in binary form. By a "cluster of ones" we mean a maximal sequence of digits which are all ones. Define function g as follows.

$$g(n) = \begin{cases} f(n) + 1 & \text{if the first two digits of } n \text{ in binary form are 11;} \\ f(n) & \text{if the first two digits of } n \text{ in binary form are 10.} \end{cases}$$

As an example, let $n = 105$. Then in binary form, n will equal 1101001, so $f(n) = 3$ and thus $g(n) = 4$. Now the following result can be stated.

Lemma 5.5.6. For a given integer n, there is a poset P of width $g(n)$ containing exactly n chains.

Proof. This proof is by construction. Consider a given n in binary form. The position of a digit d in n, $p(d)$, will refer to the number of digits to the right of d. Let d be the leftmost digit of n and construct a chain on $p(d)$ vertices. Now consider the remaining $p(d)$ digits in n. Each cluster of ones in the remaining digits will produce a construction. Given a cluster of ones, let d_1 and d_2 be the leftmost and rightmost digits of the cluster. If $p(d_2) > 0$ then construct a chain by adding $d_1 - d_2 + 1$ elements below the $p(d_2)$'th vertex from the top in the original chain. If $p(d_2) = 0$ then form an isolated chain containing d_1-d_2+1 elements. Repeat this for all clusters of ones. It is straightforward to check that the resulting partially ordered set will contain exactly n chains. Figure 5.5.1 gives examples of the construction. □



$n = 230 = 11100110_2$ $g(n) = 3$ $n = 215 = 11010111_2$ $g(n) = 4$ $n = 280 = 100011000_2$ $g(n) = 3$

Figure 5.5.1

A weaker, but perhaps more intuitive idea which results from the previous lemma is the following.

Lemma 5.5.7. For a given integer n, there is a poset P containing exactly n chains such that

$$\text{width}(P) \le \left\lceil \frac{\log_2 n}{2} \right\rceil + 1.$$

Proof. In the construction used for the proof of Lemma 5.5.6, a chain is first created giving a poset of width 1. Then for each cluster of ones in the binary form of n with its first digit removed, another minimal element is added to the original chain thereby increasing the width of the poset by 1. In the worst case, the binary form of n with its first digit removed will have $\lceil (\log_2 n)/2 \rceil$ clusters of ones. Thus in the worst case, we get the width of P to be $\lceil (\log_2 n)/2 \rceil + 1$. □

From this point on, only posets of width two will be considered. The following lemma is a bounded version of the original conjecture.

Lemma 5.5.8. For every integer n with $1 \le n \le 10{,}000$ there exists a partially ordered set of width at most two containing exactly n chains.

Proof. Because of Lemma 5.5.5 and the comment which follows it, it is sufficient to only find the posets with an odd number of chains. Generalized diagrams of the constructions which produce these posets are shown in Figure 5.5.2. In this figure, each of variables a - g represents the number of elements including endpoints in the subchain indicated by the corresponding bracket.

