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Indiscernibility and Mathematical Structuralism

by

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Abstract

This project describes a solution to a problem in Stewart Shapiro's *ante rem* structuralism, a theory in the philosophy of mathematics. Shapiro's theory proposes that the nature of mathematical objects is less important than the relations mathematical objects have to one another. Thus, mathematical objects are places in patterns and are constituted by the relations they have to the other places. However, Jukka Keränen demonstrated that there are some distinct mathematical objects which bear all the same relations to every other mathematical object. If Leibniz's Law, the Identity of Indiscernibles, were accepted, this would mean that Shapiro's theory identifies objects which can be mathematically proved to be distinct. This thesis demonstrates that this problem can be avoided by taking identity as a primitive notion, and by using Hilbert's epsilon calculus as a tool for referring to indistinguishable objects.

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Chapter 1

Introduction

During the 19th century, both Dedekind and Cauchy developed methods for constructing the real numbers out of the rational numbers. Both successfully produced the real numbers from the rationals, and both methods are still commonly accepted for doing so today. However, by constructing the real numbers out of the rationals in Dedekind's fashion, they are imbued with properties which they do not possess when we construct them via Cauchy's method. The same phenomenon occurs the other way around: reals constructed via Cauchy's method have properties which Dedekind's real numbers do not. When we construct the reals via these two methods, we 'forget' about the extra properties they add to each number, and only concern ourselves with what is essential to the numbers themselves.

During the 20th century, these techniques were taken one step further. Since the reals can be constructed out of the rationals, rationals out of the integers, and integers out of the natural numbers, questions arose about whether or not the natural numbers could be constructed out of something simpler. One of the main theses put forward was that natural numbers were really specific types of sets. However, as explicitly shown in Benacerraf (1965), any reduction of this sort also adds features to the natural numbers that are not essential to the natural numbers themselves. Reducing the natural numbers to any particular kind of set adds properties to them which are extraneous to their numberhood. This conflicts with an intuition that many mathematicians and philosophers have: that a number does not need these extra properties to be a number. That is, any of the reductions will do, as long as they preserve the right relations between the numbers. Structuralism is the view in the philosophy of mathematics that denies that numbers can be considered individually (see Hellman, 1993; Shapiro, 1997; Resnik, 1997; Chihara, 2004; Awodey, 2004). Instead, structuralists hold that numbers are places in a pattern. For example, natural numbers are places in the pattern given by $\{0, 1, 2...\}$, integers are the places in the pattern $\{... - 2, -1, 0, 1, 2...\}$, etc. Structuralism, unlike rival theories, is capable of capturing the intuition that any instantiation of the right type of pattern can be substituted as a system of natural numbers.

While most structuralists (Hellman and Awodey, for example) claim that structures are special types of mathematical objects (e.g., sets or categories), Stewart Shapiro holds that structures are abstract objects in a category of their own. For Shapiro, since structures are non-physical objects which exist, the places in them (the numbers) actually exist. Thus, when we talk about numbers, we talk about real, though non-physical, things (places in a pattern). This solves three major problem in the philosophy of mathematics which will be detailed with Shapiro's theory in chapter 2.

In his 1997, Shapiro states that "there is no more to individual numbers 'in themselves', than the relations they bear to each other" (p. 79). This view has given rise to what is called Keränen's identity problem. For most of Shapiro's competitors, it is not a problem, since they take structures to have other properties (i.e., the same properties as sets in set theory, or categories in category theory). However, since Shapiro has posited that structures are a new type of object, he must provide a method which we can use to distinguish one place in a structure from another one. There is an excellent theory for telling structures themselves apart, but not for places within those structures. For example, although Shapiro can uniquely identify the complex number structure, he cannot distinguish between the two square roots of -1, i and $-i$, within that very structure. Keränen showed that this was in fact impossible. Based on Shapiro's claim that numbers are no more then the relations they bear to each other, Keränen demonstrated that there is no relation which i bears to any number that $-i$ does not also bear. Leibniz's law states that there are no indistinguishable objects. Thus, Keränen claims that $i = -i$ under Shapiro's description. The identity problem for *ante rem* structuralism (henceforth ARS) is a generalization of this; there are many indistinguishable objects in mathematics.[1] Based on Leibniz's law Keränen claims that any objects which Shapiro cannot distinguish must be one object, and thus Shapiro's theory must be rejected. This is at the surface a metaphysical issue. It is much like giving a colourblind person a green and red ball and asking them to point to the red one. He or she cannot do it (intentionally, at least).[2] Shapiro's predicament is like that of a colourblind person: just as, in his or her context, a colourblind person cannot distinguish between the red and green ball, so in Shapiro's context, i cannot be distinguished from $-i$.[3] The identity problem is discussed in chapter 3 and the solution to it is given in chapter 4. By rejecting the principle that indistinguishable objects must be the same, abstract or otherwise, the identity problem can be avoided.

This is all well and good, but it still leaves us with no answer about how we can know that there is more than one indistinguishable object, and how we can talk about i without talking about $-i$ if we cannot tell them apart. This is the topic of chapter 5. The answer to the first question is quite straightforward: we know there is more than one object when mathematics tells us this is so. There is no plausible reason why the philosophy of mathematics should differ so extensively from the practice of mathematics that it generates results inconsistent with those generated by the practice. If mathematics can prove it, ARS must be able to prove it as well. A philosophy of mathematics is meant to explain what mathematics is, not to change mathematics. Answering the second part of the question is far more complicated. Which complex number do we mean when we

[1]There will be indistinguishable objects in any structure with non-trivial automorphisms. The definition of a non-trivial automorphism is given in footnote 1 in chapter 3.

[2]Thanks to Dr David Feder of the University of Calgary Physics Department for this example.

[3]Unfortunately, because physical objects can always be picked out by their positions in space-time, this is not a perfect example. There is a fact of the matter as to which ball is which colour, while with the two square roots of -1 there is no fact of the matter about which is i and which is $-i$.

say 'square root of -1'?[4] Do we intend to refer to i or $-i$? This is what I call the reference problem, which is the epistemic part of the identity problem. We need to be able to refer to one member of a set of many indistinguishable objects. When we make a claim like $i^2 = -1$, though it does not matter which square root i picks out, it must pick only one. It is certainly not the case that $\{i, -i\}^2 = -1$ - squaring a set is absurd! That statement is meaningless. So, then, which root are we referring to when we use 'i'? The intuitive answer is that we intend to refer to one of them. We say, 'let i be one of those two indistinguishable objects which are the square roots of -1'. This sort of thing is common in everyday language as well. Take the sentence, 'There are two copies of Shapiro's book in the library. Please bring me one'. When asked which one to bring, the answer is simply 'whichever one you want'. The two books have all their properties in common (aside from their spatial location, but numbers have no spatial locations); no one of them stands out as the uniquely desired book. We say, 'let the book you bring be one of those two indistinguishable objects'. Shapiro's solution is that indistinguishable objects behave like the books in the library. We can simply say 'pick one'. Shapiro claims that definite descriptions and anaphoric pronouns might not refer uniquely, and uses these results to postulate his theory of parameters. Although Shapiro's solution is correct (and equivalent to my own), there are pragmatic reasons why my solution is better. All that is required to solve this problem is some sort of choice function, and Hilbert's epsilon calculus serves well to formalize this. Using the epsilon function as our choice function, we can solve the problem while remaining more faithful to the intuition that we are just picking one thing out of a group of many. All that we require is that the function be able to pick out one of the indistinguishable objects. Both solutions are also discussed in chapter 5.

[4] It is a convention in the positive real numbers to say that $\sqrt{4}$ refers to the positive root, namely 2. However, though this is a convention, there are structural differences between 2 and -2, so the problem does not apply.

If Keränen is correct that Shapiro and other like-minded structuralists cannot distinguish between i and $-i$ and must claim that there is really only one root of -1, then there is a problem for everyday mathematics as well. The everyday working mathematician does not think about whether or not numbers are uniquely defined, and in fact he does not have an answer to whether they are so defined. Philosophy of mathematics should follow mathematical practice, not change it. In order to avoid forcing mathematicians to change the way they think about numbers, we must solve Keränen's problem. No mathematician in his right mind would accept that $i = -i$, and our philosophy must conform to this intuition. We are then left with two choices: either reject ARS, or solve the identity problem. This project takes that latter route. The solution presented in this project will demonstrate that there is no need to reject ARS as Keränen's problem can be solved both metaphysically and epistemologically.

Chapter 2

Ante Rem Structuralism

This chapter will explain in detail Shapiro's theory of *ante rem* structuralism (ARS). As motivation, I will begin by discussing three major problems in the philosophy of mathematics, one from Frege (1892), and one from each of Benacerraf (1965, 1973). The solutions to Frege's problem and Benacerraf's first problem (Benacerraf, 1965) are discussed in section 2.4 after a brief sketch of *ante rem* structuralism, while the solution to the second of Benacerraf's problems is discussed after a more detailed examination of *ante rem* structuralism in section 2.5.

2.1 Frege's Caesar Problem

The Caesar problem was part of Frege's motivation for not finding Hume's principle sufficient. Hume's principle is the principle that that the number of things which fall under some concept, F, is equal to the number of things that fall under some other concept, G, if and only if there is a map between them which maps each F-thing to a unique G thing, and which 'hits' all the G things.[1] Frege was right to identify it as a problem, although he famously chose the wrong solution.

Frege held that numbers are objects, and hence it was important to specify which objects they were. However, if one wants to use Hume's principle to determine whether the number of F's was equal to Julius Caesar, a problem emerges. Though Hume's principle can explain when the numbers falling under two concepts are equal, it can tell you nothing about the numbers themselves. It cannot determine whether or not numbers are objects to begin with, let alone whether or not Julius Caesar is a number. Hume's

[1] This type of map is called a bijection.

principle is only useful if we are only concerned with whether or not two numbers are equal. Frege states:

> ... we can never — to take a crude example — decide by means of our definitions whether any concept has the number Julius Caesar belonging to it[2], or whether that same familiar conqueror of Gaul is a number or is not. (Frege, 1892, §56)

Thus, because Frege insists that numbers are objects, and because Hume's principle does not specify which objects they are (or can be), Hume's principle cannot, for Frege, be the only identity criterion for numbers. There is a deeper problem about what types of objects numbers can be, and this is really what the Caesar problem points out. A successful philosophy of mathematics will have to answer this question, and will be able to pick out what type of objects numbers are, or will claim they are not objects at all. Hence, something more than Hume's principle is needed to solve the Caesar problem. For Frege, numbers were certain types of classes, which were in turn certain types of objects. However, his theory about them lead to contradiction. Section 2.4 will show that there is indeed another solution: to deny that numbers are objects at all.

2.2 Benacerraf's First Problem

Benacerraf's first problem is presented in "What Numbers Could Not Be" (Benacerraf, 1965). The article is an explanation of why numbers cannot be sets, or more generally any other object at all. This argument forces us to conclude that Frege's solution cannot be correct, as he defines numbers as a certain type of object.

Benacerraf asks us to imagine two children, Ernie and Johnny. Growing up, Ernie and

[2] When Frege states that a number n belongs to a concept F, he is claiming, in layman's terms, that there are n distinct things falling under F. So, for example, the number 1 belongs to the concept 'even prime number'.

Johnny both learn arithmetic differently from ordinary children, starting with set theory and then moving to natural numbers. Ernie learns that each natural number is a Zermelo ordinal (with the natural number series starting with $0 = \emptyset, 1 = \{\emptyset\}, 2 = \{\{\emptyset\}\}, 3 = \{\{\{\emptyset\}\}\}$), and Johnny learns that each is a von Neumann ordinal (with the natural number series beginning with $0 = \emptyset, 1 = \{\emptyset\}, 2 = \{\emptyset, \{\emptyset\}\}, 3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$). It is quite clear that for both of them, 0 and 1 are represented by the same sets, namely $\emptyset$ and $\{\emptyset\}$. Once the education of both boys is complete, we would say that they both 'know' arithmetic, and that the theorems they prove about natural numbers are true. However, an issue arises when we consider their answers to the question 'does 3 belong to 17?'. Johnny, of course, will say that it does. On the other hand, Ernie will say it does not. For Johnny, any number n has n members; for Ernie, it only has one. Benacerraf claims

> The fact that they disagree on which particular sets the numbers are is fatal to the view that each number is some particular set. For if the number 3 is in fact some particular set b, it cannot be that two correct accounts of the meaning of "3" — and therefore also its reference — assign two different sets to 3. For if it is true for some set b, $3 = b$, then it cannot be true that for some set c, different from b, $3 = c$. (Benacerraf, 1965, p. 56)

Though Benacerraf only explicitly makes reference to two possible different (yet equally good) sets which represent 3, there are infinitely many to choose from. This dilemma is not restricted to the number 3 exclusively, or even just to numbers individually. It also applies to the meaning of 'number', which will be different for both boys. If they disagree about things as fundamental as the cardinality of the sets which they take to be natural numbers, then there is a problem which needs to be solved.

Benacerraf's solution is to claim that numbers cannot be sets. Since we cannot find a particular set which everyone agrees is the right one, we cannot find any set at all. Presumably, there are practical reasons to pick one set over another. However, there seems to be no *prima facie* reason to do so. For Benacerraf, "there is little to conclude except that any feature which identifies 3 with a set is a superfluous one — and that therefore 3, and its fellow numbers, could not be sets at all" (Benacerraf, 1965, p. 69). In the final section of the paper, he proposes a tentative solution. It seems as though any recursive progression of objects would adequately model the natural number series. This suggests that what is important is not "the individuality of each element, but the structure that they jointly exhibit"(Benacerraf, 1965, p. 69). Numbers would not only not be sets, they would not be objects at all. If Benacerraf's tentative solution is correct, numbers would be nothing outside of the structure in which they occur. Benacerraf's sketch of a solution does in fact work when fleshed out in full, as we shall see in section 2.4.

