# UNIVERSITY OF CALGARY

**Altruism and the Family Firm: Some Theory**

by

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# Abstract

Private income transfers are becoming increasingly recognized as a key aspect of the U.S. economy (Cox, 1987). Private income transfers based on reciprocal altruism usually occur inter vivos (i.e. between living persons). In our modern society, altruism, especially reciprocal altruism, is usually seen in the family context, and involves two generations, parents and their kids. In recent years, more and more economists have researched topics in this area. Moreover, most of them agree that altruism, especially reciprocal altruism, is one of the most important motives for private income transfers. Therefore, reciprocal altruism is interesting in its own right, and has received considerable attention. In this essay, I am going to pose and discuss the following questions: What can be said in general about transfers from one party to the other in the presence of reciprocal altruism? What is (are) the Nash Equilibrium (N.E.) transfer(s) of reciprocal altruism in the general nature? Is (are) the Nash Equilibrium(s) Pareto-optimal? What are the circumstances in which there is no equilibrium? When two people are reciprocally altruistic, do both of them necessarily make positive transfers? If only one makes a positive transfer, what are the factors that determine the amount of the transfer and the person who makes it? How do these factors affect the amount of the transfer and determine the person who does the positive transfer? What can be said about the effect of reciprocal altruism on the work efforts of family members? How does reciprocal altruism affect their equilibrium personal utilities? I begin by addressing these questions when incomes of family members are fixed or earned in an impersonal market. Building on this base, I address the same questions when incomes are generated in a family firm.

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**To my parents, Fengqi Wu and Yingmin Song**

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# Chapter I: Introduction and Review of Literature

Reciprocal altruism is interesting in its own right, and has received considerable attention. It is also of interest in the theory of the family firm because family members are arguably altruistic toward each other. Reciprocal altruism within the family has received virtually no attention, and the purpose of this thesis is to begin the exploration of the implications of reciprocal altruism within the family firm. This would seem to be important area of research for the simple reason that a great deal of economic activity occurs within family firms. Data tell us that the family business is the typical form of American business since over 90 per cent of businesses are owned by a family (Beckhard and Dyer, 1983). According to Beckhard and Dyer, these family businesses produce an estimated 50 per cent of the Gross National Product and provide a majority of all jobs in the United States.

This essay makes use of simple tools of economic theory to construct and analyze a model of reciprocal altruism first, and then to apply it to the family firm. The main analytical tools used in my model are game theory and the Kuhn-Tucker method. For analytical simplicity, my model of reciprocal altruism includes only two individuals, called 'the parent' and 'the kid'. After discussing the existing literature concerning income transfers and reciprocal altruism in this Chapter, I set up my own model of reciprocal altruism and analyze that model when incomes are fixed and when incomes are earned in an impersonal market in Chapter II. Building on results from Chapter II, I then consider reciprocal altruism in the context of the family firm in Chapter III. From the results in Chapter II and Chapter III, a very interesting comparison can be found: in the family that does not own a family firm, reciprocal altruism only induces the person who makes positive transfers to work more but decreases his personal utility, while it makes the person who receives the transfers give up working but increases his personal utility. However, in the family owning a family firm, balanced reciprocal altruism can induce all the family members to work harder and can make the family firm more efficient in terms of personal utilities. Chapter IV is the conclusion of this essay.

The phenomenon of income or wealth transfers has been studied by a number of economists. Typically, the transfer is from the parent to the kid, and a distinction is made in the literature between transfers inter vivos (i.e., when both parent and kid are alive) and bequests (i.e., a transfer from parent to child when the parent dies). One can identify two motives for such transfers. These two motives are altruism and exchange. In other words, broadly speaking, two approaches have been used. In one, transfers are motivated by altruism, and in the other, transfers are seen as a quid pro quo for some good or service provided by the person receiving the transfer. However, substantial disagreement exists regarding the nature of private transfer motives (Cox, 1987). During the processing of the development of the Economics theories on this topic, the literature on transfers contains two separate strands of thought.

The first strand of the literature is the research focusing on the altruism. Altruism means behavior that reduces the actor's fitness while enhancing the fitness of others (Simon, 1993). In other words, the altruistic person can obtain utility from other persons' utility. The typical representative in this research area is Becker's 1974 paper on social interactions. In his paper, Becker proposed the famous Rotten-kid Theorem as "if a head exists, other members also are motivated to maximize family income and consumption, even if their welfare depends on their own consumption alone" (Becker, 1974). In the intergenerational case, the Rotten-kid Theorem can be expressed in other words as: "if the parent is altruistic, the kid won't behave in a manner that lowers the welfare of the whole family at least" (Bruce and Waldman, 1986). The term "head" in Becker's Rotten-Kid Theorem is a family member who transfers the general purchasing power to all other family members. In other words, the head of a family in Becker's model is an altruist who cares about other members' welfare. Becker introduced two terms, social income and social environment, into his general model of social interaction. The term "social income" includes the person's own income (his earnings, etc.) and the monetary value to him of his social environment, which is the level of the characteristics of other people to a person when he makes no effort. In other words, the social environment is the monetary value to a person of the relevant characteristics of others (Becker, 1974). The main point of Becker's Rotten-kid Theorem is that the redistribution of the family's income has no effect at all on the consumption or welfare of any family member as long as the head of the family continues to transfers resources to others, since it simply induces offsetting changes in transfers from the head (Becker, 1974). The Rotten-Kid Theorem was advanced in Bruce and Waldman (1990), Bergstrom (1989), Bernheim and Stark (1988), Bernheim, Shleifer and Summers (1985), Andreoni Andreoni (1989), and Becker (1976). These papers developed the basic model and theory of Becker's 1974 paper from different angles. For example, Bruce and Waldman (1990) advanced the basic one-period model in Becker's 1974 paper into the two-period model with some restrictions to check whether the Rotten-Kid Theorem was still applicable or not.

