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Locational Spread Options with Stochastic Correlation

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Locational Spread Options with Stochastic Correlation

by

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Abstract

Contrary to the common assumption, correlation between financial derivatives may not be constant across time. This thesis analyses the role of stochastic correlation in modelling for locational spread options for natural gas. We first derive a model with Ornstein–Uhlenbeck process between two spread assets with constant correlation and then a combination of the Ornstein–Uhlenbeck and Jacobi process is used to model a stochastic correlation. The Margrabe formula is employed to evaluate options prices with constant correlation, the solution for which is used to compare with Monte Carlo simulations for stochasticity. Comparing the results, we find out why stochastic correlation is more important in real markets.

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To my family and my supervisor...

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"If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is." ...

- John von Neumann

Chapter 1

Introduction

Financial instruments, such as shares and options, are traded all over the world, across different platforms. Over the years, derivatives have captured a large share of investment totalling to $200.4 trillion, including commodity derivatives ($324.5 billion) in 2022, according to the US Office of the Comptroller of Currency. Spread options are one such form of derivatives, or performance-based financial contracts. As the name suggests, these derivative contracts are established based on the difference in price of two different commodities. This difference may arise from margins, quality or creditworthiness, time, and location. This study focuses on locational spread options. Locational spread options are options where the spread is calculated using the price difference of for example, natural gas between the two distinct points of delivery. These are among the most common derivatives used in the natural gas markets. In [27], the authors established an analytic approximation for the price of a spread option for a compound exchange option and its hedge ratio. The pay-off of a spread option depends on the price difference between the two correlated derivative products. Suppose F_1 and F_2 represent the future prices, then the payoff of a spread option is $P = max[\omega(F_1 - F_2), 0]$, where $\omega = 1$ for a call and $\omega = -1$ for a put [27]. This idea is developed further in the paper.

In the commodities market, locational spread options are based on the difference in prices of a commodity, energy commodities for this study, at two different geographical locations (Carmona and Durrleman, 2003). The energy sector is far more fragmented than financial markets and does not enjoy the liquidity present in other financial markets. In fact, a strong degree of mean-reversion is present in energy commodities, such as gasoline, petroleum, natural gas and electricity, where prices tend to converge to a long-term average price level. This reversion exists partly due to higher fixed costs, such as set-up and transmission, fluctuating prices due to economic conditions and seasonal changes that are associated with the sector. Clewlow and Strickland (2000), Carmona and Durrleman (2003), Eydeland and Wolyniec (2003), and Hull (2005) all portray evidence reflecting the mean-reversion of energy products [6].

As the demand for energy commodities soars throughout the world, and with the expected rise in energy prices due to political situations, there is a need for deeper investigation of energy-based derivatives, particularly locational spread options. Moreover, prices of one energy commodity often effect the demand and supply, and thus the price, of another commodity, or in this case, the commodity in another location.

It is therefore crucial to understand what role stochastic correlations play in determining the valuation of locational spread options. This thesis attempts to answer this by identifying the impact of stochastic correlation on the value of a locational spread option for natural gas commodities in Alberta.

Spreads contracts are established on the difference in prices between two commodities and can be divided into four basic groups:

- Margin or Refining Spreads
- Quality Spreads
- Calendar or Time Spread
- Basic or Locational Spreads

The methods used to model each type of spread resembles each other. [26] describes applications of locational spreads in Natural Gas Markets. The locational spread options are one of the most common derivatives used in the natural gas markets. It represents the price difference of natural gas between the two distinct points of delivery. Margrabe [44] considered the spread as an asset price and adopted the Black-Scholes formula to give the price of a spread option.

[27] describes the analytic approximations for spread options via assuming two underlying asset prices following correlated GBM with constant volatility and constant correlation. [29] discusses Monte Carlo methods to model gas swing options on spot prices of the underlying energy commodities. [30] elaborates a Monte Carlo valuation method, which is able to include real gas price dynamics and complex physical restraints. Here they especially extend the Least Squares Monte Carlo Method for American Options to storage valuation. [31] provides a generalized concept of stochastic correlation processes(SCP) as a hyperbolic transformation of the modified Ornstein-Uhlenbeck process. They then derive a transition density function for SCP in closed form which is later applied to historical data to calibrate SCP models.

Here, the question **What is the impact of stochastic correlation on the value of a locational spread option?** we are going to answer this in our thesis.

Correlation has been widely used to measure the dependence between different financial assets [31]. A constant correlation usually initiates a correlation risk and correlation ‘frowns’ instead of ‘smiles’ are important aspects in spread option markets. When determining the correlation between the two financial derivative, different methodologies are considered for constant correlation, subsequently moving towards stochastic correlation techniques. It is a well-established fact that the correlation between financial commodities is fundamentally reflected in the pricing and estimation of financial products.

For given two random variables X_1 and X_2 with finite variances, the correlation is specified as

$$\rho_{1,2} = Corr(X_1, X_2) = \frac{Cov(X_1, X_2)}{\sigma_1 \sigma_2}, \tag{1.1}$$

with covariance given by

$$Corr(X_1, X_2) = \mathbb{E}[(X_1 - \mu_1)(X_2 - \mu_2)], \tag{1.2}$$

where μ_i and σ_i are the expectation and standard deviation of X_i, $i = 1, 2$. Also, $\rho_{1,2}$ lies in the interval $[-1, 1]$. The boundaries -1 and 1 will be attained only if X_1 and X_2 are strictly linearly dependent. If the absolute value of $\rho_{1,2}$ is bigger so the dependence between X_1 and X_2 will be stronger. There are many drawbacks of the following correlation concept (1.1), a few of them are as follows:

- If the random variables X_1 and X_2 are independent, then it follows $\rho_{1,2} = 0$, but conversely the proposition does not hold, as in (1.1) they include only first two moments. Therefore, shows a correlation coefficient identifies only linear dependencies between random variables.

- The correlation of the random variables X_1 and X_2 does not equal to the correlation of the random variables $\ln X_1$ and $\ln X_2$, which means after a transformation of the financial data the correlation is changeable.

- Normally, the provided marginal distributions and pairwise correlations of a random vector cannot define its joint distribution.

- Finally, the variances of the two random variables X_1 and X_2 have to be finite.

For further details on the disadvantages consult [32]. Though the idea of correlation (1.1) to calculate dependencies has several limitations, it has been used broadly in financial commodities. The correlation like other quantities spot prices, volatility and exchange rates, etc, in financial markets, can't be observed directly but rather determined by means of a suitable model. [31] The simplest estimator of correlation is Pearson correlation coefficient for sample defined on paired data $(x_{1,i}, x_{2,i}), \cdots, (x_{n,i}, x_{n,i}), i = 1, 2, \cdots, N$ consisting of N pairs of X_1 and X_2 as follows:

$$\bar{\rho}_{1,2} = \frac{\sum_{i=1}^{N}(x_{1,i} - \bar{\mu}_1)(x_{2,i} - \bar{\mu}_2)}{\sqrt{\sum_{i=1}^{N}(x_{1,i} - \bar{\mu}_1)^2 \sum_{i=1}^{N}(x_{2,i} - \bar{\mu}_2)^2}}, \tag{1.3}$$

where μ_1 and μ_2 are the sample means of X_1 and X_2 [31].

One of the stochastic correlation processes was proposed by [33], incorporating a limitation on the parametric range to make sure that the boundaries -1 and 1 of the correlation process are unappealing and unachievable. A modified Jacobi process is proposed in modeling stochastic correlation [34] and a more generalized stochastic process was recommended by [35], which depends on the hyperbolic transformation with the hyperbolic tangent function of any mean-reverting process with positive and negative values.

During the analysis, it is seen that when the commodity prices are modeled as OU-process no closed-form solution is obtained for $K \neq 0$. We emphasize on $K = 0$ case, in which we use a closed-form solution for our mean-reverting model via Margrabe's formula. With our closed-form solution, by using the same constant correlation we can find an estimated solution through the Monte Carlo method and we compare them together.

Margrabe [44] calculated the exchange options by considering asset prices as geometric Brownian motion, under the risk-neutral measure, and treating one of the assets as numeraire. In reality, energy commodities are not liquid, thus their spot or future prices cannot act as a numeraire, but still risk-neutral approach is popular among them. [6] dealt with forwarding price curves and evaluates a class of two asset exchange options for energy commodities. It presented a model of spot prices using an affine two-factor mean-reverting process with and without jumps. It figured out closed-form results for spread options on the forward price process and provide a calibration procedure. [38] presented a new stochastic volatility model in which the squared volatility of the asset return follows a Jacobi process and covers the Heston model as a limit case. It derives closed-form series images for option prices with discounted payoffs defined as functions of the asset price trajectory at many finite points of time.

The Jacobi process [38], also known as Wright-Fisher diffusion, was initially applied to model gene frequencies ([62]; [63]). Later on, the Jacobi process was also utilized to model financial factors [64] such as interest rates, and study moment-based techniques for pricing bonds. In [64], bond prices accepted a series declaration with regard to Jacobi polynomials. Some additional properties of the Jacobi process can be found in [65] and [66]. The multivariate Jacobi process has been considered in [67] which proposed a smooth regime shifts model and illustrated a stochastic volatility model without leverage effect as an example. The Jacobi process has been also used to model stochastic correlation matrices in [68] and credit default swap indexes in [69].

[39] provides a closed-form approximation by assuming stochastic correlation and constant volatility to measure an error for the price of several two-dimensional derivatives. Then this technique was applied to the three models of stochastic correlation while pricing the Spread Option and Quantos Options. This paper [40] describes stochastic correlation processes for modeling the credit spread. Firstly, it shows the modeling of components of spread option as correlated Ornstein-Uhlenbeck processes and stochastic correlation as Jacobi process. Secondly, uses the properties of the Jacobi process to obtain analytical solutions for credit spread options. Lastly, introduced the time change Jacobi process for correlation series and its comparison with the Jacobi process. This is one of the main papers which pro-

vokes me to work in a similar direction but with locational spread options as correlated Ornstein-Uhlenbeck processes.

In this thesis, the central focus is to model the locational spread options and price the locational spread options with the stochastic correlation process. This thesis emphasizes the stochastic correlation developed on Natural gas future prices in different regions. To model this locational spread option, we took the mean-reversion into consideration and believed that natural gas prices, return to their mean levels during the time. In order to pursue this idea, we model the elements of the locational spread process as correlated OU-processes and stochastic correlation as Jacobi process which assures the correlation between main variables is a bounded process. Then using the change of variables for the Jacobi process, we are able to derive the formation required for computing the locational spread option pricing via the Monte Carlo method.

The structure of this thesis is as follows: In chapter 2, the basic concepts and definitions of stochastic analysis are presented, which are crucial to develop our mathematical model, i.e. the preliminaries. In chapter 3, the Energy markets, energy commodities, and derivatives are discussed at length, followed by an analysis of the natural gas markets, both global and Albertan. In chapter 4, our basic mathematical model and spread options are defined. Chapter 5 follows with a consideration of the calibration model from the perspective of constant correlation and calculations of the unknown parameters using two different methods, maximum log-likelihood and multivariate linear regression. the stochastic correlation as a Jacobi process is detailed in Chapter 6, to figure out its value as well as discuss an optimization model by reducing the number of unknown parameters. Penultimately, Chapter 7 elaborates the pricing of locational spread options by Margrabe's model and Monte Carlo simulations. Finally, the ideas presented in this thesis are concluded in Chapter 8, along with ideas for future development.

Chapter 2

Fundamentals of Stochastic Processes

In this chapter, the basic concepts are explained including some definitions and theorems relating to stochastic processes. It is essential that these definitions and theorems are set up as a precursor to the model, for better understanding. These are the initial requirements of stochastic calculus in mathematical finance. This chapter defines the measurable spaces, the stochastic process then the concept of Ito Integral from one-dimensional to Multidimensional. Later in the chapter, the Geometric Brownian Motion, Ornstein-Uhlenbeck process, and Exponential Ornstein-Uhlenbeck process are explained, along with discussions regarding Risk-Neutral measures and their properties. All materials related to the basic stochastic process are taken from the textbook [1].

Definition 2.1. If Ω is a given set, then a σ-algebra $\mathscr{F}$ on Ω is a family $\mathscr{F}$ of subsets of Ω with the following properties:

(i) $\varnothing \in \mathscr{F}$.

(ii) F $\in \mathscr{F} \Rightarrow F^C \in \mathscr{F}$, where $F^C = \Omega \setminus F$ is the complement of F in Ω.

(iii) $A_1, A_2, ... \in \mathscr{F} \Rightarrow A := \bigcup_{i=1}^{\infty} \in \mathscr{F}$.

The pair $(\Omega, \mathscr{F})$ is called a ***measurable space***.

Definition 2.2. A **probability measure** P on a *measurable space* $(\Omega, \mathscr{F})$ is a function $P : \mathscr{F} \to [0,1]$ such that

1. $P(\varnothing) = 0, \quad P(\Omega) = 1.$

2. If $A_1, A_2, ... \in \mathscr{F}$ and $\{A_i\}_{i=1}^{\infty}$is disjoint (i.e. $A_i \bigcap A_j = \phi$ if $i \neq j$) then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i). \tag{2.1}$$

The triple $(\Omega, \mathscr{F}, P)$ is called a *probability space*.

Definition 2.3. A *random variable* X is an $\mathscr{F}$-measurable function $X : \Omega \to \mathbf{R}^n$. Every random variable induces a probability measure μ_X on $\mathbf{R}$, defined by

$$\mu_X(B) = P(X^{-1}(B)), \tag{2.2}$$

where μ_X is called the ***distribution*** of X.

Definition 2.4. If $\int_\Omega |X(\omega)|\, dP(\omega) < \infty$ then the number

$$E[X] := \int_\Omega X(\omega) dP(\omega) = \int_{\mathbf{R}^n} x d\mu_X(x) \tag{2.3}$$

is called the ***expectation***of X (w.r.t. P).

Definition 2.5. A **stochastic process** is a *parametrized collection of random variables*

$$\{X_t\}_{t \in T} \tag{2.4}$$

defined on a probability space $(\Omega, \mathscr{F}, P)$ and assuming values in $\mathbf{R}^n$. Note that for each $t \in T$ fixed we have a random variable $\omega \to X_t(\omega)$; $\omega \in \Omega$. On the other hand, fixing $\omega \in \Omega$ we can consider the function $t \to X_t(\omega)$; $t \in T$ which is called a *path* of X_t.

Definition 2.6. Let $f \in \mathscr{V}(S,T)$. The **Itô integral** of f (from S to T) is defined by

$$\int_S^T f(t,\omega) dB_t(\omega) = \lim_{n \to \infty} \int_S^T \phi_n(t,\omega) dB_t(\omega), \tag{2.5}$$

where B_t is one-dimensional Brownian motion and ϕ_n is a sequence of elementary functions such that

$$E\left[\int_S^T (f(t,\omega) - \phi_n(t,\omega))^2 dt\right] \to 0 \quad as \quad n \to \infty. \tag{2.6}$$

Definition 2.7. Let B_t be one-dimensional Brownian motion on $(\Omega, \mathscr{F}, P)$. A (1-dimensional) **Itô process (or stochastic integral)** is a *stochastic process* X_t on $(\Omega, \mathscr{F}, P)$ of the form

$$X_t = X_0 + \int_0^t u(s,\omega) ds + \int_0^t v(s,\omega) dB_s. \tag{2.7}$$

If X_t is an Itô process of the form (2.7) then its sometimes written in the shorter differential form as follows:

$$dX_t = udt + vdB_t. \tag{2.8}$$

Theorem 2.8. *Let X_t be an Itô process given by (2.7). Let $g(t,x) \in C^2([0,\infty)\times\mathbf{R})$ (i.e. g is twice continuously differentiable on $([0,\infty)\times\mathbf{R})$. Then*

$$Y_t = g(t,X_t) \tag{2.9}$$

is again an Itô process, and

$$dY_t = \frac{\partial g}{\partial t}(t,X_t)dt + \frac{\partial g}{\partial x}(t,X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,X_t)\cdot(dX_t)^2, \tag{2.10}$$

where $(dX_t)^2 = (dX_t).(dX_t)$ is computed according to the rules

$$dt\cdot dt = 0, \quad dt\cdot dB_t = 0 = dB_t\cdot dt, \quad dB_t\cdot dB_t = dt. \tag{2.11}$$

Definition 2.9. Multidimensional Itô Formula: Let $B_1(t), B_2(t), \cdots, B_m(t)$ be m independent Brownian motions. Consider n Itô processes $X_t^{(1)}, X_t^{(2)}, \cdots, X_t^{(n)}$ given by

$$X_t^{(i)} = X_a^{(i)} + \sum_{j=1}^{m}\int_a^t f_{ij}(s)dB_j(s) + \int_a^t g_i(s)ds, \quad 1\le i\le n, \tag{2.12}$$

where $f_{ij} \in \mathscr{L}_{ad}(\Omega, L^2[a,b])$ and $g_i \in \mathscr{L}_{ad}(\Omega, L^1[a,b])$ for all $1\le i\le n$ and $1\le j\le m$.

If we introduce the matrices

$$B(t) = \begin{bmatrix} B_1(t) \\ . \\ . \\ . \\ B_m(t)\end{bmatrix}, X_t = \begin{bmatrix} X_t^{(1)} \\ . \\ . \\ . \\ X_t^{(n)}\end{bmatrix}, f(t) = \begin{bmatrix} f_{11}(t) & . & . & . & f_{1m}(t) \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ f_{n1}(t) & . & . & . & f_{nm}(t)\end{bmatrix}, g(t) = \begin{bmatrix} g_1(t) \\ . \\ . \\ . \\ g_n^(t), \end{bmatrix}, \tag{2.13}$$

then (2.12) can be written as a matrix equation:

$$X_t = X_a + \int_a^t f(s)dB(s) + \int_a^t g(s)d(s), \quad a\le t\le b. \tag{2.14}$$

Or we have

$$d\mathbf{X}(\mathbf{t}) = \mathbf{g}(\mathbf{t})dt + \mathbf{f}(\mathbf{t})d\mathbf{B}(\mathbf{t}), \tag{2.15}$$

where $X_t, g(t), f(t)$ and $B(t)$ are defined in (2.13). Such an $X(t)$ process is called an **n-dimensional Itô** process (or just an **Itô** process).

We extend Itô's formula in the following theorems to the multidimensional case, from the book [2].

Theorem 2.10. *Let (2.15) be an n-dimensional Itô process. Let $g(t,x) = (g_1(t,x),...,g_p(t,x))$ be a C^2 map from $[0,\infty)\times\mathbf{R}^n$ into $\mathbf{R}^p$. Then the process*

$$Y(t,\omega) = g(t,X(t)) \tag{2.16}$$

is again an Itô process, whose component number k, Y_k, is given by

$$dY_k = \frac{\partial g_k}{\partial t}(t,X)dt + \sum_i \frac{\partial g_k}{\partial x_i}(t,X)dX_i + \frac{1}{2}\sum_{i,j}\frac{\partial^2 g_k}{\partial x_i \partial x_j}(t,X)dX_i dX_j, \tag{2.17}$$

where $dB_i dB_j = \delta_{ij}dt$, $dB_i dt = dt dB_i = 0$.

Theorem 2.11. *Let X_t be an Itô process given by*

$$X_t = X_a + \int_a^t f(s)dB(s) + \int_a^t g(s)ds, \quad a \le t \le b. \tag{2.18}$$

Suppose $\theta(t,x)$ is a continuous function with continuous partial derivatives $\frac{\partial\theta}{\partial t}$, $\frac{\partial\theta}{\partial x}$, and $\frac{\partial^2\theta}{\partial x^2}$. Then $\theta(t,X_t)$ is also an Itô process and

$$\begin{aligned}\theta(t,X_t) =& \theta(a,X_a) + \int_a^t \frac{\partial\theta}{\partial x}(s,X_s)f(s)dB(s) \\ &+ \int_a^t \left[\frac{\partial\theta}{\partial t}(s,X_s) + \frac{\partial\theta}{\partial x}(s,X_s)g(s) + \frac{1}{2}\frac{\partial^2\theta}{\partial x^2}(s,X_s)f(s)^2\right]ds.\end{aligned} \tag{2.19}$$

2.1 Other Definitions

In this section, we'll discuss Brownian Motion, the OU-processes, and the Exponential OU processes.

Definition 2.12. A stochastic process $B(t,\omega)$ is called a **Brownian motion** if it satisfies the following conditions:

1. $P\{\omega; B(0,\omega) = 0\} = 1$.

2. For any $0 \le s < t$, the random variable $B(t) - B(S)$ is normally distributed with mean 0 and variance $t - s$, i.e., for any $a < b$,

$$P\{a \le B(t) - B(s) \le b\} = \frac{1}{\sqrt{2\pi(t-s)}}\int_a^b e^{-x^2/2(t-s)}dx. \tag{2.20}$$

3. $B(t,\omega)$ has independent increments, i.e., for any $0 \le< t_1 < t_2 < \cdots < t_n$, the random variables

$$B(t_1), B(t_2) - B(t_1), \cdots, B(t_n) - B(t_{n-1}), \tag{2.21}$$

are independent.

