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MEAN PERIODIC FUNCTIONS AND FUNCTIONAL DIFFERENTIAL EQUATIONS

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Mean Periodic Functions and Functional Differential Equations", submitted by P. G. Laird in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

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ABSTRACT

This thesis is primarily concerned with the continuous complex-valued mean periodic functions of a real variable that were innovated by Laurent Schwartz and also considered by J. P. Kahane. An account of such functions, together with some results from a previous thesis of the author is given in Chapter 0 .

Our results concerning such functions fall into two main categories: new properties of the functions themselves and a consideration of some systems of functional differential equations (Chapter 2) and Volterra integral equations of convolution type (Chapter 3) that admit continuous mean periodic functions as solutions.

Throughout Chapters 0 - 4, frequent reference is made to the truncated convolution product of two functions f, g defined as

$$f \circledast g : t \to \int_0^t f(t - r)g(r)dr \qquad \text{for all real } t .$$

Several function spaces, including the set of continuous mean periodic functions, are identified as algebras with the operations of addition and truncated convolution. Properties of this convolution product and subalgebras are considered in Chapter 1 whereas some ideals of such algebras are described in Chapter 4 .

Chapter 5 is concerned with entire functions and the entire mean periodic functions of a complex variable z . The theory of entire mean periodic functions is due mainly to Laurent Schwartz and an outline of his theory is included in Chapter 5 . It is shown that

if f, g are entire functions, so is the 'truncated convolution' product

$$f \circledast g : z \to \int_0^z f(z - \xi)g(\xi)d\xi \quad .$$

Chapter 5 also contains other properties and applications of entire mean periodic functions that are similar to some of the properties and applications of continuous mean periodic functions given in the earlier chapters.

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Chapter Zero

PRELIMINARIES

This introductory chapter serves to familiarize the reader with some concepts of continuous mean periodic functions and relevant properties of continuous functions and measures (Sections 0.1 - 0.4) that will be required in subsequent chapters.

The term, mean periodic function, was coined by Delsarte [1] in 1935 to denote a complex-valued continuous function, f, of a real variable t that satisfies an integral equation of the form

$$\int f(t - r)\, k(r)\, dr = 0 \qquad \text{for all } t$$

where k is a not identically zero continuous function with compact support. This name is suggested by the fact that if f is a periodic function of period τ then its average is zero if and only if $\int_0^{\tau} f(t - r)\, dr = 0$ for all t.

A more complete theory was presented by L. Schwartz [1] in 1947. For any topological vector space, E, of complex-valued functions of a real variable, the function $f \in E$ is defined to be mean periodic in E if the linear combination of the translates is not dense in E. For $E = C(R)$, the space of all complex-valued continuous functions defined on the real line and equipped with the topology of convergence uniform on all compact subsets of R, Schwartz obtains properties similar to those of Delsarte and shows that the general and intrinsic definition is equivalent to the following.

A necessary and sufficient condition that $f \in C(R)$ be mean periodic is that there exists a non-zero measure, μ, with compact support for which

$$\int f(t - r)\, d\mu(r) = 0 \qquad \text{for all } t \quad .$$

The main property of mean periodic functions in $C(R)$ is that they are limits of linear combinations of exponential monomials. The means Kahane [1], [2] used to prove this are summarised in the Section 0.5 and make use of the Carleman transform. This approach to the theory of mean periodic functions in $C(R)$ relies on the theory of analytic functions of one complex variable and extensively uses the Fourier transform of a measure. In this thesis, $\hat{\mu}$ is defined as $\hat{\mu} : z \to \int e^{-zt} d\mu(t)$ instead of $\int e^{-2\pi izt} d\mu(t)$ (Schwartz) or $\int e^{-izt} d\mu(t)$ (Kahane); consequently, statements like " $\hat{\mu}$ is bounded on the real axis" as found in Schwartz and Kahane will read here as " $\hat{\mu}$ is bounded on the imaginary axis" .

The reader's attention is drawn to the fact that the contents of this chapter, and this chapter only, have been adapted from the author's M.Sc. Thesis (Laird, [1]) .

§0.1 The Space $C(R)$

We denote by $C(R)$ the complex vector space of all complex valued continuous functions defined on the real line (R) and equipped with the topology of convergence uniform on all compact subsets of R. This topology may be defined by the seminorms

$$p_k(f) = \sup\{|f(t)| : -k \leq t \leq k\} \quad .$$

As the sets $\{f : f \in C(R) \text{ and } p_k(f) < \varepsilon\}$, formed when ε ranges over all positive numbers and k ranges over the set N of positive integers, are convex and form a base of neighbourhoods at the origin, $C(R)$ is locally convex. This topology for $C(R)$ is Hausdorff, since $p_k(f) = 0$ for each $k \in N \Leftrightarrow f = 0$.

A bounded metric defining the given topology is

$$\rho(f, g) = \sum_{k=1}^{\infty} 2^{-k} \frac{p_k(f - g)}{1 + p_k(f - g)}$$

with $\rho(f_n, 0) \to 0 \Leftrightarrow p_k(f_n) \to 0$ as $n \to \infty$ for each $k \in N$.

$$\Leftrightarrow f_n \to 0 \text{ in } C(R) \text{ as } n \to \infty .$$

$$\overset{\text{def.}}{\Leftrightarrow} f_n \text{ converges "locally uniformly" to zero as } n \to \infty .$$

$C(R)$ is a complete space, for if $\{f_n\}$ is a Cauchy sequence of elements of $C(R)$, i.e., $\rho(f_m, f_n) \to 0$ as $m, n \to \infty$, then there exists an $f \in C(R)$ such that $\rho(f, f_n) \to 0$ as $n \to \infty$. This follows since for each $t \in R$, $\{f_n(t)\}$ is a Cauchy sequence of complex numbers and so convergent to $f(t)$, say. Then $\{f_n\}$

converges to f locally uniformly and so f is continuous. Thus, $C(R)$ is a Fréchet space.

The support of a function f is the closure of the set $\{t : f(t) \neq 0\}$. The set of all complex-valued continuous functions having compact supports is denoted by $C_C(R)$. As well, $C^{\infty}(R)$ denotes the set of all indefinitely differentiable complex-valued functions defined on R and $C_C^{\infty}(R)$ denotes those $C^{\infty}(R)$ functions with a compact support. $C(R_+)$ is used to denote the set of all continuous complex-valued functions defined on $R_+ = [0, \infty)$.

The truncated convolution product of two continuous functions x and y is defined as

$$x \circledast y : t \to \int_0^t x(t - r)\, y(r)\, dr$$

and $x \circledast y$ is continuous. With addition and this product, $C(R)$ is a commutative ring and an algebra over ¢ (see Erdélyi, [1], page 15 for details of $C(R_+)$ that apply to $C(R)$.)

§0.2 M_C, The Dual Space of $C(R)$

The term, "measure", will be used in this thesis to denote a continuous linear functional on the complex topological linear space $C(R)$. The set of all such measures is denoted by M_C. The relevant properties of these measures are outlined here; for full details of their theory involving integration theory and Radon measures on $C(R)$, see, for example, Edwards [1], Chapter 4.

Any such measure, μ, assigns to each $f \in C(R)$ a complex number $\mu(f)$ in such a way that:-

$$\mu(af) = a\mu(f) \qquad \text{for every complex number } a,$$

$$\mu(f + g) = \mu(f) + \mu(g) \qquad \text{for } f,\ g \in C(R), \text{ and}$$

$$\mu(f_n) \to \mu(f) \qquad \text{when } f_n \to f \text{ in } C(R) \text{ as } n \to \infty .$$

A positive measure, λ, is one for which $\lambda(f) \geq 0$ whenever $f \in C(R)$ and $f \geq 0$. It is customary and often convenient to write $\mu(f)$ as $\int f d\mu$ or $\int f(t) d\mu(t)$.

For a linear functional, μ, on $C(R)$ to be continuous, (i.e., to be a measure in the above sense), it is necessary and sufficient that there exist a compact set K and a non-negative real number C such that

$$(0.1) \quad |\mu(f)| \leq C \cdot \sup\{|f(t)| : t \in K\} \quad \text{for each } f \in C(R) .$$

For a given measure μ, there exists a smallest compact set K for which (0.1) holds for a suitable $C = C_K$, and K is called the support of μ. If a continuous function, f, is zero on K, then $\mu(f)$ is zero. The smallest closed interval containing K, i.e., the closed convex envelope of K, will be called the <u>segment of support</u> of μ.

It can be shown that, for any measure μ with compact support K, there exists a complex valued function u defined on R that is of bounded variation and constant on each component interval of $R \setminus K$ for which

$$\mu(f) = \int_K f(t)\, du(t) \qquad \text{for each } f \in C(R) .$$

Conversely, any such function u will define through the above formula a measure μ .

If v is a complex-valued Lebesgue integrable function of compact support, then a measure μ is defined by

$$\mu(f) = \int f(t)\, v(t)\, dt \qquad \text{for each } f \in C(R) \quad .$$

Such a measure is said to have density v relative to Lebesgue measure and, when there is no possibility of confusion, we use μ to denote both the function and the measure. However, the supports of v and μ may not agree when the support of a function is defined as in §0.1 .

The norm of the measure μ with compact support K is defined as

$$||\mu|| = \sup\{|\mu(f)| : f \in C(R) \text{ and } |f(t)| \leq 1 \text{ for all } t \in K\} .$$

In Chapters 2 and 3, we will make use of distributions. Let $C^n(R)$ denote the complex vector space of all complex-valued functions on R that have continuous n^{th} order derivatives, equipped with the topology defined by the semi norms

$$p_k^{(n)}(f) = \sup\{|D^p f(t)| : -k \leq t \leq k, \quad p = 0, 1, \ldots, n\}$$

for $k \in N$ and $f \in C^n(R)$.

In this thesis, the term 'distribution' will denote any linear functional on $C(R)$ that is continuous on $C^n(R)$ for at least one non-negative integer n. (This restricted concept of a distribution is consistent with the more general definition given by Edwards [1], Chapter 5). Thus, a distribution which is also a continuous linear functional on $C(R)$ is a measure. The distributional derivative, DT, of a distribution T is defined by

$$DT(f) = - T(f') \qquad \text{for each} \qquad f \in C_C^\infty(R)$$

and is also a distribution.

§0.3 The Fourier-Laplace Transform

The Fourier-Laplace transform of a measure μ is defined here as

$$\hat{\mu}(z) = \int e^{-zt}\, d\mu(t) \quad .$$

This transform proves to possess certain properties. If $M(z) = \hat{\mu}(z)$:

(i) M is an entire function. To see this, set

$$f_h : t \to [\exp(-t(z+h)) - \exp(-zt)]/h + t\exp(-tz) \quad .$$

Now $p_n(f_h) \to 0$ as $h \to 0$ for each $n \in N$ and so $\mu(f_h) \to 0$ as $h \to 0$. Hence M is complex-differentiable at each point and so analytic in the complex plane.

(ii) M is of exponential type and bounded on the imaginary axis. For if μ has support $K \subset [-L, L]$ and $z = x + iy$, $|M(z)| \leq \|\mu\| \exp(L|x|)$ and so M is of order one and type not exceeding L; also $|M(iy)| \leq \|\mu\|$.

(iii) From (i) and (ii), the zeros of M have no finite limit point; in fact, by Hadamard's factorization theorem, (see Titchmarch [1], page 250) if the zeros of $M(z)$ are $a_n \neq 0$ of orders p_n respectively,

$(n = 1, 2, \ldots)$, and if $M(z)$ has a zero of order k at the origin ($k = 0$ if $M(0) \neq 0$), then

$$M(z) = Az^k e^{cz} \prod_{n=1}^{\infty} (1 - z/a_n)^{p_n} \exp(zp_n/a_n)$$

where c is imaginary and

(0.2) $$\Sigma\, p_n/|a_n|^2 < \infty \quad .$$

(iv) From (ii), the entire function M satisfies

$$\overline{\lim_{|z|\to\infty}} \frac{\log|M(z)|}{|z|} \leq L \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\max(\log|M(iy)|,\ 0)}{1+y^2}\, dy < \infty$$

so by Levinson [1], pages 25-28,

$$\lim_{R\to\infty} \int_1^R \log|M(iy)M(-iy)| \frac{1}{y^2}\, dy$$ exists and is finite. Then

(a) the set of zeros of M has a finite density $\leq L$, ie..,

$\lim\limits_{n\to\infty} n(r)/r$ exists and does not exceed L,, where $n(r)$ is the number of zeros of M, counted according to their order, inside the circle $|z| = r$.

(b) Let the non-vanishing zeros of M be $a_n = r_n \exp(i\theta_n)$. Then $\Sigma p_n |\cos\theta_n|/r_n \to \infty$, showing that for any sector S containing the imaginary axis,

(0.3) $$\sum_{a_n \notin S} p_n/|a_n| < \infty \quad .$$

For a distribution, T, the Fourier-Laplace transform will still be an entire function of exponential type. However, $\hat{T}$ need only be of polynomial order on the imaginary axis.

Finally, we note that $\hat{\mu} = 0 \Rightarrow \mu = 0$.

§0.4 Convolution Products

The convolution product of a measure $\mu \in M_C$ and a function $f \in C(R)$ is defined as

$$\mu * f : t \to \int_K f(t - r)\, d\mu(r) = \mu(T_t \check{f})$$

where K is the support of μ and the translation operator, T_t, takes f into $T_t f : r \to f(r - t)$, and $\check{f} : t \to f(-t)$. This product is a continuous function, for

$$|\mu * f(r) - \mu * f(s)| \leq \|\mu\| \sup_{t \in K} |f(t - r) - f(t - s)| \quad .$$

The function, $(\mu, f) \to \mu * f$ is a bilinear map from $M_C \times C(R)$ into $C(R)$.

If $f_n \to 0$ in $C(R)$ as $n \to \infty$, then $\mu * f_n \to 0$ in $C(R)$. For let ε be any positive number, L be any compact subset of R and let μ have support K . An $m \in N$ can be chosen such that when $n > m$,

$$\sup\{|f_n(t - r)| : t \in L,\ r \in K\} < \varepsilon / \|\mu\|$$

whence $|\mu * f_n(t)| < \varepsilon$ for $t \in L$.

If $\alpha \in R$, the α-translate of a measure μ is defined by

$$T_\alpha \mu(f) = \mu(T_{-\alpha} f) \qquad \text{for each } f \in C(R) \quad .$$

The convolution product of two functions f, g is defined, when it exists, as

$$f * g : t \to \int_{-\infty}^{\infty} f(t - r)g(r)dr \quad .$$

The convolution product of two measures μ, λ is denoted by $\mu * \lambda$ and is a measure. For measures, this operation is associative and commutative. If μ and λ have segments of supports $[\alpha, \beta]$ and $[\gamma, \delta]$, then $\mu * \lambda$ has segment of support $[\alpha + \gamma, \beta + \delta]$. Also $(\mu * \lambda)\hat{} = \hat{\mu} \cdot \hat{\lambda}$ ($\cdot$ denoting, of course, the pointwise product). If T is any distribution of compact support, then for all $\rho \in C_C^\infty(R)$, $T * \rho \in C_C^\infty(R)$.

We will make use of the Dirac measures, either $\delta = \delta_0$ placed at the origin or δ_α placed at any point, α, on the real line. δ_α has support $\{\alpha\}$ and for each $f \in C(R)$, we have

$$\delta(f) = f(0) \qquad \delta_\alpha(f) = f(\alpha)$$

$$\delta * f = f \qquad \delta_\alpha * f = T_\alpha f$$

$$\delta_\alpha * \mu = T_\alpha \mu \qquad \delta_\alpha * \delta_\beta = \delta_{\alpha+\beta}$$

with $D\delta * f = f'$ when f is absolutely continuous.

We can show that $M_C(+, *)$ is an integral domain.

The following formulae are valid for distributions as well as measures:

$$T_\alpha(\mu * \lambda) = T_\alpha \mu * \lambda = \mu * T_\alpha \lambda$$

$$T_\alpha(\mu * f) = T_\alpha \mu * f = \mu * T_\alpha f$$

$$D(\mu * \lambda) = D\mu * \lambda = \mu * D\lambda \qquad \text{and}$$

$$D(\mu * f) = D\mu * f = \mu * f' \quad \text{when } f \text{ is absolutely continuous.}$$

§0.5 Continuous Mean Periodic Functions

For $f \in C(R)$, let V_f denote the closed subspace of $C(R)$ generated by f and its translates.

DEFINITION. *If* $f \in C(R)$ *and if* $V_f \neq C(R)$, *then* f *is mean periodic.*

From this definition, it follows that a necessary and sufficient condition for $f \in C(R)$ to be mean periodic is that there exists a non-zero measure μ such that $\mu * f = 0$, i.e.,

$$\int f(t - r)\, d\mu(r) = 0 \qquad \text{for all real } t .$$

DEFINITION. *An exponential polynomial is a finite linear combination of terms* $u_n e_a : t \to t^n e^{at}$ *where* n *is any non-negative integer and* a *is any complex number.*

If f is an exponential polynomial, then the subspace generated by f and its translates is finite dimensional and so closed. Then $V_f \neq C(R)$ and so <u>any exponential polynomial is mean periodic</u>.

In the remainder of this chapter, let f denote any mean periodic function in $C(R)$ and let μ be any non-zero measure for which $\mu * f = 0$. Put

$$f^+(t) = f(t) \quad \text{for } t \geq 0 \quad \text{and} \quad f^+(t) = 0 \quad \text{for } t < 0 .$$

Also put $g = \mu * f^+$ and $f^- = f - f^+$. Then $g = -\mu * f^-$ and

g is an integrable function of compact support contained in the segment of support of μ. As in Section 0.3, put

$$\hat{\mu}(z) = \int e^{-zt}\, d\mu(t) \,. \quad \text{Also let } \hat{g}(z) = \int e^{-zt}\, g(t)\, dt,$$

so that $\hat{\mu}(z)$ and $\hat{g}(z)$ are entire functions of exponential type.

DEFINITION. *The Kahane Transform of a continuous mean periodic function,* f, *is* $K(f)$ *where*

$$K(f)(z) = \hat{g}(z)/\hat{\mu}(z) \quad .$$

Note that $K(f)$ is independent of the choice of non-zero measure μ for which $\mu * f = 0$.

DEFINITION. *The spectral set,* S_f, *of a continuous mean periodic function,* f, *is the set of all poles of the Kahane transform,* $K(f)$. *The spectrum,* Λ_f, *of* f *is the set of pairs* (a, p) *where* $\alpha \in S_f$ *and* p *is the order of the pole* $K(f)(z)$ *at* $z = a$.

If $(a, p) \in \Lambda_f$ and $\mu * f = 0$, then $\hat{\mu}(z)$ has a zero at $z = a$ of order $\geq p$. So the conditions of Section 0.3 pertaining to the zero's of $\hat{\mu}(z)$ apply to Λ_f. Thus, if $\Lambda_f = \{(a_k, p_k)\}_{k=1}^{\infty}$, then (0.2) and (0.3) hold.

If $f^+(t)\, e^{\alpha t}$ and $f^-(t)\, e^{\beta t}$ are bounded for all real t, then the Carleman transforms

$$F^+(z) = \int f^+(t)\, e^{-zt}\, dt \quad \text{and} \quad F^-(z) = \int f^-(t)\, e^{-zt}\, dt$$

are defined, analytic, and coincide with the Kahane transform on the respective half planes $x > -\alpha$ and $x < -\beta$ when $z = x + iy$.

We now state four theorems which serve to illustrate the steps that were taken by Kahane [1], [2], to show that any mean periodic function f may be expressed as the locally uniform limit of a sequence of exponential polynomials in V_f.

THEOREM 0.1 *Let* V *be a closed translation invariant subspace of* $C(R)$. *If* β *is any fixed real number and if* $\mu * g(\beta) = 0$ *for every* $\mu \in M_C$ *that satisfies* $\mu * V = \{0\}$, *then* $g \in V$.

THEOREM 0.2 *The following are equivalent:*

a) $u_n e_a \in V_f$,

b) for each non-zero measure μ *with* $\mu * f = 0$,

$$D^k \hat{\mu}(z) = 0 \qquad \text{for } z = a \text{ and } k = 0, 1, \ldots n,$$

c) $K(f)(z)$ *has a pole at* $z = a$ *of order exceeding* n, *and*

d) $(a, p) \in \Lambda_f$ *where* $n < p$.

THEOREM 0.3 *If* f *is a mean periodic function whose spectrum is void (i.e.,* $K(f)(z)$ *is entire), then* $f = 0$.

THEOREM 0.4 f *belongs to the closed subspace generated by the exponential polynomials of* V_f.

DEFINITION. *The mean period,* L_f, *of a mean periodic function* f *is the infimum of the lengths of the segments of supports of the measures* $\mu \in M_C$ *such that* $\mu \neq 0$ *and* $\mu * f = 0$.

Some properties of the mean period given by J-P. Kahane ([2], pages 29, 33) may be summarized as:

THEOREM 0.5 *(i) mean periodic functions with the same spectrum have the same mean period.*

(ii) If f *is zero on an interval of length exceeding its mean period, then* $f = 0$.

(iii) $\Lambda_f = \{(a_n, 1) : n = 1, 2, \ldots\}$ *and* $\Sigma 1/|a_n| < \infty \Rightarrow L_f = 0$.

(iv) If f *is a continuous periodic function of period* τ *and if all Fourier coefficients of* f *are non-zero, then* $L_f = \tau$.

Some properties of mean periodic functions that have been stated and proved by the author elsewhere (Laird [1], Chapter 2, and [2]) may be summarized as follows. Let MP denote the set of all mean periodic functions in C(R) and let MP_o denote the set of all mean periodic functions with mean period zero. Also let MQ denote the set of all exponential polynomials.

THEOREM 0.6 *Let* f, $g \in MP$. *Then*

a) $e_a f$, $\int_o^t f(r)dr$ *and* f' *(when it is continuous) are mean periodic and each has the same mean period as* f,

b) If $h \in MQ$, *then* $fh \in MP$,

c) MQ, MP_o *and* MP *are subalgebras of* C(R) $(+, \circledast)$ *and each is* *dense in* C(R) . *Also*

$$MQ \subset MP_o \subset MP \subset C(R)$$

and the inclusion is proper in each case.

It is worth remarking that fg need not be mean periodic when f and g are mean periodic. This can be seen by the following example: if f and g are continuous periodic functions of periods α and β where α/β is irrational and if f and g have all Fourier coefficients non-zero, then fg is not mean periodic.

We also give some examples, due to L. Schwartz, [1], of continuous functions that are not mean periodic. They are:

a) $\exp(t^2)$

b) any non-zero absolutely integrable function (and so any $C_c(R)$ function), and

c) $g : t \to \sum_{n=1}^{\infty} a_n \exp(i\alpha_n t)$ where $\Sigma|a_n| < \infty$ and $\{\alpha_n\}$ is any sequence of real numbers with a finite limit point ($a_n \neq 0$).

This last mentioned function is a uniformly almost periodic function. Several properties of these well known functions that contrast with continuous mean periodic functions are as follows.

Uniformly almost periodic functions are uniformly continuous and bounded. The set of such functions forms a Complex Banach space with the norm $\|f\| = \sup\{|f(t)| : t \in R\}$. When f is almost periodic, $e_{i\alpha}f$ is almost periodic when α is real, f' is almost periodic when it is uniformly continuous and $\int f$ is almost periodic when it is bounded. When f and g are almost periodic, fg is almost periodic but in general $f \circledast g$ is not bounded and so not almost periodic.