In addition, altruism is one way to explain bequests from parents to kids. The typical models in this area have been introduced in Becker and Tomes (1979), Adams (1980), Menchik (1980), Menchik and David (1983 and 1985), Maria G. Perozek (1998) and Zhang (1994). There is some empirical evidence in support of the hypothesis that bequests are motivated by altruism. In his 1981 paper, Tomes found, by testing the altruistic hypothesis using bequest data, that bequests performed a compensatory role and the bequest received is inversely related to recipient income. The bequest behavior can be considered as a type of transfer, although the bequest behavior does not occur inter vivos.

The second strand of the transfer literature sees transfer as a quid pro quo for some good or service provided by the recipient of the transfer. A variety of situations involving household production and insurance has been explored. The typical representatives include Kotlikoff and Spivak (1981), Manser and Brown (1980), Kaufman (1982), McElroy (1985), Lucas and Stark (1985), Bernheim et al. (1985) and Tomes (1981). These papers support the point that transfers represent payments made in exchange for services provided by family members (Cox, 1987). Gradually, this point has been applied to the topic of bequest behavior to find the empirical support for the bequests-as-exchange model. However, Tomes (1981) found an inverse or zero relationship between recipient-decedent contact and bequests received and concluded that "this result presents prima facie evidence against the pure child-services model of inheritance (i.e., with no altruism)" (Tomes, 1981, P946). Tomes's finding suggests that the bequest-as-payments question may not be completely settled (Cox, 1987).

There is an empirical literature that attempts to determine which of the two motives is better able to explain transfers. In his 1987 paper, Cox tested the motives for the private income transfers by making use of the data from the President's Commission on Pension Policy (PCPP) survey, which contained a special module in which survey respondents reported various types of inter vivos transfers. The survey information used by Cox in his 1987 paper was collected in August 1979, and the data for income generally covered the first 8 months of 1979, which contained 3,440 households. These households were broken down into 4,605 family units, each of which contained a head and, if present, a spouse and children under the age of 18 who lived at home (Cox, 1987). Cox considered altruism and exchange as two of the most important motives for the transfers. Results in his 1987 paper are mixed. By using econometrics methods and rich data, Cox argued that bequests were motivated by altruism, and argued that transfers inter vivos were motivated by exchange by saying, "This investigation of the motives for inter vivos transfers supports the idea that inter vivos transfers are payments for services that are exchanged among family units. A key factor in making inferences about motives for private transfers is the relationship between the recipient's income and transfers, given an interior solution for transfers. A positive relationship between the recipient's income and transfers contradicts the altruist hypothesis. Such a positive relationship is consistent with the exchange hypothesis when the elasticity of services with respect to the implicit service price is less than unity in absolute value..." (Cox, 1987). Actually, these two motives are not completely independent but dependent, even consistent under some special situations.

For example, "reciprocal exchange is a necessary, but not sufficient, condition for exchange-motivated transfers. Two-way exchange is also consistent with reciprocal altruism" (Cox, 1987). Cox made use of the survey results collected by Adams (1968) for a sample of 799 individuals in Greensboro, North Carolina, to support his issues. Adams found that "35 percent of these young adults who are close to both parents see them weekly or more, 33 percent of those affectionately close to one parent see them that often, as do 34 percent of the respondents who indicate that they were affectionately distant from their parents" (Adams, 1968, P72). In short, my position is that both motives seem to be at work, and that one need not choose one or the other.

In the next chapter, I am going to construct and analyze a model of reciprocal altruism. The results about transfer behaviors based on reciprocal altruism in this model include the result of a negative relationship between the recipient's income (or wage rate) and benefactor's transfer. What is different from previous research work is that the model named reciprocal altruism includes two altruistic persons who care about the opposite's welfare.