4. Almost all sample paths of $B(t,\omega)$ are continuous functions, i.e.,

$$P\{\omega; B(\cdot,\omega)\} = 1. \tag{2.22}$$

where $B(\cdot,\omega)$ is continuous.

2.1.1 Simple Properties of Brownian Motion

Let $B(t)$ be a fixed Brownian motion. Some simple properties are sated below that follow directly from the definition of Brownian motion.

1. For any $t > 0$, $B(t)$ is normally distributed with mean 0 and variance t. For any $s,t \geq 0$, we have $E[B(s)B(t)] = \min(s,t)$.

2. (**Translation Invariance**) For fixed $t_0 \geq 0$, the stochastic process $\widetilde{B}(t) = B(t+t_0) - B(t_0)$ is also a Brownian motion.

3. (**Scaling Invariance**) For any real number $\lambda > 0$, the stochastic process $\widetilde{B}(t) = B(\lambda t/\sqrt{\lambda})$ is also a Brownian motion.

Definition 2.13. Ornstein-Uhlenbeck Process: The stochastic procedure that satisfies the following differential equation:

$$dX_t = \kappa(\theta - X_t)dt + \sigma dB_t, \tag{2.23}$$

where dB_t is a standard Brownian motion and $\kappa > 0, \theta, \sigma > 0$ are the rate of mean reversion, long-term mean and the volatility constants respectively. The equilibrium attraction and repulsion are determined by the sign of the rate of mean reversion κ.

2.1.2 Solution of the Ornstein-Uhlenbeck Process

The solution of the SDE (2.23) can be establish by applying Itô 's lemma to the $f(t,x) = e^{\kappa t}X(t)$ as follows in [3]:

$$\frac{\partial f}{\partial t} = \kappa e^{\kappa t}X(t), \quad \frac{\partial f}{\partial X} = \kappa e^{\kappa t}, \quad \frac{\partial^2 f}{\partial X^2} = 0. \tag{2.24}$$

Hence,

$$\begin{aligned} d\left(e^{\kappa t}X(t)\right) &= \kappa e^{\kappa t}X(t)dt + e^{\kappa t}dX(t) \\ &= \left(\kappa e^{\kappa t}X(t) + e^{\kappa t}\kappa(\theta - X(t)\right) + e^{\kappa t}\sigma dB(t). \end{aligned} \tag{2.25}$$

Integrating gives us

$$e^{\kappa t}X(t) - X(0) = \kappa\theta \int_0^t e^{\kappa s} ds + \sigma \int_0^t e^{\kappa s} dB(s), \tag{2.26}$$

which directs to the following with initial condition $X(0) = x(0)$:

$$x(t) = e^{-\kappa t}x(0) + \theta\left(1 - e^{\kappa t}\right) + \sigma \int_0^t e^{\kappa(s-t)} dW(s), \tag{2.27}$$

where *mean* and *variance* of $X(t)$ is provided by:

$$\mathbf{E}\left[X(t)\right] = e^{-\kappa t}x(0) + \theta(1 - e^{-\kappa t}), \tag{2.28}$$

$$\mathbf{Var}\left[X(t)\right] = \sigma^2 \int_0^t e^{2\kappa(s-t)} ds = \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t}). \tag{2.29}$$

Definition 2.14. [3] We can say that the process $S(t) = \exp X(t), t \geq 0$ or $X(t) = \ln S(t)$ is an **Exponential Ornstein-Uhlenbeck** process if $S(t)$ satisfies the the stochastic differential equation (SDE) given by

$$d(\ln S(t)) = \kappa(\theta^* - \ln S(t)) + \sigma dB(t). \tag{2.30}$$

where $dB(t)$ is a standard Brownian motion and $\kappa > 0, \theta^*, \sigma > 0$ represents the rate of mean reversion, long-term mean and the volatility constants respectively.

2.1.3 Solution of Exponential Ornstein-Uhlenbeck Process

Using Itô's lemma on $S(t) = \exp X(t)$ as follows:

$$\begin{aligned} dS(t) &= e^{X_t} dX(t) + \frac{1}{2} e^{X(t)} (dX(t))^2 \\ &= S(t)\left[\kappa(\theta^* - X(t))dt + \sigma dB(t) + \frac{1}{2}\sigma^2 dt\right], \end{aligned} \tag{2.31}$$

whereas $X(t) = \ln S(t)$, thus we have the following:

$$\frac{dS(t)}{S(t)} = \kappa\left(\theta^* - \frac{\sigma^2}{2\kappa}\right)dt - \kappa \ln S(t)dt + \sigma dB(t), \tag{2.32}$$

considering $(\theta = \theta^* - \frac{\sigma^2}{2\kappa})$ and multiplying on both sides by the integrating factor $I = e^{-\kappa t}$, then integrating the results, is:

$$S(t) = \exp\left[e^{-\kappa T}\ln S(0) + \theta(1 - e^{-\kappa t}) + \sigma\sqrt{\frac{(1 - e^{-2\kappa t})}{2\kappa}} z\right], \tag{2.33}$$

where $S(0)$ is the initial condition and $z \sim \mathbf{N}(0,1)$. With mean and variance as follows:

$$\hat{\mu} = e^{-\kappa t} \ln S_0 + \theta \left(1 - e^{-\kappa t}\right), \quad \hat{\sigma^2} = \sigma^2 \frac{\left(1 - e^{-2\kappa t}\right)}{2\kappa}. \tag{2.34}$$

In (2.34) notice how the mean is basically a weighted average between the initial value $\ln S_0$ and the long-term mean θ, whereas the variance increases as time increases and approaches $\sigma^2/2\kappa$ as the time approaches to infinity [1].

2.2 Basic Aspects of Risk-Neutral Measures

It is important to outline some basic knowledge related to risk-neutral measures that we require for our model.

Theorem 2.15. *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and let Z be an almost surely non-negative random variables with $\mathbb{E}Z = 1$. For $A \in \mathscr{F}$, define*

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega), \tag{2.35}$$

*Then $\widetilde{\mathbb{P}}$ is a **probability measure**. Furthermore, if X is a non-negative random variable, then*

$$\widetilde{\mathbb{E}}X = \mathbb{E}[XZ], \tag{2.36}$$

given Z is almost surely strictly positive, we also have

$$\widetilde{\mathbb{E}}Y = \widetilde{\mathbb{E}}\left[\frac{Y}{Z}\right], \tag{2.37}$$

for every non-negative random variable Y. The $\widetilde{\mathbb{E}}$ appearing in (2.36) is expectation under the probability measure $\widetilde{\mathbb{P}}$: i.e., $\widetilde{\mathbb{E}}X = \int_\Omega X(\omega) d\widetilde{\mathbb{P}}(\omega)$.

2.2.1 Girsanov's Theory for a Single Brownian Motion

In Theorem 2.15, the starting point is a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a non-negative random variable Z that satisfies the condition $\mathbb{E}Z = 1$. A new probability measure $\widetilde{\mathbb{P}}$ is then defined using the formula,

$$\widetilde{\mathbb{P}}A = \int_A Z(\omega) dP(\omega) \quad \forall \quad A \in \mathscr{F}. \tag{2.38}$$

Any random variable X now has two expectations, one under the original probability measure $\mathbb{P}$, which we denote with $\mathbb{E}X$, and the other under the new probability measure $\widetilde{\mathbb{P}}$, which we denote with $\widetilde{\mathbb{E}}X$. The relationship between these two can be stated as:

$$\widetilde{\mathbb{E}}X = \mathbb{E}[XZ], \tag{2.39}$$

if $\mathbb{P}\{Z > 0\} = 1$, then $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ agree which sets have a probability of zero and (2.36) has the companion formula in (2.37). We say Z is the *Radon-Nikodým derivative* of $\widetilde{\mathbb{P}}$ with respect to $\mathbb{P}$, written as:

$$Z = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}. \tag{2.40}$$

This serves as a reminder that Z is like a ratio of these two probability measures. In the case of a finite probability model, we actually have:

$$Z(\omega) = \frac{\widetilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)}, \tag{2.41}$$

where multiplying both sides of (2.41) by $\mathbb{P}(\omega)$ and summing it over ω in a set A, returns:

$$\widetilde{\mathbb{P}}(A) = \sum_{\omega \in A} Z(\omega)P(\omega) \quad for \quad all \quad A \subset \Omega. \tag{2.42}$$

In a general probability model, we cannot write (2.41) because $(\mathbb{P}, \omega)$ is typically zero for each individual ω, which leads to an analogue form of (2.42). This analogue form essentially is (2.38).
In particular, if X is a standard normal random variable on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, θ is a constant, and defined as follows:

$$Z = \exp\left\{-\theta X - \frac{1}{2}\theta^2\right\}, \tag{2.43}$$

where, under the probability measure $\mathbb{P}$ given by (2.38), the random variable $Y = X + \theta$ is standard normal. This means $\widetilde{\mathbb{E}}Y = 0$, whereas $\mathbb{E}Y = \mathbb{E}X + \theta = \theta$. By changing the probability measure, the expectation of Y also changes.

Theorem 2.16. *Let $W(t)$, $0 \le t \le T$, be a Brownian motion on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and let $\mathscr{F}, 0 \le t \le T$, be a filtration for this Brownian motion. Let $\Theta(t), 0 \le t \le T$, be an adapted process. Define*

$$Z(t) = \exp\left\{-\int_0^t \Theta(u)dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)du\right\}, \tag{2.44}$$

with

$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(u)du, \tag{2.45}$$

and assume that

$$\mathbb{E}\int_0^T \Theta^2(u)Z^2(u)du < \infty. \tag{2.46}$$

Set $Z = Z(T)$. Then $\mathbb{E}Z = 1$ and under the probability measure $\widetilde{\mathbb{P}}$ given by (2.38), the process $\widetilde{W}, 0 \le t \le T$, is a Brownian Motion.

The probability measures $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ in Girsanov 's Theorem are equivalent i.e., they agree about which sets have a probability of zero and hence about which sets have a probability of one. This is because $\mathbb{P}\{Z > 0\} = 1$.

2.2.2 Stocks Under the Risk-Neutral Measure

Let $W(t), 0 \leq t \leq T$, be a Brownian motion on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and let $\mathscr{F}(t), 0 \leq t \leq T$, be a filtration for this Brownian motion where T is a fixed final time. A differential stock price process can be written as:

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t), \quad 0 \leq t \leq T, \tag{2.47}$$

The mean rate of return $\alpha(t)$ and the volatility $\sigma(t)$ are allowed to be adapted processes. Assuming that, for all $t \in [0,T], \sigma(t)$ is almost surely not zero, the stock price, as a generalized geometric Brownian motion and an equivalent way of writing (2.47) is:

$$S(t) = S(0)\exp\left\{\int_0^t \sigma(s)dW(s) + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s)ds\right) ds\right\}. \tag{2.48}$$

In addition, suppose we have an adapted interest rate process $R(t)$. The discount process is defined as:

$$D(t) = e^{-\int_0^t R(s)ds} \tag{2.49}$$

and note that:

$$dD(t) = -R(t)D(t)dt. \tag{2.50}$$

To obtain (2.50) from (2.49), we can define $I(t) = \int_0^t R(s)ds$ such that $dI(t) = R(t)dt$ and $dI(t)dI(t) = 0$. The function $f(x) = e^{-x}$ is introduced, for which $f'(x) = -f(x), f"(x) = f(x)$, thereafter the Itô's formula is applied:

$$\begin{aligned} dD(t) &= df(I(t)) \\ &= f'(I(t))dI(t) + \frac{1}{2}f''(I(t))dI(t)dI(t) \\ &= -f(I(t))R(t)dt \\ &= -R(t)D(t)dt. \end{aligned} \tag{2.51}$$

The discounted stock price thus becomes:

$$D(t)S(t) = S(0)\exp\left\{\int_0^t \sigma(s)dW(s) + \int_0^t \left(\alpha(s) - R(s) - \frac{1}{2}\sigma^2(s)\right) ds\right\} \tag{2.52}$$

With the following differential equation:

$$\begin{aligned} d(D(t)S(t)) &= (\alpha(t) - R(t))D(t)S(t)dt + \sigma(t)D(t)S(t)dW(t) \\ &= \sigma(s)D(t)S(t)\left[\Theta(t)dt + dW(t)\right], \end{aligned} \tag{2.53}$$

where *market price of risk* is denoted as:

$$\Theta(t) = \frac{\alpha(t) - R(t)}{\sigma(t)}. \tag{2.54}$$

One can derive (2.53) either by applying the Itô formula to the right-hand side of (2.52). The volatility of the discounted stock price is equal to the volatility of the non-discounted stock price.

The probability measure $\widetilde{\mathbb{P}}$ as defined earlier in Girsanov's Theorem, 2.16, included, using the market price of risk $\Theta(t)$ presented in (2.54). Rearranging the formula in terms of the Brownian motion $W(t)$ of that theorem, we may rewrite (2.53) as

$$d\left(D(t)S(t)\right) = \sigma(t)D(t)S(t)d\widetilde{W}(t) \tag{2.55}$$

. $\widetilde{\mathbb{P}}$, the measure defined in Girsanov's Theorem, the *risk-neutral measure* because it is equivalent to the original measure $\mathbb{P}$ and it renders the discounted stock price $D(t)S(t)$ into a martingale. Infact, according to (2.55):

$$D(t)S(t) = S(0) + \int_0^t \sigma(u)D(u)S(u)d\widetilde{W}(u), \tag{2.56}$$

and under $\widetilde{\mathbb{P}}$ the process

$$\int_0^t \sigma(u)D(u)S(u)d\widetilde{W}(t), \tag{2.57}$$

is an Itô integral and hence, considered a martingale.
The undiscounted stock price $S(t)$ under $\widetilde{\mathbb{P}}$ has a mean rate of return equal to the interest rate, which is verifiable with the following replacement:

$$dW(t) = -\Theta(t)dt + d\widetilde{W}(t), \tag{2.58}$$

thus (2.47) becomes:

$$dS(t) = R(t)S(t)dt + \sigma(t)S(t)d\widetilde{W}(t). \tag{2.59}$$

Solving this equation for $S(t)$ or simply replace the Itô integral $\int_0^t \sigma(s)dW(s)$ by its equivalent $\int_0^t \sigma(s)d\widetilde{W}(s) - \int_0^t (\alpha(s) - R(s))\,ds$ in (2.48) the undermentioned:

$$S(t) = S(0)\exp\left\{\int_0^t \sigma(s)d\widetilde{W}(s) + \int_0^t \left(R(s) - \frac{1}{2}\sigma^2(s)\right)ds\right\}. \tag{2.60}$$

2.2.3 Existence of the Risk-Neutral Measure

A probability measure $\widetilde{\mathbb{P}}$ is said to be risk-neutral if

(i) $\widetilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent (i.e., for every $A \in \mathscr{F}, \mathbb{P}(A) = 0$ if and only if $\widetilde{\mathbb{P}}(A) = 0$), and

(ii) under $\widetilde{\mathbb{P}}$, the discounted stock price $D(t)S_i(t)$ is a martingale for every $i = 1, \cdots, m$.

Theorem 2.17. ***Risk Neutral Pricing Formula**: Let $\mathbb{V}(T)$ be a $\mathscr{F}_T$-a measurable random variable that represents the payoff of derivative security, and let $\widetilde{\mathbb{P}} = \widetilde{\mathbb{P}}_T$ be the risk-neutral measure above. The arbitrage-free price at time t of derivative security with payoff $\mathbb{V}(T)$ and maturity T is given as:*

$$\mathbb{V}(t) = \widetilde{\mathbb{E}}\left(\exp\left(-\int_t^T R(s)ds\right)\mathbb{V}(T)|\mathscr{F}_t\right). \tag{2.61}$$

2.2.4 European Call Option under Risk-Neutral Measure

Let S(t) be the price of an underlying asset, which is not necessarily a geometric Brownian motion, i.e., volatility may or may not be constant. With $S_0 = x$, the risk-neutral pricing formula for the price at $t = 0$ of a European call on this asset, paying $(S(T) - K)^+$ at time T is:

$$c(0, T, x, K) = \widetilde{\mathbb{E}}\left[e^{-rT}\left(S(T) - K\right)^+\right]. \tag{2.62}$$

These theorems are later employed in the paper to develop the main model.

Chapter 3

Energy Markets and Natural Gas

Commodity markets revolve around the trading for raw materials or primary economic output. These can include agricultural produce such as wheat and corn, natural resources including crude oil, and precious metals such as gold and silver. This study revolves around the energy sector commodities, specifically natural gas. This chapter examines the features of energy markets, various commodities within the energy sector and the financial derivatives associated with these commodities. The role of natural gas in energy markets is also defined, as the model focuses on this natural resource.

3.1 Energy Markets

Energy markets help in maintain our daily lives and economy, as energy is one of the most widely consumed commodities in any economy. Energy commodity markets still have their differences compared to the traditional commodity markets in the financial sector. For instance, they are far less liquid than other markets, due to the nature of fixed costs. All forms of energy require heavy investment, such as equipment, transmission, transportation, and storage costs. Certain commodities may be kept in store for years while others, such as electricity are very difficult to store or even impossible to store. Moreover, prices behave differently for each type of commodity, with different variables affecting the price. While weather conditions may affect renewable energy supply positively, they may end up having a very different or no impact on crude oil or natural gas. Commodity prices in the energy sector sometimes show strong mean reverting trends as well [6].

Energy markets all over the world are controlled by the relevant regulatory authority or government. In the early 1990s, deregulation began in energy markets, particularly for electricity and natural gas, allowing prices to be determined by the free market forces of demand and supply. This deregulation paved the way for spot and futures trading in energy commodities and their derivatives all across the world. The new laws allowed brought greater flexibility for manufacturers and consumers involved in the energy markets, to choose the risk level [4]. A significant advantage of the deregulated energy markets was that this permitted the trading of energy commodities, even after the production process had been initiated [5].

3.2 Energy Commodities

Energy commodities comprise a variety of non-renewable and renewable resources including crude oil, natural gas, coal, landfill gas, bio-gas, wind, solar, and hydro-power derived products. Energy commodities are critical to human survival and draw the attention of investors, looking to earn profit from the growing demand for energy. As a result, financial derivatives of energy commodities are widely traded globally [7].

Energy commodities are considered primary inputs in industrial applications. The tariffs on energy commodities are closely monitored by economists, investors, producers, and governmental authorities [8]. A correlation is observed between economic development and energy expenses in an economy, perhaps, because industrial output and production are heavily dependent on various forms of energy. Consequently, changes in the global economy have an impact on energy prices, whether it be economic conditions, such as a global recession, or regional wars, such as the Ukraine-Russia conflict.

In addition to production, transport industries are heavily reliant on these resources, and changes in energy prices can translate into other goods in the economy. This widespread impact is one reason why investors and governments may want to hedge against hikes in fuel prices.

Financial derivatives have become increasingly popular in the 21st century. Since the global financial crisis in 2007-08, diversification of investment and financial assets has been significantly emphasized by investment banks and hedge funds. Commodities and commodity derivatives provide one such channel for diversification. Even the popularity of crypto-currencies, which consume large quantities of electricity for mining, has increased the importance of energy derivatives. Such financial instruments have become essentials for diversifying investments and growing portfolios– either for the long term or to hold cash under more volatile or bearish markets, where stocks might not be a fruitful investment.

3.3 Energy Derivatives

An energy derivative is a derivatives contract where the underlying asset is an energy commodity. This commodity could be a natural resource, such as crude oil or natural gas, or an energy product such as electricity. The financial instruments derived from such commodities include options, futures, forwards, and swap agreements. Energy derivatives have become tools for investors and companies to hedge against fluctuations in energy prices. Derivative markets have expanded in the past few years, converting the energy commodity markets into a vast system of derivatives contracts. These derivatives can be traded either on a stock exchange or over the counter (OTC), with prices varying based on the asset value and market conditions. The exchanges, alongside energy products commonly traded on the exchanges, are presented in Table 3.1, by differing energy products [9].

Exchange	Products
New York Mercantile Exchange (NYMEX)	Coal, Crude Oil, Electricity, Natural Gas, Refined Products (www.cmegroup.com/trading/energy/)
Chicago Board of Trade (CBOT)	Ethanol (www.cmegroup.com/trading/energy/)
Intercontinental Exchange (ICE)	Coal, Crude Oil, Electricity, Emissions, LNG, Natural Gas, Refined Products (https://www.theice.com/products.jhtml)
NASDAQ OMX Commodities	Carbon, Power, Natural Gas (www.nasdaqomx.com/commodities/markets/products/)
International Commodity Exchanges (complete list)	www.commodityonline.com/commodityexchanges / global-futures-trading-exchangesand-website-address/

Table 3.1: Commodity Exchanges

3.4 Natural Gas Markets

Natural gas is a conventional fuel with a relatively low-pollution composition. The base component of both Liquid Natural gas (LNG) and Compressed Natural gas (CNG) is methane and includes lower quantities of propane, butane, and pentane. During the 18th century and at the beginning of the 19th century, natural gas was primarily utilized for lighting streets and buildings. Even today, its widespread use for heating and producing energy supplies has made the commodity indispensable. Natural gas is utilized by domestic and industrial consumers for space cooling and heating as well as to transformers for power generation. It has gradually overtaken oil and coal in the energy sector, due to lower emissions of pollutants. Gas is also used as a fuel in steam powered systems in heavy oil production and fleet vehicles.