2.3 Benacerraf's Second Problem

What I will refer to as Benacerraf's second problem is presented in his article "Mathematical Truth" (Benacerraf, 1973). The problem concerns a dilemma that philosophers of mathematics face. It emerges from a tension between the epistemology of mathematics and the nature of mathematical truth. He states:

> ...two quite distinct kinds of concerns have separately motivated accounts of the nature of mathematical truth: (1) the concern for having a homogeneous semantical theory in which semantics for the propositions of mathematics parallel the semantics of the rest of language, and (2) the concern that the account of mathematical truth mesh with a reasonable account of epistemology. (Benacerraf, 1973, p. 661)

The argument presented in the article serves to demonstrate that accounts of mathematical truth can accommodate one of these concerns, but only *at the expense of the other* (Benacerraf, 1973, p. 661). Accounts of mathematical truth either treat mathematical propositions as having mind-independent truth conditions and lose the ability to explain how mathematical knowledge is acquired (or acquirable), or they treat mathematical knowledge as everyday knowledge and lose the ability to explain why it is so universally applicable.

For Benacerraf, any account of mathematical truth should satisfy two conditions. The first is that if any condition is given for the generation and preservation of truth, the theory must explain why the condition works (in Benacerraf (1973), page 666, theoremhood is given as this condition). The second is that an account of mathematical truth must not be inconsistent with the fact that some mathematical truths are knowable. The first is because, like any ordinary theory of truth, a theory of mathematical truth must recognize only true statements as true. If, given the theoremhood condition, a theorem (properly derived) turns out to be false, then we would claim that theoremhood is not a suitable condition for truth. The second condition is motivated by the desire that an acceptable semantics for mathematics must fit into an acceptable epistemology (Benacerraf, 1973, p. 667). If the theory of mathematical truth precludes knowing any of those truths, then we gain nothing from it, and moreover may be reduced to skepticism.

Benacerraf claims these two conditions cannot both be satisfied. Two different kinds of accounts of mathematical truth can be (and have been) conceived. The first account is motivated by the first concern, namely realism (something akin to Platonism), and another account is motivated by the second concern, namely nominalism (or something close to it). These two theories are what Benacerraf calls the standard and combinatorial theories, respectively. The standard theory has an excellent grasp of what mathematical truth is. However, in getting such a firm grip on truth, it puts mathematical objects outside of our normal modes of perception (e.g. sense perception), as our normal modes of perception are not guaranteed to only perceive true things. For example, when people hallucinate they think that they are seeing something real. If mathematics were like this, we might be subject to believing 'hallucinated' (and thus false) mathematical statements. Thus, mathematical objects are outside of our normal realm of knowledge, violating the second condition. Benacerraf states: "the principal defect of the standard account is that is appears to violate the requirement that our account of mathematical truth be susceptible to integration into our over-all account of knowledge" (Benacerraf, 1973, p. 670). The nominalistic view, on the other hand, is motivated by the second concern. It starts out by treating mathematical knowledge as 'normal' knowledge. It runs into problems when it attempts to make statements about why things are true. The combinatorial theory has no satisfactory account of how the truth conditions attributed to mathematical propositions actually obtain.

The two equally desirable conditions for mathematical truth are contradictory, then. Starting with one precludes the other. This is the crux of Benacerraf's second problem.

2.4 Ante Rem Structuralism

Shapiro (1997) defines *ante rem* structuralism (ARS) as the theory that numbers[3] are not sets, but rather places or positions in structures. An *ante rem* structuralist is one for whom structures are objects in and of themselves. This is in contrast to an eliminative structuralist position, which holds that talk of structures is shorthand for talk of other objects.[4] The number 3, then, would be nothing more or less then the fourth place in the natural number structure. Structures can be considered as patterns.[5] Thus, talk of natural number systems refers to structures, and talk of individual numbers refers to

[3]In this particular instance, the 'numbers' in question are the natural numbers. However, this conception can easily be extended to the rationals and reals, as they are well ordered. With a little more work, complex numbers and higher order number systems can also be caputured, and eventually all mathematical objects.

[4]These other objects are frequently sets or categories, see Awodey (2004) and McLarty (1993). However, according to Hellman (1993), they can also be considered modal possibilities.

[5]This terminology is taken from Resnik (1997), but it lends insight to Shapiro's conception nonetheless.

places within structures. Shapiro calls himself both a realist in ontology and in truth value (Shapiro, 1997, p. 72). Thus, he holds that mathematical objects exist and that mathematical statements have objective truth values. For Shapiro, there is an intimate relationship between the practice of mathematics and the philosophy of mathematics. He says “I propose the metaphor of a partnership or healthy marriage rather then a merger or blending — a stew rather then a melting pot” (Shapiro, 1997, p. 35). Thus, philosophy of mathematics must remain faithful to the practice of mathematics and vice versa. The philosophy should not change what mathematics is capable of, only lend insight to why it is capable. Thus, if any philosophy of mathematics suggests drastic changes to the practice of mathematics, other than changes to how we think about the practice of mathematics, it should be rejected.[6] This is what I will refer to as Shapiro's faithfulness constraint.

Since numbers are now places in *sui generis* abstract structures and not objects like sets, Shapiro can solve both the Caesar problem and Benacerraf's first problem.

For our *ante rem* structuralist, Caesar is not a number. Interestingly, if the entire (chronologically ordered) list of Roman emperors were taken and copied ω times, Caesar himself could hold the place of the number 1, the first copy of Caesar (Caesar$'$) could hold the place of the number 150, etc.[7] Since numbers are not the place holders in a structure but rather the places, Caesar can hold the place of 1 in any given instantiation. Anything can *instantiate* any place in a structure, but it will never *be* a place in a structure, which is a *sui generis* object, so it can never be a number. It is the places themselves which are the *sui generis* objects, and thus the numbers. On this theory, numbers are *sui generis* objects. Taking any properly ordered sequence of objects, whichever object we choose to hold the place of the number 1 will never be the number 1 itself, but a mere instantiation

[6] Clearly, many philosophers produce mathematical results. It is not the philosopher that must concede to the mathematician, but rather the philosophy which must listen to the mathematics.

[7] Assuming, of course, that there were 149 Roman emperors.

of the number. In this sense, Caesar is not a number, but he might represent a number in one particular instantiation of the natural number structure.

On the same note, Benacerraf's first problem is also solved. Mathematical objects are *sui generis* objects: they are places in structures. The only question we could ask here is what types of objects can 'stand in' for any given natural number. But that has an easy answer: any type of object will do, even another number. Thus, Shapiro has fleshed out Benacerraf's sketch of a solution and shown that it does work. Whether or not $3 \in 17$ is no longer a question we can coherently ask about the natural number structure, but rather has to be asked about a specific instantiation of that structure. The von Neumann natural numbers and the Zermelo natural numbers are both instantiations of a natural number structure. Without taking pragmatic considerations into account, they are both on equal footing. Yet, because they are two different instantiations, we cannot expect their corresponding answers to the questions we ask to be the same.[8]

These two problems are solved because of the same characteristic of structures: numbers are the places within them, not the objects which hold those places. From this respect, "anything at all can 'be' 2 — anything can occupy that place in a system exemplifying the natural number structure" (Shapiro, 1997, p. 80). Thus, 2 can be instantiated by Julius Caesar, $\{\emptyset, \{\emptyset\}\}$ or $\{\{\emptyset\}\}$, or even 3. There is an infinitely long list of other possibilities.

[8]It is important to note that the answers to these types of questions will (can) vary only when the questions are referring to properties which are not essential to the structure (and thus the mathematical object).

2.5 ARS in Detail

2.5.1 Ontology

According to ARS, structures are *sui generis* objects, and numbers are places within a natural number structure. Shapiro is careful to emphasize that numbers are not place holders, but rather the places themselves. In fact, "each mathematical object is a place in a particular structure" (Shapiro, 1997, p. 78). Thus, mathematical objects are not objects which instantiate a structure, but are part of the structure itself.

Shapiro characterizes his structuralism as *ante rem* because it accepts the existence of universals without needing them to be instantiated. An *in re* structuralist position holds that, for example, without red things there is no universal redness. However, for Shapiro this is unsatisfactory. We can make reference to things which are never instantiated.[9] We are not capable of creating triangles whose angles sum to exactly 180 degrees, only better and better approximations. However, Euclidean geometry only refers to triangles of exactly 180 degrees and our philosophy of mathematics needs to be able to discuss them.[10] We now have an account of mathematics where mathematical objects are places in structures and structures exists in and of themselves.

So far, what has been discussed applies to mathematical structures as well as non-mathematical structures. It fits just as well with the structure of a hockey team as it does with the natural number system. We can talk about the position of goalie without making reference at all to the person who happens to be playing in net that day. A

[9] A perfect circle is a perfect example.

[10] An eliminative structuralist like Hellman claims that mathematical structures are modalized statements about mathematical possibilities. This allows talk of non-Euclidean geometry as talk about possible structures, which do not have to be instantiated to exist. Thus, though he does not hold that universals actually exist without being instantiated, he can say that they possibly exist without being instantiated. However, if it is possible that a mathematical objects exists, then it necessarily exists because of the abstract nature of mathematical objects, and so Hellman falls prey to the same trap as the *in re* structuralists. We would say that even though non-Euclidean geometry is merely possible on our actual, physical, Earth, its theorems produce necessary truths. Thus, Hellman's possibilities are really disguised necessities.

hockey team is instantiated when each of the six positions is filled by a person. One of the key moves Shapiro makes is to distinguish between mathematical and ordinary structures. He does this by claiming that mathematical structures are "*freestanding*" (Shapiro, 1997, p. 100). For example, the structure of a hockey team is not itself a hockey team, the structure of a government is not itself a government. However, in a mathematical structure, "every office[11] is characterized completely in terms of how its occupant relates to the occupants of any other office of the structure and any object can occupy any of its places." (Shapiro, 1997, p. 100). For mathematics, then, there is "no difference between simulating a structure and exemplifying it" (Shapiro, 1997, p. 100).[12] When compared to our hockey team the difference is clear. The structure of a hockey team is not itself a hockey team, the structure needs to be instantiated by people to make it such. However, the structure of the natural numbers is itself a natural number system. Anything you can do with the place 1 in a structure $\mathscr{S}$ you can do with any place holder of 1 in an instantiation of $\mathscr{S}$, and vice versa, which is not the case with a hockey goalie. Clearly putting the goalie-place in net is not the same as putting an actual physical instantiation of a goalie in net.

A mathematical structure is a collection of places, and a collection of functions and relations on those places. Two systems have the same structure if and only if there is a structure, $\mathscr{S}$, such that they are both isomorphic to full substructures of $\mathscr{S}$. A full substructure of $\mathscr{S}$ is a structure $\mathscr{P}$ which has the same objects as $\mathscr{S}$ such that every relation in $\mathscr{S}$ can be defined in terms of the relations in $\mathscr{P}$. Using Shapiro's example (Shapiro, 1997, p. 91), the natural numbers with addition and multiplication is a full substructure of the natural numbers with addition, multiplication and less than. A second order background language is assumed because, in addition to variables which

[11]Where an office is one of the 'blank places' in the structure, i.e., a mathematical object.
[12]It will become clear in subsequent chapters that the restriction of the definition of a number to solely intra-structural properties generates a major problem for Shapiro, namely the identity problem.

range over places in structures, variables which range over structures themselves are permitted. This is particularly important to Shapiro, since he needs to be able to claim that the distinguishing feature of a mathematical structure is that it is freestanding, i.e. "is freestanding" must be true of any mathematical structure.

Formally, there are several axioms of ARS, most of which are quite similar to the axioms of ZFC. For the most part, they are uncontroversial. The axioms of infinity, subtraction, subclass, addition, power-structure and replacement are all easily justified, and are very similar to the set theoretic axioms. They are (Shapiro, 1997, p. 93-94):

Infinity: There is at least one structure that has an infinite number of places.

Subtraction: If S is a structure and R is a relation of S, then there is a structure S' isomorphic to the system that consists of the places, functions, and relations of S except R. If S is a structure and f is a function of S, then there is a structure S'' isomorphic to the system that consists of the places, functions, and relations of S except f.

Subclass: If S is a structure and c is a subclass of the places of S, then there is a structure isomorphic to the system that consists of c but with no relations and no functions.

Addition: If S is a structure and R is any relation on the places of S, then there is a structure S' isomorphic to the system that consists of the places, functions, and relations of S together with R. If S is a structure and f is any function on the places of S, then there is a structure S'' isomorphic to the system that consists of the places, functions, and relations of S together with f.

Power-structure: Let S be a structure and s its collection of places. Then there is a structure T and a binary relation R such that for each subset $s' \subseteq s$ there is a place x of T such that $\forall z(z \in s' \leftrightarrow Rxz)$.

and finally,

Replacement: Let S be a structure and f a function such that for each place x of S, fx is a place of a structure which we call S_x. Then there is a structure T that is at least the size of the union of the places in the structure S_x. That is, there is a function g such that for every place z in S_x there is a place y in T such that $gy = z$.