# Chapter II: A Model of Reciprocal Altruism

In this chapter, two problems in the field of reciprocal altruism are going to be solved: one with exogenous incomes, and the other with endogenous incomes. For simplicity, just two people, whom I call parent (p) and kid (k), will be involved in the model. Moreover, I use two utility functions, *personal utility*, the argument of which is the person's own consumption and in some cases work effort (or leisure), and *total utility*, which is a weighted average of one's own personal utility and the personal utility of the other person, where relative weights capture the degree of altruism. Let $u_p(.)$ and $u_k(.)$ denote personal utilities of the parent and the kid, then total utilities of the parent and kid are written as

$$U_p = u_p(.) + a_p \cdot u_k(.) \qquad (1)$$

$$U_k = u_k(.) + a_k \cdot u_p(.) \qquad (2)$$

where $U_p$ expresses the parent's total utility and $U_k$ expresses the kid's total utility. The parameters $a_p$ and $a_k$ express the degree of altruism of the parent and the kid, respectively. The degree of altruism is the degree to which a person cares about another person's welfare relative to his own welfare. We restrict parameters $a_p$ and $a_k$ as follows

$$0 \le a_p < 1 \textit{ and } 0 \le a_k < 1$$

If the parameter $a_i$ (i=p, k) is negative, the person is envious or malevolent. If the parameter $a_i$ (i=p, k) is zero, the person is egoistic. If the parameter $a_i$ (i=p, k) is positive, the person is altruistic. A very special case happens when the value of parameter $a_i$ (i=p,

k) is one, which means that the person is perfectly altruistic because he cares about the other person's utility to the same degree as he cares about his own utility. As will become apparent, if $a_i>1$, $i=p, k$, there exist circumstances in which there is no Nash Equilibrium to the transfer game.

In Section II.1, I analyze Problem 1 in which the arguments of personal utility functions are simply consumption and incomes are fixed. In Section II.2, I analyze Problem 2 in which the arguments of personal utility functions are consumption and work effort, and incomes are earned in an impersonal market. In both cases, I focus on N.E. income transfers. In other words, who transfers what to whom?

## Section II.1: Problem 1 -- Reciprocal Altruism with Fixed Incomes

### I. The Special Case in Which the Parent and the Kid Have the Same Personal Utility Functions

Define $C_p$ and $C_k$ as the consumption of the parent and the kid and define $I_p$ and $I_k$ as their incomes, respectively. Then observe that $U_p$ and $U_k$ are defined as

$$U_p(C_p, C_k) = u_p(C_p) + a_p \cdot u_k(C_k),\ 0 \le a_p < 1 \qquad (3)$$

$$U_k(C_p, C_k) = u_p(C_p) + a_k \cdot u_k(C_k),\ 0 \le a_k < 1 \qquad (4)$$

where $u_p(C_p)$ and $u_k(C_k)$ are strictly concave so that we can get interior solutions. Define $t_p$ and $t_k$ as their income transfers. Then observe that the parent's problem is

$$\max_{C_p, t_p}\ u_p(C_p) + a_p \cdot u_k(C_k), \qquad s.t.\ \ C_p = I_p + t_k - t_p$$

Similarly, the kid's problem is

$$\max_{C_k, t_k}\ u_k(C_k) + a_k \cdot u_p(C_p), \qquad s.t.\ \ C_k = I_k + t_p - t_k$$

Using the constraints to eliminate $C_p$ and $C_k$ as choice variables, we can simplify the problems. First, define

$$U_p(t_p, t_k) \equiv u(I_p + t_k - t_p) + a_p \cdot u(I_k + t_p - t_k) \qquad (5)$$

$$U_k(t_p, t_k) \equiv u(I_k + t_p - t_k) + a_k \cdot u(I_p - t_p + t_k) \qquad (6)$$

We now have a game in which the parent's strategy and payoff function are $t_p$ and $U_p(t_p, t_k)$ and the kid's strategy and payoff function are $t_k$ and $U_k(t_p, t_k)$. Denote the Nash Equilibrium (N.E.) of this game by $(t_p^*, t_k^*)$. Any N.E. is characterized by the following conditions:

$$Condition\,1: t_p^* \; solves \max_{t_p} U_p(t_p, t_k^*)$$

$$Condition\,2: t_k^* \; solves \max_{t_k} U_k(t_k, t_p^*)$$

To simplify the problem, choose the following personal utility functions

$$u_p(C_p) = C_p^\alpha \;\; and \;\; u_k(C_k) = C_k^\alpha$$

where $0<\alpha<1$. Notice that $U_p(t_p, t_k)$ is a strictly concave function of $t_p$, and that $U_k(t_p, t_k)$ is a strictly concave function of $t_k$. Logically, there are four sorts of possibilities for $(t_p^*, t_k^*)$ calling: Case (i), the parent makes a positive transfer to the kid but the kid transfers nothing to the parent (i.e., $t_p^*>0$ and $t_k^*=0$); Case (ii), the parent transfers nothing to the kid but the kid makes a positive transfer to the parent (i.e., $t_p^*=0$ and $t_k^*>0$); Case (iii), neither the parent nor the kid makes transfers (i.e., $t_p^*=t_k^*=0$); Case (iv), both of them make positive transfers (i.e., $t_p^*>0$ and $t_k^*>0$).