An example of an almost periodic function that is not mean periodic has already been given. The exponential function is not bounded on R and provides an example of a mean periodic function that is not almost periodic.

It has been shown by Kahane [3] that a bounded uniformly continuous mean periodic function is uniformly almost periodic.

Chapter One

FURTHER PROPERTIES OF MEAN PERIODIC FUNCTIONS

In this chapter, we first discuss in Section 1.1 some interesting properties (see Propositions 1.1, 1.2 and 1.3) of Kahane's Transform of mean periodic functions (defined in Chapter 0). These properties are analogous to those of the Laplace Transform of continuous functions.

Section 1.2 is mainly concerned with the mean period of the truncated convolution product of two mean periodic functions. In Section 1.3, properties are given of some subalgebras of the algebras MP, MP_0 and MQ taken with the operations of addition and truncated convolution. These algebras are also commutative rings with no non-zero divisors of zero. In Section 1.4, it is noted that it is possible to construct fields of convolution quotients from any one of these three rings in the same manner that Mikusinski (c.f. Erdélyi, [1]) constructed a field of convolution quotients from the ring $C(R_+)$.

§1.1 Kahane's Transform of Mean Periodic Functions

When f is a non-zero mean periodic function, its Kahane Transform, $K(f)(z)$, is a meromorphic function with at least one pole. Many of the statements in this section will be expressed as identities involving meromorphic functions. If G and H are meromorphic functions, the statement "$G = H$" will indicate that if $G(z)$ does not have a pole at $z = a$, then neither does $H(z)$ and $G(a) = H(a)$, whilst if $G(z)$ has a pole of order n at $z = a$, then so has $H(z)$. To establish such an identity, it is sufficient to find an entire function, say M, such that $M \cdot G$ and $M \cdot H$ are entire and $M \cdot G = M \cdot H$.

In the proofs of these properties, we will assume without further reference, the following facts:

(i) convolution of measures with compact supports and/or functions whose supports lie on a fixed half line is an associative and commutative operation (Gel'fand and Shilov, [1], page 104); and

(ii) the Fourier-Laplace transform of a measure or function with a compact support is an entire function (Chapter 0, §0.2).

PROPOSITION 1.1 *Let* f, g *be mean periodic functions and let* a, b *be complex numbers. Then* $af + bg$, $f \circledast g$ *are mean periodic, and*

$$K(af + bg) = aK(f) + bK(g) \quad \textit{and} \quad K(f \circledast g) = K(f) \cdot K(g)$$

PROOF. Let μ, λ be non-zero measures so that $\mu * f = 0$, $\lambda * g = 0$. Then $\mu * f^+$, $\lambda * g^+$ have compact support.

As $\mu * \lambda * (af + bg) = 0$ and as $\mu * \lambda$ is a non-zero measure, $af + bg$ is mean periodic. Also $\mu * \lambda * (af + bg)^+$ has compact support and is equal to $a\lambda * (\mu * f^+) + b\mu * (\lambda * g^+)$. Hence

$$(\mu * \lambda)^\wedge \cdot K(af + bg), \quad a\hat{\lambda} \cdot \hat{\mu} \cdot K(f) + b\hat{\mu} \cdot \hat{\lambda} \cdot K(g)$$

are entire functions and equal to one another. Since

$(\mu * \lambda)\hat{} = \hat{\mu} \cdot \hat{\lambda}$ is also an entire function,
$K(af + bg) = aK(f) + bK(g)$.

For $f \circledast g$, we note that

$$(f \circledast g)^+ = f^+ * g^+ \quad \text{and} \quad (f \circledast g)^- = - f^- * g^- .$$

so that

$$\begin{aligned} \mu * \lambda * (f \circledast g) &= \lambda * (\mu * f^+ * g^+ - \mu * f^- * g^-) \\ &= \lambda * (\mu * f^+) * g = 0 \quad . \end{aligned}$$

Thus $f \circledast g$ is mean periodic. Also, $\mu * \lambda * (f \circledast g)^+$ has compact support and is equal to $\mu * f^+ * \lambda * g^+$. Hence

$$(\mu * \lambda)\hat{} \cdot K(f \circledast g) = \hat{\mu}\, K(f)\, \hat{\lambda}\, K(g)$$

and so

$$K(f \circledast g) = K(f) \cdot K(g) \quad .$$

PROPOSITION 1.2 *Let* f *be mean periodic. Then*

a) If $DF = f$, F *is mean periodic and*

$$K(F)(z) = (K(f)(z) + F(0))/z ,$$

b) If Df *is continuous, it is mean periodic and*

$$K(Df)(z) = zK(f)(z) - f(0) ,$$

c) If a *is any complex number,* $e_a f : t \to e^{at} f(t)$ *is mean periodic and* $K(e_a f)(z) = K(f)(z - a)$,

d) If γ *is any real number,* $T_\gamma f : t \to f(t - \gamma)$ *is mean periodic* *and* $K(T_\gamma f)(z) = e^{-z\gamma} K(f)(z) + H(z)$ *where* H *is an entire function, and*

e) If α *is any real number,* $f\circ\alpha : t \to f(\alpha t)$ *is mean periodic and when* $\alpha \neq 0$, $K(f\circ\alpha)(z) = \frac{1}{\alpha} K(f)(\frac{z}{\alpha})$.

PROOF. Firstly, we show that $K(e)(z) = 1/z$ where $e : t \to 1$. If β is any positive number and if λ is the measure, defined by $\lambda(g) = \int_{-\beta}^{\beta} r\, g(r)\, dr$ for each $g \in C(R)$, then $\lambda * e = 0$. A little calculation shows that

$$\hat{\lambda}(z) = 2(\sinh(\beta z) - \beta z \cosh(\beta z))/z^2 ,$$

$$\lambda * e^+(t) = \tfrac{1}{2}(t^2 - \beta^2) \quad\text{and}\quad (\lambda * e^+)\hat{}\,(z) = \hat{\lambda}(z)/z$$

and so

$$K(e)(z) = 1/z \quad .$$

If $DF = f$, then $F(t) = F(0) + \int_0^t f(r)\, dr$ or $F = e \circledast f + F(0)e$. Thus, by Proposition 1.1 , F is mean periodic and $K(F)(z) = (K(f)(z) + F(0))/z$.

In the remainder of the proof, we assume that μ is a non-zero measure and $\mu * f = 0$. If Df is continuous, then $\mu * Df = 0$ and so Df is mean periodic. As $f = f(0)e + e \circledast Df$,

$$K(f)(z) = (f(0) + K(Df)(z))/z$$

and so

$$K(Df)(z) = z\, K(f)(z) - f(0) .$$

With $\mu * f = 0$, $(e_a\mu) * (e_a f) = 0$ where

$(e_a\mu)(g) = \mu(e_a g)$ for each $g \in C(R)$. With

$(e_a\mu) * (e_a f)^+ = e_a(\mu * f^+)$ and $(e_a\mu)\hat{}(z) = \hat{\mu}(z - a)$, it follows

that $K(e_a f)(z) = K(f)(z - a)$.

Also, $(T_\gamma\mu) * f = 0$ where $(T_\gamma\mu)(g) = \mu(T_{-\gamma}g)$ for each $g \in C(R)$ and $\mu * T_\gamma f = 0$. From $(T_\gamma f)^+ = T_\gamma(f^+) + h$ where h is a function with a compact support,

$$\mu * (T_\gamma f)^+ = \mu * T_\gamma(f^+) + \mu * h = T_\gamma\mu * f^+ + \mu * h \quad .$$

Moreover, $(T_\gamma\mu)\hat{}(z) = (\delta_\gamma * \mu)(z) = e^{-\gamma z}\,\hat{\mu}(z)$ so that

$$K(T_\gamma f)(z) = (\mu * (T_\gamma f)^+)\hat{}(z) \;/\; \hat{\mu}(z)$$

$$= ((T_\gamma\mu) * f^+)\hat{}(z)\; e^{-\gamma z} \;/\; (T_\gamma\mu)\hat{}(z) + \hat{h}(z)$$

$$= e^{-\gamma z}\, K(f)(z) + H(z) \quad .$$

Here H, being the Fourier-Laplace transform of a function with a compact support, is an entire function.

For (e), by convolving μ, if need be, with a suitable twice differentiable function, we may and shall suppose that $\phi * f = 0$ where ϕ is a non-zero continuous function with support in $[0, L]$ for some $L > 0$.

With $\int_0^L f(t - r)\, \phi(r)\, dr = 0$,

if $r = \alpha s$, $\quad \alpha \int_0^{L/\alpha} f(t - \alpha s)\, \phi(\alpha s)\, ds = 0 \quad ,$

so $$\int_0^{L/\alpha} f\circ\alpha\, (t/\alpha - s)\, \psi(s)\, ds = 0$$

where $\psi = \phi o \alpha$. As $\psi \neq 0$, $fo\alpha$ is mean periodic. After some routine calculations,

$$(\psi * (fo\alpha)^{+})\hat{}(z) = \frac{1}{\alpha}(\phi * f^{+})\hat{}(\frac{z}{\alpha}) \quad \text{and} \quad \hat{\psi}(z) = \hat{\phi}(\frac{z}{\alpha})$$

and so

$$K(fo\alpha)(z) = \frac{1}{\alpha} K(f)(\frac{z}{\alpha}) \quad .$$

COROLLARY. *If* f *is mean periodic, and if* α *is a real non-zero number, then* $L_{fo\alpha} = L_f/|\alpha|$.

PROOF. In the proof of (e) above, for any $\varepsilon > 0$, L may be chosen so that $L < L_f + \varepsilon$. Since $\psi * (fo\alpha) = 0$, $\psi \neq 0$ and ψ has support in $[0, L/\alpha]$, (or $[L/\alpha, 0]$ if $\alpha < 0$), $L_{fo\alpha} < L/|\alpha|$. Hence $L_{fo\alpha} \leq L_f/|\alpha|$. With $f = fo\alpha o\alpha^{-1}$,

$$L_f \leq |\alpha| \, L_{fo\alpha} \quad \text{and so} \quad L_{fo\alpha} = L_f/|\alpha| \; .$$

PROPOSITION 1.3 *Let* f *be mean periodic with spectral set* S_f *and spectrum* Λ_f . *Then* $uf : t \to tf(t)$ *is mean periodic. If* $z \notin S_f$, *then*

$$K(uf)(z) = -\frac{d}{dz} K(f)(z)$$

and if $(a,p) \in \Lambda_f$, *then* $(a, p+1) \in \Lambda_{uf}$.

PROOF. As usual, let μ be a non-zero measure with $\mu * f = 0$. Put $\mu * f^{+} = g$ and $\Omega = C/S_f$. If $u\mu$ denotes the measure defined for each $x \in C(R)$ by $(u\mu)(x) = \mu(ux)$, then

$$\mu * (uf) = u(\mu * f) - (u\mu) * f \quad \text{and so} \quad \mu * \mu * (uf) = 0 \quad .$$

As $\mu * \mu$ is a non-zero measure, uf is mean periodic.

From $\hat{\mu}(z) = \int e^{-zt} d\mu(t)$, $\frac{d}{dz} \hat{\mu}(z) = -(u\hat{\mu})(z)$ or $D\hat{\mu} = -(u\hat{\mu})$. Also $D\hat{g} = -(u\hat{g})$. On Ω, differentiation of the formula $\hat{\mu} \cdot K(f) = \hat{g}$ yields

$$D\hat{g} = D\hat{\mu} \cdot K(f) + \hat{\mu}\, DK(f) \quad .$$

so that

$$-(u\hat{g}) = -(u\hat{\mu}) \cdot K(f) + \hat{\mu}\, DK(f)$$

Then $$\hat{\mu}\,\hat{\mu}\, DK(f) = \hat{\mu} \cdot (u\hat{\mu})\, K(f) - \hat{\mu}(u\hat{g})$$

$$= (u\hat{\mu}) \cdot \hat{g} - \hat{\mu}(u\hat{g}) = \hat{y}$$

where $$y = (u\mu) * g - \mu * (ug) \quad .$$

Now $\mu * (uf)^+ = ug - (u\mu) * f^+$ and so $\mu * \mu * (uf)^+ = -y$. As $\mu * \mu * (uf) = 0$, $(\mu * \mu)\hat{}\cdot K(uf) = -\hat{y}$. With $(\mu * \mu)\hat{} = \hat{\mu}\cdot\hat{\mu}$, we see that $K(uf)(z) = -D\,K(f)(z)$ when $z \in \Omega$.

If $(a, p) \in \Lambda_f$, we may suppose that $\hat{g}(a) \neq 0$ for if not, another non-zero measure λ can be found for which $\lambda * f = 0$ and $(\lambda * f^+)\hat{}(a) \neq 0$. With $\hat{g}(a) \neq 0$, $\hat{\mu}(z)$ has a zero of order p at $z = a$. From $\hat{y} = (u\hat{\mu}) \cdot \hat{g} - \hat{\mu} \cdot (u\hat{g}) = -D\hat{\mu} \cdot \hat{g} + \hat{\mu} \cdot D\hat{g}$, and the fact that if $H(z)$ is an entire function with a zero at $z = a$ of order p, then $DH(z)$ has a zero at $z = a$ of order $p - 1$, $\hat{y}(z)$ has a zero of $z = a$ of order $p - 1$. Since $(\hat{\mu})^2(z)$ has a zero at $z = a$ of order $2p$ and since $(\hat{\mu})^2\, K(uf) = -\hat{y}$, we see that

$K(uf)(z)$ has a pole at $z = a$ of order $p + 1$. Thus $(a, p + 1) \in \Lambda_{uf}$.

§1.2 The Truncated Convolution Product

An application of Propositions 1.1 and 1.3 is now given. As defined in Chapter 0, L_f denotes the mean period of a mean periodic function f and V_f denotes the closed translation invariant subspace generated by f.

PROPOSITION 1.4 *Let* f *be mean periodic. Then* $f \in V_{uf}$ *and* $uf \in V_{f \circledast f}$. *Moreover,*

$$L_f \leq L_{uf} \leq L_{f \circledast f} \leq 2L_f \quad .$$

PROOF. Let $\Lambda_f = ((a_k, p_k))_{k=1}^{\infty}$ so that $\Lambda_{uf} = ((a_k, p_{k+1}))_{k=1}^{\infty}$ Since f is the limit in $C(\mathbb{R})$ of a sequence of exponential polynomials $\{f_n\} \subset V_f$, and since V_{uf} is spanned by the exponential polynomials it contains, $\{f_n\} \subset V_{uf}$ and so $f \in V_{uf}$. From Proposition 1.1, $K(f \circledast f) = (K(f))^2$, so $\Lambda_{f \circledast f} = ((a_k, 2p_k))_{k=1}^{\infty}$ and as $p_k \leq p_k + 1 \leq 2p_k$ when p_k is any positive integer, we see that $uf \in V_{f \circledast f}$.

Thus, if λ is any measure with $\lambda * uf = 0$, then $\lambda * f = 0$ and so $L_f \leq L_{uf}$. Also, if $\nu \in M_c$ and $\nu * (f * f) = 0$, then $\nu * (uf) = 0$ and so $L_{uf} \leq L_{f \circledast f}$. Since $\mu * f = 0$ entails $\mu * \mu * (f \circledast f) = 0$, $L_{f \circledast f} \leq 2L_f$.

PROPOSITION 1.5 *Let* f, g *be mean periodic. If* $h = f \circledast g$, *then*

$$|L_f - L_g| \leq L_h \leq L_f + L_g \quad .$$

PROOF. For any $\varepsilon > 0$, non-zero measures μ, λ, ν may be chosen so that their supports are contained in intervals of lengths not exceeding $L_f + \varepsilon$, $L_g + \varepsilon$, $L_h + \varepsilon$ respectively and $\mu * f = 0$, $\lambda * g = 0$, $\nu * h = 0$.

With $\mu * \lambda * h = 0$, as $\mu * \lambda$ is a non-zero measure with support contained in an interval on length $L_f + L_g + 2\varepsilon$, it follows that $L_h \leq L_f + L_g + 2\varepsilon$. Hence $L_h \leq L_f + L_g$.

Now from

$$\mu * (f \circledast g) = \mu * (f^+ * g^+ - f^- * g^-) = (\mu * f^+) * g,$$

$$0 = \nu * \mu * h = \nu * (\mu * f^+) * g \quad .$$

Since $\nu * (\mu * f^+)$ is a non-zero measure with support contained in an interval of length $L_h + L_f + 2\varepsilon$, it follows that $L_g \leq L_h + L_f + 2\varepsilon$ for any $\varepsilon > 0$. Hence, $L_g \leq L_h + L_f$. Similarly $L_f \leq L_g + L_h$ and so $|L_f - L_g| \leq L_h$.

REMARK. These bounds for the mean period of $f \circledast g$ when f and g are mean periodic are the best possible in the sense that it is possible to find two sets of functions f, g, each with $L_f = L_g > 0$ and in one case, $L_h = 2L_f$ and in the other $L_h = 0$.

For our first example, we use the fact that the mean period of a continuous periodic function is equal to the period when all Fourier coefficients are non-zero. Let f be such a function of period π so that $\int_0^\pi f(t)\, e^{-2int} dt \neq 0$ for all $n \in Z$.

By Theorem 0.5, $e_i f : t \to e^{it} f(t)$ has mean period π (although $e_i f$ has period 2π). Let $h = f \circledast e_i f$ so that by Propositions 1.1 and 1.2 , $K(h)(z) = K(f)(z) \cdot K(f)(z - 1)$.

If $\mu = \delta - \delta_\pi$, then $\mu * f = 0$. Also $(\mu * f^+)\hat{}\,(z) = \int_0^\pi f(t)\, e^{-zt}\, dt = \phi(z)$, say. Then $\phi(ni) \neq 0$ for all $n \in Z$ and $K(f)(z) = \phi(z)/\hat{\mu}(z) = \phi(z)/(1 - e^{-\pi z})$. Thus

$$K(h)(z) = \phi(z)\ \phi(z - i)\ /\ (1 - e^{-\pi z})\,(1 - e^{-\pi(z-1)})$$

$$= \phi(z)\ \phi(z - i)\ /\ (1 - e^{-2\pi z}) \quad .$$

Now $1 - e^{-2\pi z} = 0$ if and only if $z = ni$ for some $n \in Z$ and $1 - e^{-2\pi z}$ has only simple zeros. With $\phi(ni) \neq 0$ it is apparent that $K(h)(z)$ has a simple pole at $z = ni$ for all integers n and $K(h)(z)$ has no other poles. So, the mean periodic function h has spectrum $((ni, 1))_{n=-\infty}^{\infty}$. Hence h is a periodic function of period 2π that has all Fourier coefficients non-zero and so h has mean period 2π .

We then have mean periodic functions f, g with $L_f = L_g > 0$ and $L_{f \circledast g} = L_f + L_g$.

Our second example is a consequence of Theorem 3.4; namely, if y is any mean periodic function, a unique mean periodic function x can be found satisfying $x + y + x \circledast y = 0$. For any $\varepsilon > 0$, we may choose a non-zero measure λ with $\lambda * x = 0$ and whose support is contained in an interval of length $L_x + \varepsilon$. Then $\lambda * y + (\lambda * x^+) * y = 0$ and as $\lambda + \lambda * x^+$ is a non-zero measure with support contained in an interval of length $L_x + \varepsilon$, $L_y \le L_x + \varepsilon$. Hence $L_y \le L_x$. Similarly, $L_x \le L_y$ and so $L_x = L_y$.

For such a pair of functions x, y with $L_x > 0$, put $f = e \circledast x + e$, $g = e \circledast y + e$ where $e : t \to 1$. Then f, g are mean periodic and $L_f = L_x = L_y = L_g$. From $x + y + x \circledast y = 0$,

$$e \circledast e \circledast x + e \circledast e \circledast y + e \circledast x \circledast e \circledast y = 0,$$

so

$$e \circledast (f - e) + e \circledast (g - e) + (f - e) \circledast (g - e) = 0$$

whence $f \circledast g = e \circledast e$. As $e \circledast e$ has mean period zero, we have $L_{f \circledast g} = |L_f - L_g| = 0$ and $L_f > 0$.

In addition to each of MP, MP_o, and MQ being subalgebras of $C(R)$, we now show

PROPOSITION 1.6 *Let* f, $g \in C(R)$ *and* $f \neq 0$. *If* f, $f \circledast g \in MP$, MP_o *or* MQ, *then* $g \in MP$, MP_o *or* MQ *respectively.*

PROOF. Let f, $f \circledast g \in MP$ and let μ, ν be non-zero measures with $\mu * f = 0$, $\nu * (f \circledast g) = 0$. With $\mu * (f \circledast g) = (\mu * f^+) * g$, $\nu * (\mu * f^+) * g = 0$. Since $f \neq 0$, $\nu * (\mu * f^+)$ is a non-zero measure and so g is mean periodic.

When f, $f \circledast g \in MP_o$, the non-zero measure $\nu * (\mu * f^+)$ can be chosen with support contained in an interval of arbitrarily small length. Thus g has mean period zero.

When f, $f \circledast g \in MQ \subset MP$, we know that $g \in MP$.

Now a mean periodic function is an exponential polynomial if, and only if, its Kahane Transform has a finite number of poles. Since $K(f \circledast g) = K(f) \cdot K(g)$ and as $K(f)$, $K(f \circledast g)$ have a finite number of poles, $K(g)$ must have a finite number of poles. Thus g is an exponential polynomial.

REMARK. We note that it is possible to have both f, g continuous and non mean periodic but $h = f \circledast g$ to be an exponential polynomial. As a consequence of Theorem 3.4, if $x \in C(R)$, there exists a unique continuous $y \in C(R)$ satisfying $x + y + x \circledast y = 0$, and $x \in MP$ if, and only if, $y \in MP$. Choosing an $x \in C(R)\setminus MP$ so that $y \in C(R)\setminus MP$ and setting $f = e \circledast x + e$, $g = e \circledast y + e$, we find as before that $f \circledast g = e \circledast e$. Then $f, g \in C(R)\setminus MP$ and $e \circledast e \in MQ$.

One may then ask if f is continuous and if $f \circledast f$ is mean periodic whether f is mean periodic. The answer is by no means affirmative, even when $f \circledast f$ is an exponential polynomial. This is illustrated by the function (one of many) $f : t \to +\sqrt{t}$ which is not mean periodic and

$$\begin{aligned} f \circledast f(t) &= \int_0^t \sqrt{t-r} \cdot \sqrt{r}\, dr \\ &= \int_0^{\frac{1}{2}\pi} 2t^2 \sin^2\theta \cos^2\theta \, d\theta \\ &= \pi t^2/8 \end{aligned}$$

which is an exponential polynomial.

§1.3 Subalgebras of MP, MP_o and MQ

As a spectral set is not defined for a non mean periodic continuous function, but a mean periodic function does have a spectral set, we are able to characterize some subalgebras of MP with this concept. Properties are given of subalgebras of each of the algebras MP, MP_o and MQ. More properties will be given for the subalgebras of MQ. However, we do not attempt to describe all of the numerous subalgebras of each of these algebras.

PROPOSITION 1.7 *Let* M *denote any one of the algebras* MP, MP_o *or* MQ *and let* A *be any set of complex numbers. Then*

$$M(A) = \{f \in M \ : \ S_f \subset A\}$$

is a subalgebra of M. *Moreover,* M(A) *is translation invariant and contains* uf *whenever* $f \in M(A)$.