The axiom of coherence is another story. This axiom states that if Φ is a coherent formula in a second order language, then there is a structure which satisfies Φ (Shapiro, 1997, p. 95). The problem is that all meaningful definitions of 'coherent' seem circular. We would like 'coherent' to mean something like 'any well formed second order formula that has an appropriate meaning'. For example, the formula $\exists x(x$ is a round square$)$ is not coherent and hence there will be no structure containing a round square thing. Thus, the set of coherent formulas must be some proper subset of the set of well formed second order formulas. In this situation, consistency does not even imply coherence. For example, there is a structure satisfying both the axioms of Peano arithmetic and the statement that Peano arithmetic is not consistent. Something more general is needed. Shapiro suggests satisfiability (Shapiro, 1997, p. 95). However, in general, a satisfiable statement is one for which there is a model, which the axioms of Peano arithmetic and the statement of its inconsistency does not have (i.e., it has a structure but no standard model). Shapiro's only hope is that he can somehow produce a definition of coherence which is not viciously circular. In the end, Shapiro takes coherence as a primitive notion.

Finally, a reflection axiom is introduced: if Φ, then there is a structure which satisfies the axioms of structure theory and Φ. He accompanies this axiom with a statement that there is no structure of all structures, in order to avoid Russell-like paradoxes. Since set theory needs to be 'part of' structure theory, we can conclude that the reason adaptations of the ZFC axioms fit so well is because structure theory must be as rich a set theory. Since Shapiro wants a minimal background, he makes it just as rich and stops there. This, along with the axiom of coherence, allows him to capture all mathematical theories as structures.

2.5.2 Epistemology

One of the biggest challenges in Shapiro's program is to provide a successful epistemology. There is no way to causally interact with mathematical objects, since they are not physical or mental, but rather abstract. The epistemology presented in Shapiro (1997) has three components. There are three ways structures are apprehended and knowledge of them is acquired: pattern recognition and abstraction, linguistic abstraction, and implicit definition.

Pattern recognition and abstraction has two levels: one for small (finite) structures and one for large structures. In the small structure case, we apprehend simple types through tokens. We know that all tokens of the 5th letter of the alphabet, namely 'E', represent the same type. This technique presupposes abilities on the part of the teacher and student, but it is clear that both groups make use of it daily, especially when the students are children. This is also the way we learn the smaller finite numbers, like 4. Learning to recognize groups of four objects (tokens) is how we come to know what is meant by the type 4. It is pattern recognition that goes beyond simple abstraction which allows us to apprehend larger structures. There are actually tokens of even the largest numbers we can conceive of. A child learning what is meant by the token "4677" will start with smaller tokens and then recognize that the pattern of numerals can be extended to larger and larger numbers, eventually realizing that the series of numerals can be extended indefinitely. This is where the natural numbers are learned: once someone realizes they can extend their pattern infinitely, they have apprehended the natural numbers. This is in line with what psychological practice leans towards. Piaget (1965) proposes, after extensive testing, that children seem to learn numbers from small to big. Thus, Shapiro's process seems to fit with how the process works in actual human beings.

Linguistic abstraction comes into play when we start with an interpreted base language, which is generally another structure. The next step is to focus on equivalence relations in the ontology given by the base language in order to generate a new language. We focus one one particular aspect of the base structure and take the classes formed by that aspect. Thus, we can generate a sublanguage of the base language for which the equivalence relation is a congruence. "The language and sublanguage together characterize a structure, the structure exemplified by the equivalence classes and relations between them formulable in the sublanguage" (Shapiro, 1997, p. 123). For example, if we take the structure of set theory, we can linguistically abstract to form the structure of the natural numbers. This is done by taking the equivalence classes of sets with the same number of elements. We let the sets with no elements be represented by 0, the sets with 1 element be represented by 1, the sets with n elements be represented by n, etc. We can linguistically abstract from set theory to get number theory.

The third and final means of apprehending structures is implicit definition. "In the present context, an implicit definition is a *simultaneous* definition of a number of items in terms of their relations *to each other*" (Shapiro, 1997, p. 130, fn. 15). Here, we start with a selected number of axioms and use that to learn about the structure they produce. Because of the axiom of coherence, any set of coherent axioms produces exactly one structure (up to isomorphism). However, the use of implicit definition as a method of apprehending structures means that coherence must be formalized. To solve these problems, Shapiro takes coherence to be primitive.

If Shapiro's theory of ARS works, he has successfully solved Benacerraf's second problem. ARS comes equipped with both a clear ontology and epistemology, thus skirting the threat of Benacerraf's problem. Thus, ARS solves all three problems mentioned in this chapter, making it a good candidate for a successful philosophy of mathematics.

Chapter 3

The Identity Problem: The Case of The Non-trivial Automorphism

Keränen (2001) presents the identity problem. Although both Geoffrey Hellman (Hellman, 2005, p. 544/5) and John Burgess (Burgess, 1999, p. 287/8) discuss this problem, the most detailed exposition is Keränen's, and so discussion in this chapter will follow his.

A distinction first needs to be made between rigid and non-rigid mathematical structures. We call a structure rigid if it has no non-trivial automorphisms.[1] For example, the field of real numbers have no non-trivial automorphisms, and so the structure they instantiate is rigid. On the other hand, a non-rigid structure is one which does have non-trivial automorphisms. The complex numbers satisfy this, as there is an automorphism which sends $a+bi$ to $a-bi$, for all a and b in $\mathbb{R}$. There is a wealth of non-rigid structures in mathematics, and this is what causes the identity problem in Shapiro's conception of structuralism.

Keränen's initial concern was that Shapiro must adopt a criterion of identity for places in structures. This does not apply to eliminativist structuralists, who believe that structures are short hand for something else, like sets or categories. Those structuralists can simply adopt the identity criteria presented by whatever they think structures actually are. For any *ante rem* structure, $\mathscr{S}$, and for any two denoting singular terms in that

[1] An automorphism, $\mathscr{A}$ is a map from a mathematical object, X onto that same mathematical object X which is one to one and onto. In other words it maps every element in X to an element in X, and every element in X has something mapped onto it. It must also preserve the relations and functions in X. Thus, for any relation R in X, $(x_1, x_2, ..., x_n) \in R$ if and only if $(\mathscr{A}(x_1), \mathscr{A}(x_2), ..., \mathscr{A}(x_n)) \in R$ and for any function $f : X \to X$ $\mathscr{A}(f(x_1, x_2, ..., x_3) = f(\mathscr{A}(x_1), \mathscr{A}(x_2), ..., \mathscr{A}(x_n))$. A non-trivial automorphism is one which does not map every element to itself.

language, say a and b, we must be able to do two things. We must be able to provide the circumstances under which $a = b$, and, more generally, we must be able to provide the circumstances under which any two (arbitrary) objects of the domain of $\mathscr{S}$ are the same. If ARS is incapable of doing this, then Keränen claims that it must be rejected as the theory must consider any two objects which it cannot tell apart as one object. Consider a colourblind person holding a ball which is half red and half green. The colourblind person sees the ball as a solid colour. He or she is unable to answer questions about which half of the ball is green. If ARS sees mathematics as the colourblind person sees the multicoloured ball, then there will be questions with obvious answers which ARS is unable to answer. Thus, ARS must provide a method to tell two given objects apart, and one to tell two arbitrary objects apart. For example, in the natural numbers, we need a method to distinguish between 1 and 5, and we also need a method to distinguish between x and y when we know x and y are natural numbers. This, claims Keränen, we do by filling in the blank in the sentence

$$\forall x \forall y (x = y \Leftrightarrow \underline{\qquad}) \tag{3.1}$$

Keränen is correct to point out that Shapiro is only capable of filling this blank with a sentence strictly about intra-structural properties.[2] Shapiro says so himself (Shapiro, 1997, p. 100): "every office is characterized completely in terms of how its occupant relates to the occupants of any other offices of the structure and any object can occupy any of its places". Keränen holds that Shapiro's claim commits him to the identity criterion above.

[2]Intra structural properties are properties which are only about the structure itself. For example, in the natural number structure, 'is the successor of 7' is an intra-structural property true of 8, while 'is the number of planets' is a property true of 8 but is not intra-structural.

Keränen holds that the blank in 3.1 cannot be filled in with any sort of haecceity condition since this would violate the intentions of *ante rem* structuralism. A haecciety is a property which belongs to one and only one object. It is the essence of that object, in a sense it is an object's "thisness". Were the identity conditions of places in structures to be given by haecceities, then there would be something more to the individual places than just their "placeness". This cannot be the case. For Shapiro, and all other advocates of ARS, mathematical objects are places in structures before they are anything else. Hence, the blank cannot be filled in by haeccities.

The only other option presented by Keränen is a general account of identity. This he gives as

$$\forall x \forall y (x = y \Leftrightarrow \forall r (r \in \mathscr{R} \Rightarrow (r(x) \Leftrightarrow r(y)))) \tag{3.2}$$

Here, $\mathscr{R}$ is the set of structural properties between the places in a structure $\mathscr{S}$.

Because of the conditions of ARS there are some fairly tight constraints on what $\mathscr{R}$ can contain and what the structural relations are.

1. No property the specification of which essentially involves an individual constant denoting an element of an instantiation of $\mathscr{S}$ may be admitted.

2. No property the specification of which essentially involves an individual constant denoting a place of $\mathscr{S}$ may be admitted. (Keränen, 2001, p. 316)

Restriction 1 is required for two reasons. The first is that being a place holder in an instantiation of $\mathscr{S}$ is not a universal property. In other words, just because Julius Caesar represents 1 in some particular instantiation of the natural number structure, the number 1 itself does not have any Caesar-esque properties. Thus, Caesar could not be used in an identity statement about numbers. So too with any object which happens to occupy a place in an instantiation of a structure. The second reason is that the relation 'being occupied by x', where x is any object, is extra-structural. In other words, it makes reference to something which is not essential to the structure, namely x. If 'being occupied by x' were allowed to be a relation in the set $\mathscr{R}$, then whatever place was occupied by x would be identical with x itself. Thus, if we used individual constants to denote an element of an instantiation of $\mathscr{S}$ in our identity criterion, we would either be referring to something non-universal or suggesting that places and place holders were identical. Neither of these is acceptable in ARS.

Restriction 2 is needed since the identity relations in $\mathscr{S}$ are also the identity relations in any instantiation of $\mathscr{S}$. If we admitted any property which has an individual constant denoting a place in $\mathscr{S}$, such as the number 1 (which is just the second place of the natural number structure), we would have to say that the place holders and the places are the same. For example, Caesar = 1 is false, since Caesar is not the number 1 itself, but just a stand-in for 1 in a given instantiation. Thus, admitting constants denoting a place in $\mathscr{S}$ would imply places and place holders are really the same thing, which contradicts ARS.

Thus, an advocate of ARS must accept that $\mathscr{R}$ contains only intra-structural relational properties. That is, $\mathscr{R}$ must contain only those properties that refer to only properties essential to the structure, and can make use of no individual constants denoting either places or place holders. This means that "only the properties that can be specified by formulae in one free variable and without individual constants may be admitted to the set [$\mathscr{R}$]" (Keränen (2001), p 317). This condition is reached after four other possibilities are considered and rejected (Keränen (2001), page 319). They are:

- Extra-structural properties, which cannot be used universally as they are particular to a (several) given instantiation(s)

- Non-structural properties, which again cannot be used universally
- Intra-structural relational properties the specification of which essentially involves an individual constant denoting an element of S (because of restriction 1)
- Intra-structural relational properties the specification of which essentially involves an individual constant denoting a place of S (because of restriction 2)

Thus, the sentence 3.2 is the right type of sentence to specify a criterion of identity for places in structures, provided that $\mathscr{R}$ contains only intra-structural properties with one free variable with no individual constants. This combined with Leibniz's law called the Identity of Indiscernibles is what causes the problem for Shapiro. The Identity of Indiscernibles (IND) is formulated as follows:

$$\forall x \forall y (x = y \leftrightarrow \forall F (Fx \leftrightarrow Fy)) \tag{3.3}$$

It can be seen that equation 3.2 is a special case of equation 3.3. It is a restriction on the properties that can be included.

3.0.3 Examples

The effects of this identity criterion are easiest to see when we consider several examples. In this section, we will look at four:

- The complex numbers
- The Euclidean Plane
- The integers modulo 3

- A "barbell" graph

The identity problem was first explicated using the field of complex numbers ($\mathbb{C}$). Burgess (1999) claims that "[o]n Shapiro's view...there seems to be *nothing* to distinguish [i and $-i$]" (page 288). This fact needs a little spelling out. The field of complex numbers is a field extension of the real numbers generated by adding the two roots of the equation $x^2 + 1 = 0$. We know nothing about the two roots, other then the fact that there are two, that they are additive inverses of each other, and that they are both square roots of -1. Thus, neither one of them has an intra-structural property that the other does not have. There is an automorphism which maps every complex number, $a + bi$ onto its complex conjugate, $a - bi$. This is a non-trivial automorphism which takes i to $-i$.[3] Leibniz's law, the identity of indiscernibles, combined with the restrictions of the properties available to ARS (the set $\mathscr{R}$), suggests that 3.2 implies that in the $\mathbb{C}$ structure, $i = -i$. This is obviously unacceptable.

If it looks like the complex numbers are a problem, then the Euclidean plane as a metric space is a disaster. Each point in the Euclidean plane has, by definition, all the same characteristics as every other one. When considering any pair of points distance d apart, each is indistinguishable. Both have the relations of being 'distance d from point x'. In fact, there are several non-trivial automorphisms in the Euclidean plane. Translating, reflecting and rotating all preserve (metric) structural properties, and hence any two points that can reach each other via one of those isometries will be indistinguishable. Unsurprisingly, any two points can be mapped to each other this way. If Keränen's claim holds, then the Euclidean plane structure only contains one point. This is simply absurd.

