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The Geometry of the Rolling Disk

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "The Geometry of the Rolling Disk" submitted by David Wayne Kemppainen in partial fulfillment of the requirements for the degree of Master of Science.

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Abstract

A disk that rolls without slipping is an example of an integrable nonholonomic mechanical system. The equations of motion for the disk can be found by the use of Lagrange multipliers and Lagrange's equations of motion. However, the configuration space of the rolling disk $R^2 \times SO(3)$ admits the group action $E(2) \times SO(2)$. In the paper *Nonholonomic Reduction* [2] by L. M. Bates and J. Śniatycki, they have a reduction theorem for nonholonomic mechanical systems. With this technique the rolling disk can be reduced to a problem on a four dimensional manifold $\bar{M}$ with a nondegenerate non-closed two-form.

By partially integrating the differential equations on $\bar{M}$, one realizes that the reduced dynamics on $\bar{M}$ is foliated by a two parameter family of Hamiltonian systems on the cotangent bundle of an open interval with one degree of freedom, where the two free parameters correspond to two conserved quantities.

Acknowledgements

As a child I remember having a talent for mathematics. I was usually the first person to finish my mathematics test in elementary school. Only when I arrived at the University of Calgary did I realize the true diversity of mathematics. In particular how intertwined mathematics and physics are in explaining our universe.

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Chapter 1

Reduction

1.1 History

Mechanics is an important part of mathematics today. Since Newton has published his three laws of motion in *Principia* in 1687, there has been steady progression in the area of mechanics.

In statics, the principle of virtual work states that a mechanical system is in equilibrium if the virtual work δW done by the force $F = (F_1, \cdots, F_n)$ for all possible virtual displacements $\delta x = (\delta x_1, \cdots, \delta x_n)$ which do not violate the given constraints is zero.

$$\delta W = \sum_{i=1}^{n} F_i \delta x_i = 0. \tag{1.1}$$

These constraints can be holonomic or nonholonomic, however, Bernoulli originally stated the principle of virtual work for holonomic constraints in 1717. Holonomic constraints can be written as a integrable one-form where as nonholonomic constraints cannot. D'Alembert realized that the virtual work done by the forces of constraint is zero. This is known as d'Alembert principle. This led to a generalization of the principle of virtual work to dynamics by including the force of inertia with the other forces. Thus Newtons equations can be written in the form

$$\sum_{i=1}^{n} (m\ddot{x}_i - F_i)\,\delta x_i = 0, \tag{1.2}$$

where the force $F = (F_1, \cdots, F_n)$ do not include the forces of constraint and the virtual displacements $\delta x = (\delta x_1, \cdots, \delta x_n)$ do not violate the given constraints. This work was done in 1743 in his book *Traité de dynamique.*

Lagrange made several contributions to the area of mechanics. One contribution is the method of Lagrange multipliers dealing with problems involving constraints. Instead of solving equation (1.2) for all possible variations δx satisfying the constraints, one can solve the following equivalent problem. If the k constraint one-forms are

$$\sum_{i=1}^{n} A_{ij} dx_i, \text{ for } j = 1, \cdots, k, \tag{1.3}$$

and the λ_j's are the Lagrange multipliers then for all possible variations δx we have

$$\sum_{i=1}^{n} \left(m\ddot{x}_i - F_i - \sum_{j=1}^{k} \lambda_j A_{ij} \right) \delta x_i = 0. \tag{1.4}$$

Another contribution is the idea of generalized coordinates and configuration space. The advantage of generalized coordinates is that the fewest number of coordinates are needed to describe the configuration of a mechanical system. A mechanical system with n degrees of freedom and k holonomic constraints can be written as a mechanical system with $n - k$ degrees of freedom and no constraints. Because nonholonomic constraints involve a relationship between the velocities, we cannot eliminate a coordinate. Another advantage of generalized coordinates is that the equations of motion are not dependent upon the coordinates chosen. Therefore, a Hamiltonian system with n degrees of freedom and k nonholonomic constraints, in generalized coordinates $q = (q_1, \cdots, q_n)$, equation (1.4) becomes

$$\sum_{i=1}^{n} \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} - F_i - \sum_{j=1}^{k} \lambda_j A_{ij} \right) \partial q_i = 0, \tag{1.5}$$

where T is the kinetic energy of the mechanical system. A variation of these equations are Hamilton's equations, whereby the system of second order differential equations is changed into a system of first order differential equations by the Legendre transformation. Most of this work was published by Lagrange in *Méchanic Analytique* in 1788.

With the development of variational calculus, we have the creation of other variational principles. Euler derived equation (1.5) by the methods of variational calculus. Thus equation (1.5) is sometimes referred to as the Euler-Lagrange's equations of motion. Hamilton's principle for a mechanical system with conservative forces and holonomic constraints states that the variation of the integral

$$\delta I = \delta \int_{t_1}^{t_2} L dt = 0, \tag{1.6}$$

where L is the Lagrangian function. Hamilton's principle does not work for nonholonomic constraints.

Before reduction was accomplished, the theory of Lie groups and group actions on symplectic manifolds were needed. These theories were developed in the 1800's.

1.2 Nonholonomic reduction

The idea of reduction is to reduce the number of degrees of freedom a mechanical system has by taking advantage of the symmetries. With holonomic constraints it was realized that for every holonomic constraint a mechanical system had the number of degrees of freedom can be reduced by one. However this is not possible for nonholonomic constraints. Due to the difficulty of handling nonholonomic constraints the first attempts at reduction were for holonomic Hamiltonian systems with symmetry. Meyer-Marsden-Weinstein [5] [4] came up with a reduction method for a holonomic Hamiltonian system with symmetry. This was accomplished by taking a Hamiltonian system on a symplectic manifold and quotienting by the group actions of the symmetries to get a reduced Hamiltonian system on a smaller symplectic manifold. This theory is fully explained in *Foundations of Mechanics* by Abraham and Marsden [1]. However, this theory does not apply to nonholonomic Hamiltonian systems.

In the paper *Nonholonomic reduction* [2] by L. M. Bates and J. Śniatycki, they have a reduction technique for nonholonomic Hamiltonian systems. A disk that rolls without slipping is an example of a nonholonomic Hamiltonian system, which is integrable. Because the configuration space Q of the disk admits a fairly large symmetry group $E(2) \times SO(2)$, it is an excellent example to test their reduction theory. In applying their method, the rolling disk can be reduced from a nonholonomic Hamiltonian system with five degrees of freedom to a problem on a four dimensional manifold $\bar{M}$ with a nondegenerate non-closed two-form. By partially integrating the differential equations on $\bar{M}$ one realizes the reduced dynamics on $\bar{M}$ is foliated by a two parameter family of Hamiltonian systems on the cotangent bundle of an open interval with one degree of freedom, where the two free parameters correspond to two conserved quantities on $\bar{M}$.

Chapter 2

The theory of nonholonomic reduction

2.1 Nonholonomic Hamiltonian systems

A holonomic Hamiltonian system (P, ω, h) consists of a manifold P, a Hamiltonian function h, and a symplectic two-form ω. The equations of motion for such a system are satisfied by integral curves of a vector field X on P, such that

$$X \lrcorner \omega = dh. \tag{2.1}$$

Usually $P = T^*Q$, where Q is the configuration space of the system and ω is the canonical symplectic two-form.

A nonholonomic Hamiltonian system (P, ω, h, ϕ^a) with k linearly independent nonholonomic constraint one-forms ϕ^a on Q can be written as

$$X \lrcorner \omega = dh + \sum_{a=1}^{k} \lambda_a \pi^* \phi^a, \tag{2.2}$$

where the constraint forms satisfy

$$\phi^1 \wedge \ldots \wedge \phi^k \neq 0. \tag{2.3}$$

Here $\pi : T^*Q \longrightarrow Q$ is the cotangent bundle projection and λ_a are the Lagrange multipliers.

In the theory developed in [2], the first goal is to write the equations of motion for a nonholonomic Hamiltonian system in such a way that we emulate equation (2.1). Weber [10] noticed the constraint one-forms define a constraint manifold and a horizontal distribution H. In [2] it is shown that restricting the two-form ω to this distribution H is non-degenerate. Then restricting equation (2.2) to this distribution H becomes

$$X \lrcorner \omega_H = d_H h, \tag{2.4}$$

where ω_H is the restriction of ω to the distribution H and $d_H h$ is the restriction of dh to the distribution H. The second goal is to develop a theory of reduction that results in a set of reduced equations on a reduced manifold $\bar{M}$ which mimics equation (2.4).

The proofs of propositions and theorems in this section are given in [2]. The propositions and theorems are stated in the notation of a nonholonomic Hamiltonian system (P, ω, h, ϕ^a).

Proposition 1 (*Conservation of energy*) *Let $c(t)$ be an integral curve for $X \in TP$. Then $h(c(t))$ is constant in t.*

Because of the constraints ϕ^a there are some states in P that are not possible. This is where Weber noticed that the constraints define quite naturally a submanifold M of P. This constraint manifold M is defined by the kernel of the k linearly independent constraint one-forms ϕ^a on Q.

$$M = \left\{p \in P \mid \left\langle \phi^a \circ \pi (p), \mathcal{L}^{-1} (p)\right\rangle = 0, a = 1, ..., k\right\}, \tag{2.5}$$

where the map $\mathcal{L} : TQ \longrightarrow P = T^*Q$ is the Legendre transformation and $\pi : P \longrightarrow Q$ is the cotangent bundle projection. Define the distribution F by

$$F = \{v \in TP \mid \langle \pi^* \phi^a, v\rangle = 0, a = 1, ..., k\}. \tag{2.6}$$

Now define a distribution H representing the set of admissible velocities and accelerations a mechanical system can have without violating the constraints ϕ^a by

$$H = F \cap TM. \tag{2.7}$$

Theorem 2 *The restriction of ω to H, denoted ω_H, is nondegenerate.*

Proposition 3 *The Hamiltonian vector field $X \in TP$ is in the distribution H.*

The above proposition implies that the dynamics of a Hamiltonian system must satisfy the constraints. Let X be the vector field satisfying

$$X \lrcorner \omega = dh + \sum_{a=1}^{k} \lambda_a \pi^* \phi^a. \tag{2.8}$$

Because of the Proposition (3), the vector field X is in the distribution H. Let ω_H be the restriction of the two-form ω to the distribution H and $d_H h$ be the restriction of the one-form dh to the distribution H. Because the distribution H is in the kernel of the constraint forms, when we restrict equation (2.8) to the distribution H, we have

$$X \lrcorner \omega_H = d_H h. \tag{2.9}$$

This makes sense, since $X \in H$. Thus equation (2.9) now mimics equation (2.1) for a holonomic Hamiltonian system.

2.2 Symmetries and reduction

Let G be a Lie group acting on P, where the action of G on P is defined by the map $\Phi : G \times P \longrightarrow P$. Suppose for all $g \in G$ the map $\Phi_g : P \longrightarrow P$ is defined by

$\Phi_g(p) = \Phi(g, p)$ for all $p \in P$. The group G is symmetry group of the nonholonomic Hamiltonian system (P, ω, h, ϕ^a), if for all $g \in G$, the map Φ_g does not violate the constraints, $\omega = \Phi_g^* \omega$, $h = h \circ \Phi_g$, and the distribution H defined by the kernel of the one-forms ϕ^a is invariant under the map Φ_g.

Let $\bar{M}$ be the space of G orbits in M. Assume $\bar{M}$ is a quotient manifold of M with projection map $\rho : M \longrightarrow \bar{M}$.

$$\bar{M} = M/G. \tag{2.10}$$

Vector fields and distributions on M pushdown to $\bar{M}$. However, the two-form ω_H need not, because there may be infinitesimal symmetries X_ζ such that

$$X_\zeta \lrcorner \omega_H \neq 0. \tag{2.11}$$

Let V be the distribution on M tangent to the group orbits of G in M, where V is spanned by the infinitesimal symmetries generated by the action Φ. Define the distribution U by

$$U = \{u \in H \mid \omega_H(u, v) = 0, \forall v \in V \cap H\}. \tag{2.12}$$

Now, U and V project to $\bar{M}$ and $\rho_* V = 0$. Define the reduced distribution $\bar{H}$ by

$$\bar{H} = \rho_* U. \tag{2.13}$$

Theorem 4 *The Hamiltonian vector field X is contained in U.*

Theorem 5 *The restriction ω_U of ω to the distribution U pushes down to a nondegenerate form $\omega_{\bar{H}} = \rho_* \omega_U$ on $\bar{H}$. Furthermore*

$$\bar{X} \lrcorner \omega_{\bar{H}} = d_{\bar{H}} \bar{h}, \tag{2.14}$$

where $\bar{h} = \rho_* h_M$ is the pushdown of the energy h restricted to M.