PROOF. Let $a, b \in C$. Also let $f, g \in M(A)$ so that the poles of the Kahane Transforms, K(f), K(g) are contained in A. By Proposition 1.1, $K(af + bg) = a K(f) + bK(g)$ and $K(f \circledast g) = K(f) \cdot K(g)$ and so the poles of $K(af + bg)$ and $K(f \circledast g)$ are contained in $S_f \cup S_g \subset A$. Hence $af + bg, \ f \circledast g \in M(A)$ and so M(A) is a subalgebra of M.

From Propositions 1.2 and 1.3, the spectral sets of $T_\alpha f$, uf and f coincide. Thus, if $f \in M(A)$, $T_\alpha f$ and uf belong to M(A).

Notes. 1. If $f \in M(A) \cap C^1(R)$, then $f' \in M(A)$ (by use of Proposition 1.2),

2. $M(A)$ is dense in $C(R)$. For if $a \in A$, the $\{u_n e_a\}_{n=1}^{\infty} \subset M(A)$ and the set of all polynomials is dense in $C(R)$,

3. If $f \in M$ and if $U(f) = \{\sum_{p=1}^{n} A_p f^{\circledast p} : A_p \in C, n=1,2,\ldots\}$ then $U(f)$ is a subalgebra of M. Here $f^{\circledast 1} = f$, $f^{\circledast p} = f \circledast f^{\circledast (p-1)}$.

If $f = A e_a$, then it is apparent that as $e_a^{\circledast (q+1)} = u_q e_a/q!$ (from $e^{\circledast (q+1)} = u_q/q!$), $U(f) = M(\{A\})$ and so $U(f)$ is translation invariant when $f = A e_a$. However, if $f = e_a + e_b$ $(a \neq b)$, then $U(f)$ does not contain

$$T_\alpha(e_a + e_b) = e^{-a\alpha} e_a + e^{-b\alpha} e_b = e^{-a\alpha}(e_a + e^{(a-b)\alpha} e_b)$$

when $\alpha \neq 0$. Thus, in general $U(f)$ is not translation invariant.

4. If $B \subset M$ and if

$$U(B) = \{\sum_{p=1}^{n} A_p f_{p1} \circledast f_{p2} \circledast \ldots f_{pn_p} : f_{pk} \in B\}$$

then $U(B)$ is a subalgebra of M. In general, $U(B)$ is not translation invariant. One exception to this is given in the following:

PROPOSITION 1.8 *If* $f \in M$ *and if* V_f *denotes the closed translation invariant subspace generated by* f, *then* $U(V_f)$ *is a translation invariant subalgebra of* M.

PROOF. If $f \in MP$, and if μ is any measure for which $\mu * f = 0$, then $\mu * V_f = 0$. Thus $V_f \subset MP$ and if $f \in MP_o$, $V_f \subset MP_o$.

If $f \varepsilon MQ$, and if U_f is the subspace spanned by f and its translates, then $U_f \subset MQ$. Since U_f is finite dimensional, it is closed and so when $f \varepsilon MQ$, $U_f = V_f \subset MQ$.

Thus, if $f \varepsilon M$, then $V_f \subset M$. So $U(V_f)$ is a subalgebra of M. To show that $U(V_f)$ is translation invariant, it suffices to show that if $x_1, x_2, \dots x_n \varepsilon V_f$, and if $g = x_1 \circledast x_2 \dots \circledast x_n$, then $T_\alpha g \varepsilon U(V_f)$.

Let $x, y \varepsilon C(R)$. Then

$$\begin{aligned} T_\alpha(x \circledast y)(t) &= \int_0^{t-\alpha} x(t - \alpha - r)y(r)dr \\ &= \int_t^{t-\alpha} x(t - \alpha - r)y(r)dr + \int_0^t x(t - \alpha - r)y(r)dr \\ &= \int_0^\alpha x(s - \alpha)y(t - s)ds + ((T_\alpha x) \circledast y)(t) \end{aligned}$$

so $T_\alpha(x \circledast y) = \psi * y + (T_\alpha x) \circledast y$

where ψ is a measure, defined for each $h \varepsilon C(R)$, by

$$\psi(h) = -\int_0^\alpha x(s - \alpha)\, h(s)\, ds \quad .$$

On repeated applications of such a formula,

$$T_\alpha g = \eta_1 * x_n + (\eta_2 * x_{n-1}) \circledast x_n + \dots (T_\alpha x_1) \circledast x_2 \dots \circledast x_n \quad .$$

If $y_j = \eta_j * x_k$ where η_j is any measure and $x_k \varepsilon V_f$, then for each measure μ satisfying $\mu * f = 0$, $\mu * y_j = 0$ and so $y_j \varepsilon V_f$.

Hence $T_\alpha g \varepsilon U(V_f)$ and so $U(V_f)$ is translation invariant.

REMARK. We note that if $f \varepsilon M$, then $U(V_f) \subset M(S_f)$ since if $x_1, x_2, \dots x_n \varepsilon V_f$, $S_{x_1}, S_{x_2}, \dots S_{x_n} \subset S_f$ and if

$g = x_1 \circledast x_2 \ldots * x_n$, then $S_g \subset S_f$ so $g \in M(S_f)$.

It may also be noted that when $a \in S_f$, $e_a \in V_f$ and so $u_q e_a \in U(V_f)$ for $q = 0, 1, 2, \ldots$. Since $M(S_f)$ also contains $u_q e_a$ for $q = 0, 1, 2, \ldots$ when $a \in S_f$ and $M(S_f)$ is translation invariant, it would be interesting to know if indeed $U(V_f) = M(S_f)$ or merely $uh \in U(V_f)$ when $h \in U(V_f)$. No answer is yet available for when f is any mean periodic function but when f is an exponential polynomial.

PROPOSITION 1.9 *Let* V *be a translation invariant subalgebra of* MQ . *If* $A = \bigcup_{x \in V} S_x$, *then*

$$V = \{x \in MQ \ : \ S_x \subset A\} \quad .$$

PROOF. It is clear that if $x \in V$, then $S_x \subset A$.

Conversely, let $x \in MQ$ and $S_x \subset A$ and suppose that $x = \sum\sum A(k,q)\, u_q e_{a_k}$ where $a_1, a_2, \ldots a_n \in A$.

It is required to show that $x \in V$. For each $a_j \in A$, there exists an $x_j \in V$ with $a_j \in S_{x_j}$. Since V is a translation invariant subspace of MQ, V contains U_{x_j} and so e_{a_j} As V is also a subalgebra, V contains $u_q e_{a_j}$ and so V contains x .

Hence $V = \{x \in MQ \ : \ S_x \subset A\}$.

§1.4 Fields of Convolution Quotients

In concluding this chapter, it is shown that the ring MP has no non-zero divisors of zero. As well, a brief account of Mikusinski's convolution quotients in $C(R_+)$ that is extracted from Erdélyi's book, [1], is included.

PROPOSITION 1.10 MP *has no non-zero divisors of zero.*

PROOF. Let $f, g \in MP$, $f \circledast g = 0$ and suppose that $f \neq 0$. From Theorem 0.5, f cannot be zero on any interval of length exceeding the mean period of f. So f^+ and f^- are both non-zero. As $(f \circledast g)^+ = f^+ * g^+$ and $(f \circledast g)^- = -f^- * g^-$ are both zero, it follows from Titchmarsh's convolution theorem (see, for example, Erdélyi, [1], page 16) that both g^+ and g^- are zero. Thus $g = 0$ and so MP has no non-zero divisors of zero.

REMARK. It is possible to show that MQ has no non-zero divisors of zero without recourse to Titchmarsh's convolution theorem.

For details of an elementary proof that the set of entire functions (and so exponential polynomials) have no non-zero divisors of zero, see Theorem 5.3 .

Mikusinski observed that the set $C(R_+)$ of continuous complex-valued functions on $R_+ = [0, \infty)$ with the operations of addition and truncated convolution formed an integral domain without an identity and an algebra over $\mathbb{C}$. Thus a quotient field F can be constructed in which the equation $x \circledast g = f$ $(g \neq 0)$, always has a solution. To do this, one defines an equivalence relation on the set of ordered pairs $\{(f, g): f, g \in C(R_+)$ and $g \neq 0\}$ by putting (f, g) and (f_1, g_1) equivalent if $f \circledast g_1 = f_1 \circledast g$.

Each equivalence class is called a convolution quotient and f/g is used to denote an equivalence class that contains (f, g). The set of such equivalence classes, $\mathcal{F}$, may be shown to be a field and a vector space over $\mathbb{C}$. Moreover, it is possible to embed both $\mathbb{C}$ and $C(R_+)$ into $\mathcal{F}$. Thus, some elements of $\mathcal{F}$ will correspond to numbers or continuous functions, still others will correspond to abstract entities including Dirac's delta function which appears as the identity in the field and the equivalence class g/g $(g \in C(R),\ g \neq 0)$. Also, $\mathcal{F}$ contains an 'extended' derivative $s = e/(e*e)$ with $s \circledast x = Dx + x(0)\delta$ for each $x \in C(R)$.

With MP having no non-zero divisors of zero, and MQ, MP_o being subsets of MP that are closed with respect to addition and truncated convolution, it is apparent that if M denotes any one of MP, MP_o or MQ, then M is an integral domain without an identity and an algebra over $\mathbb{C}$. Thus, in the same way that the quotient field $\mathcal{F}$ is constructed from $C(R_+)$, a quotient field $\mathcal{F}_M$ may be constructed from M when M is any one of MP, MP_o or MQ.

One difference between $\mathcal{F}$ and $\mathcal{F}_M$ is that $\mathcal{F}$ is complete (in the sense that $C(R_+)$ is a complete metric space) whereas $\mathcal{F}_M$ is not.

Chapter Two

ORDINARY AND FUNCTIONAL DIFFERENTIAL EQUATIONS

In this chapter, we show that if $A(t)$ is a continuous periodic matrix (i.e., a matrix whose elements are continuous periodic functions with a common period), then for certain mean periodic vector functions, $\underline{f}$, all solutions to the system of equations $\underline{x}' = A(t)\underline{x} + \underline{f}$ are mean periodic. Also, in Section 2.2, we show that if $[\lambda]$ and $[\mu]$ are matrices whose elements are certain measures, then the system of equations $\underline{x}' = [\lambda] * \underline{x}' + [\mu] * \underline{x}$ has non-trivial solutions and moreover, all solutions valid on R are mean periodic. Such functional-differential equations have been studied in a different context by J. Hale, [1], and others, and include as special cases, differential-difference equations.

As well, in Section 2.3, we give examples of ordinary and functional differential equations that admit non mean-periodic solutions.

The results given in this chapter represent a continuation of some of the author's earlier work, that included (Laird [2]), the following propositions:

1. For the system of equations,

$$\underline{x}'(t) + A\underline{x}(t) = \underline{y}(t) \quad \text{with} \quad \underline{x}(\alpha) = \underline{c}$$

where $\underline{y}$ is an n-vector valued function with continuous components and A is a constant $n \times n$ matrix, a necessary and sufficient condition that $\underline{x}$ be mean periodic is that $\underline{y}$ be mean periodic, and

2. For the differential-difference equation

$$x'(t) + \sum_{k=0}^{n} a_k\, x(t - \omega_k) = y(t)$$

where y is continuous on R, $n > 0$, $a_o, a_1, \ldots, a_n$ are non-zero complex numbers and $\omega_o < \omega_1 < \ldots \omega_n$ are real numbers, if any solution is mean periodic, then y is mean periodic. Conversely, if y is mean periodic, then all solutions valid on R are mean periodic.

§2.1 Ordinary Differential Equations

THEOREM 2.1 *Let* $A(t)$ *be a continuous periodic* $n \times n$ *matrix of period* τ . *Let* $\underline{f}$ *be an n-vector whose* i^{th} *component is of the form* $b_i g_i$ $(i = 1, 2, \ldots n)$ *where* b_i *is a continuous periodic function of period* τ_i *where each* τ_i *is commensurable with* τ *and* g_i *is any exponential polynomial. Then all solutions of the system of equations*

(2.1) $\underline{x}'(t) = A(t)\,\underline{x}(t) + \underline{f}(t)$

are mean periodic.

PROOF. An example is given in Section 2.3 that shows some restriction additional to $\underline{f}$ being mean periodic must be made to ensure that $\underline{x}$ is mean periodic. Let $Y(t)$ be a fundamental matrix for the system $\underline{x}' = A(t)\,\underline{x}$. By Floquet's Theorem, $Y(t) = P(t)\, e^{tL}$ where $P(t)$

and $P^{-1}(t)$ are continuous periodic matrices of period τ and L is a constant matrix. Then any solution of (2.1) may be written as

$$\underline{x}(t) = P(t)\, e^{tL}\, \underline{c} + P(t)\, e^{tL} \int_o^t e^{-sL}\, P^{-1}(s)\, \underline{f}(s)\, ds .$$

where $\underline{c}$ is a constant vector and the elements of the matrices e^{tL} and e^{-sL} are exponential polynomials (see, for example, Coppel, [1], pages 45-47).

Hence each component x_i of $\underline{x}$ is of the form

$$x_i = \sum_j \sum_k p_{ij}\, g_{jk}\, c_k + \sum_i \sum_j \sum_k \sum_\ell \sum_m p_{ij}\, g_{jk} \int_o^t h_{k\ell}\, g_{\ell m}\, f_m$$

where p_{ij}, $g_{\ell m}$ are continuous periodic functions of period τ and g_{jk}, $h_{k\ell}$ are exponential polynomials. Now a periodic function is mean periodic, the product of a mean periodic function with an exponential polynomial is mean periodic and a finite linear combination of mean periodic functions is mean periodic. Thus each x_i is mean periodic when the terms $p_{ij} \int_o^t h_{k\ell}\, g_{\ell m}\, f_m$ have been shown to be mean periodic. We now use $f_m = b_m\, g_m$ where b_m is a continuous periodic function of period τ_m where τ_m is commensurable with τ and the fact that an exponential polynomial is a finite linear combination of terms $u_n\, e_a : t \to t^n\, e^{at}$. Let ω be the least common multiple of $\tau, \tau_1, \tau_2, \ldots . \tau_m$. Then each function $p_{ij} \int_o^t h_{k\ell}\, g_{\ell m}\, f_m$ is mean periodic if for $n = 0, 1, 2, \ldots,$ the functions

$$F_n : t \to p(t) \int_o^t r^n\, e^{ar}\, q(r)\, dr$$

are mean periodic when p and q are complex-valued continuous periodic functions of period ω.

To show that F_n is mean periodic, we start by observing that $F_o(t-\omega) = p(t) \int_\omega^t e^{a(s-\omega)} q(s)\,ds$ so that $F_o(t) - e^{a\omega} F_o(t-\omega) = A_o p(t)$ where $A_o = \int_o^\omega e^{ar} q(r)\,dr$. This may be written as $\lambda * F_o = A_o p$ where λ is the non-zero measure $\delta - e^{a\omega}\delta_\omega$. With F_n defined as above.

$$F_n(t) - e^{a\omega} F_n(t-\omega) = A_n p(t) - \sum_{j=1}^{n} {}^nC_j(-\omega)^j F_{n-j}(t)$$

or

$$\lambda * F_n = A_n p - \sum_{j=1}^{n} {}^nC_j(-\omega)^j F_{n-j}$$

where A_n is some constant.

With $\delta_\omega * p = p$, $\lambda * p = (1 - e^{a\omega})p$. Thus $\lambda * \lambda * F_n = B_n p + B_o F_o + \ldots B_{n-2} F_{n-2}$, where B_n, B_o, ... B_{n-2} are constants. Continuing in this manner, we find that $\lambda^{*(n+1)} * F_n = Cp$ where C is another constant. If $\mu_n = \lambda^{*(n+1)} * (\delta - \delta_\omega)$, then μ_n is a non-zero measure and $\mu_n * F_n = 0$, showing that F_n is mean periodic.

Hence the solution to (2.1) is mean periodic.

REMARK. A special case of the above theorem is as follows: *Let* A(t) *be a continuous periodic* n x n *matrix of period* ω *and let* $\underline{f}$ *be a continuous periodic n-vector valued function of period* ω. *Then all solutions of the system of equations* (2.1) *are mean periodic.*

§2.2 Functional Differential Equations

Our next Theorem will involve linear systems of functional differential equations with constant coefficients that admit mean periodic functions. Before moving onto this Theorem, we give a brief account of some Functional Differential Equations considered by J. Hale and others (Hale [1], Hale and Meyer [1]). As well, we state and prove an interesting Lemma that will be used in this and the next chapter.

Let r be a fixed non-negative integer and, when x is defined on $[-r, \infty)$, let x_t be the function defined on $[-r, 0]$ by $x_t : \theta \to x(t + \theta)$ for $-r \leq \theta \leq 0$ and $t \geq 0$. Also let $\mathbb{C}^n$ denote the n-dimensional vector space over $\mathbb{C}$ and let X denote the space of continuous functions from $[-r, 0]$ into $\mathbb{C}^n$.

J. Hale ([1], page 293) has defined a linear functional differential equation with constant coefficients as any equation of the form

(2.2) $\quad \underline{x}'(t) = \underline{f}(\underline{x}_t)$

where $\underline{f}$ is any continuous linear map of X into $\mathbb{C}^n$. For such a map, he notes the existence of an $n \times n$ matrix $[\eta(\theta)]$ $(-r \leq \theta \leq 0)$ whose elements are of bounded variation and

$$\underline{f}(\underline{\phi}) = \int_{-r}^{0} [d\eta(\theta)]\, \phi(\theta)$$

for all $\underline{\phi} \in X$. Hale also observes that such equations include systems of linear differential-difference equations of retarded type and with constant coefficients; for example,

$$\underline{x}'(t) = \sum_{k=1}^{n} A_k \, \underline{x}(t - \tau_k) \qquad \tau_k \geq 0 \quad .$$

By a simple change of variable, we see that equation (2.2) may be rewritten as $\underline{x}'(t) = [\mu] * \underline{x}(t)$ or $x_j' = \sum_{k=1}^{n} \mu_{jk} * x_k$ $(j = 1, 2, \ldots n)$ where $[\mu]$ denotes an $n \times n$ matrix of measures $\{\mu_{jk}\}$ whose supports lie in $[0, r]$.

Hale and Meyer [1] have considered more general systems including those of the form

$$\underline{x}'(t) = \underline{g}(\underline{x}_t') + \underline{f}(\underline{x}_t) + \underline{y}(t)$$

where $\underline{f}$ and $\underline{g}$ are continuous linear maps of X into R^n and $\underline{y} \in X$. This equation may be written as

$$(2.3) \qquad \underline{x}'(t) = [\lambda] * \underline{x}'(t) + [\mu] * \underline{x}(t) + \underline{y}(t)$$

or $$x_j' = \sum_{k=1}^{n} \lambda_{jk} * x_k' + \sum_{k=1}^{n} \mu_{jk} * x_k + y_j \qquad (j = 1, 2, \ldots n)$$

where $[\lambda] = \{\lambda_{jk}\}$ and $[\mu] = \{\mu_{jk}\}$ are $n \times n$ matrices whose elements are measures with supports lying in $[0, r]$.

Hale and Meyer (loc. cit., page 6) note that such "neutral" equations would also include equations of advanced type unless some restriction is made on the function $\underline{g}$ or the matrix $[\lambda]$. The restriction used by Hale and Meyer is to require that $[\lambda]$ "be uniformly non-atomic at zero". Although we have nothing against equations of advanced type, some restriction on $[\lambda]$ will be needed to guard against systems of equations (2.3) that admit arbitrary solutions: for example, if $n = 2$, $\lambda_{11} = 0 = \lambda_{22}$, $\lambda_{12} = \delta = \lambda_{21}$,

$[\mu] = 0$ and $\underline{y} = 0$, then $\underline{x}' = [\lambda] * \underline{x}'$ is equivalent to $x_1' = x_2'$. Accordingly, we shall adopt Hale and Meyers' concept of the restriction of $[\lambda]$ being uniformly non-atomic at zero but shall restate it as follows:

DEFINITION: *Let* $[\lambda]$ *be an* $n \times n$ *matrix whose elements are measures with supports lying in* $[0, r]$ *for some* $r > 0$. *If there exists an* $\varepsilon \in (0, r)$ *and a function* δ *on* $[0, \varepsilon]$ *that is continuous and non-decreasing with* $\delta(0) = 0$, *and if for each element* λ *of the matrix* $[\lambda]$,

$$(2.4) \qquad \left|\int_0^s \phi(\theta)\, d\lambda(\theta)\right| \leq \delta(s) \cdot \sup\{|\phi(\theta)| : 0 \leq \theta \leq s\}$$

for all $\phi \in C([0, \varepsilon])$ *and* $s \in [0, \varepsilon]$, *then* $[\lambda]$ *is said to be uniformly non-atomic at zero.*

LEMMA 2.2 *Let* $T = \{T_{ij}\}$ *be an* $n \times n$ *matrix whose elements are distributions with compact supports and let* $\underline{x}$ *be a continuous vector function that satisfies* $T * \underline{x} = 0$. *Then* $\underline{x}$ *is mean periodic when* T *has a non-zero "determinant", or when* $\hat{T}(z)$, *the matrix whose elements are the Fourier-Laplace transforms of the elements of* T, *has a determinant that does not vanish identically.*

PROOF. We make use of the fact that the set of distributions with compact supports taken with the operations of addition and convolution forms a commutative ring with identity. Thus, certain matrix concepts and operations (as in Jacobson, [1], page 56) can be employed.

Let M_{ki} denote the cofactor of T_{ki} in the matrix T so that $\sum_{i=1}^{n} M_{ki} * T_{ij}$ is equal to $\det T$ for $k = j$ and is zero for $k \neq j$. From $\sum_{j=1}^{n} T_{ij} * x_j = 0$, we see that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} M_{ki} * T_{ij} * x_j = 0 \qquad (k = 1, 2, \dots n),$$

and so $(\det T) * x_j = 0 \quad (j = 1, 2, \dots n)$. Thus, $\underline{x}$ is mean periodic when $\det T \neq 0$.

Let $S = \det T$ so that

$$S = \sum_{\sigma \varepsilon S_n} \operatorname{sgn}\sigma \; T_{1\sigma(1)} * T_{2\sigma(2)} \cdots * T_{n\sigma(n)}$$

where σ ranges over all the permutations on $\{1, 2, \dots n\}$.

Also $(a\,T_1 + b\,T_2)\hat{} = a\,\hat{T}_1 + b\,\hat{T}_2$ and $(T_1 * T_2)\hat{} = \hat{T}_1 \cdot \hat{T}_2$ when $a, b \in \mathbb{C}$ and T_1, T_2 are distributions with compact supports. Then

$$\hat{S} = \sum_{\sigma \varepsilon S_n} \operatorname{sgn}\sigma \; \hat{T}_{1\sigma(1)} \cdot \hat{T}_{2\sigma(2)} \cdots \hat{T}_{n\sigma(n)}$$

and so if $H(z) = \det \hat{T}(z)$, $H(z) = \hat{S}(z)$. If $H(z)$ is not identically zero, then $S = \det T$ is non-zero and so $\underline{x}$ is mean periodic.