[3]Dr Peter Zvengrowski has pointed out to me that there are in fact many more automorphism of the complex numbers. However, all this example needs is that there is one, and the complex conjugation automorphism is by far the simplest.

The group $\mathbb{Z}_3$ is the integers modulo three under addition.[4] In it, the following nine identities hold:

1. $\overline{0} + \overline{0} = \overline{0}$, $\overline{0} + \overline{1} = \overline{1}$, $\overline{0} + \overline{2} = \overline{2}$

2. $\overline{1} + \overline{0} = \overline{1}$, $\overline{1} + \overline{1} = \overline{2}$, $\overline{1} + \overline{2} = \overline{0}$

3. $\overline{2} + \overline{0} = \overline{2}$, $\overline{2} + \overline{1} = \overline{0}$, $\overline{2} + \overline{2} = \overline{1}$

Each place in this structure should be definable via relations of the type in $\mathscr{R}$. Thus, each place must be definable using formulas with one free variable without individual constants. However, given 1, 2 and 3 above, with quantification over the places in $\mathbb{Z}_3$, we can generate the following. Each element is defined by the relations it has to the others.

I $\overline{0}$: $(x + x) = x$ and $\forall y(x + y = y)$

II $\overline{1}$: $(x + x) \neq x$ and $\exists y[(x + y \neq x)$ and $\exists z((z \neq y)$ and $(x + z \neq x))]$

III $\overline{2}$: $(x + x) \neq x$ and $\exists y[(x + y \neq x)$ and $\exists z((z \neq y)$ and $(x + z \neq x))]$

The conditions of $\overline{1}$ and $\overline{2}$ are exactly the same. Thus, Keränen's conditions on the set $\mathscr{R}$ generate the result that $\overline{1} = \overline{2}$. So, although the group $\mathbb{Z}_3$ under addition modulo 3 has three elements, the ARS structure has only two discernible places. This is counter-intuitive at best, and Shapiro must either justify the cardinality difference or explain why the problem does not apply. Surely Shapiro does not want to claim that even though the group $\mathbb{Z}_3$ has cardinality three, the structure only has cardinality two. He must explain why you cannot distinguish between $\overline{1}$ and $\overline{2}$ or his theory will be absurd.

The barbell graph is the graph with two nodes and the universal relation, $\mathscr{U}$ (see figure 3.1). In this graph, the nodes a and b cannot be distinguished by any relation. They are both related to themselves and to the other node. In fact, both a and b satisfy

[4] $\mathbb{Z}_3$ is the group of the remainders of all integers upon division by 3. The ring $\mathbb{Z}_3$ is a rigid structure.

$$\begin{tikzcd} a \arrow[loop left] \arrow[r, leftrightarrow] & b \arrow[loop right] \end{tikzcd}$$

Figure 3.1: The Barbell Graph

the following formula: $\forall y(x, y) \in \mathscr{U}$. Under the rules of ARS, this structure does not have enough internal relations to be able to distinguish between the two nodes. Shapiro is forced to say, by equation 3.2, that the two nodes must be identified in the structure of the graph.

The identity problem is a serious concern for Shapiro's ARS. If Shapiro cannot find a way around it, he will be forced to conclude that in ARS all indistinguishable objects are the same object. As expressed by these examples, that conclusion would lead to many unpalatable results. It does not matter whether or not, from a God's eye view, there is some fact of the matter about any number of objects being distinct. If an *ante rem* structuralist wants to suggest that her theory is the right one, she must be able to explain why i and $-i$ cannot be distinguished, and why this does not imply they must be amalgamated into one object. Otherwise, ARS must be rejected as it would generate absurd results.

Chapter 4

Solutions to the Identity Problem

The solution to the identity problem comes in two parts. This chapter will discuss why we can have indistinguishable objects. The following chapter will discuss how we can refer to these indistinguishable objects. This chapter begins by exploring two solutions which ultimately fail: weak discernibility and hybrid structuralism. Weak discernibility suggests that there may be a weaker criterion of identity then the one suggested by Keränen in chapter 3. This is rejected as the condition suggested does not apply to all structures. Hybrid structuralism is the theory that our philosophy of mathematics might be half realist and half nominalist. Although this does work, it is rejected as ARS was motivated by realist concerns. The chapter will go on to discuss Shapiro's solution, and the solution that I accept, that identity is primitive in mathematics and ARS.

Other solutions and objections can be found in Parsons (2004), Carter (2005) and MacBride (2005).

4.1 Weak Discernibility

Ladyman (2005) presents an interesting solution to the identity problem for certain structures, in particular for $\mathbb{C}$ and the Euclidean plane. According to Ladyman, when we ask that two objects be distinct if and only if there is a formula in one free variable (in a language without identity) which is true of only one of the objects in question, we are asking too much. He calls this type of discernibility "absolute discernibility". Ordinary, bread box sized, physical objects are absolutely discernible. No two objects of this sort can occupy the same place in space and time. Ladyman holds that there are two other types of discernibility: relative discernibility and weak discernibility. Two objects are relatively discernible when there is a formula in two free variables which applies to them in only one order. Thus, the real numbers with the relation $<$ are relatively but not absolutely discernible, since there is an automorphism which takes every number, x, to its negative, $-x$, but it is always the case that $-x < x$ and not the other way around (with the exception of $x = 0$, which is distinct for other reasons). Finally, two objects are weakly discernible if there is a two place irreflexive relation they satisfy. Thus, i and $-i$ are weakly discernible since they satisfy the irreflexive relation 'is the additive inverse of (and does not equal 0)'. Similarly, any two distinct points in the Euclidean plane satisfy the irreflexive relation 'is distance d>0 from'. Two objects which do not satisfy any of these discernibility criteria are called indiscernible.

It might seem as though this is an *ad hoc* solution since only abstract mathematical objects are weakly discernible and not absolutely discernible. However, Saunders (2003) suggests that fundamental quantum particles only satisfy IND if formulated in terms of weak discernibility. There is a non-trivial automorphism for each singlet of two fermions, one type of fundamental particle, in the structure of quantum physics. This automorphism switches the two particles. Saunder's argument is opposed to a widely accepted claim that fermions violate IND, and thus are not objects.[1] In this sense, Saunders' claims are a defense of IND. If it turns out that Saunders made the right assumptions, and that fundamental particles are only weakly discernible, then it is certainly plausible, and not *ad hoc*, that abstract objects might be only weakly discernible. If not, then we are still one step ahead, as the application of weak discernibility in a field outside of philosophy, namely theoretical physics, lends credibility to the hypothesis.

In response to the identity problem and Ladyman's proposed solution, MacBride claims that "either it's bad news ($i = -i$) or it's old news (*ante rem* structuralism =

[1] See French and Redhead (1988).

good old fashioned Platonism)" (MacBride, 2005, p. 582).

MacBride (2006) explains in detail why Ladyman's solution does not suffice. If mathematical objects are just amalgams of the relations they bear to each other, then they really are nothing but the bundle of those relations. Thus, "there are no independently constituted particulars lurking behind the structural facade" (MacBride, 2006, p. 65). MacBride recasts an argument from Russell (1911) to elucidate why this is a problem. Suppose, for example, that we have two indistinguishable objects, x and y, which are not in the same location. Now, because no object can be in two places at once, x and y must be numerically distinct. However, because we are still working withing the realm of ARS, this distinctness must come from some intra-structural universal relation. In other words, it must be a relation which can belong to the set $\mathscr{R}$ from chapter 3, but which also is applicable (truly or falsely) to all objects in the domain of discussion. But these two objects are indistinguishable, and hence must share all the same intra-structural relations! Russell concludes that spatial relations cannot be universals, but must be particulars, applicable to only one thing, which are capable of being indistinguishable from each other and numerically diverse.

Though this may work for Russell, it is not an option for Ladyman or Shapiro. It is unarguably the case that there is a non-reflexive relationship which obtains between i and $-i$. The question lies in whether or not this is the type of relationship which an *ante rem* structuralist may consider as intra-structural. If we concede the fact that mathematical objects are just bundles of universals, and we let the relation of additive inverse be substituted for the spatial relation above, then this problem becomes quite clear. There is nothing to prevent i and $-i$ from being identical to each other, since there is no set of bare particulars to fall back upon. A universal, by definition, is capable of being instantiated more than once in exactly the same way, and thus would be able to satisfy an irreflexive relation with two distinct instantiations of itself. However, by the definitions of ARS, two representations of the same universal are the same object. Therefore, MacBride concludes that Ladyman's solution does not work, but rather that identity facts are primitive. For MacBride, the fact that identity facts must be primitive reduces ARS to a form of Platonism.

There is another similar problem with the weak discernibility solution. It seems that defining identity via irreflexive relations is circular. Ketland (2006) rejects Ladyman's solution on these grounds. Consider 'x is the additive inverse of y'. This is symbolized as $x + y = 0$. Although we know that if x and y satisfy this relation and are natural numbers, they are either both 0 or they are distinct, it seems we are still left with no way to make sense of the identity symbol in the definition. Using weak discernibility to distinguish between i and $-i$ gives the statements $i = 0 = -i$ and $i + (-i) = 0$. It seems clear that 'being the additive inverse of' is a relation of the form '_ + _ = 0'. In this sense, i and $-i$ are discerned not only by $i + -i = 0$ but also by the much more trivial formula $i = -i$. This amounts to $i = -(-i)$. More simply, weak discernibility amounts to: $\forall x \forall y (x = y \leftrightarrow \forall R(\forall z \neg zRz \rightarrow \neg xRy))$. The formula $i = -(-i)$ clearly relies on an identity predicate, which our language does not have. However, in any sort of algebraic structure (e.g. groups, rings etc.), the identity condition is primitive, as the basic formulae are statements of equality. In a field, for example, the structure is specified by giving the domain and the two operations (+ and $\times$, and the units of each). The operations are thought of in terms of functions, and thus require a primitive notion of identity and a primitive identity predicate. Thus, algebraic structures are Quinean by construction (Ketland (2006) defines Quinean structures as those in which the places are at least weakly discernible). So, if the fact that weak discernibility is trivial is acceptable, then there is no identity problem for Quinean structures, which include all algebraic structures.

On another note, though the weak discernibility solution seems to work for structures with enough structure, it fails when we consider the barbell graph. There is no non-reflexive relation that the two nodes of the graph satisfy; there is only one irreflexive relation, the empty relation. Thus, the two nodes are still indiscernible. This is also the second objection in Ketland (2006). Since the only irreflexive relation in the barbell graph is the empty relation, this structure is not Quinean. Although this is a problem for Ladyman, it is not for Ketland. Ketland's solution is to suggest that primitive identity facts are not nearly as problematic as they may seem. Ketland mounts his defense by explaining just what we would have to give up to reject primitive notions of identity. Certainly, mathematical notions like uniqueness and functionality depend on it. This objection makes itself particularly apparent when we consider the barbell structure. Whenever two elements, x and y are not weakly discernible, we will write $x \approx y$. Taking the ordinary, first order, formula for there being at most one element, we have $\forall x \forall y(x = y)$. However, switching the identity predicate for the weakly discernible predicate, we get $\forall x \forall y(x \approx y)$. The barbell structure satisfies this! Hence, Ketland concludes that in non-Quinean structures, at least some identity facts must be primitive.

Although MacBride's assessment that ARS is a form of Platonism is correct, it is less devastating then he thinks. Shapiro himself concedes to the fact that ARS might be very similar to Platonism (Shapiro, 2006b, p. 117), and the fact that it comes equipped with a good epistemology helps it avoid many of the traditional Platonistic problems. However, MacBride's theory about weak discernibility implying primitive identity facts, and Ketland's problems with the circularity of defining identity via weak discernibility and the lack of the universality of the solution all seem to hider its acceptability.

4.2 Hybrid Structuralism

Button (2006) presents what he calls a hybrid solution to the identity problem. Button holds that there can be no primitive identity facts. That is, the structures in question do not come equipped with an identity predicate. Identity is not something elementary to a given structure, but rather must be defined from something else. This position is based on two reasons: that it would be unclear how we could have epistemic access to identify facts otehrwise and that, metaphysically, "accepting indistinguishable objects commits us to an unusual notion of objecthood" (Button (2006) page 219). Clearly, if we accept that objects can be indistinguishable yet numerically distinct, we will need to make use of primitive identity facts. Taking two indistinguishable objects, x and y, we cannot know whether or not they are identical unless we have epistemic access to primitive identity facts. However, it is unclear how we can have such access. Without access to these facts we cannot know whether x and y are distinct, let alone whether x itself is one object or many. As well, accepting indistinguishables means accepting that the two nodes in a graph with no arrows are objects. Button argues that although not logically fallacious, objects with no properties (aside from being a node in a graph) and no relations are "*metaphysically* suspicious" (page 220). This requires a weaker notion of objecthood than that which Button is willing to accept.