Most natural gas contracts are over-the-counter(OTC) transactions, where the trade takes place directly between the producer and buyer, without the intervention or regulation of an exchange [10]. Natural gas is obtained after various stages of production: evaluation and extraction, processing, transportation, storage, regional deliveries, and conversion to LNG and CNG for transportation. Before the 1990s, this was a straightforward business where evaluation and production groups searched for natural gas, extracted it and handed it over to transportation companies to provide the resource to local utility companies. These companies would then circulate the gas to local consumers. It was often converted to LNG for ease of transportation. Prices were regulated by government bodies, such as the Federal Energy

Regulatory Commission (FERC), and there was little to no competition in the market. In the early 1990s, the industry was deregulated, and in 1992, the FERC authorized pipeline systems for transporting natural gas. This allowed for greater competition and the market forces of demand and supply started to have an impact on the price.

Today, demand and supply have a much greater impact on prices. As demand increases, prices rise, signalling manufacturers to increase their production for greater profits. Contrarily, as demand shrinks, manufacturers respond by scaling down production as prices fall. This mean-reversion is an essential feature of natural gas prices, as discussed in later chapters.

3.5 Alberta's Energy Sector

Alberta has been supplying oil and natural gas to other parts of Canada and the US since the 1940s. It is one of the biggest producers of natural resources such as conventional crude oil, synthetic crude, natural gas, and gas products in Canada. It is also the world's second largest exporter of natural gas and the fourth leading manufacturer [11]. In 2018, Alberta's energy sector generated a surplus of $71.5 billion from household commodities. According to Statistics Canada, oil and gas retrieving companies managed to capture the greatest share in Canada's GDP since 1985, exceeding 7%. It even surpassed the banking and insurance sectors. This was because the companies were able to extract unconventional oil from the oil sands and achieving a record-breaking high in production in May 2018.

Two of North America's core producers of petrochemicals are situated in central and northern central parts of Alberta. By-products, such as polyethylene and vinyl, manufactured in Red Deer and Edmonton, are transported around the world. The Athabasca River territory produced oil that was used within Canada and exported, and holds the biggest approved assets of oil in the world, with the exception of Saudi Arabia. Figure 3.1 illustrates the breakdown of natural resources in Alberta [18].

3.5.1 Natural Gas in Alberta

There are numerous natural gas sites in Alberta. A significant proportion of natural gas reserves was found in 1883 near Medicine Hat. In 1999, the production of natural gas liquids such as ethane, propane, and butane reached 172.8 million barrels, valued at $2.27 billion. In 2018, Alberta was producing 69% of the marketable natural gas in all of Canada [13], and 49% of Alberta's natural gas production was used in Alberta itself [14]. Today, Alberta also delivered nearly 13% of total natural gas used in the US. Note that the AECO "C" spot price is Alberta gas-trading price is one of foremost North America's benchmarks [12]. It has one of the most massive natural gas systems in the world among its energy infrastructure, along 39,000 kilometers (24,000 miles) of energy-related pipelines [17]. Figure 3.2 [18] illustrates the main pipelines in Alberta for natural gas.

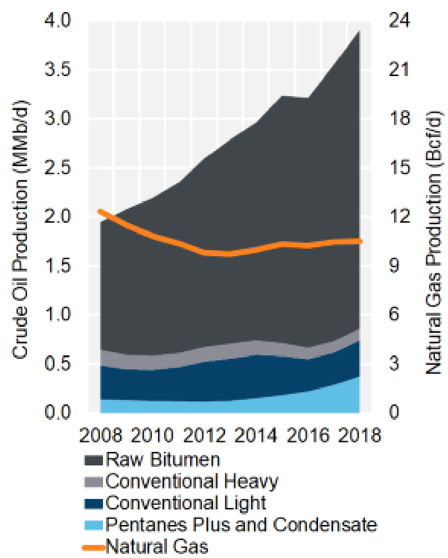


Figure 3.1: Energy Production in Alberta

The average annual consumption of natural gas in a household in Alberta is 135 GJ (38,000 kWh) [15]. It is the greatest consumer of natural gas, among Canadian provinces, at around 3.9 billion cubic feet per day [16].

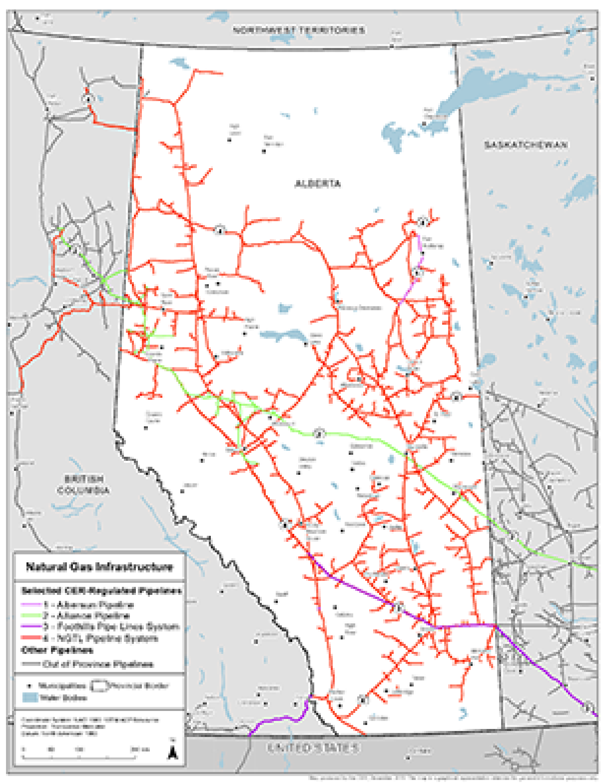


Figure 3.2: Natural Gas Infrastructure Map Alberta

Chapter 4

Modelling Locational Spread Options

This chapter explains options, types of options, and styles of options. It further details spread options and locational spread options, which is the primary focus of the study. The mathematical model is then constructed considering locational spread options as well as the components presented in earlier chapters.

4.1 Options

An option is a right, and not an obligation, to trade risky assets or financial instruments at a predetermined, mutually agreed price within a specified time period. Options are a financial instrument that allows, amongst other things, to make a bet on the rising or falling values of an underlying asset. The underlying asset could be a stock, stock indices, a parcel of shares of a company, foreign currencies, or commodities, such as oil and gas. These financial instruments can be traded, swapped, and allow for financing mobility, i.e. leveraging and risk management.

4.2 Energy Options

An energy option is a financial derivative where the underlying asset is an energy commodity. It is an agreement where the seller, or options writer provides the buyer, or the holder, the right to buy or sell an energy asset at an agreed price within the specified time. These contracts are set apart from other contracts by unusual kinds of specifications of trade such as the place of delivery, time of maturity period, conditions and size of the delivery, and storage issues.

4.3 Types of Options

There are two basic types of options.

- **Call Option**: The call option (caps) gives the holder the right to buy the energy commodity for an agreed (ceiling price) strike price K at a specific time T. The mathematical expression for the pay-off for the call option is as follows:

$$C = \max[(S_T - K)^+, 0]. \tag{4.1}$$

- **Put Option**: The put option (floors) pays a premium to give the holder the right, but not the obligation to sell the energy commodity at a strike price K at a specific time T. The mathematical expression for the pay-off for the put option is as follows:

$$P = \max[(K - S_T)^+, 0]. \tag{4.2}$$

4.3.1 Additional Key Features of Options

Some basic terminology needs to be explained regarding options, before proceeding further with the model.

- **Spot Price**: The spot price S_T, is the current market price of the commodity, at which it is traded to get instant payments and hand-in stocks.
- **Future Price:** The future price F_T, is the price agreed upon in a futures contract, which expires at time T.
- **Strike Price**: The strike price K is a specified price at which the energy can be purchased, under a call option, or sold, under a put option, by the holder upon the execution of the options contract.
- **Underlying Asset**: Options are defined as derivatives, as they infer value from the underlying asset. This asset is in principle being purchased or sold when an option's contract is executed. An underlying asset includes financial assets, such as stocks and ETFs, or commodities, such as oil and gold.
- **Pay-Off vs Profit**: The pay-off of an option is the cash earned by the investors from an option, whereas the profit is the amount leftover after the premium, paid at the time of initiation, is deducted from the pay-off.
- **Volatility**: The volatility in price refers to the variation in the price of the underlying assets in the market.

4.3.2 Option Styles

An option style depicts whether an option contract can be exercised before the date of expiry or not. There are two main types of options; European and American [9].European options can only be exercised at the time of expiration, T. American options, however, are flexible in execution such that they can be executed anytime up to and including time T. This flexibility results in the American options being traded at a higher price. This thesis restricts itself to the European-style options, based on the Black-Scholes Model.

4.3.3 Exchange-Traded and Over-the-Counter

Options can be traded using two channels [9]: Holders of exchange-traded options can exercise their contract for a cash settlement, or obtain a futures contract leading to a cash settlement. OTC options allow holders to customize the features of their options contracts, given that there is a party willing to offer all those features.

4.4 Black– Merton– Scholes Model

The Black-Scholes pricing model is used to determine the fair price or theoretical value for a call or put option based on six variables. These variables are volatility σ, type of option, underlying stock price S_1 and S_2, time T, strike price K, risk-free rate r, and continuously compounded dividend yield δ.// The formula for computing the fair value of European-style call and put options are respectively given below:

$$C = Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2), \tag{4.3}$$

and

$$P = Ke^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1), \tag{4.4}$$

Whereas:

$$d_1 = \frac{\ln(\frac{S}{K}) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}}, \tag{4.5}$$

and

$$d_2 = \frac{\ln(\frac{S}{K}) + (r - \delta - 0.5\sigma^2)T}{\sigma\sqrt{T}}. \tag{4.6}$$

Note that e^{-rT} is the present value factor, reflecting that the exercise price on the call option does not have to be paid until expiration. $N(d_1)$ and $N(d_2)$ are the probabilities under a cumulative standardized normal distribution.

Assumptions and Limitations

- There are no arbitrage opportunities.
- The market is frictionless.

  That is, there are no transaction costs (fees or taxes), the interest rates for borrowing and lending money are equal, all parties have immediate access to any information, and all securities and credits are available at any time and in any size. Further, individual trading will not influence the price.
- The underlying asset's price movement follows a log-normal distribution.
- The option is European and can only be exercised on the expiration date.

- There's no transaction cost.

- There are no tax and margin costs.

These assumptions are made for the sake of simplicity. They can be relaxed and the model can be adjusted for special circumstances when deemed necessary. In addition, we could easily use this model to price options on assets other than stocks (currencies, futures) [19].

4.5 Spread options

The value at time t of the two underlying energy commodities in a spread option is denoted by $S_1(t)$ and $S_2(t)$. For European-style options, the holder is paid the spread, $S_2(T) - S_1(T)$, at the time of maturity, T. The buyer must pay a predetermined strike price of K, at expiration time, to exercise the option. This pay-off at maturity can be expressed as [20]:

$$p(T, S_0^1, S_0^2, K) = \max[(S_T^1 - S_T^2 - K), 0]. \tag{4.7}$$

Spread trading can be used for hedging purposes or purely for trading ("arbitrage"). Examples of the types of spreads are time spreads, locational spreads, frack spreads, crack spreads and spark/dark spreads [21]. The techniques used to model each type of spread are fairly similar. Spread options are mostly carried out on equities, bonds, currencies, commodities, and so on. These sorts of financial instruments might be purchased on exchanges or traded in the Over-The-Counter markets.

Here our main focus is on **'Locational Spreads'**, which we define below:

4.5.1 Locational Spreads

A spread option covering the difference between prices of the same commodity trading at two different locations is called a *'Locational spread'*. The call (put) option defined on a locational spread is as follows:
An **European call(put)** option on the locational spread between the two locations with maturity T gives its holder the right but not the obligation to pay the price of energy commodity at location one at time T and receive the price of same energy commodity at location two. Let S_i^T be the price of energy commodity at location $i (i = 1; 2)$ at time T. Then the payoff of the **call option** $C(S_1^T; S_2^T; T)$ at time T with transportation cost K is:

$$C(S_1^T, S_2^T, K; T) = max(S_1^T - S_2^T - K, 0) \tag{4.8}$$

and the payoff of the **put option** $P(S_1^T;S_2^T;T)$ at time T with transportation cost K is [22]:

$$P(S_1^T,S_2^T,K;T) = max(K - S_1^T - S_2^T, 0). \tag{4.9}$$

4.6 Data Analysis

In previous sections, we described every single definition or concept which is required to set up our mathematical model. Now we are heading towards another important part of our thesis, that is, analyzing our data set which provides the basic characteristics of future natural gas prices. From the stock markets, we observed that energy commodity prices demonstrate more diverse behavior than other financial derivatives due to short-term trading of energy commodities, which further implies there's a distinction between the spot price and forward prices of energy commodities.

This section presents the data analysis using the model developed in the paper thus far. Time series data has been collected from Union Gas Dawn Hub (Dawn township, Ontario) and TCPL-Iroquois (Iroquois, Waddington, U.S/Canada border), from March 2002 to 2012. The data includes daily futures price data for natural gas in two locations; Waddington and Dawn township.

4.6.1 Statistical Characteristics

Summary statistics for Union Gas Dawn Hub is presented in Table 4.1 below. These statistics are based on 3655 observations for each location. The variables summarized are future prices (FP), the change in future prices (d(FP)), the logarithm of future prices (ln(FP)) and the log-returns of the future prices (dln(FP)).

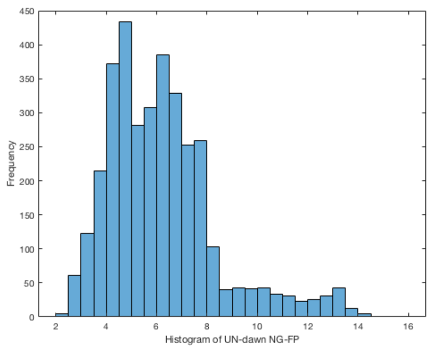
DNG-FP	Mean	Std.Dev	Skewness	Kurtosis	Min	Max
FP	6.1899	2.1919	1.2389	4.8333	2.2000	15.8800
d(FP)	0.1133	0.2593	0.5093	112.7516	-4.4300	5.1960
ln(FP)	0.5517	0.1925	-0.3538	3.2821	-0.2377	1.0171
dln(FP)	0.0107	0.0213	1.1983	44.5069	-0.2798	0.2914

Table 4.1: Statistical Characteristics of UN-dawn NG-FP

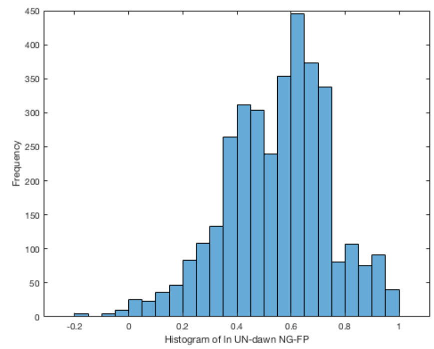
Summary statistics for TCPL-Iroquois are presented in Table 4.2 presented as follows:

DNG-FP	Mean	Std.Dev	Skewness	Kurtosis	Min	Max
FP	6.4808	2.3477	1.2976	5.5166	2.2300	20.8800
d(FP)	0.1707	0.5398	2.9844	143.8304	-9.3300	10.3600
ln(FP)	0.5755	0.1934	-0.4025	3.3310	-0.2206	1.1115
dln(FP)	0.0134	0.0293	2.1797	56.3508	-0.2470	0.5202

Table 4.2: Statistical Characteristics of TCPL-Iroquois NG-FP



(a) Histogram of UN-dawn NG-FP



(b) Histogram of log UN-dawn NG-FP

Figure 4.1: Histogram of UN-dawn natural gas future prices

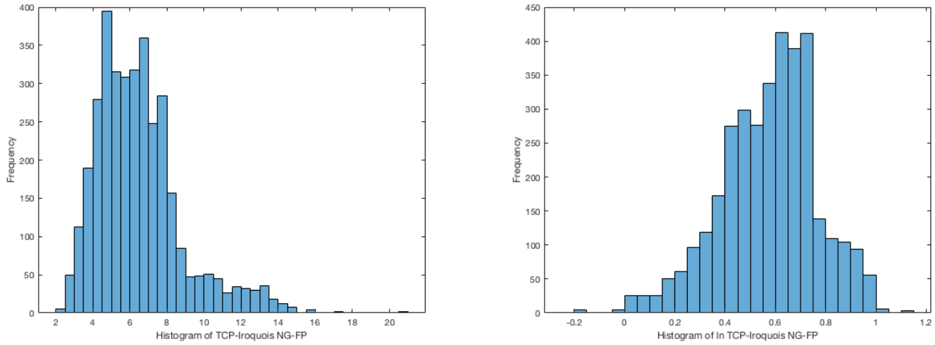
Comparing the summary statistics for the two locations, the difference in the maximum values stands out. The maximum values for Iroquois are overall higher. The logarithmic value of futures prices (ln(FP)) will be used for calibrating the model.

4.6.2 Distribution

An important assumption of our mathematical model is log-normal distribution of natural gas futures prices. A normal distribution does not catch the outliers in the data. It is evident in Figures 4.1 and 4.2, where futures prices do not spike at the extremes. The shape of the distribution is described by the skewness and kurtosis reported in summary statistics. As the shape of the distribution is not perfectly normal, there may be some bias present.

4.6.3 Mean Reversion

Mean-reversion is a process that indicates a time series may exhibit the tendency of a data series to its long-term mean over time, whenever the data series fluctuates. This is usually due to unexpected events, often beyond anyone's control, such outages, transmission constraints, weather changes or international crisis. If mean-reversion causes a spike in data greater than the next level, it may revert back to the natural average price. In Figure 4.3 and Figure 4.4, the time series of future and log returns of UN-dawn and TCP-Iroquois natural gas future prices are plotted, identifying the presence of mean-reversion. The mathematical model employs the exponential Ornstein-Uhlenbeck process to reflect



(a) Histogram of TCP-Iroquois NG-FP (b) Histogram of log TCP-Iroquois NG-FP

Figure 4.2: Histogram of TCP-Iroquois natural gas future prices

this mean-reverting process of the data series.

Comparing Figure 4.3 above with Figure 4.4, data from the two locations follow a similar trend IN prices, with a mean-reverting tendency. The latter shows the correlation between futures prices of the two series, rather than the average prices.

4.6.4 Correlation

Correlations exist in energy markets as the resources are very closely related and impact the valuation of energy commodities and derivatives. The mathematical definition of the correlation of two random variables X and Y is given as follows:

$$\rho = \frac{cov(X,Y)}{\sqrt{var(X)}\sqrt{var(Y)}} = \frac{E[XY]-E[X]E[Y]}{\sqrt{E[X^2]-E[X]^2}\sqrt{E[Y^2]-E[Y]^2}}. \tag{4.10}$$

Correlation coefficient is bounded between -1 to +1, where 1 indicates perfect positive correlation, where both variables move in the same direction, and -1 indicates perfect negative correlation, where the variables move in opposite directions.

In order to identify the correlation that exists in locational spread options, Figure 4.4 illustrates the log of futures prices for gas in the two locations. Two different scenarios are discussed in this thesis: constant correlation and time-variant stochastic correlation. Stochastic correlation is estimated using the Pearson product-moment correlation coefficient:

$$\hat{\rho}_{X,Y} = \frac{\sum_{i=1}^{N}(x_i-\overline{x})(y_i-\overline{y})}{\sqrt{\sum_{i=1}^{N}(x_i-\overline{x})^2}\sqrt{\sum_{i=1}^{N}(y_i-\overline{y})^2}}, \tag{4.11}$$

whereas x_i and y_i are the estimates of X and Y, while $\overline{x}$ and $\overline{y}$ are the respective means [23].

