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CHAOTIC DYNAMICS
IN
FUTURES MARKETS

by

G. Paul Dormaar

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Chaotic Dynamics in Futures Markets" submitted by G. Paul Dormaar, in partial fulfillment of the requirements for the degree of Master of Arts.

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Abstract

Financial market price fluctuations have been explained, both theoretically and empirically, as the consequence of stochastic processes. However, there exists simple nonlinear deterministic [chaotic] systems that are also capable of generating random looking output and can fool many tests of whiteness.

Chaotic and stochastic systems are fundamentally different and require different methods of analysis. Here, two state-of-the-art tests of nonlinearity and of chaotic dynamics are applied to thirteen various commodity and currency spot-month futures series. Results indicate evidence of nonlinearity in five series and of chaos in three. This is consistent with an underlying chaotic process generating price changes.

Acknowledgments

Large projects such as this cannot be accredited to one person alone. Other people who deserve recognition for this work are as follows:

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Any errors or deficiencies are mine alone.

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Chapter 1

INTRODUCTION

Until recently all analyses of financial market price changes have assumed that observed fluctuations are dominated by stochastic processes. Some influences are known, such as economic growth paths or time to maturity of futures contracts, but once these effects are removed the remaining fluctuations appear random and have been explained using linear stochastic methods. There has been, however, a growing interest in a new field of study which may offer an alternative explanation for this apparently random price behavior. It may be the case that the noise in asset markets is the deterministic result from inherent nonlinearities.

In asset markets, prices change to bring supply and demand into equilibrium. This implies some feedback mechanism which returns prices back to equilibrium after circumstances change. When corrections are linear, feedback is simply proportional to the amount that prices are out of equilibrium. In this case market fluctuations would be stochastic since linear processes cannot generate random looking output. There is, however, no theoretical reason why corrections of this nature must be linear. Financial markets are composed of individual buyers and sellers, each with individual motivations and reactions. It is merely a simplification that we assume the aggregate response from all market participants is a linear function. Once nonlinear correction is introduced it is possible to explain market fluctuations in a deterministic structure. In particular, chaos theory shows how simple deterministic nonlinear difference equations can generate time paths with incredibly complex but random looking behavior. Hence, an alternative explanation for the nature of asset markets.

Two attributes of chaotic time paths are random looking but bounded fluctuations and sensitive dependence on initial conditions. A time path generated under a chaotic system will never return to the same value but nevertheless remain dispersed in a bounded region. Upon visual inspection it is easy to see how one could conclude that stochastic unforecastable shocks are present. Sensitive dependence on initial conditions refers to the case where two time paths that begin infinitely close diverge from each other rather than converge, yet both are guided by the same system. For this reason long term forecasting is impossible but perhaps short term trading rules exist -- depending on the speed of divergence.

Chaos was first discovered in 1892 by the French mathematician Henri Poincaré (1892) while studying the dynamics of three celestial bodies with mutual gravitational attractions -- e.g., 2 planets and a star. He was able to show that complicated orbits were possible from various initial points. Other early pioneering attempts were made but the concepts were not appreciated in other fields for two reasons: early mathematical papers were difficult to read by researchers from other sciences and the proofs were not strong enough to be considered applicable to other sciences. It was not until the early 1960's when the meteorologist Edward Lorenz (1963) "rediscovered" chaos that interest was generated in fields other than mathematics. He was developing weather forecasting models when he noticed that small changes in initial conditions lead to large changes in solution values. Interest was still slow at first but gained momentum. Over the past fifteen years, though, proliferation has increased throughout the entire spectrum of the sciences at an amazing rate. It has been argued [Rasband (1990)] that chaos theory is the most broad based revolution, in the world view of science, in the twentieth century. This growth has been lubricated by the increasing availability of high powered computers. Researchers in all fields can now find chaotic solutions to problems of great practical importance -- such as stimulating heart cells [Glass et al. (1983)], or designing nonlinear optical devices [Hopf et al. (1982)]. Interest in economics and finance is quite natural, the chance to explain random looking market fluctuations using deterministic methods cannot be overlooked.

Some key contributions were necessary before empirical work could be conducted. In 1981 Floris Takens discovered that the attractor from the underlying *n*-dimensional system with one observable could be reconstructed using only the observable. Then in 1982 Grassberger and Procaccia used this to find dimension estimates of the underlying unknown system. Detecting existence of an underlying system, however, did not become a statistic until 1987 when Brock, Dechert, and Scheinkman devised the BDS statistic which tests the null of whiteness. The BDS statistic can be used to test for residual nonlinear structure after any linear structure has been filtered out. Nonlinearity is consistent with chaotic behavior but it is not a necessary condition so further testing is required

Lyapunov exponents are used to measure exponential diversion (positive Lyapunov exponent) or conversion (negative Lyapunov exponent) of two time paths with similar initial positions. In 1985 Wolf, Swift, Swinney, and Vastano devised an algorithm that estimates Lyapunov exponents directly from the data. However, Brock and Sayers (1988) found the results from this algorithm disappointing when applied to economic data. They claimed that Lyapunov exponents could not be defined using the Wolf et al. (1985) algorithm when stochastic noise is present. All the tests designed thus far, except for the BDS statistic, were intended for use on experimental data. Special problems had to be addressed when testing economic data. In particular, economic time series are not generated by purely deterministic systems and are shorter than those required to make suitable estimates. An empirical survey conducted by Ramsey, Sayers, and Rothman (1990) showed that no legitimate claims of chaos had yet been found, based on the weak inference methods of the day.

The year 1992 marked the emergence of a new generation of Lyapunov exponent estimators. These are nonparametric nonlinear least squares estimators built into a neural net environment. Unlike the Wolf et al. (1985) algorithm the regression method accommodates noise and the results refer to the noisy system, rather than the hypothetical underlying system that the direct method tries to estimate. Two versions are available: the Nychka, Ellner, Gallant, and McCaffrey (1992) algorithm and the Gencay and Dechert (1992) algorithm. However, the most versatile of these is that of Nychka et al. (1992). According to Barnett et al (1993), the Nychka et al (1992) approach is the only credible candidate for testing chaos. Other than this study, the only other research -- known to this author -- which used the Nychka et al (1992) Lyapunov exponent estimator on economic data was that of Barnett et al. (1993). In that study successful detection of chaos was claimed for the CE index M4 monetary aggregate.

Here, thirteen various commodity and currency spot-month futures series are considered. They include the Australian dollar, the British pound, the Canadian dollar, crude oil, copper, the Deutschemark, gold, heating oil, unleaded gas, the Japanese yen, platinum, the Swiss franc, and silver. The two methods of inference used to test for nonlineariaties and chaos are the BDS test and the Nychka et al (1992) algorithm. Results indicate successful detection of chaos in the Australian dollar, copper, and the Japanese yen.

This thesis begins on the theoretical side where both stochastic and chaotic theories are discussed. Chapter 2 starts with an outline of the stochastic models which best describe futures markets. Then chaos is defined as it applies to economics and a specific example is given to show how chaotic dynamics can be achieved. Next, I traverse into the empirical side where both the BDS statistic and the regression type Lyapunov exponent estimates are described. The data and special problems regarding financial data are discussed in chapter 4. Finally, in chapter 5, prefiltering and empirical investigation of the chosen financial time series is addressed.

Chapter 2

FAIR GAMES, SUBMARTINGALES, AND CHAOS

2.1 Introduction

Until recently empirical evidence has confirmed the notion that asset price movements follow a linear but stochastic process. There is, however, no theoretical reason why market behavior is inherently linear. Previous results have shown that market prices are independently distributed in a linear structure but have not shown nonlinear independence. Some nonlinear models have the ability to generate similar random looking behavior which is not detectable by traditional linear methods. Thus, offering an alternative explanation to random looking price changes.

The next section explains the stochastic models most representative of futures behavior. First the fair game model is described which leads to the submartingale model. The submartingale model closely represents futures behavior since it allows for prices to increase over time and for time varying volatility. In section 2.3 a precise description of chaotic behavior is given. Analysis proceeds "in general", using vector notation. Then in section 2.4 a particular univariate model is used to show a method how chaos can be achieved. The next section, 2.5, gives a diagrammatical representation of the events described in section 2.4.

2.2 Fair Games and Submartingales

Futures contracts are financial instruments pertaining to the sale of some asset, where the price is agreed on today but actual delivery takes place at some future period. This implies a forecast of the future price of the good in question. One determinant of today's price of a future contract price is the expectation of how the price of the underlying asset will change.

Markets are efficient in the sense that based on today's expectations given all known relevant information, Ω_t [where Ω_t includes all information $\Omega_{t-j},\ j \geq 0$], it is not possible to make abnormal profits. This refers to returns made in excess of that amount necessary to cover all opportunity costs [which include information retrieval costs, transfer costs, etc.].

Since buying decisions are made in terms of expected returns, the actual abnormal profit or loss next period is

$$\Pi_{t+1} = R_{t+1} - \mathrm{E}_t\left(R_{t+1} \middle| \Omega_t\right)$$

where

$$R_{t+1} = \frac{x_{t+1} - x_t}{x_t}$$

and

$$\mathrm{E}_t\left(R_{t+1} \middle| \Omega_t\right) = \frac{\mathrm{E}_t\left(x_{t+1} \middle| \Omega_t\right) - x_t}{x_t}$$

are the actual and expected rates of return given prices x_t, x_{t+1}, and information set Ω_t. The expected abnormal profit or loss next period would be

$$\begin{aligned}\mathrm{E}_t\left(\Pi_{t+1} \middle| \Omega_t\right) &= \mathrm{E}_t\left[\left(R_{t+1} - \mathrm{E}_t\left(R_{t+1} \middle| \Omega_t\right)\right) \middle| \Omega_t\right] \\ &= \mathrm{E}_t\left(R_{t+1} \middle| \Omega_t\right) - \mathrm{E}_t\left(R_{t+1} \middle| \Omega_t\right) \\ &= 0\end{aligned}$$

Hence, knowing all relevant information today does not lead to abnormal profit. In terms of probabilities [assuming the distribution is symmetric about the mean]

$$\mathrm{P}\left(\Pi_{t+1} > 0\right) = \mathrm{P}\left(\Pi_{t+1} < 0\right) = 0.5$$

which means that Π_{t+1} is a fair game with respect to the information set Ω_t. This is known as the *fair game model* where the average return of an asset is equal to its expected return.

A *submartingale* is a fair game where prices are expected to increase at a rate equal to the opportunity cost of the asset.

$$E_t\left(x_{t+1} \middle| \Omega_t\right) > x_t$$

Notice that the fair game model in terms of returns still gives

$$E_t\left(\Pi_{t+1} \middle| \Omega_t\right) = 0$$

The submartingale model, which has grown in popularity since Mandelbrot (1966) or Samuelson (1965), only requires independence of successive price changes which allows for changes in volatility of futures prices.

One implication of the submartingale theory is that price changes, beyond opportunity costs, are serially uncorrelated and appear random. Since price changes respond to new information which arrives randomly, future price changes will move in an unpredictable manner.

2.3 Chaos

Economists have, until recently, had little success explaining the random looking but bounded fluctuations of economic time series using deterministic methods. Only four types of nonlinear dynamical behavior were considered, monotonic convergent, monotonic divergent, periodic convergent, and periodic divergent. Each would converge to or diverge from some point or limit cycle so could not represent economic fluctuations. Physical scientists, however, have recently revived interest in nonlinear dynamics and in particular chaos theory.

Before the precise meaning of chaos can be given, as it applies to economics, a few preliminary definitions are necessary.

Definition:

$F^k(X_t)$ is the k^{th} *iteration* of the C^1 system [differentiable once]

$$F(X_t): I^n \to I^n, \quad I^n \subseteq R^n,$$

for all $X \in I^n$ and all integers k and l, if

1) $F^k(X_t) = F(F(\cdots F(X_t)\cdots))$, k times.

2) $F^0(X_t) = X_t$

3) $F^k(F^l(X_t)) = F^{k+l}(X_t)$

where I is an n-dimensional subset of R^n.

Using this method we can choose any system F and any initial point $X_0 \in I$ and follow the discrete iterative process generated by F. Notice that the initial point X_0 is drawn from the continuous interval I but the iterations $F^k(X_0)$ are discrete.

Definition:

When an initial state X_0 is given then the flow

$$\{F^k(X_0)\}_{k=0}^{\infty}$$

is the *trajectory* of X_0 under the function F.

The iterative process generated by F follows a time path or sequence $\{F^k(X_0)\}_{k=0}^{\infty}$ which is known as the trajectory of the initial point X_0 driven by the system F.

Definition:

X_t is a *periodic (fixed) point* of F with period k if $F^k(X_t) = X_t$. The point X_t is then known as a *period k point* or alternately a *k period point*

Here, the iterative process of $F^k(X_t)$ always returns back to the point X_t after k iterations. Since all points between X_t and $F^k(X_t)$ are also period k points the resulting sequence is known as a *period k cycle* or alternately a *k period cycle*.

Definition:

If X_t is a periodic point of F then X_t is *asymptotically stable* if there exists some neighborhood U of X_t such that for all $X_0 \in U$

$$\lim_{l \to \infty} F^{kl}(X_0) = X_t$$

The limit of the trajectory [as l goes to infinity] of any initial point X_0 in the interval I is some k period cycle.

Definition:

The set of all initial states whose trajectory asymptotically approaches the stable periodic point X_t is known as the *basin of attraction* of X_t. This means that there is some subset of I whose trajectory under the system F converges to the stable k-period cycle.

Definition:

The set I is *positive invariant* with respect to the trajectory $\{F^k(X_0)\}_{k=0}^{\infty}$ if for each $X_0 \in I$, $F^k(X_0) \in I$ for all k. When this holds for $k \in (-\infty, \infty)$ the trajectory is invariant.

This condition gives the values generated by the trajectory of X_0 both an upper limit and a lower limit.

Definition:

If the limit $\lim_{l \to \infty} F^{kl}(X_0)$ does not exist for any k or any $X_0 \in I$, but the trajectory $\{F^n(X_0)\}_{n=0}^{\infty}$ is positive invariant in I then X_t is *aperiodic*.

The function F generates some trajectory of X_0 that never converges to a periodic cycle but stays within an upper and a lower bound. In this case the trajectory appears random even though it is generated by a deterministic system.

Definition:

The set I^1 is *dense* in I if $I^1 \subseteq I$ and for any $X_0 \in I^1$, $X_t \in I$, and $\varepsilon > 0$ there exists k such that $\left|F^k(X_0) - X_t\right| < \varepsilon$.

In other words I^1 is dense in I if for any point, X_0, originating in I^1 we can find a sequence of points that start at X_0 and converge to X_t. When denseness is applied to periodic points in I under the function F, structure is implied for the sequence $\left\{F^k(X_t)\right\}$, and values in the range of F encompass all the infinite points of I. For any element, X_t, in the range of F, one can retrace the deterministic sequence of points which converge to it. Comparing this to a stochastic sequence, we find that random points are identified by their probability of occurrence in some distribution, not by their place in a deterministic sequence.

Definition:

F is *topologically transitive* if for any pair of sets $I^1, I^2 \subseteq I$ there exists k>0 such that $F^k\left(I^1\right) \cap I^2 \neq \varnothing$.

This means that, as k increases, the function $F^k(X_t)$ will take on infinitely many values within the set I. Over time, the value of $F(X_t)$ can move from any point in I to any other point in I. This condition is required to ensure that the function F maps I back into itself, without decomposing into positive invariant subsets of I. Essentially, it is the same as the ergotic character of economic time series.

Definition:

$F: I \to I$ has *sensitive dependence on initial conditions* if for all arbitrarily close initial points $X_0^1, X_0^2 \in I$ where $\left|X_0^1 - X_0^2\right| = \varepsilon$ there exists $\delta > \varepsilon$ and k>0 such that

$$\left|F^k\left(X_0^1\right) - F^k\left(X_0^2\right)\right| \geq \delta .$$

Given two initial points that are arbitrarily close to each other, their respective trajectories under the map F will diverge at some rate characteristic of F until, for all practical purposes, they are uncorrelated. Since initial conditions are not known exactly, serious doubts are introduced as to the accuracy of long term forecasting.

Definition: [Devaney (1989)]

The set I is a *strange (chaotic) attractor* under $F: I \to I$ if

1) periodic points are dense in I,
2) F is topologically transitive, and
3) F has sensitive dependence on initial conditions.

This is the definition of chaos -- in a conservative system since $I \to I$. A deterministic function F that maps the set I back into itself and results in an aperiodic but bounded trajectory with sensitive dependence on initial conditions. The set I is known as a strange [chaotic] attractor and the function F is known as *chaotic*.

Economic models that incorporate either stable aperiodic or chaotic motion do closely replicate actual economic data, but without a stochastic component. Since the first two conditions of chaos are not empirically verifiable given economic time series we have to identify chaos by searching for its sensitive dependence on initial conditions property -- following Ruelle (1989).

2.4 Route to Chaos

Consider the system

$$X_{t+1} = F(A, X_t), \quad F: R^p \times I^n \to I^n, \quad I^n \subseteq R^n$$

where A is a p-dimensional parameter set, X_t is an n-dimensional set of variables, and changes in A lead to changes in the periodicity of the function F. Here, A is known as a set of *tuning parameters*.

To make the following ideas clear, it would be easier to use a specific form of F and use it to walk through this particular route to chaos. I will use the one dimensional logistic map

$$x_{t+1} = f(a, x_t) = a x_t (1 - x_t), \quad f: (0,4) \times (0,1) \to (0,1)$$

which has one variable x and one tuning parameter a. Since the maximum of the function f is $a/4$, $0<a<4$ is required to ensure that x is mapped from (0,1) back into (0,1).

To find the periodic points of this system, x_{t+k} is equated with x_t. Period one [k=1] gives

$$x_t = a x_t (1 - x_t)$$

where the two roots are $1 - 1/a$ and 0.

Once the periodic points are found, they can be tested for stability. Discrete one dimensional systems are stable if

$$\left| \mathrm{D} f^k(x_t) \right| < 1$$

and unstable if

$$\left| \mathrm{D} f^k(x_t) \right| > 1.$$

In this particular case

$$\left| \mathrm{D} f^1(x_t) \right| = \left| a - 2x_t \right|$$

At the period one point, x_t=0, the system is stable for all values $a \in (0,1)$, and at $x_t = 1 - 1/a$ the system is stable for all values $a \in (1,3)$. This means that in state space [the state of x at time t or, alternately, variable x plotted against time] the trajectory will converge to 0 for all values $a \in (0,1)$ and will converge to some unique point in $\boldsymbol{I}$ for all values $a \in (1,3)$. In either case the trajectory is monotonic.

Now consider the second iteration of the logistic equation

$$f^2(x_t) = ax_{t+1}(1-x_{t+1})$$

where the periodic points are

$$x_t = f^2(x_t)$$

where the four roots are 0, $1-1/a$, and $\left(a+1\pm\sqrt{(a-3)(a+1)}\right)/2a$. The first two share the same stability properties as the period one points but the latter two are stable in the interval

$$a \in (3, 3.4495)$$

As a increases past three the trajectory bifurcates from being monotonic to a stable two period cycle. If this analysis is continued to $f^4(x_t)$ we will find that as the value of a passes through 3.4495, the two period cycle bifurcates into a 4 period cycle.

Increasing the tuning parameter a from 0 up, leads to a stable periodic point doubling effect on the trajectory $\left\{f^k(x_0)\right\}_{k=0}^{\infty}$ that follows Sarkovskii's ordering:

$$\begin{aligned}
&1 \prec 2 \prec 2^2 \prec 2^3 \prec 2^4 \prec \ldots \\
&\prec 2^l \cdot 9 \prec 2^l \cdot 7 \prec 2^l \cdot 5 \prec 2^l \cdot 3 \prec \ldots \\
&\prec 2^2 \cdot 9 \prec 2^2 \cdot 7 \prec 2^2 \cdot 5 \prec 2^2 \cdot 3 \prec \ldots \\
&\prec 2 \cdot 9 \prec 2 \cdot 7 \prec 2 \cdot 5 \prec 2 \cdot 3 \prec \ldots \\
&\prec 9 \prec 7 \prec 5 \prec 3
\end{aligned}$$

for all l, where $a \prec b$ indicates a precedes b. In reverse order, this is all the odds, increasing in value, except 1, then 2 times the odds, then 2^2 times the odds, and continues until all the natural numbers are used up except for 1 and the powers of 2, which are listed in a decreasing sequence. Figure 2.1 shows all possible bifurcations as the parameter a is tuned between 2.7 and 4.

The range over which a is stable between critical bifurcation values decreases as a increases until the 2^{∞} cycle, which materializes [for the logistic map] at approximately 3.5699. Feigenbaum (1978) showed that convergence to 2^{∞} is a universal feature of unimodal maps and that the interval over which a cycle remains stable converges at the geometric rate of

$$\lim_{p\to\infty}\left(\frac{a_p - a_{p-1}}{a_{p+1} - a_p}\right) \cong 4.6692$$

where a_p is some critical bifurcation value of a

Sarkovskii's theorem (1980) states that if $f\colon \boldsymbol{I} \to \boldsymbol{I}$ is a continuous mapping with a period k trajectory and $l \prec k$ in Sarkovskii's ordering, then f also has a trajectory of period l. This theorem has consequences for economic modeling, since higher periodic points imply the existence of lower ones then if no 2 period cycle can be found in some dynamic model, then there are no other periodic points. Alternately, if a period three cycle can be detected then cycles exist of every possible periodicity.

The celebrated Li and Yorke (1975) result which states that period 3 implies chaos refers to the infinite cycles within Sarkovskii's ordering. These cycles are aperiodic since they never repeat themselves but they are not chaotic since they are stable and lack the sensitive dependence of initial conditions property of chaos. For each periodic point $x_t = f^k(x_t)$ where k=1,2,...,5,3 there exists a basin of attraction from which trajectories converge to x_t. This means that points infinitely close to the period 2^{∞} trajectory can be found such that long term forecasting is feasible. This is an important note since many economic models have been developed that claim chaotic behavior based on this result -- for a survey see Boldrin and Woodford (1990). For empirical purposes, an aperiodical cycle does not necessarily imply the presence of sensitive dependence on initial conditions.

Once a increases past the interval where 3 period cycles are generated all the [natural] periodic points become unstable and the trajectory once again behaves in an aperiodic manor. However, this time the trajectory f has sensitive dependence on initial conditions and is chaotic.

Period doubling is a sufficient but not a necessary route to chaos. Yet, it lends itself well to economic modeling. Simple models are tuned into a chaotic regime, giving complicated but random looking fluctuations that fool many empirical tests.

2.5 Diagrammatical Representation

Another method used to illustrate the route to chaos is the following diagrammatical approach. Consider the phase space graph where all values of x_t are on the horizontal axis and all values of x_{t+k} are on the vertical axis. The two lines $x_{t+k} = x_t$ and $f^k(x_t)$ are plotted in this space, where $x_{t+k} = x_t$ turns out to be a 45° line [since $x_t, x_{t+k} \in \boldsymbol{I}$] and $f^k(x_t)$ is the value of f^k evaluated at each $x_t \in \boldsymbol{I}$. Points where these lines intersect are periodic points since at their intersection, $f^k(x_t) = x_{t+k} = x_t$.

Now consider only x_t, x_{t+1}, and $f(x_t)$, as represented in figure 2.2. Choose some arbitrary x_0 on the x_t axis. The first iterate is found at the height of the curve $f(x_t)$ evaluated at x_0. This value becomes x_1 on the vertical axis. To obtain the second iterate, x_1 has to be transferred to the x_t axis so the function $f(x_t)$ can be applied again. This is done by moving x_1 horizontally to the 45° line where $x_{t+1} = x_t$. Directly below this *axis transfer point* is where x_1 lies on the x_t axis. The height of $f(x_t)$ at x_1 then becomes the next iterate in the series $\{x_t\}_{t=0}^{\infty}$. To find the behavior of the entire series this algorithm can be repeated for all time, t.

Earlier it was stated that the periodic point, $f^k(x_t)$, is stable if $\left|Df^k(x_t)\right| < 1$ and unstable if $\left|Df^k(x_t)\right| > 1$. This can easily be shown by using phase diagrams. Figure 2.3 shows a stable periodic point. The slope of $f(x_t)$, $\left[Df(x_t)\right]$, is greater than -1 and less than 1 when it crosses the 45° line. Each time an iteration is transferred to the horizontal axis the axis transfer point gets closer to the intersection of $f(x_t)$ and $x_{t+1} = x_t$. If $f(x_t)$ is unstable as in figure 2.4 then after each iteration the axis transfer point is moved farther away from the intersection of $f(x_t)$ and $x_{t+1} = x_t$.

Figures 2.5 and 2.6 give phase and state space representations of the logistic map at $a = 0.8$ and $x_0 = 0.5$. Figure 2.5 has the relevant interval ***I*** on both the horizontal and the vertical axis and Figure 2.6 has ***I*** on the vertical axis and time on the horizontal axis. Notice that the only fixed point is at $f(x_t) = x_t = 0$. However, since the interval ***I*** -- (0,1) in this case -- is open the 0 endpoint cannot be included as a possible initial point which leads to chaos. The next two figures, 2.7 and 2.8, show what happens to $f(x_t)$ and $f^2(x_t)$ when the parameter a is increased to 2.6. There are now two fixed points, 0 is unstable and point A is stable. The slope, in absolute value, of both $f(x_t)$ and $f^2(x_t)$ as they pass through $x_{t+k} = x_t$ at point A is less than one. From Figure 2.8 we see that the trajectory approaches some positive unique value between 0 and 1. Figures 2.9 and 2.10 picture the results after a is increased to 3.3. Point A, in Figure 2.9, where $f(x_t)$, $f^2(x_t)$, and $x_{t+k} = x_t$ intersect has become unstable for both $f(x_t)$ and $f^2(x_t)$. However, points B and D have become stable for $f^2(x_t)$ indicating a period 2 cycle. The asymptotically stable cycle in this case is the box BCDE. In Figure 2.10 the upper and lower limits correspond to points C and D of Figure 2.9, respectively. In Figures 2.11 and 2.12 the parameter a has been increased to 3.52 giving a period 4 cycle. The line $f^2(x_t)$ has become unstable at all points of intersection with the 45° line , but following Sarkovskii's ordering, $f^4(x_t)$ has become stable at points A, B, C, and D. A period 3 cycle is shown in Figures 2.13 and 2.14. Between the period one cycle and the period 3 cycle, all other possible period k

cycles have had their cycles bifurcated into and out of stability. Beyond the period 3 cycle the slope of $f^k(x_t)$ becomes unstable, for all k, at all points of intersection with the 45° line. The trajectory $\{f^k(x_t)\}$ becomes chaotic.

To demonstrate sensitive dependence on initial conditions of the chaos pictured in figures 2.15 and 2.16, figure 2.17 has two overlapping trajectories, both with $a = 3.9$, as before, but one with initial condition $x_0 = 0.5$ and the other $x_0 = 0.5001$. Notice that the trajectories overlap for the first 20 or so iterations, begin to diverge, and eventually become incomparable. Long run forecasting is feasible only if the initial condition x_0 is known with precise accuracy.

It is easy to see how one could conclude by linear testing that the system is linear but disturbed by random shocks. However, such complex dynamics come from a very simple yet very deterministic system.

2.6 Conclusion

Chaotic dynamics show how simple nonlinear difference equations can yield deterministic time paths that mimic the output of stochastic systems. Thus, offering an alternative explanation for the behavior of asset prices. This chapter began with some traditional beliefs regarding price changes, then showed how these changes can be replicated using nonlinear dynamics. We now know that it is possible to mimic stochastic looking systems using chaos theory. To make the connection complete we now have to find chaotic dynamics in actual time series.