COROLLARY: *If* T *is an* $n \times n$ *matrix whose elements are measures, then* $\det T$ *is a measure.*

THEOREM 2.3 *For the system of linear functional differential equations*

$$(2.3) \qquad \underline{x}' = [\lambda] * \underline{x}' + [\mu] * \underline{x} + \underline{y}$$

where (1) $[\lambda]$ *is an* $n \times n$ *matrix whose elements are measures with compact supports lying in* $[0, \infty)$ *and* $[\lambda]$ *is uniformly non-atomic at zero,*

(2) $[\mu]$ *is an* $n \times n$ *matrix whose elements are any measures with compact supports, and*

(3) $\underline{y}$ *is a continuous n-vector function on* R, *then*

(a) if $\underline{y} = 0$, *there is at least one non-trivial exponential solution on* R *and all continuous solutions valid on* R *are mean periodic,*

(b) if $\underline{y}$ *is mean periodic, all continuous solutions valid on* R *are mean periodic, and*

(c) if any continuous solution $\underline{x}$ *is mean periodic, then* $\underline{y}$ *is mean periodic.*

PROOF. Let T denote the $n \times n$ matrix

$$I\, D\delta - [D\lambda] - [\mu]$$

where I is the unit $n \times n$ matrix, $D\delta$ is the distributional derivative of the Dirac measure, and $[D\lambda]$ is the distributional derivative of the matrix $[\lambda]$. Then T is an $n \times n$ matrix whose elements are distributions with compact supports and equation (2.3) may be written as $T * \underline{x} = \underline{y}$.

For a), we show that $T * \underline{x} = \underline{0}$ has at least one non-zero solution of the form $\underline{x} = \underline{d}\, e^{zt}$.

Now $\dot{T} * (\underline{d}\, e^{zt}) = \hat{T}(z)\ \underline{d}\, e^{zt}$

where $\hat{T}(z) = Iz - z[\hat{\lambda}(z)] - [\hat{\mu}(z)]$.

Here $[\hat{\lambda}(z)]$, $[\hat{\mu}(z)]$ are matrices whose elements are the Fourier-Laplace transforms of the elements of $[\lambda]$, $[\mu]$ respectively. Thus there exists a non-zero vector $\underline{d}$ with $T * (\underline{d}\, e^{zt}) = 0$ if, and only if, $h(z) \equiv \det \hat{T}(z) = 0$ for some $z \in \mathbb{C}$. (Incidentally, the equation $h(z) = 0$ is known as the characteristic equation of (2.3)) .

We now show, indirectly, that $h(z)$ has at least one zero. Suppose that $h(z)$ does not have any zeros. As h is an entire function, there exists an entire function g so that $h(z) = e^{g(z)}$. Moreover, as h is of order one and of polynomial order on the imaginary axis, $h(z) = c\, e^{\alpha z}$ where c is complex and α is real.

It is well known that if A is any $n \times n$ matrix,

$$\det(zI - A) = z^n + z^{n-1}\, p_1 + \dots z^{n-k}\, p_k + \dots p_n$$

where each p_k involves a finite number, say η_k, of sums of terms that are ± 1 times a product of k elements of A (for example, $p_1 = -\,\text{trace}\, A$ and $p_n = (-1)^n \det A)$. If we regard A as $z[\hat{\lambda}(z)] + [\hat{\mu}(z)]$, then from $h(z) = c\, e^{\alpha z}$ and

$$h(z) = z^n + z^{n-1}\, p_1 + \dots z^{n-k}\, p_k + \dots p_n \quad ,$$

$$(2.5) \quad |z|^n \le |z^{n-1}\, p_1| + \dots |z^{n-k}\, p_k| + \dots |p_n| + |c\, e^{\alpha z}| \quad .$$

Now let μ be any measure and let λ be any measure with support lying in $[0, r]$ belonging to the matrix $[\lambda]$ satisfying a uniformly non-atomic at zero condition. Suppose also that μ has support (α, β) and norm ℓ and λ has norm m. If $z = x + iy$ and $x > 0$, then

$$|\hat{\mu}(z)| \leq \ell \cdot \sup\{|e^{-zt}| : \alpha \leq t \leq \beta\} \leq \ell\, e^{-\alpha x} \quad .$$

Also, if $\lambda_s(\phi) = \int_0^s \phi(\theta)d\lambda(\theta)$ for each $\phi \in C([0, r])$ and $0 < s < r$, λ_s and $\nu_s = \lambda - \lambda_s$ are measures.

By equation (2.4),

$$|\hat{\lambda}_s(z)| \leq \delta(s) \cdot \sup\{|e^{-zt}| : 0 \leq t \leq r\} = \delta(s)$$

where $z = x + iy$ and $x > 0$.

As well,

$$|\hat{\nu}_s(z)| \leq m \cdot \sup\{|e^{-zt}| : s \leq t \leq r\} = me^{-sx}$$

when $x > 0$ and so

$$|\hat{\lambda}(z)| \leq \delta(s) + m\, e^{-xs} \qquad \text{when } x, \ s > 0 \quad .$$

Let $A = \{a_{jk}\}$ and $a_{jk} = z\hat{\lambda}_{jk}(z) + \hat{\mu}_{jk}(z)$.

Also let μ_{jk} have norm ℓ_{jk}, λ_{jk} have norm m_{jk}, and μ_{jk} have support $[\alpha_{jk}, \beta_{jk}]$. Put

$$L = \max\{\ell_{jk} : 1 \leq j, k \leq n\},$$

$$M = \max\{m_{jk} : 1 \leq j, k \leq n\},$$

and $\alpha = \min\{\alpha_{jk} : 1 \leq j, k \leq n\}$.

As $[\lambda]$ is uniformly non-atomic at zero and each λ_{jk} has support in $[0, r]$, it follows that

$$|a_{jk}| \leq |z| \ (\delta(s) + Me^{-xs}) + Le^{-x\alpha}$$

for $x > 0$ and $j, \ k = 1, 2, \dots n$.

Next, choose any $\zeta \varepsilon (0, 1)$. By referring again to the uniformly non-atomic at zero definition, an $s > 0$ can be chosen so that $\delta(s) < \frac{1}{2}\zeta$. With this s, fix $x > 0$ so that $Me^{-xs} < \frac{1}{2}\zeta$. Putting $\xi = Le^{-x\alpha}$, we then have $|a_{jk}| \leq |z|\zeta + \xi$.

Recalling the definition of $p_1, p_2, \dots p_n$, we see that when $z = x + iy$ and $x > 0$ is chosen as above,

$$|p_1| \leq \eta_1(|z|\zeta + \xi) \qquad \text{and}$$

$$|p_k| \leq \eta_k(|z|\zeta + \xi)^k \leq \eta\zeta \ |z|^k + q_k \qquad .$$

Here $\eta = \max\{\eta_1, \eta_2, \dots \eta_k\}$ is dependent only on n and q_k is a polynomial in $|z|$ of degree $k - 1$.

From equation (2.5), we obtain

$$|z|^n \leq n\eta\zeta \ |z|^n + \sum_{k=1}^{n} |z^{n-k}|q_k + |ce^{\alpha z}| \quad .$$

Now $P(|z|) = \sum_{k=1}^{n} |z^{n-k}| \ q_k$ is a polynomial of degree $n - 1$ in $|z|$. If we choose $\zeta \ \varepsilon(0, 1)$ so that $n\eta\zeta < \frac{1}{2}$, and consequently choose $x > 0$, then $\frac{1}{2}|z|^n \leq P(|z|) + |c| \ e^{\alpha x}$.

But this is contradictory for large values of $|y|$.

Hence $h(z)$ has at least one zero and so the homogeneous equation (2.3) with $\underline{y} = 0$ has at least one non-trivial solution.

As well, $h(z)$ is not identically zero. For if we assume otherwise, we have, as above, $\frac{1}{2}|z|^n \le P(|z|)$ which is contradictory.

Since $h(z) = \det(\hat{T}(z))$ is not identically zero, Lemma 2.2 shows that all continuous $\underline{x}$ satisfying $T * \underline{x} = \underline{0}$ are mean periodic.

For b), if $\underline{y}$ is mean periodic, let ν be a non-zero measure with $\nu * \underline{y} = \underline{0}$ and put $\underline{z} = \nu * \underline{x}$ (i.e., $\nu * y_i = 0$ and $z_i = \nu * x_i$ for $i = 1, 2, \ldots n$) . From equation (2.3), and $\underline{z}' = \nu * \underline{x}'$, we obtain

$$\underline{z}' = [\lambda] * \underline{z}' + [\mu] * \underline{z} \quad .$$

If $\underline{x}$ is any continuous solution on R, then $\underline{z}$ is continuous. From part a), any continuous solution $\underline{z}$ to this equation is mean periodic. Hence $\underline{z}$, and so $\underline{x}$ is mean periodic when $\underline{x}$ is a continuous solution on R and $\underline{y}$ is mean periodic.

For c), if $\underline{x}$ is any continuous mean periodic solution to (2.3), let ρ be any non-zero measure with $\rho * \underline{x} = \underline{0}$. Then $\rho * \underline{x}' = 0$ and so $\rho * \underline{y} = \underline{0}$ showing that $\underline{y}$ is mean periodic.

REMARKS. We note that a considerably shorter proof can be given for the above theorem when $[\lambda] = 0$, that is, when (2.3) is replaced by

$$\underline{x}' = [\mu] * \underline{x} + \underline{y} \quad .$$

Also, we note that the theorem includes as a special case the following concerning linear differential difference equations with constant coefficients.

Let $a_{jk\ell}$, $b_{jk\ell}$ $(1 \le j,\ k \le n)$ be complex numbers. Let ω_ℓ be positive real numbers and let τ_ℓ $(1 \le \ell \le m)$ be any real numbers. Also let $\underline{x}$, $\underline{y}$ have components $x_1, x_2 \ldots x_n$ and $y_1, y_2 \ldots y_n$ respectively, and $\underline{y}$ be continuous. Then for the system of equations

$$x_j'(t) = \sum_{\ell=1}^{m} \sum_{k=1}^{m} a_{jk\ell}\, x_k'(t - \omega_\ell) + \sum_{\ell=1}^{m} \sum_{k=1}^{m} b_{jk\ell}\, x_k(t - \tau_\ell) + y_j(t)$$

$(j = 1, 2, \ldots n)$, the conclusions of Theorem 2.3 hold.

§2.3 Counter examples

We now give examples of ordinary and differential difference equations that admit non mean periodic solutions. All of these examples will be scalar equations. The claims made for most of these equations may be verified by making use of the properties of Kahane Transforms shown in Chapter 1 and properties of spectral sets of mean periodic functions.

Our first example complements Theorem 2.1 of this chapter.

EXAMPLE A *Let* g *be a continuous periodic function of period* $2\pi/\beta$. *Let* $\hat{g}(n)$ *(the* n^{th} *Fourier coefficient of* g*) be non-zero for* $n = 1, 2, \ldots$ *and let* β *be irrational. Then there are no mean periodic solutions to the equation*

$$(2.6) \qquad x'(t) = e^{it}\, x(t) + g(t)$$

Clearly $x = 0$ is not a solution of (2.6). If we suppose that (2.6) has a non-zero mean periodic solution x, then on taking Kahane Transforms, we obtain, with $f(z) = K(x)(z)$

(2.7) $$z\, f(z) - x(0) = f(z - i) + K(g)(z) \quad .$$

Since $\hat{g}(n) \neq 0$, $K(g)(z)$ has a pole at $z = in\beta$ so $K(g)(in\beta)$ is infinite for $n = 1, 2, \ldots$. From (2.7), we see that at least one of $f(in\beta)$ and $f(in\beta - i)$ is infinite. If $f(in\beta)$ is infinite, with $in\beta \neq i$ and $K(g)(in\beta + i)$ finite, $f(in\beta + i)$ is infinite. Continuing in this manner, and using $in\beta + im \neq ip$ for any integer p, we conclude that $f(in\beta + im)$ is infinite for $m = 1, 2, \ldots$. Also, if for a fixed value of n, $f(in\beta - i)$ is infinite, then $f(in\beta - im)$ is infinite for $m = 1, 2, \ldots$.

Thus, $f(x)$ has poles at points including $\{z = n\beta i + \varepsilon_n im : m, n = 1, 2, \ldots\}$ where $\varepsilon_n = \pm 1$. Since

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{|n\beta i + \varepsilon_n im|^2} \geq \sum_m \sum_n \frac{1}{|n\beta + m|^2} = \infty \quad ,$$

we see, by (0.2), that $f(z) = K(x)(z)$ cannot be the Kahane Transform of any mean periodic function, x .

EXAMPLE(S) B *When* $a(t)$ *is any one of the exponential polynomials* $2t$, e^t *or* $\alpha \cos \alpha t + \beta \cos \beta t$ *(where* α/β *is irrational) any non-zero solution to the equation* $x'(t) = a(t)\, x(t)$ *is not mean periodic.*

If $a(t) = 2t$ and $x'(t) = a(t)\,x(t)$, then $x(t) = c\,e^{t^2}$. That e^{t^2} is not mean periodic has been noted by Laurent Schwartz [1].

If $x'(t) = e^t\,x(t)$, suppose that x is mean periodic. Taking the Kahane transform of this equation, from Proposition 1.2, we obtain with $f(z) = K(x)(z)$,

$$z\,f(z) - x(0) = f(z-1) \quad .$$

If $x \neq 0$ and x is mean periodic, then the spectral set S_x of x contains at least one point, say a, and so $f(a)$ is infinite. Then $(a+1)\,f(a+1)$ is infinite and so $f(a+1)$ is infinite. Thus $f(a+n)$ is infinite and so $a+n \in S_x$ for all $n \in N$. Since $\sum_{n=1}^{\infty} |a+n|^{-1} = \infty$, S_x cannot be the spectral set of a mean periodic function. Hence $x'(t) = e^t\,x(t)$ has no non-zero mean periodic solutions.

Now if $x'(t) = (\alpha \cos \alpha t + \beta \cos \beta t)\,x(t)$, then $x(t) = c \cdot e^{\sin\alpha t} \cdot e^{\sin\beta t}$ for some constant c. As $e^{\sin\alpha t}$, $e^{\sin\beta t}$ are two periodic functions with non-zero Fourier coefficients of respective periods $2\pi/\alpha$, $2\pi/\beta$ and as α/β is irrational, the product of these two functions is not mean periodic, (see remark following Theorem 0.6). Hence if $c \neq 0$, x is not mean periodic.

EXAMPLE C *The solutions to the equation* $x'(t) = 1 - x^2(t)$ *are not mean periodic.*

For a solution to this equation is $x(t) = \tanh\,(t+c)$. Since $x'(t) = \operatorname{sech}^2\,(t+c)$ is an absolutely integrable function, it is not mean periodic and so $x(t)$ is not mean periodic.

Before giving examples of differential difference equations that admit non mean periodic solutions, we may observe that the equation $x'(t) = a(t)\,x(t)$ may have all solutions mean periodic but $a(t)$ is not mean periodic. Two simple examples are

i) $a(t) = b + n/t$ (n a non-negative integer), a function that is discontinuous at the origin with $x(t) = c\,t^n e^{bt}$, an exponential polynomial, and

ii) $a(t) = \tanh t$, a non mean periodic function with $x(t) = \cosh t$, an exponential polynomial.

EXAMPLE D *The equation*

(2.8) $\quad x'(t) = -\,x(t/k) \qquad$ with $k > 1$

has no non-zero mean periodic functions.

This equation has been considered by G. Morris, [1] On $(0, \infty)$ it is seen to be a differential difference equation of retarded type as it may be written as

$$x'(t) = -\,x(t - \tau(t)) \quad \text{where} \quad \tau(t) = t(k-1)/k > 0 \quad .$$

Morris shows that any non-zero solution to this equation on R_+ oscillates unboundedly, and also, from seeking a solution $x(t) = \sum_{n\varepsilon Z} c_n\, e^{\alpha_n t}$, one obtains

$$x(t) = c\,'\sum_{n\varepsilon Z} k^{\frac{1}{2}(n+\alpha)^2} \exp(-t\, k^{n+\alpha+\frac{1}{2}}) \quad .$$

Suppose that equation (2.8) has a non-zero mean periodic function x. Then the Kahane transform of this equation, with $f(z) = K(x)(z)$ is

$$(2.9) \qquad z\, f(z) - x(0) = -\, k\, f(kz) \quad .$$

Now as x is supposed mean periodic, the spectral set, S_x, of x is non empty. Moreover, $S_x \neq \{0\}$ since there is no constant solution to (2.8). So there exists an $a \neq 0$ with $a \in S_x$. From (2.9), it is clear that $f(z)$ has a pole at $z = b$ if, and only if, $f(z)$ has a pole at $z = kb$. Thus $ak^n \in S_x$ for all integers n. Since $\sum\limits_{n \in z} |ak^n|^{-2} = \infty$, S_x cannot be the spectral set of a mean periodic function. Thus we have a contradiction and so equation (2.8) has no non-zero mean periodic solutions.

Our next example concerns a special case of the differential-difference equation

$$(2.10) \qquad x'(t) = p(t)\, x(t - \omega) \quad \text{with } p(t + \omega) = p(t), \quad p \text{ continuous.}$$

Systems of such equations have been considered by A. Stokes [1], and A. Halany [1].

Put $\int_0^\omega p(r)\, dr = \omega L$ so that $\int_0^t p(r)\, dr - Lt$ is periodic of period ω. If x satisfies both (2.10) and $x(t + \omega) = zx(t)$, then $x'(t) = p(t)x(t)/z$. Thus $x(t) = c \exp(\frac{1}{z} \int_0^t p(r)\, dr)$ and if

$$q(z, t) = c \exp\left(\frac{1}{z} \left(\int_0^t p(r)dr - Lt\right)\right) \quad ,$$

then $x(t) = q(z, t)\, e^{Lt/z}$, where $q(z, t)$ is periodic in t of period ω. Hence $x(t + \omega) = zx(t)$ and x is a solution of (2.10) if, and only if, $e^{\omega L/z} = z$.

When $\omega L \neq 0$, it is known that the equation $e^{\omega L/z} = z$ has an infinity of solutions. Consequently, equation (2.10) will admit many solutions or Floquet terms of the form $x(t) = q(z,t)\, e^{Lt/z}$. Any finite linear combination of such terms is a solution of (2.10) and is also mean periodic. However, there exist solutions of (2.10) that are limits of infinite series of such terms, and it remains unknown as to whether or not such limits are mean periodic.

EXAMPLE E *When* $\int_0^\omega p(r)\, dr = 0$, *the equation*

(2.10) $\quad x'(t) = p(t)\, x(t - \omega), \quad p(t + \omega) = p(t), \quad p$ continuous,

has precisely one family of periodic solutions

$$x(t) = c \exp\left(\int_0^t p(r)\, dr\right) \qquad (c \text{ constant})$$

and all other solutions are non mean periodic.

To show this, we use the terminology and a Theorem of A. Stokes, [1] . With $\int_0^\omega p(r)\, dr = 0$, $x(t) = c \exp\left(\int_0^t p(r)\, dr\right)$ is clearly a solution of (2.10) and it is periodic. Moreover, there is no other solution to (2.10) that also satisfies $x(t + \omega) = zx(t)$ for any complex z.

Thus, the period map T mapping the Banach space $X = C([-\omega, 0])$ into X defined by $Tx(r) = x(\omega + r)$ $(-\omega \leq r \leq 0)$ is a completely continuous operator that has only one non-zero eigen-value. If E is the corresponding eigen manifold and if P is the projection of X onto E, then by Theorem 1, Stokes [1], if ϕ is any function in X, if $R(\phi) = \phi - P(\phi)$ and if y is the solution agreeing with $R(\phi)$ on $[-\omega, 0]$, we have $e^{\alpha t}\| y\| \to 0$ as $t \to \infty$.

To complete the proof, it is only necessary to show that such a function cannot be mean periodic when it is non-zero. Suppose that y is non-zero and mean periodic, then, for each real α, there exists a positive M_α such that $|y(t)| \leq M_\alpha e^{-\alpha t}$ for $t \geq 0$.

In the half plane $Re(z) > -\alpha$, the Laplace transform of y exists, is analytic and coincides with the transform of Kahane. As α is any real number, it is clear that the transform of Kahane is an entire function and so y must be zero.

EXAMPLE F *The equation* $x'(t) = e^t x(t - \omega)$ *has no non-zero mean periodic solutions.*

Suppose that x is mean periodic. By use of Proposition 1.2 the Kahane transform of this equation yields with $f(z) = K(x)(z)$,

$$z\, f(z) = e^{-\omega z} f(z - 1) + H(z)$$

where $H(z)$ is an entire function. If $x \neq 0$ and x is mean periodic, the spectral set S_x contains at least one point, say a. Then $f(z)$ is infinite and so $(a + 1)\, f(a + 1)$ is infinite. Hence $f(a + 1)$ is infinite. Continuing in this manner, we find that $f(a + n)$ is infinite and so $a + n \in S_x$ for all $n \in N$.

Since $\Sigma|a + n|^{-1} = \infty$, S_x cannot be the spectral set of a mean periodic function (Chapter 0, (0.3) and §0.5) . Hence $x'(t) = e^t\, x(t - \omega)$ has no non-zero mean periodic solutions.

Chapter Three

SYSTEMS OF INTEGRAL EQUATIONS

Our main results in this chapter (Section 3.2; Theorem 3.4) concern the following system of integral equations

$$x_{ik}(t) - \sum_{j=1}^{n} \int_{o}^{t} g_{ij}(t - r)\, x_{jk}(r)\, dr = f_{ik}(t) \quad (1 \le i \le n,\ 1 \le k \le m)$$

or

$$X - G \circledast X = F$$

where $G = \{g_{ij}\}$ is an $n \times n$ matrix whose elements are continuous functions and F is an $n \times m$ matrix whose elements are continuous functions. We prove the existence of a unique continuous solution to this equation. Moreover, when F and G have certain properties such as mean periodicity, so has the solution. As well, we consider the system

$$X' - G \circledast X = F$$

and give (Section 3.3) a brief discussion of the system

$$G \circledast X = F \quad .$$

Throughout this chapter, the term 'continuous matrix' will denote a matrix whose elements are complex valued continuous functions defined on R . The term 'mean periodic matrix' will denote a continuous matrix whose elements are also mean periodic functions. By a matrix with mean period zero is meant a mean periodic matrix whose elements have mean period zero.

A sequence of $\ell \times m$ continuous matrices $\{F_p\}_{p=1}^{\infty}$ will be said to tend, locally uniformly as $p \to \infty$, to a matrix F if $F = \{f_{ij}\}$ is a $\ell \times m$ matrix, $F_p = \{f_{pij}\}$, and for $1 \leq i \leq \ell$, $1 \leq j \leq m$, each $f_{pij} \to f_{ij}$ locally uniformly as $p \to \infty$. In this case, we write $F_p \to F$ locally uniformly as $p \to \infty$ and we observe that F is a continuous matrix.

The statement and proof of our Theorem 3.4 are preceded by Section 3.1 that contains three propositions that are relevant to the proof of this theorem.

For more information about the linear functional equation, $x - g \circledast x = f$, and systems of such equations, the reader is referred to Bellman and Cooke ([1], Chapters 7, 8) who also give references to certain applications of these equations, and call such functional equations "renewal equations".

§3.1 Preliminaries

Propositions 3.1 and 3.2 are concerned with continuous matrices. These propositions may be regarded as extensions of properties of continuous functions given by Erdélyi ([1], §4.1) .

PROPOSITION 3.1 *Let* $G = \{g_{ij}\}$ *be an* $n \times n$ *continuous matrix and let* ℓ *be a positive constant. If* c *is another constant with* $|g_{ij}(t)| \le c$ *for* $|t| \le \ell$ *and* $1 \le i, j \le n$ *and if* $G_1 = G$, $G_{m+1} = G \circledast G_m = \{g_{m+1,ij}\}$, *then*

$$|g_{m+1,ij}(t)| \le c(cn|t|)^m/m! \le c(cn\ell)^m/m! \tag{3.1}$$

for $|t| \le \ell$.