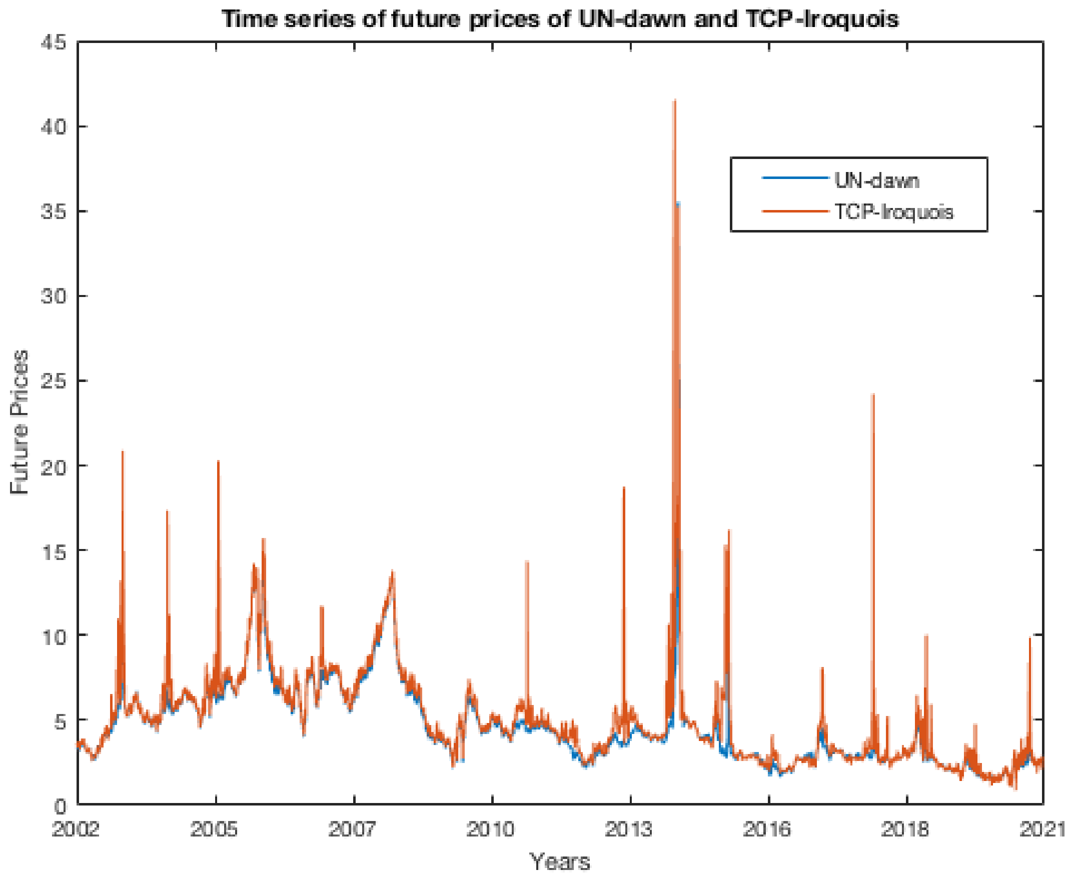


Figure 4.3: UN-dawn and TCP-Iroquois NG-FP from March 22, 2002, to May 21, 2021

4.6.5 Bivariate Normal Distribution

Assuming that Z_1 and Z_2 are two independent random variables, with a standard normal distribution $N \sim (0,1)$. The joint probability density function $f(z_1,z_2)$ of Z_1 and Z_2 for all values of z_1 and z_2 is described as follows:

$$f(z_1,z_2) = \frac{1}{2\pi}\exp\left[-\frac{1}{2}(z_1^2+z_2^2)\right]. \tag{4.12}$$

For conditional constants μ_1, μ_2, σ_1, σ_2 and ρ, defined by $-\infty < \mu_i < \infty$, $\sigma_i > 0 \quad (i = 1,2)$ and $-1 < \rho < 1$, two new dependent random variables are defined X_1 and X_2 by **Choleskydecomposition**, with the given correlation of the independent normal variables Z_1 and Z_2, as follows:

$$\begin{aligned} X_1 &= \sigma_1 Z_1 + \mu_1, \\ X_2 &= \sigma_2\left[\rho Z_1 + \sqrt{(1-\rho^2)}Z_2\right] + \mu_2. \end{aligned} \tag{4.13}$$

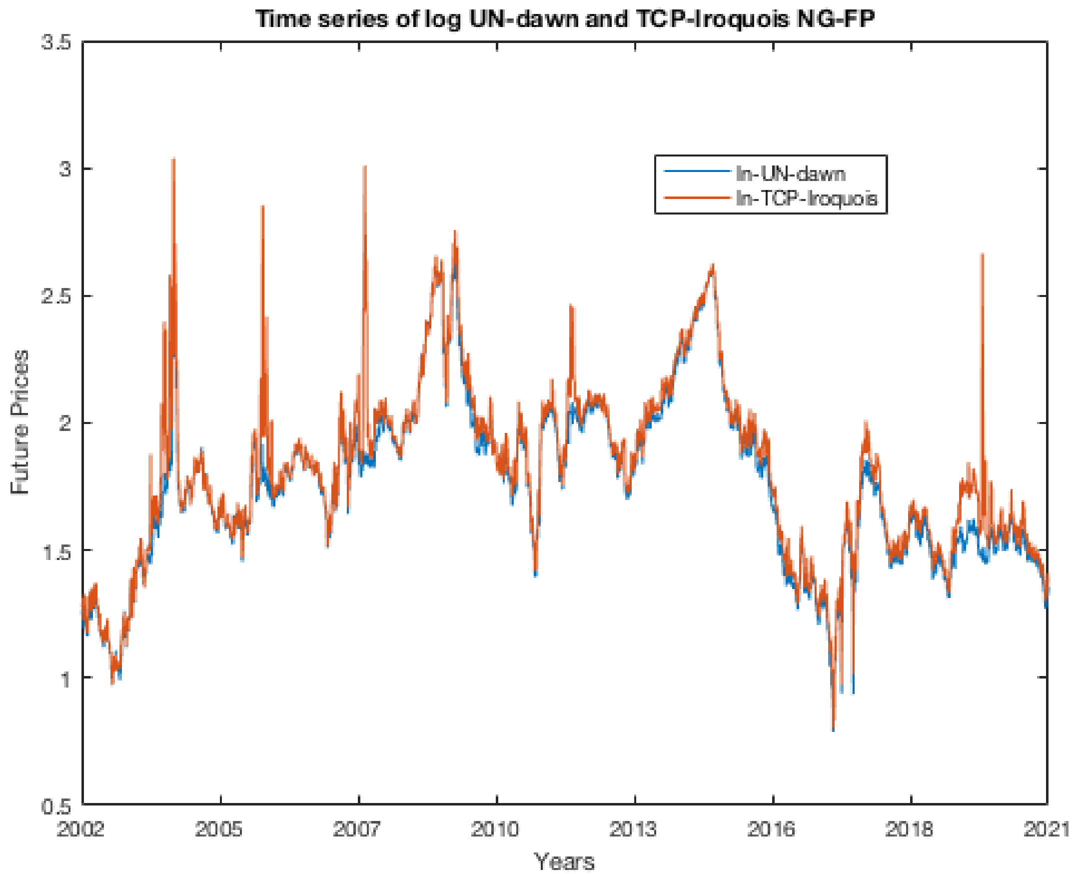


Figure 4.4: Log of UN-dawn and TCP-Iroquois NG-FP from March 22, 2002, to May 21, 2021

The joint probability density function $F(x_1,x_2)$ of X_1 and X_2 X2 can then be written as:

$$F(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[(\frac{x_1-\mu_1}{\sigma_1})^2 - 2\rho(\frac{x_1-\mu_1}{\sigma_1})(\frac{x_2-\mu_2}{\sigma_2}) + (\frac{x_2-\mu_2}{\sigma_2})^2\right]\right\}. \tag{4.14}$$

Therefore the joint probability distribution indicates that X_1 and X_2 is defined by (4.14), then it means X_1 and X_2 follows a **bi-variate normal distribution** with $E(X_i) = \mu_i$ and $Var(X_i) = \sigma_i^2$ for $i = 1,2$ and $Cov(X_1,X_2) = \rho\sigma_1\sigma_2$.

4.7 Mathematical Model Establishment

Consider $(\Omega, \mathscr{F}, P)$ to be a probability space with information filtration $(\mathscr{F})$. Thus, under the physical measure P, the logarithms of two natural gas future assets, denoted by X_1 and X_2 that is $X_1(t) = \ln F_1(t)$ and $X_2(t) = \ln F_2(t)$, follows the correlated Ornstein–Uhlenbeck processes, which is given as follows:

$$dX_1(t) = \kappa_1(\theta_1 - X_1(t))dt + \sigma_1 dW_1(t), \tag{4.15}$$

$$dX_2(t) = \kappa_2(\theta_2 - X_2(t))dt + \sigma_2 dW_2(t), \tag{4.16}$$

with

$$dW_1(t)dW_2(t) = \rho dt, \tag{4.17}$$

where, $\kappa_1 > 0$, $\kappa_2 > 0$, θ_1, θ_2, σ_1 and $\sigma_2 \in \mathbb{R}$, with $W_1(t)$ and $W_2(t)$ are two dependent Brownian motions with correlation coefficient ρ. To price the locational spread options, we should perform under the risk-neutral measure Q. We suppose that under this measure Q, the stochastic processes of X_1 and X_2 are assumed correlated with Ornstein–Uhlenbeck processes, however with different coefficients to account for the risk premium. Therefore (4.15), (4.16) and (4.17) become:

$$dX_1(t) = \kappa_1^Q(\theta_1^Q - X_1(t))dt + \sigma_1 dW_1^Q(t), \tag{4.18}$$

$$dX_2(t) = \kappa_2^Q(\theta_2^Q - X_2(t))dt + \sigma_2 dW_2^Q(t), \tag{4.19}$$

with

$$dW_1^Q(t)dW_2^Q(t) = \rho dt. \tag{4.20}$$

With new assumed measure Q, the vector $X(T) = (X_1(T), X_2(T))$ at time T, exhibits bivariate normal distribution with mean $\boldsymbol{\mu}$ and co-variance matrix $\boldsymbol{\Sigma}$, defined in previous section. Since κ_1^Q, κ_2^Q, θ_1^Q, θ_2^Q are constants,, these symbols are continued to maintain the simplicity of the model. Risk-neutrality has already been detailed in Chapter 2.

In order to discretize the Ornstein–Uhlenbeck processes, let $t_i = t_0 + i\Delta t$, with $i = 0, 1, 2, \cdots, N$, where $k = 1, 2$ and $i = 0, 1, \cdots, N$ and also ($\theta^* = \theta - \sigma^2/2\kappa$). Therefore:

$$X_1(i) = e^{-\kappa_1 \Delta t} X_1(i-1) + (1 - e^{-\kappa_1 \Delta t})\theta_1^* + \sigma_1 \sqrt{\frac{(1 - e^{-2\kappa_1 \Delta t})}{2\kappa_1}} \Delta W_{1,i}, \tag{4.21}$$

$$X_2(i) = e^{-\kappa_2 \Delta t} X_2(i-1) + (1 - e^{-\kappa_2 \Delta t})\theta_2^* + \sigma_2 \sqrt{\frac{(1 - e^{-2\kappa_2 \Delta t})}{2\kappa_2}} \Delta W_{2,i}, \tag{4.22}$$

and

$$\Delta W_{k,i} = W_k(t_i) - W_k(t_{i-1}). \tag{4.23}$$

The next step is to calibrate this model in Chapter 5, using the multivariate linear regression and maximum likelihood methods to obtain prices for locational spread options.

Chapter 5

Calibration with Constant Correlation

This chapter revolves around the calibration of the mathematical model that has been developed in the paper thus far. This calibration process will comprise of finding parameter values using a specific estimation method on historical market values. Through this process, estimates for the parameters of locational spread commodities with constant correlation under Ornstein–Uhlenbeck process will be determined using two methods. Multivariate linear regression is the first estimation method to predict a single regression model with more than one response parameter. The maximum likelihood method, which maximizes the log-likelihood of the actual data will be the second model. After the two estimations are completed, the resulting parameters will then be discussed.

5.1 Multivariate Linear Regression

Multivariate regression is a technique that allows for multiple variables to be included in a regression model to estimate a resultant variable.

Definition 5.1. The *Multivariate Linear Regression Model*

$$y_i = \mathbf{B}^T x_i + \varepsilon_i, \qquad i = 1, \cdots, n \tag{5.1}$$

has $m \geq 2$ response variables $Y_1, \cdots, Y_m$ and p predictor variables $x_1, x_2, \cdots, x_p$ where $x_1 \equiv 1$ is the trivial predictor. The ith case is $(x_i^T, y_i^T) = (1, x_{i2}, \cdots, x_{ip}, Y_{i1}, \cdots, Y_{im})$, where 1 could be omitted. The matrix form of the model can be expressed As:

$$\mathbf{Z} = \mathbf{XB} + \mathbf{E}, \tag{5.2}$$

where the matrices are defined below. The model has $E(\varepsilon_k) = 0$ and $Cov(\varepsilon_k) = \sum_\varepsilon = (\sigma_{ij})$ for $k = 1, \cdots, n$. Then the

$p \times m$ coefficient matrix $\mathbf{B} = [\beta_1 \beta_2 \cdots \beta_m]$ and the $m \times m$ covariance matrix $\sum_\varepsilon$ are to be estimated, and $E(\mathbf{Z}) = \mathbf{XB}$ while $E(Y_{ij}) = x_i^T \beta_j$. The ε_i are assumed to be iid (independent and identically distributed).

The data matrix $\mathbf{W} = [\mathbf{X} \quad \mathbf{Z}]$ except the first column 1 of $\mathbf{X}$. The $n \times m$ matrix is as follows:

$$\mathbf{Z} = \begin{bmatrix} Y_{1,1} & Y_{1,2} & \cdots & Y_{1,m} \\ Y_{2,1} & Y_{2,2} & \cdots & Y_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{n,1} & Y_{n,2} & \cdots & Y_{n,m} \end{bmatrix} = \begin{bmatrix} \mathbf{Y}_1 & \mathbf{Y}_2 & \cdots & \mathbf{Y}_m \end{bmatrix} = \begin{bmatrix} y_1^T \\ \vdots \\ y_n^T \end{bmatrix}. \tag{5.3}$$

The $n \times p$ design matrix of predictor variables is

$$\mathbf{X} = \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,p} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1} & x_{n,2} & \cdots & x_{n,p} \end{bmatrix} = \begin{bmatrix} \upsilon_1 & \upsilon_2 & \cdots & \upsilon_p \end{bmatrix} = \begin{bmatrix} x_1^T \\ \vdots \\ x_n^T \end{bmatrix}, \tag{5.4}$$

where $\upsilon_1 = 1$. The $p \times m$ matrix

$$\mathbf{B} = \begin{bmatrix} \beta_{1,1} & \beta_{1,2} & \cdots & \beta_{1,m} \\ \beta_{2,1} & \beta_{2,2} & \cdots & \beta_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{n,1} & \beta_{n,2} & \cdots & \beta_{n,m} \end{bmatrix} . = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_m \end{bmatrix}. \tag{5.5}$$

The $n \times m$ matrix

$$\mathbf{E} = \begin{bmatrix} \varepsilon_{1,1} & \varepsilon_{1,2} & \cdots & \varepsilon_{1,m} \\ \varepsilon_{2,1} & \varepsilon_{2,2} & \cdots & \varepsilon_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{n,1} & \varepsilon_{n,2} & \cdots & \varepsilon_{n,m} \end{bmatrix} = \begin{bmatrix} \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_m \end{bmatrix} = \begin{bmatrix} \varepsilon_1^T \\ \vdots \\ \varepsilon_n^T \end{bmatrix}. \tag{5.6}$$

Considering the ith row of $\mathbf{Z}$, $\mathbf{X}$ and $\mathbf{E}$ shows that $y_i^T = x_i^T \mathbf{B} + \varepsilon_i^T$.

5.1.1 Least Square Estimation

Least squares are the classical method for fitting multivariate linear regression. The **least squares estimators** are

$$\hat{\mathbf{B}} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{Z} = \begin{bmatrix} \hat{\beta}_1 & \hat{\beta}_2 & \cdots & \hat{\beta}_m \end{bmatrix}. \tag{5.7}$$

The predicted values or fitted values

$$\hat{\mathbf{Z}} = \mathbf{X}\hat{\mathbf{B}} = \begin{bmatrix} \hat{\mathbf{Y}}_1 & \hat{\mathbf{Y}}_2 & \cdots & \hat{\mathbf{Y}}_m \end{bmatrix} = \begin{bmatrix} \hat{Y}_{1,1} & \hat{Y}_{1,2} & \cdots & \hat{Y}_{1,m} \\ \hat{Y}_{2,1} & \hat{Y}_{2,2} & \cdots & \hat{Y}_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{Y}_{n,1} & \hat{Y}_{n,2} & \cdots & \hat{Y}_{n,m} \end{bmatrix}. \tag{5.8}$$

The *residuals* $\hat{\mathbf{E}} = \mathbf{Z} - \hat{\mathbf{Z}} = \mathbf{Z} - \mathbf{X}\hat{\mathbf{B}}$ are written as:

$$\hat{\mathbf{E}} = \begin{bmatrix} \hat{\varepsilon}_1^T \\ \hat{\varepsilon}_2^T \\ \vdots \\ \hat{\varepsilon}_n^T \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \cdots & \mathbf{r}_m \end{bmatrix} = \begin{bmatrix} \hat{\varepsilon}_{1,1} & \hat{\varepsilon}_{1,2} & \cdots & \hat{\varepsilon}_{1,m} \\ \hat{\varepsilon}_{2,1} & \hat{\varepsilon}_{2,2} & \cdots & \hat{\varepsilon}_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\varepsilon}_{n,1} & \hat{\varepsilon}_{n,2} & \cdots & \hat{\varepsilon}_{n,m} \end{bmatrix}. \tag{5.9}$$

These quantities can be found as: $\hat{\beta}_j = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y}_j$, $\hat{\mathbf{Y}}_j = \mathbf{X}\hat{\beta}_j$ and $\mathbf{r}_j = \mathbf{Y}_j - \hat{\mathbf{Y}}_j$ for $j = 1, \cdots, m$. Hence $\hat{\varepsilon}_{i,j} = Y_{i,j} - \hat{Y}_{i,j}$ where $\hat{Y}_j = (\hat{Y}_{1,j}, \cdots, \hat{Y}_{n,j})^T$. Also,

$$\hat{\mathbf{E}} = [\mathbf{I} - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}]\mathbf{Z}. \tag{5.10}$$

All this material is covered in [24].

5.1.2 Calibrating with Multivariate Linear Regression

Here the model by using least square estimation method is as follows:

$$\mathbf{X} = A\mathbf{Y} + B + C\mathbf{Z}, \tag{5.11}$$

with

$$X = \begin{bmatrix} X_{1n} \\ X_{2n} \end{bmatrix}, \quad Y = \begin{bmatrix} Y_{1n} \\ Y_{2n} \end{bmatrix}, \quad A = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad C = \begin{bmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{bmatrix}, \quad Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix},$$

and whereas we define:

$$X_{1n} = a_1Y_{1n} + b_1 + c_{11}z_1, \qquad X_{2n} = a_2Y_{2n} + b_2 + c_{21}z_1 + c_{22}z_2, \tag{5.12}$$

with residuals Δ_1 and Δ_2:

$$\Delta_1 = X_{1n} - a_1Y_{1n} - b_1, \qquad \Delta_2 = X_{1n} - a_2Y_{2n} - b_2, \tag{5.13}$$

and

$$\Delta = \begin{bmatrix} \Delta_1 & \Delta_2 \end{bmatrix} = CZ. \tag{5.14}$$

While comparing with (4.15) or (4.21) and (4.16) or (4.22), we have

$$a_1 = e^{-\kappa_1 \Delta t}, \qquad a_2 = e^{-\kappa_2 \Delta t}, \tag{5.15}$$

or solving

$$\kappa_1 = \frac{-\ln a_1}{\Delta t}, \qquad \kappa_2 = \frac{-\ln a_2}{\Delta t}, \tag{5.16}$$

and

$$b_1 = (1 - e^{-\kappa_1 \Delta t})\theta_1, \qquad b_2 = (1 - e^{-\kappa_2 \Delta t})\theta_2, \tag{5.17}$$

while simplifying:

$$\theta_1 = \frac{b_1}{(1 - a_1)}, \qquad \theta_2 = \frac{b_2}{(1 - a_2)}, \tag{5.18}$$

and last ones

$$C_{11} = \sigma_1 \sqrt{\frac{1 - e^{-2\kappa_1 \Delta t}}{2\kappa_1}} = \hat{\sigma}_1, \qquad C_{21}^2 + c_{22}^2 = \sigma_2 \sqrt{\frac{1 - e^{-2\kappa_2 \Delta t}}{2\kappa_2}} = \hat{\sigma}_2, \tag{5.19}$$

with co-variance matrix:

$$C = \begin{bmatrix} \hat{\sigma}_1 & 0 \\ \rho \hat{\sigma}_2 & \hat{\sigma}_2 \sqrt{1 - \rho^2} \end{bmatrix}. \tag{5.20}$$

Now

$$\begin{aligned} \varepsilon^2 &= \|X - AY - b\|^2 \\ &= X^T X + Y^T A^T A Y + \|b\|^2 - 2X^T A Y + 2b^T A Y - 2b^T X \\ &= X_1^2 + X_2^2 + a_1^2 Y_1^2 + a_2^2 Y_2^2 + b_1^2 + b_2^2 - 2(a_1 X_1 Y_1 + a_2 X_2 Y_2) \\ &\quad + 2(b_1 a_1 Y_1 + b_2 a_2 Y_2) - 2(b_1 X_1 + b_2 X_2). \end{aligned} \tag{5.21}$$

Partial derivative with respect to a_1, a_2, b_1, b_2 as follows (First order derivative test), and equating them equals to zero to figure out their values:

$$\frac{\partial \varepsilon^2}{\partial a_1} = 2a_1Y_1^2 - 2X_1Y_1 + 2b_1Y_1 = 0, \tag{5.22}$$

$$\frac{\partial \varepsilon^2}{\partial a_2} = 2a_2Y_2^2 - 2X_2Y_2 + 2b_2Y_2 = 0, \tag{5.23}$$

$$\frac{\partial \varepsilon^2}{\partial b_1} = 2b_1 + 2a_1Y_1 - 2X_1 = 0, \tag{5.24}$$

$$\frac{\partial \varepsilon^2}{\partial b_2} = 2b_2 + 2a_2Y_2 - 2X_2 = 0. \tag{5.25}$$

Four equations and four unknowns a_1, a_2, b_1 and b_2 can easily be estimated and replaced in (5.16), (5.18) to figure out the results, using Matlab. In order to calculate the remaining three values σ_1, σ_2 and ρ we proceed as follows, knowing that:

$$\Delta\Delta^T = CZZ^TC^T. \tag{5.26}$$

Then taking expectations on both sides and applying its properties:

$$\begin{aligned} \mathbb{E}\left[\Delta\Delta^T\right] &= \mathbb{E}\left[CZZ^TC^T\right] \\ &= C\mathbb{E}\left[ZZ^T\right]C^T \\ &= C\mathbf{I}C^T = CC^T, \end{aligned} \tag{5.27}$$

whereas multiplying C with it transpose matrix C^T, we get

$$CC^T = \begin{bmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} \\ 0 & C_{22} \end{bmatrix} = \begin{bmatrix} C_{11}^2 & C_{11}C_{21} \\ C_{21}C_{11} & C_{22}^2 \end{bmatrix}. \tag{5.28}$$

Using (5.19), (5.20) and (5.28) we will find the remaining three values too.