We first find the best response functions (BRFs) for this special case. Denote the parent's best response and the kid's best response by $\hat{t}_p$ and $\hat{t}_k$. Given the concavity of $U_p(t_p, t_k)$ in $t_p$, the parent's best response is

$$\hat{t}_p = 0 \; if \; \frac{\partial U_p(t_p = 0, t_k^*)}{\partial t_p} < 0$$

In addition, if

$$\frac{\partial U_p(t_p = 0, t_k^*)}{\partial t_p} \geq 0$$

then the parent's best response satisfies the standard first-order condition

$$\frac{\partial U_p(\hat{t}_p, t_k)}{\partial t_p} = 0$$

These observations yield the parent's BRF to be

$$\boxed{\begin{array}{l} \hat{t}_p = 0, \textit{ if } t_k < \dfrac{1}{1+A_p}(A_p \cdot I_k - I_p) \\ \hat{t}_p = t_k + \dfrac{1}{1+A_p}(I_p - A_p \cdot I_k), \textit{ otherwise} \\ \textit{where } A_p = a_p^{\frac{1}{\alpha-1}} \end{array}}$$

Similarly, the kid's BRF is

$$\boxed{\begin{array}{l} \hat{t}_k = 0, \textit{ if } t_p < \dfrac{1}{1+A_k}(A_k \cdot I_p - I_k) \\ \hat{t}_k = t_p + \dfrac{1}{1+A_k}(I_k - A_k \cdot I_p), \textit{ otherwise} \\ \textit{where } A_k = a_k^{\frac{1}{\alpha-1}} \end{array}}$$

Now, let us look at Case (i). Algebraically,

$$\frac{\partial U_p(t_p^*, t_k^* = 0)}{\partial t_p} = -\alpha(I_p + t_k - t_p)^{\alpha-1} + a_p \cdot \alpha \cdot (I_k + t_p - t_k)^{\alpha-1} = 0 \quad (7)$$

$$\frac{\partial U_k(t_p^*, t_k^* = 0)}{\partial t_k} = -\alpha(I_k + t_p - t_k)^{\alpha-1} + a_k \cdot \alpha \cdot (I_p - t_p + t_k)^{\alpha-1} \leq 0 \quad (8)$$

Figure 1(a) and 1(b) depict transfers of the parent and the kid respectively. Then deriving from the first-order condition for $t_p$*, and from the second that is a parameter restriction, we can get the result of case (i) as

$$\boxed{\begin{array}{c} \text{If } \dfrac{I_p}{I_k} > A_p, \\ \text{then } t_p^* = \dfrac{1}{1+A_p}(I_p - A_p I_k) \text{ and } t_k^* = 0 \\ \text{where } A_p = a_p^{\frac{1}{\alpha-1}} \end{array}}$$

Therefore, their equilibrium personal utilities are

$$u_p^* = (I_p - \frac{1}{1+A_p} I_p + \frac{A_p}{1+A_p} I_k)^\alpha$$

$$u_k^* = (I_k + \frac{1}{1+A_p} I_p - \frac{A_p}{1+A_p} I_k)^\alpha$$

Since their equilibrium personal utilities, if they are not altruistic, are $(I_p)^\alpha$ and $(I_k)^\alpha$ respectively, and since $I_p/I_k > A_p$, we learn that when there is a transfer from the parent to the kid, the equilibrium personal utility of the parent in the case with reciprocal altruism is less than that in the case without altruism, but the equilibrium personal utility of the kid

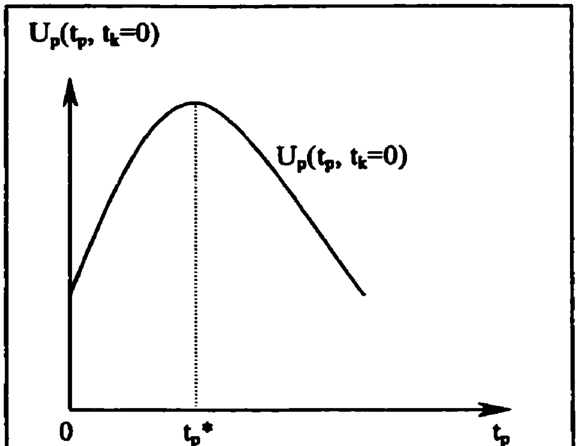

Figure 1(a)

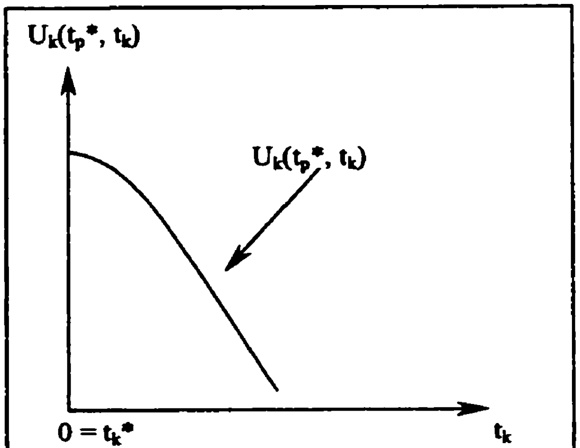

Figure 1(b)

in the case with reciprocal altruism is greater than that in the case without altruism.