PROOF. The formula is true for $m = 1$. By assuming

$$|g_{m,ij}(t)| \le c(cn|t|)^{m-1} / (m-1)! \quad ,$$

it follows that as $g_{m+1,ij}(t) = \sum_{p=1}^{n} g_{ip} \circledast g_{m,pj}$,

$$|g_{m+1,ij}(t)| \quad \le \quad \frac{n\ c\ c\ (cn)^{m-1}}{(m-1)!} \int_0^{|t|} r^{m-1}\, dr = \frac{c\ (cn\ell)^m}{m!}$$

when $|t| \le \ell$. So, by induction, the statement is true for all positive integers m .

COROLLARY. $G_m \to 0$ *locally uniformly as* $m \to \infty$.

PROPOSITION 3.2 *Let* a) $\{F_p\}$ *be a sequence of* $\ell \times m$ *continuous matrices and* $F_p \to F$ *locally uniformly as* $p \to \infty$,

b) $\{G_p\}$ *be a sequence of* $m \times n$ *continuous matrices and* $G_p \to G$ *locally uniformly as* $p \to \infty$, *and*

c) $H_p = F_p \circledast G_p$ *and* $H = F \circledast G$. *Then* $H_p \to H$ *locally uniformly as* $p \to \infty$.

PROOF. Throughout this proof, it will be assumed that the indices i, j, k will take the values $1 \leq i \leq \ell$, $1 \leq j \leq m$ and $1 \leq k \leq n$.

Let t_o be any positive number and choose an $\varepsilon > 0$. As $\{f_{pij}\}$ and $\{g_{pjk}\}$ are sequences of continuous functions that converge uniformly on $[-t_o, t_o]$ to f_{ij}, g_{jk} respectively, as $p \to \infty$, there exists a positive constant c such that

$$|f_{pij}(t)| \leq c \quad \text{and} \quad |g_{pjk}(t)| \leq c \quad \text{for} \quad |t| \leq t_o \quad \text{and all} \quad p \ .$$

Moreover, there exists an integer n_o such that

$$|f_{ij}(t) - f_{pij}(t)| < \varepsilon/2n\, t_o c \quad \text{and} \quad |g_{jk}(t) - g_{pjk}(t)| < \varepsilon/2n\, t_o c$$

whenever $|t| \leq t_o$ and $p > n_o$. As

$$h_{ik} - h_{pik} = \sum_j f_{ij} \circledast (g_{jk} - g_{pjk}) + \sum_j (f_{ij} - f_{pij}) \circledast g_{pjk} \ ,$$

and after applications of a well known and elementary estimate for the absolute value of an integral, it follows that

$$|h_{ik}(t) - h_{pik}(t)| < \varepsilon \quad \text{whenever} \quad p > n_o \quad \text{and} \quad |t| \leq t_o \ .$$

Hence $F_p \circledast G_p \to F \circledast G$ locally uniformly as $p \to \infty$.

COROLLARY. *If* F *and* G *are* $n \times n$ *continuous matrices and if* $F = F \circledast G$ *or* $F = G \circledast F$, *then* $F = 0$.

PROOF. If $F = F \circledast G$, then $F = (F \circledast G) \circledast G$ so $F = F \circledast G_m$ for all positive integers m, where G_m is defined as in Proposition 3.1 . As $G_m \to 0$ locally uniformly as $m \to \infty$, $F = 0$.

PROPOSITION 3.3 *Let* $x \in C^n(R)$ *and* $y \in C^{n-1}(R)$ *where* $n = 1, 2, \ldots$. *Then* $x \circledast y \in C^n(R)$ *and*

$$(3.2) \quad D^n(x \circledast y) = x(0)\, D^{n-1}y + Dx(0) \cdot D^{n-2}y + \ldots D^{n-1}x(0) \cdot y + (D^n x) \circledast y$$

PROOF. A proof may be given by induction. If $x \in C^1(R)$ and $y \in C(R)$, put $h = x(0)y + (Dx) \circledast y$. Then $h \in C(R)$ and if $e : t \to 1$,

$$\begin{aligned} e \circledast h &= x(0)\; e \circledast y + (Dx \circledast e) \circledast y \\ &= x(0)\; e \circledast y + (x - x(0)e) \circledast y = x \circledast y \end{aligned}$$.

Since $h \in C(R)$, $e \circledast h \in C^1(R)$ and so $x \circledast y \in C^1(R)$. Also $D(x \circledast y) = D(e \circledast h) = h$ so the formula is true for $n = 1$.

Now assume that the formula is true for $n = k$ where k is a positive integer. Assume also that $x \in C^{k+1}(R)$ and $y \in C^k(R)$ so that if

$$g = x(0)\, D^{k-1}y + \ldots D^{k-1}x(0) \cdot y + (D^k x) \circledast y \quad ,$$

then $g = D^k(x \circledast y)$. Moreover, since $D^k x \in C\ (R)$ and $D^{k-1}y \in C^1(R)$,

$(D^k x) \circledast y \in C^1(R)$ and $D((D^k x) \circledast y) = D^k x(0) \cdot y + (D^{k+1}x) \circledast y)$. Thus $g \in C^1(R)$ and

$$Dg = x(0) \cdot D^k y + \ldots D^{k-1}x(0) \cdot Dy + D^k x(0) \cdot y + (D^{k+1}x) \circledast y .$$

As $Dg = D^{k+1}(x \circledast y)$, $D^{k+1}(x \circledast y) \in C(R)$ and the forumla is true for $n = k + 1$. Hence, by induction, the statement is proved for all positive integers n .

Alternatively, one may verify the statement by showing that if $x \in C^k(R)$ and $y \in C^{k+1}(R)$, and if g is defined as above, then $e^{\circledast k} \circledast g = x \circledast y + P$ where P is some polynomial of degree less than k .

COROLLARY. *Let* F *be a* $\ell \times m$ *matrix whose elements are* $C^n(R)$ *functions and let* G *be an* $m \times n$ *matrix whose elements are* $C^{n-1}(R)$ *functions where* $n = 1, 2, \ldots$. *Then* $F \circledast G$ *is a* $\ell \times n$ *matrix whose elements are* $C^n(R)$ *functions.*

§3.2 Systems of Volterra Integral Equations of Convolution Type: Second Kind

THEOREM 3.4 *Let* F *be an* $n \times m$ *continuous matrix and* G *be an* $n \times n$ *continuous matrix. Then each of the systems of equations*

$$X - G \circledast X = F$$

$$X' - G \circledast X = F \quad , \qquad X(0) = C$$

has a unique continuous $n \times m$ *matrix solution. Furthermore* *a)* X *depends continuously on* F, G *(and* C*),*

b) if F *and* G *are mean periodic, then* X *is mean periodic,*

c) if F *and* G *have mean period zero, then* X *has mean period zero,*

d) if F *and* G *are indefinitely differentiable, then* X *is indefinitely differentiable, and*

e) if the elements of F *and* G *are exponential polynomials, then the elements of* X *are exponential polynomials.*

PROOF. The system of equations $X - G \circledast X = F$ is equivalent to m systems of the form $\underline{x} - G \circledast \underline{x} = \underline{f}$ and we shall consider this system. The system $X' - G \circledast X = F$, $X(0) = C$ is also equivalent to m systems of the form $\underline{x}' - G \circledast \underline{x} = \underline{f}$, $\underline{x}(0) = \underline{c}$ and this system shall be considered after $\underline{x} - G \circledast \underline{x} = \underline{f}$ has been dealt with.

Setting $G_1 = G$ and $G_{q+1} = G \circledast G_q$ put $\underline{x}_q = \underline{f} + G_1 \circledast \underline{f} + \ldots G_q \circledast \underline{f}$ for $q = 1, 2, 3, \ldots$. If x_{qi} denotes the i^{th} component of $\underline{x}_q$, from (3.1), it is apparent that $\{x_{qi}\}_{q=1}^{\infty}$ is a Cauchy sequence in $C(R)$, and so, has a limit, say x_i, in $C(R)$ for $i = 1, 2, \ldots n$. From

$$\underline{x}_q - G \circledast \underline{x}_q = \underline{f} - G_{q+1} \circledast \underline{f} \quad ,$$

it follows from Proposition 3.2 that $G \circledast \underline{x}_q$ and $G_{q+1} \circledast \underline{f}$ tend locally uniformly to $G \circledast \underline{x}$ and $\underline{0}$ respectively as $q \to \infty$. Thus, there exists a continuous solution $\underline{x}$ to the equation $\underline{x} - G \circledast \underline{x} = \underline{f}$.

That there is only one solution to this equation may be shown by assuming $\underline{z}$ to be the difference between any two solutions.

Then $\underline{z} = G \circledast \underline{z}$ and so $\underline{z} = G \circledast (G \circledast z) = G_q \circledast z$ for any integer q. As $G_q \to 0$ locally uniformly as $q \to \infty$, $\underline{z} = 0$.

For (a), we show that if $G_p \to G$ and $\underline{f}_p \to \underline{f}$ locally uniformly as $p \to \infty$ and if $\underline{x}_p - G_p \circledast \underline{x}_p = \underline{f}_p$, then $\underline{x}_p \to \underline{x}$ locally uniformly as $p \to \infty$ where $\underline{x} - G \circledast \underline{x} = \underline{f}$. From

$$\underline{x} - \underline{x}_p - G \circledast \underline{x} + G_p \circledast \underline{x} - G_p \circledast \underline{x} + G_p \circledast \underline{x}_p = \underline{f} - \underline{f}_p ,$$

if $\underline{z}_p = \underline{x} - \underline{x}_p$ and if $\underline{v}_p = \underline{x} \circledast (G - G_p) + \underline{f} - \underline{f}_p$, then $\underline{z} - G_p \circledast \underline{z}_p = \underline{v}_p$.

Now by Proposition 3.2, $\underline{v}_p \to \underline{0}$ locally uniformly as $p \to \infty$. If ℓ is any positive number, a positive constant c can be chosen so that if $G_p = \{g_{pij}\}$, then

$$|g_{pij}(t)| \leq c \quad \text{for} \quad |t| \leq \ell, \quad 1 \leq i, j \leq n \quad \text{and} \quad p = 1, 2, \ldots .$$

Also, if an $\varepsilon > 0$ is chosen, an integer n_o can be found for which $|v_{pi}(t)| < \varepsilon$ whenever $|t| \leq \ell$ and $p > n_o$. By expanding $\underline{z}_p$ as a series as above and by the estimates (3.1), we find that

$$|z_{pi}(t)| \leq \varepsilon + \varepsilon cn\ell + \ldots \quad \varepsilon(cn\ell)^q/q! + \ldots..$$

so

$$|z_{pi}(t)| \leq \varepsilon\, e^{cn\ell} \quad \text{when} \quad |t| \leq \ell \quad \text{and} \quad p > n_o \quad \text{for}$$

$i = 1, 2, \ldots n$.

Hence $\underline{z}_p \to 0$ or $\underline{x}_p \to \underline{x}$ locally uniformly as $p \to \infty$.

To prove (b), it suffices to take a single non-zero measure λ such that $\lambda * g_{ij} = 0 = \lambda * f_i$ for $i, j = 1, 2, \ldots n$. (That this can be done: if $y_1, y_2, \ldots y_n$ are mean periodic functions with $\lambda_i * y_i = 0$, $\lambda_i \neq 0$, λ_i a measure for $i = 1, 2, \ldots k$, put $\lambda = \lambda_1 * \lambda_2 * \ldots * \lambda_k$ so that λ is a non-zero measure and $\lambda * y_i = 0$).

With $\lambda * (g_{ij} \circledast x_j) = (\lambda * g^+_{ij}) * x_j$,

$$\lambda * x_i - \sum_{j=1}^{n} (\lambda * g^+_{ij}) * x_j = 0 \quad .$$

Let N be the $n \times n$ matrix whose elements are the measures

$$n_{ii} = \lambda - \lambda * g^+_{ii} \quad \text{and} \quad n_{ij} = -\lambda * g^+_{ij} \quad (i \neq j) \quad .$$

Then $N * \underline{x} = \underline{0}$. Now the expansion of $\det N$ gives $\lambda^{*n} * (\delta + h)$ where h is a sum of convolution products of functions (g^+_{ij}) that are zero on $(-\infty, 0)$. Then h is a function that is zero on $(-\infty, 0)$ and so $h + \delta \neq 0$. As $\lambda \neq 0$, $\lambda^{*n} * (\delta + h) \neq 0$. Thus the measure $\det N$ is non-zero and as $N * \underline{x} = \underline{0}$, $\underline{x}$ is mean periodic by Proposition 2.1 .

Part (c) is shown by observing that in the proof of (b), when $\underline{f}$ and G have mean period zero, it is possible to choose λ to have support contained in an interval of arbitrarily small length. Thus a non-zero measure, $\det N$, can be found so that $\det N * \underline{x} = 0$ with $\det N$ having support in an interval of arbitrarily small length, showing that $\underline{x}$ has zero mean period.

For (d), one may show that $\underline{x}$ is indefinitely differentiable when $\underline{f}$ and $\underline{G}$ are indefinitely differentiable by use of Proposition 3.3 and induction. Assuming that $\underline{x}$ is n times continuously differentiable leads to $G \circledast \underline{x}$, and so $\underline{x}$, being $n + 1$

times continuously differentiable. Hence $\underline{x}$ has derivatives of all orders.

When $\underline{f}$ and G have elements that are exponential polynomials (and so indefinitely differentiable), we may choose a non-zero linear differential operator, $L(D)$, with constant coefficients so that $L(D)f_i = 0$, $L(D)g_{ij} = 0$ for $i, j = 1, 2, \ldots n$. From Proposition 3.3, if $x, y \in C^\infty(R)$,

$$L(D)\ (x \circledast y) = M(D)x + x \circledast L(D)y$$

where $M(D)$ is a linear ordinary differential operator with constant coefficients depending only on $L(D)$ and y and whose order is less than that of $L(D)$. From

$$x_i - \sum_{j=1}^{n} g_{ij} \circledast x_j = f_i \qquad (i = 1, 2, \ldots n) \quad ,$$

we obtain $L(D)x_i - \sum_{j=1}^{n} M_{ij}(D)\, x_j = 0$

or

$$T\underline{x} = 0 \quad .$$

Here $T = \{T_{ij}\}$ is an $n \times n$ matrix of linear differential operators, and if $S(D) = \det T$, $S(D)\, x_i = 0$ for $i = 1, 2, \ldots n$.

As $T_{ii} = L(D) - M_{ii}(D)$ and $T_{ij} = -M_{ij}(D)$ $(i \neq j)$, an examination of the expansion of $\det T$ shows that there is only one term containing $(L(D))^n$ and all other terms contain operators $M_{ij}(D)$. Since the order of each $M_{ij}(D)$ is less than the order of $L(D)$, it is clear that $S(D) = \det T$ is a non-zero linear differential operator.

Hence $\underline{x}$ has exponential polynomial components.

Turning now to the second equation, or

$$\underline{x}' - G \circledast \underline{x} = \underline{f} \quad \text{with} \quad \underline{x}(0) = \underline{c} \quad ,$$

we may integrate this equation to get

$$\underline{x} - H \circledast \underline{x} = \underline{v} + \underline{c}$$

where $H = \{h_{ij}\}$, $h_{ij}(s) = \int_0^s g_{ij}(r)\, dr$, and $v_i(s) = \int_0^s f_i(r)\, dr$. As $\underline{f}$ and G are continuous, $\underline{v} + \underline{c}$ and H are continuous and so there exists a unique continuous solution $\underline{x}$.

If $\underline{f}_p \to \underline{f}$ and $G_p \to G$ locally uniformly as $p \to \infty$ and if $v_{pi}(t) = \int_0^t f_{pi}(r)\, dr$ and $h_{pij}(t) = \int_0^t g_{pij}(r)\, dr$, then $\underline{v}_p \to \underline{v}$ and $H_p \to H$ locally uniformly as $p \to \infty$. If as well, $\underline{c}_p \to \underline{c}$ as $p \to \infty$, we see that if $\underline{x}_p$ is the solution of

$$\underline{x}'_p - G_p \circledast \underline{x}_p = \underline{f}_p \quad , \quad \underline{c}_p = \underline{x}_p(0) \quad ,$$

then $\underline{x}_p \to \underline{x}$ locally uniformly as $p \to \infty$.

When $\underline{f}$ and G are mean periodic, $\underline{v} + \underline{c}$ and H are mean periodic and so $\underline{x}$ is mean periodic. Also, when $\underline{f}$ and G have zero mean period; $\underline{v} + \underline{c}$ and H have mean period zero and so $\underline{x}$ has zero mean period. It is also clear that when $\underline{f}$ and G are indefinitely differentiable, so is $\underline{x}$, and when $\underline{f}$ and G have elements that are exponential polynomials, so does $\underline{x}$.

§3.3 Systems of Volterra Integral Equations of Convolution Type: First Kind

THEOREM 3.5 *Let* G *be an* $n \times n$ *continuously differentiable matrix with* $\det G(0) \neq 0$. *Let* F *be an* $n \times m$ *continuously differentiable matrix. Then the system of equations* $G \circledast X = F$ *admits a continuous* $n \times m$ *matrix solution if, and only if,* $F(0) = 0$. *Moreover,*

a) if F *and* G *are mean periodic, then* X *is mean periodic;*

b) if F *and* G *have zero mean period, then* X *has zero mean period;*

c) if F *and* G *are indefinitely differentiable,* X *is indefinitely differentiable; and*

d) if F *and* G *are matrices of exponential polynomials,* X *is a matrix of exponential polynomials.*

PROOF. From $\sum_{j=1}^{n} g_{ij} \circledast x_{jk} = f_{ij}$, the condition that $F(0) = 0$ is readily seen to be necessary. To show that it is sufficient along with the other hypotheses, let $A = G(0)$ so that A is non-singular. Write $B = A^{-1}$ and consider the system of equations

$$\underline{X} + (B \cdot DG) \circledast \underline{X} = B \cdot DF \quad .$$

Since $B \cdot DG$ and $B \cdot DF$ are continuous, Theorem 3.4 guarantees the existence of a continuous solution $\underline{x}$. On integration of this system, we obtain, with $F(0) = 0$,

$$B(G \circledast X) = BF$$

and so X satisfies $G \circledast X = F$.

The remaining statements are proved by directly appealing to Theorem 3.4 .

In the above theorem, it may be noted that the condition $\det G(0) \neq 0$ is a sufficient but not a necessary condition for the existence of a continuous solution. (An example of this is given for the scalar equation $g \circledast x = f$ in the next chapter.)

When $G \circledast X = F$ has a continuous solution, it is possible to give a sufficient condition for X to be mean periodic (cf. Proposition 1.6). In the following theorem, $\det G$ denotes a determinant formed from the commutative ring of continuous functions with the operations of addition and truncated convolution.

THEOREM 3.6 *Let* F *be an* $n \times m$ *mean periodic matrix and let* G *be an* $n \times n$ *mean periodic matrix with* $\det G \neq 0$. *If the system of equations* $G \circledast X = F$ *admits a continuous solution, it is mean periodic.*

PROOF. Let $F = \{f_{ik}\}$ and $G = \{g_{ij}\}$. Choose a non-zero measure μ for which $\mu * f_{ik} = 0$, $\mu * g_{ij} = 0$ and put $\nu_{ij} = \mu * \overset{+}{g}_{ij}$ for $i, j = 1, 2, \ldots n$ and $k = 1, 2, \ldots m$. Also let V denote the matrix $\{\nu_{ij}\}$ of measures and let $V_{\ell i}$ denote the cofactor of $\nu_{\ell i}$ in this matrix (here, multiplication is convolution of measures). As well, put $\nu = \det V$.

From $\sum_{j=1}^{n} g_{ij} \circledast x_{jn} = f_{ik}$,

$$\sum_{j=1}^{n} \nu_{ij} * x_{jk} = 0$$

so $\sum_{i=1}^{n} V_{\ell i} * \sum_{j=1}^{n} \nu_{ij} * x_{jk} = 0$ and $\nu * x_{ik} = 0$ for $i = 1, 2, \dots n$, $k = 1, 2, \dots m$. Thus, to show X is mean periodic, it suffices to show that the measure ν is non-zero. To do this, we make use of the definition and properties of Kahanes transform given in Theorem 1.1, and also, the fact that if f is mean periodic, $K(f) = 0$ if, and only if $f = 0$.

From $\nu = \det V$, $\hat{\nu} = \det \hat{V}$. Also

$$\hat{\nu} = \det(\mu * \overset{+}{g}_{ij})^{\wedge} = \det\,(\hat{\mu} \cdot K(g_{ij}))$$

$$= (\hat{\mu})^n \det K(g_{ij}) = (\hat{\mu})^n K(\det G) \quad .$$

Since by hypothesis, $\det G \neq 0$, $K(\det G) \neq 0$ and so $\hat{\nu}$ is non-zero. Hence ν is non-zero and so X is mean periodic.

REMARK. A simpler method of proof is as follows. Let G_{ij} denote the cofactor of g_{ij} in the matrix G, so that each G_{ij} is mean periodic. From

$$\sum_{j=1}^{n} g_{ij} \circledast x_{jk} = f_{ik} \quad ,$$

we obtain $\det G \circledast x_{jk} = \sum_{i=1}^{n} G_{ji} \circledast f_{ik}$.

By Proposition 1.6, with $\det G \neq 0$, we find that each x_{jk} is mean periodic.

Chapter Four

RINGS ASSOCIATED WITH MEAN PERIODIC FUNCTIONS

We have noted in Chapter 0 that the function space $C(R)$ is an algebra over $\mathbb{C}$ and a commutative ring with the operations of addition and truncated convolution. It was also noted (Theorem 0.6) that each of MQ, MP_o and MP are subalgebras of $C(R)$.

In Section 4.1, we describe some of the ideals to be found in the ring $C(R)$, along with some of the ideals of the sets MP and MP_o, each regarded as ring and an algebra. In discussing ideals, we shall find it convenient to let V denote any one of the rings $C(R)$, MP or MP_o taken with the operations of addition and truncated convolution. Thus V will be a commutative ring and an algebra over $\mathbb{C}$. Consequently, an ideal of V will be a subspace of V that contains $x \circledast y$ whenever x belongs to V and y belongs to the ideal.

Section 4.2 contains a brief description of some of the ideals in the rings $C^\infty(R)$, $C^\infty(R) \cap MP$ and $C^\infty(R) \cap MP_o$.

For the ring of exponential polynomials, it will be shown that an unusual situation occurs in that all of the ideals in this ring form a single descending chain. As this is not the case in any of the other rings considered in Sections 4.1, 4.2, discussion of ideals in the ring MQ is given separately in Section 4.3.

Section 4.4 concerns the embedding of the ring of exponential polynomials into a ring that has an identity and is a Euclidean Domain.

The results of Sections 4.3, 4.4 have already been incorporated into a paper of the author (Laird, [3]) .

§4.1 Ideals in the Rings $C(R)$, MP, MP_0

With V denoting any one of these three rings, a restatement of parts of Theorem 3.4 gives:

If $f, g \in V$, *then the equation* $x - g \circledast x = f$ *has a unique solution,* x, *in* V .

From this, we obtain

PROPOSITION 4.1 *Let* $f, g \in V \quad C^n(R)$. *If*

$$\begin{array}{llll} & D^k f(0) = 0 & \textit{for} & k = 0, 1, \ldots m, \\ & D^k g(0) = 0 & \textit{for} & k = 0, 1, \ldots m-1 , \\ \textit{and} & D^m g(0) \neq 0 & \textit{where} & m < n, \end{array}$$

then the integral equation $x \circledast g = f$ *has a solution,* x, *in* V .