5.1.3 Results

Δt	κ_1	κ_2	θ_1	θ_2	σ_1	σ_2	ρ
1.00	0.0056	0.0141	1.17718	1.8115	0.0206	0.0394	0.6030

Table 5.1: Parametric Estimation Valuation via Multivariate Linear Regression.

The results are shown in table 5.1 we get after calibrating our model with respect to multivariate linear regression with future prices of natural gas with data length $L = 3500$.

5.2 The Maximum Log-Likelihood Method

Maximum log-likelihood estimation (MLE) is one of the most important techniques that helps estimate parameters using the likelihood function.

Definition 5.2. Assume that the random sample $X_1, \cdots, X_n \sim F$, where $F = F_\theta$ is a normal distribution depending on a parameter $\theta = (\mu, \sigma)$, the mean and the variance. The probability density function (PDF) is employed to find the most likely parameter for simplicity. In other words, PDF $p = p_\theta$ will also be determined by the parameter θ. The independence property dictates the joint PDF of the random sample $X_1, \cdots, X_n$:

$$p_{X_1,\cdots,X_n}(x_1, \cdots, x_n) = \prod_{i=1}^{n} p_\theta(x_i), \tag{5.29}$$

because $p_\theta(x)$ also changes when θ changes, we rewrite it as $p(x;\theta) = p_\theta(x)$. Thus, the joint PDF can be rewritten as

$$p_{X_1,\cdots,X_n}(x_1, \cdots, x_n) = \prod_{i=1}^{n} p_\theta(x_i;\theta). \tag{5.30}$$

Having observed $x_1 = X_1, \cdots, x_n = X_n$, how can the likeliness be determined? MLE proposes that, based on the joint PDF and and $x_1 = X_1, \cdots, x_n = X_n$, we can rewrite the joint PDF as a *function of parameter* θ:

$$L(\theta \mid X_1, \cdots, X_n) = \prod_{i=1}^{n} p_\theta(x_i;\theta). \tag{5.31}$$

5.2.1 Calibration using maximum log-likelihood

We consider the correlated exponential Ornstein–Uhlenbeck processes $X(t) = (X_1(t), X_2(t))$ at any time t, follows a bivariate normal distribution with mean $\boldsymbol{\mu}= (\mu_1, \mu_2)$ and variance $\boldsymbol{\eta}^2= (\eta_1^2, \eta_2^2)$ given by:

$$\mu_1 = e^{-\kappa_1 t} X_1(n-1) + \theta_1(1 - e^{-\kappa_1 t}) = \theta_1 + (X_1(n-1) - \theta_1)e^{-\kappa_1 t}, \tag{5.32}$$

$$\mu_2 = e^{-\kappa_2 t} X_2(n-1) + \theta_2(1 - e^{-\kappa_2 t}) = \theta_2 + (X_2(n-1) - \theta_2)e^{-\kappa_2 t}, \tag{5.33}$$

$$\eta_1^2 = \frac{\sigma_1^2}{2\kappa_1}(1 - e^{-2\kappa_1 t}), \tag{5.34}$$

$$\eta_2^2 = \frac{\sigma_2^2}{2\kappa_2}(1 - e^{-2\kappa_2 t}). \tag{5.35}$$

The joint probability density function of bivariate $f(X_1,X_2)$ with mean and variance defined in (5.32), (5.33), (5.34) and (5.35) for n sequential data sets where $t_{n-1} < t_n$ is stated by

$$f(X_1(t),X_2(t)) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2}Q(X_1(t),X_2(t))}, \tag{5.36}$$

where

$$\begin{aligned} Q(X_1(t),X_2(t)) = \frac{1}{(1-\rho^2)}\{ & \frac{2\kappa_1(X_1(n)-\theta_1-e^{-\kappa_1\Delta t}(X_1(n-1)-\theta_1))^2}{\sigma_1^2(1-e^{-2\kappa_1\Delta t})} \\ & -2\rho\frac{(X_1(n)-\theta_1-e^{-\kappa_1\Delta t}(X_1(n-1)-\theta_1))(X_2(n)-\theta_2-e^{-\kappa_2\Delta t}(X_2(n-1)-\theta_2))}{\sqrt{\frac{\sigma_1^2}{2\kappa_1}(1-e^{-2\kappa_1\Delta_t})}\sqrt{\frac{\sigma_2^2}{2\kappa_2}(1-e^{-2\kappa_2\Delta_t})}} \\ & +\frac{2\kappa_2(X_2(n)-\theta_2-e^{-\kappa_2\Delta t}(X_2(n-1)-\theta_2))^2}{\sigma_2^2(1-e^{-2\kappa_2\Delta t})}\}. \end{aligned} \tag{5.37}$$

Thus,

$$\begin{aligned} f(X_1,X_2) = & \frac{2\sqrt{\kappa_1\kappa_2}}{2\pi\sigma_1\sigma_2\sqrt{(1-e^{-2\kappa_1\Delta t})(1-e^{-2\kappa_2\Delta t})}\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\right. \\ & \left(\frac{2\kappa_1(X_1(n)-\theta_1-e^{-\kappa_1\Delta t}(X_1(n-1)-\theta_1))^2}{\sigma_1^2(1-e^{-2\kappa_1\Delta t})}\right. \\ & +\frac{2\kappa_2(X_2(n)-\theta_2-e^{-\kappa_2\Delta t}(X_2(n-1)-\theta_2))^2}{\sigma_2^2(1-e^{-2\kappa_2\Delta t})} - 4\rho\sqrt{\kappa_1\kappa_2} \\ & \left.\left.\frac{(X_1(n)-\theta_1-e^{-\kappa_1\Delta t}(X_1(n-1)-\theta_1))(X_2(n)-\theta_2-e^{-\kappa_2\Delta t}(X_2(n-1)-\theta_2))}{\sigma_1\sigma_2\sqrt{(1-e^{-2\kappa_1\Delta t})(1-e^{-2\kappa_2\Delta t})}}\right)\right\}. \end{aligned} \tag{5.38}$$

Eliminating '2' from the above equation, and simplifying, the following result is obtained:

$$\begin{aligned} f(X_1,X_2) = & \frac{\sqrt{\kappa_1\kappa_2}}{\pi\sigma_1\sigma_2\sqrt{(1-e^{-2\kappa_1\Delta t})(1-e^{-2\kappa_2\Delta t})}\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{(1-\rho^2)}\right. \\ & \left(\frac{\kappa_1(X_1(n)-\theta_1-e^{-\kappa_1\Delta t}(X_1(n-1)-\theta_1))^2}{\sigma_1^2(1-e^{-2\kappa_1\Delta t})}\right. \\ & +\frac{\kappa_2(X_2(n)-\theta_2-e^{-\kappa_2\Delta t}(X_2(n-1)-\theta_2))^2}{\sigma_2^2(1-e^{-2\kappa_2\Delta t})} - 2\rho\sqrt{\kappa_1\kappa_2} \\ & \left.\left.\frac{(X_1(n)-\theta_1-e^{-\kappa_1\Delta t}(X_1(n-1)-\theta_1))(X_{2(n)}-\theta_2-e^{-\kappa_2\Delta t}(X_2(n-1)-\theta_2))}{\sigma_1\sigma_2\sqrt{(1-e^{-2\kappa_1\Delta t})(1-e^{-2\kappa_2\Delta t})}}\right)\right\}. \end{aligned} \tag{5.39}$$

The log-likelihood function of the given data sets $(X_1(0),X_1(1),...X_1(n))$ and $(X_2(0),X_2(1),...X_2(n))$ can then be obtained from the joint probability density function as follows:

$$L = \log\prod_{n=1}^{N} f_{X_1,X_2}(x_1(n),x_2(n);\boldsymbol{\mu},\boldsymbol{\eta}). \tag{5.40}$$

Using (5.39) where N refers to the total size of the data, , the following is arrived at:

$$L = \log \prod_{n=1}^{N} \left\{ \frac{\sqrt{\kappa_1 \kappa_2}}{\pi \sigma_1 \sigma_2 \sqrt{(1-e^{-2\kappa_1 \Delta t})(1-e^{-2\kappa_2 \Delta t})}\sqrt{1-\rho^2}} exp \left\{ \frac{-1}{(1-\rho^2)} \right.\right.$$
$$\left\{ \frac{\kappa_1 (X_1(n) - \theta_1 - e^{-\kappa_1 \Delta t}(X_1(n-1) - \theta_1))^2}{\sigma_1^2 (1-e^{-2\kappa_1 \Delta t})} \right.$$
$$+ \frac{\kappa_2 (X_2(n) - \theta_2 - e^{-\kappa_2 \Delta t}(X_2(n-1) - \theta_2))^2}{\sigma_2^2 (1-e^{-2\kappa_2 \Delta t})} - 2\rho \sqrt{\kappa_1 \kappa_2}$$
$$\frac{(X_1(n) - \theta_1 - e^{-\kappa_1 \Delta t}(X_1(n-1) - \theta_1))(X_2(n) - \theta_2 - e^{-\kappa_2 \Delta t}(X_2(n-1) - \theta_2))}{\sigma_1 \sigma_2 \sqrt{(1-e^{-2\kappa_1 \Delta t})(1-e^{-2\kappa_2 \Delta t})}}.$$

(5.41)

Solving to simplify as follows:

$$\begin{aligned} L = & \frac{N}{2}\log(\kappa_1 \kappa_2) - N\log(\pi \sigma_1 \sigma_2) - \frac{N}{2}\log(1-e^{-2\kappa_1 \Delta t}) - \frac{N}{2}\log(1-e^{-2\kappa_2 \Delta t}) \\ & - \frac{N}{2}\log(1-\rho^2) - \frac{1}{(1-\rho^2)} \sum_{n=1}^{N} \{ \frac{\kappa_1 (X_{1(n)} - \theta_1 - e^{-\kappa_1 \Delta t}(X_1(n-1) - \theta_1))^2}{\sigma_1^2 (1-e^{-2\kappa_1 \Delta t})} \\ & \qquad + \frac{\kappa_2 (X_2(n) - \theta_2 - e^{-\kappa_2 \Delta t}(X_2(n-1) - \theta_2))^2}{\sigma_2^2 (1-e^{-2\kappa_2 \Delta t})} \\ & - 2\rho \sqrt{\kappa_1 \kappa_2} \frac{(X_1(n) - \theta_1 - e^{-\kappa_1 \Delta t}(X_1(n-1) - \theta_1))(X_2(n) - \theta_2 - e^{-\kappa_2 \Delta t}(X_2(n-1) - \theta_2))}{\sigma_1 \sigma_2 \sqrt{(1-e^{-2\kappa_1 \Delta t})(1-e^{-2\kappa_2 \Delta t})}} \}. \end{aligned} \tag{5.42}$$

The log-likelihood in (5.42) is maximized using the first order derivative with respect to each underlying parameters and then equating it to zero. However, it is not possible to differentiate independently, therefore the maximization of the negative '$-L$' log-likelihood function in (5.42),) is established, we use the MATLAB build-in function *'fmincon()'* via discrete time steps to estimate these underlying parameters.

5.2.2 Results

Δt	κ_1	κ_2	θ_1	θ_2	σ_1	σ_2	ρ
1.00	0.0063	0.0072	1.7705	1.8113	0.0349	0.0568	0.6032

Table 5.2: Parametric Estimation Valuation via maximum likelihood estimation.

The results shown in Table 5.2 we got after calibrating our model with respect to maximum log-likelihood, with our daily data, length $L = 3500$.

Chapter 6

Calibration with Stochastic Correlation

Stochastic volatility models, such as Heston's stochastic volatility model, assume constant correlation. However, in reality, the correlation may not remain constant overtime and is likely to be stochastic. Another assumption of such models is that correlation is the most unstable of all parameters in option pricing models. This chapter identifies techniques for calibration with stochastic correlation. Since the correlation between two variables is limited between $[-1,1]$ [25], only two methods are considered: the optimization technique by reducing the number of unknown parameters and changing the regression. The Jacobi process is then applied to model stochastic correlation.

6.1 Generating a Time Series of Correlation Reducing Unknown parameters

In order to generate a series of correlation, the following steps need to be followed.

1. Initialize/ choose appropriate data.
2. Compute initial parameters P_0.
3. Divide the data into equal window sizes of $n_T = 30$, monthly time step.
4. Calculate Rho ρ by using the first derivative test i.e $\frac{\partial L}{\partial \rho} = 0$ to get the implicit function of ρ in terms of other parameters.
5. Using this maximum rho ρ_{max}, choose one of the best remaining six parameters from all windows.
6. Then with the help of these best parameters we will generate our best rho that is explicit value, ρ_{best} in the time series.

6.1.1 Mathematical Description

Differentiating (5.42) with respect to ρ gives the following result:

$$
\begin{aligned}
\frac{\partial L}{\partial \rho} = & -\frac{-2N\rho}{2(1-\rho^2)} - \left[\frac{2\rho}{(1-\rho^2)^2} \right. \\
& \sum_{i=1}^{N} \left\{ \frac{\kappa_1 (X_1(n) - \theta_1 - (X_1(n-1) - \theta_1) e^{-\kappa_1 \Delta t})^2}{\sigma_1^2 (1 - e^{-2\kappa_1 \Delta t})} \right. \\
& + \frac{\kappa_2 (X_2(n) - \theta_2 - (X_2(n-1) - \theta_2) e^{-\kappa_2 \Delta t})^2}{\sigma_2^2 (1 - e^{-2\kappa_2 \Delta t})} - 2\rho \sqrt{\kappa_1 \kappa_2} \\
& \frac{(X_1(n) - \theta_1 - (X_1(n-1) - \theta_1) e^{-\kappa_1 \Delta t})(X_2(n) - \theta_2 - (X_2(n-1) - \theta_2) e^{-\kappa_2 \Delta t})}{\sigma_1 \sigma_2 \sqrt{(1 - e^{-2\kappa_1 \Delta t})(1 - e^{-2\kappa_2 \Delta t})}} - \frac{1}{(1-\rho^2)} \sum_{i=1}^{N} \\
& \left\{ -2\sqrt{\kappa_1 \kappa_2} \frac{(X_1(n) - \theta_1 - (X_1(n-1) - \theta_1) e^{-\kappa_1 \Delta t})(X_2(n) - \theta_2 - (X_2(n-1) - \theta_2) e^{-\kappa_2 \Delta t})}{\sigma_1 \sigma_2 \sqrt{(1 - e^{-2\kappa_1 \Delta t})(1 - e^{-\kappa_2 \Delta t})}} \right\},
\end{aligned}
\tag{6.1}
$$

Or

$$
\begin{aligned}
\frac{\partial L}{\partial \rho} = & \frac{N\rho}{(1-\rho^2)} - \frac{2\rho}{(1-\rho^2)} \\
& \sum_{i=1}^{N} \left\{ \frac{\kappa_1 (X_1(n) - \theta_1 - (X_1(n-1) - \theta_1) e^{-\kappa_1 \Delta t})^2}{\sigma_1^2 (1 - e^{-2\kappa_1 \Delta t})} \right. \\
& + \frac{\kappa_2 (X_2(n) - \theta_2 - (X_2(n-1) - \theta_2) e^{-\kappa_2 \Delta t})^2}{\sigma_2^2 (1 - e^{-2\kappa_2 \Delta t})} - 2\rho \sqrt{\kappa_1 \kappa_2} \\
& \frac{(X_1(n) - \theta_1 - (X_1(n-1) - \theta_1) e^{-\kappa_1 \Delta t})(X_2(n) - \theta_2 - (X_2(n-1) - \theta_2) e^{-\kappa_2 \Delta t})}{\sigma_1 \sigma_2 \sqrt{(1 - e^{-2\kappa_1 \Delta t})(1 - e^{-2\kappa_2 \Delta t})}} + \frac{2}{(1-\rho^2)} \sqrt{\kappa_1 \kappa_2} \\
& \sum_{i=1}^{N} \left\{ \frac{(X_1(n) - \theta_1 - (X_1(n-1) - \theta_1) e^{-\kappa_1 \Delta t})(X_2(n) - \theta_2 - (X_2(n-1) - \theta_2) e^{-\kappa_2 \Delta t})}{\sigma_1 \sigma_2 \sqrt{(1 - e^{-2\kappa_1 \Delta t})(1 - e^{-\kappa_2 \Delta t})}} \right\}.
\end{aligned}
\tag{6.2}
$$

Therefore, by simplifying:

$$
\begin{aligned}
\frac{\partial L}{\partial \rho} &= \frac{N\rho}{(1-\rho^2)} - \frac{2\rho}{(1-\rho^2)^2} \\
&\sum_{i=1}^{N} \left\{ \frac{\kappa_1 (X_1(n) - \theta_1 - (X_1(n-1) - \theta_1) e^{-\kappa_1 \Delta t})^2}{\sigma_1^2 (1 - e^{-2\kappa_1 \Delta t})} \right. \\
&+ \frac{\kappa_2 (X_2(n) - \theta_2 - (X_2(n-1) - \theta_2) e^{-\kappa_2 \Delta t})^2}{\sigma_2^2 (1 - e^{-2\kappa_2 \Delta t})} \\
&+ \frac{4\rho}{(1-\rho^2)^2} \sqrt{\kappa_1 \kappa_2} \\
&\sum_{i=1}^{N} \left\{ \frac{(X_1(n) - \theta_1 - (X_1(n-1) - \theta_1) e^{-\kappa_1 \Delta t})(X_2(n) - \theta_2 - (X_2(n-1) - \theta_2) e^{-\kappa_2 \Delta t})}{\sigma_1 \sigma_2 \sqrt{(1 - e^{-2\kappa_1 \Delta t})(1 - e^{-\kappa_2 \Delta t})}} \right\} \\
&+ \frac{2}{(1-\rho^2)} \sqrt{\kappa_1 \kappa_2} \\
&\sum_{i=1}^{N} \left\{ \frac{(X_1(n) - \theta_1 - (X_1(n-1) - \theta_1) e^{-\kappa_1 \Delta t})(X_2(n) - \theta_2 - (X_2(n-1) - \theta_2) e^{-\kappa_2 \Delta t})}{\sigma_1 \sigma_2 \sqrt{(1 - e^{-2\kappa_1 \Delta t})(1 - e^{-\kappa_2 \Delta t})}} \right\}.
\end{aligned}
\tag{6.3}
$$