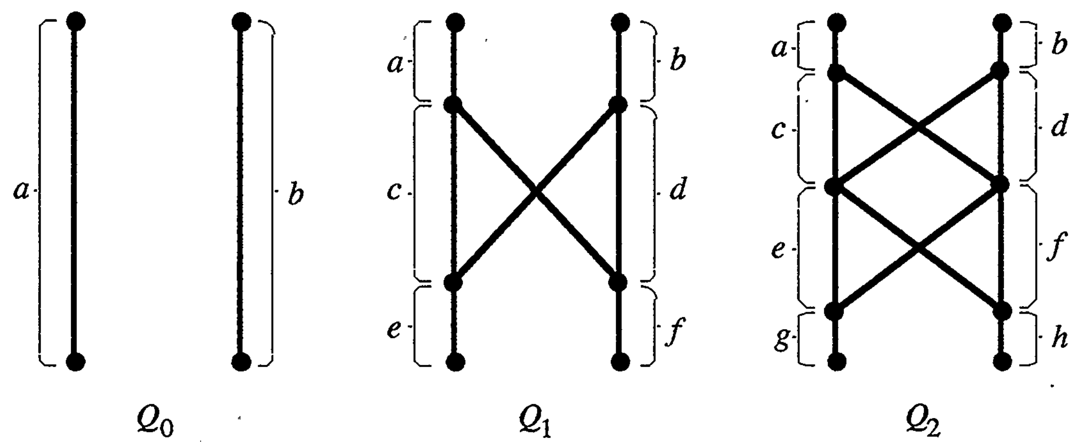


Figure 5.5.2

Note that Q_1 has constraints $a, b, e, f \geq 1$ and $c, d \geq 2$, and Q_2 has constraints $a, b, g, h \geq 1$ and $c, d, e, f \geq 2$. Enumerating the chains in each of these posets produces the following results.

$$c(Q_0) = 2^a + 2^b - 1$$

$$c(Q_1) = 2^{a+c+e-2} + 2^{b+d+f-2} - 1 + (2^a - 1)(2^f - 1) + (2^b - 1)(2^e - 1)$$

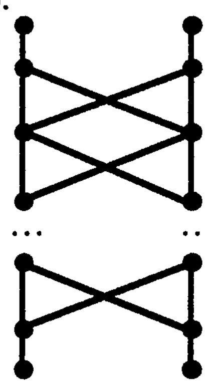
$$\begin{aligned} c(Q_2) = {} & 2^{a+c+e+g-3} + 2^{b+d+f+h-3} - 1 + (2^a - 1)(2^{f+h-1} - 1) + (2^{a+c-1} - 1)(2^h - 1) \\ & - (2^a - 1)(2^h - 1) + (2^b - 1)(2^{e+g-1} - 1) + (2^{b+d-1} - 1)(2^g - 1) \\ & - (2^b - 1)(2^g - 1) + (2^a - 1)(2^g - 1) + (2^b - 1)(2^h - 1). \end{aligned}$$

By creating a computer program to generate the chain counts of these types of posets, it is found that for n an odd number, posets of the form Q_0 produce $1 \leq n \leq 11$, posets of the form Q_1 produce $13 \leq n \leq 469$ and posets of the form Q_2 produce $471 \leq n \leq 9999$, with the exception of $n = 541$ and $n = 9073$. 541 is produced by a poset of the form Q_1, and $9073 = 43 \cdot 211$ so by Lemma 5.5.5 a poset with 9073 chains can be found by taking the linear sum of the posets having 23 and 211 chains. Thus all cases have been covered. □

Note that most values of n are given by more than one of the above types. For instance, all odd integers n with $195 \leq n \leq 469$ are produced by both posets of the form Q_1 and Q_2.

Given the previous results it is reasonable to strengthen Conjecture 5.5.1 to the following:

Conjecture 5.5.9. For every integer $n \geq 1$ there exists a partially ordered set of the form Q_m for some integer $m \geq 0$, containing exactly n chains, where Q_m is the poset containing m stacked crosses as in Figure 5.5.3.



Q_m

Figure 5.5.3

The previous lemma shows that the conjecture holds for all "small" values of n. Now it will be shown that the conjecture holds for some larger values of n.

Lemma 5.5.10. For every odd positive integer n containing 4 or fewer ones in its binary form, there is a partially ordered set of width at most 2 containing exactly n chains.

Proof. This is demonstrated by constructing the partially ordered sets shown in Figure 5.5.4.

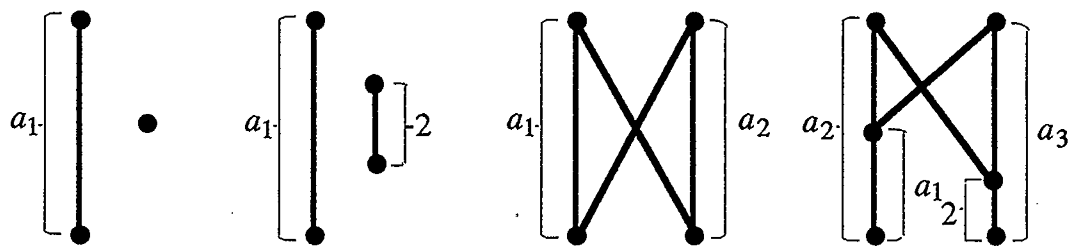


Figure 5.5.4

Enumerating the chains in each of these posets produces the following.