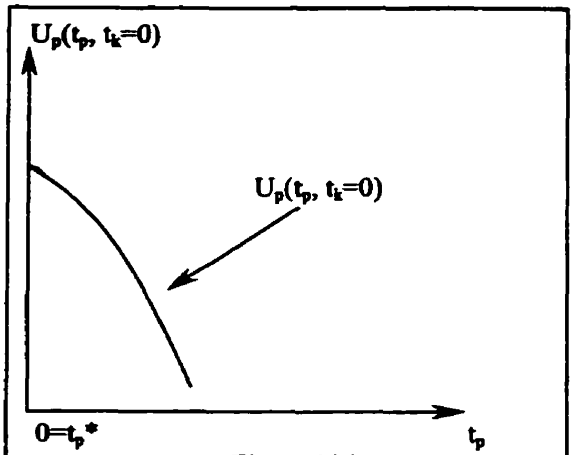

Figure 1(c)

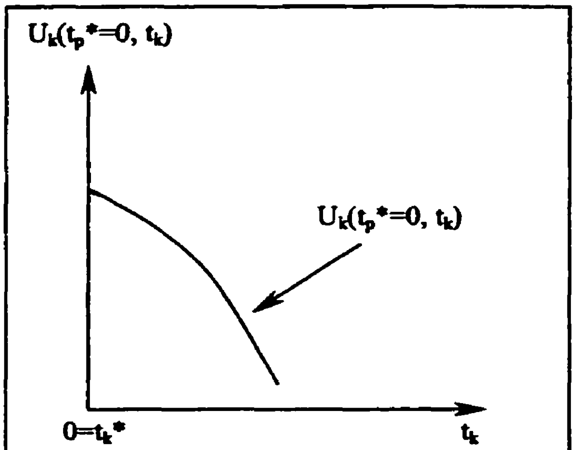

Figure 1(d)

Case (ii) is analogous, but roles are reversed. The result of Case (ii) is

$$\begin{gathered}
If\ \frac{I_p}{I_k} < \frac{1}{A_k}, \\
then\ t_k^* = \frac{1}{1+A_k}(I_k - A_k \cdot I_p)\ and\ t_p^* = 0 \\
u_p^* = (I_p + \frac{1}{1+A_k} I_k - \frac{A_k}{1+A_k} I_p)^\alpha\ and\ u_k^* = (I_k - \frac{1}{1+A_k} I_k + \frac{A_k}{1+A_k} I_p)^\alpha \\
where\ A_k = a_k^{\frac{1}{\alpha-1}}
\end{gathered}$$

Now, let us look at Case (iii). Figure 1(c) and 1(d) depict transfers of the parent and the kid respectively. Algebraically,

$$\frac{\partial U_p(t_p^*, t_k^* = 0)}{\partial t_p} = -\alpha(I_p + t_k - t_p)^{\alpha-1} + a_p \cdot \alpha \cdot (I_k + t_p - t_k)^{\alpha-1} \leq 0 \quad (9)$$

$$\frac{\partial U_k(t_p^* = 0, t_k^*)}{\partial t_k} = -\alpha(I_k + t_p - t_k)^{\alpha-1} + a_k \cdot \alpha \cdot (I_p - t_p + t_k)^{\alpha-1} \leq 0 \quad (10)$$

Then the result of case (iii) can be derived as

$$\boxed{\begin{array}{l} \textit{If } \dfrac{1}{A_k} \le \dfrac{I_p}{I_k} \le A_p, \textit{ then } t_p^* = t_k^* = 0 \\ \textit{and } u_p^* = (I_p)^\alpha, u_k^* = (I_k)^\alpha \end{array}}$$

Their equilibrium personal utilities in Case (iii) are the same as those in the case without altruism.

Given $0 \le a_i < 1$, $(i=p,k)$, and then $A_p > 1$ and $1/A_k < 1$, results in above three cases can be summarized in Figure 2

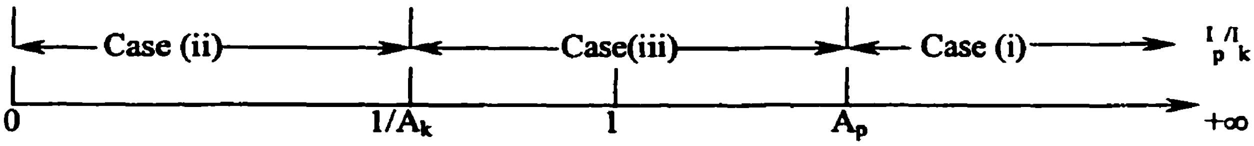


Figure 2

We observe that there are no circumstances in which both transfers are positive. In other words, Case (iv) is empty. Suppose however that either $a_p > 1$ and/or $a_k > 1$. Then it is possible that $A_p < 1/A_k$, as illustrated in Figure 3

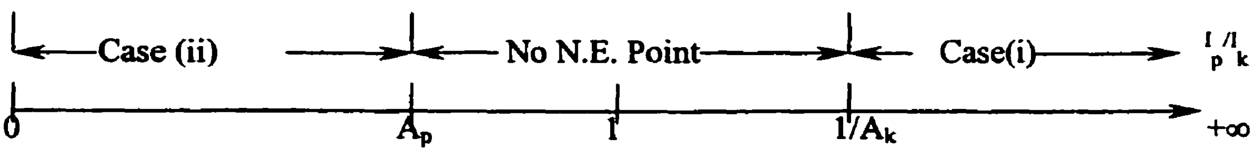


Figure 3

Notice that in the middle of Figure 3, both of them want to make positive transfers so that there is no N.E.