PROOF. When $f, g \in C^n(R) \cap V$, $m < n$ and $a = D^m g(0) \neq 0$, the equation

$$ax + (D^{m+1}g) \circledast x = D^{m+1}f$$

has a solution, x, in V . On convolving this equation with e (i.e., on integration of the equation from 0 to t),

$$ax \circledast e - (D^m g - ae) \circledast x = D^m f - D^m f(0) \cdot e \quad .$$

As $D^m f(0) = 0$, $(D^m g) \circledast x = D^m f$.

With m further integrations, using

$$D^k g(0) = 0 = D^k f(0) \qquad \text{for} \quad k = 0, 1, \dots m-1 ,$$

$$x \circledast g = f \qquad .$$

REMARK. When f, g are continuous functions and when $g(0) \neq 0$, the condition that $f(0) = 0$ is clearly necessary to ensure that x is continuous when $x \circledast g = f$. Although the requirement of f, g being continuously differentiable may appear extraneous to the existence of a continuous function x with $x \circledast g = f$, one cannot assume that f, g are merely continuous. One reason for this is that if x is "locally integrable" (and not necessarily continuous) and if g is continuous, then $f = x \circledast g$ is continuous (see, for example, Erdélyi, [1], pages 9, 27).

PROPOSITION 4.2 *If* V *denotes any one of the rings* $C(R)$, MP *or* MP_o, *the following are ideals in* V *for* $n = 0, 1, 2, \dots$.

a) $I(x) = \{x \circledast y : y \in V\}$ *when* $x \in V$

b) $J(x) = \{\lambda x + x \circledast y : \lambda \in ¢, y \in V\}$ *when* $x \in V$

c) $W = C^1(r) \cap V$

d) $X_n = \{x \in C^n(R) \cap V : D^k x(0) = 0$ *for* $k = 0, 1, \dots n\}$

e) $Y_n = \{x \in C^{n+1}(R) \cap V : D^k x(0) = 0$ *for* $k = 0, 1, \dots n\}$

f) $Z_n = \{x \in C^{n+2}(R) \cap V : D^k x(0) = 0$ *for* $k = 0, 1, \dots n\}$

g) $I_{\alpha,\beta} = \{x \in V : x(t) = 0$ for $\alpha \leq t \leq \beta$ *where* $\alpha < 0 < \beta$.

PROOF. It is apparent that each of these subsets of V are also subspaces of V. To show that W, X_n, Y_n, Z_n $(n = 0, 1, 2, \dots)$

are ideals of V, we may use Proposition 3.3 that when restated says that if $x \in C^n(R)$ and if $y \in C^{n-1}(R)$, then $x \circledast y \in C^n(R)$ and

$$D^k(x \circledast y) = x(0)D^{k-1}y + \dots D^{\ell-1}x(0)D^{k-\ell}y + \dots D^{k-1}x(0) \cdot y + (D^k x) \circledast y$$

for $k \le n$. Thus, if $x \in W = C^1(R) \cap V$ and if $y \in V$, then $x \circledast y \in W$ showing that the subspace W of V is an ideal of V. If $x \in X_n$, $y \in V$, then $(x \circledast y)(0) = 0$ and

$D(x \circledast y) = x(0)y + (Dx) \circledast y = (Dx) \circledast y$. Also, if $k \le n$, if $x \in X_n$, and if $y \in V$, $D^k(x \circledast y) = (D^k x) \circledast y$ showing that $x \circledast y \in C^n(R) \cap V$ and $D^k(x \circledast y)(0) = 0$. Thus $x \circledast y \in X_n$ and so X_n is an ideal of V for $n = 0, 1, 2, \dots$.

If $x \in Y_n = X_n \cap C^{n+1}(R)$ and if $y \in V$, then

$$D^n(x \circledast y) = (D^n x) \circledast y \quad \text{so} \quad D^{n+1}(x \circledast y) = D^n x(0) \cdot y + (D^{n+1}x) \circledast y = (D^{n+1}x) \circledast y$$

Thus $x \circledast y \in Y_n$ and so Y_n is an ideal of V for $n = 0, 1, 2, \dots$.

If $x \in Z_n = Y_n \cap C^{n+2}(R)$ and if $y \in V$, then

$$D^{n+1}(x \circledast y) = (D^{n+1}x) \circledast y \quad \text{so} \quad D^{n+2}(x \circledast y) = D^{n+1}x(0) \cdot y + (D^{n+2}x) \circledast y .$$

Thus $x \circledast y \in Y_n \cap C^{n+2}(R) = Z_n$ and so Z_n is an ideal of V for $n = 0, 1, 2, \dots$.

For (g), when $\alpha < 0 < \beta$, $I_{\alpha,\beta}$ is seen to be an ideal of V for if $x \in I_{\alpha,\beta}$ and if $y \in V$, then $x \circledast y : t \to \int_0^t x(r)y(t-r)dr$ is zero for $\alpha \le t \le \beta$ and

$x \circledast y \in V$. Thus $x \circledast y \in I_{\alpha,\beta}$ and so $I_{\alpha,\beta}$ is an ideal of V .

REMARKS. It is clear that $I(x) \subset J(x)$. When $x \neq 0$, $x \notin I(x)$ and $I(x) \neq J(x)$.

It may be noted that neither of $\{x \in V : x(0) = 0 = Dx(0)\}$ nor $V \cap C^2(R)$ are ideals of V . The former set is not well defined and for the latter, if $V = C(R)$, if $x = e : t \to 1$ and if $y \in C(R)\backslash C^1(R)$, then $x \in C^2(R)$. However, $D(x \circledast y) = y \notin C^1(R)$ so $x \circledast y \notin C^2(R)$ and so the subspace $C^2(R)$ is seen not to be an ideal of $C(R)$.

For the same reasons, neither of

$$\{x \in V \cap C^n(R) : D^k x(0) = 0 \quad \text{for } k = 0, 1, 2, \ldots n+\ell\}$$

nor

$$\{x \in V \cap C^{n+2+\ell}(R) : D^k x(0) = 0 \quad \text{for } k = 0, 1, \ldots n\}$$

is seen to be an ideal in V when ℓ is a positive integer and n is any non-negative integer.

In the proof of the above proposition, we have noted that $X_n \supset Y_n \supset Z_n$. If is apparent that $Y_o = W \cap X_o$ and $Y_{n+1} = Z_n \cap X_{n+1}$, also $X_1 \subset Y_o \subset W$ and $X_{n+2} \subset Y_{n+1} \subset Z_n$. The relationship between the ideals is illustrated as follows:

$$
\begin{array}{ccccccc}
V & \supset & W & & & & \\
\cup & & \cup & & & & \\
X_o & \supset & Y_o & \supset & Z_o & & \\
 & & \cup & & \cup & & \\
 & & X_1 & \supset & Y_1 & \supset & Z_1 \\
 & & & & \cup & & \cup
\end{array}
$$

$$
\begin{array}{ccccccc}
\cup & & \cup & & & & \\
X_n & \supset & Y_n & \supset & Z_n & & \\
 & & \cup & & \cup & & \\
 & & X_{n+1} & \supset & Y_{n+1} & \supset & Z_{n+1} \\
 & & & & \cup & & \cup \\
 & & & & X_{n+2} & &
\end{array}
$$

and $\bigcap_{n=0}^{\infty} X_n = \bigcap_{n=0}^{\infty} Y_n = \bigcap_{n=1}^{\infty} Z_n = {}_{\alpha}\bigcap_{\beta} \{I_{\alpha,\beta} : \alpha < 0 < \beta\}$.

Also, if $\alpha' < \alpha < 0$ and $0 < \beta < \beta'$,

$$
\begin{array}{ccc}
I_{\alpha',\beta'} & \subset & I_{\alpha,\beta'} \\
\cap & & \cap \\
I_{\alpha',\beta} & \subset & I_{\alpha,\beta}
\end{array} \quad .
$$

PROPOSITION 4.3 *Let* $g \in V$. *If* K *is any one of the ideals* W, X_n, Y_n, Z_n $(n = 0, 1, 2, \ldots)$ *or* $I_{\alpha,\beta}$ $(\alpha < 0 < \beta)$ *defined in Proposition 4.2, then*

$$g \in K \Leftrightarrow J(g) \subset K$$

Also, if $g \in W$, *then* $J(g) = W \Leftrightarrow g(0) \neq 0$,
and, if $g \in Z_n$, *then* $J(g) = W_n \Leftrightarrow D^{n+1}g(0) \neq 0$.

PROOF. If $g \in K$, where K is any one of the above ideals, $g \circledast y \in K$ for all $y \in V$. As K is a subspace of V, it follows that

$$J(g) = \{\lambda g + g \circledast y : \lambda \in \mathbb{C}, y \in V\} \subset K \quad .$$

It is clear that if $J(g) \subset K$, then $g \in K$.

Suppose that $g \in W$ and $g(0) \neq 0$. If $z \in W$, we may choose $b \in \mathbb{C}$ so that $f(0) = 0$ where $f = z - bg$. As f, $g \in W = C^1(R) \cap V$, and as $f(0) = 0$ and $g(0) \neq 0$, Proposition 4.1 guarantees the existence of an $x \in V$ with $g \circledast x = f$. Thus $z = bg + g \circledast x$ and so $z \in J(g)$. Hence $W \subset J(g)$ and as $J(g) \subset W$, $J(g) = W$.

Conversely, if $g \in U$ and $g(0) = 0$, $g \subset Y_o$ so $J(g) \subset Y_o$. Since $Y_o \subset W$ and $Y_o \neq W$, we see that $J(g) = W$ entails $g \notin Y_o$ or $g(0) \neq 0$.

Similarly, if $g \in Z_n/Y_{n+1}$ $(n = 0, 1, 2, \ldots)$ so that $g \in C^{n+2}(R)$, $D^k g(0) = 0$ for $k = 0, 1, 2, \ldots n$ and $D^{n+1}g(0) \neq 0$, let $z \in X_n$ and choose b so that $D^{n+1}f(0) = 0$ where $f = z - bg$. As $f,\ g \in C^{n+2}(R)$, $D^k f(0) = 0 = D^k g(0)$ for $k = 0, 1, \dots n$, there exists an $x \in V$ with $g \circledast x = f$ by Proposition 4.1 . Thus $z = bg + g \circledast y$ so $z \in J(g)$. Hence $V_n \subset J(g)$ and as $J(g) \subset V_n$, $J(g) = V_n$.

Conversely, if $g \in Y_{n+1} \subset Z_n$, $J(g) \subset Y_{n+1}$. As $Y_{n+1} \neq Z_n$, we see that $J(g) = Z_n$ entails $g \notin Y_{n+1}$.

§4.2 Ideals in the Rings $C^\infty(R)$, $C^\infty(R) \cap MP$, $C^\infty(R) \cap MP_o$

It is a consequence of Proposition 3.3 that $C^\infty(R)$ $(+, \circledast)$ is a ring and an algebra over $\mathbb{C}$. Thus $C^\infty(R) \cap MP$ and $C^\infty(R) \cap MP_o$ are rings and algebras over $\mathbb{C}$. If W denotes any one of these three rings, it follows from Section 4.1 and Theorem 3.4 that:

a) If $f,\ g \in W$, there exists a unique $x \in W$ satisfying $x - g \circledast x = f$,

b) If $f,\ g \in W$, if $D^k f(0) = 0$ for $k = 0, 1, 2, \dots m$ and if $D^m g(0) \neq 0$, then there exists an $x \in W$ satisfying $g \circledast x = f$,

c) If $W_n = \{x \in W : D^k x(0) = 0$ for $k = 0, 1, 2, \dots n\}$, then W_n is an ideal of W for $n = 0, 1, 2, \dots$, and

d) If $\alpha \le 0 \le \beta$, $I_{\alpha,\beta} = \{x \in U : x(t) = 0$ for $\alpha \le t \le \beta\}$ is an ideal of W .

With the above properties of W and by modifying the proof of Proposition 4.3, if

$$J(g) = \{\lambda g + x \circledast g : \lambda \in \mathbb{C},\ x \in W\} \quad \text{for } g \in W, \text{ then}$$

i) $g \in W_n \Leftrightarrow J(g) \subset W_n$

ii) $g \in I_{\alpha,\beta} \Leftrightarrow J(g) \subset I_{\alpha,\beta}$

iii) $g(0) \neq 0 \Leftrightarrow J(g) = W$

iv) If $D^k g(0) = 0$ for $k = 0, 1, \ldots n$ and if $D^{n+1} g(0) \neq 0$, then $J(g) = W_n$.

§4.3 Ideals in the Ring of Exponential Polynomials

The remainder of this chapter shall be concerned only with the ring X (and algebra over $\mathbb{C}$) of exponential polynomials with the operations of addition and truncated convolution.

DEFINITION. *Let* x *be an exponential polynomial. Then the* *__degree__* *of* x *is zero if* $x(0) \neq 0$ *and* n *if* $D^k x(0) = 0$ *for* $k = 0, 1, 2, \ldots n-1$ *but* $D^n x(0) \neq 0$. *When* $x \in X$ *has degree* n, *we write* $\deg x = n$.

We note that as any exponential polynomial is an entire function, if $x \in X$ and $x \neq 0$, then x has finite degree. If $x = 0$, we regard the degree of x as being infinite.

PROPOSITION 4.4 *Let* $f, g \in X$. *Then*

a) there exists a unique $x \in X$ *satisfying* $x - g \circledast x = f$, *and*

b) when $g \neq 0$, *a necessary and sufficient condition for the equation* $x \circledast g = f$ *to have a solution* x *is that* $\deg f > \deg g$.

PROOF. Part a) is a special case of Theorem 3.4. The sufficiency condition for part b) is shown as for Proposition 4.1, and by using a).

Conversely, suppose that $\deg f = m$ and that $m \le \deg g$. Suppose also that there exists an $x \in X$ for which $x \circledast g = f$. If $m = 0$ so that $f(0) \neq 0$, there is a contradiction of $x \circledast g(0) = 0$. If $m > 0$, the relation $x \circledast g = f$ leads, after differentiating m times, to a similar contradiction. Hence it is necessary that $\deg f > \deg g$ for $x \circledast g = f$ to have a solution in X.

PROPOSITION 4.5 *Let* f *be any non-zero exponential polynomial of degree* n. *If*

$$I_f = \{f \circledast g : g \in X\}, \qquad \textit{and if}$$

$$Y_n = \{y \in X : \deg y > n\},$$

then

$$I_f = Y_n \quad .$$

PROOF. We firstly show that if $f,\ g \in X$, $f,\ g \neq 0$, then

$$\deg(f \circledast g) = \deg f + \deg g + 1 . \tag{4.1}$$

From $f \circledast g(0) = 0$, $\deg(f \circledast g) \ge 1$. From Proposition 3.3,

$$D^k(f \circledast g) = f(0) \cdot D^{k-1}g + \ldots\ D^{k-1}f(0) \cdot g + (D^k f) \circledast g$$

so that

$$D^k(f \circledast g)(0) = \sum_{\ell=1}^{k} D^{k-\ell}f(0) \cdot D^{\ell-1}g(0) \quad .$$

Let $\deg f = m$ and $\deg g = n$. If $m = 0 = n$, then $f(0) \neq 0$, $g(0) \neq 0$ so $D(f \circledast g)(0) = f(0)\, g(0) \neq 0$. As $f \circledast g(0) = 0$, $\deg(f \circledast g) = 1$. If $m + n > 0$ and $m + n \geq k > 0$, then $D^k(f \circledast g)(0) = 0$. However, $D^{m+n+1}(f \circledast g)(0) = D^m f(0) \cdot D^n g(0) \neq 0$, and so $\deg(f \circledast g) = \deg f + \deg g + 1$.

Hence, if $h = f \circledast g \in I_f$, $\deg h > \deg f = n$ and so $I_f \subset Y_n$. The reverse inclusion is a consequence of Proposition 4.4 since if $h \in Y_n$, $\deg h > \deg f$ and so $h = x \circledast f$ for some $x \in X$.

PROPOSITION 4.6 *Let* I *be any non-trivial ideal of* X . *Then* $I = Y_n$ *for some non-negative integer* n .

PROOF. It is trivial that each Y_n is an ideal of X. If J is any ideal of X that contains an element x for which $x(0) \neq 0$ and $z(0) = bx(0)$ where z is any element of X, then, by Proposition 4.4, the equation $Dz - bDx = (Dx) \circledast y + x(0)y$ has a solution y in X. On integration, this equation yields $z = bx + x \circledast y$ and so $z \in J$. Hence $J = X$.

Let I be any non-trivial proper ideal of X and x be any element of I so that $x(0) = 0$. Thus $x = e \circledast Dx$ and so $x \in I_{Dx}$.

Now let $h \in I_{Dx}$ with $h = g \circledast Dx$ where $g \in X$. With $x(0) = 0$,

$$h = g \circledast Dx = D(g \circledast x) = g(0)x + (Dg) \circledast x$$

and so $h \in I$. Thus $x \in I_{Dx} \subset I$ for all $x \in I$.

By Proposition 4.5, $I_{Dx} = Y_{n(x)}$ for some non-negative integer $n(x)$. Hence $I = \bigcup_{x \varepsilon I} Y_{n(x)}$ and as

$$\ldots \subset Y_m \subset Y_{m-1} \quad \subset Y_1 \subset Y_o \quad ,$$

we see that $I = Y_n$ for some non-negative integer n.

Notes: Since $X \supset Y_o \supset Y_1 \supset \ldots . Y_n \supset Y_{n+1} \supset \ldots$, by definition of each Y_n, Theorem 4.6 states that these are the only ideals in X. With the observations that $Y_n = I_{u_n}$ where $u_n : t \to t^n$ and that the trivial ideal is none other than I_o, we see that any proper ideal is of the form $I_f = \{g \circledast f : g \varepsilon X\}$ for some $f \varepsilon X$.

It is clear that no non-trivial proper ideal of X is semiprime or prime since for any $f \varepsilon X$, I_f contains $f \circledast f$ but not f. Thus, the prime radical of X, being the intersection of all prime ideals of X, is zero.

However, if I is a non-trivial proper ideal of X so that $I = Y_n$ for some non-negative integer n, then $x^{\circledast(n+2)}$ belongs to Y_n for all $x \varepsilon X$. Adopting the definition of the radical of an ideal given by Jacobson ([1], page 173) as the set

$$\{x \varepsilon X : x^{\circledast m} \varepsilon I \text{ for some positive integer } m\} ,$$

we see that the radical of any non-trivial ideal is X, and so all ideals of X are primary.

§4.4 An Euclidean Domain

The ring X may be embedded in a ring U consisting of ordered pairs (a, x) where $a \in ¢$ and $x \in X$ and where addition and multiplication are defined as

$$(a, x) + (b, y) = (a + b, x + y)$$

$$(a, x) \circledast (b, y) = (ab, ay + bx + x \circledast y) .$$

(See, for example, Jacobson, [1], page 85 for applicable details that show U is a ring with identity (1, 0)) . Moreover, U is an algebra over C .

The following propositions give details about U .

PROPOSITION 4.7 U *is an integral domain. An element* $(a, x) \in U$ *is a unit if, and only if,* $a \neq 0$.

PROOF. Let $(a, x) \in U$ and $a \neq 0$. By Proposition 4.4, there exists $y \in X$ such that

$$ay + \frac{1}{a}x + x \circledast y = 0$$

and so $(a, x) \circledast (\frac{1}{a}, y) = (1, 0)$. So, if $a \neq 0$, (a, x) has an inverse in U and so (a, x) is a unit.

Clearly, there is no $(b, y) \in X$ such that $(0, x) \circledast (b, y) = (1, 0)$ and so (a, x) is a unit if, and only if, $a \neq 0$.

To show that U is an integral domain, let $(a, x) \circledast (b, y) = (0, 0) = 0$ and suppose that $(a, x) \neq 0$. If

$a \neq 0$, (a, x) is a unit and so $(b, y) = 0$. If $a = 0$, then $x \neq 0$ and from $(a, x) \circledast (b, y) = 0$, $bx + x \circledast y = 0$. Since $x \neq 0$, bx must be zero, and so $b = 0$ and $x \circledast y = 0$. Since X has no non-zero divisors of zero, $y = 0$.

Thus $(b, y) = 0$ and so U has no non-zero divisors of zero.

PROPOSITION 4.8 *If* I *is any ideal of* X, *then* $(0, I)$ *is a proper ideal of* U. *If* J *is any ideal of* U, *then* $J = (0, I)$ *for some ideal and subspace* I *of* X. *Also, if* $(0, x) \in J$ *where* $x(0) \neq 0$, *then* $J = (0, X)$.

PROOF. Let I be an ideal of X. If $x \in I$ and if $(b, y) \in U$, then $(0, x) \circledast (b, y) = (0, bx + x \circledast y) \in (0, I)$. Hence $(0, I)$ is a proper ideal of U.

Let J be any proper ideal of U. Then J contains no units. By Proposition 4.7, $(a, x) \in J$ entails $a = 0$ and so, $J = (0, S)$ for some subset S of X. If $(0, x)$, $(0, y) \in J$, then $(0, x + y) \in J$ so $x, y \in S \Rightarrow x + y \in J$. Also $(0, x) \circledast (c, 0) = (0, cx) \in J$ so that $cx \in S$ for all $c \in ¢$ and $x \in S$. Thus S is a subspace of X. Now if $(0, x) \in J$ and $(b, z) \in U$, $bx + x \circledast z \in S$ and so $x \circledast z \in S$, that is, if $x \in S$ and $z \in X$, $x \circledast z \in S$. Thus S is an ideal and so $J = (0, I)$ for some ideal and subspace I of X.

Suppose J is proper ideal of U that contains $(0, x)$ where $x(0) \neq 0$. Let z be any element of X. Choosing $b \in ¢$ so that $z(0) = bx(0)$, Proposition 4.4 guarantees the existence of a $y \in X$ so that $z - bx = x \circledast y$. Thus $(0, z) = (b, y) \circledast (0, x)$ and so $(0, X) \subset J$. Hence $J = (0, X)$.

An immediate application of the preceding work is that all ideals of U form a single ascending chain

$$U \supset (0, X) \supset (0, Y_o) \supset \ldots \; (0, Y_n) \supset (0, Y_{n+1}) \supset \ldots$$

It is also easy to see that each of these ideals are principal ideals. Since

$$(1, -e) \circledast (0, x + e_1 \circledast x) = (0, x) \quad \text{for all} \quad x \in X,$$

$(0, X)$ is generated by $(1, -e)$.

As $(0, u_n) \circledast (0, n!\, D^{n+1}x) = (0, n!\, u_n D^{n+1}x) = (0, x)$ for each $x \in Y_n$, $(0, Y_n)$ is generated by $(0, u_n)$.

Thus U is a principal ideal domain.

Jacobson ([1], page 122) defines an Euclidean Domain U as an integral domain with an identity where there exists a function $\delta(A)$ defined on U for which

1) $\delta(A)$ is a non-negative integer, $\delta(A) = 0 \Leftrightarrow A = 0$;
2) $\delta(A \cdot B) = \delta(A) \cdot \delta(B)$;
3) If A, $B \in U$, and if $B \neq 0$ and if A is arbitrary, there exists Q, $R \in U$ such that $A = BQ + R$ where $\delta(R) < \delta(B)$.