$c(R_0) = 2^{a_1} + 2^0$;

$c(R_1) = 2^{a_1} + 2^1 + 2^0$;

$c(R_2) = 2^{a_2} + 2^{a_1} + 2^0$ for $a_2, a_1 > 0$;

$c(R_3) = 2^{a_3} + 2^{a_2} + 2^{a_1} + 2^0$ for $a_3 > 2$, $a_2 > a_1 > 0$.

Note that the poset containing 2^0 chains is the empty poset, so we get a poset of width two or less for all odd values of n containing 4 or fewer ones in binary form. □

Attempts were made to find a general construction for posets which could produce the sum of any number of powers of two. Such a poset was found, however constraints on the combination of powers of 2 allowed, prevent it from covering all values of n. This construction is given in the following lemma.

Lemma 5.5.11. For an odd integer $n \geq 1$ there is a poset of width two containing exactly n chains if n can be written in one of the following forms:

a) $n = 2^{b_m} + 2^{b_{m-1}} + \ldots + 2^{b_1} + 2^c + 2^{c-1} + \ldots + 2^0$

where $b_1, \ldots, b_m$, c and m are non-negative integers subject to the constraints

i) $b_m > b_{m-1} > \ldots. > b_1 \geq m-1 > 1$,

ii) $b_1 \geq c+2$.

b) $n = 2^{b_m} + 2^{b_{m-1}} + \ldots + 2^{b_1} + 2^{p+m-1} + 2^{p+m-2} + \ldots + 2^{m-1} + 2^c + 2^{c-1} + \ldots + 2^0$

where $b_1, \ldots, b_m, m, c$ and p are non-negative integers subject to the constraints

i) $b_{m-1} > b_{m-2} \geq b_{m-3} \geq \ldots. \geq b_1 > p+m-1$,

ii) $b_m \geq c + m$,

iii) $m > c+2$.

Proof. Consider poset P shown in Figure 5.5.5.

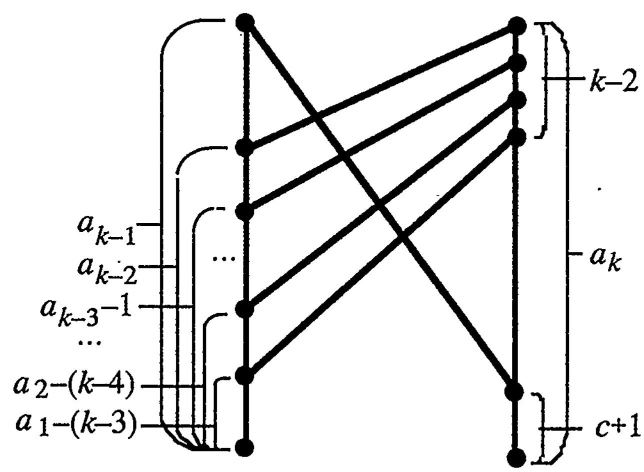


Figure 5.5.5

A systematic enumeration of the chains in P gives

$$c(P) = 2^{a_k} + 2^{a_{k-1}} + \ldots + 2^{a_1} + 2^{c+1} - 2^{k-2} - 2^0. \qquad (5.5.1)$$

Thus we get that for every positive integer n for which there exist integers $a_k, a_{k-1}, \ldots, a_1$, c, and k with $a_{k-1} > a_{k-2} \geq a_{k-3} \geq \ldots \geq a_1 \geq k-2$, $c + k - 1 \leq a_k$, $k \geq 3$ and $c \geq 0$ such that n can be written in the form of equation 5.5.1, then we can find a poset with exactly n chains. It will be shown that the set of n's satisfying equation 5.5.1 is the same set of n's satisfying the forms in Lemma 5.5.11.