Note the original equation was

$$X \lrcorner \omega_H = d_H h. \tag{2.15}$$

Using the symmetries to reduce the problem, we have an equation of the same type as (2.15) on the reduced manifold $\bar{M}$

$$\bar{X} \lrcorner \omega_{\bar{H}} = d_{\bar{H}} \bar{h}. \tag{2.16}$$

2.3 The 2-dimensional Kepler problem

Consider the 2-dimensional Kepler problem of the motion of two bodies in the plane under a mutual gravitational attraction. Suppose one body is fixed at the origin of R^2. In polar coordinates (r, θ), the Hamiltonian system (P, ω, h) for the Kepler problem is given on the phase space

$$P = T^*Q = T^*\left(R^+ \times SO(2)\right), \tag{2.17}$$

with the canonical symplectic two-form

$$\omega = d\theta \wedge dp_\theta + dr \wedge dp_r, \tag{2.18}$$

and the total energy

$$h = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{\mu}{r}. \tag{2.19}$$

Here, μ is proportional to the gravitational constant. The two-dimensional Kepler problem is an example of a holonomic system with symmetry. Therefore, we hope the reduction method in [2] to give the same reduced Hamiltonian system as we would expect with the reduction method of Meyer-Marsden-Weinstein.

Because there are no constraints, $M = P$, $H = TP$, $\omega_H = \omega$, and $d_H h = dh$. Suppose X is the Hamiltonian vector field satisfying

$$X \lrcorner \omega_H = d_H h. \tag{2.20}$$

Since $\omega_H = \omega$ and $d_H h = dh$, equation (2.20) becomes

$$X \lrcorner \omega = dh, \tag{2.21}$$

which is the same equation we expect for a holonomic Hamiltonian system. Now the Hamiltonian vector field X satisfying equation (2.20) is given by

$$X = \frac{1}{m} p_r \partial_r + \left(\frac{p_\theta^2}{r^3} + \frac{\mu}{r^2} \right) \partial_{p_r} + \frac{1}{mr^2} p_\theta \partial_\theta, \tag{2.22}$$

where the integral curves of X are solutions to Hamilton's equations.

The Kepler problem has an S^1 symmetry which is given by the action $S^1 \times P \longrightarrow P$ defined by

$$(\alpha, r, p_r, \theta, p_\theta) \longrightarrow (r, p_r, \theta + \alpha, p_\theta). \tag{2.23}$$

Thus the infinitesimal generators of this action is the distribution

$$V = \operatorname{span}\{\partial_\theta\}. \tag{2.24}$$

By inspection

$$\begin{aligned} U &= \{u \in TP \mid \omega(u, v) = 0, \forall v \in V \cap TP\} \\ &= \operatorname{span}\{\partial_r, \partial_{p_r}, \partial_\theta\}. \end{aligned} \tag{2.25}$$

From equation (2.22), we see $X \in U$.

Let $\rho : M \longrightarrow \bar{M} = P/S^1$ be the projection map defined by

$$\rho(r, p_r, \theta, p_\theta) = (r, p_r, p_\theta), \tag{2.26}$$

where $\bar{M}$ is the reduced manifold. Then $\rho_* V = 0$ and

$$\bar{H} = \rho_* U = \operatorname{span}\{\partial_r, \partial_{p_r}\} \tag{2.27}$$

Thus we have the following Hamiltonian system $\left(\bar{M}, \omega_{\bar{H}}, \bar{h}\right)$on the reduced manifold $\bar{M}$, where the reduced two-form $\omega_{\bar{H}}$ on $\bar{M}$ is given by

$$\omega_{\bar{H}} = dr \wedge dp_r, \tag{2.28}$$

and the total energy $\bar{h}$ on $\bar{M}$ is given by

$$\bar{h} = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{\mu}{r}. \tag{2.29}$$

The reduced vector field $\bar{X}$ on $\bar{M}$ satisfying equation (2.14) is given by

$$\bar{X} = \frac{1}{m} p_r \partial_r + \left(\frac{p_\theta^2}{r^3} + \frac{\mu}{r^2}\right) \partial_{p_r}, \tag{2.30}$$

and $\bar{X} = \rho_* X$. Thus $\bar{X}$ is a vector field on the three dimensional manifold $\bar{M} = T^*R^+ \times R$. Since $\dot{p}_\theta = 0$, p_θ is a constant. Therefore, we have a constant of motion on $\bar{M}$. Suppose we define a map $f_c : N_c \longrightarrow \bar{M}$ by

$$f_c(r, p_r) = (r, p_r, c), \tag{2.31}$$

where $N_c = T^*R^+$. If we pull back the Hamiltonian system $\left(\bar{M}, \omega_{\bar{H}}, \bar{h}\right)$ to N_c by the map f_c we have the Hamiltonian system $\left(N_c, \tilde{\omega}, \tilde{h}\right)$, where $\tilde{\omega} = f_c^* \omega_{\bar{H}} = \omega_{\bar{H}}$ and

$$\tilde{h} = h \circ f_c = \frac{1}{2m}\left(p_r^2 + \frac{c^2}{r^2}\right) - \frac{\mu}{r}.$$

Thus the reduced dynamics on $\bar{M}$ is foliated by a one-parameter family of Hamiltonian system $\left(N_c, \tilde{\omega}, \tilde{h}\right)$ with one degree of freedom, where the Hamiltonian system $\left(N_c, \tilde{\omega}, \tilde{h}\right)$ is the one given by the reduction of Meyer-Marsden-Weinstein.

2.4 The nonholonomic free particle

A simple example of a nonholonomic system is the nonholonomic free particle. A particle is allowed to move freely in R^3 with coordinates x, y, and z, with the constraint $p_z = yp_x$. This example was given in [2], where the particle is assumed to have unit mass. Thus the nonholonomic free particle is given by the nonholonomic Hamiltonian system (P, ω, h, ϕ), where the phase space

$$P = T^*R^3, \tag{2.32}$$

the canonical two-form

$$\omega = dx \wedge dp_x + dy \wedge dp_y + dz \wedge dp_z, \tag{2.33}$$

the total energy

$$h = \frac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right), \tag{2.34}$$

and the one linearly independent constraint on R^3

$$\phi = dz - ydx. \tag{2.35}$$

The constraint manifold M is defined by

$$M = \left\{(x, y, z, p_x, p_y, p_z) \in T^*R^3 \mid p_z = yp_x\right\}. \tag{2.36}$$

Now x, y, z, p_x, and p_y are a set of global coordinates on M, because M is a graph over these coordinates. Suppose $i : M \longrightarrow T^*R^3$ is the inclusion map defined by

$$i(x, y, z, p_x, p_y) = (x, y, z, p_x, p_y, yp_x). \tag{2.37}$$

Note this map is in harmony with constraint $p_z = yp_x$. Then pulling the nonholonomic Hamiltonian system (P, ω, h, ϕ) back to M by the inclusion map i, we have the nonholonomic Hamiltonian system (M, ω_M, h_M, ϕ), where the two-form $\omega_M = i^*\omega$

$$\omega_M = dx \wedge dp_x - p_x dy \wedge dz + dy \wedge dp_y + y dz \wedge dp_x, \tag{2.38}$$

the total energy $h_M = h \circ i$

$$h_M = \frac{1}{2}\left(\left(1 + y^2\right) p_x^2 + p_y^2\right), \tag{2.39}$$

and the constraint one-form $\phi = i^*\phi$ on R^3

$$\phi = dz - y dx. \tag{2.40}$$

By definition, on M

$$\begin{aligned} H &= \ker\{dz - y dx\} \\ &= \operatorname{span}\left\{y\partial_z + \partial_x, \partial_y, \partial_{p_x}, \partial_{p_y}\right\} \end{aligned} \tag{2.41}$$

The vector field X on M satisfying equation (2.9) is given by

$$X = p_x\partial_x + p_y\partial_y + yp_x\partial_z - \frac{y}{1 + y^2} p_x p_y \partial_{p_x}. \tag{2.42}$$

Thus integral curves of X are solutions to the equations of motion for the nonholonomic free particle.

The nonholonomic free particle admits the group action of the two dimensional translation group in the x-z plane. On P the action $R^2 \times P \longrightarrow P$ is defined by

$$(x_1, z_1, x, y, z, p_x, p_y, p_z) \longrightarrow (x + x_1, y, z + z_1, p_x, p_y, p_z). \tag{2.43}$$

Note this action is in harmony with constraint $p_z = yp_x$. Thus on M the action $R^2 \times M \longrightarrow M$ is defined by

$$(x_1, z_1, x, y, z, p_x, p_y) \longrightarrow (x + x_1, y, z + z_1, p_x, p_y). \tag{2.44}$$

Therefore, the infinitesimal generators of this action on M is given by the distribution

$$V = \text{span}\left\{\partial_x, \partial_z\right\}. \tag{2.45}$$

Now

$$V \cap H = \text{span}\left\{y\partial_z + \partial_x\right\}, \tag{2.46}$$

and by definition

$$\begin{aligned} U &= \left\{u \in H \mid \omega_H(u, v) = 0, \forall v \in V \cap H\right\} \\ &= \text{span}\left\{y\partial_z + \partial_x, \left(1+y^2\right)\partial_y - yp_x\partial_{p_x}, \partial_{p_y}\right\}. \end{aligned} \tag{2.47}$$

From equation (2.42), we see $X \in U$, because

$$X = p_x\left(y\partial_z + \partial_x\right) + \frac{p_y}{1+y^2}\left(\left(1+y^2\right)\partial_y - yp_x\partial_{p_x}\right). \tag{2.48}$$

Let $\rho : M \longrightarrow \bar{M}$ be the projection map defined by

$$\rho\left(x, y, z, p_x, p_y\right) = \left(y, p_x, p_y\right), \tag{2.49}$$

where the reduced manifold $\bar{M}$ is defined by

$$\bar{M} = M/R^2 = \left\{\left(y, p_x, p_y\right) \in R^3\right\}. \tag{2.50}$$

Thus $\rho_* V = 0$ and on the reduced manifold $\bar{M}$ we have the distribution

$$\bar{H} = \rho_* U = \text{span}\left\{\left(1+y^2\right)\partial_y - yp_x\partial_{p_x}, \partial_{p_y}\right\}. \tag{2.51}$$

Thus we have a generalized Hamiltonian system $\left(\bar{M}, \omega_{\bar{H}}, \bar{h}\right)$on $\bar{M}$, where $\omega_{\bar{H}}$ is defined by the two spanning vector fields on $\bar{H}$

$$\omega_{\bar{H}}\left(\left(1+y^2\right)\partial_y - yp_x\partial_{p_x}, \partial_{p_y}\right) = 1 + y^2, \tag{2.52}$$

and the total energy $\bar{h}$ on $\bar{M}$ is

$$\bar{h} = \frac{1}{2}\left(\left(1+y^2\right)p_x^2 + p_y^2\right). \tag{2.53}$$

Then the reduced vector field $\bar{X}$ on $\bar{M}$ satisfying equation (2.16) is given by

$$\bar{X} = p_y\partial_y - \frac{y}{1+y^2}p_x p_y \partial_{p_x} \tag{2.54}$$

and $\bar{X} = \rho_* X$. Note that $\dot{p}_y = 0$. Thus p_y is a conserved quantity.

If we look at the differential equation that is given by the vector field $\bar{X}$ and realize that

$$\dot{p}_x = p_y\frac{dp_x}{dy} = -\frac{y}{1+y^2}p_x p_y, \tag{2.55}$$

then we have the differential equation

$$\frac{dp_x}{dy} = -\frac{y}{1+y^2}p_x. \tag{2.56}$$

Integrating this equation, we get

$$p_x = \frac{c}{\sqrt{1+y^2}}. \tag{2.57}$$

Suppose we define a map $f_c : N_c \longrightarrow \bar{M}$ by

$$f_c\left(y, p_y\right) = \left(y, \frac{c}{\sqrt{1+y^2}}, p_y\right), \tag{2.58}$$

where c is a constant and $N_c = T^*R$. Pulling the generalized Hamiltonian system $\left(\bar{M}, \omega_{\bar{H}}, \bar{h}\right)$ on $\bar{M}$ back to N_c by the map f_c, we get the Hamiltonian system $\left(N_c, \tilde{\omega}, \tilde{h}\right)$, where

$$\tilde{\omega} = f_c^*\omega_{\bar{H}} = dy \wedge dv_y, \tag{2.59}$$

and

$$\tilde{h} = f_c^*\bar{h} = \frac{1}{2}\left(c^2 + p_y^2\right). \tag{2.60}$$

Moreover,

$$\tilde{X} = p_y \partial_y \tag{2.61}$$

is the vector field satisfying the equation

$$\tilde{X} \lrcorner \tilde{\omega} = d\tilde{h}, \tag{2.62}$$

and

$$\begin{aligned} f_{c*}\tilde{X} &= p_y \partial_y - cy\left(1+y^2\right)^{-\frac{3}{2}} p_y \partial_{v_x} \\ &= p_y \partial_y - \tfrac{y}{1+y^2} p_x p_y \partial_{v_x} \\ &= \bar{X}. \end{aligned} \tag{2.63}$$

Thus the reduced dynamics on $\bar{M}$ is foliated by a one-parameter family of Hamiltonian systems $\left(N_c, \tilde{\omega}, \tilde{h}\right)$ with one degree of freedom.