Figure 2.1 Bifurcation Diagram for the Logistic Equation

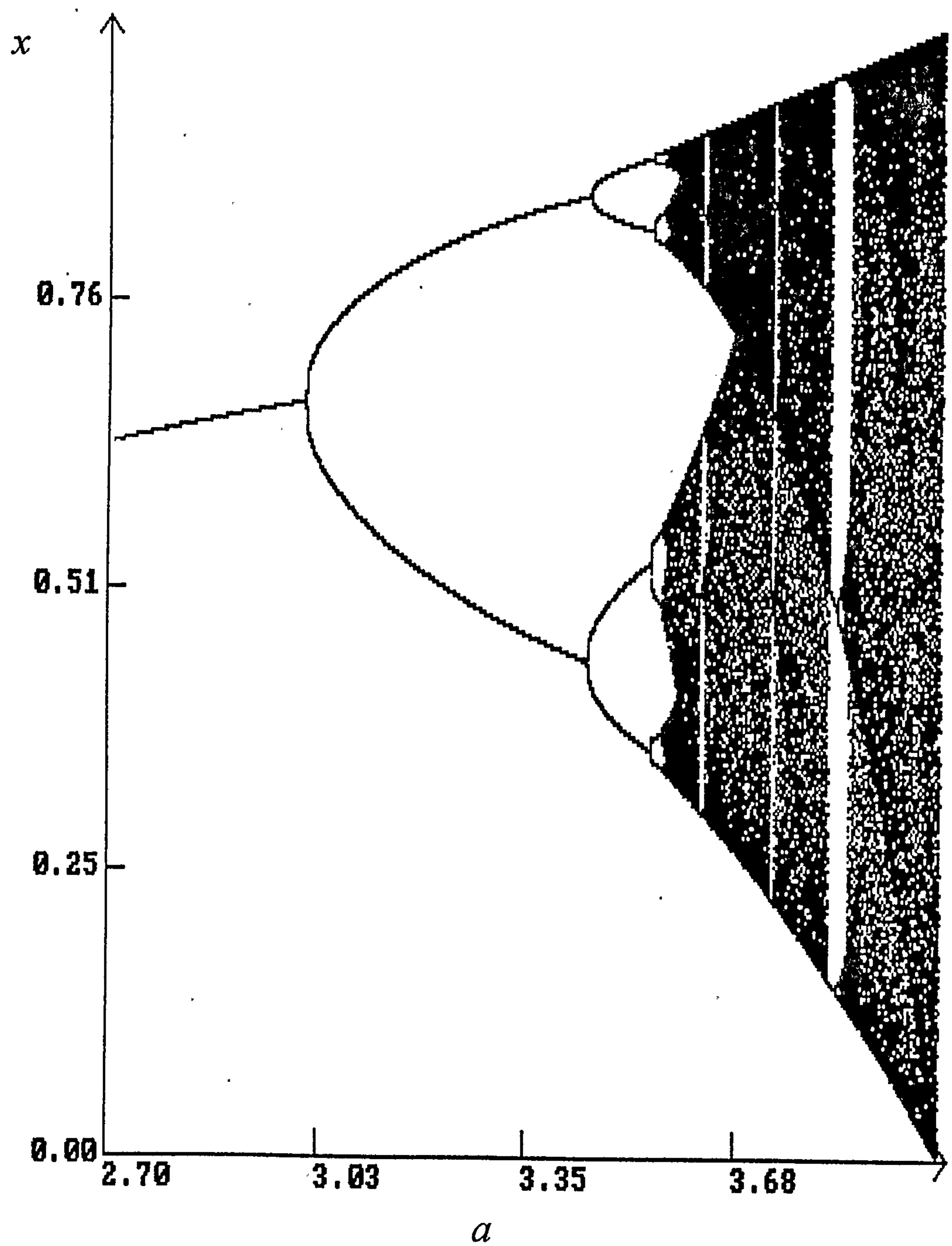


Figure 2.2 Dynamic Process

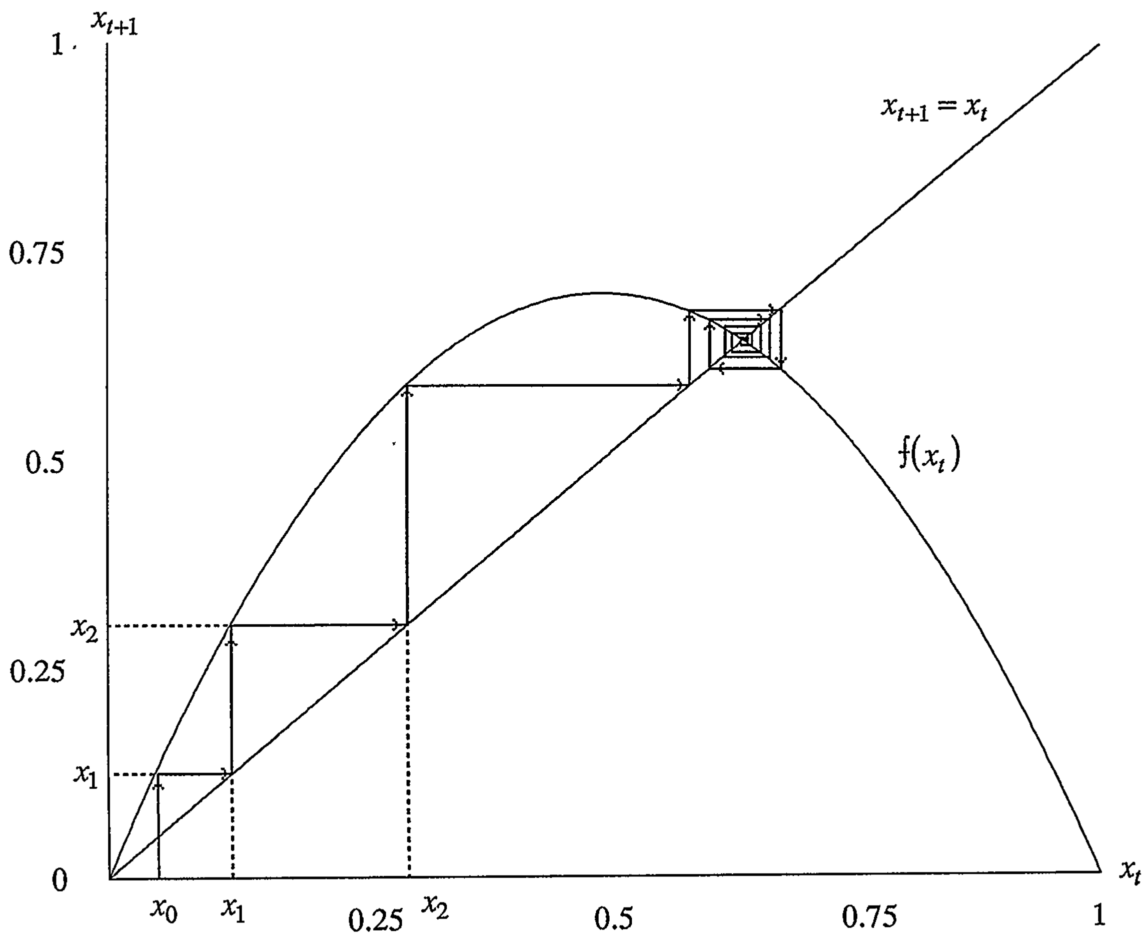


Figure 2.3 Stable Dynamics

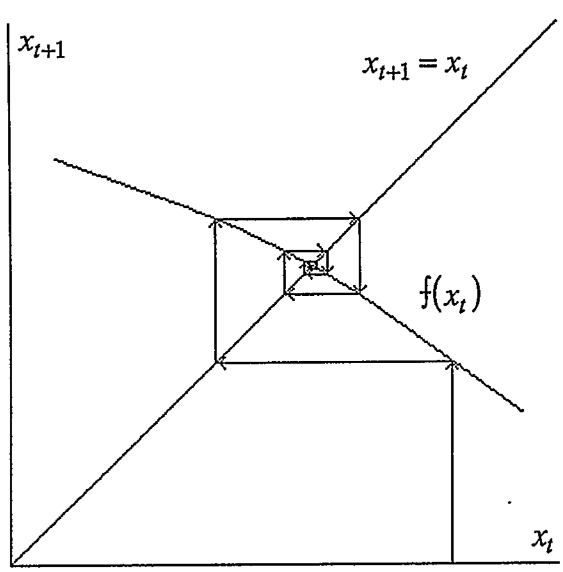


Figure 2.4 Unstable Dynamics

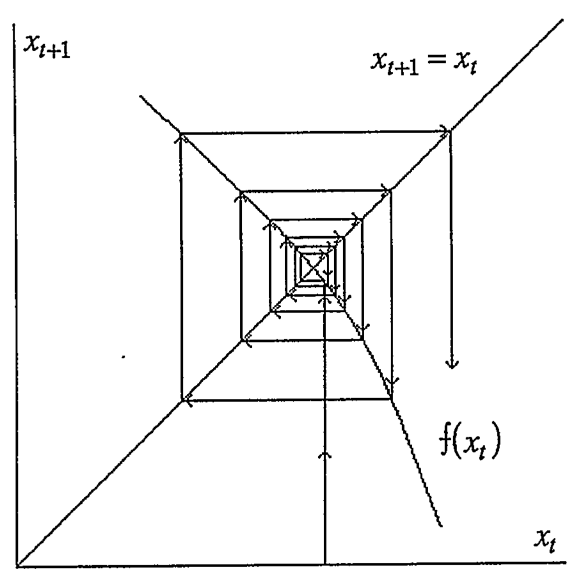


Figure 2.5 Phase Diagram for the Logistic Equation, Period 1 Cycle

$a = 0.8$

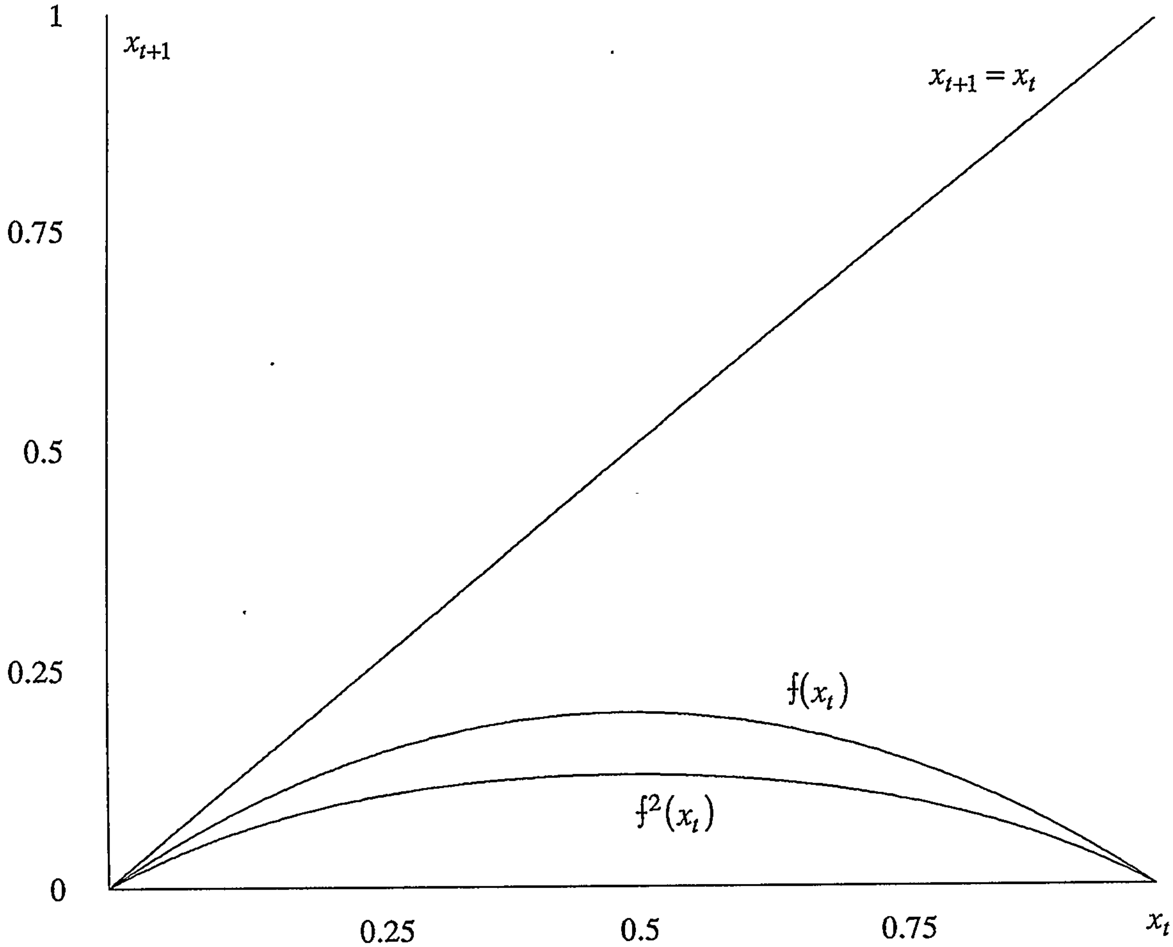


Figure 2.6 State Diagram for the Logistic Equation, Period 1 Cycle

$x_0 = 0.5, \quad a = 0.8$

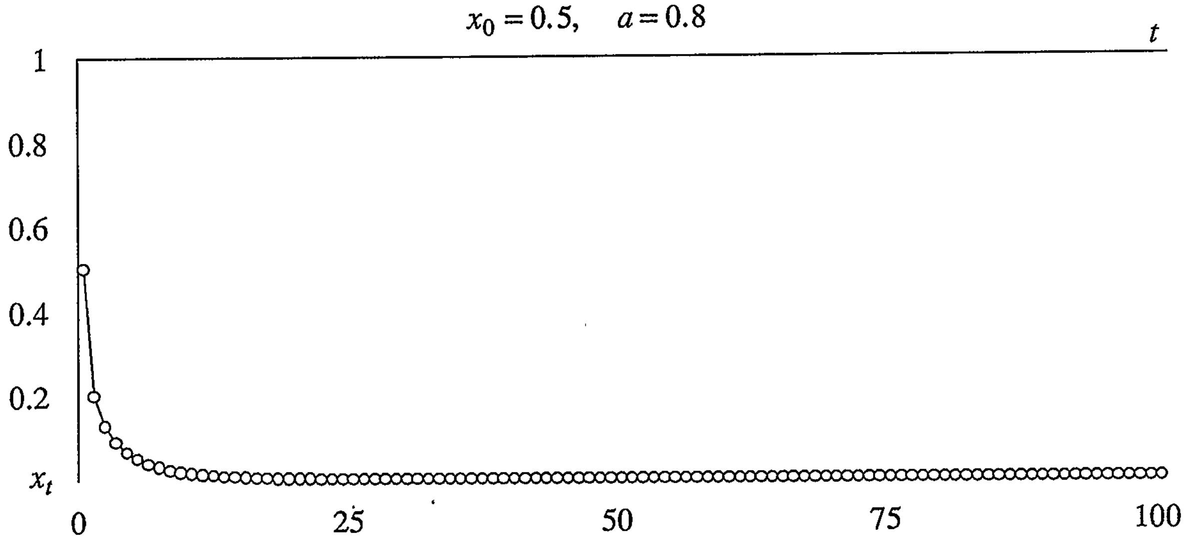


Figure 2.7 Phase Diagram for the Logistic Equation, Period 1 Cycle

$a = 2.6$

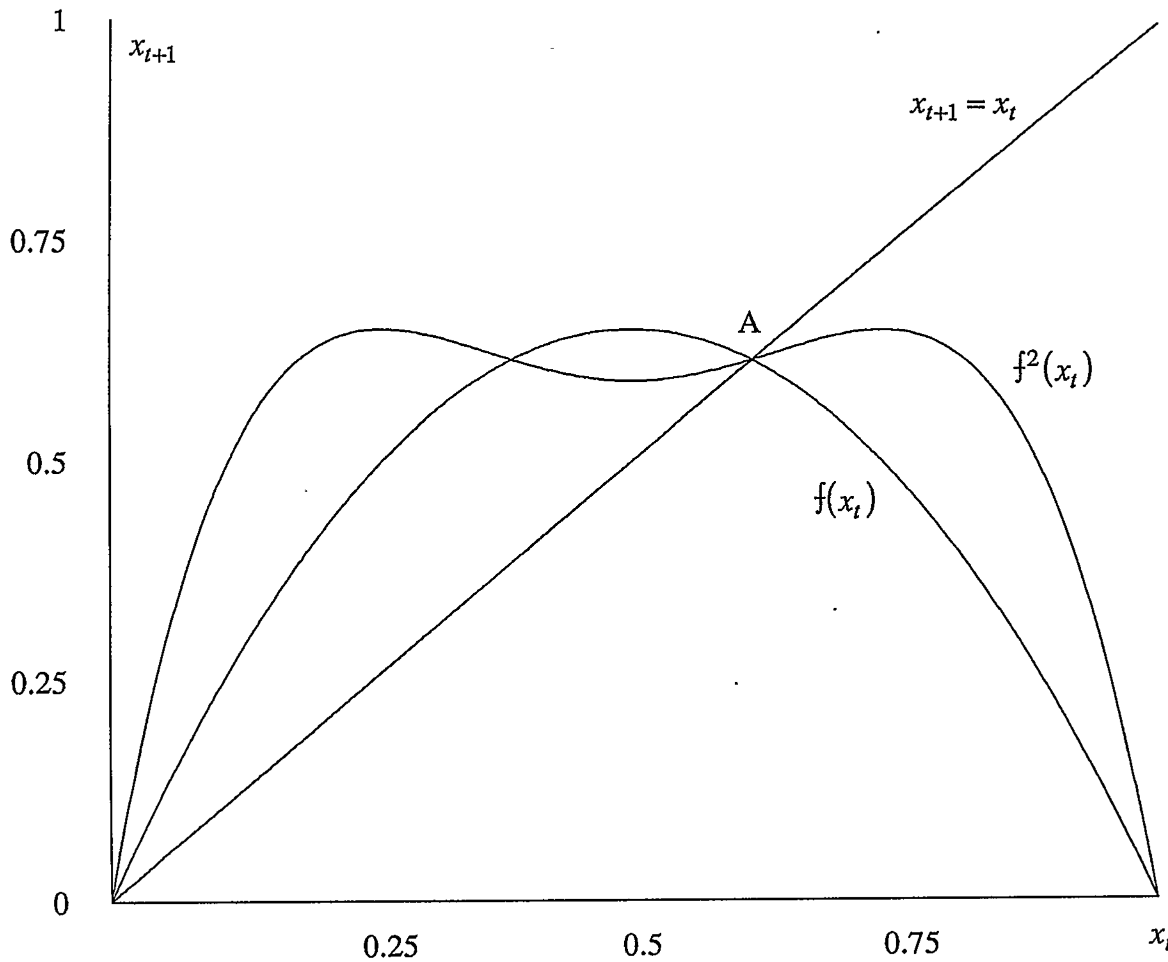


Figure 2.8 State Diagram for the Logistic Equation, Period 1 Cycle

$x_0 = 0.5, \quad a = 2.6$

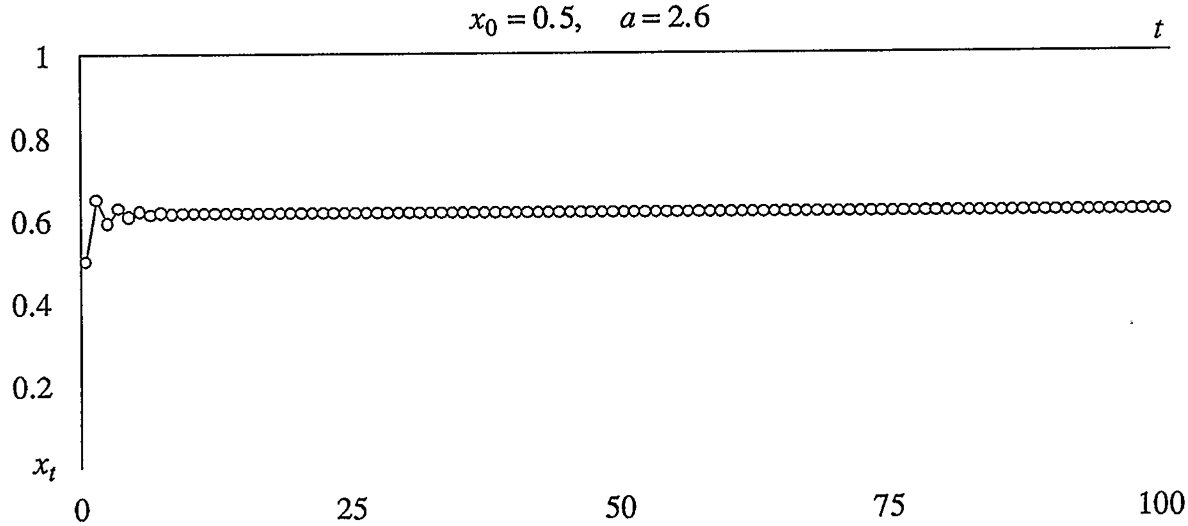


Figure 2.9 Phase Diagram for the Logistic Equation, Period 2 Cycle

$a = 3.3$

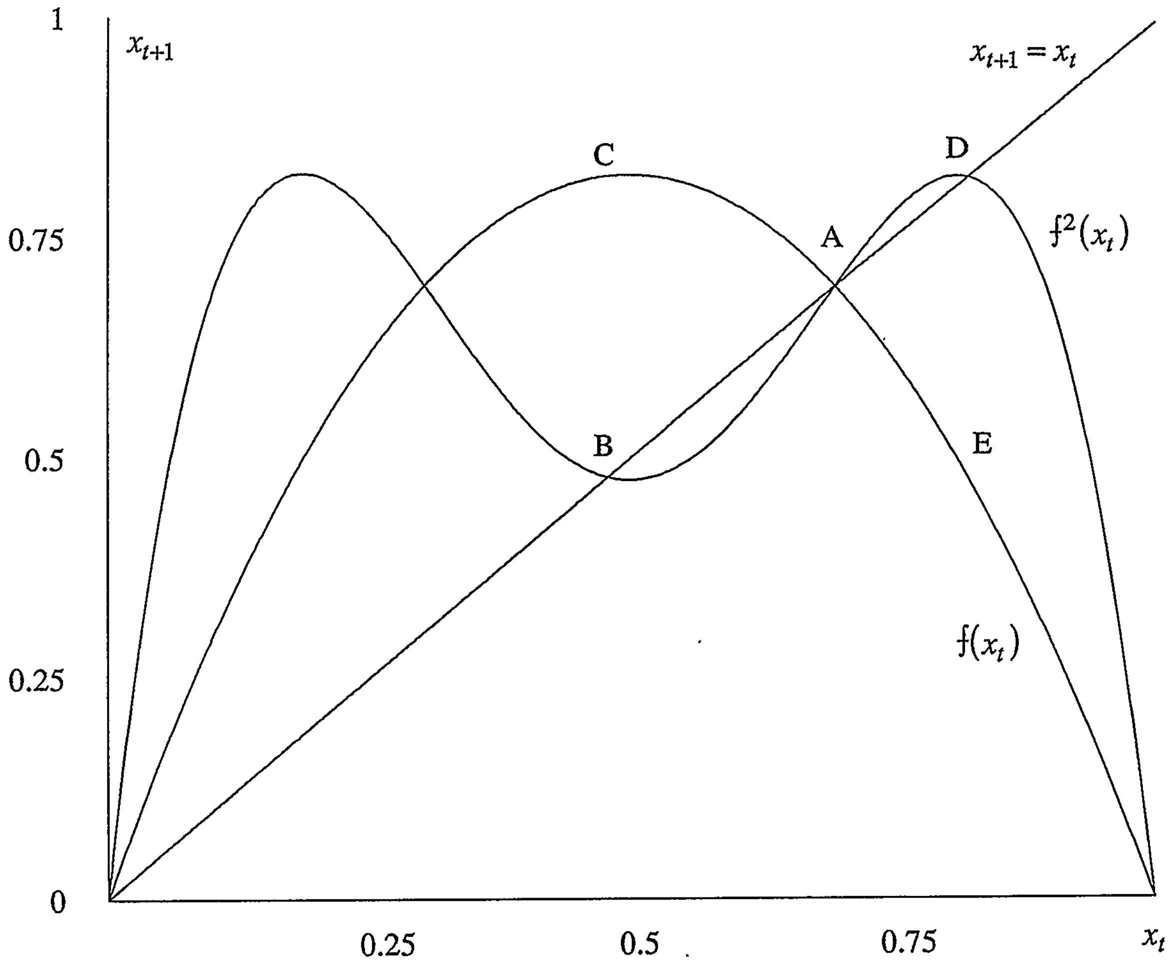


Figure 2.10 State Diagram for the Logistic Equation, Period 2 Cycle

$x_0 = 0.5, \quad a = 3.3$

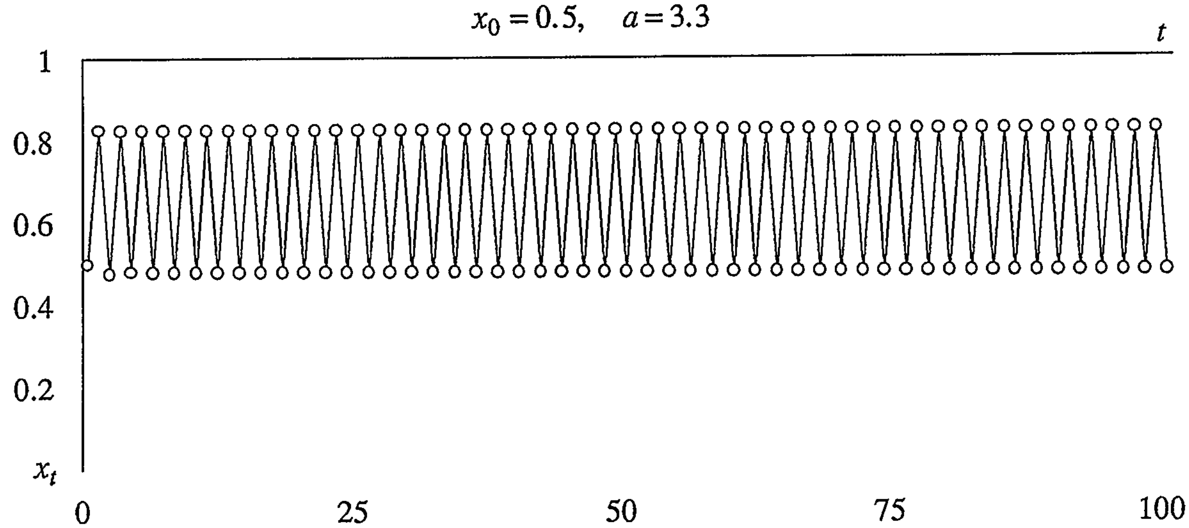


Figure 2.11 Phase Diagram for the Logistic Equation, Period 4 Cycle

$a = 3.52$

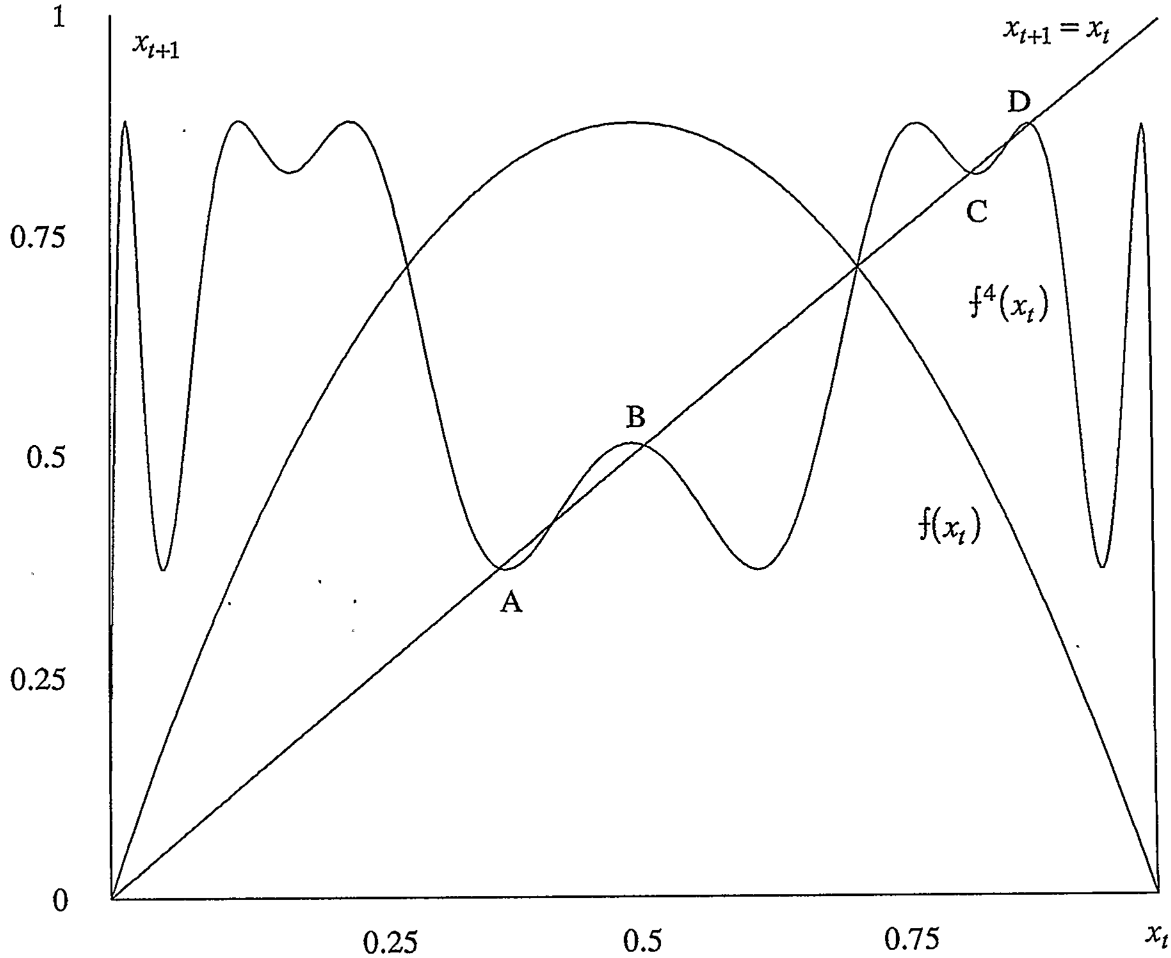


Figure 2.12 State Diagram for the Logistic Equation, Period 4 Cycle

$x_0 = 0.5, \quad a = 3.52$

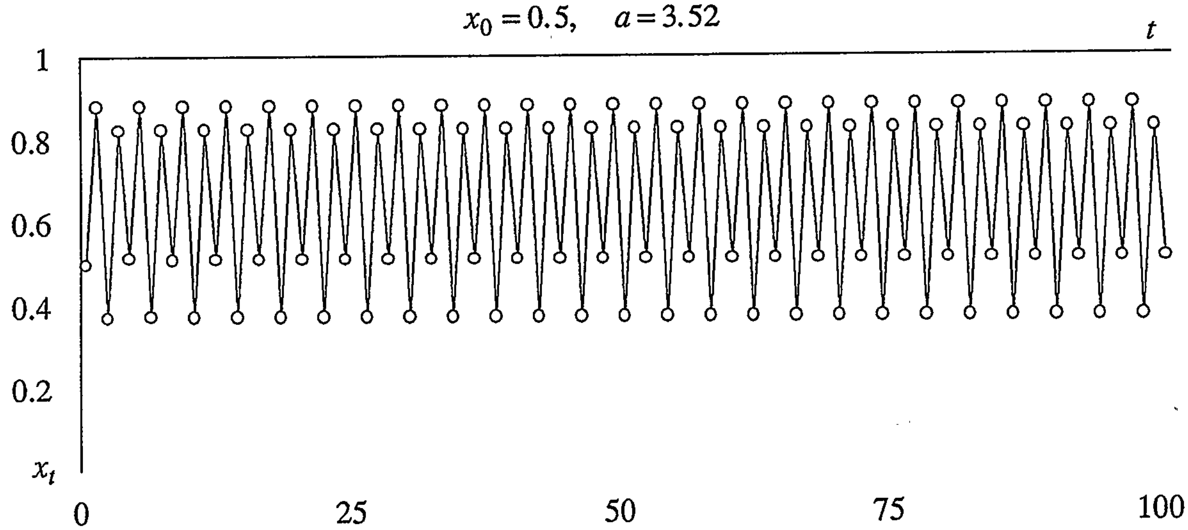


Figure 2.13 Phase Diagram for the Logistic Equation, Period 3 Cycle

$a = 3.839$

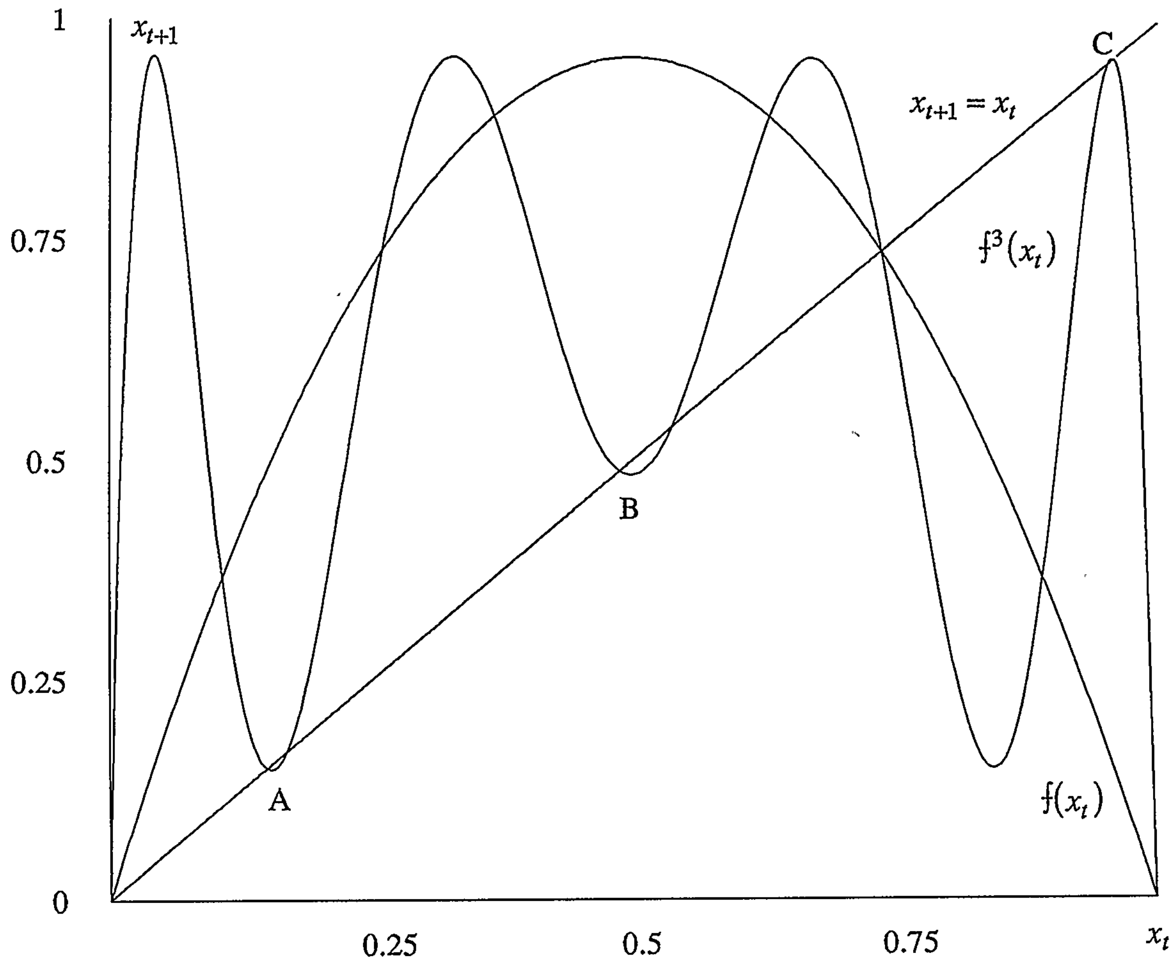


Figure 2.14 State Diagram for the Logistic Equation, Period 3 Cycle

$x_0 = 0.5, \quad a = 3.839$

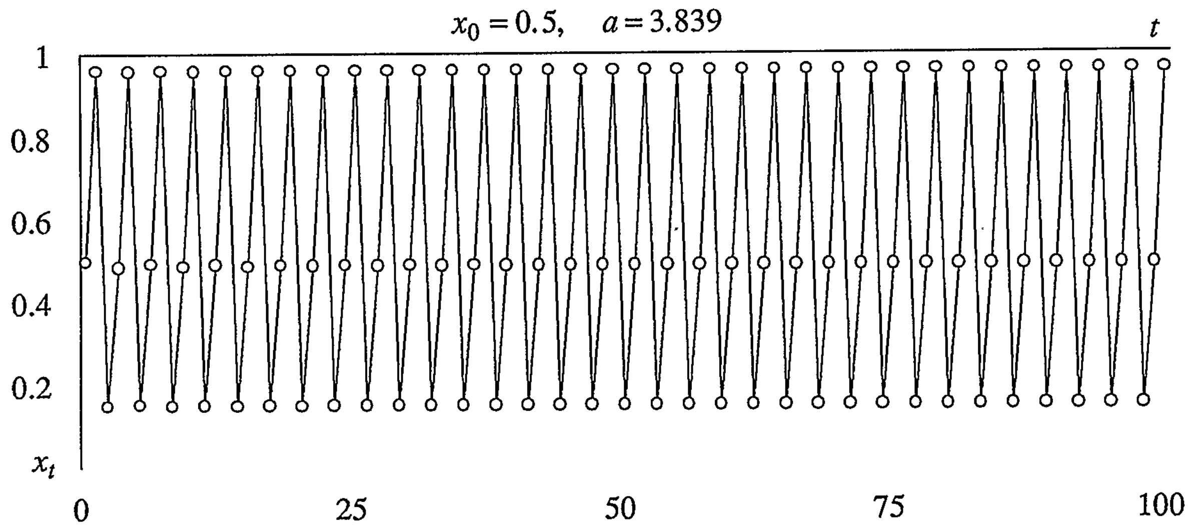


Figure 2.15 Phase Diagram for the Logistic Equation, Chaos

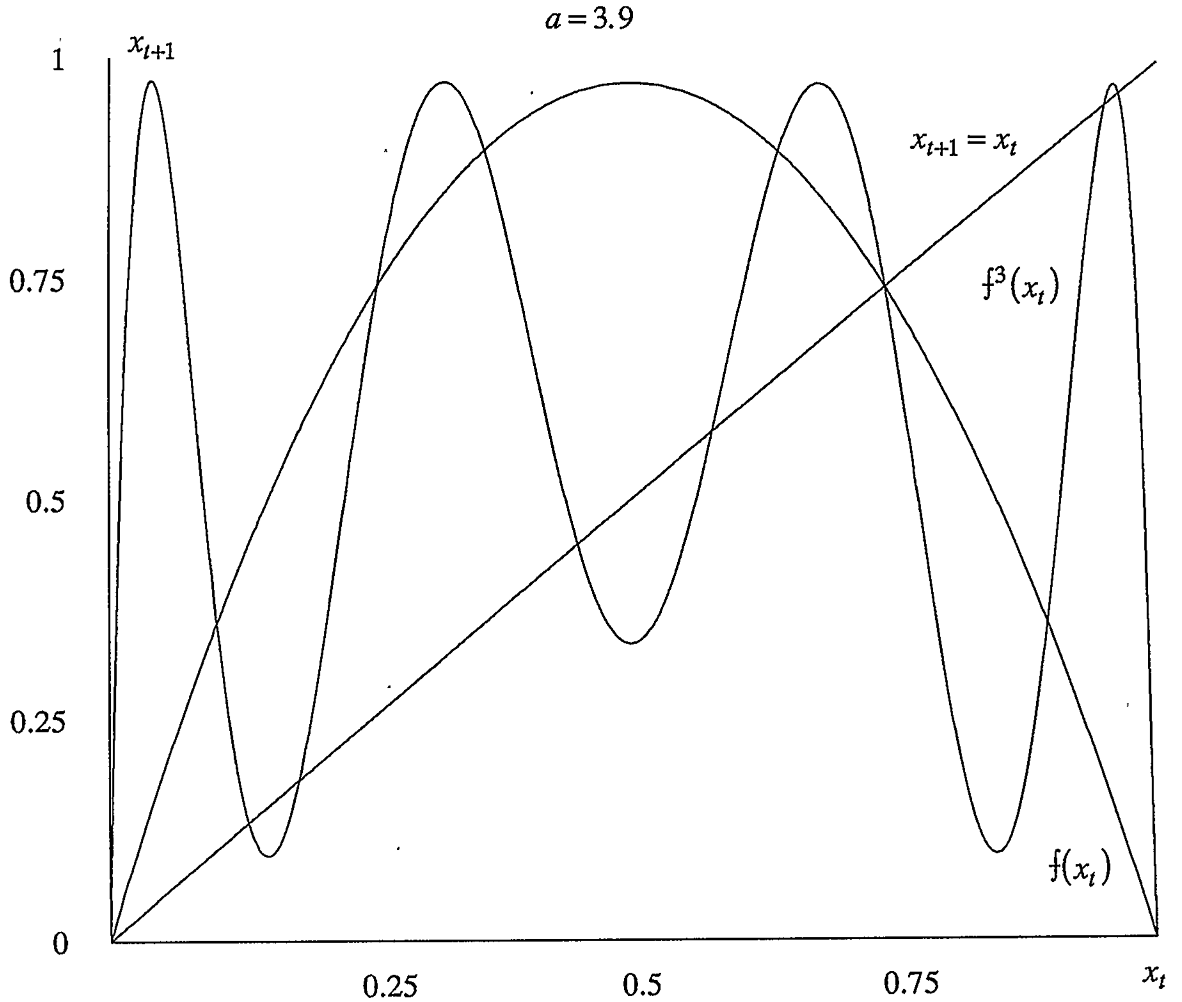


Figure 2.16 State Diagram for the Logistic Equation, Chaos

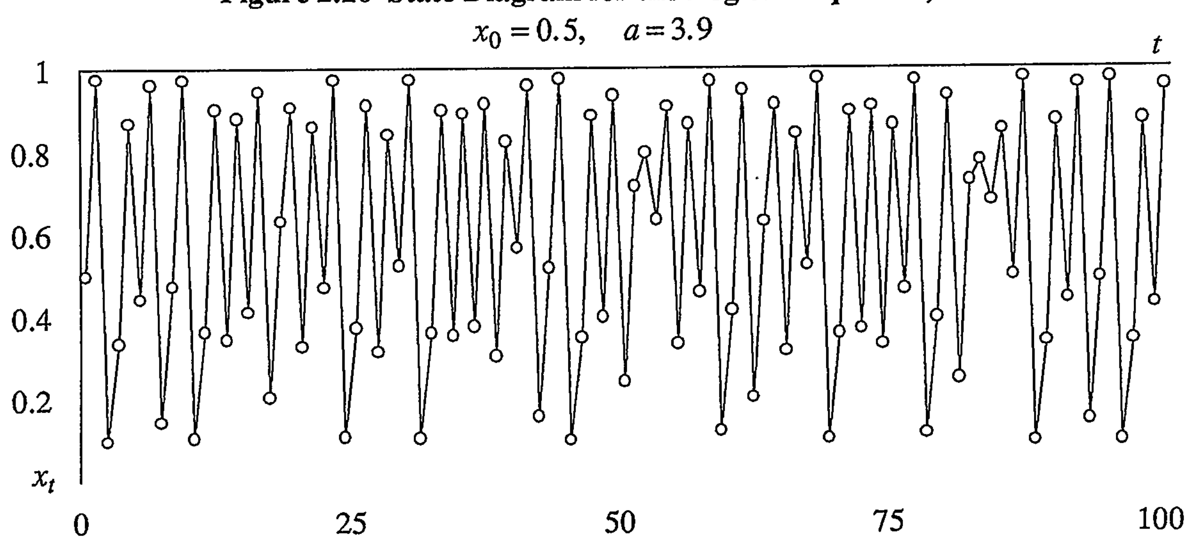
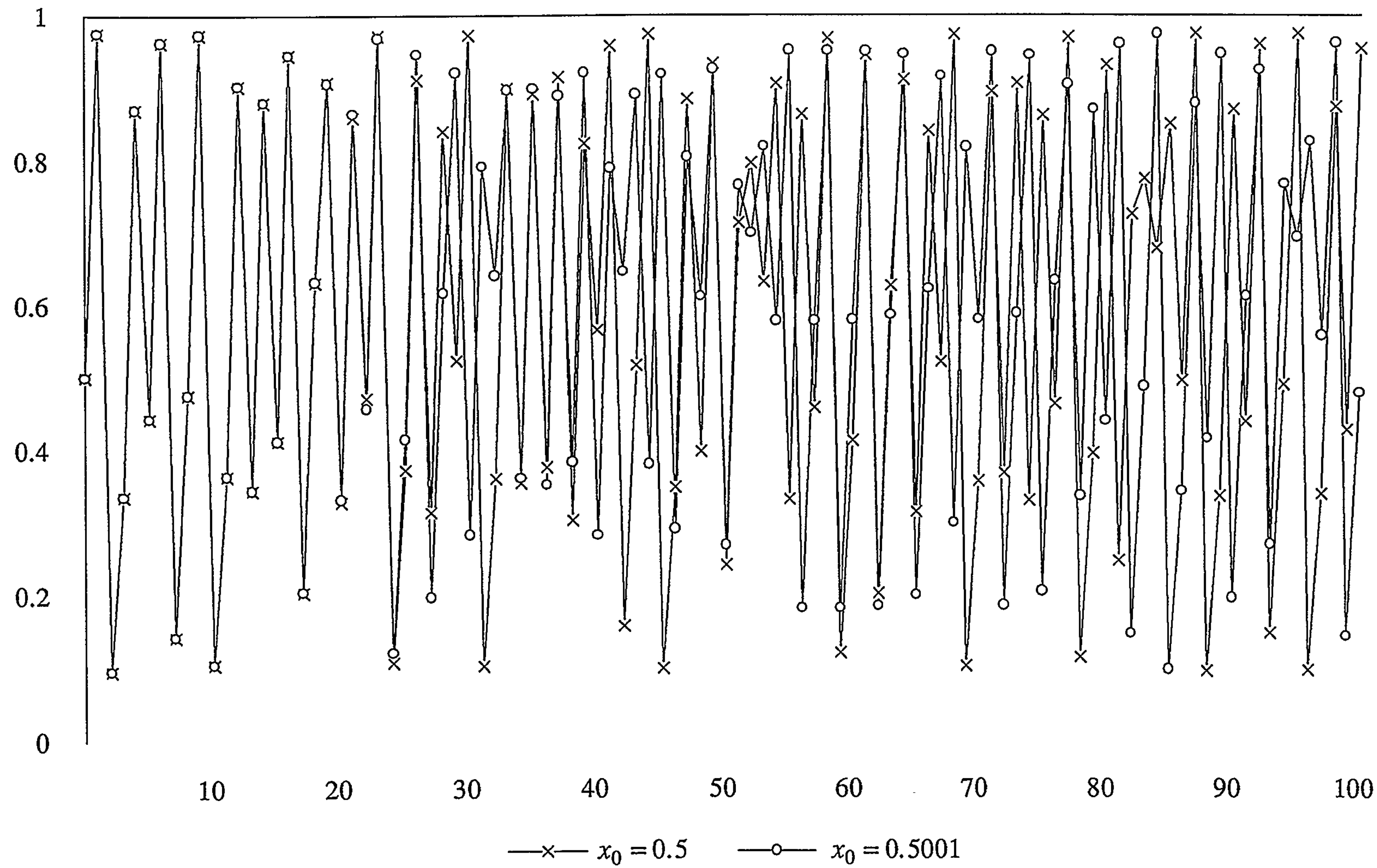


Figure 2.17 Sensitive Dependence on Initial Conditions, Logistic Equation

$a = 3.9$



Chapter 3

TESTING METHODOLOGY

3.1 Introduction

One of the most important contributions for the empirical analyses of nonlinear dynamics was that of Floris Takens in 1981. He found that for nonlinear dynamical systems with one observable, the entire underlying attractor can be rebuilt using only the observed variable. In the study of financial time series it is important to expose the underlying system -- if there is one -- which drives the series, and to reveal its various properties. If it can be established that the underlying system is chaotic then price changes can be explained as a deterministic outcome.

Section 3.2 explores, more formally, how the underlying attractor can be revealed. Then in section 3.3 it is shown how this information can be used to find the dimension of the underlying attractor. This knowledge is then, in section 3.4, used to derive a statistic which tests for nonlinear dependence, known as the BDS statistic [Brock, Dechert, and Sheinkman (1987)]. Since nonlinearity is a necessary but not a sufficient condition for chaos, more information is required to conclude chaos. We cannot empirically test finite systems for denseness or topological transitivity, so we follow Ruelle (1989), and rely on sensitive dependence on initial conditions as a testable condition for chaos. A measure of sensitive dependence, known as the dominant Lyapunov exponent, is outlined in section 3.5. Then section 3.6 shows how to empirically quantify this measure from a given time series.

3.2 Phase Space Embedding

All that we have available to us is the observed time series

$$\{x_t\}_{t=1}^{N}, x_t \in \boldsymbol{R}.$$

The true unknown dynamical system

$$Y_t = G(Y_{t-1}), G: \boldsymbol{R}^n \to \boldsymbol{R}^n$$

is seen through the observation function

$$x_t = h(Y_t), \quad h: \boldsymbol{R}^n \to \boldsymbol{R}.$$

The observed time series $\{x_t\}$ can be embedded into a series of m-dimensional vectors

$$X_t = (x_t, x_{t-1}, \cdots, x_{t-m+1})^{\mathrm{T}}$$

giving the series

$$\{X_t\}_{t=m}^{N}.$$

where each X_t is known as an *m-history* of the series $\{x_t\}_{t=1}^{N}$. For example, the set of 3-histories for the series $\{x_1, \cdots, x_6\}$ would be

$$X_3 = (x_3, x_2, x_1)^{\mathrm{T}}$$
$$X_4 = (x_4, x_3, x_2)^{\mathrm{T}}$$
$$X_5 = (x_5, x_4, x_3)^{\mathrm{T}}$$
$$X_6 = (x_6, x_5, x_4)^{\mathrm{T}}$$

The number of m-histories in the series $\{X_t\}_{t=m}^{N}$ is $T = N - (m-1)$.

Generically, the trajectory X_t may be written as

$$X_t = F(X_{t-1}), F: \boldsymbol{R}^m \to \boldsymbol{R}^m \tag{1}$$

Also, if the basin of attraction is a compact set [closed and bounded] and $m \geq 2n+1$, then X_t can be written in terms of Y_{t-m+1}.

$$X_t = H(Y_{t-m+1}) = \left(h\left(G^{m-1}(Y_{t-m+1})\right), h\left(G^{m-2}(Y_{t-m+1})\right), \cdots, h(Y_{t-m+1})\right).$$

Notice that

$$X_{t+1} = H(Y_{t-m+2}) = H\left(G(Y_{t-m+1})\right) \tag{2}$$

and from (1) and (2)

$$H\left(G(Y_{t-m+1})\right) = F\left(H(Y_{t-m+1})\right)$$

Assuming that H is a homeomorphism [continuous bijection with a continuous inverse], F and G are topologically equivalent and share many dynamic properties. This result [Takens (1981)] allows us to use the series of m-histories to analyze the true underlying dynamics of the observed series. If the elements of the true series $\{Y_t\}$ are on an attractor then the geometric object created by plotting the x_t's in m-dimensional phase space is congruent to the true attractor. Theiler (1990) also notes that as long as $m>n$ the reconstructed object almost always has the same dimension as the true attractor.

Figures 3.1 and 3.2 show the distribution of 2000 numbers produced by a standard random number generator and 2000 consecutive chaotic iterations from the logistic equation. For the logistic map the parameter a and initial value x_0 are the same as in figure 2.16 from the previous chapter. When these observations are embedded in 2-dimensional phase space the portraits 3.3 and 3.4 are produced. Notice that the series of randomly generated numbers cloud all available space when embedded in 2-space but the logistic iterations fall on the shape of their respective function.

3.3 Correlation Integral

If all the points $\{Y_t\}$ are on an attractor then any two points of the series $\{X_t\}$ are spatially correlated. Rather than being plotted randomly in phase space, the m-histories show a clustering effect. The probability that the distance between any two points X_t and X_s is less than some arbitrary small radius (ε) of an m-dimensional ball centered on one of the points [i.e., $\|X_t - X_s\| < \varepsilon$] is greater if clustering exists than if the points were plotted randomly -- as seen in Figures 3.1 through 3.4. The *correlation integral* for $t \neq s$ is

$$C_m(\varepsilon,T) = \frac{\text{number of distances less than } \varepsilon}{\text{total number of distances}}$$

$$= \frac{1}{T(T-1)} \sum_{t \neq s} \mathrm{H}\left(\varepsilon - \|X_t - X_s\|\right)$$

$$= \frac{2}{T(T-1)} \sum_{m \leq t < s \leq N} \mathrm{H}\left(\varepsilon - \|X_t - X_s\|\right)$$

where T is the total number of m-histories. and H is the Heaviside function.

$$\mathrm{H}(z) = \begin{cases} 1 & \text{if} \quad z > 0 \\ 0 & \text{otherwise} \end{cases}$$

The result $C_m(\varepsilon,T)$ is independent of any two norms $\|\cdot\|$ [Brock (1986)] so we may use any convenient form. The type most often used is the max-norm which is more convenient for computer applications.

$$\|X_t - X_s\| = \max_{k \in [0,m-1]} \left\{ |x_{t+k} - x_{s+k}| \right\}$$

where $|\cdot|$ is Euclidean distance. Using this norm the correlation integral may be written as

$$C(m,\varepsilon,T) = \frac{2}{T(T-1)} \sum_{m \leq t < s \leq N} \prod_{k=0}^{m-1} \mathrm{H}\left(\varepsilon - |x_{t+k} - x_{s+k}|\right)$$

since

$$\mathrm{H}\left(\varepsilon - \|X_t - X_s\|\right) = \prod_{k=0}^{m-1} \mathrm{H}\left(\varepsilon - |x_{t+k} - x_{s+k}|\right)$$

i.e., if any $|x_{t+k} - x_{s+k}| \geq \varepsilon$ then $\mathrm{H}(\cdot) = 0$.

$C(m,\varepsilon,T)$ is interpreted as the proportion of T m-histories that are within ε of each other.

Many authors have used the form

$$C_m(\varepsilon,T)=\frac{1}{T^2}\sum_{t,s=m}^{N}\mathrm{H}(\varepsilon-\|X_t-X_s\|)$$

but Grassberger (1982) shows that the inclusion of $t=s$ is unjustified and may lead to wrong conclusions when testing finite time series.

Grassberger and Procaccia (1982) have shown that the relationship

$$C(m,\varepsilon,T)=k\varepsilon^{\alpha} \qquad (3)$$

holds, where k is some constant and α is a dimension measure of the attractor. Taking logs of (3) gives

$$\ln C(m,\varepsilon,T)=\ln k+\alpha\ln\varepsilon$$

or

$$\alpha=-\frac{\ln k}{\ln\varepsilon}+\frac{\ln C(m,\varepsilon,T)}{\ln\varepsilon}.$$

As ε gets smaller and T gets larger α becomes

$$D(m)=\lim_{\varepsilon\to 0}\lim_{T\to\infty}\frac{\ln C(m,\varepsilon,T)}{\ln\varepsilon}$$

which is the *correlation dimension*. The correlation dimension of the true attractor is

$$D=\lim_{m\to\infty}D(m).$$

Let the dimension of a particular attractor be $\tilde{D}$. For $m<\tilde{D}$, $\tilde{D}(m)$ increases as m increases but levels off for $m\geq\tilde{D}$. However, if there is no clustering then the probability that two points being within ε of each other decreases as m increases so $\tilde{D}(m)$ increases endlessly with m. For finite data sets we cannot find $D(m)$ so we must be content with

$$D(m,\varepsilon,T)=\frac{\ln C(m,\varepsilon,T)}{\ln\varepsilon}$$

where T is large and ε is close to zero.