Now, let us look at the indifference curves of $U_p(t_p,t_k)$ and $U_k(t_p,t_k)$. We need to calculate the marginal rate of substitution (MRS) to prepare for checking the Pareto-optimality of the N.E.s in the above three cases. The parent's MRS is

$$\frac{dt_p}{dt_k}=-\frac{\dfrac{\partial U_p(t_p,t_k)}{\partial t_k}}{\dfrac{\partial U_p(t_p,t_k)}{\partial t_p}}=-\frac{\alpha(I_p+t_k-t_p)^{\alpha-1}+a_p\cdot\alpha(I_k+t_p-t_k)^{\alpha-1}}{-\alpha(I_p+t_k-t_p)^{\alpha-1}-a_p\cdot\alpha(I_k+t_p-t_k)^{\alpha-1}}=1$$

Similarly, the kid's MRS is 1. Thus, what can be learned from the parent's and the kid's MRS is that all of their indifference curves are straight lines with slope of 1. Now, let us make use of Figure 4 to consider the Pareto-optimality of the N.E. in Case (i) for example. In Figure 4, straight line AB expresses the parent's BRF and Curve OCD expresses the kid's BRF in Case (i). Point A is the N.E. point. The 45-degree lines express the indifference curves of the parent and of the kid. At the same time, indifference curve AB is the parent's best indifference curve and indifference curve CD is the kid's best indifference curve. If AB were moved leftward, the welfare of the parent would be damaged while the welfare of the kid would be advanced. On the contrary, if AB were moved rightward, both the welfare of the parent and the welfare of the kid would be damaged. If CD were moved rightward, the welfare of the kid would be damaged while the welfare of the parent would be advanced. Thus, there is no way to make both the parent and the kid better off. According to the definition of Pareto-efficiency, the N.E. in Case (i) is Pareto-optimal. Similarly, the N.E. in Case (ii) is also Pareto-optimal.

The Pareto-Optimality of N.E. in Case (iii) can be illustrated by Figure 5. In Figure 5, curve OIJ is the kid's BRF line and Curve OKL is the parent's BRF line in this case.