Chapter 3

Nonholonomic reduction for the rolling disk

3.1 The rolling disk

A disk that rolls without slipping is a non-holonomic Hamiltonian system with symmetry, which is integrable. Thus it is an excellent example to apply the non-holonomic reduction techniques given in Chapter 2. In the book, *A Treatise on Analytic Dynamics* [6], Pars uses Lagrange's equations and the method of undetermined Lagrange multipliers for the non-holonomic constraints to find the equations of motion for the rolling disk. Therefore, we will be comparing our derivation with that of Pars.

The coordinates Pars uses to describe the rolling disk are ξ, η, θ, φ, and ψ. We will use the coordinates x and y for the center of the disk instead of ξ and η. As in the following diagram, the tilt of the disk is determined by the angle θ as measured between the x-y plane and the plane containing the disk. When $\theta = 0$ or π, the disk is lying flat on the x-y plane. Thus it is expected that our equations of motion should not be defined for these values of θ. The direction the disk is rolling is measured by the angle φ with respect to the positive x-axis. The rotation of the disk about the z_1-axis, which is perpendicular to the plane containing the disk, is measured by the angle ψ. The configuration space Q for the disk is $R^2 \times SO(3)$ and we will work on the open submanifold given by $0 < \theta < \pi$, which will still be denoted by Q.

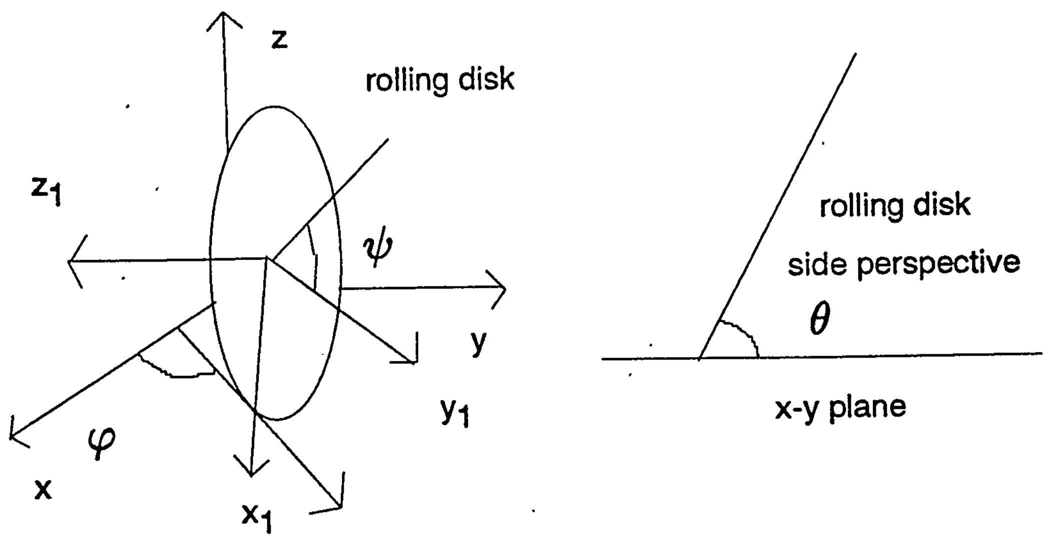

Figure 3.1: Coordinates for the rolling disk

The translational kinetic energy of the disk on TQ is given by

$$KE = \frac{1}{2}m\left(v_x^2 + v_y^2 + r^2 v_\theta^2 \cos^2\theta\right). \tag{3.1}$$

The rotational kinetic energy for the disk on TQ is given by

$$RE = \frac{1}{2}A\left(v_\theta^2 + v_\varphi^2 \sin^2\theta\right) + \frac{1}{2}C\left(v_\psi + v_\varphi \cos\theta\right)^2. \tag{3.2}$$

Here, A is the moment of inertia about the x_1-axis or the y_1-axis, which goes through the center of the disk and C is the moment of inertia about the z_1-axis, which goes through the center of the disk. For our equations to be valid, we will assume a mass distribution such that the center of mass for the disk occurs at the center of the disk. Then the potential energy on TQ is then given by

$$PE = mgr\sin\theta, \tag{3.3}$$

where m is the mass of the disk and g is the acceleration due to gravity. Therefore, the Lagrangian $l : TQ \longrightarrow R$ for the rolling disk is

$$
\begin{aligned}
l = \; & \tfrac{1}{2}m\left(v_x^2 + v_y^2 + r^2 v_\theta^2 \cos^2\theta\right) + \tfrac{1}{2}A\left(v_\theta^2 + v_\varphi^2 \sin^2\theta\right) \\
& + \tfrac{1}{2}C\left(v_\psi + v_\varphi \cos\theta\right)^2 - mgr\sin\theta.
\end{aligned} \tag{3.4}
$$

Consequently, the Legendre transformation $\mathcal{L} : TQ \to T^*Q$ is given by

$$
\begin{aligned}
p_x &= mv_x, \\
p_y &= mv_y, \\
p_\theta &= (A + mr^2\cos^2\theta)\, v_\theta, \\
p_\varphi &= Av_\varphi \sin^2\theta + C\cos\theta\left(v_\psi + v_\varphi\cos\theta\right), \\
p_\psi &= C\left(v_\psi + v_\varphi\cos\theta\right).
\end{aligned} \tag{3.5}
$$

Thus the Hamiltonian $h : P = T^*Q \longrightarrow R$ for the rolling disk is

$$
\begin{aligned}
h = \; & \tfrac{1}{2m}\left(p_x^2 + p_y^2\right) + \tfrac{1}{2}\left(A + mr^2\cos^2\theta\right)^{-1} p_\theta^2 + \tfrac{1}{2A\sin^2\theta}\left(p_\varphi - p_\psi\cos\theta\right)^2 \\
& + \tfrac{1}{2C}p_\psi^2 + mgr\sin\theta.
\end{aligned} \tag{3.6}
$$

The two non-holonomic rolling constraints for the disk are

$$
\begin{aligned}
\phi^1 &= \cos\varphi\, dx + \sin\varphi\, dy - r\sin\theta\, d\theta, \\
\phi^2 &= -\sin\varphi\, dx + \cos\varphi\, dy + r\cos\theta\, d\varphi + r\, d\psi.
\end{aligned} \tag{3.7}
$$

These are the non-holonomic constraints derived by Pars in [6] on p. 120, where r is the radius of the disk instead of a. These constraints are linearly independent, since

$$\phi^1 \wedge \phi^2 = dx \wedge dy + \text{other terms} \neq 0.$$

Therefore, the nonholonomic Hamiltonian system (P, ω, h, ϕ^a) for the rolling disk is given on the phase space $P = T^*Q$, with the canonical two-form

$$\omega = dx \wedge dp_x + dy \wedge dp_y + d\theta \wedge dp_\theta + d\varphi \wedge dp_\varphi + d\psi \wedge dp_\psi, \tag{3.8}$$

where

$$h = \tfrac{1}{2m}\left(p_x^2+p_y^2\right)+\tfrac{1}{2}\left(A+mr^2\cos^2\theta\right)^{-1}p_\theta^2+\tfrac{1}{2A\sin^2\theta}\left(p_\varphi-p_\psi\cos\theta\right)^2 + \tfrac{1}{2C}p_\psi^2+mgr\sin\theta, \tag{3.9}$$

and

$$\begin{aligned}\phi^1 &= \cos\varphi\,dx+\sin\varphi\,dy-r\sin\theta\,d\theta,\\ \phi^2 &= -\sin\varphi\,dx+\cos\varphi\,dy+r\cos\theta\,d\varphi+r\,d\psi.\end{aligned} \tag{3.10}$$

3.2 The equations of motion for the rolling disk

None of the theory in [2] depends upon being in phase space $P = T^*Q$. In the rolling disk there is an interesting case when the Legendre transformation is singular. This occurs when the moment of inertia $C = 0$. For this reason I have chosen to work in the tangent bundle TQ. This way I do not have to invert the Legendre transformation $\mathcal{L} : TQ \longrightarrow T^*Q$. It is also much easier to work out the constraint manifold M in TQ then T^*Q.

Let the constraint manifold M be a submanifold of TQ defined by the zero set of the constraints in 3.7. Thus

$$M = \{u\in TQ \mid \langle \phi^a \circ \pi(u), u\rangle = 0, a=1,2\}, \tag{3.11}$$

where $\pi : TQ \longrightarrow Q$ is the tangent bundle projection. Suppose $u \in TQ$, where

$$u = v_x\partial_x+v_y\partial_y+v_\theta\partial_\theta+v_\varphi\partial_\varphi+v_\psi\partial_\psi, \tag{3.12}$$

then

$$\begin{aligned}\langle \phi^1\circ\pi(u),u\rangle &= v_x\cos\varphi+v_y\sin\varphi-rv_\theta\sin\theta=0,\\ \langle \phi^2\circ\pi(u),u\rangle &= -v_x\sin\varphi+v_y\cos\varphi+rv_\varphi\cos\theta+rv_\psi=0.\end{aligned} \tag{3.13}$$

Solving for v_x and v_y, we have

$$v_x = rv_\theta \sin\theta\cos\varphi + rv_\varphi \cos\theta\sin\varphi + rv_\psi \sin\varphi, \tag{3.14}$$

$$v_y = rv_\theta \sin\theta\sin\varphi - rv_\varphi \cos\theta\cos\varphi - rv_\psi \cos\varphi.$$

Thus we can use x, y, θ, φ, ψ, v_θ, v_φ, and v_ψ as a set of global coordinates for M, since M is a graph over the variables x, y, θ, φ, ψ, v_θ, v_φ, and v_ψ.

Suppose we define the inclusion map $i : M \longrightarrow TQ$ by

$$i(x, y, \theta, \varphi, \psi, v_\theta, v_\varphi, v_\psi) = (x, y, \theta, \varphi, \psi, v_x, v_y, v_\theta, v_\varphi, v_\psi), \tag{3.15}$$

where v_x and v_y are defined by 3.14. To eliminate as many variables as possible, we can pull the nonholonomic Hamiltonian system (P, ω, h, ϕ^a) back to M by the Legendre transformation$\mathcal{L}$ and the inclusion map i. Thus we have the nonholonomic Hamiltonian system $(M, \omega_M, h_M, \phi^a)$,where the symplectic two-form ω_M is defined by

$$\begin{aligned}
\omega_M &= i^*\mathcal{L}^*\omega \\
&= mr\,(v_\theta\cos\theta\cos\varphi - v_\varphi\sin\theta\sin\varphi)\,dx\wedge d\theta \\
&\quad + mr\,(-v_\theta\sin\theta\sin\varphi + v_\varphi\cos\theta\cos\varphi + v_\psi\cos\varphi)\,dx\wedge d\varphi \\
&\quad + mr\sin\theta\cos\varphi\,dx\wedge dv_\theta + mr\cos\theta\sin\varphi\,dx\wedge dv_\varphi + mr\sin\varphi\,dx\wedge dv_\psi \\
&\quad + mr\,(v_\theta\cos\theta\sin\varphi + v_\varphi\sin\theta\cos\varphi)\,dy\wedge d\theta \\
&\quad + mr\,(v_\theta\sin\theta\cos\varphi + v_\varphi\cos\theta\sin\varphi + v_\psi\sin\varphi)\,dy\wedge d\varphi \\
&\quad + mr\sin\theta\sin\varphi\,dy\wedge dv_\theta - mr\cos\theta\cos\varphi\,dy\wedge dv_\varphi - mr\cos\varphi\,dy\wedge dv_\psi \\
&\quad + (A + mr^2\cos^2\theta)\,d\theta\wedge dv_\theta + (A\sin^2\theta + C\cos^2\theta)\,d\varphi\wedge dv_\varphi \\
&\quad + C\,d\psi\wedge dv_\psi + C\cos\theta\,d\varphi\wedge dv_\psi + C\cos\theta\,d\psi\wedge dv_\varphi \\
&\quad + Cv_\varphi\sin\theta\,d\theta\wedge d\psi + \sin\theta\,(2\,(A - C)\,v_\varphi\cos\theta - Cv_\psi)\,d\varphi\wedge d\theta,
\end{aligned} \tag{3.16}$$

the Hamiltonian $h_M : M \longrightarrow R$ is defined by

$$h_M = i^*\mathcal{L}^*h = \tfrac{1}{2}(A+mr^2)\,v_\theta^2 + \tfrac{1}{2}(C+mr^2)\,(v_\psi + v_\varphi \cos\theta)^2 + \tfrac{1}{2}Av_\varphi^2 \sin^2\theta + mgr\sin\theta, \tag{3.17}$$

and the constraints ϕ^a on Q are

$$\begin{aligned} \phi^1 &= \cos\varphi\, dx + \sin\varphi\, dy - r\sin\theta\, d\theta, \\ \phi^2 &= -\sin\varphi\, dx + \cos\varphi\, dy + r\cos\theta\, d\varphi + r\, d\psi. \end{aligned} \tag{3.18}$$