PROPOSITION 4.9 U *is an Euclidean Domain.*

PROOF. For $A = (a, x) \in U$, define $\delta(0) = 0$, $\delta(a, x) = 1$ if $a = 0$ and if $a \neq 0$, $\delta(a, x) = 2^{(\deg x+1)}$. It is clear that

δ as defined satisfies the first of the above properties. For the second, let $A = (a, x)$ and $B = (b, y)$ and note that if $A = 0$ or $B = 0$, then $A \circledast B = 0$ and $\delta(A) \cdot \delta(B) = 0 = \delta(A \circledast B)$. Suppose now that A and B are non-zero. If A and B are both units, then $A \circledast B$ is a unit and $\delta(A \circledast B) = 1 = \delta(A) \cdot \delta(B)$. If only one of A, B are units, say A with $a \neq 0$, then $b = 0$ and $y \neq 0$. Also $A \circledast B = (0, ay + x \circledast y)$ and as $\deg(ay + x \circledast y) = \deg y$ when $a \neq 0$, we see that $\delta(A \circledast B) = \delta(B) = \delta(A) \cdot \delta(B)$. When A and B are neither units nor zero, then $a = 0 = b$ and $x \neq 0 \neq y$. By (4.1), $\deg(x \circledast y) = \deg x + \deg y + 1$ and so, $\delta(A \circledast B) = \delta(A) \cdot \delta(B)$.

To show the third property, let $B = (b, y)$ be non-zero and let $A = (a, x) \varepsilon U$. If $b \neq 0$, B is a unit and so there is always a $C \varepsilon U$ such that $A = B \circledast C$. Now consider the cases when $b = 0$ and $y \neq 0$. If $a \neq 0$ or if $a = 0$ and $\deg y > \deg x$, then $\delta(A) < \delta(B)$. Since $A = B \circledast 0 + A$, we may put $Q = 0$ and $R = A$. The only remaining cases are when $a = 0 = b$, $x \neq 0 \neq y$ and $\deg y \leq \deg x$. If $\deg x = n = \deg y$, choose $c \neq 0$ so that $\deg(x - cy) > n$ (i.e., $D^n x(0) = c\, D^n y(0)$) and solve for $g \varepsilon X$ satisfying $x - cy = y \circledast g$. Then $A = B \circledast Q$ where $Q = (c, g)$. But if $\deg y < \deg x$, one can find $g \varepsilon X$ so that $y \circledast g = x$ and if $Q = (0, g)$, then $A = B \circledast Q$.

Thus, for all $B \neq 0$ and $A \varepsilon U$, elements Q and R of U can be found so that $A = B \circledast Q + R$ and $\delta(R) < \delta(B)$.

Hence U is an Euclidean Domain.

We have already noted that U is a principal ideal domain and so U is a unique factorization domain. An element $A = (a, x) \varepsilon V$ is said to be <u>irreducible</u> if (a, x) is not a unit and has no proper factors. Two elements, $A, B \varepsilon U$ are <u>associates</u> if there exists a unit C and $A \circledast C = B$.

PROPOSITION 4.10 *Let* $x, y \varepsilon X$ *and let* $x, y \neq 0$. *Then* $A = (0, x)$ *and* $B = (0, y)$ *are associates if, and only if,* $\deg x = \deg y$.

PROOF. We have already noted in the proof of Proposition 4.9 that if $\deg x = \deg y$, then there is a non-zero $c \varepsilon ¢$ and $g \varepsilon X$ such that $x - cy = y \circledast g$ and so, if $Q = (c, g)$, Q is a unit and $A = B \circledast Q$.

Conversely, if A and B are associates, there exists a unit (c, z) with $c \neq 0$ such that $(0, x) = (c, z) \circledast (0, y)$. Then $x = cy + y \circledast z$ and as $c \neq 0$, $\deg x = \deg y$.

Chapter Five

ENTIRE MEAN PERIODIC FUNCTIONS

This chapter is concerned with entire functions (Section 5.1) and the mean periodic entire functions (Section 5.2) introduced by Laurent Schwartz ([1], §4).

It is shown (Theorem 5.1) that it is possible to define a 'truncated convolution' product of two entire functions f, g as

$$f \circledast g : z \to \int_0^z f(z - \xi)g(\xi)d\xi$$

and that $f \circledast g$ is entire. With addition and this product, H, the set of entire functions is an algebra. Moreover, the set of entire mean periodic functions is a subalgebra of H (Theorem 5.4).

Section 5.3 contains new properties of entire mean periodic functions that resemble <u>some</u> of the properties given earlier in this thesis for continuous mean periodic functions of a real variable. However, Kahane's definition of a transform for a continuous mean periodic function does not apply to entire mean periodic functions. For this, and other reasons, some of the results in earlier chapters of this thesis will not have counterparts in this chapter.

Differential and differential-difference equations that admit entire mean periodic solutions are discussed in Section 5.4 .

§5.1 Entire Functions

We denote by H the topological vector space of all entire functions equipped with the topology of convergence uniform on all compact subsets of $\mathbb{C}$. This topology may be defined by the seminorms $\{p_k\}_{k=1}^{\infty}$ where, for $f \in H$,

$$p_k(f) = \sup\{|f(z)| : |z| \leq k\} .$$

A metric defining this topology is

$$d(f, g) = \sum_{k=1}^{\infty} 2^{-k} \frac{p_k(f - g)}{1 + p_k(f - g)}$$

with $d(f_n, f) \to 0 \Leftrightarrow p_k(f_n - f) \to 0$ as $n \to \infty$ for each $k \in N$;

$\Leftrightarrow f_n \to f$ in H as $n \to \infty$;

$\Leftrightarrow f_n \to f$ "locally uniformly" as $n \to \infty$.

H is a complete space, for if $\{f_n\}$ is a Cauchy sequence of elements of H, then there exists a function $f \in H$ such that $f_n \to f$ locally uniformly as $n \to \infty$. Like $C(R)$, H is a Fréchet space. However, H has an additional property in that if $\{f_n\} \subset H$ and if $f_n \to f$ locally uniformly as $n \to \infty$,
then $D^k f_n \to D^k f$ locally uniformly as $n \to \infty$

for all positive integers k. (See, for example, Ahlfors, [1], page 138).

When f, g are entire functions, and z is any complex number, $f(z - \xi)g(\xi)$ is an entire function of ξ. The integral

$\int_o^z f(z-\xi)g(\xi)d\xi$ is then independent of the path of integration. Accordingly, we may define a 'truncated convolution' product of two entire functions, f, g as

$$f \circledast g(z) = \int_o^z f(z-\xi)g(\xi)d\xi \quad .$$

THEOREM 5.1 *Let* f, g *be entire functions. Then*

$$f \circledast g : z \to \int_o^z f(z-\xi)g(\xi)d\xi$$

is an entire function and

$$(5.1) \quad D^n(f \circledast g) = f(0)D^{n-1}g + Df(0)D^{n-2}g + \ldots (D^{n-1}f(0))g + (D^n f) \circledast g$$

PROOF. We use the fact that any entire function is the locally uniform limit of a sequence of polynomials. Let $\{f_n\}$, $\{g_n\}$ be sequences of polynomials that converge, locally uniformly, to f, g respectively. Let $h_n = f_n \circledast g_n$. If $u_p : z \to z^p$, then

$$u_p \circledast u_q(z) = \int_0^z (z-\xi)^p \xi^q d\xi$$

$$= \sum_{j=0}^{p} {}^pC_j \; z^{p-j}(-1)^j \int_o^z \xi^{j+q}d\xi$$

$$= A_{p,q} z^{p+q+1}$$

where $A_{p,q}$ is a constant. Thus $\{h_n\}$ is a sequence of polynomials.

We now show that $h_n \to h$ locally uniformly as $n \to \infty$. Let r be any positive number and let $S = \{z : |z| \le r\}$. As $f_n \to f$,

$g_n \to g$ uniformly on S as $n \to \infty$, and as each f_n, g_n is continuous, there exists a positive constant c such that

$$|f_n(z)|,\ |g_n(z)|,\ |f(z)|,\ |g(z)| \text{ are bounded by } c$$

when $z \in S$ and $n = 1, 2, \ldots$. Now, for any $\varepsilon > 0$, there exists an integer m for which

$$|f(z) - f_n(z)| < \varepsilon \quad \text{and} \quad |g(z) - g_n(z)| < \varepsilon$$

whenever $z \in S$ and $n > m$. As

$$h(z) - h_n(z)$$
$$= \int_0^z (f(z-\xi) - f_n(z-\xi))g(\xi)d\xi + \int_0^z f_n(z-\xi)(g(\xi) - g_n(\xi))d\xi ,$$

it follows that

$$|h(z) - h_n(z)| < 2cr\varepsilon$$

whenever $n > m$, $z \in S$ and the paths of integration of the integrals are chosen to be the line joining 0 to z . Hence $h_n \to h$ uniformly on S as $n \to \infty$.

Since $\{h_n\}$ is a sequence of polynomials and $h_n \to h$ locally uniformly as $n \to \infty$, h is an entire function.

To prove the identity (5.1) when f, g are entire, it is only necessary to note that both sides of the identity are entire functions, and, by Proposition 3.3, agree on $\mathbb{R}$ (the set of reals). Hence (5.1) holds on $\mathbb{C}$.

THEOREM 5.2 *The set* H *of entire functions with the operations of addition and truncated convolution is a commutative ring without divisors of zero and an algebra over* $\mathbb{C}$.

PROOF. It is clear that H is a vector space over $\mathbb{C}$. To show the other properties, we may use the facts that:

i) $C(R_+)$ $(+, \circledast)$ is a commutative ring without divisors of zero and an algebra over $\mathbb{C}$ (Erdélyi, [1], Chapter 2); and,

ii) If an entire function vanishes on a half line, it is identically zero.

Let f, g, h ε H and let f, g, h be the respective restrictions of these functions to R_+ . To show that $f \circledast g = g \circledast f$, observe that

$$(f \circledast g - g \circledast f)\big|_{R_+} = f \circledast g - g \circledast f = 0 \quad .$$

As $f \circledast g - g \circledast f$ is an entire function that vanishes on R_+, $f \circledast g = g \circledast f$. Similarly,

$$((f \circledast g) \circledast h - f \circledast (g \circledast h))\big|_{R_+} = (f \circledast g) \circledast h - f \circledast (g \circledast h) = 0 \quad .$$

Thus $\circledast$ is an associative operation in H . In this manner, it may be shown that $H(+, \circledast)$ is a commutative ring and an algebra over $\mathbb{C}$.

We now show that H has no non-zero divisors of zero in a manner that does not require the use of Titchmarsh's convolution theorem. Firstly, suppose that g, h ε H and $g \circledast h = h$. Then $h = g \circledast (g \circledast h) = g^{\circledast n} \circledast h$ for $n = 0, 1, 2, \ldots$ and as $g^{\circledast n} \to 0$

locally uniformly as $n \to \infty$, it follows that $h = 0$ (cf. Proofs of Propositions 3.1 and 3.2 that are also valid for entire functions).

Now suppose that $f \circledast g = 0$ and $g \neq 0$. By (5.1)

$$0 = D(f \circledast g) = f(0)g + (Df) \circledast g$$

and as $g \neq 0$, $f(0) = 0$. Inductively, $D^n f(0)$ is zero for $n = 0, 1, 2, \ldots$ and as f is an entire function, $f = 0$.

Hence $H(+, \circledast)$ has no non-zero divisors of zero.

The next theorem is concerned with certain integral equations involving entire functions. These include

$$w_{jp}(z) - \sum_{k=1}^{n} \int_{o}^{z} g_{jk}(z - \xi) w_{kp}(\xi) d\xi = f_{jp}(z) \quad (1 \leq j \leq n,\ 1 \leq p \leq m)$$

or

$$W - G \circledast W = F$$

where $G = \{g_{jk}\}$ and $F = \{f_{jp}\}$ are matrices whose elements are entire functions.

THEOREM 5.3 *Let* F, G *be* $n \times m$, $n \times n$ *matrices respectively whose elements are entire functions. Then each of the matrix systems of equations*

$$W - G \circledast W = F$$

$$W' - G \circledast W = F \quad , \qquad W(0) = C$$

has as a solution, a unique $n \times m$ *matrix whose elements are entire functions. Moreover, if* $\det G(0) \neq 0$, *then the system of equations*

$G \circledast W = F$ *admits an* $n \times m$ *matrix of entire functions as a solution if, and only if,* $F(0) = 0$.

PROOF. The proofs of Propositions 3.1, 3.2 and Theorems 3.4, 3.5 given for systems of integral equations involving continuous functions hold, with due alteration of details (such as given in the proof of Theorem 5.1 and using $\mathcal{H}$ as a complete metric space) for systems involving entire functions.

§5.2 Laurent Schwartz's Theory

In this section, we outline properties of entire mean periodic functions due to Laurent Schwartz ([1], §4), who was the first to introduce such functions. Our outline is adapted from this reference but some changes are made in notation.

A complex translate of an entire function f is $T_a f : z \to f(z - a)$ where a is any complex number. If W_f denotes the closed linear subspace of $\mathcal{H}$ spanned by f and its complex translates, then f is said to be mean periodic in $\mathcal{H}$ if $W_f \neq \mathcal{H}$. This may be shown to be equivalent to the existence of a continuous linear functional L on $\mathcal{H}$ such that

$$L(T_z \check{f}) = 0 \qquad \text{for all } z \in \mathbb{C}$$

where $\check{f} : \xi \to f(-\xi)$.

The main property of any entire mean periodic function f is that it is the limit in $\mathcal{H}$ of a sequence of exponential polynomials belonging to W_f . Here, by an exponential polynomials is meant a finite linear combination of terms $u_n e_a : z \to z^n e^{az}$ where n is a non-negative integer and a is any complex number.

If L is any continuous linear functional on H, (i.e., if $L \in H'$, the dual-space of H) it may, by virtue of the Hahn-Banach Theorem, be extended to a continuous linear functional on $C(R^2)$, the space of continuous complex-valued functions defined on R^2 with the topology of locally uniform convergence. Thus, there exists at least one measure μ with a compact support K in the plane such that, for $f \in H$.

$$(5.2) \qquad L(f) = \iint_K f(x + iy)\,d\mu(x, y) \quad .$$

We may write (5.2) as $L(f) = \int f(\xi)\,d\mu(\xi)$. It is noted that for a given $L \in H'$, there exists an infinity of such measures, the difference between any two being orthogonal to H. Moreover, if $L, M \in H'$ and L, M are extended by measures μ, λ respectively, $\mu * \lambda$ defines a 'convolution' $L * M$ of L and M. $L * M$ belongs to H' and is independent of the choice of measures that extend L, M. (Since the set of complex measures with compact support in the plane forms an integral domain with the operations of addition and convolution, H' with addition and convolution defined above is also an integral domain.)

Use will be made of the "Fourier-Laplace transform" of L defined by

$$(5.3) \qquad \Phi(z) = L(e_{-z}) = \int e^{-\xi z}\,d\mu(\xi)$$

where $e_{-z} : \xi \to e^{-z\xi}$. This transform is independent of the measure μ that extends L . Moreover, $\Phi(z)$ is an entire function of exponential type. Conversely, any entire function of exponential type is the Fourier-Laplace transform of an element of H' . (For if $\Phi(z)$ is an entire function of exponential type, then from the Paley-Wiener Theorem, $\Phi(z)$ is the Fourier-Laplace transform of a distribution T of compact support in R^2 . Thus $T = \sum_{k=0}^{n} D^k \mu_k$ where $\mu_0, \mu_1, \ldots \mu_n$ are measures with compact supports and hence T is the extension of an element of H' to $C^n(R^2)$.)

According to a theorem of Borel, there exists a unique function $\phi(\xi)$ that is analytic for $|\xi|$ large enough, zero at infinity, and such that

$$\Phi(z) = \frac{1}{2\pi i} \oint_C e^{-\xi z} \phi(\xi) d\xi \quad .$$

Here, C is a simple closed curve that encloses all the singularities of ϕ . As well,

$$\phi(\xi) = \int_0^{"\infty"} e^{\xi z} \Phi(z) dz$$

where the limit at infinity and the path of integration is chosen so as to ensure the convergence of the integral.

If $\Phi(z) = \sum_{n=0}^{\infty} a_n z^n / n!$

and if

$$L = \limsup_{n \to \infty} \sqrt[n]{|a_n|} \quad ,$$

$\Phi(z)$ is an entire function of exponential type L . Then

$$\phi(\xi) = -\sum_{n=0}^{\infty} (-1)^n a_n/\xi^{n+1}$$

and $\phi(\xi)$ is analytic on $\{\xi : |\xi| > L\}$. Moreover,

$$a_n = -\frac{1}{2\pi i} \oint_C (-\xi)^n \phi(\xi) d\xi = L * u_n(0)$$

where $u_n : z \to z^n$ and $L * u_n(0) = L(\check{u}_n) = (-1)^n L(u_n)$.

As $L * f(z) = L(T_z \check{f})$ and f has a Taylor series development $f = \sum_{n=0}^{\infty} D^n f(0) u_n/n!$, we see that

$$(5.4) \qquad L * f(z) = \sum_{n=0}^{\infty} a_n D^n f(z)/n!$$

The formula (5.4) shows that the equation $L * f = 0$ is none other than a linear ordinary differential equation with constant coefficients and of infinite order in f. Thus:

An entire function f *is mean periodic in* H *if, and only if,* $\sum_{n=1}^{\infty} a_n D^n f(z)/n! = 0$ *for some sequence* $\{a_n\} \subset ¢$ *with* $\limsup_{n \to \infty} \sqrt[n]{|a_n|} < \infty$.

For any entire mean periodic function f, with $W_f \neq H$, the spectral set is defined as $S_f = \{a_k : e_{a_k} \in W_f\}$. The spectrum, Λ_f, is the set of ordered pairs (a_k, p_k) such that $a_k \in S_f$ and $u_j e_{a_k} \in W_f$ for $j \leq p_k - 1$ but not $j = p_k$.

It is noted that the spectrum, Λ_f, of an entire mean periodic function f is less restricted than that of a continuous mean periodic function, because if $\Phi(z)$ is the Fourier-Laplace transform of L with $L * f = 0$, then $\Phi(z)$ of exponential type but is not required to be bounded on the imaginary axis. Accordingly, if

$$\Lambda_f = \{(a_k, p_k)\}_{k=1}^{\infty}, \quad \text{then} \quad \sum_{k=1}^{\infty} p_k |a_k|^{-2} < \infty$$

but condition (0.3) on the spectrum of a continuous mean periodic function does not apply to the spectrum of an entire mean periodic function.

The distribution, $D\delta$, with $D\delta(f) = f'(0)$ for $f \in H$, is a continuous linear functional on H, and $f' = D\delta * f$. If $L \in H'$, then $(D\delta * L) * f = L * f'$.

The function $f : z \to e^{z^2}$ is noted as not being mean periodic in H. It is also noted that if

$$g : z \to \sum_{n=1}^{\infty} a_n \exp(i\alpha_n z) \tag{5.5}$$

where the sequence $\{\alpha_n\}$ is real and has infinite density and the a_n decreases rapidly enough so that g is entire, then g is not mean periodic in H.

However, if f is entire and is such that its restriction to R is mean periodic in C(R), then f is mean periodic in H. But it is possible that f is mean periodic in H and its restriction to R is not mean periodic in C(R). (An example of this is $f : z \to \exp(\exp z)$.)

Another characterization of mean periodicity in H is given. For $f \in H$, let $\mathcal{D}_f$ denote the closed subspace of H generated by the derivatives of f. Then:

$\mathcal{D}_f = W_f$ *and so* f *is mean periodic in* H *if, and only if,* $\mathcal{D}_f \neq H$.

For $Df = \lim_{\xi \to 0} (T_{-\xi} f - f)/\xi \in W_f$ whence $D^n f \in W_f$ for $n = 1, 2, \ldots,$ and so $\mathcal{D}_f \subset W_f$. Taylor's formulae, $f(z - \xi) = \Sigma(-1)^n \xi^n D^n f(z)/n!$ shows that $T_\xi f \subset \mathcal{D}_f$. Hence $W_f \subset \mathcal{D}_f$ and so $W_f = \mathcal{D}_f$.

§5.3 Further Properties

We now give new properties of entire mean periodic functions. Throughout this section, we shall assume that H' with the operations of addition and 'convolution' ($*$ as defined in §5.2) has the properties of an integral domain. The set of all entire mean periodic functions shall be denoted by MH .

THEOREM 5.4 MH *is a subalgebra of* H ($+, \circledast$).

PROOF. If $a, b \in \mathbb{C}$, if $f, g \in MH$ and if $L * f = 0$, $M * g = 0$ where L, M are non-zero elements of H', then

$$L * M * (af + bg) = aM * (L * f) + bL * (M * g) = 0 \quad .$$

As $L * M$ is a non-zero element of H', $af + bg \in MH$. Thus MH is a subspace of H .

We now show that $f \circledast g \in MH$. Let $\{g_m\}$ be a sequence of exponential polynomials that converge, locally uniformly, to g and such that $g_m \in W_g$ for each $m \in N$.

Now let a be any complex number, n be any non-negative integer and $u_n e_a : z \to z^n e^{az}$. If $L(D) = (D - aI)^{n+1}$, $L(D)$ is a non-zero differential operator with constant coefficients and $L(D)u_n e_a = 0$. From (5.1),

$$L(D)(u_n e_a \circledast f) = M(D)f + (L(D)u_n e_a) \circledast f = M(D)f$$

where $M(D)$ is a non-zero differential operator with constant coefficients. So,

$$\begin{aligned} L(D)(L * (u_n e_a \circledast f)) &= L * (L(D)(u_n e_a \circledast f) \\ &= L * (M(D)f) \\ &= M(D)(L * f) \\ &= 0 \quad . \end{aligned}$$

Thus $L * (u_n e_a \circledast f) = P_n e_a$ where P_n is a polynomial of degree not exceeding n .

Hence $L * (g_m \circledast f) \in W_g$

so that $$M * L * (g_m \circledast f) = 0 \qquad \text{for } m = 1, 2, \ldots \; .$$

As $g_n \circledast f \to g \circledast f$ locally uniformly as $m \to \infty$,

$$M * L * (g \circledast f) = 0 \text{ showing that } f \circledast g \in MH \; .$$

COROLLARY *If* $f \in MH$, *then* $e \circledast f : z \to \int_0^z f(\xi)d\xi \in MH$.

PROPOSITION 5.5 *If* $f \in MH$ *and if* g *is an exponential polynomial, then* $fg \in MH$. *Also, if* $c \in ¢$, $h : z \to f(cz) \in MH$.

PROOF. Since MH is a subspace of H, it is only necessary to show that $u_n e_a f \in MH$ when $f \in MH$. In turn, this reduces to showing that $e_a f$, $uf \in MH$ when $f \in MH$.

Let μ be any non-zero measure such that $\int f(z - \xi)d\mu(\xi) = 0$. If λ is the non-zero measure defined by $\lambda(g) = \mu(e_a g)$ for all $g \in C(R^2)$, then

$$\lambda * (e_a f)(z) = \int e^{a(z-\xi)} f(z - \xi) e^{a\xi} d\mu(\xi) = e^{az} 0 = 0 \quad .$$

So $e_a f$ is mean periodic in H .

If ν is the non-zero measure defined by $\nu(g) = \mu(ug)$ for all $g \in C(R^2)$, then

$$\mu * (uf)(z) = \int (z - \xi) f(z - \xi) d\mu(\xi) = z\, 0 - \nu * f(z) \quad .$$

So $\mu * \mu * uf = 0$ and as $\mu * \mu \neq 0$, uf is mean periodic in H . Hence $fg \in MH$.