The correlation dimension can be used to detect the presence of an attractor. Plot the correlation dimension of a series for various ε's against m, if the slope of the lines level off at some point then an attractor is detected, otherwise the series is white noise. This method has been used in the literature [Frank, Gencay, and Stengos (1988)] but is not a statistic.

3.4 BDS Statistic

To deal with finite data samples Brock, Dechert, and Scheinkman (1987) devised a statistic which tests the null hypothesis of independent and identically distributed [*iid*] observations of a time series. $C(m,\varepsilon,T)$ has two possibilities. If the m-histories are *iid* then for all m they will be embedded randomly in m-space. However, if the points are not *iid*, then as m increases the points will arrange themselves on the attractor which will take shape at the correlation dimension of the true attractor. In this case, as m increases the probability that the points are within ε of each other decreases until the correlation dimension is reached, after which it levels off. Data not arranged in m-histories appears random so the value $C(1,\varepsilon,T)^m$ would be close to the value $C(m,\varepsilon,T)$ given *iid* observations. For a stochastic series the statistic

$$C(m,\varepsilon,T)-C(1,\varepsilon,T)^m$$

asymptotically [as $T \to \infty$] follows a normal distribution with zero mean and σ_w^2/T sample variance [Brock, Hsieh, and Lebaron (1991)]. Where

$$\sigma_w = 4\left(K^m + 2\sum_{j=1}^{m-1} K^{m-j}C^{2j} + (m-1)^2 C^{2m} - m^2 K C^{2(m-1)}\right)$$

$$C = \mathrm{E}\left[\mathrm{H}\left(\varepsilon - |x_t - x_s|\right)\right]$$

$$K = \mathrm{E}\left[\mathrm{H}\left(\varepsilon - |x_t - x_s|\right)\mathrm{H}\left(\varepsilon - |x_s - x_r|\right)\right]$$

and E is the expectations operator.

Consistent estimators for C and K are:

$$\hat{C} = C(1,\varepsilon,T)$$

$$\hat{K} = \frac{6}{T(T-1)(T-2)} \sum_{1 \le t < s < r \le T} \mathrm{H}(\varepsilon - |x_t - x_s|)\mathrm{H}(\varepsilon - |x_s - x_r|).$$

The BDS test may be written as

$$W(m,\varepsilon,T) = \frac{\sqrt{T}\left[C(m,\varepsilon,T) - C(1,\varepsilon,T)^m\right]}{\sigma_w} \xrightarrow{T \to \infty} \mathrm{N}(0,1).$$

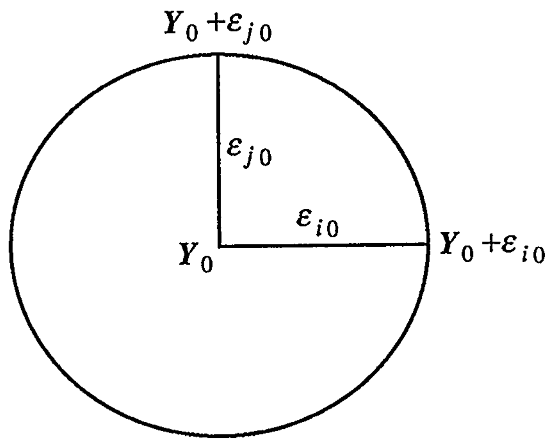
Rejection of the null implies that the data is either linear deterministic, nonlinear deterministic [chaotic]. or nonlinear stochastic, [Hseih, 1991].

As m increases both $C(m,\varepsilon,T)$ and $C(1,\varepsilon,T)^m$ decrease until the correlation dimension of the attractor is reached, if there is one, beyond which $C(m,\varepsilon,T)$ levels off and $C(1,\varepsilon,T)^m$ continues to decline.

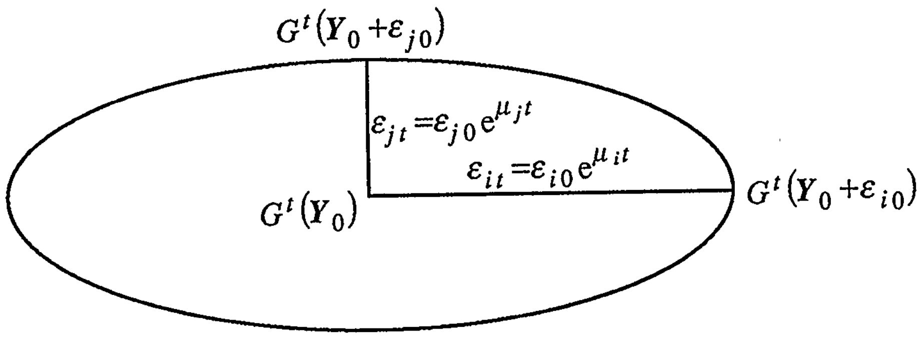
3.5 Lyapunov Exponents

The main characteristic of chaotic motion is its sensitivity to initial conditions. Lyapunov exponents are used to quantify this concept, thus differentiating between regular or nonregular motion of an attractor. Since G maps the interval $\boldsymbol{I}$ back into itself divergence of trajectories cannot go beyond the interval $\boldsymbol{I}$. In general, chaotic motion must then consist of exponential stretching and shrinking along various axes and then folding of the attractor by $Y_{t+1} = G(Y_t)$. Two points which are initially close together may get closer as in regular motion or they could be stretched far apart from each other. This Stretching determines the attractors sensitivity to initial conditions. Picture an n-

dimensional sphere, stretching and contracting along its various axes distorts it into an n-dimensional ellipsoid. After t iterations a two dimensional circle



becomes



which is folded over itself t times. μ_i gives the amount of stretching(μ_i>0) or contracting(μ_i<0) per period.

At time T

$$\varepsilon_{iT} = \varepsilon_{i0} e^{\mu_i T} \tag{4}$$

which can be written as

$$\mu_i = \frac{1}{T} \ln \left| \frac{\varepsilon_{iT}}{\varepsilon_{i0}} \right| \tag{5}$$

The absolute value $\left|\frac{\varepsilon_{iT}}{\varepsilon_{i0}}\right|$ is used since stretching or shrinking can occur in either direction along an axes.

Now consider the linear system

$$\varepsilon_{it+1} = q_i \varepsilon_{it}$$

whose solution at time T is

$$\varepsilon_{iT} = q_i^T \varepsilon_{i0} = \mathrm{e}^{\ln q_i^T}\, \varepsilon_{i0}.$$

Looking back at (4) it is clear that

$$\ln q_i^T = \mu_i T.$$

Substituting this into (4) gives:

$$\varepsilon_{iT} = \varepsilon_{i0} e^{\ln q_i^T} \quad \text{or} \quad \ln q_i^T = \ln\left(\frac{\varepsilon_{iT}}{\varepsilon_{i0}}\right)$$

$$\text{or} \quad q_i^T = \frac{\varepsilon_{iT}}{\varepsilon_{i0}}$$

Substituting this into (5) gives

$$\mu_i = \frac{1}{T} \ln\left|q_i^T\right|$$

which can be written as

$$\mu_i = \frac{1}{T} \sum_{t=0}^{T-1} \ln\left|q_i\right|.$$

μ_i is interpreted as the time average of the log of the absolute value of the slope value q_i. The i^{th} Lyapunov exponent of the system $G(Y_t)$ is

$$\mathrm{LE}_i = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \ln\left|q_i\right|, \quad i = 1, \cdots, n$$

which is the limit of μ_i as T goes to infinity.

More formally, a linear approximation of

$$\varepsilon_T = (\varepsilon_{1T}, \cdots, \varepsilon_{nT})$$

is

$$\varepsilon_T \cong \mathrm{J}^T(Y_0)\varepsilon_0$$

where $\mathrm{J}^T(Y_0)$ is the Jacobian of the system $G^T(\cdot)$, evaluated at Y_0. Recall that

$$G^T(\cdot) = G(G(\cdots G(\cdot)\cdots)),\ T \text{ times.}$$

By the chain rule

$$\begin{aligned}
\mathrm{J}^T(Y_0) &= \left.\frac{\partial G^T}{\partial G^{T-1}}\right|_{Y=Y_0} \left.\frac{\partial G^{T-1}}{\partial G^{T-2}}\right|_{Y=Y_0} \cdots \left.\frac{\partial G}{\partial Y}\right|_{Y=Y_0} \\
&= \left.\frac{\partial G}{\partial Y}\right|_{Y=Y_{T-1}} \left.\frac{\partial G}{\partial Y}\right|_{Y=Y_{T-2}} \cdots \left.\frac{\partial G}{\partial Y}\right|_{Y=Y_0} \\
&= \mathrm{J}(Y_{T-1})\mathrm{J}(Y_{T-2})\cdots\mathrm{J}(Y_0) \\
&= \prod_{t=0}^{T-1}\mathrm{J}(Y_t) \qquad (6)
\end{aligned}$$

diagonalizing $\mathrm{J}(Y_t)$ gives

$$\mathrm{J}(Y_t) = P\Lambda_t P^{-1}$$

where Λ_t is the diagonal matrix of eigenvalues $[\lambda_{it}, i = 1, \cdots, n]$ of $\mathrm{J}(Y_t)$ and P is the associated matrix of eigenvectors.

$$\Lambda_t = \begin{pmatrix} \lambda_{1t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{nt} \end{pmatrix}$$

Also

$$\ln\mathrm{J}(Y_t) = \ln P + \ln \Lambda_t - \ln P = \ln \Lambda_t.$$

Taking logs of (6) gives

$$\begin{aligned}
\ln \prod_{t=0}^{T-1}\mathrm{J}(Y_t) &= \sum_{t=0}^{T-1}\ln \mathrm{J}(Y_t) \\
&= \sum_{t=0}^{T-1}\ln \Lambda_t
\end{aligned}$$

The i^{th} Lyapunov exponent becomes

$$\mathrm{LE}_i = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \ln\left|\lambda_{i,t}\right|$$

which is a time average for the absolute value of the i^{th} eigenvalue from the system $G(\cdot)$. From the multiplicative ergotic theory [Eckman and Ruelle (1985)] this limit exists for most initial conditions Y_0.

The usual explanation of eigenvalues for discrete systems also applies here.

$$|\lambda_i| \quad \begin{cases} <1 \Rightarrow \mathrm{LE}_i < 0 & \text{(contraction)} \\ >1 \Rightarrow \mathrm{LE}_i > 0 & \text{(expansion)} \end{cases}$$

Since the attractor requires n dimensions there are n Lyapunov exponents. These can be listed in descending order, i.e., $\mathrm{LE}_1 \geq \mathrm{LE}_2 \geq \cdots \geq \mathrm{LE}_n$ which is called the Lyapunov spectrum. The signs of the spectrum are $(+,\cdots,+,0,\cdots,0,-,\cdots,-)$. Various forms of underlying attractors can be explained as follows:

$(-,\cdots,-)$ attractor contracts to a stationary point.

$(0,\cdots,0,-,\cdots,-)$ contracts in directions indicated by $(-)$'s and is stationary periodic in directions indicated by (0)'s.

$(+,\cdots,+,0,\cdots,0,-,\cdots,-)$ presence of stretching indicated by (+)'s.

The sum of all Lyapunov exponents is interpreted as follows:

$$\sum_{i=1}^{n} \mathrm{LE}_i \quad \begin{cases} >0 \Rightarrow G(\cdot) & \text{is} & \text{expansionary} \\ =0 \Rightarrow G(\cdot) & \text{is} & \text{conservative} \\ <0 \Rightarrow G(\cdot) & \text{is} & \text{dissipative} \end{cases}$$

Chaotic attractors are presented here as conservative systems that include stretching, shrinking, and folding. Therefore, the Lyapunov spectrum would include at least one positive element but the sum of all elements would be zero.

3.6 Estimating The Largest Lyapunov Exponent

Earlier in this chapter it was shown that the series of m-histories

$$X_t = F(X_{t-1}), F: R^m \to R^m \qquad (1)$$

is congruent to the true series

$$Y_t = G(Y_{t-1}), G: R^n \to R^n$$

as long as $m \geq 2n+1$. As m increases between n and $2n+1$ the n largest Lyapunov exponents of F are the same as those of G, while the remaining m-n exponents diverge to $-\infty$ [Gencay and Dechert (1992)]. Since n is unknown we cannot test whether the true attractor is dissipative or explosive but we can reveal the sign and magnitude of the largest Lyapunov exponent, which is a test for chaos.

Equation (1) may be written more generally as

$$\begin{pmatrix} x_t \\ x_{t-L} \\ \vdots \\ x_{t-mL+L} \end{pmatrix} = \begin{pmatrix} f(x_{t-L}, \cdots, x_{t-mL}) \\ x_{t-L} \\ \vdots \\ x_{t-mL+L} \end{pmatrix} + \begin{pmatrix} e_t \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

which reduces to

$$x_t = f(x_{t-L}, \cdots, x_{t-mL}) + e_t$$

where m is the length of the embedding, L is the number of lags between observations and $\{e_t\}$ is a sequence of zero mean, unknown constant variance, independent random shocks.