Let H be the distribution on M defined by

$$H = \{v \in TM \mid \langle i^*\pi^*\phi^a, v\rangle = 0, a = 1,2\}, \tag{3.19}$$

where $i : M \longrightarrow TQ$ is the inclusion map. The distribution H represents the set of admissible variations that do not violate the constraints. In the coordinates x, y, θ, φ, ψ, v_θ, v_φ, and v_ψ on M, we have

$$H = \mathrm{span}\left\{ X_\theta, X_\varphi, X_\psi, \partial_{v_\theta}, \partial_{v_\varphi}, \partial_{v_\psi} \right\}, \tag{3.20}$$

where

$$\begin{aligned} X_\theta &= r\sin\theta\cos\varphi\,\partial_x + r\sin\theta\sin\varphi\,\partial_y + \partial_\theta, \\ X_\varphi &= r\cos\theta\sin\varphi\,\partial_x - r\cos\theta\cos\varphi\,\partial_y + \partial_\varphi, \\ X_\psi &= r\sin\varphi\,\partial_x - r\cos\varphi\,\partial_y + \partial_\psi. \end{aligned} \tag{3.21}$$

The distribution H is not integrable, because

$$\begin{aligned} [X_\varphi, X_\psi] &= r\cos\varphi\,\partial_x + r\sin\varphi\,\partial_y, \\ [X_\varphi, [X_\varphi, X_\psi]] &= -r\sin\varphi\,\partial_x + r\cos\varphi\,\partial_y, \end{aligned} \tag{3.22}$$

are not linear combinations of vector fields in H. However, taking all possible linear combinations of the Lie brackets of the vector fields in H, we do obtain the distribution

$$TM = \mathrm{span}\left\{\partial_x, \partial_y, \partial_\theta, \partial_\varphi, \partial_\psi, \partial_{v_\theta}, \partial_{v_\varphi}, \partial_{v_\psi}\right\}. \tag{3.23}$$

Hamilton's equations on M is a vector field $X \in TM$, such that

$$X \lrcorner \omega_M = dh_M + \sum_{a=1}^{2} \lambda_a i^* \pi^* \phi^a, \tag{3.24}$$

where

$$\begin{aligned} dh_M = \; & \Big(Av_\varphi^2 \cos\theta \sin\theta + mgr\cos\theta - (C + mr^2)(v_\psi + v_\varphi \cos\theta) v_\varphi \sin\theta\Big)\, d\theta \\ & + (A + mr^2) v_\theta dv_\theta + (Av_\varphi \sin^2\theta + \cos\theta (C + mr^2)(v_\psi + v_\varphi \cos\theta))\, dv_\varphi \\ & + (C + mr^2)(v_\psi + v_\varphi \cos\theta)\, dv_\psi, \end{aligned} \tag{3.25}$$

the λ_a's are the unknown Lagrange multipliers and

$$\begin{aligned} \phi^1 &= \cos\varphi\, dx + \sin\varphi\, dy - r\sin\theta\, d\theta, \\ \phi^2 &= -\sin\varphi\, dx + \cos\varphi\, dy + r\cos\theta\, d\varphi + r\, d\psi. \end{aligned} \tag{3.26}$$

By Theorem 3, we know $X \in H$. Thus let

$$X = \dot\theta X_\theta + \dot\varphi X_\varphi + \dot\psi X_\psi + \dot v_\theta \partial_{v_\theta} + \dot v_\varphi \partial_{v_\varphi} + \dot v_\psi \partial_{v_\psi}. \tag{3.27}$$

From the constraints, we already know that

$$\begin{aligned} \dot x = v_x &= rv_\theta \sin\theta\cos\varphi + rv_\varphi \cos\theta\sin\varphi + rv_\psi \sin\varphi, \\ \dot y = v_y &= rv_\theta \sin\theta\sin\varphi - rv_\varphi \cos\theta\cos\varphi - rv_\psi \cos\varphi. \end{aligned} \tag{3.28}$$

Therefore, the only unknown variables are $\dot\theta$, $\dot\varphi$, $\dot\psi$, $\dot v_\theta$, $\dot v_\varphi$,and $\dot v_\psi$. Because $X_\theta = \partial_\theta$+other terms, similarly for X_φ and X_ψ, we are justified in stating equation (3.27). By contracting on both sides of equation (3.24) with vectors in H, the following equations hold

$$
\begin{aligned}
\langle X \lrcorner \omega_M, X_\theta \rangle = \; & A v_\varphi^2 \cos\theta \sin\theta + mgr\cos\theta \\
& -(C+mr^2)(v_\psi + v_\varphi \cos\theta) v_\varphi \sin\theta, \\
\langle X \lrcorner \omega_M, X_\varphi \rangle &= 0, \\
\langle X \lrcorner \omega_M, X_\psi \rangle &= 0, \\
\langle X \lrcorner \omega_M, \partial_{v_\theta} \rangle &= (A+mr^2) v_\theta, \\
\left\langle X \lrcorner \omega_M, \partial_{v_\varphi} \right\rangle &= A v_\varphi \sin^2\theta + \cos\theta (C+mr^2)(v_\psi + v_\varphi \cos\theta), \\
\left\langle X \lrcorner \omega_M, \partial_{v_\psi} \right\rangle &= (C+mr^2)(v_\psi + v_\varphi \cos\theta).
\end{aligned}
\tag{3.29}
$$

Since the one-forms $i^*\pi^*\phi^a$ annihilate the distribution H, the terms involving the Lagrange multipliers in (3.24) disappear. Now the linear equations in (3.29) are then solvable for

$$
\begin{aligned}
\dot\theta &= v_\theta, \\
\dot\varphi &= v_\varphi, \\
\dot\psi &= v_\psi, \\
\dot v_\theta &= \tfrac{1}{A+mr^2}\left(A v_\varphi^2 \cos\theta\sin\theta - (C+mr^2)(v_\psi + v_\varphi\cos\theta) v_\varphi \sin\theta - mgr\cos\theta\right), \\
\dot v_\varphi &= \tfrac{v_\theta}{A\sin\theta}\left(C\left(v_\psi + v_\varphi\cos\theta\right) - 2A v_\varphi \cos\theta\right), \\
\dot v_\psi &= \tfrac{C+2mr^2}{C+mr^2} v_\theta v_\varphi \sin\theta - \tfrac{v_\theta\cos\theta}{A\sin\theta}\left(C\left(v_\psi + v_\varphi\cos\theta\right) - 2A v_\varphi\cos\theta\right).
\end{aligned}
\tag{3.30}
$$

since the restriction of ω_M to the distribution H is nondegenerate. The equations given by Pars in [6] on p. 122 are

$$
\begin{aligned}
&(A+mr^2)\ddot\theta = A v_\varphi^2\cos\theta\sin\theta - (C+mr^2)(v_\psi + v_\varphi\cos\theta) v_\varphi\sin\theta - mgr\cos\theta, \\
&(C+mr^2)\frac{d}{dt}(v_\psi + v_\varphi\cos\theta) = mr^2 v_\theta v_\varphi \sin\theta, \\
&\frac{d}{dt}(A v_\varphi \sin^2\theta) = C(v_\psi + v_\varphi\cos\theta) v_\theta \sin\theta.
\end{aligned}
\tag{3.31}
$$

We will now verify that (3.30) and (3.31) are equivalent. The first equation in (3.31) is easily derived from (3.30) since

$$\begin{aligned}\left(A+mr^2\right)\ddot{\theta} &= \left(A+mr^2\right)\dot{v}_\theta \\ &= Av_\varphi^2\cos\theta\sin\theta-\left(C+mr^2\right)\left(v_\psi+v_\varphi\cos\theta\right)v_\varphi\sin\theta \\ &\quad -mgr\cos\theta.\end{aligned} \tag{3.32}$$

The second equation in (3.31) is obtained as follows

$$\begin{aligned}\left(C+mr^2\right)\frac{d}{dt}\left(v_\psi+v_\varphi\cos\theta\right) &= \left(C+mr^2\right)\left(\dot{v}_\psi+\dot{v}_\varphi\cos\theta-v_\theta\sin\theta\right) \\ &= \left(C+mr^2\right)\left(\tfrac{C+2mr^2}{C+mr^2}v_\theta v_\varphi\sin\theta-v_\theta\sin\theta\right) \\ &= mr^2 v_\theta v_\varphi\sin\theta.\end{aligned} \tag{3.33}$$

The third equation in (3.31) comes from

$$\begin{aligned}\frac{d}{dt}\left(Av_\varphi\sin^2\theta\right) &= A\dot{v}_\varphi\sin^2\theta+2Av_\theta v_\varphi\sin\theta\cos\theta \\ &= \left(C\left(v_\psi+v_\varphi\cos\theta\right)-2Av_\varphi\cos\theta\right)v_\theta\sin\theta+2Av_\theta v_\varphi\sin\theta\cos\theta \\ &= C\left(v_\psi+v_\varphi\cos\theta\right)v_\theta\sin\theta.\end{aligned} \tag{3.34}$$

Therefore, the equations derived by Pars and our equations agree.

Now, if we define ω_H to be the restriction of ω_M to the distribution H and $d_H h_M$ to be the restriction of dh_M to the distribution H, then

$$X \lrcorner \omega_H = d_H h_M, \tag{3.35}$$

Thus equation (3.35) resembles the equations for a Hamiltonian system.

3.3 Reduction by $E(2)\times SO(2)$

On configuration space Q, the rolling disk admits the symmetry groups of the Euclidean group $E(2)$ and the rotation group $SO(2)$. The Euclidean group involves translations and rotations of the disk in the x-y plane and the rotation group involves rotations about the axis perpendicular to the plane containing the disk. If the action $\Phi : E(2) \times SO(2) \times Q \longrightarrow Q$ is defined by

$$\begin{aligned} &\Phi(x_1, y_1, \alpha, \beta, x, y, \theta, \varphi, \psi) \\ &= (x\cos\alpha + y\sin\alpha + x_1, x\sin\alpha - y\cos\alpha + y_1, \theta, \varphi + \alpha, \psi + \beta), \end{aligned} \tag{3.36}$$

then the infinitesimal generators of this action on Q is given by the distribution

$$\operatorname{span}\{\partial_x, \partial_y, -y\partial_x + x\partial_y + \partial_\varphi, \partial_\psi\}. \tag{3.37}$$

Lifting these infinitesimal generators to TQ, we have the distribution

$$\operatorname{span}\left\{\partial_x, \partial_y, -v_y\partial_{v_x} + v_x\partial_{v_y} + \partial_\varphi, \partial_\psi\right\}. \tag{3.38}$$

On M, the infinitesimal generators of our symmetries are given by the distribution

$$V = \operatorname{span}\{\partial_x, \partial_y, \partial_\varphi, \partial_\psi\}, \tag{3.39}$$

since

$$\begin{aligned} i_*(\partial_x) &= \partial_x, \\ i_*(\partial_y) &= \partial_y, \\ i_*(\partial_\varphi) &= -v_y\partial_{v_x} + v_x\partial_{v_y} + \partial_\varphi, \\ i_*(\partial_\psi) &= \partial_\psi. \end{aligned} \tag{3.40}$$