In order to avoid having to make use of primitive identity facts to solve the identity problem, Button distinguishes between basic and constructed structures. For him, basic structures are treated realistically (on a par with Shapiro's structures), and constructed structures are treated eliminativistically.[2] A basic structure is one without indistinguishable places (it is rigid), and a constructed structure is treated as a universal generalization over the positions and relations in a basic structure. A universal generalization of a basic structure is the basic structure with some of the relations removed or changed. For example, we can construct the system $\mathbb{Z}_3$ from the basic system which is $\mathbb{Z}_3$ with the usual ordering on the natural numbers ($0 < 1 < 2$). In this way, a constructed structure

[2]cf. Chihara (2004), Awodey (2004)

is a type of metaphor, and talk of constructed structures is just shorthand for talk about universal generalizations of basic structures. For Button then, once we accept Ladyman's weak discernibility, we can construct all mathematical structures. In order to show that all structures with indiscernible places can be constructed, we can simply take a structure $\mathscr{S}$ which contains at least two indiscernible objects. Then, structure $\mathscr{T}$ which has all the same objects as $\mathscr{S}$, but whose objects are all well-ordered, can be used to construct $\mathscr{S}$. In this way, the indistinguishable objects in $\mathscr{S}$ will be distinguishable in $\mathscr{T}$, and we can make use of that fact to distinguish them in $\mathscr{S}$. Thus, the structure of the barbell graph would be comparable to a universal generalization over a non-problematic structure: a barbell graph with an ordering on the nodes.

This solution works as long as one is willing to have an only partially realistic theory. However, it violates the goals of Shapiro's theory, in that it is partially eliminativistic. Thus, although it does work for those willing to accept a hybrid theory, it does not work for Shapiro's ARS.

4.3 A Step in the Primitive Identity Direction

Ketland (2006) explicitly rejects the need to supply an identity criterion for places in structures. This is very similar to the view Shapiro takes, and the view on which I will ultimately settle. This view is also upheld in Ladyman and Leitgeb (2008).

The main goal of Ladyman and Leitgeb (2008) is to show that the identity (or difference) of places in a structure is accounted for solely by the structure itself. They claim that

> The identity relation for positions in a structure is a relation that ought to be viewed as an integral component of a structure in the same way as, for example, the successor relation is an integral component of the structure of

the natural numbers. (p. 390)

Thus, for Ladyman and Leitgeb, identity is just one of many structural relations. There is no possible way for it to be a non-structural relation since it is literally contained in the structure itself. This, it seems, is on par with mathematical practice. In order to solve the identity problem for the barbell graph, they take the (non-)identity relation as a structural relation. Thus, the (non-)identity relation belongs to the set $\mathscr{R}$ that Keränen described. Clearly, if the (non-)identity relation is counted as one of the relations a and b bear to each other, the problem is resolved. If we accept this solution, another problem arises. How do we know there are two nodes in the barbell graph? This is answered in a quite straightforward manner: because the basic axioms of graph theory are coherent, and we can create a graph in those axioms that looks exactly like the barbell graph, the barbell graph structure must have two nodes. This problem will be discussed further in section 5.1.

In response to the claims in Button (2006) that there are no primitive identity facts, it is suggested in Ladyman and Leitgeb (2008) that "there is no reason to expect mathematical objects to be like those with which we are familiar, and furthermore that the suspicions of metaphysicians weigh much less heavily with us then the implications of mathematical practice." (p. 395). This offers a response to Button's worries about metaphysical issues arising from the existence of primitive identity facts. In response to Button's epistemic concerns, it is shown that the barbell graph can be apprehended by abstracting from a similarly shaped directed graph, where the arrow between a and b is replaced by an arrow in only one direction, from a to b (or vice versa).

Ladyman and Leitgeb successfully avoid the identity problem by postulating that identity relations are just as integral to structures generally as the successor relation is to the natural numbers. Though this does work, it is restricted solely to ARS. Shapiro's final solution can be extended to all realist mathematics.

Shapiro articulates his first response to Keränen (2001) in Shapiro (2006b). Although he provides solutions to two potential problems with ARS, this section will only focus on his response to Keränen.[3]

The first problem Shapiro spots has to do with the size of the language philosophers (and mathematicians) have to make use of. The real numbers, for example, are usually described in a countable language, but there are uncountably many reals. If Keränen wants a specific formula to individuate each number, then he will at some point run out of formulae to do so. In fact, it will not be possible to individuate the majority of real numbers. There is a way around this, if we allow sets of formulae to be admitted, as some sort of quasi-properties, to $\mathscr{R}$. For example, any real number, r, can be individuated by the set of formulae containing $x < s$, where s is rational and greater than r, and $t < x$ where t is rational and less than r. In this case, r, and only r, satisfies all of these formulae. However, for any countable language, there will only ever be at most continuum many sets of formulae. A theory with a countable language and more than continuum many elements will not contain enough formulae to distinguish all its elements, even if we do allow sets of formulae to count as distinguishers. At some point this trick will run its course and no longer be useful. Even in set theory, the axiom of extensionality is dependent on when any two members of a set in question are identical. Thus, it runs into the same cardinality problem. Although philosophers seem to favour a countable language, it is still the case that even if an uncountable language were postulated, and each real number was given a name, we would be left with the problem of identifying names in an uncountable language. It seems that this is a never ending problem. Suppose we let our language be larger then the continuum. Say it has size κ, for $\kappa > \aleph_0$. Then we can identify several more structures. However, the problem repeats itself. There will

[3]The first half of the paper is devoted to a response to Jonathon Kastin, who proposed that there was a problem with cross-structural identity. Kastin (1998) questioned whether the natural number 2 was the same as the real number 2. In the end, Shapiro concedes that they are not the same mathematical object.

be structures with too many elements for us to identify. Thus, we must pick the biggest cardinal number. However, this does not exist, as proved by Cantor. Thus, it seems we cannot have a language big enough to pick out every mathematical object.

Keränen (2006) responds to this cardinality problem by making a distinction between a criterion for identity which explicitly characterizes each object, and an implicit criterion which is an identity schema. Thus, instead of requiring an identity criterion for each object individually, one formula or criterion with one free variable would be applied to all objects in question. A haecceity is a property specific to one and only one object. The theory of Haecceitism postulates that every object is individuated by its haecceity, or 'this-ness'. Thus, every object, a can be uniquely distinguished by the formula 'being identical to a'. Keränen agrees that even if haecceities were accepted (which he will go on to suggest is a move Shapiro cannot make) there is still no way to explicitly individuate all mathematical objects, since there are only countably many singular terms in any given mathematical language. Thus, we could have only countably many instances of the formula $\forall x(x = a \Leftrightarrow x = a)$ where a is the haecceity of x. To make his explicit/implicit distinction clear, Keränen takes the example of the Axiom of Extensionality in Zermelo-Fraenkel set theory. The axiom, $\forall x \forall y(x = y \Leftrightarrow \forall z(z \in x \Leftrightarrow z \in y))$, is the criterion of identity for sets. It is possible to explicitly provide a criterion for small sets. Take, for example, the set $\{1, 2\}$. The explicit criterion is $\forall x(x = \{1, 2\} \Leftrightarrow (1 \in x \wedge 2 \in x \wedge \forall z(z = 1 \wedge z = 2 \Rightarrow z \notin x))$. However, it is certainly not the case that all sets can be explicitly described in this way. Some are simply too large. In these situations, the Axiom of Extensionality provides an implicit criterion of identity. It can be hypothetically applied to any two sets to determine whether or not they are one and the same. Thus, the Axiom of Extensionality is a non-trivial account of identity which does not rely on the size of the language in question. Keränen concludes that the identity problem is not avoided simply because of language size restraints.

Keränen's restrictions to the set of properties an *ante rem* structuralism can talk about imply that we must specify what can count as a non-explicit individuating property. These properties cannot make explicit reference to particular places (which are objects) in a given structure. Thus, the property of 'being the successor of the 2-place' in the natural number structure would not be allowed in the description of 3. So what is allowed? Keränen's example is that in the set theoretic hierarchy, the Axiom of Extensionality is permitted. This axiom depends on the set membership relation $\in$. However, when discussing identity criteria, we may only do so with properties, since we can only have formulae with one free variable. Thus, relations must be reduced. Taking $\in$ we must find a way to reduce it to a one place property. Shapiro suggests the property M_r apply to t whenever $r \in t$ (Shapiro (2006b) page 167). Listing all the M_x properties for all elements of t, we can uniquely pick out t. However, it seems that M_r explicitly refers to an element of the set theoretic hierarchy, namely r. This is problematic, and contradicts restriction 1 from chapter 3. The set theoretic hierarchy is rigid, and so according to Keränen we should be able to uniquely pick out each object without making reference to any particular object in the structure. If he cannot provide an identity scheme for set theory, then why should ARS be required to?

If, for a moment, we accept that there is no problem with set theory and the axiom of extensionality, then we can construct non-rigid structures which meet Keränen's identity criteria. If we take a structure isomorphic to the integers under 'less than', we can construct an identity criterion similar enough to the axiom of extensionality that they must either both be accepted or both rejected:

$$\forall x \forall y (x = y \Leftrightarrow \forall z (z < x \Leftrightarrow z < y)) \tag{4.1}$$

We can also make the same relations-to-properties move. We let L_r be the property such that $L_r s$ if and only if $r < s$ (Shapiro (2006b), page 168). Thus, each place in the structure can be properly individuated by which L_x properties it has. This is very much in the same vein as the set theoretic hierarchy with the axiom of extensionality. Our system, however, has one small difference: it is not rigid. There is an automorphism, Φ_n, which takes x to $x + n$ for each n. There is only one structural relation, and we know $x < y$ if and only if $x + n < y + n$. However, this structure has an identity scheme Keränen would accept (equation 4.1), and satisfies equation 3.1. Thus, he must either accept that it is a suitable criterion of identity, and thus that rigid structures can have identity criteria, or he must deny that the axiom of extensionality is not a suitable criterion of identity, and hence that set theory falls prey to the identity problem.

Keränen proposes that what is important about a criterion of identity is not that it list all the properties each object has which individuate it from the rest, but rather that it provide an intensional scheme which allows someone to pick out an object of his or her choice. Thus, there is not a particular formula for each object, but rather one formula (with one free variable) which we can use, by replacing the free variable with our desired object, to uniquely pick out each object. As opposed to an extensional criterion, Keränen explicitly demands that it be intensional. This move is made mainly to avoid problems with the cardinality of languages describing uncountable sets. If it works, we will not have the cardinality problem described above, as we will not need to uniquely describe any of the (possibly more then continuum many) elements in our structure. However, as Shapiro points out, it is not nearly as successful as Keränen thinks. Automorphisms preserve the truth (or satisfaction) of formulae in the language in question. In other words, if there is an automorphism of a structure $\mathscr{S}$ which takes x to y then any one place formula $A(z)$ which applies to x also applies to y. However, in order to avoid cardinality problems, Keränen suggests that we don't need a formula which can pick out an individual element. Instead, we must use intensional schemes, which can be reduced to properties. The problem is that automorphisms do not preserve one place properties, but one place formulas instead. In particular, the properties L_x are not preserved under automorphism. Automorphisms only preserve properties which are definable in the language. Take as an example the automorphism from $\mathbb{Z}$ to $\mathbb{Z}$, which takes each integer x to its negative $-x$. This is clearly an automorphism, but does not preserve the L_x properties above.

Before discussing the next problem presented by Shapiro, a brief digression needs to be made. It is important to note that there are other realist structuralist alternatives to ARS. The two main competitors are *in re* structuralism and eliminative structuralism. Although they are both quite similar to ARS, there are some crucial differences between them, and some of these differences have a bearing on the identity problem. *In re* structuralism is the same as ARS in all respects except one: while ARS holds that structures exist independently of their instantiations, *in re* structuralism holds that universals only exist in abstraction from their instantiations. Thus, *in re* structuralism is more Aristotelian then its *ante rem* counterpart. It is easy to see that if the identity problem is a problem for ARS then it is also a problem for *in re* structuralism. Because *in re* structuralism still holds that structures exist, even if it is only within the systems exemplifying them, each place in a structure must be properly individuated. Thus, though it may appear that i and $-i$ are properly individuated by the real number pairs $\langle 0, 1\rangle$ and $\langle 0, -1\rangle$, there is still no way of knowing whether i is associated with $\langle 0, 1\rangle$ or with $\langle 0, -1\rangle$. Without knowing which square root of -1 i actually is, we cannot know which real pair it is individuated by. We can send either i or $-i$ to the real pair $\langle 0, 1\rangle$. Thus, even with access to this map, *in re* structuralism is just as blind as ARS.

The eliminative structuralist, on the other hand, has a large background ontology in which to find a criteria for identity. An eliminative structuralist holds that structures are not *sui generis* objects, but are merely tools for paraphrasing talk about different types of mathematical objects. Most often, these objects are sets or categories. Thus, talking about the structure of natural numbers is shorthand for talk of some set or category which models the natural numbers. It seems that an eliminative structuralist might be able to avoid the identity problem, since they have access to the criterion of identity of whatever mathematical theory they think structures are really part of. However, in order to model all of mathematics, the background ontology must be so huge (as big as the ontology of mathematics) that the only criterion available is a trivial one. This argument is similar to the cardinality argument presented above. Since the ontology is so huge (bigger then the continuum) and we only have access to a countable language, we cannot possibly individuate all objects. For an eliminative structuralist, talk of structures is shorthand for talk of some other mathematical object. Thus, eliminative structuralists hold that structures are not objects in themselves, but rather some other type of object. The most common of these objects are categories, but they are also frequently sets or modal possibilities. Keränen holds that the identity problem does not affect eliminative structuralism because it can make use of whatever identity criteria the base theory has. For example, those who base the theory in sets can make use of the axioms of extensionality as the criterion of identity. Shapiro disagrees.