To derive equation a), consider the situation when $k-2=a_1$. Substituting into equation (5.5.1) we get

$$c(P) = 2^{a_k} + 2^{a_{k-1}} + \ldots + 2^{a_2} + 2^c + 2^{c-1} + \ldots + 2^0.$$

Letting $m = k{-}1$ and $b_i = a_{i+1}$ for $1 \le i \le m$ produces equation a). The constraint $a_{k\text{-}1} > a_{k-2} \ge a_{k-3} \ge \ldots \ge a_1 \ge k-2$ along with constraint $k \ge 3$ and initial condition $k{-}2 = a_1$ produces $b_{m-1} > b_{m-2} \ge b_{m-3} \ge \ldots \ge b_1 \ge m-1 \ge 1$. The constraint $c + k - 1 \le a_k$ becomes $c + m \le b_m$. The first constraint can be strengthened to $b_m > b_{m-1} > b_{m-2} > b_{m\text{-}3} > \ldots > b_1 \ge m-1 \ge 1$ and then the second constraint strengthened to $b_1 \ge c{+}1$, since examination of the equations shows that the new constraints won't exclude values included in the original constraints.

For equation b), consider the situation when $a_1 > k{-}2 > c{+}1$. Equation (5.5.1) can then be rewritten as follows:

$$c(P) = 2^{a_k} + 2^{a_{k-1}} + \ldots + 2^{a_2} + 2^{a_1-1} + 2^{a_1-2} + \ldots + 2^{k-2} + 2^c + 2^{c-1} + \ldots + 2^0.$$

As before, let $m = k{-}1$ and $b_i = a_{i+1}$ for $1 \le i \le m$ and also let $p = a_1 - k + 1$ to produce equation b). The first constraint combined with initial condition $a_1 > k{-}2$ becomes $b_{m-1} > b_{m-2} \ge b_{m-3} \ge \ldots \ge b_1 > p + m - 1 > m-1$, which can be rewritten as $b_{m-1} > b_{m-2} \ge b_{m\text{-}3} \ge \ldots \ge b_1 > p + m - 1$ and $p \ge 0$. The initial condition $k{-}2 > c{+}1$ becomes $m > c{+}2$ and constraint $a_k \ge c + k - 1$ becomes $b_m \ge c + m$. Constraint $k \ge 3$ becomes $m \ge 2$ which is implied by $m \ge c + 2$ and $c \ge 0$ and so can be omitted. Thus the desired constraints have been found. □

Note that in the previous proof, only the cases where $a_1 = k-2$ and $a_1 > k{-}2 > c{+}1$ were considered. It is then natural to ask whether there are values of n which fit neither equation a) nor b) of Lemma 5.5.11 yet which can be produced by equation 5.5.1. By considering the remaining case which is $a_1 > k-2$, $c{+}\,1 \ge k-2$, it is found that all values of n fitting this case will also fit either equation a) or b) in Lemma 5.5.11. Thus all possible values for $c(P)$ achievable by a poset P of the form shown in Figure 5.5.5 are described by Lemma 5.5.11. As a matter of interest, the smallest n containing more than four powers of two which is not satisfied by equations a) or b) is $n = 245 = 2^7 + 2^6 + 2^5 + 2^4 + 2^2 + 2^0$.

Lemma 5.5.6 provides a width restriction on the set of all posets which still allows all values of n to occur as the number of chains in P for some poset P in that family. The work described in this section strengthens the idea that Conjecture 5.5.1 must be true. Perhaps further work in this area will yield a conclusive result.

This problem marks the completion of this summary of results dealing with the counting of chains, antichains and linear extensions of partially ordered sets. Though such results abound, there remain many open problems in need of solutions and conjectures in need of validation. Further work in this area may eventually close some of the gaps in our knowledge of this topic.

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