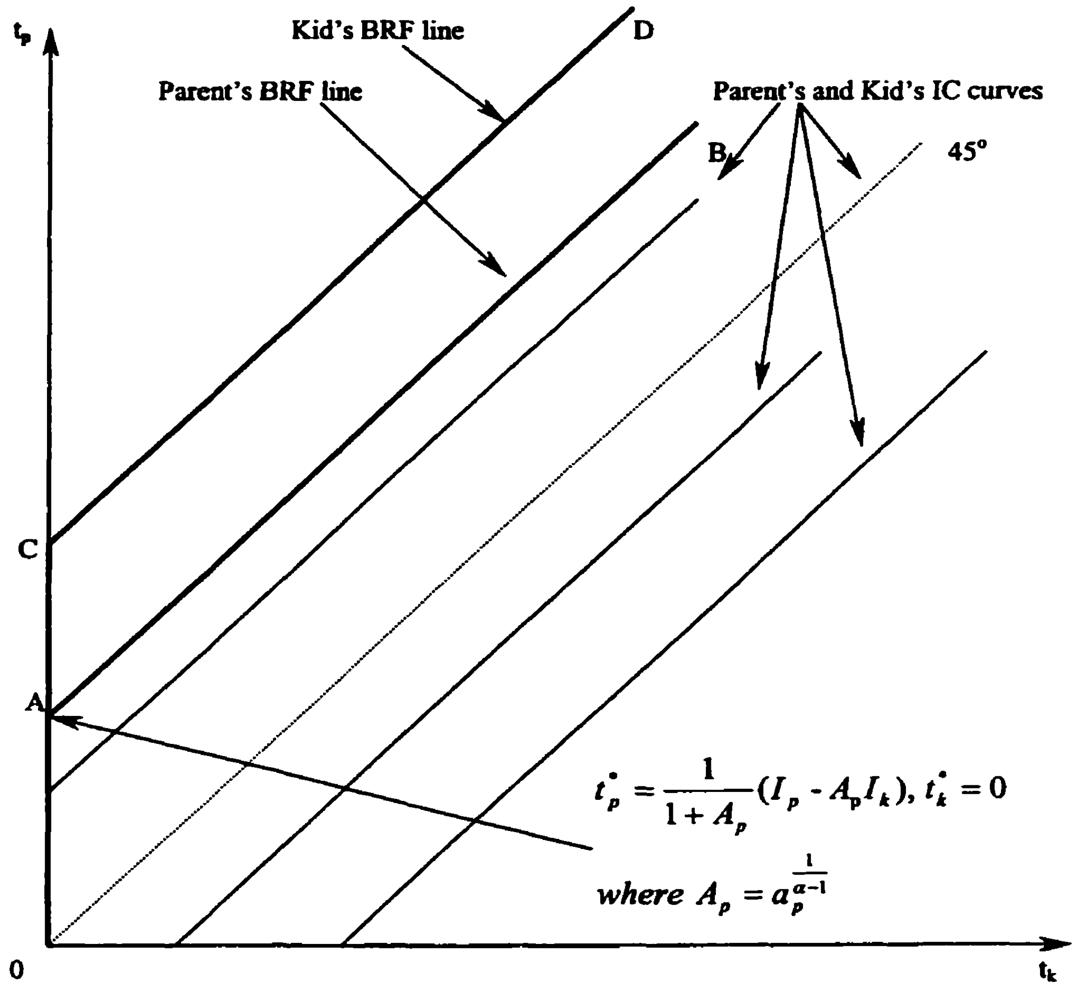


Figure 4

The original point O is the N.E. point. The 45-degree lines express the indifference curves of the parent and of the kid. At the same time, indifference curve IJ is the best indifference curve for the kid and indifference curve KL is the best indifference curve for the parent. The line OM is their indifference curves that pass through the N.E. point. If

OM were moved rightward, the welfare of the parent would be damaged while the welfare of the kid would be advanced. On the contrary, if OM were moved leftward, the welfare of the kid would be damaged while the welfare of the parent would be advanced. Thus, the N.E. in Case (iii) is also Pareto-optimal.

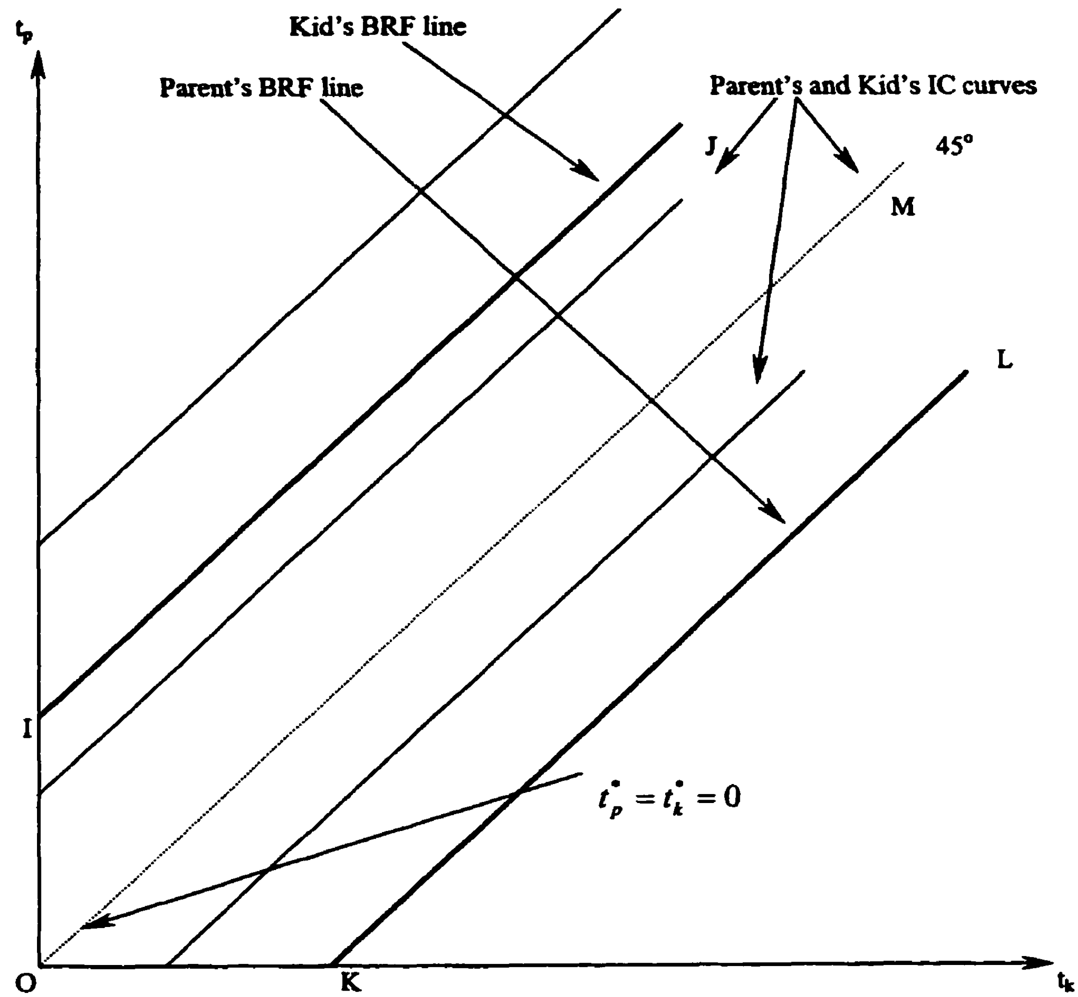


Figure 5

Now, let us summarize this special case in Problem 1. As calculated above, the N.E. transfers between the parent and the kid depend on the values of model parameters $I_p$, $I_k$,