In Keränen's opinion, the eliminative structuralist position is free to accept haecceitism (Keränen, 2006, p. 161, fn. 15). Clearly, by adopting haecceitism, the eliminative structuralist avoids the identity problem. Although Keränen does not elaborate on why he holds this to be the case, the rest of the text does suggest he has an answer in mind. An eliminative structuralist is capable of doing two things a realist cannot. She can ground her theory in another, and she does not have to take mathematical statements at face value. Thus, singular terms in mathematical statements are shorthand for bound variables, ranging over the elements in the base theory. Since identity in the base theory can be determined by haecceities, identity in structures can also be determined by haecceities. This solution does work, provided that the base theory does not fall prey to the identity problem. We will see, in section 4.4, that it is possible that even a theory as robust as Zermelo-Frankel set theory may not meet Keränen's requirements for avoiding the identity problem.

The bulk of Shapiro's argument against the identity problem comes from his rejection of IND for ARS. Recall Equation 3.2:

$$\forall x \forall y (x = y \Leftrightarrow \forall r (r \in \mathscr{R} \Rightarrow (r(x) \Leftrightarrow r(y))))$$

It is clear that this criterion for identity implies the identity of indiscernibles (i.e. $\forall F \forall x \forall y ((Fx \leftrightarrow Fy) \rightarrow x = y))$ for at least some specific set of F-properties. Accepting 3.2 amounts to accepting IND for ARS, since the only properties available to ARS are those in $\mathscr{R}$. The $\mathscr{R}$-properties are the only properties 'in' the ARS universe. This, for Shapiro, is not an acceptable move. There are two possible ways of viewing IND, both of which lead to unattractive results. The first is the metaphysical construction. In this construction, the universe comes pre-packaged into objects. Each object can be uniquely characterized by some property. However, if it is the case that each object be characterized by a property, there seems to be no reason to think that any of our languages should be capable of capturing all of those properties. This follows from the cardinality argument. The idea that we might be able to construct a language capable of capturing all possible properties seems to suggest that we might be able to construct a perfect language capable of capturing and proving all truths. This is not possible. If objects cannot be individuated by formulae in some language, then they must be individuated by properties (or propositional functions). Unless we accept haecceities, there seems to be no reason to think that the realm of properties/propositions matches up perfectly with the realm of objects. This assumes that there is a pre-packaged property which individuates each object, and thus it assumes IND is true. There is no reason to suppose that just because the universe is divided nicely into objects that the realm of properties is also divided nicely to match the objects. Therefore, taking objecthood as a metaphysical primitive does not work. The other option is to not take objecthood as a metaphysical primitive, but to suggest that whenever two objects are indistinguishable, we must identify them. This Shapiro calls the Quine-Kraut approach, or just the Quinean approach. Unlike IND in its metaphysical form, here all we can do is distinguish two objects, as opposed to characterizing each one uniquely. However, because of theories like Euclidean geometry and complex analysis, which violate the Quine-Kraut construction, revisions would have to be made to mathematics in order to keep it. This violates Shapiro's faithfulness requirement, and more generally is a move any philosopher of mathematics, realist or not, cannot make.

Keränen (2006) holds that Leibniz's laws, in particular IND, holds for mathematical objects. Contra Shapiro's dismissal of the metaphysical principle, Keränen makes several points. First, he argues that cardinality concerns should not affect the identity criterion of a given language. Shapiro rejects this possibility for the same reasons he originally rejects the identity problem, that our countable language cannot describe uncountably many objects. However, Keränen has already dealt with this possibility. Also, since the universe in the metaphysical conception is already pre-packaged into objects (we are, after all, in a realist setting), it is not dependent on our language whether two objects are really one. Second, Keränen responds to Shapiro's suggestion that it may be properties or propositional functions that individuate objects and that these may not be able to individuate all objects. His first response is to give the example of Zermelo-Frankel set theory and the axiom of extensionality as given above, where properties of the form 'r belongs to' are used to individuate sets. However, an example of a structure where we can individuate all objects based on properties is certainly not an answer to whether or not we can do so for all structures. Third, he suggests that Shapiro must provide an answer for why the two realms, that of properties and that of objects, may not match up as we wish. Keränen suggests Shapiro needs to provide support for this view since it is generally accepted in metaphysical discussions that these two realms do match. Once again, I hold that this point does not seem to have enough force to derail Shapiro's point. Most people at one point assumed Earth was the centre of the universe; just because something is assumed by most does not make it true. As a final rejection of the metaphysical construal of IND, Keränen suggests that even if 'being identical to itself' counts as a property, Shapiro is confronted with a Benacerrafian style problem. If 'being identical to itself' counts as an $\mathscr{R}$-property, then there will be a difference between the natural number structure created from the von Neumann ordinals and the natural number structure created from the Zermelo ordinals. Each one is identical to itself and to no other, so there is an $\mathscr{R}$-property which distinguishes them, and hence they must be different structures. Because one of the main goals of ARS is to do away with Benacerraf-style problems, Keränen suggests that this means $\mathscr{R}$ cannot contain properties of the form 'being identical to itself'. Keränen goes on to make a quick remark against Shapiro's rejection of the so called Quinean version of IND. Shapiro rejects this form of the law for the same reasons he presents the cardinality problem, that there cannot be enough formulae in our language to distinguish more than continuum many pairs of objects. Thus, Keränen has the same response open to him: we do not provide a formula which distinguishes each pair of objects on a case by case basis, but rather specify an intensional criterion for identity.

Shapiro has several possible responses open to him at this point. The dialogue presented in Black (1952) poses a whole new set of problems for supporters of IND. Black's dialogue show that IND is not necessary. He constructs a possible universe, which all non-modally challenged[4] philosophers will agree is really possible. In this universe there are two identical spheres. There is nothing else. If we accept that IND is only applicable to properties which we can express, we must accept that these two spheres are really one sphere. They are both the same size and shape, they are each the same distance apart from each other, in fact they have all their expressible properties in common. And yet, there are two of them. Any addition to this universe (for example, someone standing between the spheres, with one on his left and one on his right) would affect the symmetry, and defeat the purpose of the example.[5] The defender of IND in Black's dialogue has one main concern: if we deny IND how are we to know that we have only two hands? We may have twenty. Although Black's objector to IND suggests that we can still know that we have exactly two hands since we can verify that there are exactly two, it seems that we do not even need this much. Clearly, we can only ever interact with two of our hands at any given time, yet there are ways we may in fact have more hands. If we really were Lewisian time-slices, for example, some would claim we have many hands (two for each time slice)! Though Black may have gone too far in trying to show that on some occasions we can verify that there are exactly two things, what is important is that "we could know [that] two things existed without there being any way to distinguish one from the other" (Black, 1952, p. 163).

While Black's world is merely a possibility, Saunders (2003) presents a physical application. As discussed in section 4.1, Saunders holds that fundamental particles only satisfy weak indiscernibility. In fact, two bosons can switch places without it making any difference to the system as a whole. This switch may or may not occur at any given

[4]This terminology is taken from Zimmerman (1997)

[5]There have been some suggestions that in a non-Euclidean space such as a hyperbolic space, these two spheres would *actually* be one sphere. However, for each space we can come up with a similar example.

moment, and there is no way for us to know. The two bosons are linked by the fact that they have all their properties in common (spin etc.), and if we accept IND then we must accept that there are not two bosons but one. The impact of this type of collapse on our everyday, physical, actual world would be immense.

It is Black's presentation of a physically possible world where the laws do not hold, and Saunder's example of particles in our actual world which do not adhere to the principle, that suggests Shapiro might be able to ignore IND for any restricted set of relations. This would mean that although IND might apply in general, because ARS in incapable of capturing all possible properties, it is not responsible for distinguishing all possible objects. If it is unclear that the law applies to physical objects in all possible situations, then we have no right to suppose that the law applies to abstract objects with which we have no causal interactions. IND holds, as there will inevitably be *something* which causes two objects to be distinct. However, assuming that we can express this difference, or even know what it is, is assuming too much. Since IND is not a necessary law, it might be the case that it does not apply to abstract objects. Shapiro holds that this is the case.

Finally, Shapiro comments on what he calls an 'embarrassing remark' in his book (Shapiro, 2006b, p. 140). Shapiro (1997) states "Quine's thesis is that within a given theory, language or framework, there should be a definite criteria for identity among its objects. There is no reason for structuralism to be the single exception to this" (p. 92). This, Shapiro admits, sounds like he is requiring a criterion of the identity of places within structures. What he claims to have actually meant (or rather what he claims he should have meant) is that there needs to be a criterion of identity of the structures themselves, which Shapiro has already established (cf. section 2.5). Thus, Shapiro is free to claim that the criterion of identity for places within structures might be trivial. In other words, it might be satisfied by the sentence: $\forall x \forall y(x = y \leftrightarrow x = y)$, or by suggesting that identity is primitive.

Keränen (2006) responds to Shapiro's thought that ARS might be able to adopt a trivial account of identity. There are two apparent ways of doing this: either accept haecceitism, or accept that distinct objects can have all their properties in common. Keränen's response to Shapiro being able to accept haecceitism is straightforward. If Shapiro were to accept haecceitism as a suitable criterion for identity, he must accept that objects have essential properties which are prior to the structure they are in. This would mean that it is the object, not the structures, which are the primitive objects, contrary to ARS's account of primitiveness. Turning to the second possibility, Keränen notices that it amounts to rejecting IND, namely that two objects with all the same properties are really one. For Keränen, this is absurd, and leads to haecceitism in the following manner. Suppose that two objects, a and b are indistinguishable. What makes them two objects as opposed to one object? For Keränen, there must be some fact about the world that makes them distinct. This could be nothing but 'being identical to itself', thus leading to haecceitism. If each object is identical to itself and not another object, there must be something that makes it so. The only possible candidate for the property that makes it the case that $a = a$ and not any other thing is the haecceity of a. What Keränen neglects to consider is that though there may be a fact that distinguishes a and b, there is no reason to expect that we (can) know it.

Keränen concludes by suggesting that this problem applies to realist mathematics in general. This takes it a step too far. In fact, Keränen labels himself a realist! Though there are certainly theories about realist mathematics to which it does apply, it does not apply to all of them. Keränen as much as says it himself when he claims that eliminative structuralism has an identity criterion.[6]

Shapiro (2006b) suggests that non-rigid structures can be embedded in rigid ones.

[6] An eliminative structuralist is free to be a realist about whatever they think structures really are.

For example, the cardinal 3 structure, which is a structure with three objects and no relations, could be embedded in any structure with three objects and a linear order. This is a move Shapiro makes because even though some non-rigid structures might have criteria for identity that Keränen would accept, it is not the case that they all do. Those that definitely do not include the complex numbers and the Euclidean plane, two crucial structures in mathematics. We could then embed the complex numbers in $\mathbb{R}^2$. Thus, we would embed a non-rigid structure (the complex numbers) into a rigid one ($\mathbb{R}^2$) by providing an isomorphism (a map which maps everything to something, and maps something onto everything) from the non-rigid structure to the rigid one. We would then use the identity criterion of the rigid structure as the identity criterion of the non-rigid structure and distinguish the indistinguishable places by seeing where they were mapped to. This is very similar to Button's hybrid structuralism proposal, and can be expected to fail for the same reason. There is no way to show that every non-rigid structure could be embedded in a rigid one, and for this reason the embedding solution does not work.

4.4 Primitive Identity

Like Ketland (2006) and Ladyman and Leitgeb (2008), Shapiro ultimately rejects the need to provide a general criterion of identity. Shapiro makes two attempts at a solution, each of which will be discussed in turn. I will ultimately accept the second one, based on the discussion from section 4.3.

In Shapiro (2006a), Shapiro presents a response to Keränen (2006). A more complete defense of his solution can be found in Shapiro (2008) and will be discussed in this section.

Shapiro (2006b) is Shapiro's second reply to Keränen. The paper is written is response to Keränen (2006). Shapiro first deals with several of Keränen's objections, and then suggests that a general solution to the identity problem might be found by embedding non-rigid structures into rigid ones, as discussed in section 4.3.

Shapiro (2008) presents the primitive identity solution. Though he does not provide enough reason to reject IND for abstract objects in an ARS setting, based on the discussion in section 4.3, he is correct in doing so. His final conclusion is that philosophers of mathematics do not need to provide a criterion for identity because identity is primitive in mathematics.

Shapiro's solution comes in two parts. First, IND does not apply to abstract objects. This allows him to avoid the identity problem. Second, he demonstrates that identity is primitive in mathematics, and so no identity criteria are needed in general. The first step was discussed in detail in section 4.3, and so the remainder of this chapter will focus on how we can take identity to be primitive in mathematical theories.

Shapiro's faithfulness restraint on philosophies of mathematics requires that if mathematics presupposes identity then ARS must as well. Thus, it suffices to show that mathematics presupposes identity. First, it is important to notice that mathematics cannot define identity "in full generality in a non-circular manner" (Shapiro, 2008, p. 292). Take any first order language without identity, $\mathscr{L}$, and a relation which is meant to stand for identity, I. If α is the collection of sentences which is meant to implicitly define identity, then if α has a model at all, it has one where it does not define identity (Shapiro, 2008, p. 292). This result does not apply to first order languages with a fixed interpretation, because there Ketland's Quinean indiscernibility formula from section 4.3 holds. However, all we need is for this trick to work for at least one first order language without identity, which it clearly does.

In any first order language with identity, the identity relation is presupposed. This extends to mathematical practice as well. In Peano arithmetic, '$=$' is taken to be a primitive symbol. If a and b are both successors of c, then we say $a = b$. This primitive identity is what makes successor a function to begin with. For any given map f, there is no way to say it is injective without invoking identity (or invoking something which presupposes identity). More crucially, not taking identity as primitive affects theories as simple as group theory. Without invoking identity we cannot state that two groups are isomorphic, since this relies on there being an injective function from one group to another. In fact, we cannot even specify when a given system satisfies the axioms of group theory and is a group. This is unacceptable. It seems clear that mathematical practice takes identity to be primitive, and thus philosophies of mathematics should as well.