Equating $\frac{\partial L}{\partial \rho}$ to zero, gives the following reduced form:

$$
\begin{aligned}
&\frac{N\rho}{(1-\rho^2)} - \frac{2\rho}{(1-\rho^2)^2} \\
&\sum_{i=1}^{N} \left\{ \frac{\kappa_1 (X_1(n) - \theta_1 - (X_1(n-1) - \theta_1) e^{-\kappa_1 \Delta t})^2}{\sigma_1^2 (1 - e^{-2\kappa_1 \Delta t})} \right. \\
&+ \frac{\kappa_2 (X_2(n) - \theta_2 - (X_2(n-1) - \theta_2) e^{-\kappa_2 \Delta t})^2}{\sigma_2^2 (1 - e^{-2\kappa_2 \Delta t})} \\
&+ \frac{4\rho}{(1-\rho^2)^2} \sqrt{\kappa_1 \kappa_2} \\
&\sum_{i=1}^{N} \left\{ \frac{(X_1(n) - \theta_1 - (X_1(n-1) - \theta_1) e^{-\kappa_1 \Delta t})(X_2(n) - \theta_2 - (X_2(n-1) - \theta_2) e^{-\kappa_2 \Delta t})}{\sigma_1 \sigma_2 \sqrt{(1 - e^{-2\kappa_1 \Delta t})(1 - e^{-\kappa_2 \Delta t})}} \right\} \\
&+ \frac{2}{(1-\rho^2)} \sqrt{\kappa_1 \kappa_2} \\
&\sum_{i=1}^{N} \left\{ \frac{(X_1(n) - \theta_1 - (X_1(n-1) - \theta_1) e^{-\kappa_1 \Delta t})(X_2(n) - \theta_2 - (X_2(n-1) - \theta_2) e^{-\kappa_2 \Delta t})}{\sigma_1 \sigma_2 \sqrt{(1 - e^{-2\kappa_1 \Delta t})(1 - e^{-\kappa_2 \Delta t})}} \right\} = 0.
\end{aligned}
\tag{6.4}
$$

Multiplying throughout by $(1-\rho^2)^2$, and simplifying:

$$
\begin{aligned}
&N\rho(1-\rho^2)-2\rho\sum_{i=1}^{N}\left\{\frac{\kappa_1(X_1(n)-\theta_1-(X_1(n-1)-\theta_1)e^{-\kappa_1\Delta t})^2}{\sigma_1^2(1-e^{-2\kappa_1\Delta t})}\right.\\
&+\left.\frac{\kappa_2(X_2(n)-\theta_2-(X_2(n-1)-\theta_2)e^{-\kappa_2\Delta t})^2}{\sigma_2^2(1-e^{-2\kappa_2\Delta t})}\right\}+\sqrt{\kappa_1\kappa_2}\left\{4\rho^2+2(1-\rho^2)\right\}\\
&\sum_{i=1}^{N}\left\{\frac{(X_1(n)-\theta_1-(X_1(n-1)-\theta_1)e^{-\kappa_1\Delta t})(X_2(n)-\theta_2-(X_2(n-1)-\theta_2)e^{-\kappa_2\Delta t})}{\sigma_1\sigma_2\sqrt{(1-e^{-2\kappa_1\Delta t})(1-e^{-\kappa_2\Delta t})}}\right\}=0,
\end{aligned}
\tag{6.5}
$$

which can be written as follows in cubical form:

$$
\begin{aligned}
&N\rho-N\rho^3-2\rho\sum_{i=1}^{N}\left\{\frac{\kappa_1(X_1(n)-\theta_1-(X_1(n-1)-\theta_1)e^{-\kappa_1\Delta t})^2}{\sigma_1^2(1-e^{-2\kappa_1\Delta t})}\right.\\
&+\left.\frac{\kappa_2(X_2(n)-\theta_2-(X_2(n-1)-\theta_2)e^{-\kappa_2\Delta t})^2}{\sigma_2^2(1-e^{-2\kappa_2\Delta t})}\right\}+2\rho^2\sqrt{\kappa_1\kappa_2}\\
&\sum_{i=1}^{N}\left\{\frac{(X_1(n)-\theta_1-(X_1(n-1)-\theta_1)e^{-\kappa_1\Delta t})(X_2(n)-\theta_2-(X_2(n-1)-\theta_2)e^{-\kappa_2\Delta t})}{\sigma_1\sigma_2\sqrt{(1-e^{-2\kappa_1\Delta t})(1-e^{-\kappa_2\Delta t})}}\right\}\\
&+2\sum_{i=1}^{N}\left\{\frac{(X_1(n)-\theta_1-(X_1(n-1)-\theta_1)e^{-\kappa_1\Delta t})(X_2(n)-\theta_2-(X_2(n-1)-\theta_2)e^{-\kappa_2\Delta t})}{\sigma_1\sigma_2\sqrt{(1-e^{-2\kappa_1\Delta t})(1-e^{-\kappa_2\Delta t})}}\right\}=0,
\end{aligned}
\tag{6.6}
$$

Dividing by $-N$, the final result for ρ is as follows:

$$
\begin{aligned}
&\rho^3-\frac{2}{N}\rho^2\sum_{i=1}^{N}\left\{\sqrt{\kappa_1\kappa_2}\frac{(X_1(n)-\theta_1-(X_1(n-1)-\theta_1)e^{-\kappa_1\Delta t})(X_2(n)-\theta_2-(X_2(n-1)-\theta_2)e^{-\kappa_2\Delta t})}{\sigma_1\sigma_2\sqrt{(1-e^{-2\kappa_1\Delta t})(1-e^{-\kappa_2\Delta t})}}\right\}\\
&+\frac{\rho}{N}\left[\sum_{i=1}^{N}\left\{\frac{\kappa_1(X_1(n)-\theta_1-(X_1(n-1)-\theta_1)e^{-\kappa_1\Delta t})^2}{\sigma_1^2(1-e^{-2\kappa_1\Delta t})}\right.\right.\\
&\qquad+\left.\left.\frac{\kappa_2(X_2(n)-\theta_2-(X_2(n-1)-\theta_2)e^{-\kappa_2\Delta t})^2}{\sigma_2^2(1-e^{-2\kappa_2\Delta t})}\right\}-N\right]\\
&-\frac{2}{N}\sum_{i=1}^{N}\left\{\sqrt{\kappa_1\kappa_2}\frac{(X_1(n)-\theta_1-(X_1(n-1)-\theta_1)e^{-\kappa_1\Delta t})(X_2(n)-\theta_2-(X_2(n-1)-\theta_2)e^{-\kappa_2\Delta t})}{\sigma_1\sigma_2\sqrt{(1-e^{-2\kappa_1\Delta t})(1-e^{-\kappa_2\Delta t})}}\right\}=0.
\end{aligned}
\tag{6.7}
$$

6.1.2 Results

The conclusion of the calibration process for stochastic correlation leads us to the results in Table 6.1. These are the best parameters, or optimal values, of the constants, κ_1, κ_2, θ_1, θ_2, σ_1 and σ_2.

The statistical data of the stochastic correlation ρ_{best} is given in table 6.2. The time series of the best correlation is shown in figure 6.1.

κ_1	κ_2	θ_1	θ_2	σ_1	σ_2
0.0081	0.0068	1.7837	1.8490	-0.0342	-0.0482

Table 6.1: Best Parameters choosing from maximizing the loq-likelihood through ρ_{max}

mean	std	skewness	kurtosis	min	max
-0.0286	0.0616	-3.8834	19.9427	-0.3647	0.1145

Table 6.2: Data Statistics of ρ_{best}

6.2 Construction of Time Series for the Correlation by Daily Change of Regression

This section constructs the stochastic correlation time series using the daily change of regression method. The process comprises of the undermentioned steps.

1. For a given time frame $\eta_T = 30$, we use the daily change of regression at time t is used using lagged future prices $X_1(t)$ and $X_2(t)$, i.e. the previous days future prices $X_1(t-\Delta t)$ and $X_2(t-\Delta t)$. The regression equations are as follows

$$\Delta X_1(t) = X_1(t) - X_1(t-\Delta t) = a_1 + b_1 X_1(t-\Delta t) + \varepsilon_1(t), \tag{6.8}$$

$$\Delta X_2(t) = X_2(t) - X_2(t-\Delta t) = a_2 + b_2 X_2(t-\Delta t) + \varepsilon_2(t). \tag{6.9}$$

2. For a given time frame $\eta_T = 30$, the correlation coefficient $\rho(t)$ is calculated by the following formula between $\varepsilon_1(t)$ and $\varepsilon_2(t)$, given as follows:

$$\rho(t) = \frac{\sum_{i=1}^{\eta_T}(\varepsilon_1(t-i\Delta t) - \overline{\varepsilon_1(t)})(\varepsilon_2(t-i\Delta t) - \overline{\varepsilon_2(t)})}{\sqrt{\sum_{i=1}^{\eta_T}(\varepsilon_1(t-i\Delta t) - \overline{\varepsilon_1(t)})^2(\varepsilon_2(t-i\Delta t) - \overline{\varepsilon_2(t)})^2}}, \tag{6.10}$$

where $\quad \overline{\varepsilon_1(t)} = \frac{1}{\eta_T}\sum_{i=1}^{\eta_T}\varepsilon_1(t-i\Delta t) \quad$ and $\quad \overline{\varepsilon_2(t)} = \frac{1}{\eta_T}\sum_{i=1}^{\eta_T}\varepsilon_2(t-i\Delta t).$

Then we move time next to $(t+\Delta t)$ and so on to retrieve a time series of the correlation coefficient.

Statistics	Mean	Std.Dev	Skewness	Kurtosis	Min	Max
$\rho(t)$	0.7160	0.2211	-2.2740	7.4322	-0.2255	0.8500

Table 6.3: Statistical summary of Correlation time series

Figure 6.2 shows the correlation between two log future prices of our natural gas assets. The correlation coefficient

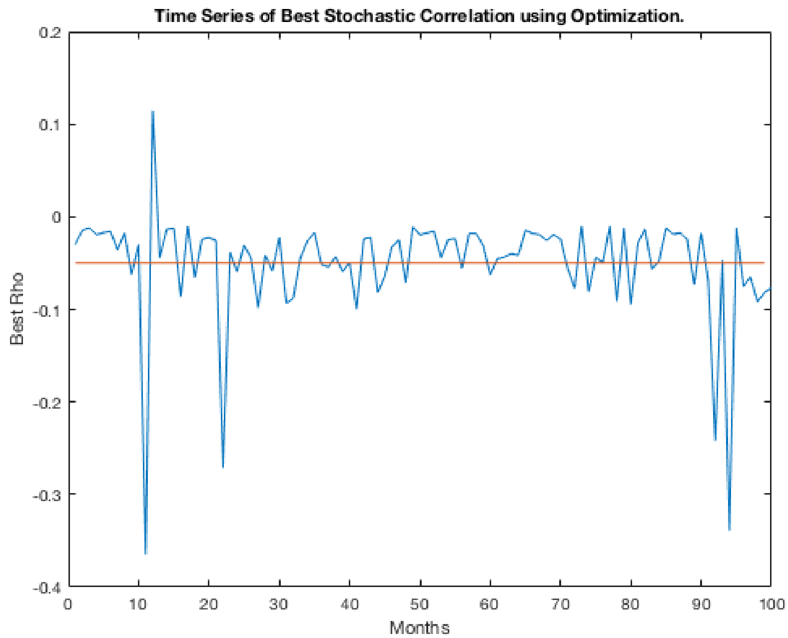


Figure 6.1: Time Series of Stochastic Correlation between UN-dawn and TCP-Iroquois NG-FP

series is presented in Figure 6.2 below which shows a more pronounced cyclicality in the data, compared to Figure 6.1. In this case, the fluctuation or correlation seems to be higher, with a natural convergence towards perfectly positive correlation. Table 6.3, presents the summary statistics for this data series also reflects this information. The daily change or regression method presents a series of correlation between futures prices where, on average, there is a high positive correlation. The density function for the correlation time series $\rho(t)$ has negative skewness and consider to have a fat tail. The average correlation between Union Gas Dawn Hub and TCPL-Iroquois natural gas future pricing is 0.7160. With a much higher mean, and standard deviation, and a greater range of values, between -0.22 and 0.85, it is safe to conclude that there is greater variation in the data series generated through the daily change of regression method.

6.2.1 Stochastic Correlation via Jacobi Process

Definition 6.1. The Jacobi Process

Let $Y(t)$ be the Jacobi process solution of the following SDE

$$dY(t) = \kappa(\theta - Y(t))dt + \sigma\sqrt{Y(t)(1-Y(t))}dW(t), \tag{6.11}$$

with $\kappa > 0$, $\sigma > 0$, $0 < \theta < 1$ and $W(t)$ is a standard Brownian motion, unrelated with $W_1(t)$ and $W_2(t)$ defined in

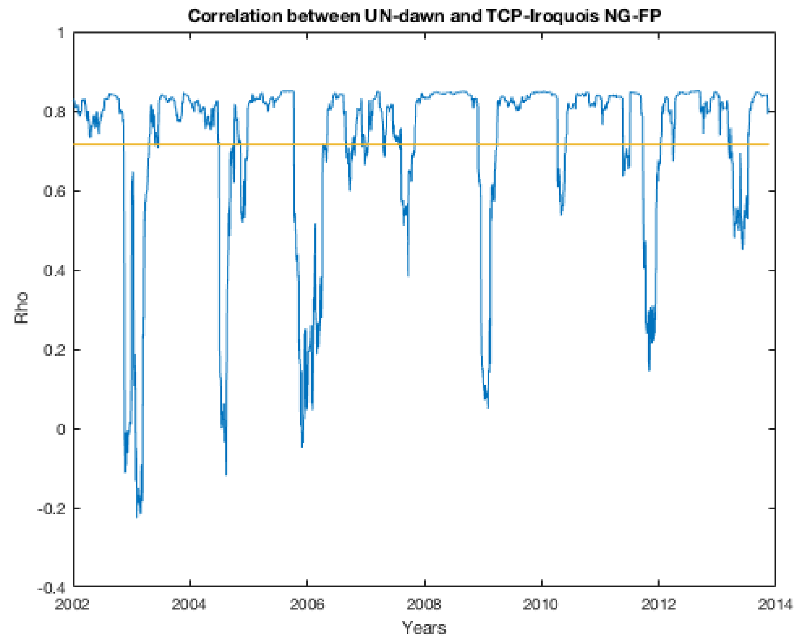


Figure 6.2: Correlation between the log future prices of UN-dawn and TCP-Iroquois Natural gas

(4.18) and (4.19). This is a stationary process with a value ranging between 0 and 1. Where κ is the mean-reverting parameter, θ is the mean of the process and σ is the volatility coefficient.

6.2.2 Construction Of Jacobi Process

In Figure 6.2, we plot the rate of change of correlation $\rho(t)$ with a time difference of $\eta_T = 30$. The graph we can clearly indicates that the correlation between future prices changes unexpectedly with time and that it does not remain constant. However, the correlation exhibits strong mean-reversion features, as presented in Figures 6.1 and 6.2. Correlation ρ is meant to be restricted between -1 and 1, according to the Jacobi process the interval is transformed to $(0,1)$ by using $p = 2Y - 1$. The correlation series as a Jacobian process, ranging from ρ_m and ρ_M can be expressed as:

$$\rho(t) = \rho_m + (\rho_M - \rho_m)Y(t), \tag{6.12}$$

where $Y(t)$ is defined by (6.11).

Following [41] it is ensured that the corresponding condition will make boundaries 0 and 1 unattainable for the process $Y(t)$:

$$\frac{\sigma^2}{2\kappa} \leq \theta \leq 1 - \frac{\sigma^2}{2\kappa^Q}, \qquad \frac{\sigma^2}{2\kappa^Q} \leq \theta^Q \leq 1 - \frac{\sigma^2}{2\kappa^Q}. \tag{6.13}$$

6.2.3 Jacobian Solution Using Change of Variables

Normally there is no closed-form expression for the transition density function for the Jacobi process [47]. However, we can use a change of variables and approximate the transition density function of the Jacobi process as follows:

$$Y(t) = \sin^2\Theta(t); \qquad \Theta(t) = \sin^{-1}\sqrt{Y(t)}, \qquad \Theta_i(t) \in [0, \frac{\pi}{2}], \tag{6.14}$$

and

$$\rho_i \in [-1,1], \qquad Y_i = \frac{\rho_i + 1}{2} \in [0,1], \qquad \rho_i = 2Y_i - 1, \tag{6.15}$$

with

$$\frac{d\Theta}{dY} = \frac{1}{\sin 2\Theta}; \qquad \frac{d^2\Theta}{dY^2} = -\frac{2\cos 2\Theta}{\sin^3 2\Theta}. \tag{6.16}$$

Now using Ito's Lemma, as defined in Chapter 2, the following result is derived:

$$d\Theta(t) = \frac{1}{\sin 2\Theta(t)}\left\{\kappa(\Theta(t) - \sin^2\Theta(t))dt + \frac{\sigma}{2}\sin 2\Theta(t)dW(t)\right\} - \frac{\sigma^2\cos 2\Theta(t)}{4\sin 2\Theta(t)}dt. \tag{6.17}$$

This can be rearranged as:

$$d\Theta(t) = \frac{\sigma^2}{4}\left[(A - \frac{1}{2})\cot\Theta(t) - (B - \frac{1}{2})\tan\Theta(t)\right]dt + \frac{\sigma}{2}dW(t), \tag{6.18}$$

where A and B are constants.

6.2.4 Calibrating the Jacobi Process

A transition density function is required in order to estimate the Jacobi process. To obtain an approximation, the Backward Euler's method is employed then the loglikelihood method is used to calibrate.

Definition 6.2. Backward Euler's Method

A Backward Euler's Method also known as the Implicit Euler's Method, is stated as

$$\begin{aligned} \frac{dy}{dt} &= F(t,y), \\ y(0) &= y_0 \\ \frac{y_{n+1} - y_n}{\Delta t} &= F(t_{n+1}, y_{n+1}). \end{aligned} \tag{6.19}$$

Now we will calibrate this process using Backward Euler's Method as follows:

$$\begin{aligned} g(\Theta_{i+1}) &= \Theta_{i+1} - \frac{\sigma^2}{4}\Delta t\left[(A-\frac{1}{2})\cot\Theta_{i+1} - (B-\frac{1}{2})\tan\Theta_{i+1}\right] \\ &= \Theta_i + \frac{\sigma^2}{2}\Delta t Z_i = \Theta_i + \hat{\sigma} Z_i, \end{aligned} \tag{6.20}$$

where $\hat{\sigma} = \frac{\sigma}{2}\Delta t$.