Now that we have our infinitesimal generators of our symmetry, we can proceed with the reduction. Therefore,

$$V \cap H = \operatorname{span}\left\{ \begin{array}{l} r\cos\theta\sin\varphi\,\partial_x - r\cos\theta\cos\varphi\,\partial_y + \partial_\varphi, \\ r\sin\varphi\,\partial_x - r\cos\varphi\,\partial_y + \partial_\psi \end{array} \right\}, \tag{3.41}$$

and

$$U = \{u \in H \mid \omega_H(u,v) = 0, \forall v \in V \cap H\}. \tag{3.42}$$

Thus

$$U = \operatorname{span}\{Y_\theta, Y_\varphi, Y_\psi, \partial_{v_\theta}\}, \tag{3.43}$$

where

$$\begin{aligned}
Y_\theta = \; & r\sin\theta\cos\varphi\,\partial_x + r\sin\theta\sin\varphi\,\partial_y + \partial_\theta \\
& + \tfrac{1}{A\sin\theta}\left((C+mr^2)(v_\psi + v_\varphi\cos\theta) - 2Av_\varphi\cos\theta\right)\partial_{v_\varphi} \\
& - \tfrac{1}{A\sin\theta}\left((C+mr^2)(v_\psi + v_\varphi\cos\theta) - Av_\varphi(1+\cos^2\theta)\right)\partial_{v_\psi}, \\
Y_\varphi = \; & r\cos\theta\sin\varphi\,\partial_x - r\cos\theta\cos\varphi\,\partial_y + \partial_\varphi - \tfrac{mr^2}{A\sin\theta}v_\theta\cos\theta\,\partial_{v_\varphi} \\
& + \left(\tfrac{mr^2}{C+mr^2} + \tfrac{mr^2\cos^2\theta}{A\sin^2\theta}\right)v_\theta\sin\theta\,\partial_{v_\psi}, \\
Y_\psi = \; & r\sin\varphi\,\partial_x - r\cos\varphi\,\partial_y + \partial_\psi - \tfrac{mr^2}{A\sin\theta}v_\theta\,\partial_{v_\varphi} + \tfrac{mr^2}{A\sin\theta}v_\theta\cos\theta\,\partial_{v_\psi}.
\end{aligned} \tag{3.44}$$

The distribution U is nonintegrable, because

$$\begin{aligned}
&[\partial_{v_\theta}, Y_\psi] = \tfrac{mr^2}{A\sin\theta}\left(\cos\theta\,\partial_{v_\psi} - \partial_{v_\varphi}\right), \\
&[\partial_{v_\theta}, Y_\varphi] = \tfrac{mr^2}{A\sin\theta}\left(\cos\theta\,\partial_{v_\psi} - \partial_{v_\varphi}\right) + \tfrac{mr^2}{C+mr^2}\sin\theta\,\partial_{v_\psi}, \\
&[Y_\varphi, Y_\psi] = r\cos\varphi\,\partial_x + r\sin\varphi\,\partial_y, \\
&[Y_\varphi, [Y_\varphi, Y_\psi]] = -r\sin\varphi\,\partial_x + r\cos\varphi\,\partial_y,
\end{aligned} \tag{3.45}$$

are not linear combinations of vector fields in U. Taking all possible linear combinations of Lie brackets of the above vector fields in U, we get the distribution

$$TM = \operatorname{span}\left\{\partial_x, \partial_y, \partial_\theta, \partial_\varphi, \partial_\psi, \partial_{v_\theta}, \partial_{v_\varphi}, \partial_{v_\psi}\right\}. \tag{3.46}$$

Let $\bar{M} = M/G$ be the reduced manifold with the projection map $\rho : M \longrightarrow \bar{M}$ defined by

$$\rho(x, y, \theta, \varphi, \psi, v_\theta, v_\varphi, v_\psi) = (\theta, v_\theta, v_\varphi, v_\psi). \tag{3.47}$$

The reduced distribution $\bar{H}$ is defined by

$$\begin{aligned}\bar{H} &= \rho_* U \\ &= \operatorname{span}\left\{\begin{array}{l} \partial_\theta + \frac{1}{A\sin\theta}\left(\left(C+mr^2\right)\left(v_\psi + v_\varphi\cos\theta\right) - 2Av_\varphi\cos\theta\right)\partial_{v_\varphi} \\ \quad -\frac{1}{A\sin\theta}\left(\left(C+mr^2\right)\left(v_\psi + v_\varphi\cos\theta\right) - Av_\varphi\left(1+\cos^2\theta\right)\right)\partial_{v_\psi}, \\ -\frac{mr^2}{A\sin\theta}v_\theta\cos\theta\partial_{v_\varphi} + \left(\frac{mr^2}{C+mr^2} + \frac{mr^2\cos^2\theta}{A\sin^2\theta}\right)v_\theta\sin\theta\partial_{v_\psi}, \\ -\frac{mr^2}{A\sin\theta}v_\theta\partial_{v_\varphi} + \frac{mr^2}{A\sin\theta}v_\theta\cos\theta\partial_{v_\psi}, \\ \partial_{v_\theta} \end{array}\right\}\end{aligned} \tag{3.48}$$

Taking all possible linear combinations of the above vector fields we see that

$$\bar{H} = \operatorname{span}\left\{\partial_\theta, \partial v_\theta, \partial v_\varphi, \partial v_\psi\right\}. \tag{3.49}$$

The restriction ω_U of ω to the distribution U pushes down to a non-degenerate form $\omega_{\bar{H}} = \rho_* \omega_U$. The values of the two form $\omega_{\bar{H}}$ are given by

$$\begin{array}{c|cccc|c} \omega_M(A,B) & Y_\theta & Y_\psi & \partial_{v_\theta} & Y_\varphi & \mathrm{B} \\ \hline Y_\theta & 0 & 0 & (A+mr^2) & 0 & \\ Y_\psi & 0 & 0 & 0 & mr^2 v_\theta\sin\theta & \\ \partial_{v_\theta} & -(A+mr^2) & 0 & 0 & 0 & \\ Y_\varphi & 0 & -mr^2 v_\theta\sin\theta & 0 & 0 & \\ \hline \mathrm{A} & & & & & \end{array} \tag{3.50}$$

Note that $\omega_{\bar{H}}$ is nondegenerate even if $C = 0$. Thus we have the generalized Hamiltonian system $\left(\bar{M}, \omega_{\bar{H}}, \bar{h}\right)$, where the reduced two form $\omega_{\bar{H}}$ on $\bar{M}$ is given by

$$\omega_{\bar{H}} = \left(A + mr^2\right) d\theta \wedge dv_\theta + a\, d\theta \wedge dv_\varphi + b\, d\theta \wedge dv_\psi + c\, dv_\psi \wedge dv_\varphi, \tag{3.51}$$

$$
\begin{aligned}
a &= \tfrac{C+mr^2}{v_\varphi mr^2}\left(\left(C+mr^2\right)\left(v_\psi + v_\varphi \cos\theta\right) - Av_\varphi\left(1+\cos^2\theta\right)\right),\\
b &= \tfrac{C+mr^2}{v_\varphi mr^2}\left(C\left(v_\psi + v_\varphi\cos\theta\right) - 2Av_\varphi\cos\theta\right),\\
c &= \tfrac{C+mr^2}{v_\varphi mr^2} A\sin\theta.
\end{aligned}
\tag{3.52}
$$

and the reduced Hamiltonian $\bar{h}$ on $\bar{M}$ is given by

$$
\begin{aligned}
\bar{h} = {} & \tfrac{1}{2}\left(A+mr^2\right)v_\theta^2 + \tfrac{1}{2}\left(C+mr^2\right)\left(v_\psi + v_\varphi\cos\theta\right)^2\\
& + \tfrac{1}{2}Av_\varphi^2 \sin^2\theta + mgr\sin\theta.
\end{aligned}
\tag{3.53}
$$

Let $\bar{X}$ be the reduced vector field on $\bar{M}$ satisfying

$$
\bar{X} \lrcorner \omega_{\bar{H}} = d_{\bar{H}}\bar{h}, \tag{3.54}
$$

where $d_{\bar{H}}\bar{h}$ is given by

$$
\begin{aligned}
d_{\bar{H}}\bar{h} = {} & \left(Av_\varphi^2\cos\theta\sin\theta + mgr\cos\theta - \left(C+mr^2\right)\left(v_\psi + v_\varphi\cos\theta\right)v_\varphi\sin\theta\right)d\theta\\
& + \left(A+mr^2\right)v_\theta dv_\theta\\
& + \left(Av_\varphi\sin^2\theta + \cos\theta\left(C+mr^2\right)\left(v_\psi+v_\varphi\cos\theta\right)\right)dv_\varphi\\
& + \left(C+mr^2\right)\left(v_\psi + v_\varphi\cos\theta\right)dv_\psi.
\end{aligned}
\tag{3.55}
$$

Then the reduced vector field $\bar{X}$ is given by

$$
\bar{X} = \dot{\theta}\partial_\theta + \dot{v}_\theta \partial v_\theta + \dot{v}_\varphi \partial v_\varphi + \dot{v}_\psi \partial v_\psi, \tag{3.56}
$$

where

$$
\begin{aligned}
\dot{\theta} &= v_\theta,\\
\dot{v}_\theta &= \tfrac{1}{A+mr^2}\left(Av_\varphi^2\cos\theta\sin\theta - mgr\cos\theta - \left(C+mr^2\right)\left(v_\psi+v_\varphi\cos\theta\right)v_\varphi\sin\theta\right),\\
\dot{v}_\varphi &= \tfrac{v_\theta}{A\sin\theta}\left(C\left(v_\psi+v_\varphi\cos\theta\right) - 2Av_\varphi\cos\theta\right),\\
\dot{v}_\psi &= \tfrac{C+2mr^2}{C+mr^2}v_\theta v_\varphi\sin\theta - \tfrac{v_\theta\cos\theta}{A\sin\theta}\left(C\left(v_\psi+v_\varphi\cos\theta\right) - 2Av_\varphi\cos\theta\right).
\end{aligned}
\tag{3.57}
$$

Moreover, $\bar{X} = \rho_* X$.

3.4 New coordinates for the reduced manifold $\bar{M}$

Suppose we perform the following change of coordinates on $\bar{M}$, where

$$\begin{aligned} \rho_\theta &= (A+mr^2)\,v_\theta, \\ \rho_\varphi &= (C+mr^2)\,(v_\psi + v_\varphi\cos\theta)\cos\theta + Av_\varphi\sin^2\theta, \\ \rho_\psi &= (C+mr^2)\,(v_\psi + v_\varphi\cos\theta)\,. \end{aligned} \tag{3.58}$$

The idea for this change of coordinates came from the Legendre transformation. However, we use the Hamiltonian function h_M on M instead of the Hamiltonian function h on P. Note that the map $(\theta, v_\theta, v_\varphi, v_\psi) \longrightarrow (\theta, \rho_\theta, \rho_\varphi, \rho_\psi)$ is nonsingular even when $C = 0$. In these new coordinates we find that the reduced distribution $\bar{H}$ is given by

$$\bar{H} = \operatorname{span}\left\{\begin{array}{l} \partial_\theta + \frac{C+mr^2}{A\sin\theta}\rho_\psi\,(\cos\theta - 1)\left(\partial_{\rho_\psi} + \cos\theta\,\partial_{\rho_\varphi}\right), \\ \frac{mr^2}{(A+mr^2)}\rho_\theta\sin\theta\,\partial_{\rho_\psi}, \\ -\frac{mr^2}{(A+mr^2)}\rho_\theta\sin\theta\,\partial_{\rho_\varphi}, \\ (A+mr^2)\,\partial_{\rho_\theta}. \end{array}\right\} \tag{3.59}$$

and taking all possible linear combinations of vector fields in $\bar{H}$ gives the distribution

$$T\bar{M} = \bar{H} = \operatorname{span}\left\{\partial_\theta, \partial_{\rho_\theta}, \partial_{\rho_\varphi}, \partial_{\rho_\psi}\right\}. \tag{3.60}$$

Using (3.58), the reduced two form $\omega_{\bar{H}}$ is given by

$$\omega_{\bar{H}} = d\theta \wedge d\rho_\theta - \frac{A+mr^2}{mr^2\rho_\theta\sin\theta}d\rho_\varphi \wedge d\rho_\psi \tag{3.61}$$

Note that the reduced two form $\omega_{\bar{H}}$ is not closed, since

$$d\omega_{\bar{H}} = \frac{(A+mr^2)\cos\theta}{mr^2\rho_\theta\sin^2\theta}d\theta \wedge d\rho_\varphi \wedge d\rho_\psi + \frac{A+mr^2}{mr^2\rho_\theta^2\sin\theta}d\rho_\theta \wedge d\rho_\varphi \wedge d\rho_\psi \tag{3.62}$$

is non zero. In these new coordinates the reduced energy $\bar{h}$ on $\bar{M}$ is given by

$$\bar{h} = \frac{1}{2(A+mr^2)}\rho_\theta^2 + \frac{1}{2(C+mr^2)}\rho_\psi^2 + \frac{1}{2A\sin^2\theta}(\rho_\varphi - \rho_\psi\cos\theta)^2 + mgr\sin\theta, \quad (3.63)$$

where $d_{\bar{H}}\bar{h}$ is given by

$$\begin{aligned} d_{\bar{H}}\bar{h} = \; & \left(mgr\cos\theta + \tfrac{1}{A\sin\theta}\rho_\psi(\rho_\varphi - \rho_\psi\cos\theta) - \tfrac{\cos\theta}{A\sin^3\theta}(\rho_\varphi - \rho_\psi\cos\theta)^2\right)d\theta \\ & + \tfrac{1}{A+mr^2}\rho_\theta\, d\rho_\theta + \tfrac{1}{A\sin^2\theta}(\rho_\varphi - \rho_\psi\cos\theta)\, d\rho_\varphi \\ & + \left(\tfrac{1}{(C+mr^2)}\rho_\psi - \tfrac{\cos\theta}{A\sin^2\theta}(\rho_\varphi - \rho_\psi\cos\theta)\right)d\rho_\psi. \end{aligned} \quad (3.64)$$