The estimate $\hat{f}$ is derived by fitting the equation

$$\hat{f}(X_t, \theta) = \varphi + \sum_{j=1}^{k} \beta_j \psi(\omega_j + \gamma_j{}^{\mathrm{T}} X_t)$$

by nonlinear least squares [Nychka et al (1992), Gencay and Dechert (1992)]

$$\mathrm{Z}(\theta) = \sum_{t=mL+1}^{N} \left(x_t - \hat{f}(X_{t-1}, \theta) \right)^2,$$

where $\theta = (\varphi, \beta, \omega, \gamma)$ is the parameter vector.

$\hat{f}$ consists of k activation units

$$\psi(u) = \frac{\exp(-u)}{1+\exp(-u)}$$

each with its respective unit weight $\beta_j \in \boldsymbol{R}$, and input weights $\omega_j \in \boldsymbol{R}^k$ and

$$\gamma_j = \left(\gamma_{1j}, \gamma_{2j}, \cdots, \gamma_{mj}\right)^{\mathrm{T}}.$$

Once $\hat{f}$ is found we can build $\hat{F}$ and the largest Lyapunov exponent becomes

$$\hat{\mathrm{LE}}_1 = \frac{1}{T} \sum_{t=0}^{T-1} \ln \left| v^{\mathrm{T}} \hat{\Lambda}_t v \right|$$

where v is a fixed m-by-1 vector of norm 1 [$v = (1,0,\cdots,0)^{\mathrm{T}}$], and $\hat{\Lambda} \in \boldsymbol{R}^m$.

3.7 Conclusion

We now have the tools with which we can test for chaos. The BDS test, which is an application of the correlation integral, is used as a test of whiteness and the dominant Lyapunov exponent is used as a test of chaos. When analyzing the underlying system we have to show both existence and stability, where stability refers to the convergence or divergence of nearby trajectories. Here, the BDS test is used to show existence and the value of the dominant Lyapunov exponent is used to show stability.

The next issue of concern is data quality. In order to be able to explain a data set using chaotic dynamics, we require at least that the data be chaotic looking.

Figure 3.1 Distribution of 2000 Random Numbers

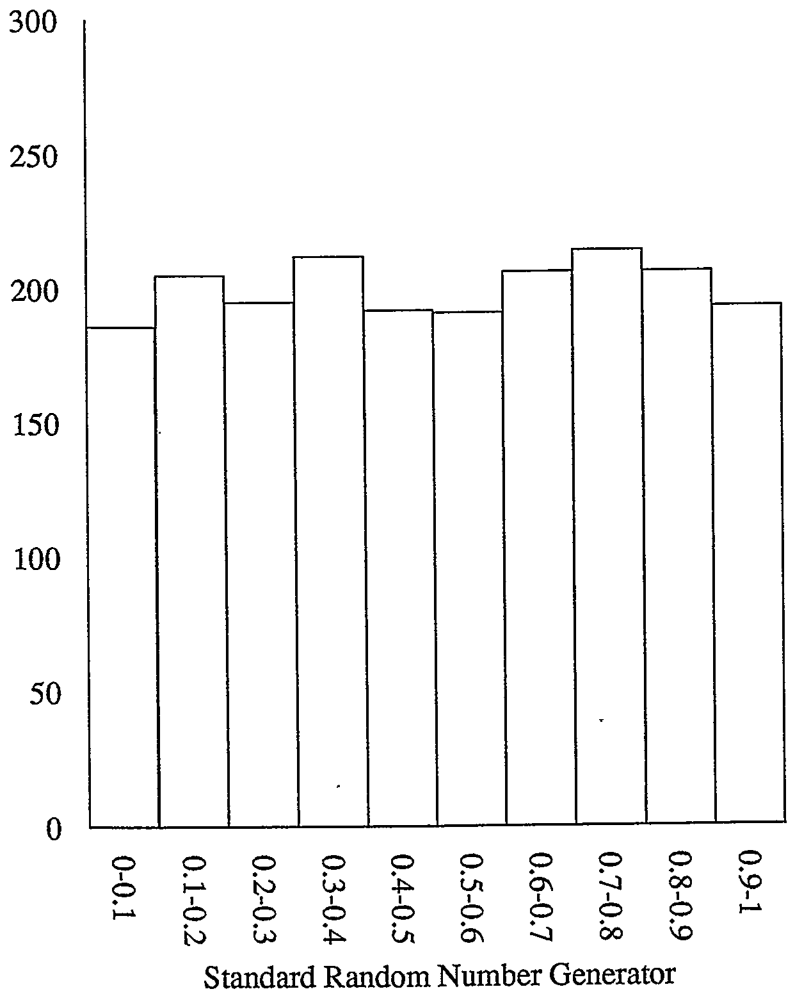


Figure 3.2 Distribution of 2000 Chaotic Iterations

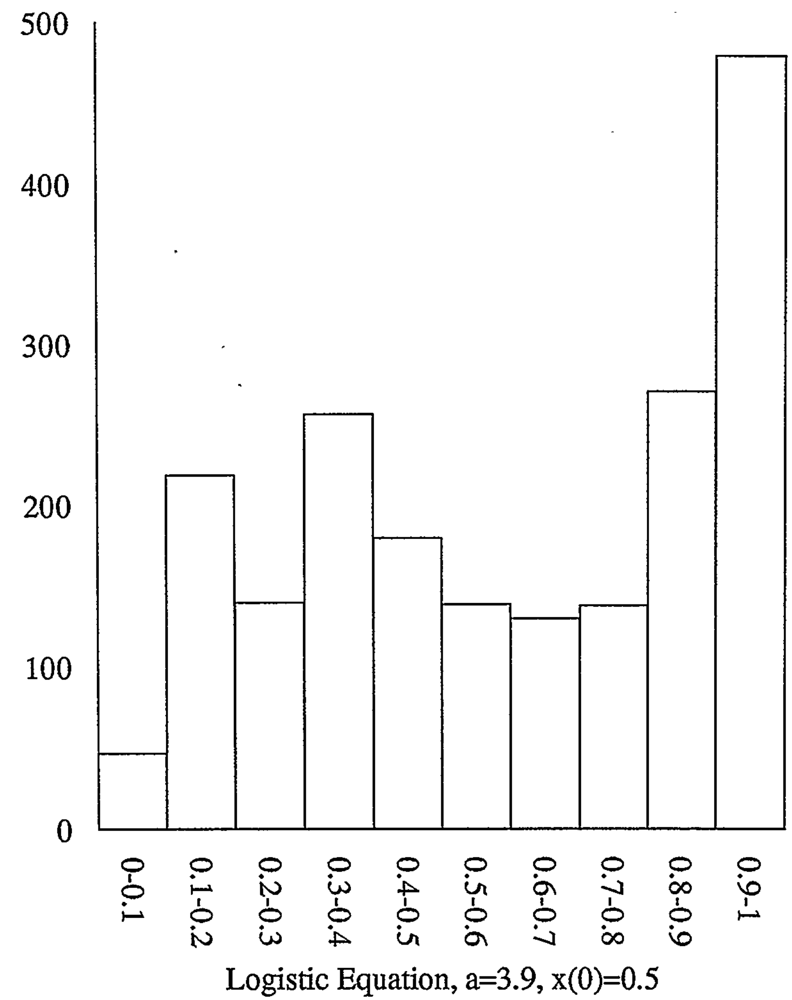


Figure 3.3 2000 Random Numbers Embedded in 2-dimensional Phase Space

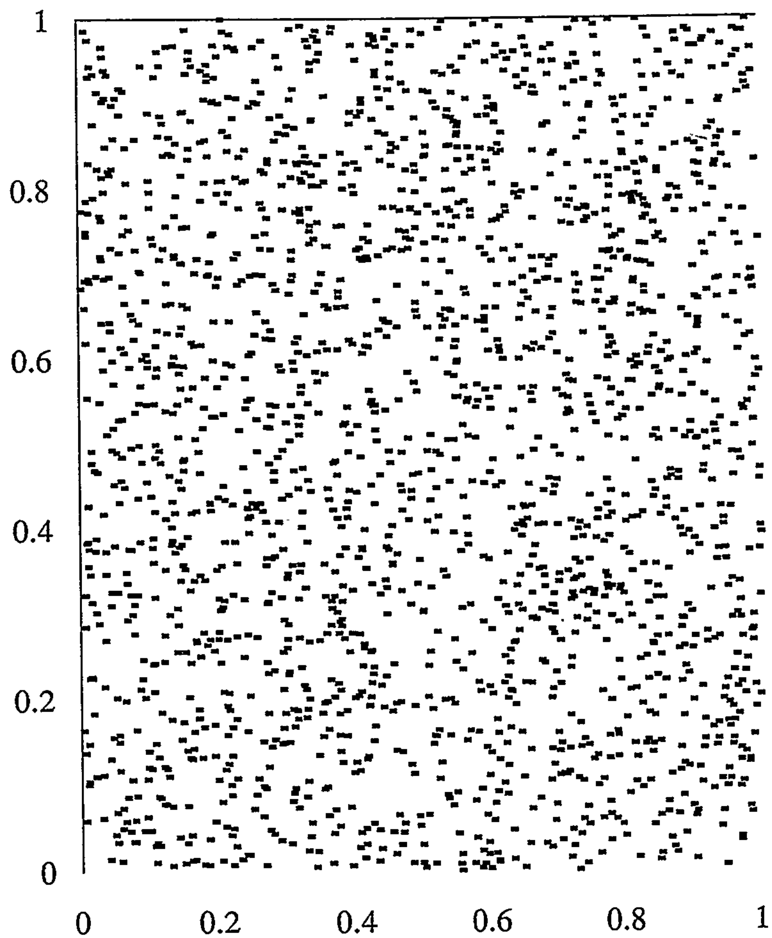
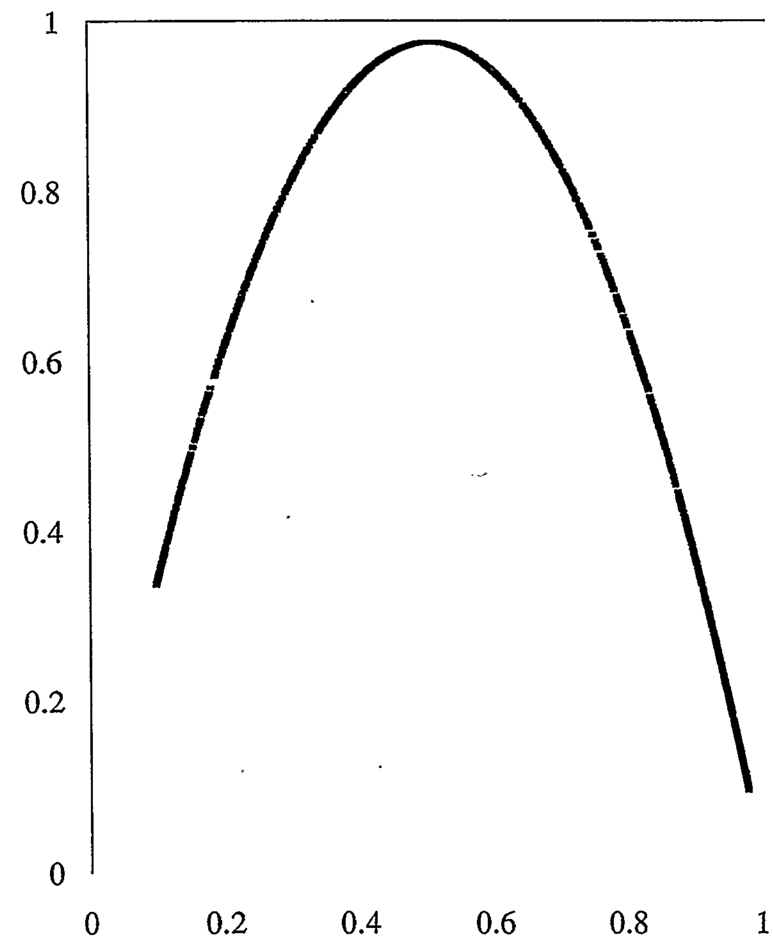


Figure 3.4 2000 Chaotic Iterations Embedded in 2-dimensional Phase Space



Chapter 4

DATA

4.1 Introduction

When testing for nonlinearities, and in particular for chaotic nonlinearities, data quality and quantity is very important. Tests developed in the physical sciences require large amounts of noiseless data. Much of the power of these tests is lost when analyzing finite and noisy economic time series. The problem with noise is that it clouds the underlying attractor, making it difficult to detect, and it imposes large measurement errors making forecasting impossible. If enough noise is present the underlying attractor might be completely dispersed and undetectable. Economic time series are aggregated over markets and over market participants. Both of these operations introduce noise into the data. There is little we can do about aggregation over participants but for aggregation over markets we can minimize noise by considering only individual markets. However, each individual asset is a compliment to some goods and a substitute for others, so is therefore inseparable from other assets.

Data frequency is also a concern. Financial data is available in tick by tick intervals. However, this frequency is too short since the series becomes dependent upon micromarket structures such as the sequential executions of limit orders as market prices pass through the respective limit prices. Frequencies must be decreased to average out these artificial dependencies. Since data sets have to be as large as possible, when the data is extended further back in time, other influences become stronger such as nonstationarity or conditional heteroscedasticity.

Section 4.2 describes the data chosen for this study and section 4.3 gives some summary statistics about each series.

4.2 Data and Sources

Relative to other economic time series, financial data has greater abundance and quality. For this study testing was conducted using weekly observations on spot-month futures prices of thirteen various commodities and currencies. They include the Australian dollar, the British pound, the Canadian dollar, crude oil, copper, the Deutschemark, gold, heating oil, unleaded gas, the Japanese yen, platinum, the Swiss franc, and silver. All the currencies are traded on the International Monetary Market at Chicago, copper, gold, and silver are traded on the New York Commodities Exchange, and crude oil, heating oil, platinum, and unleaded gas are traded on the New York Mercantile Exchange. Each series was collected by Tick Data Incorporated.

The sample period and number of observations for each series is shown in Table 4.1. They range from January 13/1987, to June 2/1993, [330 observations] in the case of the Australian dollar from July 2/1971, to June 2/1993, [1140 observations] in the case of silver. A graphical representation of each series is provided in figures 4.1 to 4.13.

4.3 Summary Statistics

Tests of normality in distribution of the data include mean, standard deviation, skewness, and excess kurtosis. The results of each are outlined in the last four columns of Table 4.1. Measures of mean and standard deviation are derived from logged differences of the raw data. A distribution's form is given by its skewness and excess kurtosis. *Skewness* refers to the proportion of observations which fall on either side of the mean. If the value of skewness is negative (positive) then more observations fall on the lower (higher) side of the mean. When the distribution is higher, narrower, with fatter tails than the normal distribution it is known to have *excess kurtosis*. A measure of zero means that the curve is normal. Notice that all the series tested have excess kurtosis and all are skewed to one side or the other except perhaps unleaded gas.

4.4 Conclusion

Quality data is crucial when testing for nonlinearities and chaos. Test results do not depend on whether chaotic output appears random but whether random looking data appears chaotic, upon close inspection. If a time series is found to be chaotic then it must also be consistent with chaotic structure. For this to happen we need data as noise free and as chaotic looking as possible. To reduce noise in time series we can only choose data as disaggregated as possible with an appropriate frequency such that the chaotic signal is detectable.

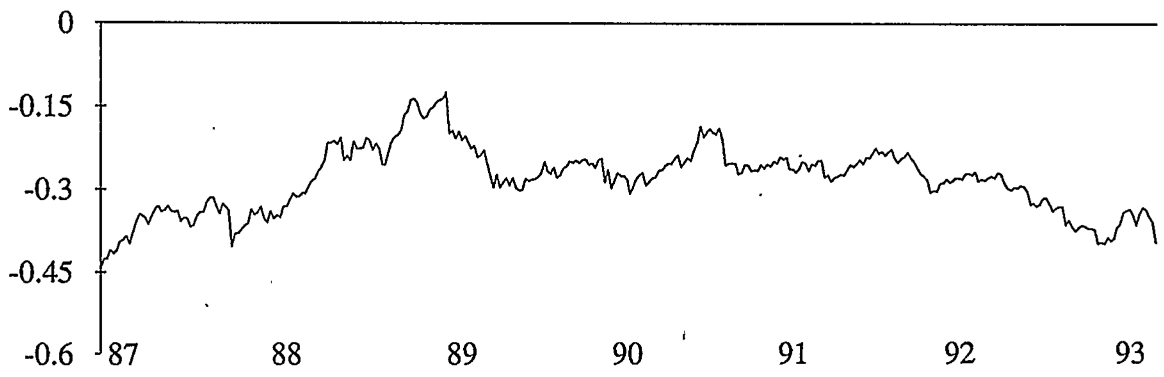
Known dependencies, such as nonstationarity or conditional heteroscedasticity, that interfere with the chaotic signal must be filtered out. The next chapter shows various prefiltering techniques so that the data can be tested for nonlinearity and chaos.

Table 4.1 Summary Statistics For Weekly Futures Prices (Logged Differences)

Series	# Obs.	Sample Period	Mean	S.Dev.	Excess Kurtosis	Skewness
Australian Dollar	330	01/13/87 - 06/02/93	.014 (.073)	1.329	4.625 (.268)	-1.307 (.134)
British Pound	955	02/13/75 - 06/02/93	-.043 (.054)	1.660	3.558 (.158)	-.364 (.079)
Canadian Dollar	854	01/17/77 - 06/02/93	-.026 (.021)	.628	3.424 (.167)	-.438 (.084)
Crude Oil	530	03/30/83 - 06/02/93	-.072 (.207)	4.767	8.834 (.212)	-.806 (.106)
Copper	905	08/22/72 - 12/27/89	.079 (.136)	4.078	2.640 (.162)	-.107 (.081)
Deutschemark	955	02/13/75 - 06/02/93	.038 (.053)	1.633	1.713 (.158)	.092 (.079)
Gold	961	01/02/75 - 06/02/93	.079 (.098)	3.046	13.855 (.158)	.221 (.079)
Heating Oil	734	03/06/79 - 06/02/93	-.009 (.170)	4.616	3.726 (.180)	-.469 (.090)
Unleaded Gas	439	12/03/84 - 06/02/93	-.045 (.248)	5.199	3.076 (.233)	-.070 (.117)
Japanese Yen	865	11/03/76 - 06/02/93	.117 (.053)	1.573	2.362 (.166)	.453 (.083)
Platinum	1072	08/22/72 - 06/02/93	.085 (.133)	4.345	4.409 (.149)	-.237 (.075)
Swiss Franc	955	02/13/75 - 06/02/93	.056 (.061)	1.896	1.234 (.158)	.322 (.079)
Silver	1140	07/29/71 - 06/02/93	.079 (.145)	4.894	10.457 (.145)	-.659 (.072)

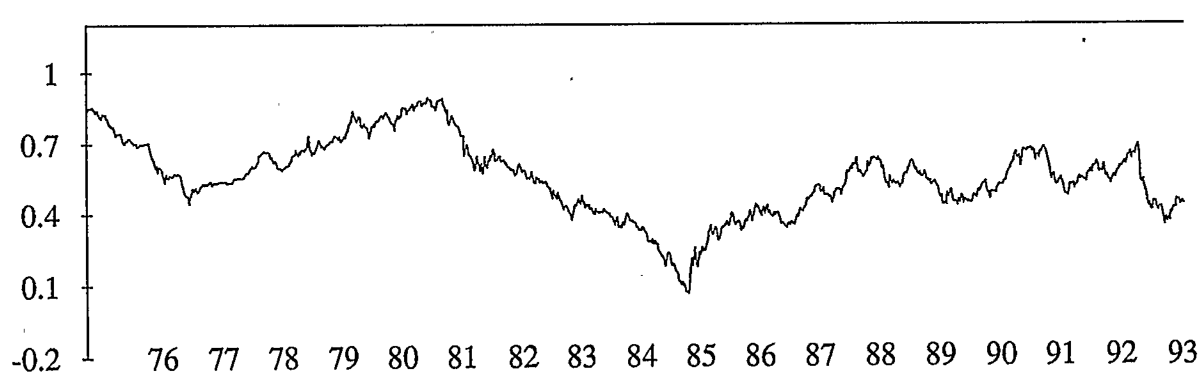
Note: Numbers in parentheses are standard errors.

Figure 4.1 Australian Dollar



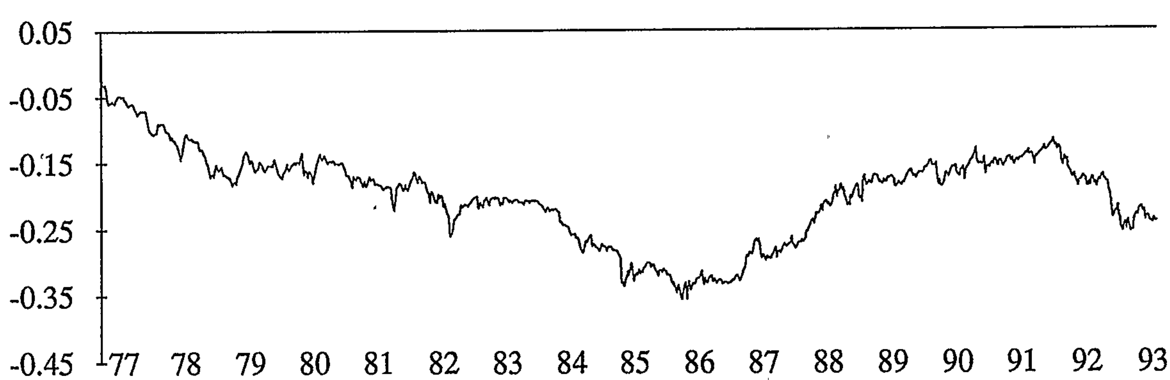
Note: Logged nearby futures prices, weekly data from 1/13/87 to 6/2/93, 330 observations.

Figure 4.2 British Pound



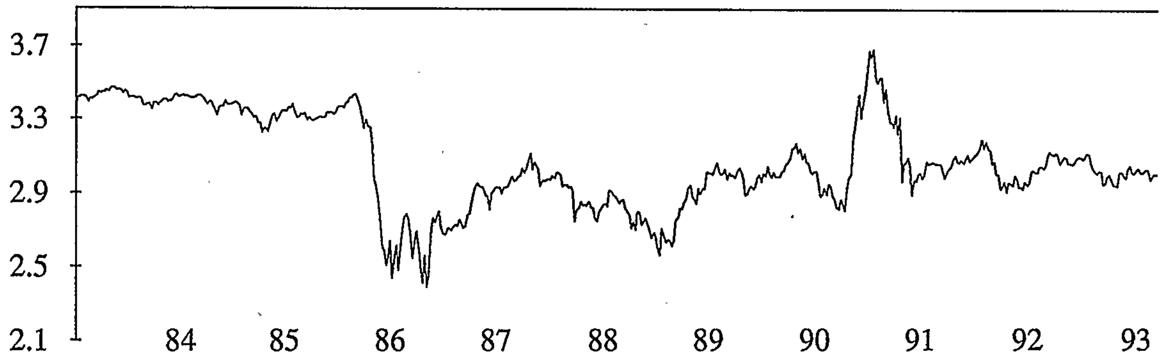
Note: Logged nearby futures prices, weekly data from 2/13/75 to 6/2/93, 955 observations.

Figure 4.3 Canadian Dollar



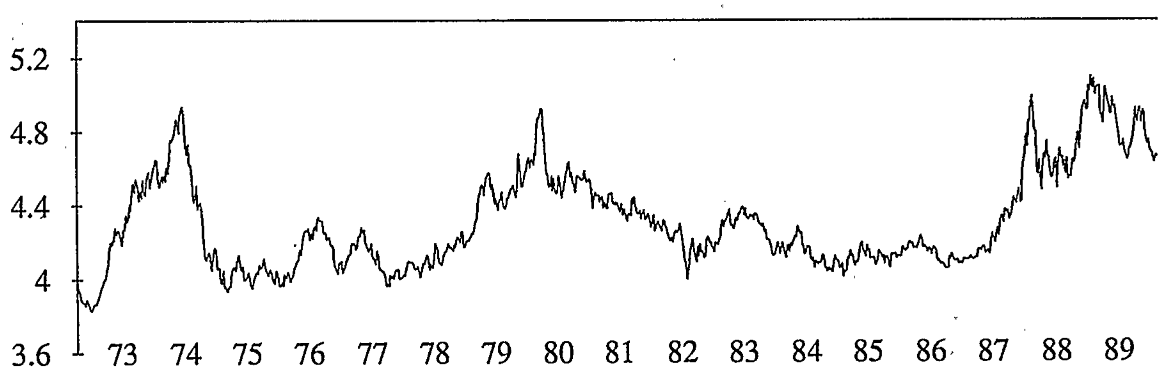
Note: Logged nearby futures prices, weekly data from 1/17/77 to 6/2/93, 854 observations.

Figure 4.4 Crude Oil



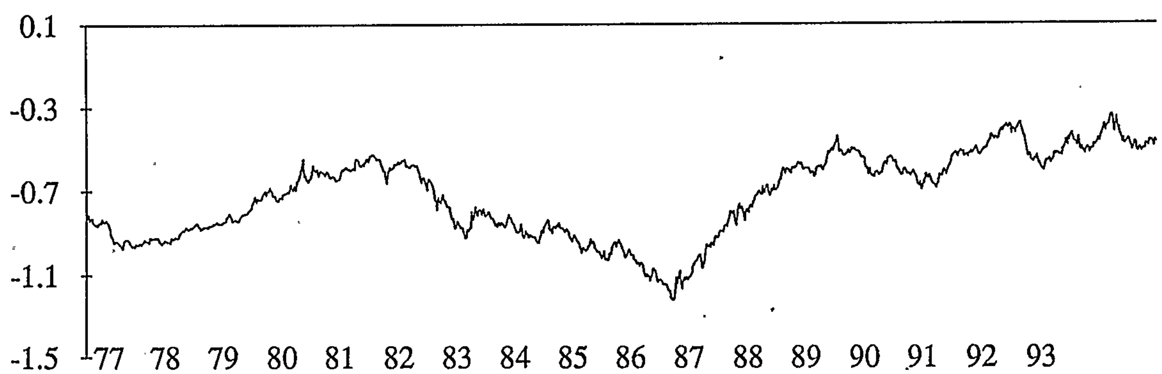
Note: Logged nearby futures prices, weekly data from 3/30/83 to 6/2/93, 530 observations.

Figure 4.5 Copper



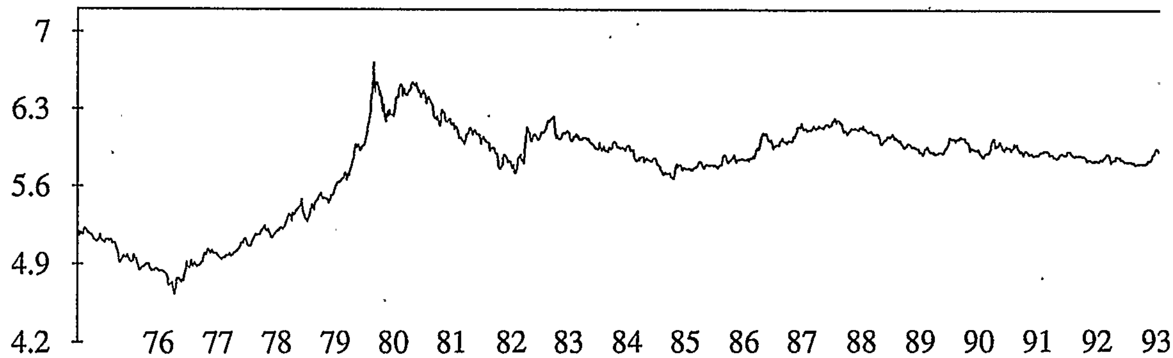
Note: Logged nearby futures prices, weekly data from 8/22/72 to 12/27/89, 905 observations.

Figure 4.6 Deutschemark



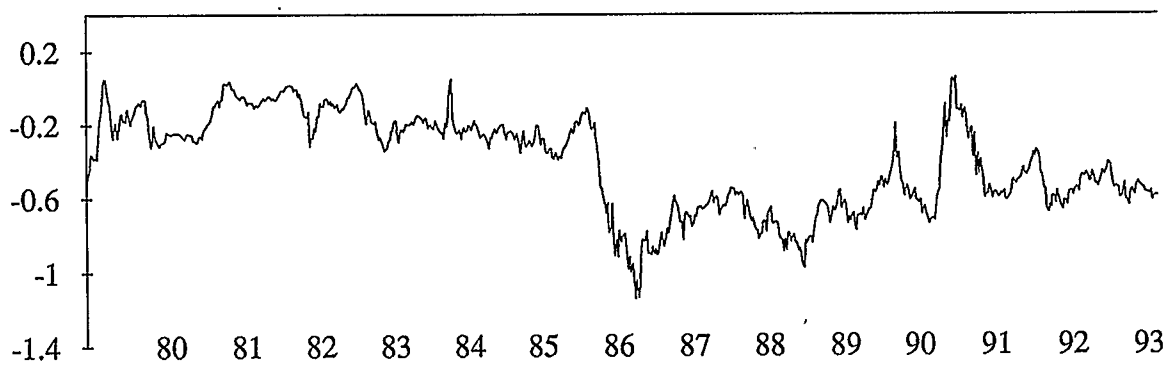
Note: Logged nearby futures prices, weekly data from 2/13/75 to 6/2/93, 955 observations.

Figure 4.7 Gold



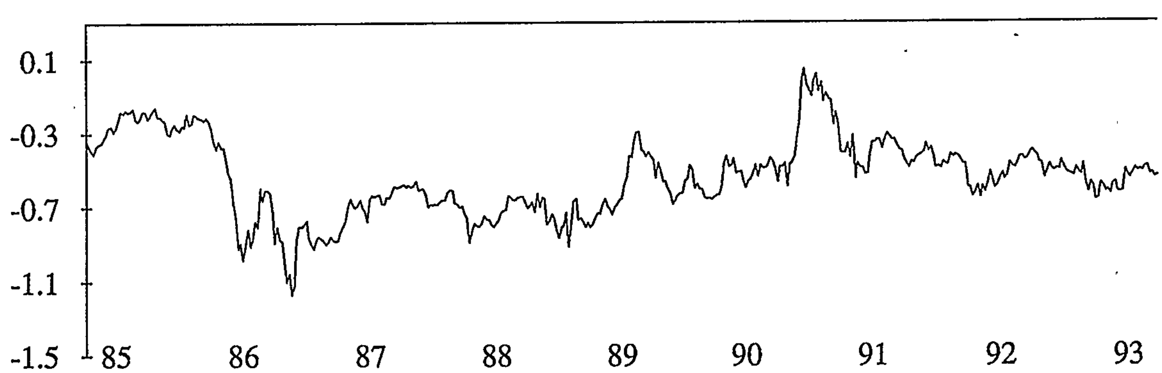
Note: Logged nearby futures prices, weekly data from 1/2/75 to 6/2/93, 961 observations.

Figure 4.8 Heating Oil



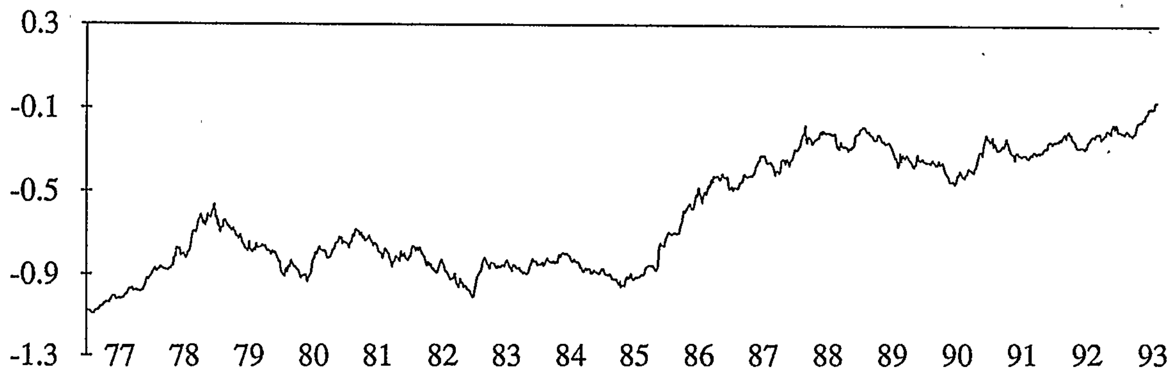
Note: Logged nearby futures prices, weekly data from 3/6/79 to 6/2/93, 734 observations.