Let

$$g(\Theta) = \Theta - \hat{\sigma}^2\left[(A-\frac{1}{2})\cot\Theta - (B-\frac{1}{2})\tan\Theta\right], \tag{6.21}$$

and

$$g^{'}(\Theta) = 1 + \hat{\sigma}^2\left[(A-\frac{1}{2})\csc^2\Theta + (B-\frac{1}{2})\sec^2\Theta\right]. \tag{6.22}$$

The probability density function is defined as:

$$\begin{aligned} \mathbb{P}[\Theta_{i+1} \leq \Theta \mid \Theta = \Theta^{'}] &= \mathbb{P}[g(\Theta_{i+1}) \leq g(\Theta) \mid \Theta = \Theta^{'}] \\ &= \mathbb{P}[\Theta^{'} + \hat{\sigma} Z_i \leq g(\Theta)] \\ &= \mathbb{P}[Z_i \leq \frac{g(\Theta) - \Theta^{'}}{\hat{\sigma}}] \\ &= \Phi(\frac{g(\Theta) - \Theta^{'}}{\hat{\sigma}}), \end{aligned} \tag{6.23}$$

so

$$\mathbb{P}_{\Theta_{i+1}|\Theta_i}(\Theta, \Theta^{'}) = \frac{g^{'}(\Theta)}{\hat{\sigma}}\Phi(\frac{g(\Theta) - \Theta^{'}}{\hat{\sigma}}). \tag{6.24}$$

Using the maximum log-likelihood defined in section 5.2, the corresponding function becomes:

$$\begin{aligned} L &= \prod_{i=1}^{N} \frac{g^{'}(\Theta_{i+1})}{\hat{\sigma}}\Phi(\frac{g(\Theta) - \Theta^{'}}{\hat{\sigma}}) \\ &= \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\hat{\sigma}^2}}\frac{g^{'}(\Theta_{i+1})}{\hat{\sigma}}\exp(-\frac{1}{2\hat{\sigma}^2}(g(\Theta_{i+1}) - \Theta_i^{'})^2). \end{aligned} \tag{6.25}$$

With a logarithmic transformation:

$$
\begin{aligned}
LL = \log L &= \sum_{i=1}^{N} \left[\log(\frac{1}{2\pi\hat{\sigma}^2})^{1/2} + \log\exp(-\frac{1}{2\pi\hat{\sigma}^2}(g(\Theta_{i+1}) - \Theta_i^{'})^2) + \log(\frac{g^{'}(\Theta_{i+1})}{\hat{\sigma}}) \right] \\
&= \log(\frac{1}{2\pi\hat{\sigma}})^{N/2} - \frac{1}{2\hat{\sigma}^2}\sum_{i=1}^{N}(g(\Theta_{i+1}) - \Theta_i^{'})^2 + \frac{1}{\hat{\sigma}}\sum_{i=1}^{N} log(g^{'}(\Theta_{i+1})),
\end{aligned} \tag{6.26}
$$

where,

$$
\begin{aligned}
(g(\Theta_{i+1}) - \Theta_i^{'})^2 &= [\Theta_{i+1} - \hat{\sigma}^2 \left\{ (A - \frac{1}{2})\cot\Theta_{i+1} - (B - \frac{1}{2})\tan\Theta_{i+1} \right\} - \Theta_i^{'}]^2 \\
&= \Theta_{i+1}^2 + \hat{\sigma}^4 \left\{ (A - \frac{1}{2})\cot\Theta_{i+1} - (B - \frac{1}{2})\tan\Theta_{i+1} \right\}^2 + \Theta_i^{'2} \\
&- 2\Theta_{i+1}\hat{\sigma}^2 \left\{ (A - \frac{1}{2})\cot\Theta_{i+1} - (B - \frac{1}{2})\tan\Theta_{i+1} \right\} - 2\Theta_{i+1}\Theta_i^{'} \\
&+ 2\Theta_i^{'}\hat{\sigma}^2 \left\{ (A - \frac{1}{2})\cot\Theta_{i+1} - (B - \frac{1}{2})\tan\Theta_{i+1} \right\}.
\end{aligned} \tag{6.27}
$$

Therefore,

$$
\begin{aligned}
LL = &-\frac{N}{2}\log(2\pi\hat{\sigma}^2) - \frac{1}{2\hat{\sigma}^2}\left[\Theta_{i+1}^2 + \Theta_i^{'2} + \hat{\sigma}^4 \left\{ (A - \frac{1}{2})\cot\Theta_{i+1} - (B - \frac{1}{2})\tan\Theta_{i+1} \right\}^2 \right. \\
&- 2\Theta_{i+1}\hat{\sigma}^2 \left\{ (A - \frac{1}{2})\cot\Theta_{i+1} - (B - \frac{1}{2})\tan\Theta_{i+1} \right\} - 2\Theta_{i+1}\Theta_i^{'} \\
&+ 2\Theta_i^{'}\hat{\sigma}^2 \left\{ (A - \frac{1}{2})\cot\Theta_{i+1} - (B - \frac{1}{2})\tan\Theta_{i+1} \right\} \\
&+ \frac{1}{\hat{\sigma}}\sum_{i=1}^{N}\log\left[1 + \hat{\sigma}^2 \left\{ (A - \frac{1}{2})\csc^2\Theta_{i+1} - (B - \frac{1}{2})\sec^2\Theta_{i+1} \right\} \right] \Bigg].
\end{aligned} \tag{6.28}
$$

Differentiating Equation (6.29) with respect to the unknown parameters A, B and $\hat{\sigma}$, gives:

$$
\begin{aligned}
\frac{\partial LL}{\partial A} = &-\frac{1}{2\hat{\sigma}^2}\sum_{i=1}^{N}\left[2\hat{\sigma}^4(A - \frac{1}{2})\cot^2\theta - 2(B - \frac{1}{2})\cot\theta\tan\theta \right. \\
&- 2\theta_{i+1}\hat{\sigma}^2\cot\theta + 2\hat{\sigma}^2\theta_i^{\prime}\cot\theta + \frac{1}{\hat{\sigma}}\sum_{i=1}^{N}\log(\hat{\sigma}^2\csc^2\theta),
\end{aligned} \tag{6.29}
$$

$$
\begin{aligned}
\frac{\partial LL}{\partial B} = &-\frac{1}{2\hat{\sigma}^2}\sum_{i=1}^{N}\left[2\hat{\sigma}^4(B - \frac{1}{2})\tan^2\theta - 2(A - \frac{1}{2})\cot\theta\tan\theta \right. \\
&+ 2\theta_{i+1}\hat{\sigma}^2\cot\theta + 2\hat{\sigma}^2\theta_i^{\prime}\tan\theta + \frac{1}{\hat{\sigma}}\sum_{i=1}^{N}\log(\hat{\sigma}^2\sec^2\theta),
\end{aligned} \tag{6.30}
$$

$$
\begin{aligned}
\frac{\partial LL}{\partial \hat{\sigma}} &= -\frac{n}{\hat{\sigma}} + \frac{1}{\hat{\sigma}^3}\sum_{i=1}^{N}\Bigg[\theta_{i+1}^2 + \theta_i'^2 + \hat{\sigma}^4\left\{(A-\frac{1}{2})\cot^2\theta + (B-\frac{1}{2})\tan^2\theta - 2(A-\frac{1}{2})(b-\frac{1}{2})\cot\theta\tan\theta\right\} \\
&- 2(\theta_{i+1})\hat{\sigma}^2\left\{(A-\frac{1}{2})\cot\theta - (B-\frac{1}{2})\tan\theta\right\} - 2(\theta_{i+1})(\theta_i') + 2\hat{\sigma}^2\theta_i'\left\{(A-\frac{1}{2})\cot\theta - (B-\frac{1}{2})\tan\theta\right\} \\
&- \frac{1}{2\hat{\sigma}^2}\sum_{i=1}^{N}\log\left[1+\hat{\sigma}^2\left[(A-\frac{1}{2})\csc^2\theta + (B-\frac{1}{2})\sec^2\theta\right]\right. - \frac{1}{2\hat{\sigma}^2}\sum_{i=1}^{N}\Bigg\{4\hat{\sigma}^3\Bigg((A-\frac{1}{2})\cot^2\theta + (B-\frac{1}{2})\tan^2\theta \\
&- 2(A-\frac{1}{2})(B-\frac{1}{2})\cot\theta\tan\theta - 4\theta_{i+1}\hat{\sigma}\left((A-\frac{1}{2})\cot\theta - (B-\frac{1}{2})\tan\theta\right) + 4\theta_i'\hat{\sigma}\left[(A-\frac{1}{2})\cot\theta - (B-\frac{1}{2})\tan\theta\right] \\
&+ \frac{1}{\hat{\sigma}}\sum_{i=1}^{N}\left(\frac{2\hat{\sigma}\left((A-\frac{1}{2})\csc^2\theta + (B-\frac{1}{2})\sec^2\theta\right)}{1+\hat{\sigma}^2\left((A-\frac{1}{2})\csc^2\theta + (B-\frac{1}{2})\sec^2\theta\right)}\right).
\end{aligned}
\tag{6.31}
$$

These are three equations with three unknowns, where equating to zero can provide the optimal values for the unknown parameters.

6.2.5 Simulating the Jacobi Process

Definition 6.3. Newton-Raphson's Method

Newton-Raphson's method is established on the simple idea of linear approximation. Let $f(x)$ be a well-mannered function, and let r be a root of the equation $f(x) = 0$. We begin with an estimate of x_0 of r. Assuming x_n is the current estimate, the general equation for Newton-Raphson's method suggests that the estimated x_{n+1} is given by:

$$
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{6.32}
$$

The Newton–Raphson's method is used to solve the Backward Euler's Method. Initial guess ϑ_1 we will get it through Beta distribution. Next our function $f(\Theta_{i+1})$ and $f^{'}(\Theta_{i+1})$ are defined in (6.21) and (6.22) respectively:

$$
f(\Theta_{i+1}) = \Theta_{i+1} - \hat{\sigma}^2\left[(A-\frac{1}{2})\cot\Theta_{i+1} - (B-\frac{1}{2})\tan\Theta_{i+1}\right] - \Theta_i - \hat{\sigma}Z_i, \tag{6.33}
$$

and

$$
f^{'}(\Theta_{i+1}) = 1 + \hat{\sigma}^2\left[(A-\frac{1}{2})\csc^2\Theta_{i+1} + (B-\frac{1}{2})\sec^2\Theta_{i+1}\right]. \tag{6.34}
$$

Therefore, using Newton Raphson's Methods in equation (6.31) aand replacing the unknown values, a solution for Θ is obtained.

$$
\Theta_{i+1} = \Theta_i - \frac{\Theta_{i+1} - \hat{\sigma}^2\left[(A-\frac{1}{2})\cot\Theta_{i+1} - (B-\frac{1}{2})\tan\Theta_{i+1}\right] - \Theta_i - \hat{\sigma}Z_i}{1+\hat{\sigma}^2\left[(A-\frac{1}{2})\csc^2\Theta_{i+1} + (B-\frac{1}{2})\sec^2\Theta_{i+1}\right]}. \tag{6.35}
$$

6.2.6 Results

A	B	$\hat{\sigma}$
2.4142	2.4142	1.0000

Table 6.4: Unknown parameters of Jacobi Process

The following table 6.4 shows the values of unknown parameters A, B, and $\hat{\sigma}$ of the Jacobi Process estimated using the maximum likelihood method.

Statistics	Mean	Std.Dev	Skewness	Kurtosis	Min	Max
$\rho(t)$	0.5063	0.2555	-1.0052	5.1957	-0.5937	0.9777

Table 6.5: Statistical summary of Correlation time series

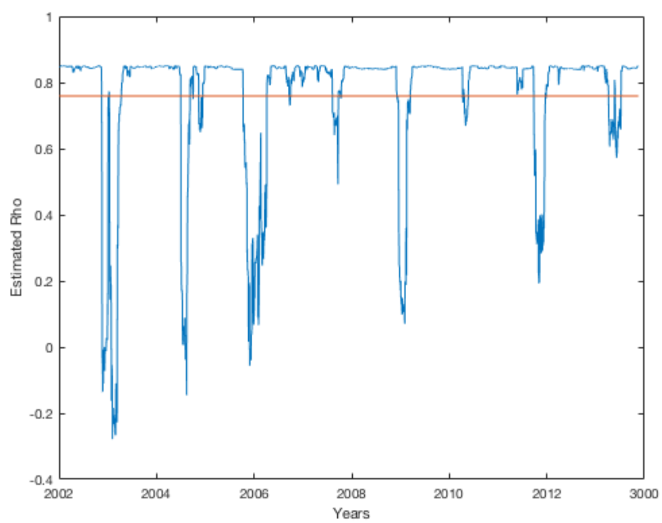


Figure 6.3: Time Series of Stochastic Correlation using Jacobi Process

Figure 6.3 illustrates the correlation series generated by the Jacobi process and its stochasticity. This seems to be little to no cyclicality in this series, and it seems to be more random. Unlike the previous series, the Jacobi process series tends to revert to the mean, represented in the figure by the horizontal line.

Table 6.5 presents summary statistics for correlation time series estimated through the Jacobi process. The long-term average for this data series is 0.51, which lies between the averages of 0.11 and 0.72 of the earlier correlation series. There is greater variation in this series, compared to previous ones, with a higher standard deviation and a greater range, with a minimum of -0.59 and a maximum value of 0.98.

In Figure 6.4 we see the difference(error) between the stochastic correlation versus the estimated time series of correlation generated by the Jacobi process and it's very stochastic while in Figure 6.5 we see the

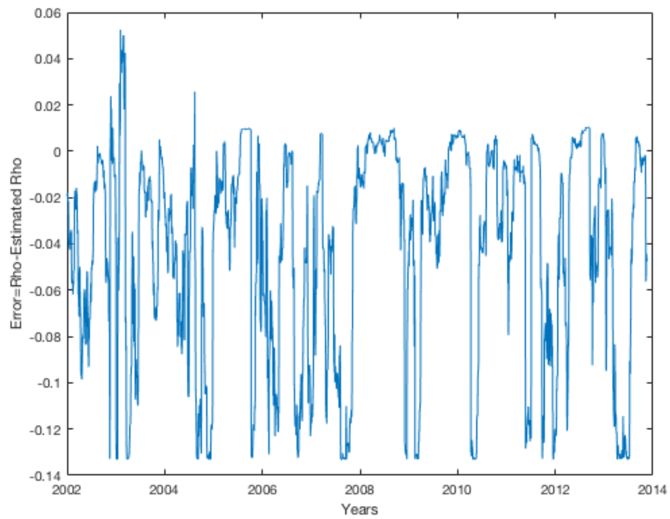


Figure 6.4: Error between Time Series of Stochastic Correlation and other using Jacobi

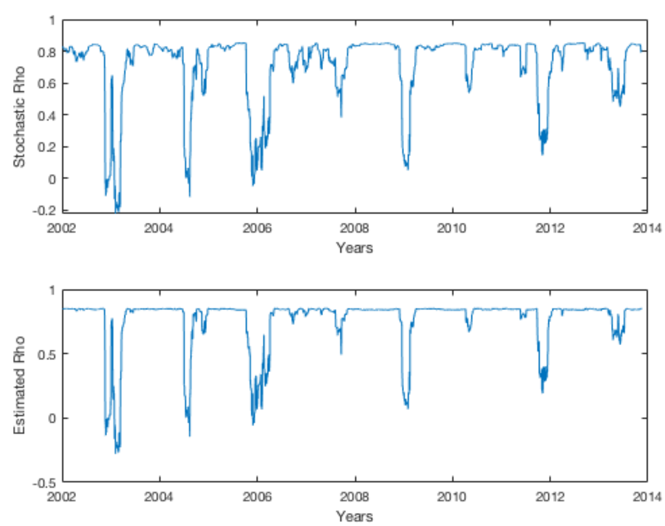


Figure 6.5: Time Series of Stochastic Correlation and correlation using Jacobi Series

Chapter 7

Spread Option Pricing

In this chapter, we will discuss option pricing models, which are useful mathematical models designed to value the options. An option is a contract between the two counterparties that gives a right to one party (not an obligation) to buy or sell a particular asset at a prearranged price before or at the expiration date. There are two main types of options: Calls and Puts, in this chapter we'll discuss only call options. Options can also be categorized according to their exercised time: European or American. European options are traded at a reduced price as compared to their American counterparts. Nowadays, the options market is not restricted to only basic call or put options but there's a collection of multi-dimensional options for investors, such as sparks, cracks, locational spreads, etc. These kinds of option contracts have a wide range of applications in the financial industry as one can lower their risk when the stock moves exceptionally against them. Spread options, being the simplest of multi-dimensional options, attained a special place in the finance literature.

Spread options represent an unusual challenge due to unachievable analytical solutions for most market models. Margrabe [44] developed a closed-formed formula for the value of the option when exchanging one risky asset for another. The formula he derived is based on the Black-Scholes model. So following the footsteps of Margarbe's formula we will produce a closed form for our Locational Spread options with constant correlation. Then we discuss the Monte Carlo simulation for stochastic correlation. Since Monte Carlo is a more refined method to value an option and it helps us to demonstrate the impact of risk and uncertainty in prediction and forecasting models. In this method, we simulate the possible spot prices and later used them to find the discounted expected option payoff for stochastic correlation. Lastly, we will compare our results and predict how they help in the context of stochastic correlation.

7.1 The Margrabe Formula

7.1.1 Definition

The Margrabe formula gives a first approximation of the cost of the spread option and can be exercised as the baseline for more complex models [43]. In general, this is an empirical formula. In this, we assume that spot prices S_1 and S_2 follow geometric Brownian motion: $dS_i(t) = S_i(t)[rdt + \sigma_i dW_i(t)]$, $i = 1,2$, under the money-market measure, whereas the correlation ρ between the Brownian motions is assumed to be a constant, with $dW_1 dW_2 = \rho dt$. Now considering, $K = 0$, the value of the spread option $V(T, S_1(0), S_2(0))$ is given by:

$$V(T, S_1(0), S_2(0)) = e^{-rT}\left[(S_1(T) - S_2(T), 0)\right]^+, \tag{7.1}$$

or it can be written as:

$$V = e^{-rT}\left[S_1(0)N(d_1) - S_2(0)N(d_2)\right], \tag{7.2}$$

whereas $N(d_1)$and $N(d_2)$ are probabilities, estimated by using a cumulative standardized normal distribution and d_1 and d_2 is given as:

$$d_{1,2} = \frac{\ln(S_1(0)/S_2(0)) \pm \frac{1}{2}\hat{\sigma}^2}{\hat{\sigma}}, \tag{7.3}$$

with

$$\hat{\sigma}^2 = Var[\sigma_1 W_1(T) - \sigma_2 W_2(T)] = T(\sigma_1^2 - \sigma_2^2 - 2\rho\sigma_1\sigma_2). \tag{7.4}$$

Margrabe's formula is basically an application of Black and Scholes's formula considering S_1 as the underlying asset and S_2 as the strike, with the equivalent volatility $\hat{\sigma}$ the variance of $\ln[S_1/S_2]$. The result can be easily proven by using a change of numeraire.

7.1.2 Margrabe's Formula for two log normally distributed variables

Suppose that $S_i(T) = e^{rT}e^{X_i}$, with $X_i \sim N(m_i - b_i^2/2, b_i^2)$ under some risk-neutral measure Q whereas the correlation between X_1 and X_2 is ρ. The swap (exchange) value is given as follows:

$$\begin{aligned} V &= e^{-rT}\mathbb{E}_Q\left[(S_1(T) - S_2(T))^+\right] \\ &= \mathbb{E}_Q\left[(e^{X_1} - e^{X_2})^+\right] \\ &= \mathbb{E}_Q\left[e^{X_2}(e^{X_1 - X_2} - 1)^+\right]. \end{aligned} \tag{7.5}$$

We can write $X_2 = m_2 - b_2^2/2 + b_2 z_2$ and $X_1 = m_1 - b_1^2/2 + b_1(\rho z_2 + \bar{\rho} z_1)$, where z_1 and z_2 are iid $N(0,1)$ and $\bar{\rho} = \sqrt{(1-\rho^2)}$. Then

$$\begin{aligned} Y = X_1 - X_2 &= m_1 - m_2 + (b_2^2 - b_1^2)/2 + b_1\bar{\rho} z_1 + (b_1\rho - b_2) z_2 \\ &= m_1 - m_2 + (b_2^2 - b_1^2)/2 - \hat{b}\hat{z}. \end{aligned} \tag{7.6}$$

Since $X_1 \geq X_2$ when $\hat{z} \leq \frac{m_1-m_2}{\hat{b}} + \frac{b_2^2-b_1^2}{z\hat{b}}$, where as $\hat{b} = \sqrt{b_1^2 + b_2^2 - \rho b_1 b_2}$ and $\hat{z} = -\frac{1}{\hat{b}}(b_1\bar{\rho} z_1 + (b_1\rho - b_2)z_2) \sim N(0,1)$. In terms of $\hat{z}$, we can write: $z_2 = \frac{b_1\rho - b_2}{\hat{b}}\hat{z} + \frac{\bar{\rho} b_1}{\hat{b}}\hat{w}$, for some $\hat{w} \sim N(0,1)$, independent of $\hat{z}$. Then:

$$X_2 = m_2 - \frac{b_2^2}{2} - \frac{b_2(b_1\rho - b_2)}{\hat{b}}\hat{z} + \frac{\bar{\rho} b_1 b_2}{\hat{b}}\hat{w}. \tag{7.7}$$

We can now calculate the option value with $z \sim N(0,1)$ and $w \sim N(0,1)$ as follows:

$$\begin{aligned} V &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{m_2 - b_2^2/2 - \frac{b_2(b_1\rho-b_2)}{\hat{b}}\hat{z} + \frac{\bar{\rho}b_1b_2}{\hat{b}}w}\left(e^{m_1-m_2+\frac{b_2^2-b_1^2}{2}} - 1 - \hat{b}z\right)^{+}\varphi(z)\varphi(w)\,dz\,dw \\ &= \int_{-\infty}^{\infty}\left(e^{m_2-b_2^2/2+\frac{\bar{\rho}b_1b_2}{\hat{b}}w}\right)\varphi(w)\,dw + \int_{-\infty}^{e^{\frac{m_1-m_2}{\hat{b}}+\frac{b_2^2-b_1^2}{2\hat{b}}}}\left(e^{m_1-m_2+\frac{b_2^2-b_1^2}{2}+(\frac{-b_2(b_1\rho-b_2)}{\hat{b}}-\hat{b})z} - e^{\frac{-b_2(b_1\rho-b_2)}{\hat{b}}z}\right)\varphi(z)\,dz. \end{aligned} \tag{7.8}$$

Now simplifying $-b_2(b_1\rho - b_2) - \hat{b} = b_1(b_2\rho - b_1)$ and solving the integral we get the following result:

$$\begin{aligned} V &= e^{m_2-b_2^2/2+\frac{\bar{\rho}^2b_1^2b_2^2}{2\hat{b}^2}}\left(e^{m_1-m_2+\frac{b_2^2-b_1^2}{2}}\int_{-\infty}^{e^{\frac{m_1-m_2}{\hat{b}}+\frac{b_2^2-b_1^2}{2\hat{b}}}} e^{\frac{b_1(b_2\rho-b_1}{\hat{b}}}\varphi(z)\,dz - \int_{-\infty}^{e^{\frac{m_1-m_2}{\hat{b}}+\frac{b_2^2-b_1^2}{2\hat{b}}}} e^{\frac{b_1(b_2\rho-b_1}{\hat{b}}}\varphi(z)\,dz\right) \\ &= e^{m_2-b_2^2/2+\frac{\bar{\rho}^2b_1^2b_2^2}{2\hat{b}^2}}(e^{m_1-m_2+\frac{b_2^2-b_1^2}{2}}e^{\frac{b_1^2(b_2\rho-b_1)^2}{2\hat{b}^2}}N\left(\frac{m_1-m_2}{\hat{b}}+\frac{b_2^2-b_1^2}{2\hat{b}}-\frac{b_1(b_2\rho-b_1}{\hat{b}}\right) \\ &\quad - e^{\frac{b_2^2(b_1\rho-b_2)^2}{2\hat{b}^2}}N\left(\frac{m_1-m_2}{\hat{b}}+\frac{b_2^2-b_1^2}{2\hat{b}}-\frac{b_1(b_2\rho-b_1}{\hat{b}}\right)). \end{aligned} \tag{7.9}$$

Now,

$(m_2 - b_2^2 + \frac{\bar{\rho}b_1^2b_2^2}{2\hat{b}^2} + m_1 - m_2 + \frac{b_2^2-b_1^2}{2} + \frac{b_1^2(b_2\rho-b_1)^2}{2\hat{b}^2}) = (m_1 + \frac{b_1^2}{2\hat{b}^2}(\bar{\rho}^2b_2^2 - \hat{b}^2 + ((b_2\rho-b_1)^2)) = m_1$

and

$(m_2 - \frac{b_2^2}{2} + \frac{\bar{\rho}^2b_1^2b_2^2}{2\hat{b}^2} + \frac{b_2^2(b_1\rho-b_2)^2}{2\hat{b}^2}) = m_2.$

Thus the final value is:

$$V = e^{m_1}N(\frac{m_1-m_2}{\hat{b}} + \frac{\hat{b}}{2}) - e^{m_2}N(\frac{m_1-m_2}{\hat{b}} - \frac{\hat{b}}{2}). \tag{7.10}$$

where N represents the cumulative distribution function of the standard normal distribution.