The reduced Hamiltonian vector field $\bar{X}$ on $\bar{M}$, satisfying

$$\bar{X} \lrcorner\, \omega_{\bar{H}} = d_{\bar{H}}\bar{h}, \quad (3.65)$$

is given by

$$\bar{X} = \dot{\theta}\partial_\theta + \dot{\rho}_\theta\partial_{\rho_\theta} + \dot{\rho}_\varphi\partial_{\rho_\varphi} + \dot{\rho}_\psi\partial_{\rho_\psi} \quad (3.66)$$

where

$$\begin{aligned} \dot{\theta} &= \tfrac{1}{A+mr^2}\rho_\theta, \\ \dot{\rho}_\theta &= \tfrac{\cos\theta}{A\sin^3\theta}(\rho_\varphi - \rho_\psi\cos\theta)^2 - \tfrac{1}{A\sin\theta}\rho_\psi(\rho_\varphi - \rho_\psi\cos\theta) - mgr\cos\theta, \\ \dot{\rho}_\varphi &= \tfrac{mr^2}{A(A+mr^2)}\rho_\theta\cot\theta(\rho_\varphi - \rho_\psi\cos\theta) - \tfrac{mr^2\sin\theta}{(C+mr^2)(A+mr^2)}\rho_\theta\rho_\psi, \\ \dot{\rho}_\psi &= \tfrac{mr^2}{A(A+mr^2)}\rho_\theta\csc\theta(\rho_\varphi - \rho_\psi\cos\theta). \end{aligned} \quad (3.67)$$

3.5 Integrating the equations

From the differential equations (3.67) in the previous section, we can find a second order differential equation for ρ_ψ as a function of θ. In Rosenberg's book, *Analytic Dynamics* [8] the derivation of these equations are described as follows. Using θ as a new time parameter we obtain from (3.67) the equations

$$\begin{aligned}
\dot{\rho}_\varphi &= \tfrac{1}{A+mr^2}\rho_\theta \frac{d\rho_\varphi}{d\theta} = \tfrac{mr^2}{A(A+mr^2)}\rho_\theta \cot\theta\,(\rho_\varphi - \rho_\psi\cos\theta) - \tfrac{mr^2\sin\theta}{(C+mr^2)(A+mr^2)}\rho_\theta\rho_\psi, \\
\dot{\rho}_\psi &= \tfrac{1}{A+mr^2}\rho_\theta \frac{d\rho_\psi}{d\theta} = \tfrac{mr^2}{A(A+mr^2)}\rho_\theta \csc\theta\,(\rho_\varphi - \rho_\psi\cos\theta).
\end{aligned} \tag{3.68}$$

Therefore,

$$\begin{aligned}
\frac{d\rho_\varphi}{d\theta} &= \tfrac{mr^2}{A}\cot\theta\,(\rho_\varphi - \rho_\psi\cos\theta) - \tfrac{mr^2\sin\theta}{(C+mr^2)}\rho_\psi, \\
\frac{d\rho_\psi}{d\theta} &= \tfrac{mr^2}{A}\csc\theta\,(\rho_\varphi - \rho_\psi\cos\theta).
\end{aligned} \tag{3.69}$$

Differentiating the following equation

$$\sin\theta\frac{d\rho_\psi}{d\theta} = \frac{mr^2}{A}(\rho_\varphi - \rho_\psi\cos\theta), \tag{3.70}$$

we obtain

$$\begin{aligned}
\sin\theta\frac{d^2\rho_\psi}{d\theta^2} + \cos\theta\frac{d\rho_\psi}{d\theta} &= \frac{mr^2}{A}\left(\frac{d\rho_\varphi}{d\theta} - \cos\theta\frac{d\rho_\psi}{d\theta} + \rho_\psi\sin\theta\right) \\
&= \frac{mr^2}{A}\left(-\tfrac{mr^2\sin\theta}{(C+mr^2)}\rho_\psi + \rho_\psi\sin\theta\right) \\
&= \tfrac{Cmr^2}{A(C+mr^2)}\rho_\psi\sin\theta.
\end{aligned} \tag{3.71}$$

Thus

$$\frac{d^2\rho_\psi}{d\theta^2} + \cot\theta\frac{d\rho_\psi}{d\theta} - \frac{Cmr^2}{A(C+mr^2)}\rho_\psi = 0, \tag{3.72}$$

which is an equation of the Legendre type.

Let $z = \cos^2\theta$, then the differential equation (3.72) becomes

$$z(1-z)\frac{d^2\rho_\psi}{dz^2} + \left(\frac{1-3z}{2}\right)\frac{d\rho_\psi}{dz} - \frac{Cmr^2}{4A(C+mr^2)}\rho_\psi = 0 \tag{3.73}$$

This is the same differential equation as in [8] on p. 339. Here (3.73) is satisfied by hypergeometric functions $F(a,b;c;z)$ and $F(a+1,b+1;c;z)$ where

$$a = \frac{1+\gamma}{4}, b = \frac{1-\gamma}{4}, c = \frac{1}{2}, \text{ and } \gamma = \sqrt{1 - \frac{Cmr^2}{A(C+mr^2)}}. \tag{3.74}$$

Recall $F(a,b;c;z)$ is defined by

$$F(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}z^n, \tag{3.75}$$

where

$$(a)_n=a(a+1)(a+2)\cdots(a+n-1), \tag{3.76}$$

similarly for $(b)_n$ and $(c)_n$. Therefore, a general solution to the differential equation (3.72) is

$$\rho_\psi=f_1F\left(\frac{1+\gamma}{4},\frac{1-\gamma}{4};\frac{1}{2};\cos^2\theta\right)+f_2\cos\theta F\left(\frac{5+\gamma}{4},\frac{5-\gamma}{4};\frac{3}{2};\cos^2\theta\right), \tag{3.77}$$

where f_1 and f_2 are integration constants. Using equations (3.69), we can solve for ρ_φ, where

$$\rho_\varphi=\frac{A}{mr^2}\sin\theta\frac{d\rho_\psi}{d\theta}+\rho_\psi\cos\theta. \tag{3.78}$$

Let $N_{(f_1,f_2)}$ be the cotangent bundle of an open interval $(0,\pi)$. Now define a map $f_{(f_1,f_2)}:N\to\bar{M}$ by

$$f_{(f_1,f_2)}(\theta,\rho_\theta)=(\theta,\rho_\theta,\rho_\varphi,\rho_\psi), \tag{3.79}$$

where

$$\begin{aligned}\rho_\varphi&=\tfrac{A}{mr^2}\sin\theta\frac{d\rho_\psi}{d\theta}+\rho_\psi\cos\theta,\\ \rho_\psi&=f_1F\left(\tfrac{1+\gamma}{4},\tfrac{1-\gamma}{4};\tfrac{1}{2};\cos^2\theta\right)+f_2\cos\theta F\left(\tfrac{5+\gamma}{4},\tfrac{5-\gamma}{4};\tfrac{3}{2};\cos^2\theta\right).\end{aligned} \tag{3.80}$$

Suppose we pullback the generalized Hamiltonian system $\left(\bar{M},\omega_{\bar{H}},\bar{h}\right)$ with the non-degenerate non-closed two-form $\omega_{\bar{H}}$ to $N_{(f_1,f_2)}$ by the map $f_{(f_1,f_2)}$. Then we have the Hamiltonian system $\left(N_{(f_1,f_2)},\tilde{\omega},\tilde{h}\right)$, where $\tilde{\omega}$ is a nondegenerate closed two-form defined by

$$\tilde{\omega}=f^*\omega_{\bar{H}}=d\theta\wedge d\rho_\theta, \tag{3.81}$$

and the Hamiltonian $\tilde{h}$ on $N_{(f_1,f_2)}$ is defined by

$$\tilde{h} = \frac{1}{2\left(A+mr^2\right)}\rho_\theta^2 + \frac{1}{2\left(C+mr^2\right)}\rho_\psi^2 + \frac{1}{2A\sin^2\theta}\left(\rho_\varphi - \rho_\psi\cos\theta\right)^2 + mgr\sin\theta, \quad (3.82)$$

where ρ_φ and ρ_ψ are given by equations (3.80).

Let $\tilde{X}$ be the vector field such that

$$\tilde{X} \lrcorner \tilde{\omega} = d\tilde{h}. \quad (3.83)$$

Solving for $\tilde{X}$, we get

$$\tilde{X} = \dot{\theta}\,\partial_\theta + \dot{\rho}_\theta\,\partial_{\rho_\theta}, \quad (3.84)$$

where

$$\begin{aligned} \dot{\theta} &= \tfrac{1}{\left(A+mr^2\right)}\rho_\theta, \\ \dot{\rho}_\theta &= \tfrac{\cos\theta}{A\sin^3\theta}\left(\rho_\varphi - \rho_\psi\cos\theta\right)^2 - \tfrac{1}{A\sin\theta}\rho_\psi\left(\rho_\varphi - \rho_\psi\cos\theta\right) - mgr\cos\theta, \end{aligned} \quad (3.85)$$

and $\bar{X} = f_{(f_1,f_2)*}\tilde{X}$. Again ρ_φ and ρ_ψ are given by equations (3.80). Therefore the reduced dynamics on $\bar{M}$ is foliated by a two parameter family of Hamiltonian systems $\left(N_{(f_1,f_2)}, \tilde{\omega}, \tilde{h}\right)$ with one degree of freedom, which are completely integrable. Since f_1 and f_2 are integration constants, which can be written as a function of θ, ρ_θ, ρ_φ, and ρ_ψ, we have two conserved quantities on $\bar{M}$. Once the Hamiltonian system $\left(N_{(f_1,f_2)}, \tilde{\omega}, \tilde{h}\right)$ is integrated we can reconstruct the dynamics on $\bar{M}$ by performing two more integrations, because the reduced vector field $\bar{X}$ on $\bar{M}$ is given by $\bar{X} = f_{(f_1,f_2)*}\tilde{X}$.

Chapter 4

Orbits for C=0

4.1 Equations of motion for C=0

The equations of motion for the rolling disk are still valid even if $C = 0$, because the two-form ω_H is nondegenerate for $C = 0$. Therefore, everything that has been stated previously is still valid even if $C = 0$. Suppose $C = 0$, then the differential equation in (3.72) becomes

$$\frac{d^2\rho_\psi}{d\theta^2} + \cot\theta \frac{d\rho_\psi}{d\theta} = 0. \tag{4.1}$$

From the differential equation in (4.1) and equation (3.78), we have

$$\begin{aligned} \rho_\varphi &= f_1 \cos\theta + f_2 \left(1 + \tfrac{mr^2}{A} \cos\theta \ln|\csc\theta - \cot\theta|\right), \\ \rho_\psi &= f_1 + f_2 \tfrac{mr^2}{A} \ln|\csc\theta - \cot\theta|. \end{aligned} \tag{4.2}$$

Thus the Hamiltonian system $\left(N_{(f_1,f_2)}, \tilde{\omega}, \tilde{h}\right)$ on $N_{(f_1,f_2)}$ is defined by the two-form

$$\tilde{\omega} = f^*\omega_{\bar{H}} = d\theta \wedge d\rho_\theta, \tag{4.3}$$

and by the Hamiltonian function $\tilde{h}$ on $N_{(f_1,f_2)}$

$$\begin{aligned} \tilde{h} = &\tfrac{1}{2(A+mr^2)}\rho_\theta^2 + \tfrac{1}{2mr^2}\left(f_1 + f_2 \tfrac{mr^2}{A} \ln|\csc\theta - \cot\theta|\right)^2 \\ &+ \tfrac{1}{2A\sin^2\theta} f_2^2 + mgr\sin\theta. \end{aligned} \tag{4.4}$$