Figure 4.9 Unleaded Gasoline



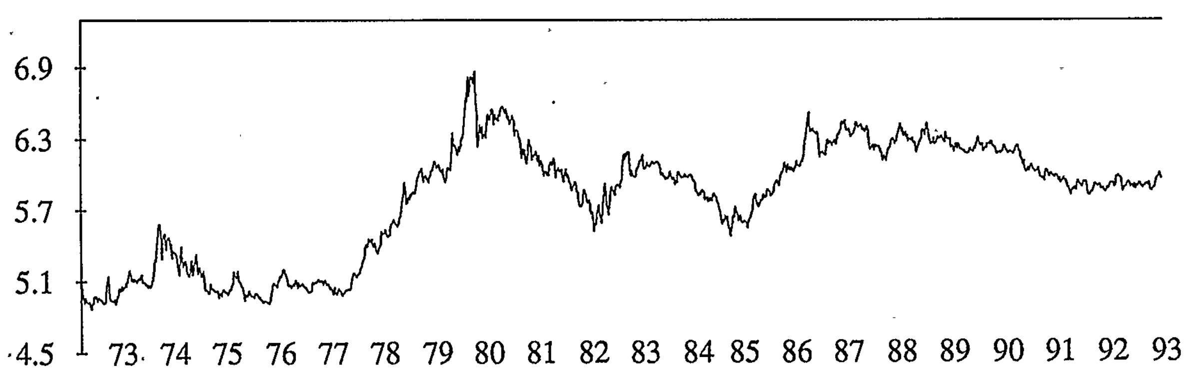
Note: Logged nearby futures prices, weekly data from 12/3/84 to 6/2/93, 439 observations.

Figure 4.10 Japanese Yen



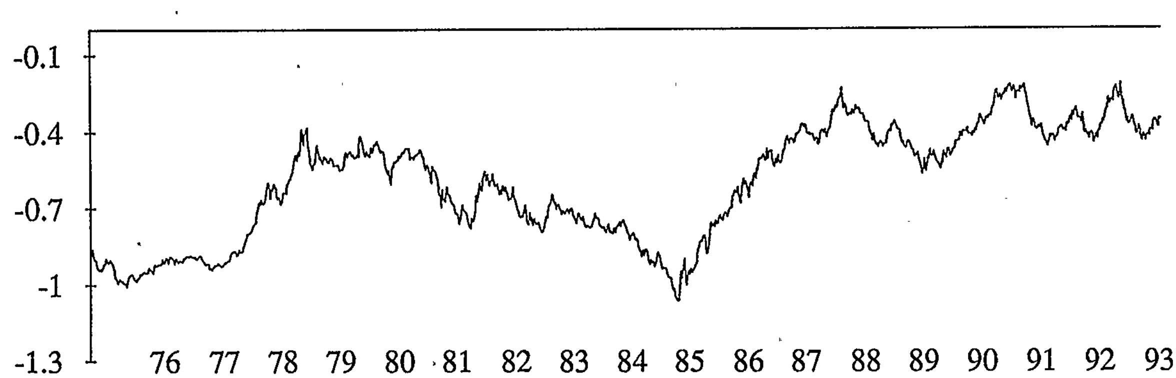
Note: Logged nearby futures prices, weekly data from 11/3/76 to 6/2/93, 865 observations.

Figure 4.11 Platinum



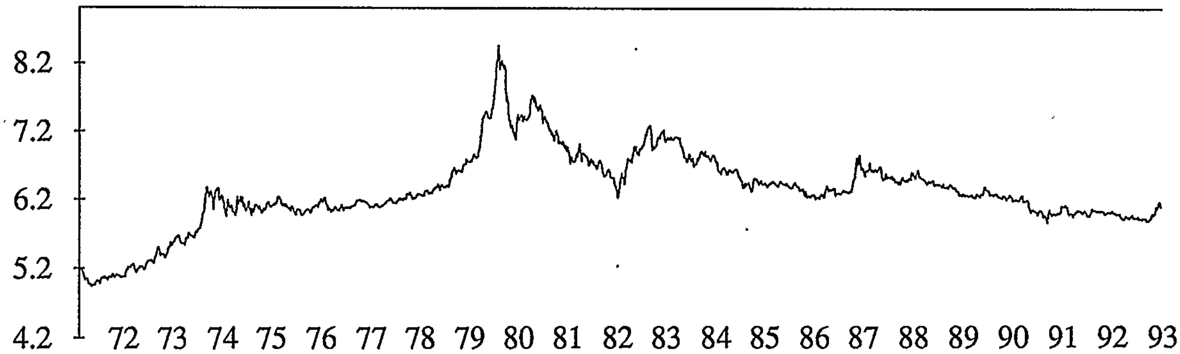
Note: Logged nearby futures prices, weekly data from 8/22/72 to 6/2/93, 1072 observations.

Figure 4.12 Swiss Franc



Note: Logged nearby futures prices, weekly data from 2/13/75 to 6/2/93, 955 observations.

Figure 4.13 Silver



Note: Logged nearby futures prices, weekly data from 7/29/71 to 6/2/93, 1140 observations.

Chapter 5

EMPIRICAL TESTING

5.1 Introduction

This chapter confronts theoretical chaos with the facts. If the underlying system of a series is linear then the residuals from the best fit linear model are white noise. Chaotic nonlinearities cannot be detected by linear methods so if chaos is present then its effects will be hidden in the residuals of the best fit linear model. Economists are now armed with two highly reputable tests of nonlinearity and chaos: the BDS test and the Lyapunov exponent estimator of Nychka et al. (1992). For each series the best fit linear model is formed then the residuals are tested for the existence of nonlinearities and for chaos.

Before conducting nonlinear dynamical analysis the data must be rendered stationary, linearized, and purged of conditional heteroscedasticity. In section 5.1 augmented Dickey-Fuller unit root tests are used to check stationarity. The data is then linearized by lagging the stationary series enough times to remove autocorrelation. This procedure is outlined in section 5.3. Conditional heteroscedasticity refers to variances that change over time. Section 5.4 describes a test for conditional heteroscedasticity and a method of removing it from the data. Once each series is linearly filtered we can conduct tests of nonlinearity and chaos. Results from these tests are given in sections 5.5 and 5.6 respectively.

5.2 Preliminary Analysis of the Data

Before we test for chaotic nonlinearities, we must first remove all linear dependencies and render each series stationary. A series is stationary if its mean and variance are constant but finite through time and the covariance between any two values depends only on the distance between them, not on time itself. In other words all observations are independently and identically distributed. More formally, a series $\{x_t\}$ is stationary if for all t

$$E(x_t) = \bar{x}$$

$$\mathrm{cov}(x_t, x_{t-s}) = E[(x_t - \bar{x})(x_{t-s} - \bar{x})] = \begin{cases} \sigma^2 < \infty & \text{for} \quad s = 0 \\ \gamma_s \quad \text{for} & s \neq 0 \end{cases}$$

Many economic time series are not stationary but can be easily rendered so.

There are two types of underlying trends and a different procedure is used to detrend each.

The first type

$$x_{t+1} = a + x_t + e_t$$
$$E(x_{t+1}) = a + x_t$$

is known as a *difference stationary process*. Shocks to this type of system have a permanent affect on the time series. Notice that the expectation of x_{t+1} is a function of x_t. In this case $\{x_t\}$ is detrended simply by taking first differences, $\Delta x_t = a + e_t$.

The second form

$$x_t = a + bt + e_t$$
$$E(x_t) = a + bt$$

is known as a *trend stationary process*. Shocks have a temporary effect since expected values are not functions of their own past values. The series $\{x_t\}$ is detrended by regressing it on both a constant and a time trend, then obtaining the residuals $\{e_t\}$.

One test used to determine which method should be used to render the series stationary, which is the one used in this study, is the augmented Dickey-Fuller test (ADF) -- Dickey and Fuller (1981)

$$\Delta \ln x_t = a_0 + a_1 t + a_2 \ln x_{t-1} + \sum_{j=1}^{l} c_j \Delta \ln x_{t-j} + e_t$$

where $\sum_{j=1}^{l} c_j \Delta \ln x_{t-j}$ corrects for autocorrelation in the series $\{\Delta \ln x_t\}$ and $a_1 t$ removes trend stationary processes. Natural logs are used so the series $\{\Delta \ln x_t\}$ is represented in terms of growth rates rather than differences. If $a_2 = 0$ then $\{x_t\}$ follows a difference stationary process and has to be differenced to become stationary. We test the null that $a_2 = 0$ [difference stationary process] against the alternative that a_2 is negative [trend stationary process]. If the null cannot be rejected the series is differenced and tested again. This process continues until the null cannot be rejected.

A series is *integrated of order d* -- I(*d*) -- if it has to be differenced *d* times to become stationary, it is then known to have *d unit roots*. Most economic and financial time series have only a single unit root -- see Nelson and Plosser (1982).

Results of the Augmented Dickey-Fuller tests on the futures data is presented in Table 5.1. In all cases we cannot reject the null of a difference stationary process in levels but reject it in first differences indicating a single unit root. Hence, we first difference the logarithms to render the series stationary, this is consistent with the evidence reported in Nelson and Plosser (1982).

5.3 Linearizing the Data

Now that the trend has been detected we start by describing the data in a linear framework. In particular we use the following autoregression (AR) model.

$$\Delta \ln x_t = b_0 + \sum_{j=1}^{q} b_j \Delta \ln x_{t-j} + e_t$$

The residuals, $\{e_t\}$, from a correctly specified AR model have zero correlation with each other. Otherwise they follow a pattern indicating some left out influence. The *autocorrelation coefficient* between e_t and e_{t-s} is

$$\rho_s = \frac{\operatorname{cov}(e_t, e_{t-s})}{\sigma_{e_t} \sigma_{e_{t-s}}}$$

Since $\rho_s = 0$ if and only if $\operatorname{cov}(e_t, e_{t-s}) = 0$, we can use ρ_s to detect autocorrelation.

As s varies, values of ρ_s also vary, giving the sequence $\{\rho_s\}_{s=0}^{k}$ which is known as the *autocorrelation function.* This provides a measure of how much correlation exists for a group of residuals of size k. For a white noise process ρ_s is approximately zero for all $s \neq 0$. Box and Pierce (1970) showed that and for sufficiently high k the Q-*statistic*

$$Q = N\sum_{s=1}^{k} \hat{\rho}_s^{\,2}$$

follows a chi-square distribution with k degrees of freedom -- $\chi^2(k)$ -- and can be used to test the joint null hypothesis that all k autocorrelation coefficients are zero. Recall that N is the length of the series $\{e_t\}$. This test uses the null of $Q = 0$ where the residuals are uncorrelated against the alternative $Q > 0$ where patterns exist between residuals. In order to consider the series free of autocorrelation $\Delta \ln x_t$ has to be regressed on enough lags to yield the Q-statistic insignificant.

The first two columns of Table 5.2 show the optimal number of AR lags based on the Q-statistic and their respective values. All series except for crude oil, gold, the Japanese yen, and silver require only one lag to remove autocorrelation. For the remaining series, crude oil requires 4 lags, gold requires 10 lags, the Japanese yen requires 2 lags, and silver requires 8 lags. In each case we cannot reject the null of no autocorrelation given a 5% critical value of 35.173. Tests of excess kurtosis and skewness were applied to the residuals of the optimal AR model. Notice that all series still have excess kurtosis and all are skewed except for unleaded gas.

5.4 <u>Heteroscedasticity</u>

Financial time series are usually studied as homoscedastic processes. Engle (1982), however, suggested the autoregressive conditional heteroscedasticity [ARCH] model as an alternative to the usual time series process -- this is consistent with Mandelbrot's (1963) observation that asset markets are characterized by time varying volatility. Before using ARCH type models, one can test for ARCH effects by regressing $\hat{e}_t^2$ on $\hat{e}_{t-1}^2$ as in

$$\hat{e}_t^2 = b_0 + b_1\hat{e}_{t-1}^2 + \zeta_t$$

The statistic $N\mathbf{R}^2$, where N is the number of observations and $\mathbf{R}^2$ is the coefficient of determination from the above regression, is distributed as a chi square with one degree of freedom, $\chi^2(1)$. If the null of $b_1 = 0$ is rejected then the series $\{\hat{e}_t\}$ increases (decreases) over time causing the variance of the original series, $\{x_t\}$, to increase (decrease) over time.

The third column of table 5.2 presents results for ARCH effects in the optimal AR model. Clearly, all series except for the Australian dollar, the Canadian dollar, and the Japanese yen have significant ARCH effects.

Given the evidence of significant ARCH effects the model used to correct for time varying variances is Bollerslev's (1986) generalized autoregressive conditional heteroscedasticity [GARCH] model. This is a generalization of Engel's (1982) ARCH model and has the variance as a function of past squared residuals as well as of past values of itself. The GARCH(l,k) model is

$$\Delta \ln x_t = b_0 + \sum_{j=1}^{q} b_j \Delta \ln x_{t-j} + e_t, \quad e_t | \mathrm{I}_{t-1} \sim \mathrm{N}(0, h_t)$$

$$h_t = w_0 + \sum_{j=1}^{k} \alpha_j e_{t-j}^2 + \sum_{j=1}^{l} \beta_j h_{t-j}$$

where the last equation is known as the *variance equation*. GARCH models are useful in correcting for conditional heteroscedasticity while still preserving any nonlinear deterministic structure [Lamareux and Lastrapes (1990)]. The most common specification used to capture the time varying variances is the GARCH(1,1) process [see Engle and Bollerslev (1986) or Hseih (1985)] When the only dependence is linear, the standardized residuals from a properly specified GARCH model are white noise and follow a standard normal distribution with mean 0 and variance σ^2. However, if other nonlinear dependencies exist, they may be hidden in the GARCH residuals.

In the first three columns of table 5.3 are the estimated values of the parameters, w_0, α_1, and β_1, and their respective t-ratios. In each case the t-ratio for the parameter β_1 is significant and in each case except for the Australian dollar, the t-ratio for α_1 is significant.

Column 4 and 6 show both the Q-statistic and the ARCH test results for the standardized residuals of the GARCH model. Notice that in each case we cannot reject the nulls of no autocorrelation and no ARCH effects. The lag structure of the autocorrelation was the same as in the previous AR model. Tests of normality on the standardized residuals show excess kurtosis and skewness in all cases.

5.5 BDS Results

The BDS statistic was calculated using the BDS program provided by W. Davis Dechert. Tests are the same as outlined in the previous chapter for both the AR and GARCH residuals and for embedding dimensions 2 to 5 and epsilon of 0.5, 1, 1.5, and 2 standard deviations of the data. Brock (1986) showed that the asymptotic distribution of the BDS test statistic is unaltered by using residuals rather than raw data in linear models. It is to be noted that when testing finite data sets the power of the BDS test weakens if epsilon is too small or too large. If epsilon is too small then there are not enough data points in $C(m,\varepsilon,T)$ to accurately compare with $C(1,\varepsilon,T)^m$ and if epsilon is too large then almost all points are in $C(m,\varepsilon,T)$ making it approximately equal to $C(1,\varepsilon,T)^m$.

Results from the BDS tests are shown in Table 5.4. Hsieh (1991) found, through Monte Carlo testing, that the BDS critical values for GARCH standardized residuals are biased upwards so his critical values were used for these residuals in this study.

In all cases except for the Australian dollar the null of independent and identically distributed residuals is rejected for the AR model. In the case of the Australian dollar we also reject the null for the GARCH standardized residuals. For gold, heating oil, platinum, the Swiss franc, and silver, we reject the null for all standardized residuals from the

GARCH model indicating conditional heteroscedasticity as the only explanation for non-*iid* residuals from the AR model. Further results refer only to the GARCH standardized residuals. The null is rejected at epsilon equal to 0.5 for the British pound, Canadian dollar, crude oil, Deutschemark, and the Japanese yen indicating nonlinear influences. In the case of the Japanese yen the null is also rejected for epsilon equaling 1 and in the case of crude oil, and unleaded gas the null is rejected for epsilon equaling 2. For copper the null is rejected for all values of epsilon.

Results from the BDS tests applied to the AR residuals indicate strong nonlinear dependence. However, when applied to the GARCH(1,1) standardized residuals, the results show that much of the nonlinear dependence is explained by conditional heteroscedasticity. Unknown nonlinear dependence remains for the British pound, the Canadian dollar, crude oil, copper, and the Japanese yen.

Now that nonlinearities have been found in the data we may go to the next step and test for chaotic nonlinearities. If positivity of the largest Lyapunov exponent for some embedding dimension of the GARCH standardized residuals is found then we can say that the nonlinearity found is chaotic.

5.6 Dominant Lyapunov Exponent Results

The dominant Lyapunov exponent was estimated using LENNS, a program written by Nychka et al. (1992) and provided by A. Ronald Gallant. In particular, LENNS estimates the parameters of $\hat{f}$ from Chapter 3 by nonlinear least squares in a neural net framework. For each triple, (L,m,k), the program fits the model with 200 different initial conditions, then polishes the best 20 with a more stringent convergence criterion. Recall that L is the time delay, m is the dimension of the embedding, and k is the number of parameter units in $\hat{f}$.

Presently there are two methods of Lyapunov exponent estimation. The direct method [e.g. Wolf et al. (1985)] is sensitive to the degree of noise in the data [see Brock and Sayers (1988)] while the regression method accommodates noise. Results from the regression method refer to the noisy series rather than the hypothetical underlying system which the direct method tries to estimate. According to Barnett et al. (1993) the Nychka et al. (1992) approach, which is a version of the direct method, is the only credible candidate for testing chaos. The various forms of $\hat{f}$ tested here include embedding dimensions from 1 to 10, lags from 1 to 3, and parameter units from 1 to 3. The best fits are chosen by minimizing the Bayesian Information Criterion [BIC] -- see Schwartz (1978) -- on the residuals of $\hat{f}$. Due to the large amount of calculations required, the LENNS program is limited to 500 or so data points. For series longer than 500 observations the test is applied to both the first 500 observations and the last 500 observations.

Results from the LENNS test are displayed in Table 5.5. The estimated point values of the best fit dominant Lyapunov exponent and the respective triple, (L,m,k), are given in columns 2, 3, and 4. BIC values [minimized over all triples] are displayed in columns 5, 6, and 7. Notice that sensitive dependence on initial conditions, indicated by positivity of the dominant Lyapunov exponent, is found for the Australian dollar, copper, and the Japanese yen.

5.7 Conclusion

This chapter began with 13 financial time series of unknown processes and ended in chaos. In sections 5.2, 5.3, and 5.4 the data was made stationary, linearized, and was properly adjusted for time varying variances. Each series was transformed such that they were chaotic looking. They were bounded from above, below, and fluctuated in a random looking manner between these two extremes. After nonlinear analysis we find the BDS test rejects the null of independent and identical distribution for all the AR residuals except for the Australian dollar. In the standardized residuals of the GARCH(1,1) process, nonlinear dependence was found for only the British pound, the Canadian dollar, crude oil, copper, and the Japanese yen. When the Nychka et al. (1992) Lyapunov exponent estimator was applied to these residuals, evidence of sensitive dependence on initial conditions was found in the Australian dollar, the Japanese yen, and copper. Conflicting results in the case of the Australian dollar indicates limited robustness across inference methods -- also found in Barnett et al. (1993).

These findings have important consequences for financial economics. They offer an alternative explanation for the behavior of futures prices and they offer potential for short run trading rules.

Table 5.1 Augmented Dickey-Fuller Unit Root Tests on Logged Weekly Futures Prices

$$\Delta \ln x_t = a_0 + a_1 t + a_2 \ln x_{t-1} + \sum_{j=1}^{l} c_j \Delta \ln x_{t-j} + e_t$$

Series	Levels	First Diff.	Crit. (10%)
Australian Dollar	-2.42	*-6.77	-3.13
British Pound	-1.92	*-5.96	-3.12
Canadian Dollar	-1.87	*-6.92	-3.12
Crude Oil	-2.54	*-5.46	-3.12
Copper	-2.50	*-5.24	-3.12
Deutschemark	-1.65	*-5.03	-3.12
Gold	-1.71	*-4.74	-3.12
Heating Oil	-2.85	*-6.53	-3.12
Unleaded Gas	-2.73	*-5.13	-3.13
Japanese Yen	-2.15	*-4.79	-3.12
Platinum	-1.79	*-5.31	-3.12
Swiss Franc	-2.20	*-5.25	-3.12
Silver	-2.15	*-5.89	-3.12

Note: An asterisk indicates significance at the 10% level.

Table 5.2 Summary Statistics for the Optimal Autoregressive Model Residuals Under the Q(23) Test Statistic

$$\Delta \ln x_t = b_0 + \sum_{j=1}^{q} b_j \Delta \ln x_{t-j} + e_t, \quad e_t | \mathrm{I}_{t-1} \sim \mathrm{N}(0, \gamma_0)$$

Series	AR Lag	Q-statistic	ARCH test	Excess Kurtosis	Skewness
Australian Dollar	1	14.715	.071	4.729 (.268)	-1.320 (.134)
British Pound	1	22.053	*12.369	3.609 (.158)	-.362 (.079)
Canadian Dollar	1	24.317	3.404	3.447 (.167)	-.447 (.084)
Crude Oil	4	33.517	*12.004	8.551 (.213)	-.749 (.107)
Copper	1	30.343	*8.316	2.628 (.162)	-.107 (.081)
Deutschemark	1	19.403	*31.924	1.685 (.158)	.100 (.079)
Gold	10	31.053	*158.025	12.834 (.158)	.152 (.079)
Heating Oil	1	30.609	*58.959	3.408 (.180)	-.398 (.090)
Unleaded Gas	1	34.084	*13.231	3.115 (.233)	-.061 (.117)
Japanese Yen	2	29.200	3.335	2.517 (.166)	.457 (.083)
Platinum	1	24.256	*56.168	4.370 (.149)	-.241 (.075)
Swiss Frank	1	17.811	*22.210	1.203 (.158)	.331 (.079)
Silver	8	33.963	*34.953	8.352 (.145)	-.594 (.073)

Note: Numbers in Parentheses are standard errors. The Q-statistic is distributed as $\chi^2(23)$ on the null of no autocorrelation and the ARCH test is distributed as $\chi^2(1)$ on the null of a stationary variance. An asterisk next to a test statistic indicates significance at the 5% critical level which is 35.173 in the case of the Q-statistic and 3.842 in the case of the ARCH statistic.

Table 5.3 GARCH(1,1) Parameter Estimates and Residual Diagnostics

$$\Delta \ln x_t = b_0 + \sum_{j=1}^{q} b_j \Delta \ln x_{t-j} + e_t, \quad e_t | \mathrm{I}_{t-1} \sim \mathrm{N}(0, h_t), \quad h_t = w_0 + \alpha_1 e_{t-1}^2 + \beta_1 h_{t-1}$$

Series	w_0	α_1	β_1	Q-statistic	ARCH	Excess Kurtosis	Skewness
Australian Dollar	.061 (0.7)	.011 (0.9)	.956 (17.4)	14.158	.170	4.893 (.268)	-1.355 (.134)
British Pound	.150 (2.9)	.123 (4.6)	.831 (23.7)	18.468	1.246	2.566 (.158)	-.320 (.079)
Canadian Dollar	.093 (3.3)	.161 (3.8)	.615 (6.6)	24.021	.197	3.925 (.167)	-.754 (.084)
Crude Oil	.231 (1.6)	.226 (4.3)	.802 (20.6)	27.489	.269	3.392 (.213)	-.584 (.107)
Copper	.499 (3.2)	.093 (4.3)	.877 (33.7)	20.678	2.734	1.110 (.123)	.107 (.081)
Deutschemark	.164 (3.4)	.153 (5.2)	.793 (21.8)	22.672	.116	.981 (.158)	.167 (.079)
Gold	.238 (3.3)	.198 (6.4)	.790 (28.2)	33.096	.093	1.693 (.158)	-.166 (.079)
Heating Oil	1.189 (4.0)	.312 (6.3)	.666 (16.4)	21.332	.623	2.369 (.181)	-.305 (.091)
Unleaded Gas	2.580 (3.0)	.213 (4.0)	.700 (11.2)	30.283	.164	1.427 (.233)	-.212 (.177)
Japanese Yen	.048 (3.5)	.025 (2.9)	.957 (86.9)	27.474	.310	3.492 (.166)	.631 (.083)
Platinum	.423 (3.0)	.095 (5.4)	.881 (44.6)	25.816	.477	1.456 (.149)	.161 (.045)
Swiss Franc	.025 (1.8)	.070 (5.0)	.927 (67.0)	24.549	.598	1.083 (.158)	.391 (.079)
Silver	.379 (3.4)	.121 (6.2)	.865 (45.5)	27.012	.032	1.618 (.145)	.188 (.023)

Note: Numbers in parentheses next to the GARCH(1,1) parameter estimates are t-ratios and next to excess kurtosis and skewness values are standard errors. The Q-statistic is distributed as $\chi^2(23)$ on the null of no autocorrelation and the ARCH test is distributed as $\chi^2(1)$ on the null of a stationary variance. An asterisk next to a test statistic indicates significance at the 5% critical level which is 35.173 in the case of the Q-statistic and 3.842 in the case of the ARCH statistic.

Table 5.4 BDS Results for AR Residuals and GARCH(1,1) Standardized Residuals for Weekly Futures Prices (Dimensions 2 Through 5 and ε Equaling 0.5, 1, 1.5, and 2 Standard Deviations)

	$\varepsilon=0.5$		$\varepsilon=1$		$\varepsilon=1.5$		$\varepsilon=2$	
m	AR	GARCH	AR	GARCH	AR	GARCH	AR	GARCH
				Australian Dollar				
2	0.216	-0.179	0.318	-0.194	0.444	-0.582	0.808	-0.910
3	0.332	-0.174	0.042	-0.452	0.183	-0.626	0.678	-0.422
4	0.709	0.089	0.074	-0.344	0.041	-0.726	0.542	-0.384
5	1.054	0.323	0.158	-0.202	-0.011	-0.706	0.393	-0.410
				British Pound				
2	*4.124	0.132	*3.625	-0.574	*3.762	-0.536	*4.301	0.009
3	*6.613	0.888	*5.284	-0.534	*4.972	-0.966	*5.371	-0.565
4	*9.824	*2.222	*6.607	-0.308	*5.989	-0.832	*6.022	-0.681
5	*13.879	*3.739	*7.971	0.131	*6.894	-0.472	*6.641	-0.447
				Canadian Dollar				
2	*5.260	1.248	*5.261	1.393	*4.929	0.879	*4.040	0.170
3	*6.669	1.449	*6.319	1.274	*5.583	0.596	*4.315	-0.135
4	*9.628	*2.596	*7.650	1.678	*6.173	0.784	*4.659	0.134
5	*11.853	*3.046	*8.756	1.911	*6.476	0.711	*4.702	-0.042

Table 5.4 (cont'd)

m	$\varepsilon=0.5$ AR	$\varepsilon=0.5$ GARCH	$\varepsilon=1$ AR	$\varepsilon=1$ GARCH	$\varepsilon=1.5$ AR	$\varepsilon=1.5$ GARCH	$\varepsilon=2$ AR	$\varepsilon=2$ GARCH
				Crude Oil				
2	*7.565	-0.001	*9.503	-0.567	*10.099	-1.081	*7.679	-1.050
3	*10.265	0.465	*11.066	-0.568	*11.139	-1.169	*8.680	*-1.420
4	*12.450	0.295	*11.900	-0.860	*11.799	*-1.449	*9.482	*-1.596
5	*16.005	0.787	*13.370	-0.648	*12.859	*-1.263	*10.635	*-1.464
				Copper				
2	*3.675	*-1.663	*3.962	*-2.070	*4.215	*-2.037	*4.119	*-1.636
3	*5.820	-1.515	*6.267	*-2.146	*6.534	*-2.156	*6.188	*-1.766
4	*7.560	-1.035	*7.979	*-1.857	*8.217	*-1.974	*7.592	*-1.699
5	*9.356	-1.071	*9.070	*-1.845	*9.200	*-2.001	*8.637	*-1.753
				Deutschemark				
2	*6.293	1.145	*5.802	0.687	*6.214	0.504	*6.175	0.242
3	*8.499	1.190	*7.465	0.515	*7.871	0.399	*7.984	0.032
4	*12.309	2.390	*9.530	1.263	*9.031	0.952	*8.783	0.423
5	*15.385	*3.055	*11.033	1.432	*9.940	1.146	*9.408	0.701

Table 5.4 (cont'd)

m	$\varepsilon=0.5$ AR	$\varepsilon=0.5$ GARCH	$\varepsilon=1$ AR	$\varepsilon=1$ GARCH	$\varepsilon=1.5$ AR	$\varepsilon=1.5$ GARCH	$\varepsilon=2$ AR	$\varepsilon=2$ GARCH
				Gold				
2	*8.290	-0.198	*8.894	-0.236	*8.327	-0.234	*6.244	-0.147
3	*10.383	-0.299	*10.251	-0.281	*9.491	-0.486	*7.619	-0.325
4	*13.403	0.652	*12.177	0.439	*10.837	0.176	*9.049	0.027
5	*16.590	0.991	*13.858	0.718	*11.673	0.378	*9.594	0.077
				Heating Oil				
2	*9.301	0.816	*9.554	0.156	*10.098	-0.253	*9.803	-0.523
3	*11.431	1.224	*11.417	0.425	*11.892	-0.017	*11.162	0.003
4	*13.478	1.205	*12.694	0.355	*12.988	0.006	*11.968	0.244
5	*16.185	0.552	*14.089	-0.009	*13.859	-0.168	*12.548	0.262
				Unleaded Gas				
2	*4.086	-0.411	*5.023	-0.441	*4.978	-0.473	*4.323	-0.640
3	*5.195	0.187	*6.257	-0.272	*6.347	-0.359	*5.457	-0.676
4	*5.627	0.094	*6.630	-0.662	*6.653	-0.691	*5.704	-1.072
5	*6.673	0.186	*7.302	-0.794	*7.094	-0.823	*6.007	*-1.241

Table 5.4 (cont'd)

m	$\varepsilon=0.5$ AR	$\varepsilon=0.5$ GARCH	$\varepsilon=1$ AR	$\varepsilon=1$ GARCH	$\varepsilon=1.5$ AR	$\varepsilon=1.5$ GARCH	$\varepsilon=2$ AR	$\varepsilon=2$ GARCH
				Japanese Yen				
2	*3.914	*2.234	*3.254	1.577	*2.615	0.849	*2.158	0.687
3	*6.471	*3.336	*4.733	*2.141	*3.852	1.332	*3.179	1.007
4	*7.147	*3.433	*5.735	*2.474	*4.634	1.685	*3.748	1.217
5	*8.268	*3.942	*6.678	*2.879	*5.295	2.141	*4.238	1.606
				Platinum				
2	*7.570	1.312	*8.681	1.244	*9.496	1.115	*9.486	0.648
3	*9.137	1.755	*9.877	1.475	*10.508	1.421	*10.584	0.989
4	*10.860	2.139	*10.707	1.655	*11.295	1.423	*11.517	1.180
5	*12.107	1.506	*11.373	1.389	*11.541	1.148	*11.622	0.996
				Swiss Franc				
2	*4.057	0.178	*4.571	0.781	*4.769	1.085	*5.048	1.630
3	*5.742	0.397	*6.385	1.014	*6.453	1.136	*6.418	1.350
4	*7.883	0.870	*8.194	1.566	*8.029	1.720	*7.700	1.766
5	*10.623	1.260	*9.773	1.831	*8.901	1.902	*8.327	1.844

Table 5.4 (cont'd)

m	$\varepsilon=0.5$ AR	$\varepsilon=0.5$ GARCH	$\varepsilon=1$ AR	$\varepsilon=1$ GARCH	$\varepsilon=1.5$ AR	$\varepsilon=1.5$ GARCH	$\varepsilon=2$ AR	$\varepsilon=2$ GARCH
				Silver				
2	*10.867	1.384	*11.039	1.445	*11.188	1.252	*11.196	0.667
3	*13.578	1.101	*13.354	1.083	*13.176	0.876	*13.068	0.582
4	*16.791	1.332	*15.535	1.354	*14.697	1.205	*14.162	0.913
5	*20.411	1.246	*17.595	1.026	*15.748	0.941	*14.740	0.909

Note: An asterisk indicates significance at the 5% critical level. The standard normal critical values were used in the case of the AR residuals and Hsieh's (1991) table XIII of simulated BDS critical values was used in the case of the GARCH standardized residuals.