With this result, we can define the barbell graph with the axiom: $\exists a \exists b(a = b \wedge (a, b) \in \mathscr{U} \wedge (b, a) \in \mathscr{U} \wedge (a, a) \in \mathscr{U} \wedge (b, b) \in \mathscr{U} \wedge \forall c(c = a \vee c = b))$. The complex number structure can be defined axiomatically as well, entailing: $\exists x \exists y(x = y \wedge x^2 = y^2 = -1 \wedge \forall z(z^2 = -1 \Rightarrow z = x \vee z = y))$.

Mathematics presupposes identity, and therefore ARS must as well. The identity problem does not, and cannot, apply. It would breach the faithfulness constraint to demand ARS provide a non-circular definition of identity. Since mathematical objects are abstract, we cannot expect them to behave like ordinary, physical, objects. Since mathematics takes identity as a primitive relation, we cannot ask philosophies of mathematics to not do the same.

Chapter 5

The Reference Problem

Although much of the work done to evade the identity problem has been covered in the previous two chapters, this still leaves us with no answer as to how we can know that there is more than one indistinguishable object, and how we can talk about i and $-i$ if we cannot tell them apart. The answer to the first question is quite straightforward: we know there is more than one object when mathematicians discover that this is what the theory proves. There is no plausible reason why the philosophy of mathematics should differ so extensively from the practice of mathematics that it generates results inconsistent with those generated by the practice. This will be discussed in section 5.1. The second question is harder to answer. How can we refer to i without referring to $-i$? Do we always have to refer to both of them? How can that be if our language can express $i = -i$? The intuitive answer is that we intend to refer to one of them. We say, 'let i be one of those two indistinguishable objects'.

Sections 5.2 and 5.3 of this chapter explain the mathematical apparatus necessary to accomplish 'picking one'. I begin by considering Shapiro's proposal in section 5.2. Shapiro suggests that definite descriptions and anaphoric pronouns might not denote uniquely, and then applies this to an apparatus for 'picking one' object out of a set of many. This approach is rejected as it is less intuitive than it could be. I suggest Hilbert's epsilon calculus as an appropriate tool for picking indistinguishable objects.

5.1 Knowledge of Numerical Diversity

It is clear that being able to solve the identity problem requires being able to determine when there is more than one object with any given set of properties. In ARS, this has a fairly simple solution. There is more than one object with a given set of properties whenever mathematics can prove this is the case. For example, there is more than one object with the property 'is a square root of -1' because the fundamental theorem of algebra implies that every non-zero real number has two distinct square roots in the complex number field. Thus, the barbell graph actually has two nodes because that is how it is constructed in mathematics. Shapiro has access to this solution because of the faithfulness constraint on ARS. It is a direct consequence of the constraint that any candidate for a philosophy of mathematics must be, in some sense, sound and complete with respect to mathematics. That is, if the theory proves something that ordinary mathematical practice cannot, it must be rejected. Thus, if ARS did prove that $i = -i$, it would have to be rejected. More generally, if Shapiro's theory suggested that the number of objects with any given set of properties depended on something non-mathematical, and if there were ever a case where the practice of mathematics diverged from that number, Shapiro would be in direct contradiction to his faithfulness constraint.[1] Since the constraint was one of the motivating factors for adopting ARS, he cannot disregard it. Thus, there is more than one object with a given set of properties just in case mathematics claims that there is more than one object that satisfies the given set of properties.

[1]Again, it is not the case that philosophers are restricted to only practicing philosophy. Should a philosopher produce a mathematical proof, it would have just as much weight as one produced by a mathematician.

5.2 Shapiro's Solution

Shapiro first considers a solution to the problem of referring to indistinguishable objects in Shapiro (2008). However, a fuller treatment of the problem occurs in Shapiro (2009), and the majority of the discussion in this section will be taken from the latter.

The goal of Shapiro (2009) is to show, assuming the solution from section 4.4 works, that we really do have a way to refer to one object in a set of indistinguishable objects. It seems clear that in most cases, just 'picking one' will do. However, although this works for everyday mathematical practice, there are deep philosophical concerns about how this trick functions semantically. A crucial point is made in Black (1952), and I will quote him at length. The universe under discussion here consists of nothing but two identical spheres.

> A: Consider one of the spheres, a,...
>
> B: How can I since there is no way of telling them apart? *Which* one do you want me to consider?
>
> A: This is very foolish. I mean either of the two spheres, leaving you to decide which one you want me to consider. If I were to say to you "Take any book off the shelf" it would be foolish on your part to reply "Which?"
>
> B: It's a poor analogy. I know how to take a book off a shelf, but I do not know how to identify one of two spheres supposed to be alone in space and so symmetrically placed with respect with each other that neither has quality or character that the other does not also have. (Black, 1952, p. 156/7)

This, then, is the crux of the problem. Although it is simple enough to say 'pick one' when talking about ordinary physical objects, in our ordinary universe, it is quite another thing to say 'pick one' when referring to abstract objects like the square roots of -1 or objects in a possible universe. The 'pick one' ability, when ordinary bread box sized objects are in question is simple: locate the group of objects in space-time and select an individual object. However, we cannot locate abstract objects in space-time, so how do we choose them?

Shapiro's solution is to rethink the theory of Russellian definite descriptions. A Russellian definite description is of the form 'the something'. For example, 'the largest office at the University of Calgary' and 'the third planet in our solar system' are both definite descriptions. Russellian definite descriptions denote a unique referent. Thus, 'the largest office at the University of Calgary' denotes what is likely the presidents's office, and 'the third planet from the Sun in our solar system' denotes Earth. On the other hand, 'the present king of France' does not denote, and so although it looks like a definite description, it is not. Shapiro's trick is two fold: first, he shows how definite descriptions might be thought to denote non-uniquely, and second he shows how we can apply this to the identity problem.

Shapiro follows Roberts (2003) in his discussion of the possibility that definite descriptions denote, but do not denote uniquely. It will help to have some running examples:

Herbs and spices are in the cabinet to the right of the stove.

(5.1)

If a bishop meets another bishop, he blesses him.

(5.2)

and finally

Everyone who bought a sage plant here bought at least eight others along with it.

(5.3)

We start our analysis with example 5.1. According to Russell, if this is true then there must be one, and only one, cabinet to the right of the stove. However, imagine someone, upon hearing this information, makes his way to the kitchen anf finds that there are several cabinets to the right of the stove. If he were an ardent Russellian when it came to definite descriptions, he would call 5.1 false, and not be able to find the herbs and spices. However, because this is ordinary discourse, it is far more likely that he will start by checking the cabinet immediately to the right of the stove, and in all likelihood, he will find the herbs and spices there, and say that 5.1 is true.

Example 5.2 is an example of what is called 'the problem of indistinguishable participants' in linguistics. Most people have the intuition that when two bishops meet, they bless each other. There is no unique referent of the anaphoric pronouns 'he' and 'him', in fact it seems that both bishops must be the referent of each one. In fact, if a group of bishops met, this sentence implies that they must all bless each other. No one of them is the privileged 'he' who blesses everyone, nor the privileged 'him', who is blessed by everyone.

Lastly, consider the sage plant sentence, example 5.3. Here, we can ask what the referent of 'it' is. Which sage plant was *the* sage plant that was purchased with eight others? The answer seems to be 'any one' or 'pick one'. Although 5.3 seems to contain a hidden definite description, it is very different from the ones Russell considered. This description is indiscriminately picking one sage plant out of the nine available. In some sense, it is picking one object out of a set of indiscernible objects.

It is important to notice that examples 5.2 and 5.3 are not definite descriptions but rather anaphoric pronouns (pronouns which refer back to textual antecedents). This is because they are clearer examples of 'picking one'. However, though there are differences between the two, there are enough similarities that we can make the same claims about definite descriptions. First of all, both sentences can be transformed into sentences which contain definite descriptions. 5.2 becomes 'if a bishop meets another bishop, the bishop blesses the other bishop', while 5.3 becomes 'everyone who bough a sage plant here bought eight others along with the one they bought'. Secondly, both anaphoric pronouns and definite descriptions act like variables and do not themselves refer. Thus, *the* bishop does not refer to a specific bishop, but rather ranges over all the meeting bishops, and *the* sage plant does not refer to a specific sage plant, but rather is a variable which ranges over all nine sage plants.

An appropriate question to ask now is how this applies to the identity problem. Although these are interesting examples of anaphoric pronouns and definite descriptions which do not denote uniquely, it seems as though they must be particular to a given conversation, with certain conversational assumptions in place. Were Russellian conventions in place, all three examples would be false sentences. They need to be considered in the right conversational context. However, the use of terms like 'i' and '$-i$' are not particular to any given mathematical conversation, but rather 'permanent fixtures' in mathematics (Shapiro, 2009, p. 24). Perhaps, though, we can think of the referents of mathematical terms conversationally. Certainly there was a point in our history where mathematicians first discovered the algebraic closure of the real numbers. Thus, if our mathematical conversation started at that point, 'an algebraic closure of the reals' would be translated to 'the algebraic closure of the reals' along the same lines as example 5.3 above. Although we do not really mean the one unique algebraic closure (there are no doubt many isomorphic copies), we at least mean the one to which all previous conversations have referred to. What happens if at some point one conversation was derailed and accidentally split in two, each referring to the square root of -1 as a different root?

At this point, it would be impossible for any two mathematicians to know whether they meant the same thing when they uttered 'i'. We would not be able to tell, for example, whether the first mathematician's $4 - 6i$ was half the second mathematician's $8 - 12i$. This must be solved. A principal square root is needed. In the reals, it is common practice to identify *the* square root as the positive one, so the square root of 4 is 2 and not -2. In the negative reals, we may say the square root is positive if its lacks a negative sign, so the square root of -4 is $2i$, and the other root is $-2i$. This is still unsatisfactory, since we can easily switch which root is positive via the automorphism which makes them indiscernible because $\mathbb{C}$ is an unordered field. Even ignoring this, we still have a problem: what is the square root of $-2i$? The two roots are $1 - i$ and $-1 + i$. Is either of these positive in the sense described above? Here is seems we need to resort to the root with the smaller counterclockwise angle from the x-axis when considering them in polar form. Thus, $-1 + i$ is the square root, and $1 - i$ is the other, since $-1 + i$ has an angle of $3\pi/4$ and $1 - i$ has an angle of $7\pi/4$. In this situation, since i lies at $\pi/2$ radians from the x-axis and $-i$ lies at $3\pi/2$ radians from the x-axis we call i the square root. However, this assumes that somehow counterclockwise measurements are privileged. Without making this assumption, we cannot tell which angle is bigger, for both i and $-i$ are $\pi/2$ radians from the x-axis in different directions. Thus, to assume one of them is *the* square root, we must simply assert it as a brute fact. Thus, like sage plants and bishops, we take i and $-i$ to be variables which range over the set of square roots of -1.

Shapiro's solution suggests two things. The first is that definite descriptions might not denote uniquely, and the second is that mathematics is more conversational than we might think. Shapiro uses these results to develop a method for picking one object out of a set of many. He analyzes existential elimination using what he calls parameters. Usually, the existential elimination rule allows one to derive $A(b)$ from $\exists x A(x)$ where b is a singular term that does not occur in $\exists x A(x)$ (Shapiro, 2009, p. 31). Usually, $A(b)$

is assumed as an assumption to be discharged later. Though b is a singular term, it is certainly not a proper name. It would be very strange to use the number 17, for example, even if the number has not appeared previously in the deduction. If A were 'is even', assuming $A(17)$ would be false, even if this assumption would be discharged later. Thus, in mathematics, we introduce arbitrary singular terms. However, defining what arbitrary means can generate some problems. Shapiro introduces a new category of singular terms for existential elimination, which he calls parameters (Shapiro, 2009, p. 32). This class of terms also works for universal introduction. When we know $\exists x A(x)$ is true, we know that there is some element in the domain which satisfies A. Thus, we let b denote one of the A's. In this sense, b is a constant. However, it also functions like a variable. Since we cannot specify which object in our domain b is, there is a sense in which b is a variable ranging over the A's. When using the existential elimination rule, instead of $A(b)$ being an assumption to be discharged later, we think of $A(b)$ as an inference. The rule then becomes: from $\exists x A(x)$ infer $A(b)$ where b does not occur previously in the deduction. Here, $A(b)$ rests on the same assumptions as $\exists x A(x)$ (Shapiro, 2009, p. 33). This new existential elimination rule seems to fit well with mathematical practice as well. It is natural in proof to say 'let b be a A' and not 'assume $A(b)$' when deriving something from an existential statement.

Applying the theory of parameters to the identity problem is fairly straightforward. Suppose there are two indistinguishable elements with the property A. Then we know that $\exists x A(x)$ is true. Thus, we let b be one such A. For example, we know that there are two square roots of -1, thus $\exists x(x^2 = -1)$. We let i be one such root. Since $i^2 = -1$ we know $-i$ is the other root. The only difference between the parameter i and the parameter b is that i is a permanent conversational fixture.

Although Shapiro's parameters solution does work, and is equivalent to the epsilon calculus solution presented below, the epsilon calculus solution is more faithful to the intuition that we are just picking one thing out of a group of many, since it makes use of a choice function.