To show that $h: z \to f(cz)$ is mean periodic in H, we use the fact that f is mean periodic in H if, and only if

$$\sum_{n=o}^{\infty} a_n D^n f(z)/n! = 0 \quad \text{for all complex } z$$

where $L = \limsup_{n \to \infty} \sqrt[n]{|a_n|} < \infty$.

If $c = 0$, then h is constant and so h is mean periodic.

If $c \neq 0$ and $f \in MH$, then

$$\Sigma a_n c^n D^n h(z)/n! = \Sigma a D^n \cdot f(cz)/n! = 0$$

for all complex z. Since

$$\limsup_{n \to \infty} \sqrt[n]{|a_n c^n|} = L|c| < \infty \quad ,$$

L is mean periodic in H.

We note that just as the product of two continuous mean periodic functions of a real variable need not be mean periodic, the product of two entire mean periodic functions need not be mean periodic. A specific example of two such entire mean periodic functions is $\exp(\exp(i\alpha z))$, $\exp(\exp(i\beta z))$ where α/β is irrational. For if h denotes their product, then

$$h(z) = \sum_{m,\ n \varepsilon Z}\sum \frac{1}{m!n!} \exp(im\alpha z + in\beta z)$$

and since $\{\alpha m + \beta n : m, n \varepsilon Z\}$ has infinite density, h is not mean periodic in H (see §5.2 (5.5)).

The next result is a counterpart of a part of Proposition 1.6. However, the short method of proof used in Proposition 1.6 for continuous mean periodic functions does not appear to be applicable to entire mean periodic functions.

PROPOSITION 5.6 *Let* $f, g \in H$, $f \neq 0$ *and* $f, f \circledast g \in MH$. *Then* $g \in MH$.

PROOF. Let L, N be non-zero elements of H' such that $L * f = 0$, $N * (f \circledast g) = 0$. If we can show that there is a non-zero $M \in H'$ for which

$$(5.6) \qquad L * (f \circledast g) = M * g, \quad \text{then}$$

$$M * N * g = L * N * (f \circledast g) = 0$$

and so g will be mean periodic in H .

We now show (5.6). First consider the function $h = L * (f \circledast e_\eta)$ where η is any complex number. It follows from

$$Dh = L * D(f \circledast e_\eta) = L * (f + \eta f \circledast e_\eta)$$

and the assumption $L * f = 0$ that $Dh = \eta h$. So $h = N(\eta)e_\eta$ for some number $N(\eta)$ and $N(\eta) = h(0)$.

As both convolution products $*$, $\circledast$ are linear,

$$L * (f \circledast (e_{\eta+\xi} - e_\eta)/\xi)(0) = (N(\eta + \xi) - N(\eta))/\xi$$

for any complex numbers, η and ξ .

As $\xi \to 0$, $(e_{\eta+\xi} - e_\eta)/\xi \to ue_\eta$ locally uniformly in H . Then

$$f \circledast (e_{\eta+\xi} - e_\eta)/\xi \to f \circledast ue_\eta \quad \text{in } H \text{ as } \xi \to 0$$

and so

$$L * (f \circledast (e_{\eta+\xi} - e_\eta)/\xi)(0) \to L * (f \circledast ue_\eta)(0)$$

as $\xi \to 0$. Hence $\lim_{\xi \to 0} (N(\eta + \xi) - N(\xi))/\xi$ exists and so $N(\eta)$ is an entire function of η .

Since $L \in H'$, L may be extended to a measure with a compact support in the plane. So there exists positive constants, c, T, such that

$$|L * h(0)| \leq cp(h)$$

where $p(h) = \sup\{|h(z)| : |z| \leq T\}$ for all $h \in H$.

As $|f * e_\eta(z)| \leq T\, p(f)p(e_\eta)$ for all $|z| \leq T$,

and as $p(e_\eta) = \sup\{|e^{\eta z}| : |z| \leq T\} \leq e^{2|\eta|T}$,

$$|N(\eta)| = |L * (f \circledast e_\eta)(0)| \leq cTp(f)e^{2|\eta|T} .$$

So, the entire function $N(\eta)$ is of exponential type. Hence there exists an $M \in H'$ whose Fourier-Laplace transform is N and so $M * e_\eta = e_\eta N(\eta)$ for all $\eta \in \mathbb{C}$.

We have now shown that under the initial assumptions, there exists an $M \in H'$ such that

$$L * (f \circledast e_\eta) = M * e_\eta \qquad \text{for all } \eta \in \mathbb{C} .$$

From this, it readily follows by another use of the linearity and continuity of L, M and $*$ that

$$L * (f \circledast ue_\eta) = M * ue_\eta$$

and so $L * (f \circledast u_n e_\eta) = M * u_n e_\eta$ for $n = 0, 1, 2, \ldots$.

Thus $L * (f \circledast P) = M * P$

for all polynomials P . As any entire function g is the locally uniform limit of a sequence of polynomials $\{g_n\}$, and as $f \circledast g_n \to f \circledast g$ in H as $n \to \infty$,

$$L * (f \circledast g_n) \to L * (f \circledast g)$$

and

$$M * g_n \to M * g \quad \text{as} \quad n \to \infty \quad .$$

Thus $L * (f \circledast g) = M * g$ for all $g \varepsilon H$.

It remains to show that M is non-zero. Suppose otherwise so that $L * (f \circledast g) = 0$ for all $g \varepsilon H$, and all $f \varepsilon H$ for which $L * f = 0$. As $f \neq 0$, W_f contains at least one exponential, say e_a, and $L * e_a = 0$. Now a sequence $\{g_n\}$ of functions may be chosen so that $g \circledast e_a = u_n e_a$ for $n = 1, 2, \ldots$. Thus $L * (u_n e_a) = 0$ for $n = 0, 1, \ldots$. But since the subspace spanned by $\{u_n e_a\}_{n=1}^{\infty}$ is dense in H, $L * h = 0$ for all $h \varepsilon H$, and so $L = 0$, a contradiction.

Hence $M \neq 0$ where $M \varepsilon H'$ and $L * (f \circledast g) = M * g$.

COROLLARY $f, g \varepsilon MH \Rightarrow f \circledast g \varepsilon MH$.

PROOF. If $L * f = 0$, then $L * (f \circledast g) = M * g$ whence $f \circledast g \varepsilon MH$.

PROPOSITION 5.7 *Let* $f, g \varepsilon MH$. *Then the integral equation*

$$(5.7) \qquad h - g \circledast h = f$$

has a unique solution, h, *in* MH .

PROOF. From Theorem 5.3, this equation has a unique entire function as a solution. If $e : z \to z$, equation (5.7) is equivalent to

$$h \circledast (e - e \circledast g) = e \circledast f \quad .$$

Now $e \circledast f$ and $e - e \circledast g$ are entire mean periodic functions by Proposition 5.4 . So, by Proposition 5.6, h is an entire mean periodic function.

Proposition 5.7 has been stated and proved for a single integral equation and not systems of integral equations. For systems of integral equations (cf. Theorem 3.4(b)), the details used in the proof of Theorem 3.4(b) showing the existence of continuous mean periodic solutions do not apply to entire mean periodic functions.

As in §4.3 for exponential polynomials, we may define the <u>degree</u> of an entire function f to be zero if $f(0) \neq 0$ and n if $D^k f(0) = 0$ for $k = 0, 1, \ldots n-1$ but $D^n f(0) \neq 0$. When $f \in H$ and f has degree n, we write $\deg f = n$ and note that if $f \neq 0$, $\deg f$ is finite.

Now if X denotes either of the commutative rings H or MH with the operations of addition and truncated convolution, X has no non-zero divisors of zero and is an algebra over $\mathbb{C}$. Moreover, part of Theorem 5.3 and Proposition 5.7 may be restated as follows:

If $f, g \in X$, then the integral equation $h - g \circledast h = f$ has a unique solution, h, in X.

The proof of Proposition 4.4(b) may then be modified so as to give:

Let $f, g \in X$ and $f \neq 0$. Then a necessary and sufficient condition for the equation $g \circledast h = f$ to have a solution, h, in $\mathcal{X}$ is that $\deg f > \deg g$.

Thus, if $I_f = \{f \circledast g : g \in \mathcal{X}\}$ for $f \in \mathcal{X}$,

and if $$Y_n = \{g \in \mathcal{X} : \deg g > n\} ,$$

Proposition 4.5 may be changed to:

If $f \in \mathcal{X}$ and if $\deg f = n$, then $I_f = Y_n$.

From this, it follows that Theorem 4.6 holds with $\mathcal{X}$ replacing X, i.e.,

Let I be any non-trivial proper ideal of $\mathcal{X}$. Then $I = Y_n$ for some non-negative integer n.

The remarks following Theorem 4.6 in §4.3 concerning the ring of exponential polynomials therefore apply to the ring $\mathcal{X}$. In particular, the set of all ideals of $\mathcal{X}$ form a single ascending chain,

$$\subset Y_n \subset Y_{n-1} \ldots \subset Y_1 \subset Y_0 \subset \mathcal{X} .$$

As well, $\mathcal{X}$ may be embedded in a ring with an identity that is also an Euclidean Domain.

It is also apparent that the ring $\mathcal{X}$ may be embedded in a field of convolution quotients (cf. §1.4). The embedding of the ring $\mathcal{H}$ into a field of convolution quotients yields a field that is complete (in the sense that $\mathcal{H}$ is a complete metric space).

§5.4 Functional-Differential Equations

We begin this section by considering a simple system of differential equations and its connection with entire mean periodic functions. Thus, if $\underline{f} = (f_1, f_2, \dots f_n)$ is mean periodic, then $L_j * f_j = 0$ where $L_1, L_2, \dots L_n$ are non-zero elements of H', and so $L * \underline{f} = \underline{0}$ where $L = L_1 * L_2 * \dots L_n$.

PROPOSITION 5.8 *For the system of equations*

$$\text{(5.8)} \qquad \underline{w}'(z) = A\underline{w}(z) + \underline{f}(z) \qquad \textit{with} \qquad \underline{w}(0) = \underline{c}$$

where $\underline{f}$ *is an n-vector whose components are entire functions and* A *is a constant* n x n *matrix, a necessary and sufficient condition that* $\underline{w}$ *be mean periodic is that* $\underline{f}$ *be mean periodic.*

PROOF. Suppose that $\underline{w}$ is mean periodic and $M * \underline{w} = \underline{0}$ where M is a non-zero element of H' . Then $M * \underline{w}' = \underline{0}$ and $M * A\underline{w} = A\, M * \underline{w} = \underline{0}$. Thus $M * \underline{f} = \underline{0}$ and so $\underline{f}$ is mean periodic.

Conversely, suppose that $\underline{f}$ is mean periodic. The solution to (5.8) is

$$\underline{w}(z) = e^{Az}\underline{d} + e^{Az}\int_0^z e^{-A\xi}\underline{f}(\xi)\,d\xi$$

and the components of $\underline{w}$ are entire functions. As the elements of the matrices $e^{-A\xi}$, e^{Az} are exponential polynomials and the components of $\underline{f}$ are mean periodic, it follows by use of Propositions 5.4 and 5.5 that the components of $\underline{w}$ are mean periodic.

Our next result is an application of Theorem 2.1 . We say that an entire function f has a period c, where c is a non-zero complex number, if $f(z + c) = f(z)$ for all complex z .

THEOREM 5.9 *Let* $A(z)$ *be an* $n \times n$ *matrix whose elements are entire functions with a complex period* c . *Let* $\underline{f}$ *be an n-vector whose* j^{th} *component is of the form* $g_j h_j$ *where each* g_j *is an exponential polynomial and* h_j *is an entire function with complex period* d_j . *If each* d_j *is a real, positive and rational multiple of* c, *then all entire solutions to the system of equations*

$$(5.9) \qquad \underline{w}'(z) = A(z)\underline{w}(z) + \underline{f}(z)$$

are mean periodic in H .

PROOF. Let $\underline{w}(z)$ be an entire solution of (5.9) . Let $\tau = |c|$, $c = \tau e^{i\theta}$, $\tau_j = |d_j|$ so that $d_j = \tau_j e^{i\theta}$ for $j = 1, 2, \ldots n$. Put $B(t) = A(te^{i\theta})$ so that $B(t)$ is a continuous periodic matrix of real period τ . Also put $b_j(t) = h_j(te^{i\theta})$ so that each b_j is a continuous periodic function of period τ_j where each τ_j is commensurable with τ . If $\underline{x}(t) = \underline{w}(te^{i\theta})$ and if the j^{th} component of $\underline{y}$ is $b_j g_j$, then

$$\underline{x}'(t) = B(t)\underline{x}(t) + \underline{y}(t)$$

where $B(t)$, $\underline{y}(t)$ satisfy the condition of Theorem 2.1 . By this theorem, $\underline{x}(t)$ is mean periodic in $C(R)$. But $\underline{x}(t)$ is the restriction of $\underline{w}(ze^{i\theta})$ to the real line and so $\underline{w}(ze^{i\theta})$ is mean periodic in H . So, by Proposition 5.4, $\underline{w}(z)$ is mean periodic in H .

REMARK. The scalar differential equation, $w'(z) = e^z w(z)$ has solutions $w(z) = ce^{e^z}$ (c constant) . As a special case of the above theorem, such solutions are mean periodic in H. This may be compared with Example B, Chapter 2, where it was shown that the differential equation $x'(t) = e^t x(t)$ has solutions that are not mean periodic in $C(R)$.

The next theorem concerns systems of linear differential difference equations that admit entire mean periodic functions as solutions. A counterpart to Theorem 2.3 involving functional-differential equations is not given here. For if one were to replace in Theorem 2.3 measures with compact support in the real line with measures with compact support in the complex plane, the Fourier-Laplace transform of the latter need not be bounded on the imaginary axis, but, the proof of Theorem 2.3 is dependent on the Fourier-Laplace transform of a measure (with compact support in R) being bounded on the imaginary axis.

Differential difference equations that admit entire mean periodic functions as solutions, but not including the following theorem, have been considered by H. S. Shapiro, [1].

As before, a mean periodic vector function is one that has mean periodic components. As well, reference to a vector that is entire will indicate that the components of the vector are entire functions.

THEOREM 5.10 *Consider the system of equations*

$$(5.10) \qquad \underline{w}'(z) = \sum_{\ell=1}^{m} A_\ell \, \underline{w}'(z - a_\ell) + \sum_{\ell=1}^{m} B_\ell \, \underline{w}(z - b_\ell) + \underline{f}(z)$$

where i) for $\ell = 1, 2, \dots m$, A_ℓ, B_ℓ *are* $n \times n$ *matrices of* *complex numbers,* a_ℓ *are non-zero complex numbers lying in a sector* $\{z = re^{i\theta} : |\theta + \alpha| < \pi/2$ *for some fixed* α *and* b_ℓ *are any complex numbers, and*

ii) the components of $\underline{f}$ *are entire functions. Then*

a) the homogeneous equation has at least one non-zero entire solution.

b) If $\underline{f}$ *is mean periodic, then all solutions that are entire are also mean periodic, and*

c) If any entire solution $\underline{w}$ *is mean periodic, then* $\underline{f}$ *is mean periodic.*

PROOF. Let T denote the $n \times n$ matrix

$$ID\delta - \sum_{\ell=1}^{m} A_\ell D\delta_{a_\ell} - \sum_{\ell=1}^{m} B_\ell \delta_{b_\ell}$$

Then T is a matrix whose elements belong to H' and equation (5.10) may be written as $T * \underline{w} = \underline{f}$.

Now let $\hat{T}(z)$ denote the matrix whose elements are the Fourier-Laplace transforms of the elements of T, i.e.,

$$\hat{T}(z) = Iz - \sum_{\ell=1}^{m} A_\ell z \exp(-a_\ell z) - \sum_{\ell=1}^{m} B_\ell \exp(-b_\ell z) \quad .$$

An examination of the expansion of the determinant of $\hat{T}(z)$ shows that it is equal to $\sum_{p=0}^{n} z^p g_p(z)$ where each g_p is a finite linear combination of exponentials. In particular,

$$g_n(z) = \det(I - \sum_{\ell=1}^{m} A_\ell \exp(-a_\ell z)) \quad .$$

If $S = \{z = re^{i\theta} : |\theta - \alpha| < \pi/2$, then by assumption, $-a_1, -a_2, \ldots -a_m \in S$. Hence $g_n(z) = 1 + \sum_{p=1}^{q} c_p \exp(d_p z)$, say, where $c_1, c_2, \ldots c_q \in S$ and so $g_n(z)$ is not identically zero.

We now show that $h(z) = \det \hat{T}(z)$ has at least one zero and also that $h(z)$ is not identically zero. If $h(z)$ has no zeros, then as $h(z)$ is an entire function of exponential type, $h(z) = \alpha e^{\beta z}$ for some complex numbers, $\alpha(\neq 0)$ and β. Thus,

$$(5.11) \qquad \sum_{p=0}^{n} z^p g_p(z) = \alpha e^{\beta z} \qquad \text{for all complex } z \quad .$$

But this is contradictory since $g_n(z)$ is not identically zero and the left side of (5.1) is a finite linear combination of exponential monomials which is linearly independent over $\mathbb{C}$. Hence $h(z)$ has at least one zero. If we assume that $h(z)$ is identically zero, then a similar contradiction results in (5.11) with $\alpha = 0$.

Let $h(c) = 0$. If $\underline{w}(z) = \underline{d}e^{cz}$, $T * \underline{w}(z) = e^{cz}\,\hat{T}(c)\underline{d}$. As $\det \hat{T}(c) = 0$, a non-zero vector $\underline{d}$ may be chosen so that $\hat{T}(c)\underline{d} = 0$ and so $T * \underline{w} = \underline{0}$ has a non-zero entire solution, $\underline{w}(z) = \underline{d}e^{cz}$.

Also, any entire solution to $T * \underline{w} = \underline{0}$ is mean periodic in H since $\det \hat{T}(z)$ is not identically zero and the arguments used in Proposition 2.2 are valid when the matrix T has elements belonging to H' rather than distributions with compact support lying in the real line.

For part b), if $\underline{f}$ is mean periodic in H with $M * \underline{f} = \underline{0}$ where M is a non-zero element of H', put $\underline{g} = M * \underline{w}$, where $\underline{w}$ is any entire solution to (5.10) . From $T * \underline{w} = \underline{f}$,

$$T * \underline{g} = T * (M * \underline{w}) = M * (T * \underline{w}) = M * \underline{f} = \underline{0} \quad .$$

Since $\underline{w}$ is entire, $\underline{g}$ is entire. By the preceding paragraph, $\underline{g}$ is mean periodic in H . Hence $\underline{w}$ is mean periodic in H .

For c) let $\underline{w}$ be any entire mean periodic solution to (5.10) . Since the derivative and any translate of an entire mean periodic function is mean periodic, it readily follows that $\underline{f}$ is mean periodic.

REMARK. To ensure that the conclusions of the above theorem hold, it is necessary to place some restriction on the numbers $a_1, a_2, \ldots a_m$. This is shown by the following example. Let a be a fixed non-zero complex number, $m = n = 2$, and $w_1'(z) = w_2'(z - a)$, $w_2'(z) = w_1'(z + a)$ (a special case of equation (5.10)) . Then w_1 may be any entire function.

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APPENDIX

This appendix concerns properties and applications of the Kahane transform of a mean periodic function suggested by Dr. H. K. Farahat, University of Calgary, subsequent to the writing of the text of this thesis.

By its definition, (page 12), the Kahane transform is a mapping from the ring $MP(+, \circledast)$ into the field MER of meromorphic functions with the operations of addition and pointwise multiplication. From Theorem 0.3, the Kahane transform is a 1-1 mapping and from Proposition 1.1, this mapping is a ring homomorphism. Thus

$$K : MP \to MER$$

is a monomorphism.

From this observation, it readily follows that MP has no non-zero divisors of zero (cf. Proposition 1.10). For if $f, g \in MP$ and if $f \circledast g = 0$, then

$$K(f) \cdot K(g) = K(f \circledast g) = 0,$$

and so $K(f) = 0$ or $K(g) = 0$. Thus $f = 0$ or $g = 0$.

It would be interesting to know the images of MP and MP_o under K in MER .

The remainder of this appendix is concerned with exponential polynomials. In the proof of Proposition 1.2 it is shown that $K(e)(z) = 1/z$ and so $K(e_a)(z) = 1/(z - a)$. Since

$u_n e_a = n!\, e_a^{\circledast(n+1)}$, $K(u_n e_a)(z) = n!/(z - a)^{n+1}$. Thus, if f is any exponential polynomial, say

$$\text{(A.1)} \qquad f = \sum_{k=1}^{n} \sum_{q=0}^{p_k} A(k, q)\, u_q e_{a_k} \quad , \qquad (a_k\text{'s distinct})$$

then,

$$\text{(A.2)} \qquad K(f)(z) = \sum_{k=1}^{n} \sum_{q=0}^{p_k} A(k, q) \frac{q!}{(z - a_k)^{q+1}} \quad .$$

For a polynomial, $p(z)$, of a complex variable z, denote by $d(p)$, the degree of $p(z)$, (i.e., if $p(z) = \sum_{k=0}^{n} b_k z^k$, $b_n \neq 0$, $d(p) = n$. Note that this usual definition of the degree of a polynomial differs from the definition given on page 78 of the degree of an exponential polynomial.). Now, let

$$RA = \{\frac{p(z)}{q(z)} : p, q \text{ are polynomials}, \quad q \neq 0, \quad d(p) < d(q)\}$$

so that RA, with the operations of addition and pointwise multiplication, is a ring. Moreover, any element of RA admits a unique decomposition into partial fractions.

From (A.2), if $f \in MQ$, $K(f)(z) \in RA$. So the Kahane transform, K, is a ring homomorphism from MQ into RA. If $K(f)(z)$ is identically zero, then

$$A(k, q) = 0 \quad \text{for} \quad q = 0, 1, \ldots p_k; \quad k = 1, 2, \ldots n$$

and so $f = 0$. This shows that the K is a 1-1 mapping without recourse to Theorem 0.3. Moreover, K maps MQ onto RA and so

MQ is isomorphic to RA.

Now let

$$SA = \{\frac{p(z)}{q(z)} : \; p, q \text{ are polynomials}, \; q \neq 0, \; d(p) \leq d(q)\} \; .$$

Then SA is a subring of the field of rational functions in z and RA is an ideal of the ring SA . A non-zero element $p(z)/q(z)$ of SA is a unit in SA if, and only if, $d(p) = d(q)$ and so RA is the unique maximal ideal of SA . Obviously every non-zero element of SA can be expressed uniquely in form $\frac{1}{z^n} u$ where u is a unit of SA . Thus SA is a unique factorization domain with only one prime divisor. The ideals of SA are simply

$$SA \supset \frac{1}{z} SA = RA \supset \frac{1}{z^2} SA \supset \frac{1}{z^3} SA \supset \cdots \quad .$$

Refering now to Section 4.4, if (a, x) belongs to the ring U defined on page 82, and if

$$L(a, x) = a + K(x) \quad ,$$

then it may be verified that L is an isomorphism from U onto SA . Since SA is an Euclidean Domain with a function ϕ defined on SA by

$$\phi(0) = 0, \quad \text{and} \quad \phi(p/q) = 2^{d(q)-d(p)}$$

when $p/q \in SA$, $p/q \neq 0$, U is an Euclidean Domain.