Table 5.4 (cont'd)

5% Critical Values for the BDS Test

m	$\varepsilon=0.5$	$\varepsilon=1$	$\varepsilon=1.5$	$\varepsilon=2$	N(0,1)
			<u>2.5% critical point</u>		
2	-1:61	-1.52	-1.52	-1.49	-1.96
3	-1.65	-1.29	-1.29	-1.29	-1.96
4	-1.63	-1.17	-1.17	-1.12	-1.96
5	-1.94	-1.11	-1.00	-0.99	-1.96
			<u>97.5% critical point</u>		
2	2.11	1.96	1.85	1.88	1.96
3	2.34	2.14	2.01	2.00	1.96
4	2.49	2.25	2.17	2.14	1.96
5	2.90	2.40	2.28	2.22	1.96

Table 5.5 Dominant Lyapunov Exponent Point Estimates for the GARCH(1,1) Standardized Residuals of Weekly Futures Prices

		Dominant Lyapunov Exponent Point Estimate			Value of Minimized BIC		
Series	# Obs.	First 500	Last 500	All Obs.	First 500	Last 500	All Obs.
Australian Dollar	330			0.043 (3,7,2)			1.3717
British Pound	955	-0.484 (1,4,2)	-0.003 (3,7,2)		1.428	1.421	
Canadian dollar	854	-0.114 (2,5,2)	-0.014 (1,10,2)		1.4118	1.3975	
Crude Oil	530			-0.319 (1,6,2)			1.4166
Copper	905	0.057 (2,10,2)	-3.939 (1,1,1)		1.4355	1.4412	
Deutschemark	955	-5.519 (1,1,1)	-0.941 (3,2,1)		1.4397	1.4372	
Gold	961	-0.019 (3,7,2)	-1.301 (3,2,1)		1.4377	1.4376	
Heating Oil	734	-0.025 (3,7,2)	-0.124 (2,6,2)		1.4366	1.4312	
Unleaded Gas	439			-4.952 (1,1,1)			1.4432
Japanese Yen	865	0.019 (3,9,2)	0.023 (3,8,2)		1.4099	1.4095	
Platinum	1072	-3.485 (1,1,1)	-0.680 (2,3,1)		1.4382	1.4407	
Swiss Franc	955	-0.22 (3,5,1)	-1.096 (2,2,1)		1.4323	1.4375	
Silver	1140	-10.767 (2,1,1)	-0.416 (3,4,1)		1.4258	1.4276	

Note: In parentheses beside each dominant Lyapunov exponent is the best fit parameter triple (L,m,k) chosen by minimizing the BIC information criterion over all triples [L from 1 to 3, m from 1 to 10, and k from 1 to 3], where L is the time delay parameter, m is the dimension of the embedding, and k is the number of parameter units in the nonlinear regression.

Figure 5.1 Estimated Largest Lyapunov Exponent for Each Triple (L,m,k) for the Australian Dollar, All Observations

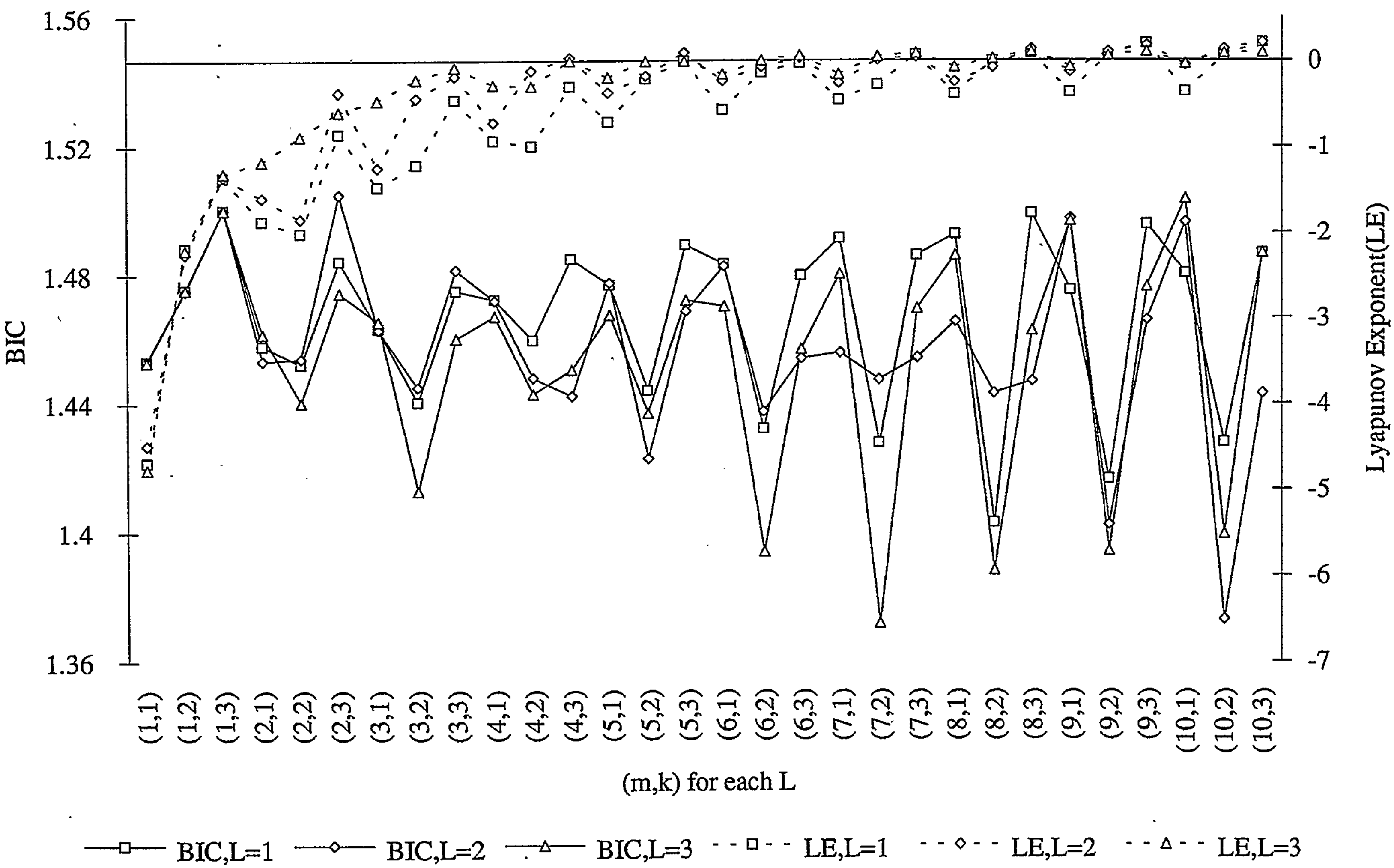


Figure 5.2.A Estimated Largest Lyapunov Exponent for Each Triple (L,m,k) for the British Pound, First 500 Observations

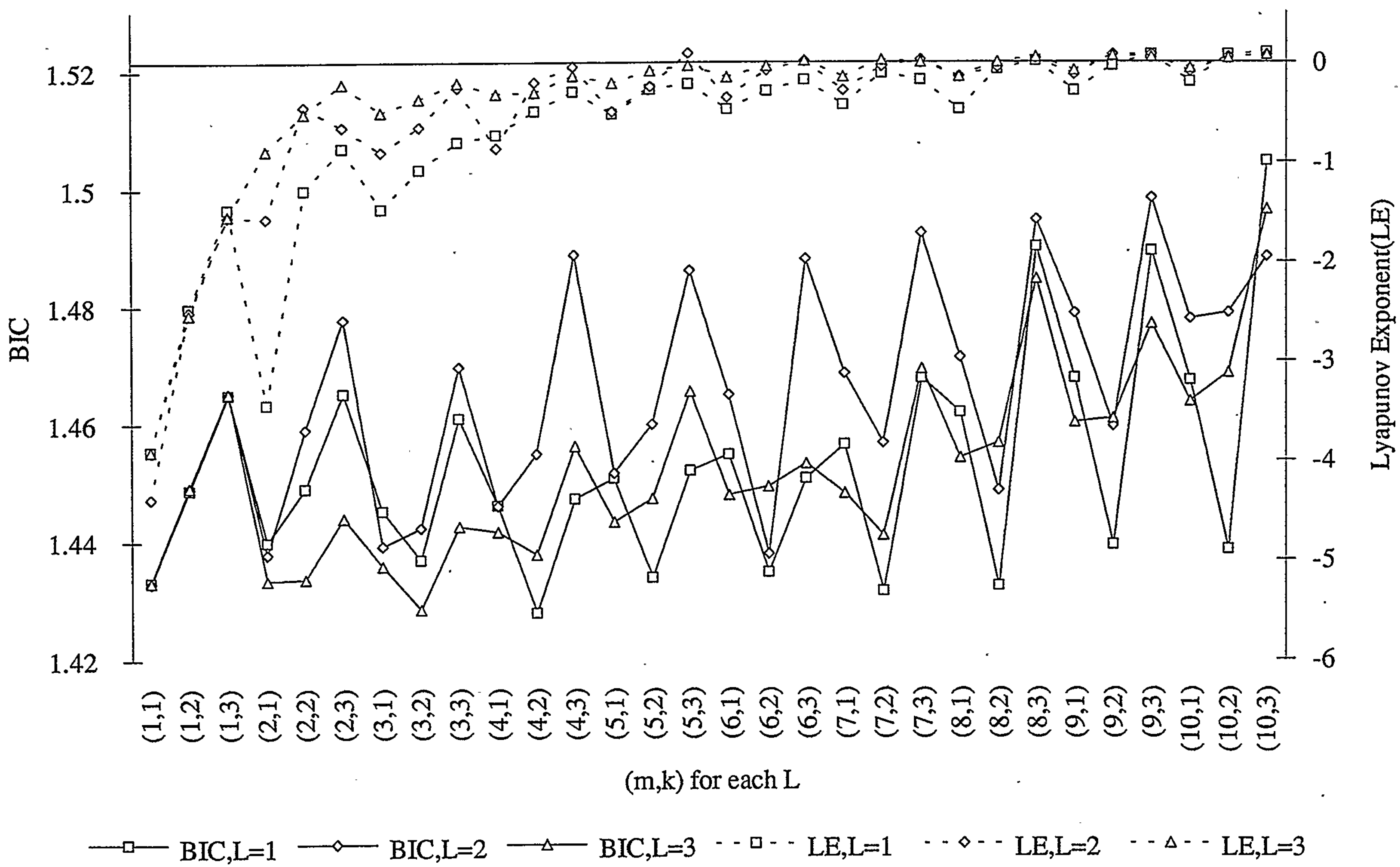


Figure 5.2.B Estimated Largest Lyapunov Exponent for Each Triple (L,m,k) for the British Pound, Last 500 Observations

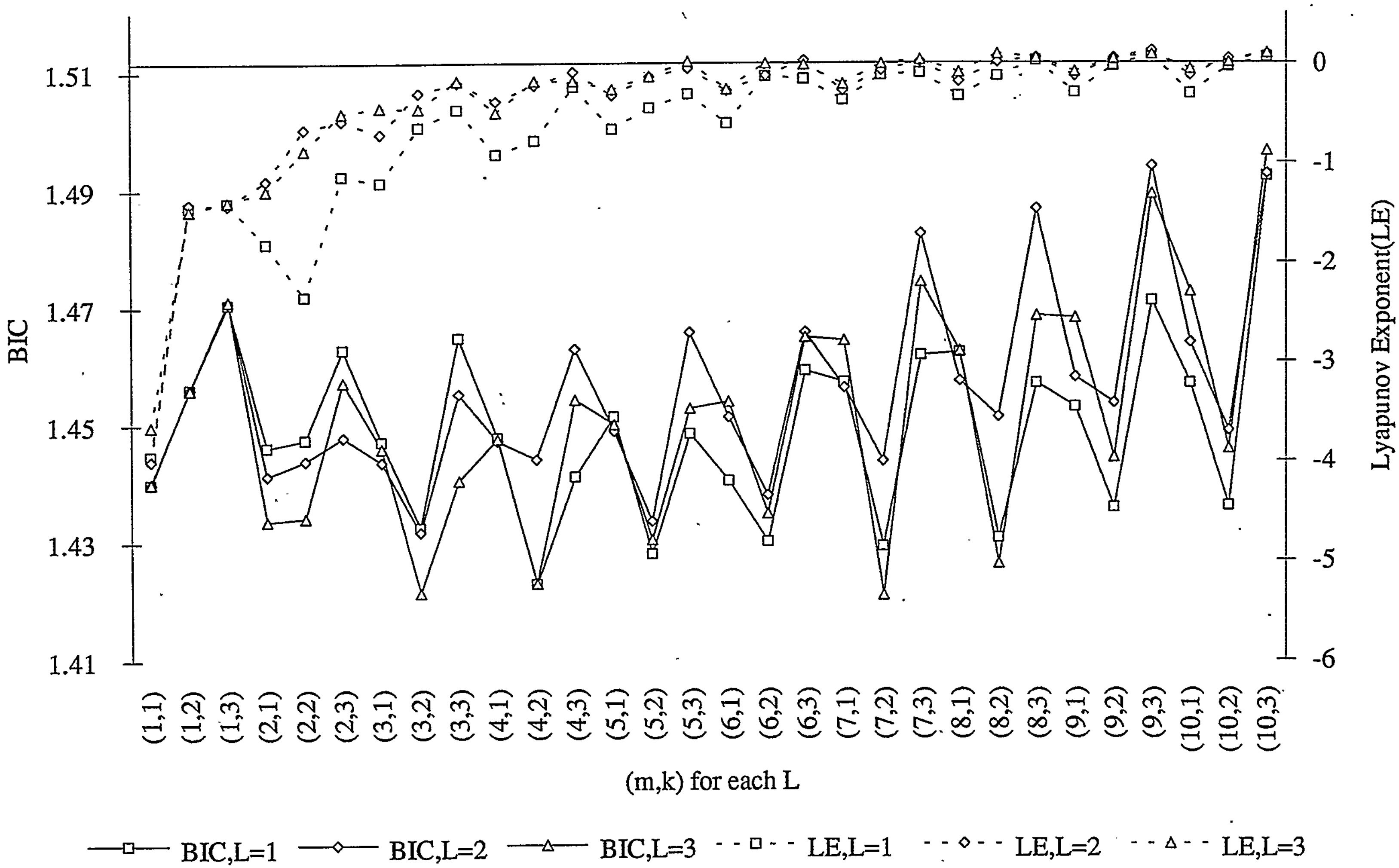


Figure 5.3.A Estimated Largest Lyapunov Exponent for Each Triple (L,m,k) for the Canadian Dollar, First 500 Observations

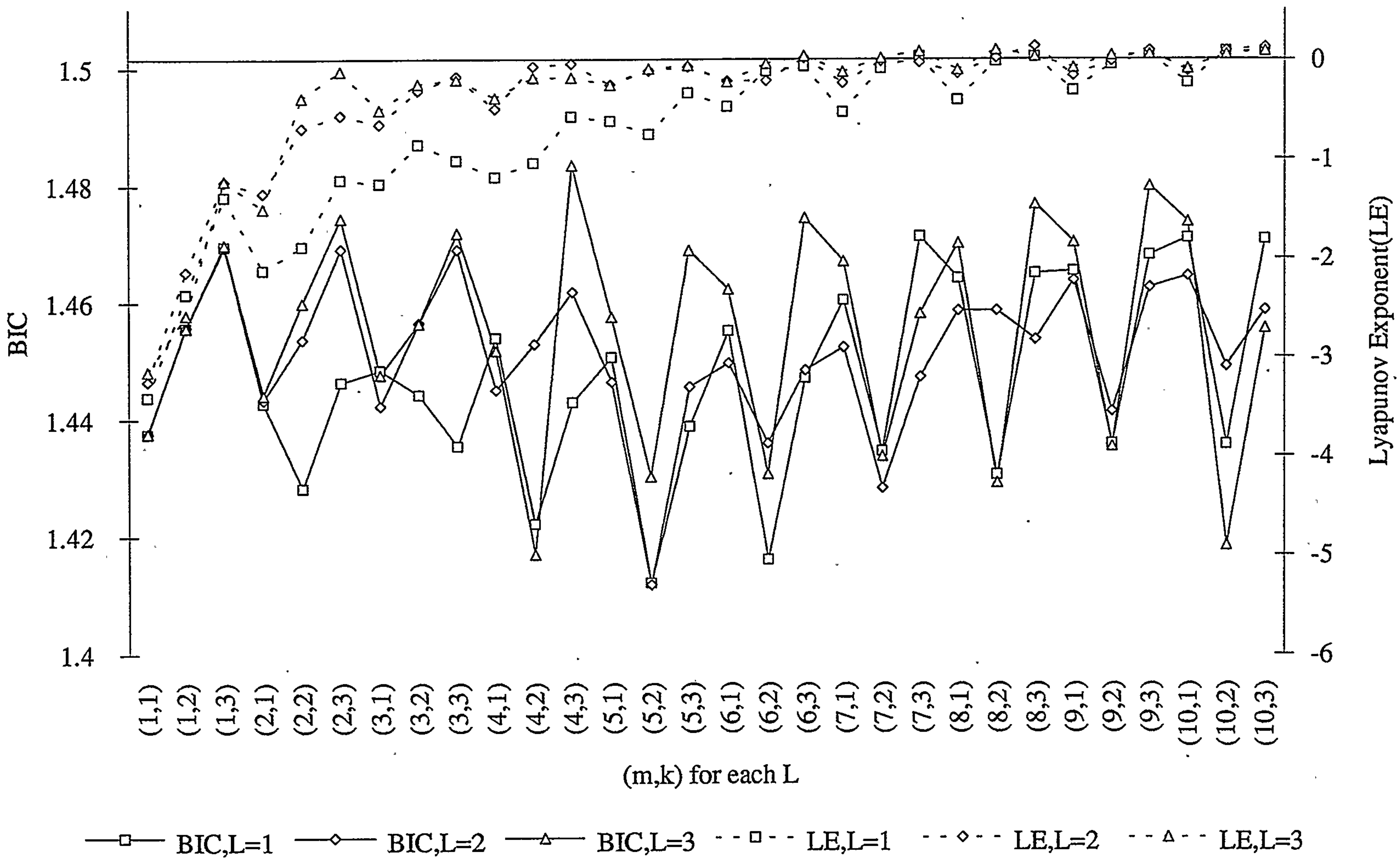


Figure 5.3.B Estimated Largest Lyapunov Exponent for Each Triple (L,m,k) for the Canadian Dollar, Last 500 Observations

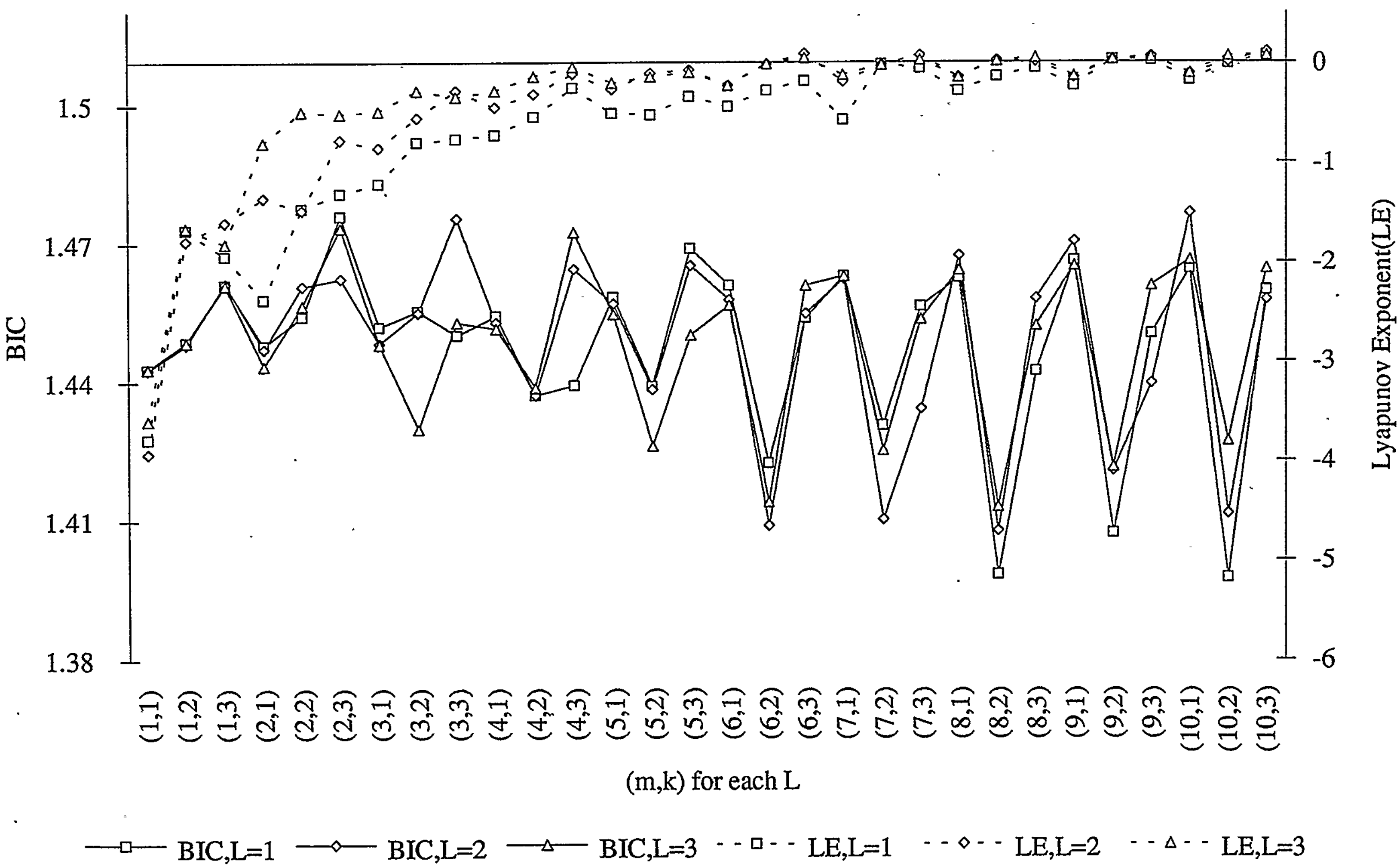


Figure 5.4 Estimated Largest Lyapunov Exponent for Each Triple (L,m,k) for Crude Oil, All Observations

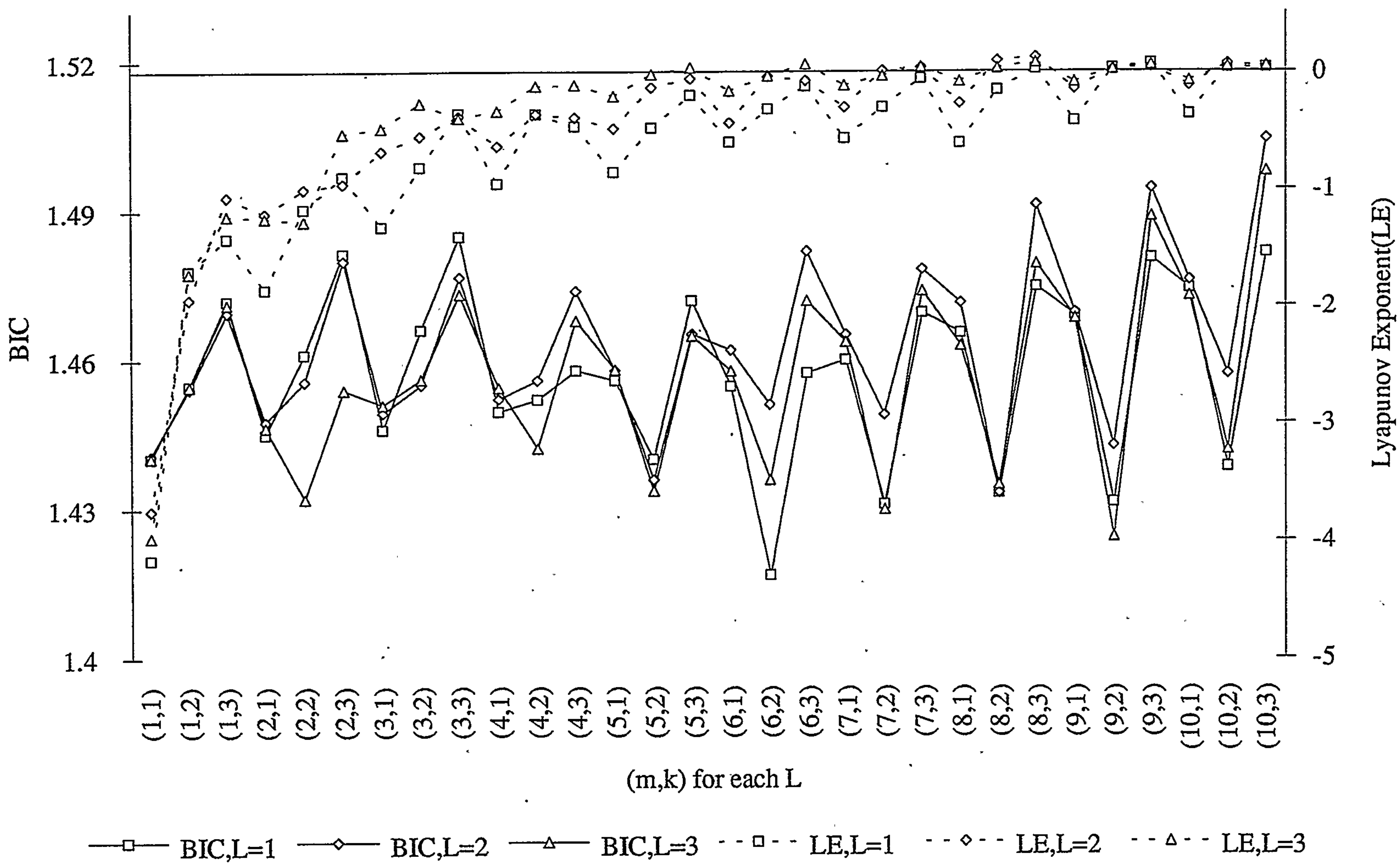


Figure 5.5.A Estimated Largest Lyapunov Exponent for Each Triple (L,m,k) for Copper, First 500 Observations

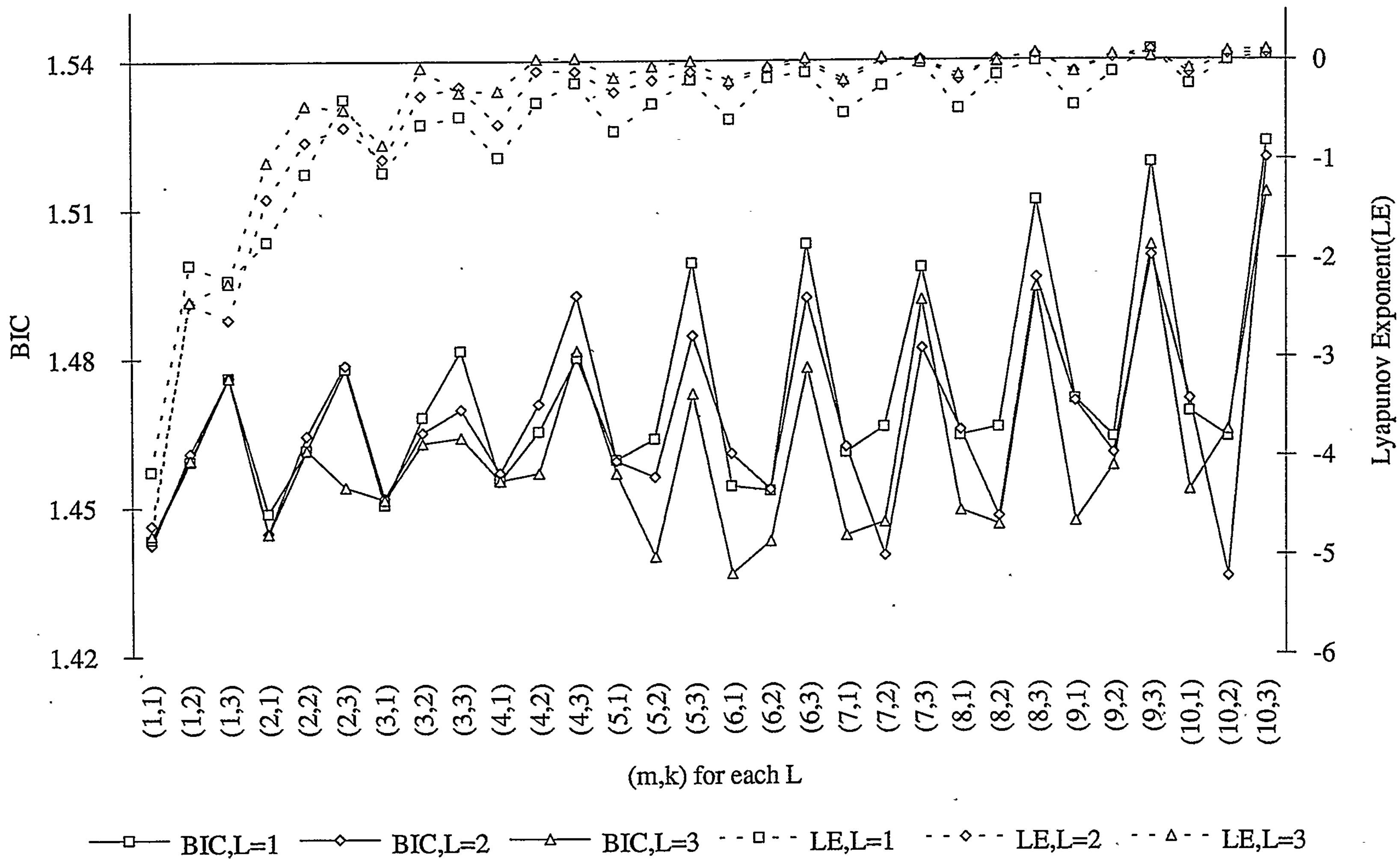


Figure 5.5.B Estimated Largest Lyapunov Exponent for Each Triple (L,m,k) for Copper, Last 500 Observations