5.3 Epsilon Calculus

David Hilbert originally developed the epsilon calculus to aid his foundational program in the philosophy of mathematics. Hilbert's program was intended to axiomatize mathematics and provide a finitary proof theory. Famously, Gödel's incompleteness theorems show that this project cannot be carried out, as no sufficiently strong finite system can prove its own consistency. Although this result made the letter of Hilbert's program impossible to fulfill, the development of the program still lead to some positive results, and to the development of some useful tools. Hilbert's program was thought by many, including Wilhelm Ackermann and Paul Bernays, to be auspicious. The program was intuitive and appealing. On account of this, the very unintuitive Gödelian incompleteness results did not completely deter the foundationalists, and work on extensions and restrictions of the the theory continued.

The epsilon calculus was developed as a tool to aid the proof theory developed for Hilbert's program. The original application of it was to first order logic, but it can be extended to second order logic as well, in order to accommodate Shapiro's ARS. Hilbert's intended use of the calculus was to eliminate quantifiers from first order proofs using the epsilon substitution method. We have that $\epsilon_x Ax$ is equivalent to 'some x such that A holds of it'. This is a choice function. It simply picks one of the elements of the domain that satisfies the formula A. More generally, we have that $\exists x Ax \equiv A(\epsilon_x Ax)$ and $\forall x Ax \equiv A(\epsilon_x \neg Ax)$. Thus, $\exists x Ax$ is true if and only if A is true of some x such that A.

Shapiro (2009) suggests that Hilbert's epsilon is an equivalent solution to the reference problem. Although it is mathematically the same as the solution Shapiro proposes (cf section 5.2), I claim it is superior, as it remains faithful to the intuition that we are just picking one object. In some sense, making use of Hilbert's epsilon calculus might be too restrictive. All that is required is some sort of choice function, and I make use of the epsilon calculus to fulfill this need.

The application of the epsilon choice function to the reference problem is quite straightforward. I will make use of the sage plant example from section 5.2, as it is particularly illuminating. Recall

Everyone who bought a sage plant here bought at least eight others along with it.

The inclination is that the answer to the question 'which sage plant was *the* sage plant?' is that it does not matter. The epsilon calculus is capable of capturing this intuition. If being a sage plant is true of the object picked out by the formula ϵ_x(x is a sage plant and x was purchased) then we have successfully captured the intuition behind this sentence, and the 'pick one' answer to the aforementioned question. The formula picks out, in layman's terms, some sage plant which was bought along with eight others. So long as that thing is also a sage plant, we have successfully 'picked' an arbitrary sage plant matching the description we desired. This is the same result as any other choice function applied to the nine sage plants. It is entirely arbitrary which sage plant the epsilon function picks, but that suits our purposes since the nine sage plants are indistinguishable. Thus, like Shapiro's definite descriptions, we have variables which range over a specific set and which do not refer.

In order to capture all of mathematics in this new 'ARS plus epsilon calculus' theory, we need to replace mathematical terms by epsilon terms. For most of our everyday mathematical objects, this is not a problem, since they can be described uniquely up to isomorphism. Thus, if the description for some object is A, we know that $\epsilon_x Ax$ will pick out one of the isomorphic copies of that object. We will continue to treat indistinguishable objects as variables ranging over a given set of objects, in the case which the set of objects which are indistinguishable. Thus, we can deny that i refers at all. We can then use definite descriptions to refer to indistinguishable objects with epsilon terms, and use the epsilon calculus to make this process rigorous. We can let $\sqrt{-1} = \epsilon_x(x^2 = -1) = i$. Thus, $-i$ would just be the 'other' root, i.e. $-i = (\epsilon_x(x^2 = -1 \wedge x = \epsilon y_{(}y^2 = -1)))$. This solution is ideal, because it picks out one of the two indistinguishable roots, assigns it the name i, and leaves everything else alone. The epsilon function acts as our intuition does. It does not care which of the two indistinguishable objects it picks and assigns to the name i, and neither do we. In fact, the formula $A(\epsilon_x(Ax \wedge \epsilon_y(Ay \wedge x = y)))$ states that there are at least two objects with the property A. Thus, $\epsilon_x(x^2 = -1 \wedge \epsilon_y(y^2 = -1 \wedge x = y))$ states that there are at least two roots of -1. The same can be said about the Euclidean plane, which was another problematic structure, as any arbitrary point can be selected by the formula ϵ_x(x is a point in the Euclidean plane). Distinguishing between two Euclidean points is simple and can be done using the formula above and substituting 'is a point in the Euclidean plane' for A.[2]

[2]A special thank you to Dr Stewart Shapiro who pointed out to me that this solution would not work for an intuitionist. The epsilon calculus assumes that either there is an object or there is no object which satisfies a certain predicate, in order to assign epsilon terms values. This allows us to derive $\exists x(E \to Kx)$ from $E \to \exists x Kx$, which intuitionist logic cannot derive. The proof is as follows:

$$
\begin{array}{r|l}
1 & E \to \exists x Kx \\
2 & \quad \mid\ E \\
3 & \quad \mid\ \exists x Kx \\
4 & \quad \mid\ K\epsilon_x Kx \\
5 & E \to K\epsilon_x Kx \\
6 & \exists x(E \to Kx)
\end{array}
$$

To fix this, an intuitionist must not be able to access the epsilon term $K\epsilon_x Kx$ once the assumption E has been discharged. Shapiro calls this a 'scope island', which was originally a linguistic definition.

5.3.1 The Complex Numbers Revisited

As an example, I will demonstrate how we can construct the complex numbers and refer to i and $-i$. First, we will need to more rigorously define the semantics of the epsilon calculus and the complex numbers.

Formally, we define a model of the epsilon calculus as $(\mathscr{M}, V)$, where $\mathscr{M}$ represents the domain, and V the valuation. Here, f represents any n-place function, R any n-place relation, t_i terms, and x variables. V is a function from the set of variables to $\mathscr{M}$. We say $(R)^{\mathscr{M}} \subseteq \mathscr{M}^n$ and $(f)^{\mathscr{M}} \in \mathscr{M}^{\mathscr{M}^n}$. We also need an extensional choice function from the power set of $\mathscr{M}$ to $\mathscr{M}$. We call it C, and define C such that $C(X) \in X$ whenever $X = \emptyset$, and an arbitrary element otherwise. Formally, we have the following rules (adapted from Zach, 2009) for the value of a term ($val_{\mathscr{M},V,C}$) and the satisfaction relation ($\mathscr{M}, V, C \models$):

term: $val_{\mathscr{M},V,C}(x) = V(x)$

***f*,*R*:** $val_{\mathscr{M},V,C}(f(t_1, t_2, ...t_n) = (f)^{\mathscr{M}}(val_{\mathscr{M},V,C}(t_1), val_{\mathscr{M},V,C}(t_2), ...val_{\mathscr{M},V,C}(t_n))$

$\mathscr{M}, V, C \models R(t_1, t_2, ...t_n)$ if and only if $(val_{\mathscr{M},V,C}(t_1), val_{\mathscr{M},V,C}(t_2), ...val_{\mathscr{M},V,C}(t_n)) \subseteq (R)^{\mathscr{M}}$

True: $\mathscr{M}, V, C \models \top$

False: $\mathscr{M}, V, C \not\models \bot$

$\wedge$,$\neg$: $\mathscr{M}, V, C \models A \wedge B$ if and only if $\mathscr{M}, V \models A$ and $\mathscr{M}, V \models B$

$\mathscr{M}, V, C \models \neg A$ if and only if $\mathscr{M}, V \not\models A$

$\forall$,$\exists$: $\mathscr{M}, V \models \forall x A x$ if and only if for all $m \in \mathscr{M}$ $\mathscr{M}(m/x), V \models Ax$, where $\mathscr{M}(m/x) = (\mathscr{M}, \overline{V})$ and $\overline{V}(x) = m$ and $\overline{V}(y) = V(y)$ for all $y = x$

$\mathscr{M}, V \models \exists x A x$ if and only if for some $m \in \mathscr{M}$ $\mathscr{M}(m/x), V \models Ax$, where $\mathscr{M}(m/x)$ is defined as above

ϵ: $val_{\mathscr{M},V,C}(\epsilon_x Ax) = C(val_{\mathscr{M},V,C}(A(x)))$ where $val_{\mathscr{M},V,C}(A(x)) = \{m | \mathscr{M}(m/x), \overline{V}, C \models Ax\}$

The only axiom is $A(t) \rightarrow A\epsilon x Ax$ for any term t, and the two rules of inference are modus ponens and substitution (from $A(x)$ conclude $A(t)$ for any term t).

We now turn to the complex numbers. We know that the complex number field is a field extension of the reals, and that we can construct the complex numbers from the real numbers. Since there are no non-trivial automorphisms of the real numbers, and thus no indistinguishable real elements, it suffices to show that this construction can be captured in the ϵ-calculus, and that we can refer to to distinct roots of -1. In ordinary mathematical thinking, we think of $\mathbb{C}$ as having a basis $\{1, i\}$ in $\mathbb{R}$. However, because of our current predicament, we do not know which square root of -1 i is meant to be. Yet, we know there is at least one, and we know that the constructions with each of them will be isomorphic. Then, we can take our basis to be $\{1, \epsilon_x(x^2 = -1)\}$. Taking it one step further, we can even construct the real numbers in epsilon calculus by letting each real number correspond to its Dedekind cut. Thus, for any real number r, we let $r = \epsilon_r(r$ is an appropriate subset of the rational numbers). Thus, 1 is $\epsilon_r(r$ is the Dedekind cut of $(\infty, 1))$. Thus, the basis for $\mathbb{C}$ is $\{\epsilon_r(r$ is the Dedekind cut of $(\infty, 1)), \epsilon_x(x^2 = -1)\}$. From this we can construct the complex numbers, when we refer to i we are really referring to the formula $\epsilon_x(x^2 = -1)$.

5.3.2 Second Order Epsilon Calculus

Though Hilbert's epsilon calculus did not work for the project he had in mind, it does contain first order logic. In fact, it is strictly stronger then first order logic. There is even a second order variant of the epsilon calculus, which satisfies Shapiro's second order background language requirement.

The second order epsilon calculus has epsilon terms which are both elements and sets.

Thus, we still have terms of the form $\epsilon_x Ax$ but now we can also include $\epsilon_X AX$ where X is some set. Thus, we can pick sets themselves. This parallels the first order case in that we have $\exists XAX \equiv A(\epsilon_X AX)$ and $\forall XAX \equiv A(\epsilon_X \neg AX)$. $\epsilon_X AX$ is a set of which A is true if there is one. For example, $\epsilon_X(\forall x(x \notin X))$ picks out the empty set, which $\epsilon_X(\emptyset \subset X)$ picks out an arbitrary set. Because the second order epsilon calculus is meant to mirror second order logic, we have the comprehension scheme $\exists X \forall x(A(x) \leftrightarrow x \in X)$ for any second order formula A. Thus, any second-order formula defines a set. In Shapiro's case, we also claim that any coherent second order formula defines a structure. This means that we can develop second order arithmetic (analysis) in much the same way that we accomplish this in second order logic with comprehension. Where the only axiom in the first order epsilon calculus is $A(t) \to A\epsilon_x Ax$ for some term t, we add $A(\epsilon_X AX) \to AX$ as an axiom in the second order case (Zach, 2003, p. 244).

Shapiro's solution to the reference problem was to rethink how definite descriptions and anaphoric pronouns refer, and then to use those results to refer to indistinguishable objects with parameters. Although this solution works, the epsilon calculus solution more accurately models the intuition that we are just picking one thing from a group of many. Thus, though the solutions are mathematically equivalent, I advocate the epsilon calculus solution.

Chapter 6

Conclusion

In my opinion, ARS is an excellent candidate for a philosophy of mathematics. It is capable of solving three of the largest problems raised in the last two centuries, and has intuitive appeal. The idea that mathematical objects themselves are less important then the relations those objects have to one another captures the intuitions behind the way mathematics is practiced.

This thesis presented a solution to one potential problem with ARS. The identity problem, as presented by Keränen, suggests that any advocate of ARS must claim that $i = -i$, which is simply absurd. Keränen makes this claim based on Shapiro's statement that mathematical objects are nothing more then the relations they have to other mathematical objects and one of Leibniz's laws, the Identity of Indiscernibles. Were Keränen correct in claiming that ARS implies $i = -i$, ARS would have had to be rejected as a potential theory of the philosophy of mathematics. Thus, the identity problem needs to be avoided. Two potential solutions for accomplishing this were weak discernibility and hybrid structuralism. Both were rejected, the first because it did not apply to all structures, and the second because it did not remain true to the motivating factors of ARS. Thus, a third solution, the primitiveness of identity in mathematics, was adopted. This solution was accepted because it worked universally, remained true to the spirit of ARS, and allowed Shapiro to claim that it was not the case that ARS implied $i = -i$.

Finally, after initially solving the identity problem, the reference problem was discussed. If ARS accepts the existence of indistinguishable objects, there must be some way to refer to one indistinguishable object out of a set of many. Shapiro's solution is to claim that definite descriptions and anaphoric pronouns do not always refer uniquely, and to use this to establish his theory of parameters. Although this solution is equivalent to my own, I claim that the epsilon calculus solution is more intuitive and remains more faithful to the intuition that we are simply picking one indistinguishable object out of a set of many.

Thus, the identity problem has a solution, and Keränen's claims that ARS implies $i = -i$ are dismissed. I claim, then, that ARS is still a good theory of the philosophy of mathematics, and though it may suffer from other problems that cause it to be rejected, the identity problem is not one of them.

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