7.1.3 Margrabe's Formula for Ornstein-Uhlenbeck Process

As we know that the mean and variance of Ornstein-Uhlenbeck process are: $\mu = \ln S(0)e^{-\kappa t} + \theta^*(1-e^{-\kappa t})$ and $\hat{\sigma}^2 = \sigma^2 \frac{(1-e^{-2\kappa t})}{2\kappa}$. So the value of the swap option with two underlying assets $X_1 = \ln S_1$ and $X_2 = \ln S_2$ will be, by using (7.10):

$$\begin{aligned} V = e^{\mu_1} N\left(\frac{\mu_1-\mu_2}{\sqrt{\hat{\sigma}_1^2+\hat{\sigma}_2^2-2\rho\hat{\sigma}_1\hat{\sigma}_2}} + \frac{\sqrt{\hat{\sigma}_1^2+\hat{\sigma}_2^2-2\rho\hat{\sigma}_1\hat{\sigma}_2}}{2}\right) \\ - e^{\mu_2} N\left(\frac{\mu_1-\mu_2}{\sqrt{\hat{\sigma}_1^2+\hat{\sigma}_2^2-2\rho\hat{\sigma}_1\hat{\sigma}_2}} - \frac{\sqrt{\hat{\sigma}_1^2+\hat{\sigma}_2^2-2\rho\hat{\sigma}_1\hat{\sigma}_2}}{2}\right). \end{aligned} \tag{7.11}$$

Here ρ is the correlation between the two above processes. In the simplified form, we can write the pay-off function as:

$$V = e^{\mu_1} N(d_1) - e^{\mu_2} N(d_2), \tag{7.12}$$

where

$$d_1 = \left(\frac{\mu_1-\mu_2}{\sqrt{\hat{\sigma}_1^2+\hat{\sigma}_2^2-2\rho\hat{\sigma}_1\hat{\sigma}_2}} + \frac{\sqrt{\hat{\sigma}_1^2+\hat{\sigma}_2^2-2\rho\hat{\sigma}_1\hat{\sigma}_2}}{2}\right) \tag{7.13}$$

and

$$d_2 = \left(\frac{\mu_1-\mu_2}{\sqrt{\hat{\sigma}_1^2+\hat{\sigma}_2^2-2\rho\hat{\sigma}_1\hat{\sigma}_2}} - \frac{\sqrt{\hat{\sigma}_1^2+\hat{\sigma}_2^2-2\rho\hat{\sigma}_1\hat{\sigma}_2}}{2}\right). \tag{7.14}$$

7.1.4 Results

In Figure 7.1 we conclude that Spread option pricing got from Margrabe's formula decreases as rho increases from -1 to 1. Here we kept other parameters constant and Margrabe becomes a function of rho. The parameters used in this model are $\kappa_1 = 1.000e-03$, $\kappa_2 = 1.000e-03$, $\theta_1 = 1.00$, $\theta_2 = 1.00$ and $\sigma_1 = 2.00$, $\sigma_2 = 4.00$.

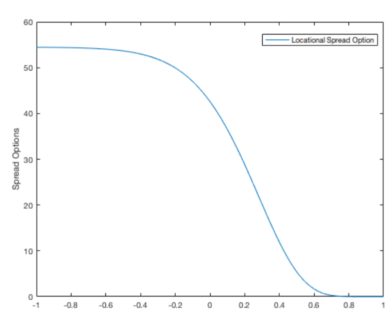


Figure 7.1: Spread Option pricing with constant Rho ρ going from -1 to 1 through Margrabe's formula with $\kappa_1 = 1.000e-03$, $\kappa_2 = 1.000e-03$, $\theta_1 = 1.00$, $\theta_2 = 1.00$ and $\sigma_1 = 2.00$, $\sigma_2 = 4.00$.

7.2 Monte Carlo Simulations

7.2.1 Definition

Monte Carlo option pricing is a process frequently used in Mathematical Finance to calculate the value of an option with multiple sources of uncertainty or with complicated features which would make it difficult to evaluate through a straightforward Black-Scholes style or any lattice-based computation. Monte Carlo methods are ideal for pricing options where the payoff is dependent on paths, for example, Asian option or the Spread option. Monte Carlo simulations turn out to be simple, flexible, and innovative. In this section, we use Monte Carlo methods to price the Spread option with stochastic correlation.

Theoretically, Monte Carlo valuation depends on risk-neutral valuation. In it, the price of the option is its discounted expected cost. The method is applied as follows:

1. We generate all possible large numbers of random price paths for the asset through simulation.
2. We calculate the corresponding exercise price of the option for each path.
3. We averaged these payoffs.
4. Then in the end discounted it to today's value, which is the price of the option.

7.2.2 Mathematical Model

Here we consider the correlated Ornstein-Uhlenbeck process for stochastic variables at time t, $X_1(t) = \ln S_1(t)$ and $X_2(t) = \ln S_2(t)$ with correlation ρ as follows, stated earlier in (4.18), (4.19) and (4.20) [70]:

$$dX_1(t) = \kappa_1[\theta_1 - X_1(t)]dt + \sigma_1 dW_1(t), \tag{7.15}$$

$$dX_2(t) = \kappa_2[\theta_2 - X_2(t)]dt + \sigma_2 dW_2(t), \tag{7.16}$$

with

$$dW_1(t)dW_2(t) = \rho dt. \tag{7.17}$$

7.2.3 Monte Carlo Simulations with Constant Correlation

These stochastic differential equations can be explicitly solvable and have the following solution in terms of stochastic integrals, with $(\theta^* = \theta - \sigma^2/2\kappa)$ [71]:

$$X_1(t) = X_1(0)e^{-\kappa_1 T} + (1 - e^{-\kappa_1 T})\theta_1^* + \sigma_1 e^{-\kappa_1 T}\int_0^T e^{\kappa_1 t} dz_1(t), \tag{7.18}$$

$$X_2(t) = X_2(0)e^{-\kappa_2 T} + (1 - e^{-\kappa_2 T})\theta_2^* + \sigma_2 e^{-\kappa_2 T}\int_0^T e^{\kappa_2 t} dz_2(t). \tag{7.19}$$

These variables $X_1(T)$ and $X_2(T)$ follow normal distribution with mean and variance as follows [72]:

$$\mathbb{E}[X_1(T)] = X_1(0)e^{-\kappa_1 T} + \theta_1^*(1 - e^{-\kappa_1 T}), \quad Var[X_1(T)] = (1 - e^{-2\kappa_1 T}) \cdot \frac{\sigma_1^2}{2\kappa_1}, \tag{7.20}$$

and

$$\mathbb{E}[X_2(T)] = X_2(0)e^{-\kappa_2 T} + \theta_2^*(1 - e^{-\kappa_2 T}), \quad Var[X_2(T)] = (1 - e^{-2\kappa_2 T}) \cdot \frac{\sigma_2^2}{2\kappa_2}. \tag{7.21}$$

In order to perform the simulation, it is important to get the discrete-time equation for the above processes (7.15) and (7.16) [72], we let $t_i = t_0 + i\Delta t$, with $i = 0, 1, 2, \cdots, N$ where $k = 1, 2$ and $i = 0, 1, \cdots, N$, are as follows:

$$X_1(i) = e^{-\kappa_1 \Delta t} X_1(i-1) + (1 - e^{-\kappa_1 \Delta t})\theta_1^* + \sigma_1 \sqrt{\frac{(1 - e^{-2\kappa_1 \Delta t})}{2\kappa_1}} Z_1, \tag{7.22}$$

$$X_2(i) = e^{-\kappa_2 \Delta t} X_2(i-1) + (1 - e^{-\kappa_2 \Delta t})\theta_2^* + \sigma_2 \sqrt{\frac{(1 - e^{-2\kappa_2 \Delta t})}{2\kappa_2}} \hat{Z}_2, \tag{7.23}$$

where

$$\hat{Z}_2 = \rho Z_1 + \sqrt{(1 - \rho^2)} Z_2, \tag{7.24}$$

with random variables Z_1, Z_2 and $\hat{Z}_2 \sim N(0, 1)$.

The payoff of Locational spread option with Monte Carlo Simulation is:

$$V = E[e^{-rt} max(X_1 - X_2, 0)]. \tag{7.25}$$

7.2.4 Results of Monte Carlo Simulations with constant Correlation

In figure 7.2, we can see that with constant correlation $\rho = 0.7030$ we get the value of locational spread option with Margrabe formula is $V = 1.6483$ which lies in the 95% confidence interval of Monte Carlo simulation with constant correlation, which is $[1.5601, 1.6681]$. The other parameters we used to evaluate our values are $\kappa_1 = 0.0056, \kappa_2 = 0.0141, \theta_1 = 1.1772$, $\theta_2 = 0.8115$ and $\sigma_1 = 0.1888$ $\sigma_2 = 0.2345$.

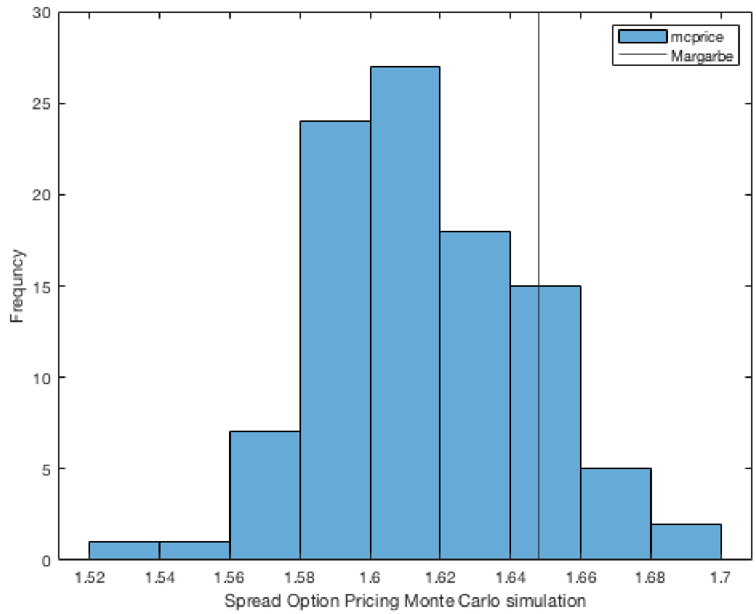


Figure 7.2: Spread Option using Monte Carlo and Margrabe's Formula with constant Correlation $\rho = 0.7030$ with $\kappa_1 = 0.0056$, $\kappa_2 = 0.0141$, $\theta_1 = 1.1772$, $\theta_2 = 0.8115$ and $\sigma_1 = 0.1888$, $\sigma_2 = 0.2345$.

7.2.5 Monte Carlo Simulations with Stochastic Correlation

In order to get Monte Carlo simulations with stochastic correlation for Spread options we model the stochastic correlation as a Jacobi process:

$$\rho(t) = \rho_m + (\rho_M - \rho_m)Y(t), \tag{7.26}$$

where $Y(t)$ is defined as:

$$dY(t) = \kappa(\theta - Y(t))dt + \sigma\sqrt{Y(t)(1 - Y(t))}dW(t), \tag{7.27}$$

These equations along with 7.22, 7.23, 7.24 and 7.26 gives us the Monte Carlo spread option value with stochastic correlation.

7.2.6 Results with Stochastic Correlation

Figure 7.3 depicts the comparison of spread option value with constant correlation with spread option value with average stochastic correlation with 5 billions samples. Here we can see the value of spread option with constant correlation and spread option with stochastic correlation goes down as correlation moves from -1 to 1. Also, we can

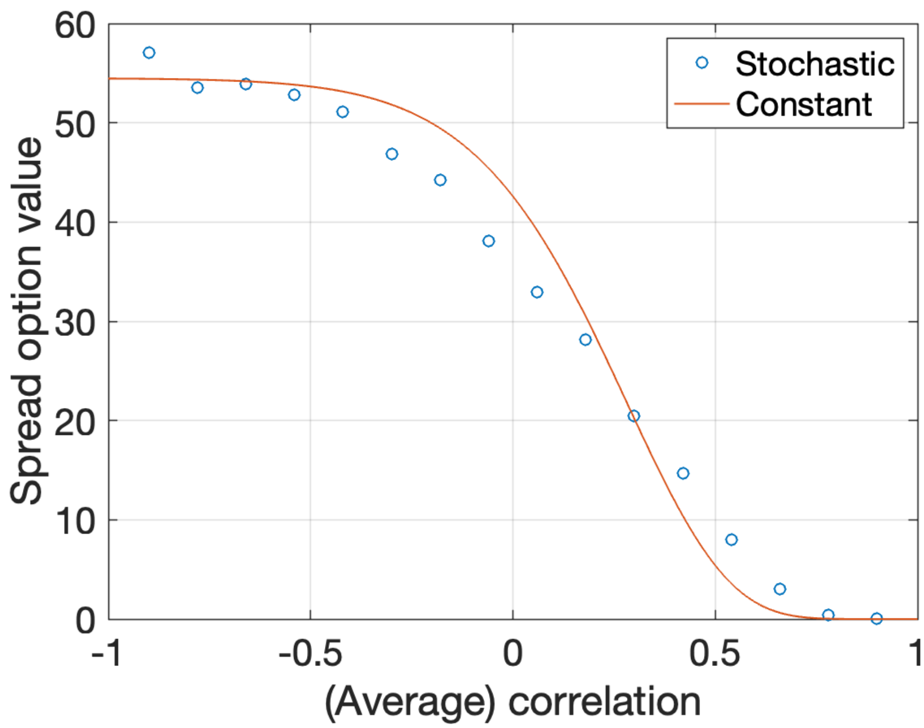


Figure 7.3: Spread Option pricing with constant correlation and stochastic correlation

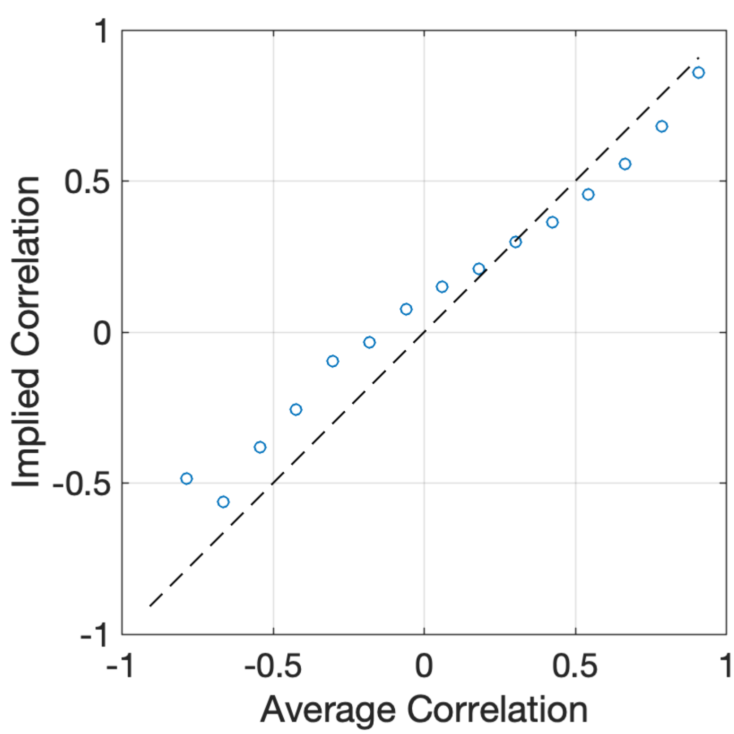


Figure 7.4: Comparison between Implied Correlation and Average Stochastic Correlation

see that the value of spread option with stochastic correlation lies in the range of that value of spread option with constant correlation. The parameters used to derive these results are shown in the following Table 7.1.

κ_1	κ_2	θ_1	θ_2	σ_1	σ_2
1.000e-03	1.000e-03	2.000	2.000	4.000	8.000

Table 7.1: Parameters choose to find the Spread Option Valuation

Later, we do another test as the check how stochastic correlation works in general. For this we calculate implied correlation with comparison to Stochastic correlation and figured out how it turns out to be. Figure 7.4 is related to implied correlation verses the average stochastic correlation. From the graph we can see that as Implied correlation behaves similar to average stochastic correlation as it goes from -1 to 1.

However, some differences are apparent. It is clear that the implied correlation calculated from Monte Carlo estimates using stochastic correlation will tend to be below the constant correlation values (using average correlation) when the Margrabe formula is a concave-up function of the correlation, and above those values when the formula is concave-down.

Chapter 8

Conclusion

This thesis introduces the idea of a stochastic correlation between the underlying energy commodities to model prices of the locational spread options. The locational spread option is based on natural gas, which is widely available in Alberta and is one of Canada's biggest export, and is the energy commodity. The spread is calculated by taking a difference between the prices found in Dawn hub and Iroquois.

These assets are modeled for constant correlation using Ornstein-Uhlenbeck process and for stochastic correlation using the Jacobi process. To evaluate the unknown parameters of locational spread commodities, the model is first calibrated with constant correlation; using two methods: the multivariate linear regression and the maximum log-likelihood. The comparison of these techniques showed that they exhibit the same results as expected.

Similar to constant correlation, we calibrate the model with stochastic correlation using two techniques as well; the reduced-form optimization technique and the Jacobi process. For the Jacobi process, the backward Euler's method is employed to approximate the transition density function, and then Newton Raphson's method to solve the equations to get unknown parameters. The results from these two methods predict the existence of stochastic correlation differently.

Finally, Margrabe's formula for two log-normally distributed asset prices is adopted to give a closed-form solution for locational spread options with constant correlation. The findings from this process indicate that its easier to evaluate locational spread option valuation. The results from Margrabe's formula are then compared with the Monte Carlo simulations for spread option pricing assuming constant correlation. The comparison suggests that the value of Margrabe's lies in the confidence interval of Monte Carlo with constant correlation. Then Monte Carlo simulations are also employed for option pricing with stochastic correlation to check its impact on the pricing. These simulations demonstrate the value of the spread option with stochastic correlation differs from the value that would be computed using constant correlation (at the average value of the stochastic correlation), and that the difference depends on the shape of the Margrabe formula at that point.

Further research in this direction would proceed in various ways: firstly, we can use a modified Jacobi process to model stochastic correlation as discussed in [34] or a more generalized stochastic process such as that recommended in [35] which depends on the hyperbolic transformation with the hyperbolic tangent function of any mean-reverting process with positive and negative values. Secondly in this thesis, we only got the estimated values of spread option pricing but we didn't compare them with market value. Thus, we can compare our results with market value and explain the authenticity of the model.

Furthermore, the framework presented in this paper can have several different applications in three key areas; different commodities, different financial instruments, and different locations. This thesis is restricted to the natural gas commodities in Alberta. Further implementation of this model could potentially include locational spread options in other energy commodities traded elsewhere on the US-Canada border, such as crude oil or petroleum. The model has the potential to be implemented to other types of financial instruments or other types or styles of options contracts. Moreover, the model could further be implemented in natural gas spread option pricing elsewhere in the world.

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