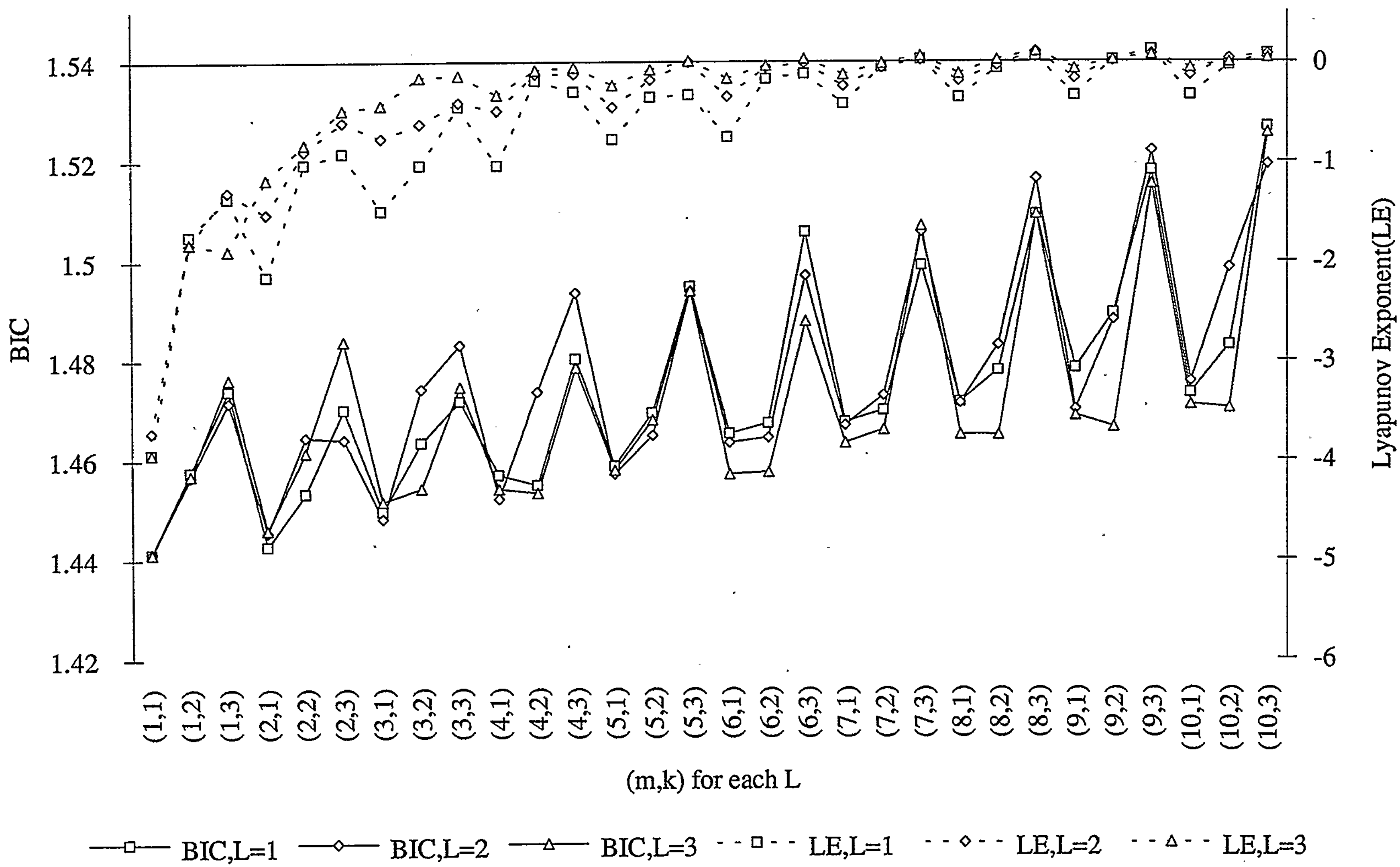


Figure 5.6.A Estimated Largest Lyapunov Exponent for Each Triple (L,m,k) for the Deutschemark, First 500 Observations

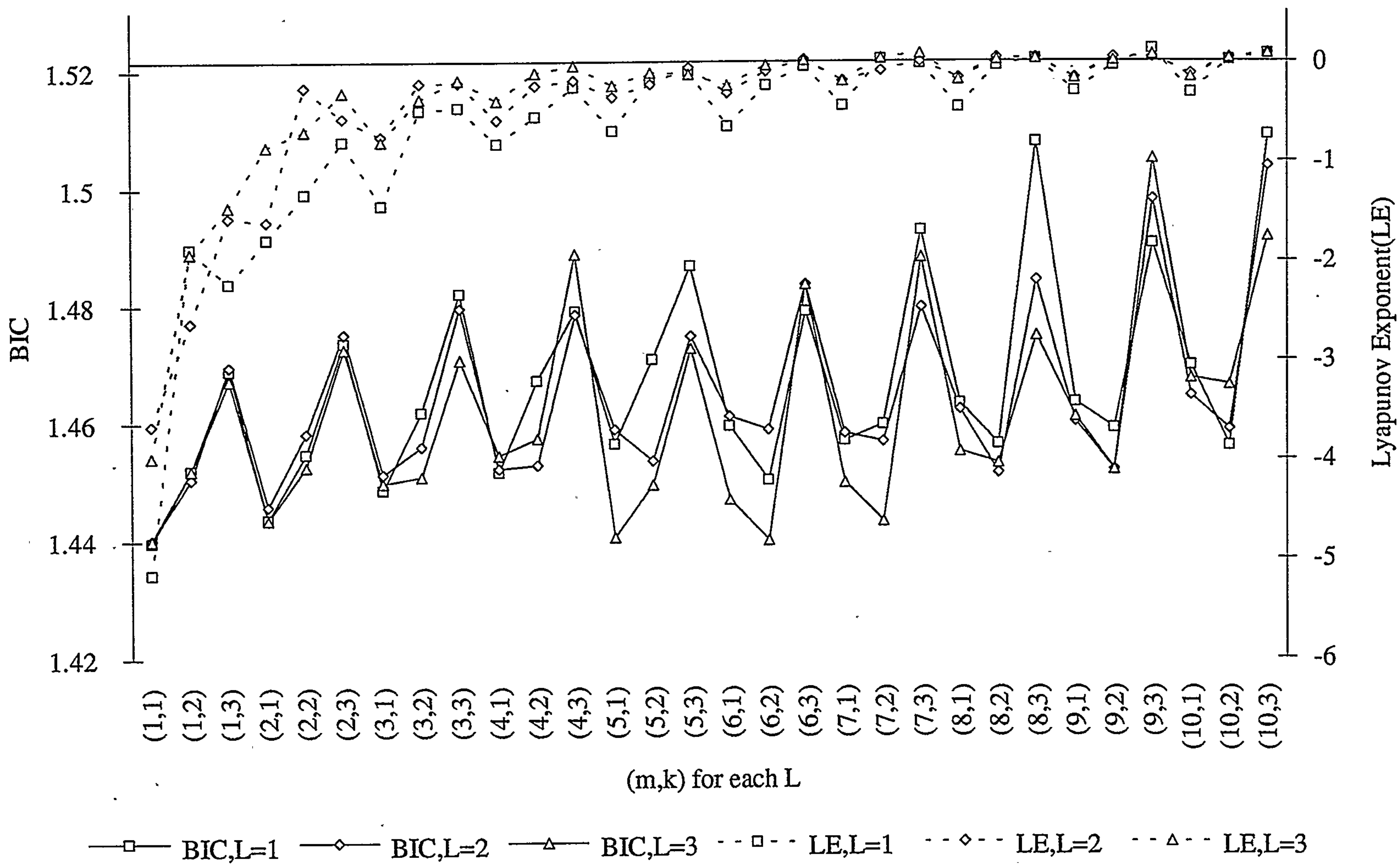


Figure 5.6.B Estimated Largest Lyapunov Exponent for Each Triple (L,m,k) for the Deutschemark, Last 500 Observations

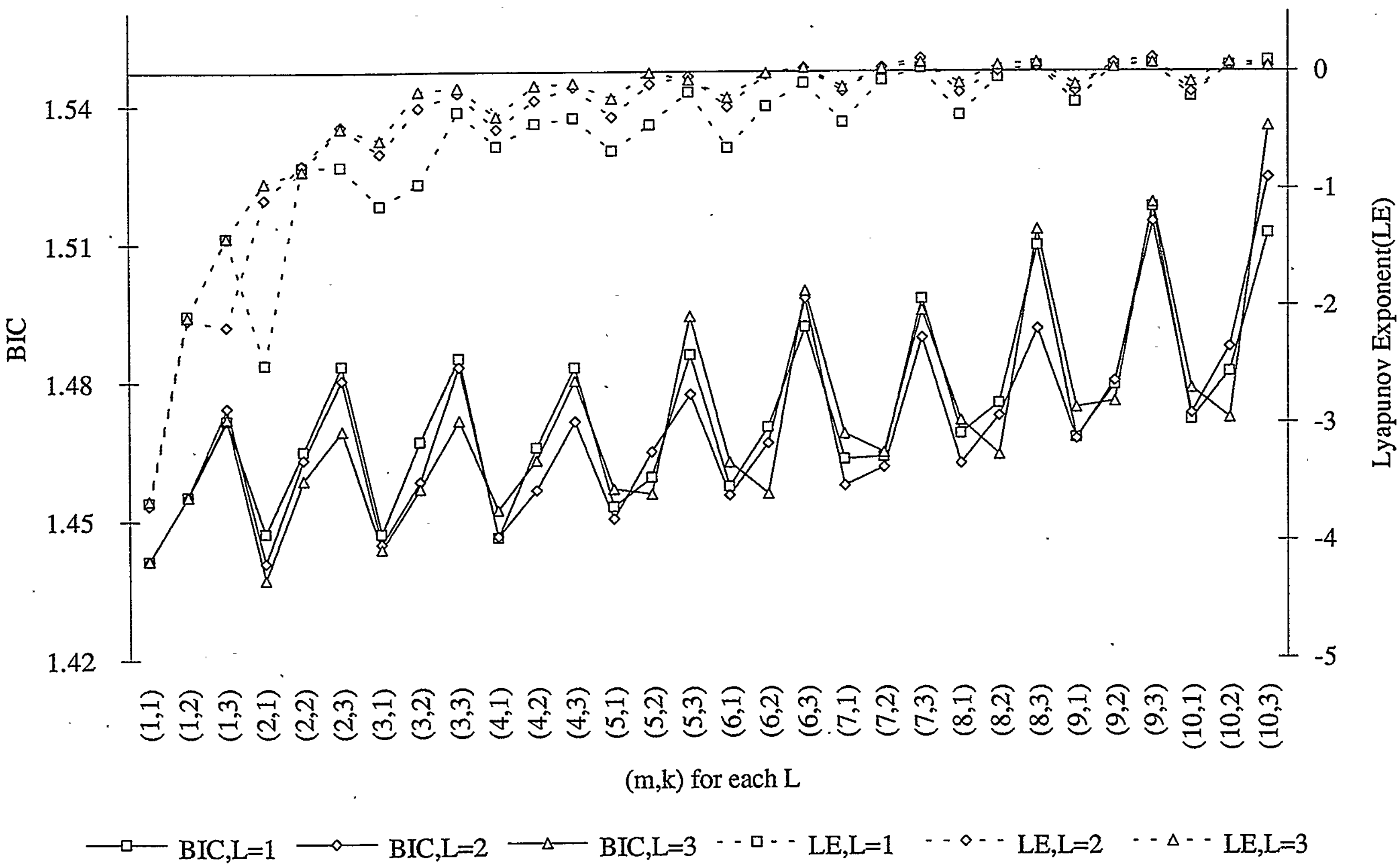


Figure 5.7.A Estimated Largest Lyapunov Exponent for Each Triple (L,m,k) for Gold, First 500 Observations

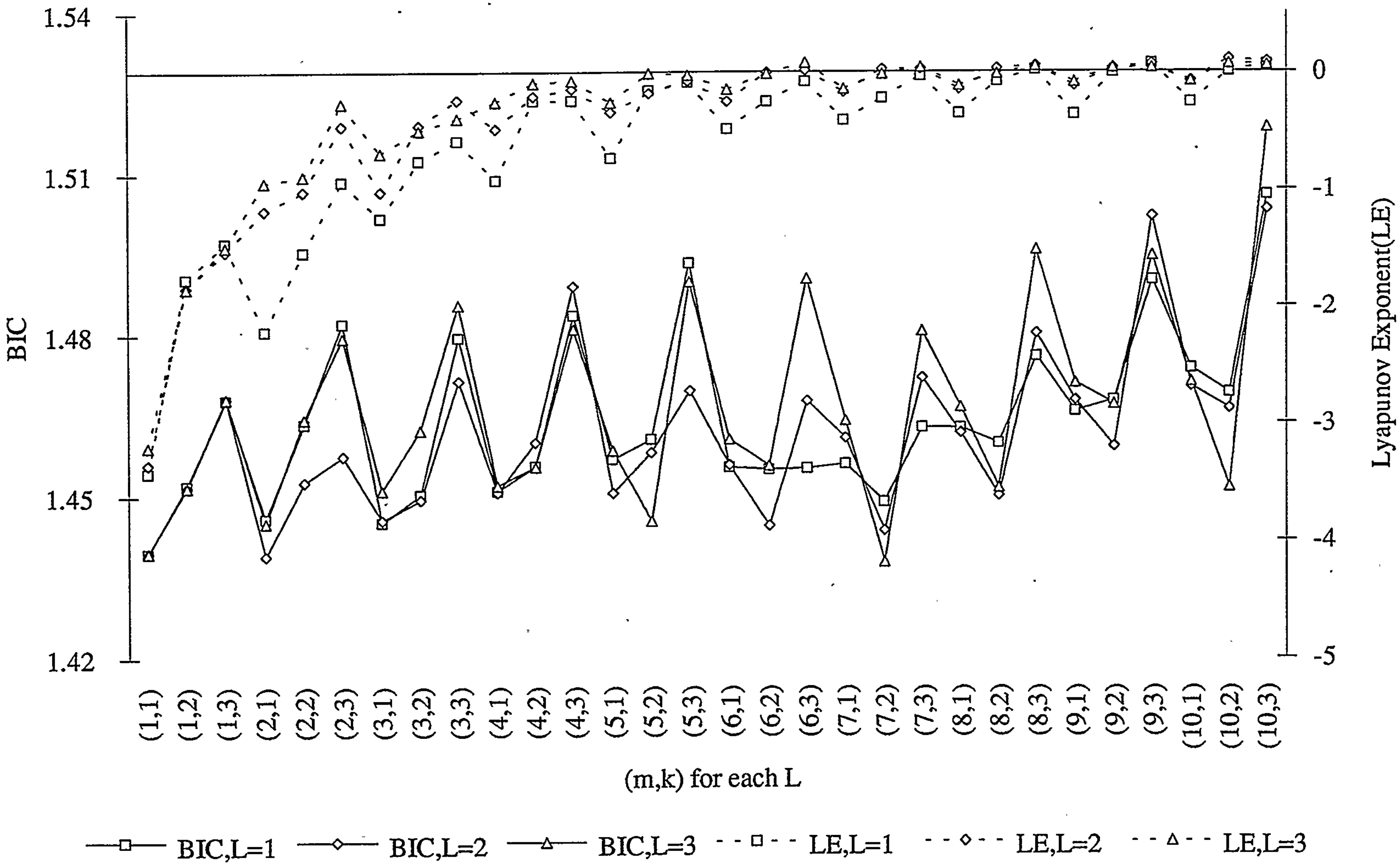


Figure 5.7.B Estimated Largest Lyapunov Exponent for Each Triple (L,m,k) for Gold, Last 500 Observations

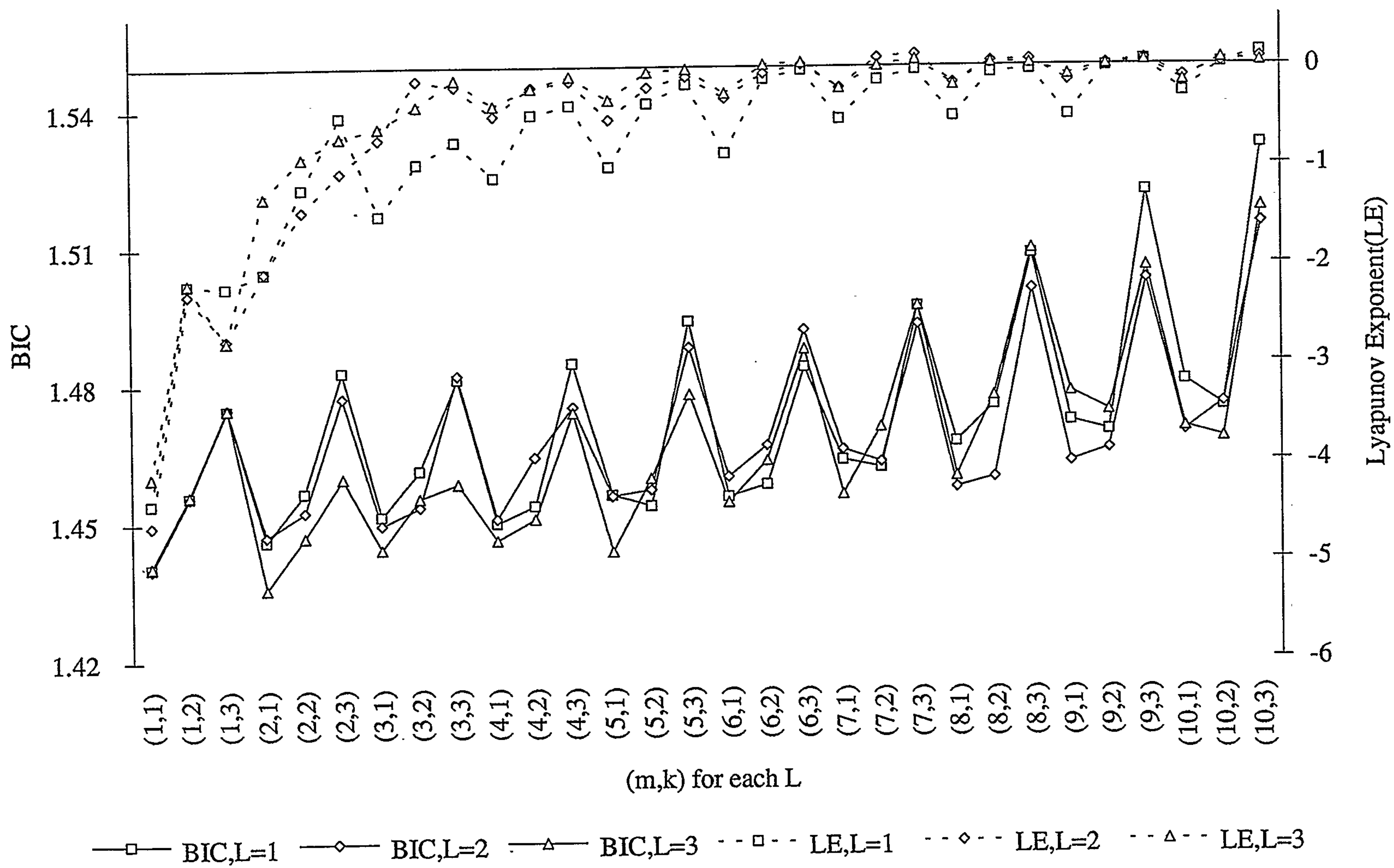


Figure 5.8.A Estimated Largest Lyapunov Exponent for Each Triple (L,m,k) for Heating Oil, First 500 Observations

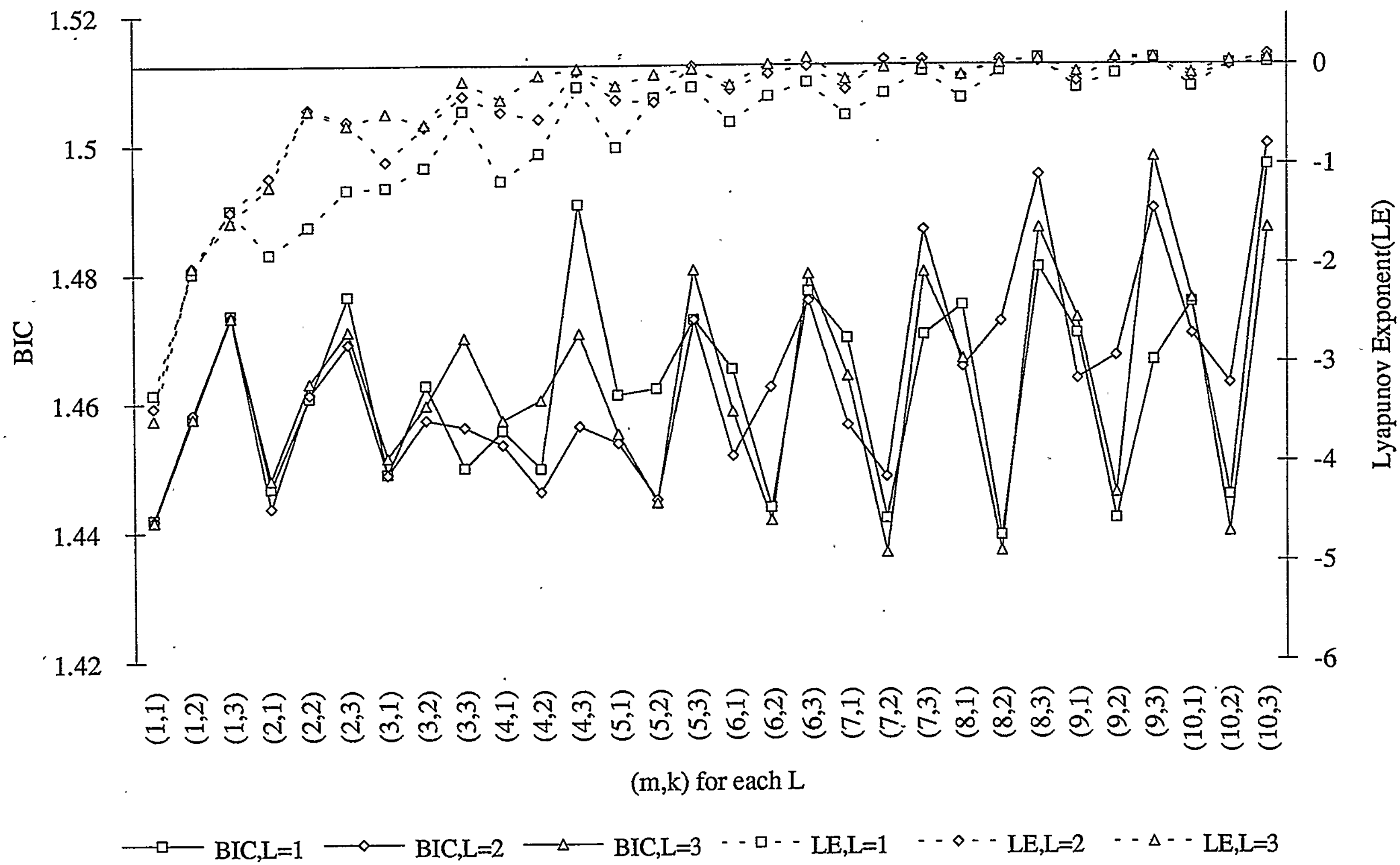


Figure 5.8.B Estimated Largest Lyapunov Exponent for Each Triple (L,m,k) for Heating Oil, Last 500 Observations

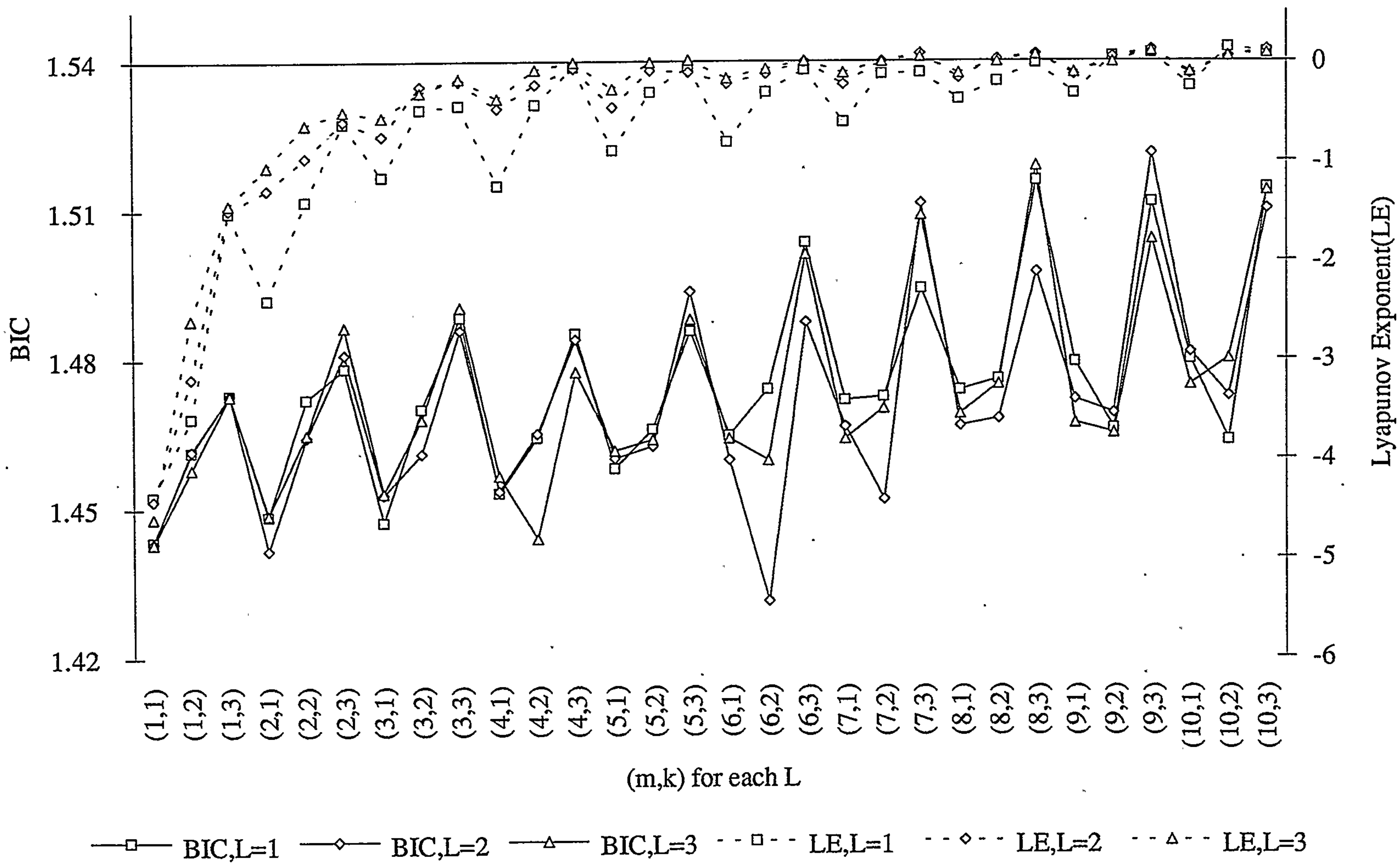


Figure 5.9 Estimated Largest Lyapunov Exponent for Each Triple (L,m,k) for Unleaded Gas, All Observations

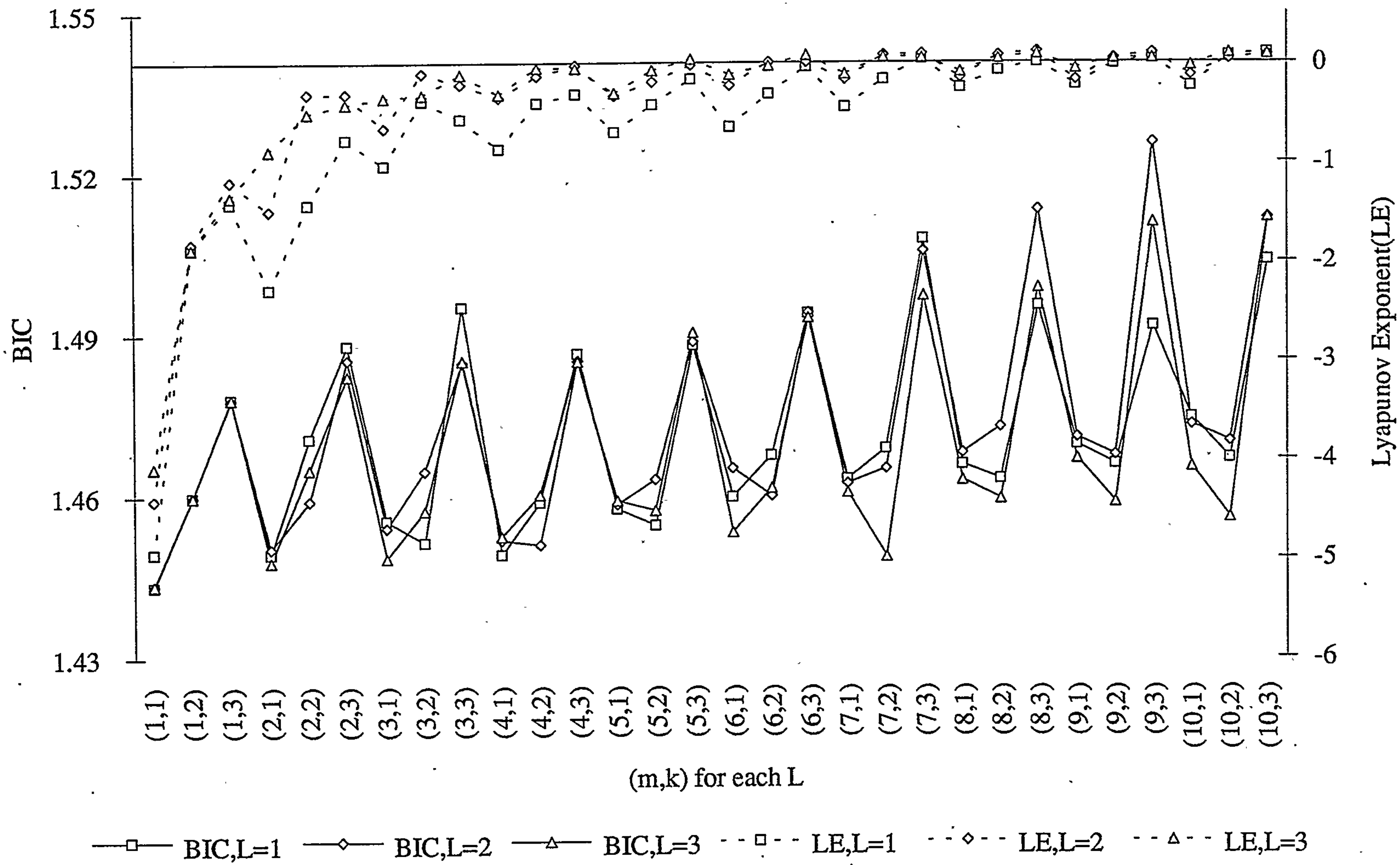


Figure 5.10.A Estimated Largest Lyapunov Exponent for Each Triple (L,m,k) for the Japanese Yen, First 500 Observations

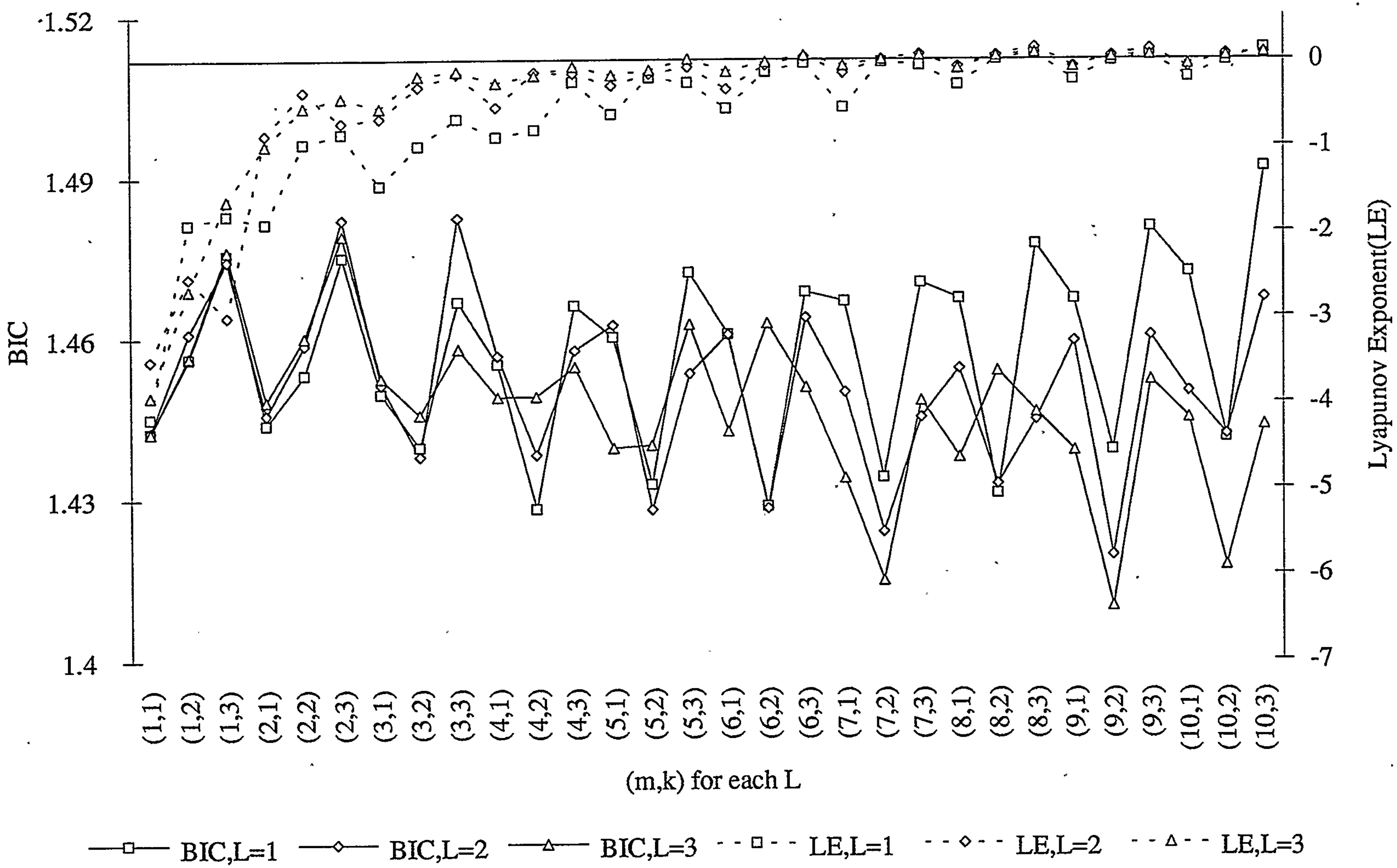


Figure 5.10.B Estimated Largest Lyapunov Exponent for Each Triple (L,m,k) for the Japanese Yen, Last 500 Observations