The vector field $\tilde{X}$ satisfying

$$\tilde{X} \lrcorner \tilde{\omega} = d\tilde{h}, \tag{4.5}$$

is given by

$$\tilde{X} = \dot{\theta}\, \partial_\theta + \dot{\rho}_\theta\, \partial_{\rho_\theta}, \tag{4.6}$$

where

$$\begin{aligned} \dot{\theta} &= \tfrac{1}{(A+mr^2)}\rho_\theta, \\ \dot{\rho}_\theta &= \tfrac{\cos\theta}{A\sin^3\theta} f_1^2 - \tfrac{1}{A\sin\theta} f_1 \left(f_1 \tfrac{mr^2}{A} \ln|\csc\theta - \cot\theta| + f_2 \right) - mgr\cos\theta. \end{aligned} \tag{4.7}$$

For the rolling the Legendre transformation is singular for $C = 0$, however, the singular case $C = 0$ is just the limit of the nonsingular case $C \neq 0$. As $C \longrightarrow 0$, $\gamma \longrightarrow 1$. Thus

$$\begin{aligned} \rho_\varphi &= \lim_{\gamma \longrightarrow 0} \left(\tfrac{A}{mr^2} \sin\theta \frac{d\rho_\psi}{d\theta} + \rho_\psi \cos\theta \right) \\ &= f_1 \cos\theta + f_2 \left(1 + \tfrac{mr^2}{A} \cos\theta \ln|\csc\theta - \cot\theta| \right), \\ \rho_\psi &= \lim_{\gamma \longrightarrow 0} \left(f_1 F\left(\tfrac{1+\gamma}{4}, \tfrac{1-\gamma}{4}; \tfrac{1}{2}; \cos^2\theta \right) + f_2 \cos\theta F\left(\tfrac{5+\gamma}{4}, \tfrac{5-\gamma}{4}; \tfrac{3}{2}; \cos^2\theta \right) \right) \\ &= f_1 + f_2 \tfrac{mr^2}{A} \ln|\csc\theta - \cot\theta|. \end{aligned} \tag{4.8}$$

4.2 Infinitesimal symmetries for the constants $f_1$and f_2

As stated earlier, the integration constants f_1 and f_2 can be written as a function on $\bar{M}$. Thus we obtain

$$\begin{aligned} f_1 &= \rho_\psi - \tfrac{mr^2}{A} (\rho_\varphi - \rho_\psi \cos\theta) \ln|\csc\theta - \cot\theta| \\ f_2 &= \rho_\varphi - \rho_\psi \cos\theta \end{aligned} \tag{4.9}$$

A straightforward calculation shows that $\mathcal{L}_{\tilde{X}} f_1 = \mathcal{L}_{\tilde{X}} f_2 = 0$. Therefore f_1, f_2 are two constants of motion for the generalized Hamiltonian system $\left(\bar{M}, \omega_{\bar{H}}, \bar{h}\right)$ on $\bar{M}$. In addition, we have conservation of the total energy $\bar{h}$, where

$$\begin{aligned} \bar{h} = \; & \tfrac{1}{2(A+mr^2)} \rho_\theta^2 + \tfrac{1}{2mr^2} \left(f_1 + f_2 \tfrac{mr^2}{A} \ln|\csc\theta - \cot\theta| \right)^2 \\ & + \tfrac{1}{2A\sin^2\theta} f_2^2 + mgr\sin\theta. \end{aligned} \tag{4.10}$$

Thus we have a total of three constants of motions. Since $\omega_{\bar{H}}$ is nondegenerate on $\bar{M}$, it makes sense to define vector fields on $\bar{M}$ such that

$$\begin{aligned} \bar{X}_{f_1} \lrcorner\, \omega_{\bar{H}} &= df_1, \\ \bar{X}_{f_2} \lrcorner\, \omega_{\bar{H}} &= df_2. \end{aligned} \tag{4.11}$$

Using the definitions of f_1, f_2, and $\omega_{\bar{H}}$, we obtain the vector fields

$$\begin{aligned} \bar{X}_{f_1} &= \tfrac{mr^2}{A \sin\theta} \left(\rho_\varphi - \rho_\psi \cos\theta\right) \partial\rho_\theta - \tfrac{mr^2 \rho_\theta \sin\theta}{A+mr^2} \partial\rho_\varphi - \tfrac{mr^2}{A} \ln\left|\csc\theta - \cot\theta\right| \bar{X}_{f_2}, \\ \bar{X}_{f_2} &= -\rho_\psi \sin\theta\, \partial\rho_\theta + \tfrac{mr^2 \rho_\theta \sin\theta}{A+mr^2} \left(\partial\rho_\psi + \cos\theta\, \partial\rho_\varphi\right). \end{aligned} \tag{4.12}$$

Taking the Lie bracket of these two vector fields we see that the span of X_{f_1} and X_{f_2} form an integrable distribution, because

$$\left[\bar{X}_{f_1}, \bar{X}_{f_2}\right] = \rho_\psi \frac{\sin\theta}{\rho_\theta} \bar{X}_{f_1} + \frac{mr^2}{A\rho_\theta} \left(\rho_\psi \sin\theta \ln\left|\csc\theta - \cot\theta\right| + \csc\theta \left(\rho_\varphi - \rho_\psi \cos\theta\right)\right) \bar{X}_{f_2}. \tag{4.13}$$

Because the distribution $\bar{V}$ =span$\left\{\bar{X}_{f_1}, \bar{X}_{f_2}\right\}$ represents a set of infinitesimal generators of a symmetry group on $\bar{M}$, one can question whether this symmetry group on $\bar{M}$ has any physical significance or can be lifted to M or P.

4.3 Orbits

For C=0, consider the Hamiltonian system $(N_{(f_1,f_2)}, \tilde{\omega}, \tilde{h})$, where $N_{(f_1,f_2)}$ is the cotangent bundle of the open interval $(0, \pi)$, where canonical two-form $\tilde{\omega}$ on $N_{(f_1,f_2)}$ is given by

$$\tilde{\omega} = d\theta \wedge d\rho_\theta, \tag{4.14}$$

and the Hamiltonian function $\tilde{h}$ on $N_{(f_1,f_2)}$ is given by

$$\begin{aligned} \tilde{h} = &\tfrac{1}{2(A+mr^2)} \rho_\theta^2 + \tfrac{1}{2mr^2} \left(f_1 + f_2 \tfrac{mr^2}{A} \ln\left|\csc\theta - \cot\theta\right|\right)^2 \\ &+ \tfrac{1}{2A \sin^2\theta} f_2^2 + mgr \sin\theta. \end{aligned} \tag{4.15}$$

Our Hamiltonian vector field $\tilde{X}$ on $N_{(f_1,f_2)}$ is given by

$$\tilde{X} = \dot{\theta}\,\partial_\theta + \dot{\rho}_\theta\,\partial_{\rho_\theta}, \tag{4.16}$$

where

$$\begin{aligned}\dot{\theta} &= \frac{1}{(A+mr^2)}\rho_\theta,\\ \dot{\rho}_\theta &= \frac{\cos\theta}{A\sin^3\theta}f_2^2 - \frac{1}{A\sin\theta}f_2\left(f_1 + f_2\tfrac{mr^2}{A}\ln|\csc\theta-\cot\theta|\right) - mgr\cos\theta.\end{aligned} \tag{4.17}$$

Suppose $f_2 = 0$, then the Hamiltonian function $\tilde{h}$ on $N_{(f_1,f_2)}$ is

$$\tilde{h} = \frac{1}{2(A+mr^2)}\rho_\theta^2 + \frac{1}{2mr^2}f_1^2 + mgr\sin\theta, \tag{4.18}$$

and the Hamiltonian vector field $\tilde{X}$ on $N_{(f_1,f_2)}$ is

$$\tilde{X} = \frac{1}{(A+mr^2)}\rho_\theta\partial_\theta - mgr\cos\theta\partial_{\rho_\theta}, \tag{4.19}$$

For the case $f_2 = 0$, the potential energy for the Hamiltonian system $(N_{(f_1,0)}, \tilde{\omega}, \tilde{h})$ is given by Figure 4.1 and the corresponding phase diagram in Figure 4.2. From the phase diagram we can see the motion of the disk on $N_{(f_1,0)}$ is similar to the motion of a pendulum, except that the motion of the disk is restricted to the interval $(0,\pi)$.

We can reconstruct the motion of the disk on $\bar{M}$, because we have the map $f_{(f_1,f_2)} : N_{(f_1,f_2)} \longrightarrow \bar{M}$ Thus from equations (3.58) and (3.74), we have

$$\begin{aligned}v_\varphi &= \frac{f_2}{A\sin^2\theta},\\ v_\psi &= \frac{f_1}{mr^2} + \frac{f_2}{A}\left(\ln|\csc\theta-\cot\theta| - \frac{\cos\theta}{\sin^2\theta}\right).\end{aligned} \tag{4.20}$$

If $f_1 = f_2 = 0$, then $v_\varphi = v_\psi = 0$ and the disk is not rolling or spinning. From Figure 4.1, we can see there is a critical point at $\theta = \pi/2$, $\rho_\theta = 0$, which is a saddle. If $\theta = \pi/2$, $\rho_\theta = 0$, then the disk is standing vertically. If $\theta \neq \pi/2$, $\rho_\theta \neq 0$, then

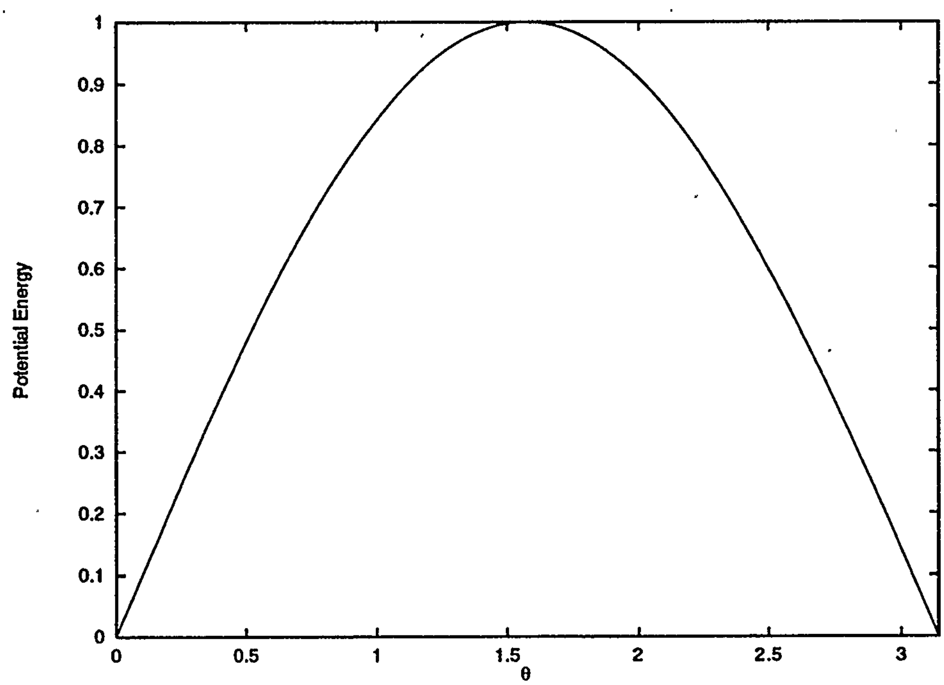

Figure 4.1: Potential energy for $f_2 = 0$

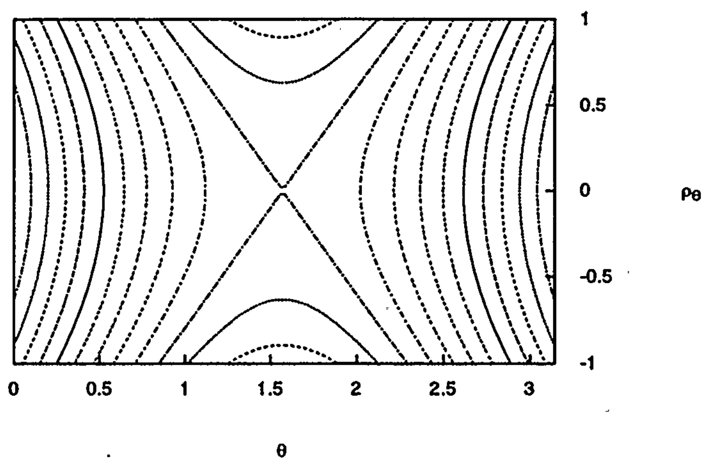

Figure 4.2: Phase diagram for $f_2 = 0$

the disk is falling over to one side or the other. However, if $f_1 \neq 0$, then $v_\varphi = 0$, and $v_\psi = \dfrac{f_1}{mr^2} \neq 0$. In this case the disk is rolling in a straight line with constant velocity. If $\theta = \pi/2$, $\rho_\theta = 0$, then the disk is standing vertically. If $\theta \neq \pi/2$, $\rho_\theta \neq 0$, then the disk is falling over to one side or the other, while rolling. For this case there are no periodic orbits.

If $f_2 \neq 0$, the Hamiltonian function $\tilde{h}$ on $N_{(f_1,f_2)}$ is given by

$$\begin{aligned}\tilde{h} = {} & \tfrac{1}{2(A+mr^2)}\rho_\theta^2 + \tfrac{1}{2mr^2}\left(f_1 + f_2\tfrac{mr^2}{A}\ln|\csc\theta - \cot\theta|\right)^2 \\ & +\tfrac{1}{2A\sin^2\theta}f_2^2 + mgr\sin\theta.\end{aligned} \tag{4.21}$$

and the Hamiltonian vector field $\tilde{X}$ on $N_{(f_1,f_2)}$ is given by

$$\tilde{X} = \dot{\theta}\,\partial_\theta + \dot{\rho}_\theta\,\partial_{\rho_\theta}, \tag{4.22}$$

where

$$\begin{aligned}\dot{\theta} &= \frac{1}{(A+mr^2)}\rho_\theta, \\ \dot{\rho}_\theta &= \tfrac{\cos\theta}{A\sin^3\theta}f_1^2 - \tfrac{1}{A\sin\theta}f_1\left(f_1\tfrac{mr^2}{A}\ln|\csc\theta - \cot\theta| + f_2\right) - mgr\cos\theta.\end{aligned} \tag{4.23}$$

For this case we have three possibilities. In the first case we have a weak gravitational field, where the potential energy is given by Figure 4.3, and the phase diagram is given by Figure 4.4. From the phase diagram in Figure 4.4, we see the rolling disk has one critical point, a center, and that all other orbits are periodic. Suppose the critical point occurs at $\theta = \pi/2$ and $\rho_\theta = 0$. If $f_1 = 0$, then $v_\varphi = \frac{f_2}{A\sin^2\theta}$ and $v_\psi = 0$. This corresponds to the disk spinning at one point. If $f_1 \neq 0$, then $v_\varphi \neq 0$ and $v_\psi \neq 0$. Thus the disk is rolling around in circles while standing vertically. If the critical point does not occurs at $\theta = \pi/2$ and $\rho_\theta = 0$, then $v_\varphi \neq 0$ and $v_\psi \neq 0$. Here the disk is rolling around in circles with a constant angle $\theta \neq \pi/2$.

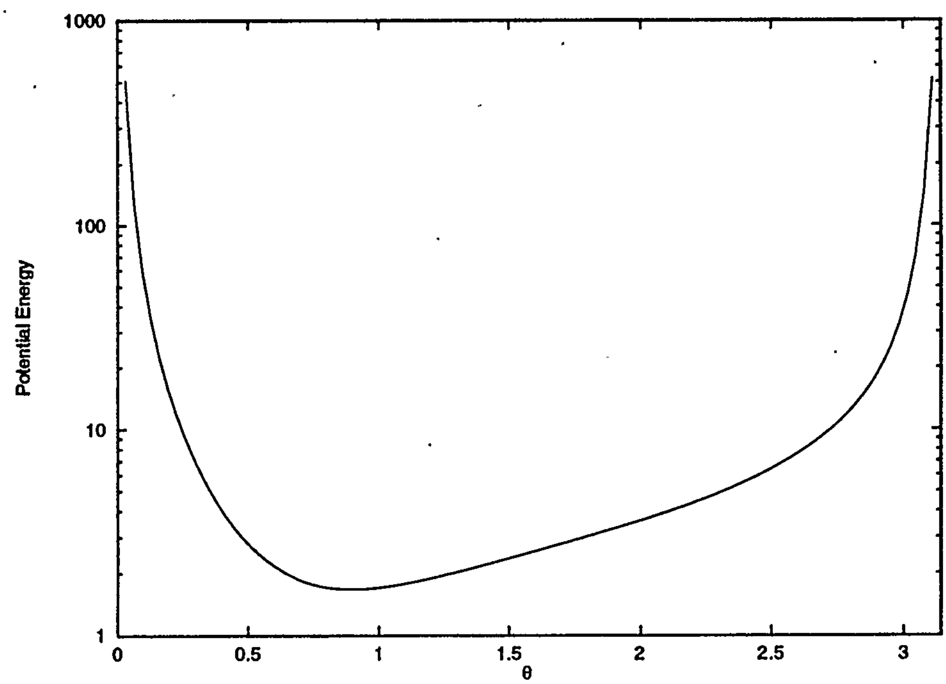

Figure 4.3: Potential energy for $f_2 \neq 0$ and a weak gravitational field

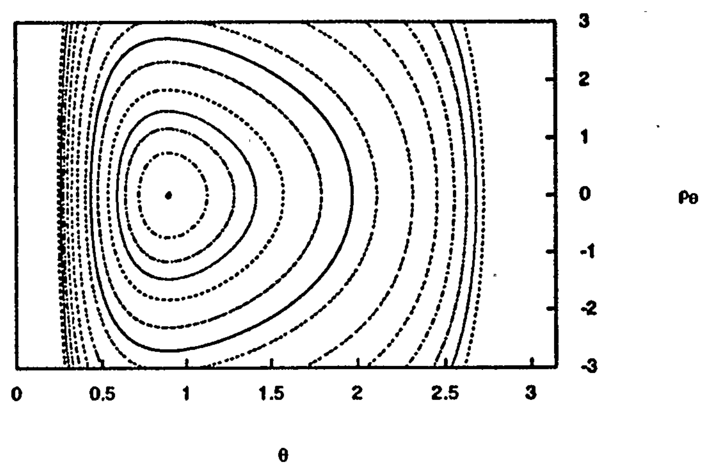

Figure 4.4: Phase diagram for $f_2 \neq 0$ and a weak gravitational field

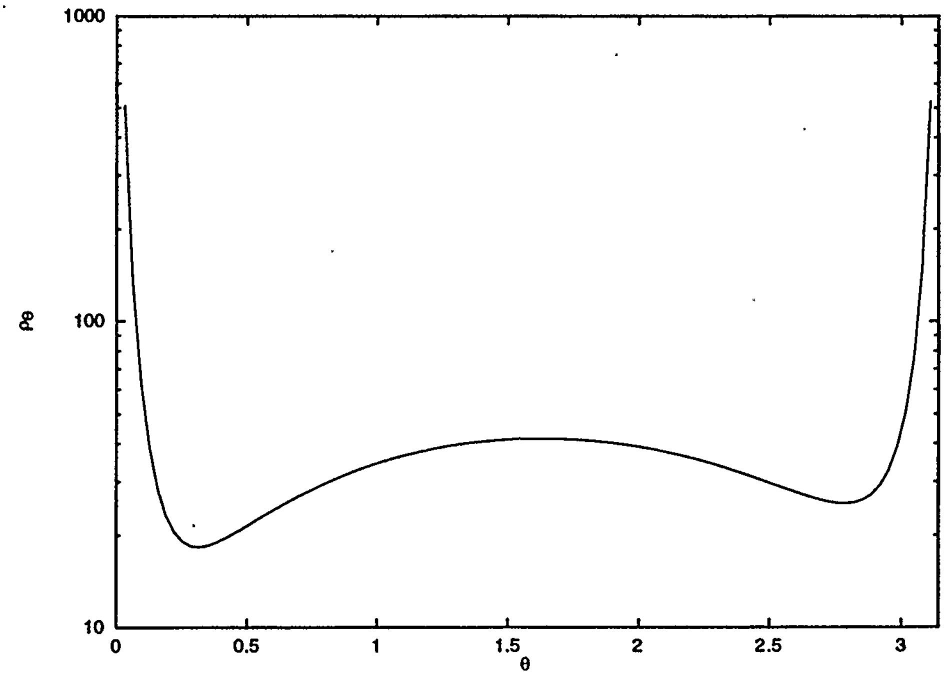

Figure 4.5: Potential energy for $f_2 \neq 0$ and a strong gravitational field

As the gravitational field increases, the potential energy of the Hamiltonian system $\left(N_{(f_1,f_2)}, \tilde{\omega}, \tilde{h}\right)$, will eventually have a strict local minimum and a horizontal inflection point. Therefore, we will have two critical points, where one is a center and the other is a cusp. This is the second possibility.

With a strong gravitational field, the cusp will bifurcate into two critical points, where one is a saddle and the other is a center. Thus there is a total of three critical points. This is the third and last possibility. The potential energy for this example is given by Figure 4.5, and the phase diagram is given by the Figure 4.6. Two of the critical points are centers while the third critical point is a saddle. This case is particularly interesting, since there are two nonperiodic orbits. Along these

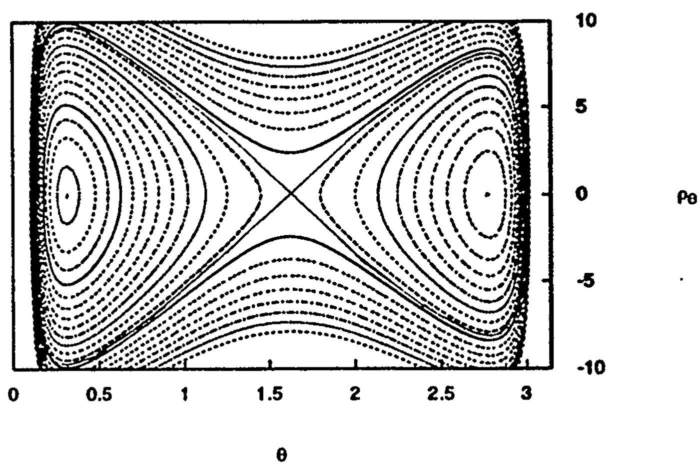

Figure 4.6: Phase diagram for $f_2 \neq 0$ and a strong gravitational field

orbits θ approaches a constant and $\rho_\theta \longrightarrow 0$ as $t \longrightarrow \pm\infty$. Therefore, the path of the disk becomes circular, if the saddle does not occur at $\theta \neq \pi/2$. If the saddle point occurs at $\theta = \pi/2$ and $f_1 = 0$, the disk will then approach the motion of a disk spinning at one point vertically since $\theta \longrightarrow \pi/2$, $\rho_\theta \longrightarrow 0$, $v_\varphi \longrightarrow \dfrac{f_2}{A\sin^2\theta}$ a constant and $v_\psi \longrightarrow 0$. If the saddle point is at $\theta = \pi/2$ and $f_1 \neq 0$, then the disk will asymptotically approach the motion of a disk going in circles standing up vertically since $\theta \longrightarrow \pi/2$, $\rho_\theta \longrightarrow 0$, and v_φ, v_ψ both approach a non-zero constant.

Chapter 5

Conclusion

5.1 Summary

From the first example, the two-dimensional Kepler problem, we see that the nonholonomic reduction techniques discussed in [2] agrees with the holonomic reduction of Meyer-Marsden-Weinstein.

Using the reduction method given in [2], the rolling disk reduces from a nonholonomic Hamiltonian system (P, ω, h, ϕ^a) with five degrees of freedom to a generalized Hamiltonian system $\left(\bar{M}, \omega_{\bar{H}}, \bar{h}\right)$ on a four dimensional reduced manifold $\bar{M}$ with a nondegenerate non closed two-form $\omega_{\bar{H}}$. By partially integrating the differential equations for the rolling disk on $\bar{M}$, one sees the reduced dynamics on $\bar{M}$ is foliated by a two parameter family of Hamiltonian systems $\left(N_{(f_1,f_2)}, \tilde{\omega}, \tilde{h}\right)$ with one degree of freedom, where $N_{(f_1,f_2)}$ is the cotangent bundle of an open interval.

Because the two-form ω_H was nondegenerate for the case $C = 0$, the equations of motion for the rolling disk are still valid even though the Legendre transformation was singular. From here, we were able to establish some of the dynamics on $\bar{M}$ by using the map $f_{(f_1,f_2)} : N_{(f_1,f_2)} \longrightarrow \bar{M}$ to reconstruct the dynamics from the completely integrable Hamiltonian system $\left(N_{(f_1,f_2)}, \tilde{\omega}, \tilde{h}\right)$. Since the equations of motion for the rolling disk where still valid even though the Legendre transformation was singular, one should check to see if the theory could be modified to handle nonholonomic Hamiltonian systems with a singular Legendre transformation.

As well, nothing is known about the dynamics of the rolling disk on $\bar{M}$ for the case $C \neq 0$. For further investigations, one should see whether the symmetry group defined by the set of infinitesimal generators $\bar{V}$ =span$\left\{\bar{X}_{f_1}, \bar{X}_{f_2}\right\}$ on $\bar{M}$ has any physical significance and whether this symmetry can be lifted to M or P.

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