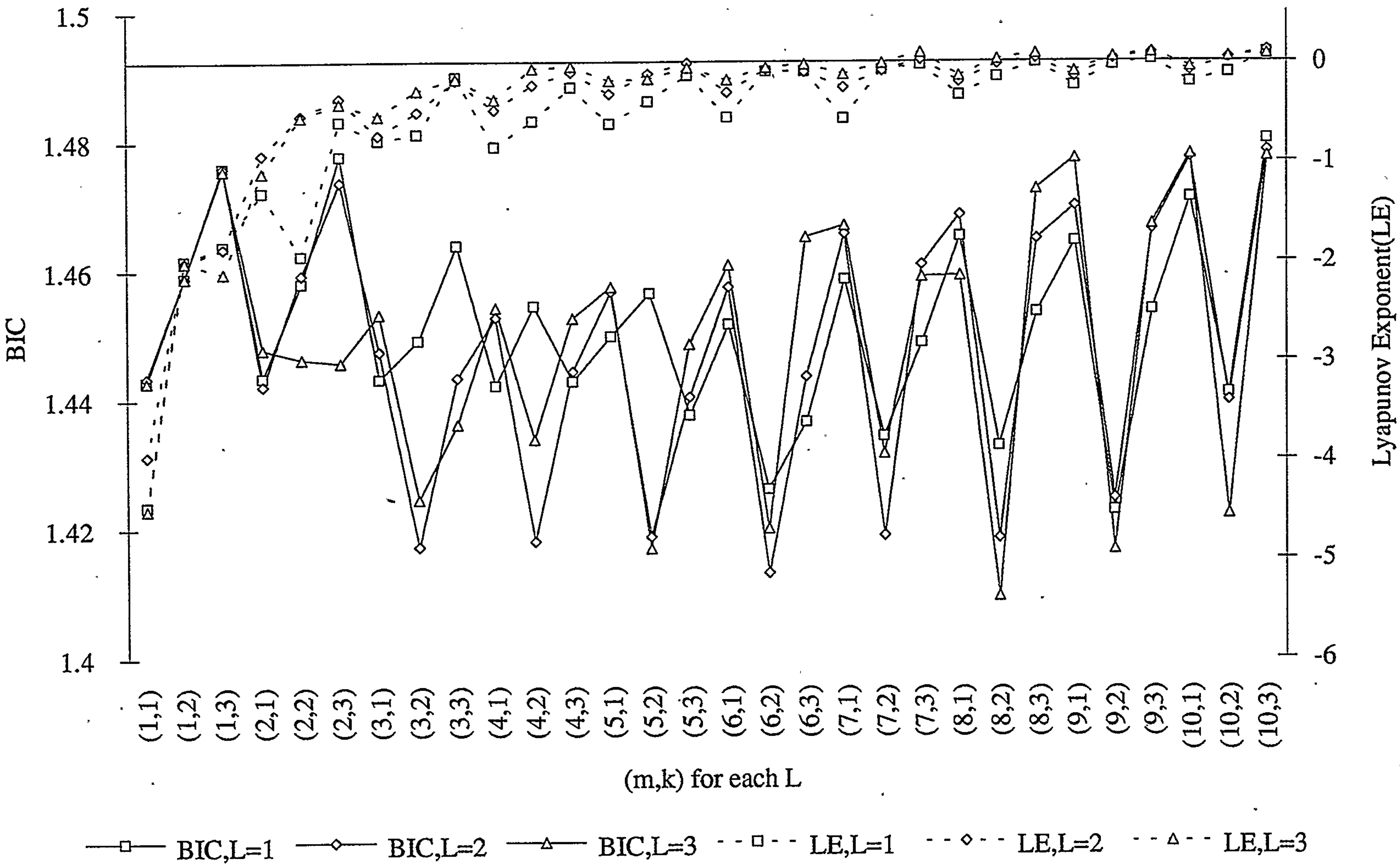


Figure 5.11.A Estimated Largest Lyapunov Exponent for Each Triple (L,m,k) for Platinum, First 500 Observations

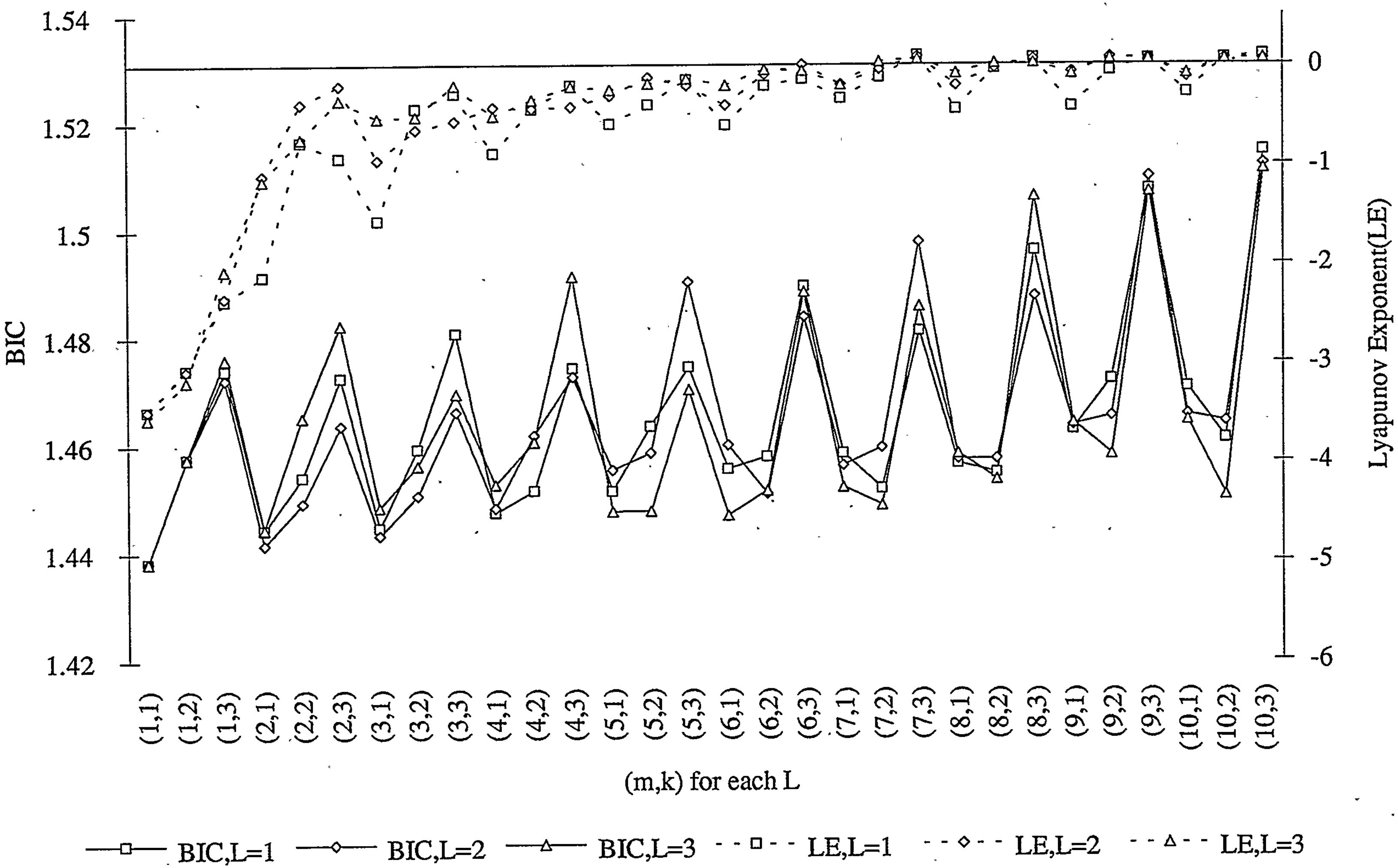


Figure 5.11.B Estimated Largest Lyapunov Exponent for Each Triple (L,m,k) for Platinum, Last 500 Observations

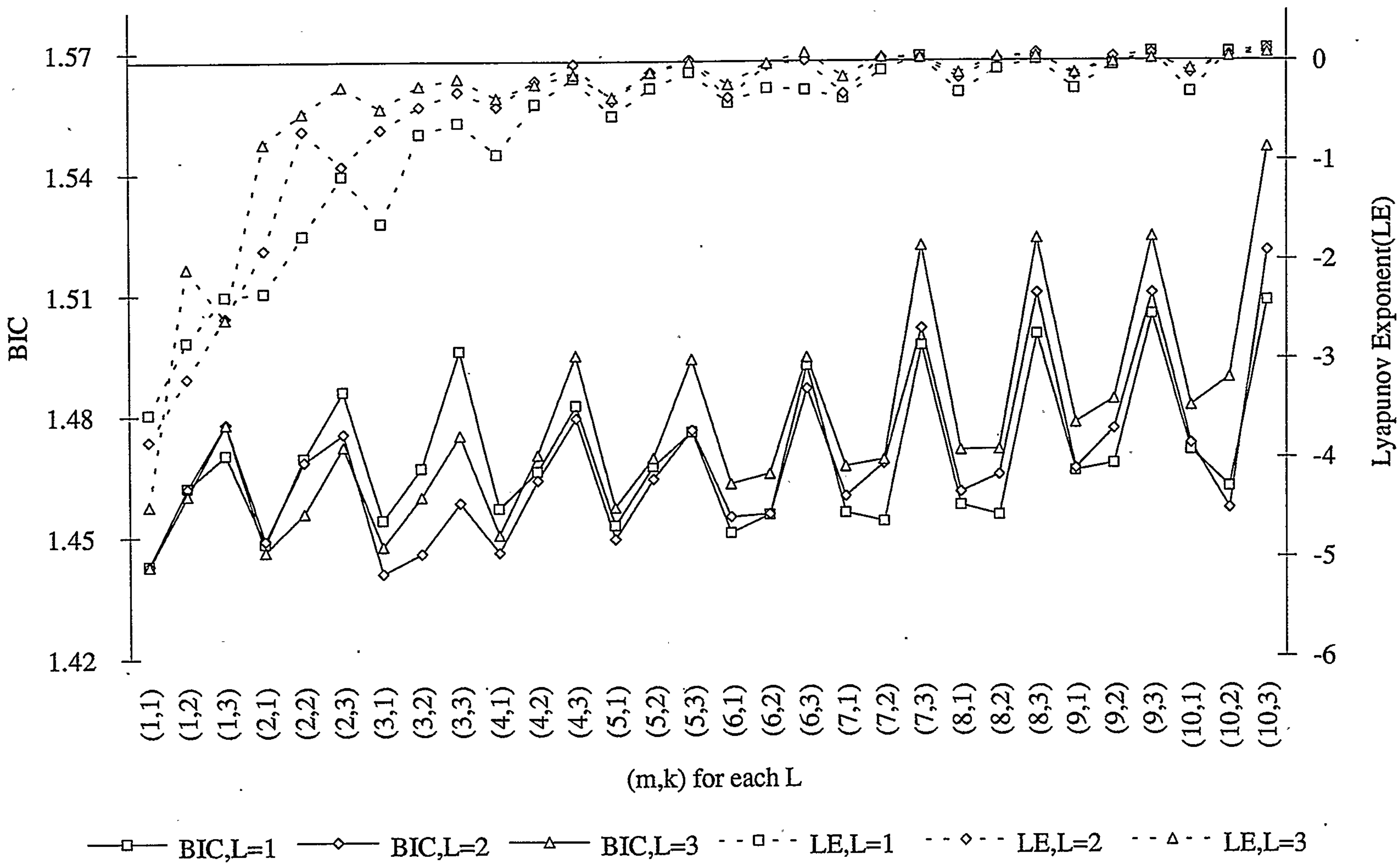


Figure 5.12.A Estimated Largest Lyapunov Exponent for Each Triple (L,m,k) for the Swiss Franc, First 500 Observations

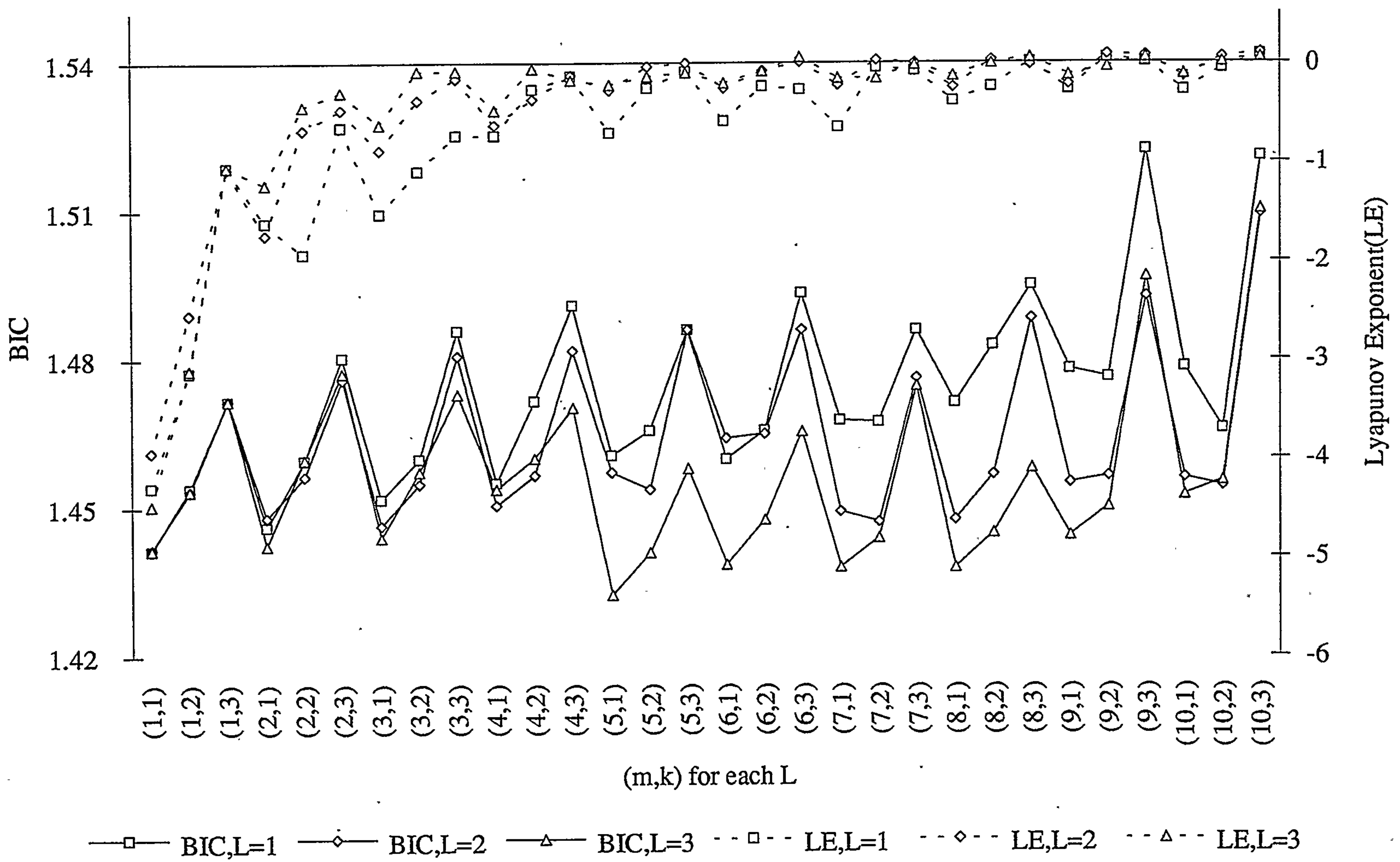


Figure 5.12.B Estimated Largest Lyapunov Exponent for Each Triple (L,m,k) for the Swiss Franc, Last 500 Observations

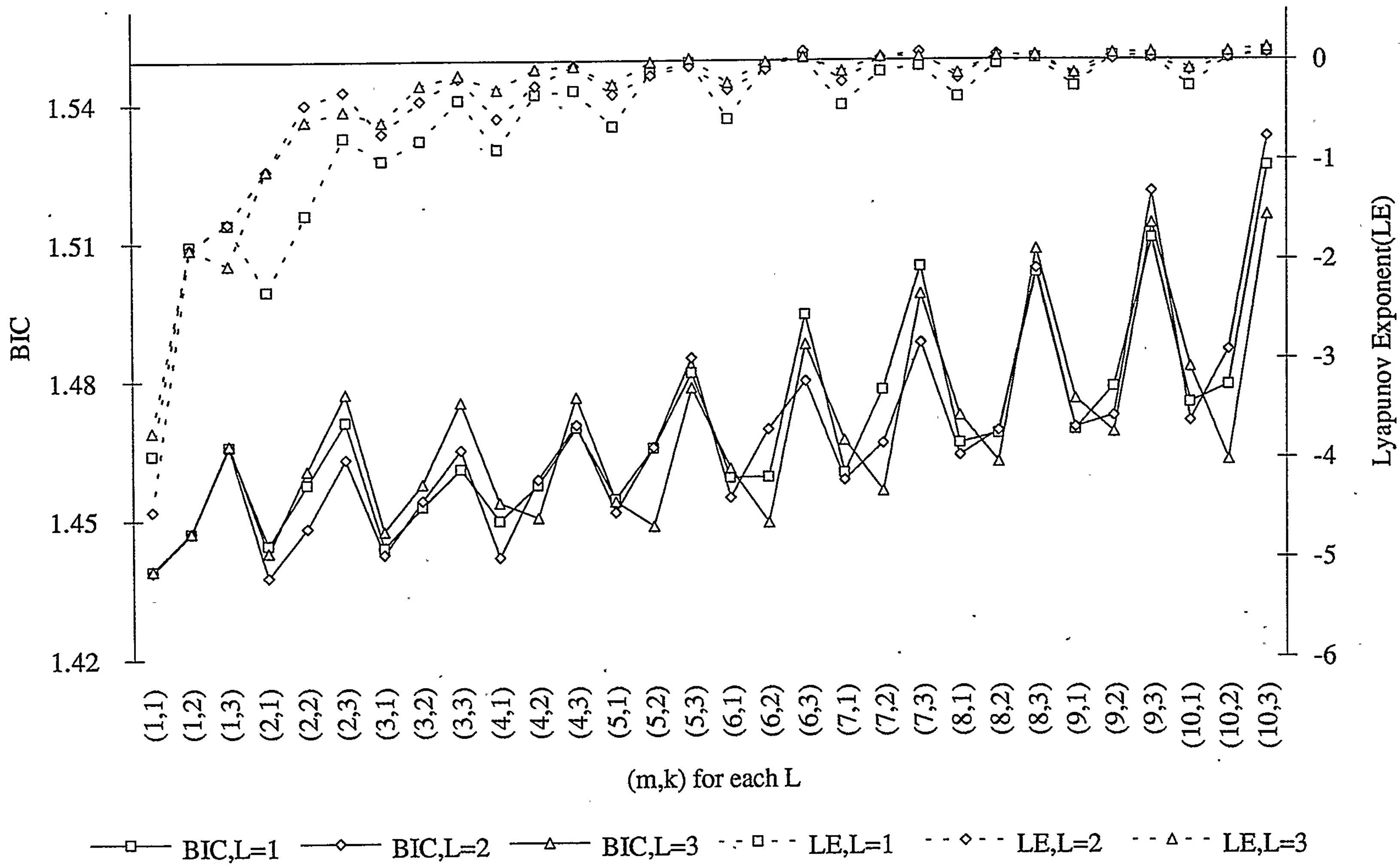


Figure 5.13.A Estimated Largest Lyapunov Exponent for Each Triple (L,m,k) for Silver, First 500 Observations

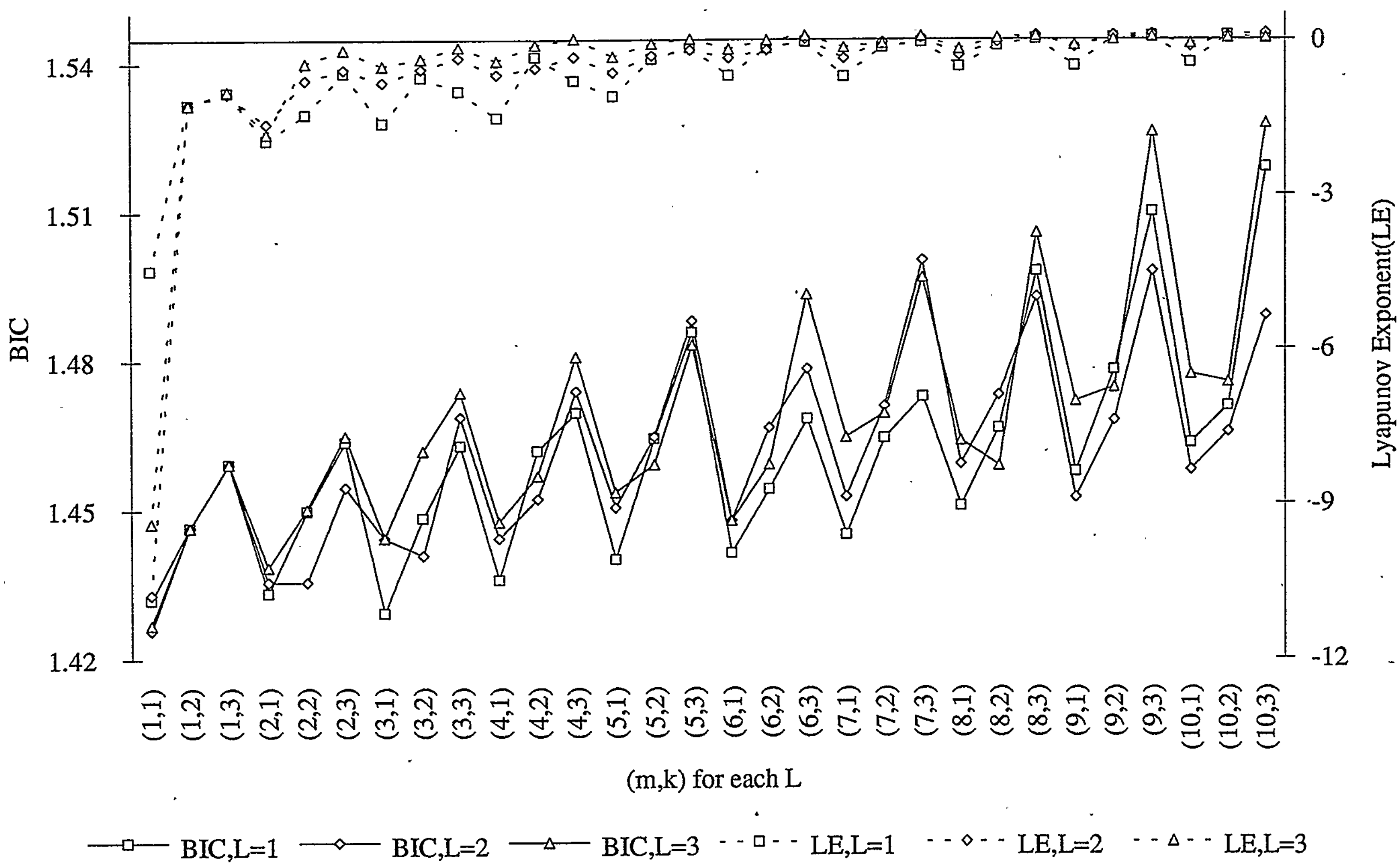
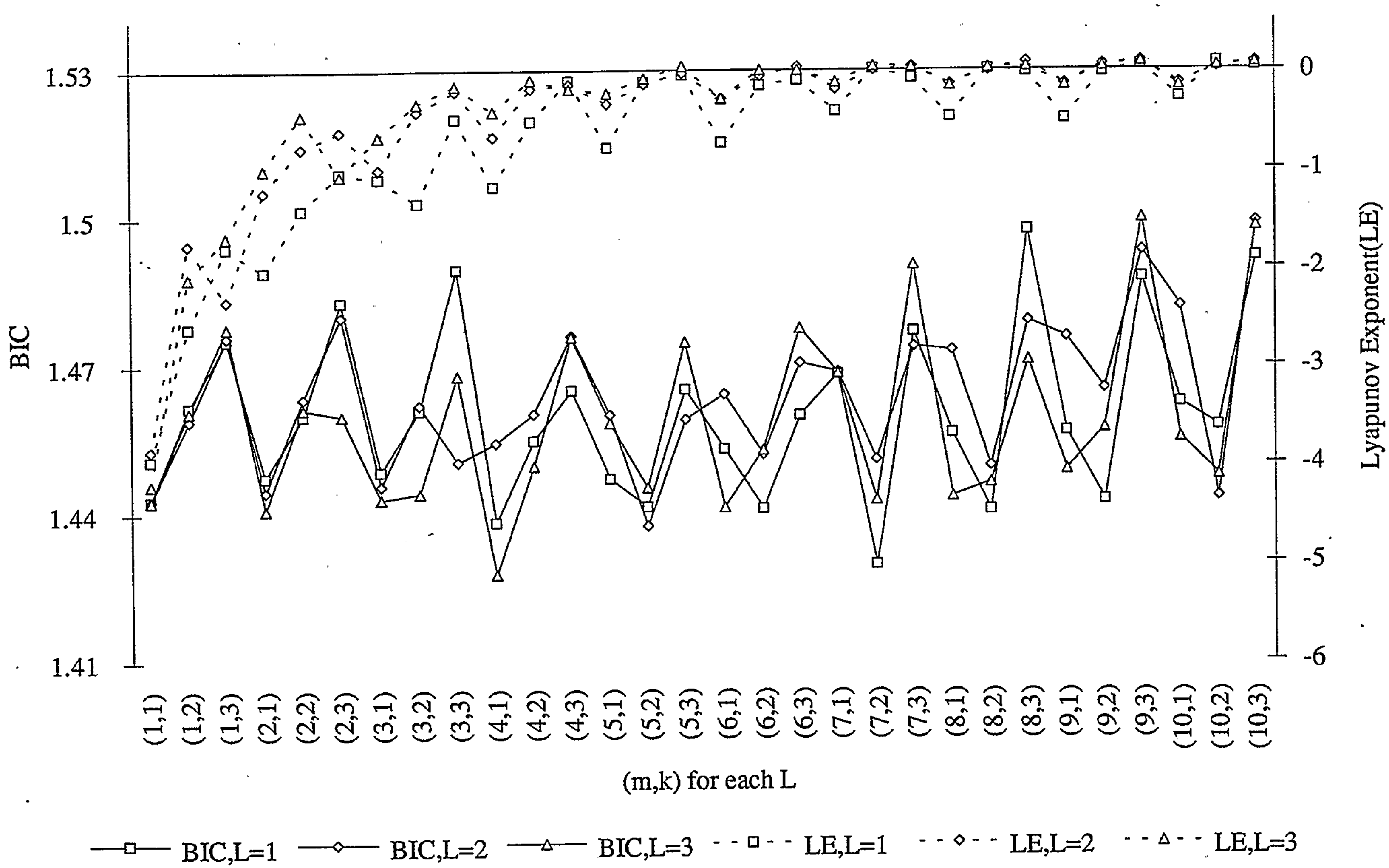


Figure 5.13.B Estimated Largest Lyapunov Exponent for Each Triple (L,m,k) for Silver, Last 500 Observations



Chapter 6

CONCLUSION

This thesis has presented some general concepts from the study of chaotic dynamics and has shown that they are applicable, both theoretically and empirically, to the interpretation of asset price changes. Chapter 2 began by outlining the traditional models that best explain asset price behavior. The fair game model shows that the average return of an asset is equal to its expected return. When this model is supplemented with the submartingale approach, prices are expected to increase at a rate equal to the opportunity cost of the underlying asset. The main characteristic of submartingale theory is that price changes, beyond opportunity costs, are serially uncorrelated and appear random.

Another explanation of random looking price changes has come from the study of nonlinear dynamics. In particular, using nonlinear chaotic dynamics we can show how simple nonlinear difference equations can yield deterministic time paths that mimic the output of stochastic systems. A precise definition of chaos was given, then a specific example, the logistic equation, was used to show how chaotic dynamics can be generated. Loosely speaking, a chaotic function is one which maps some interval back into itself and generates an aperiodic time path that has sensitive dependence on initial conditions.

If chaotic structure can be shown to exist in actual asset market time series then the traditional explanations of market behavior will be called into question. For this to happen we need good tests of nonlinearity and of chaos. Using the technique of phase space embedding [Takens (1981)] we can rebuild the underlying attractor [if it exists] of the actual driving mechanism and test it for various qualities. Using this knowledge we can find the BDS statistic which provides a diagnostic test for the presence of nonlinear structure. However, nonlinearity is a necessary but not a sufficient condition of chaos so we need to go further and test for sensitive dependence on initial conditions. Positivity of the dominant Lyapunov exponent is used as a measure of sensitive dependence. Chapter 3 describes the derivation of both the BDS statistic and the dominant Lyapunov exponent estimate. When applying these empirical tests on actual time series, data quality is extremely important. In order for chaos to be found, the data has to be at least chaotic looking, with all the known dependencies filtered out. Chapter 4 explores these problems and analyzes the data chosen for this study.

In Chapter 5 various data filtering techniques are outlined. The augmented Dickey-Fuller test is used to check for stationarity. Then the Q-statistic and the ARCH test is used to check for serial correlation and for conditional heteroscedasticity, respectively. Logged differences are then fit to a GARCH [general autoregressive conditional heteroscedasticity] model which adjusts both serial correlation and conditional heteroscedasticity, again based on the Q-statistic and the ARCH test. Traditionally, researchers would stop here and assume that the remaining fluctuations are stochastic. However, for this study, we go further and test the residual fluctuations for hidden nonlinear dependencies. Results include evidence of nonlinearity in the British pound, the Canadian dollar, crude oil, copper, and the Japanese yen, based on the BDS test. Evidence of chaos was found in the Australian dollar, copper, and the Japanese yen, based on the Nychka et al. dominant Lyapunov exponent estimator. Contradictory evidence in the case of the Australian dollar indicates limited robustness across inference methods. The true source of price changes may be some combination of both random and chaotic elements. However, the amount of noise present has a strong negative impact on the results of nonlinear testing.

The presence of chaotic nonlinearities implies the existence of deterministic trading rules. However, the ability to take advantage of these rules depends on both the approximation of initial conditions, which are obscured by noise, and the degree of sensitive dependence on initial conditions.

Chaos theory applied to economics is still at the conception stage. Current research is primarily focused on whether chaos exists at all in economic data. Many studies have found evidence of nonlinearities and very recent studies have found evidence of chaotic nonlinearities. Many questions remain to be addressed, both theoretical and empirical.

Theoretical considerations suggest two directions of future research. One is to build financial models that encompass nonlinear chaotic solutions. The other direction is to analyze the dynamics when agents take advantage of short term deterministic trading rules. This is a case where the solution becomes exogenous to the system. As computer hardware and software improve we will be able to identify more precisely the nature of the nonlinearities. Empirical considerations would question the source of the nonlinearities and identify both the structure and the prediction horizon.

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