$a_p$, $a_k$, and $\alpha$. Moreover, all of the three N.E. transfers are Pareto-optimal. The above results and conditions can be tabulated as

| Conditions | Case | Nash Equilibrium | Pareto-optimality |
|---|---|---|---|
| $1/A_k \leq I_p/I_k \leq A_p$ | $t_p=t_k=0$ | $t_k^*=t_p^*=0$ | Yes |
| $I_p/I_k > A_p$ | $t_p>0$ and $t_k=0$ | $t_p^*=(I_p-A_pI_k)/(1+A_p)$<br>$t_k^*=0$ | Yes |
| $I_p/I_k<1/A_k$ | $t_p=0$ and $t_k>0$ | $t_k^*=(I_k-A_kI_p)/(1+A_k)$<br>$t_p^*=0$ | Yes |

where $A_p=(a_p)^{1/(\alpha-1)}$ and $A_k=(a_k)^{1/(\alpha-1)}$. Notice that $A_p>1$ and $1/A_k<1$ since $0\leq a_i<1$, $(i=p, k)$ and $0<\alpha<1$.

The result of Case (i) means that the parent will make a positive transfer to the kid but the kid will transfer nothing to the parent when the parent's income is high enough relative to the kid's income. Case (ii) is the reverse of Case (i). The result of Case (iii) means that neither of them will make a transfer to the other person when the parent's income is neither high nor low enough relative to the kid's income. Moreover, the equilibrium personal utility of the person who makes a positive transfer to the other one in the case with reciprocal altruism is less than that in the case without altruism, while the equilibrium personal utility of the person who receives the transfer in the case with reciprocal altruism is greater than that in the case without altruism.

## II. The General Case with Different Utility Functions

After analyzing the special case with the same personal utility functions, we expand our analysis to the general case in which the parent and the kid have different personal utility functions. In the general case, we can check whether the intuitive understanding and main results derived from the special case still hold. Let us assume that the parent and the kid have separate personal utility functions, as strictly concave functions of consumption, called $u_p(C_p)$ and $u_k(C_k)$ respectively. We can write their total utility functions as:

$$U_p(t_p, t_k) = u_p(C_p) + a_p \cdot u_k(C_k),\ 0 \le a_p < 1 \quad (11)$$

$$U_k(t_p, t_k) = u_k(C_k) + a_k \cdot u_p(C_p),\ 0 \le a_k < 1 \quad (12)$$

Then observe that the parent's problem is:

$$\max_{C_p, t_p} u_p(C_p) + a_p \cdot u_k(C_k), \qquad s.t.\ C_p = I_p + t_k - t_p$$

Similarly, the kid's problem is:

$$\max_{C_k, t_k} u_k(C_k) + a_k \cdot u_p(C_p), \qquad s.t.\ C_k = I_k + t_p - t_k$$

Using the constraints to eliminate $C_p$ and $C_k$ as choice variables, we can simplify the problems to be

$$\max_{t_p} U_p(t_p, t_k) = u_p(I_p + t_k - t_p) + a_p \cdot u_k(I_k + t_p - t_k) \quad (13)$$

$$\max_{t_k} U_k(t_p, t_k) = u_k(I_k + t_p - t_k) + a_k \cdot u_p(I_p - t_p + t_k) \quad (14)$$

In the general case, there are two possibilities for the transfer from parent to kid: one is a positive transfer depicted by Figure 6(a), and the other is a zero transfer depicted by Figure 6(b).

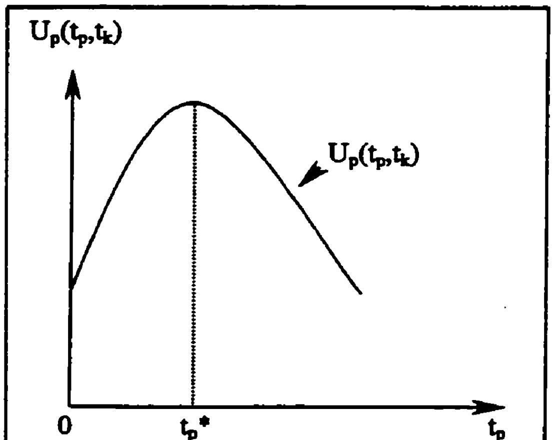

Figure 6(a)

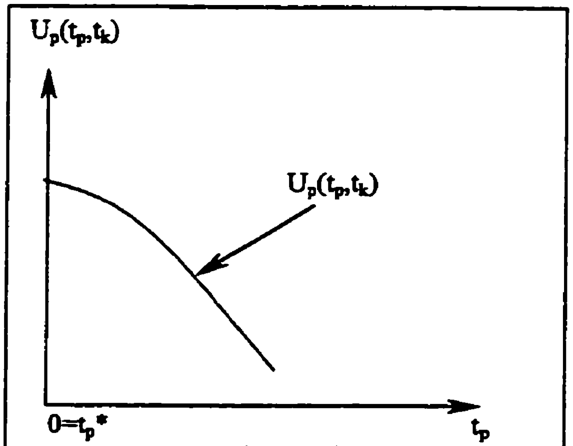

Figure 6(b)

Notice that the parent's best response is

$$\hat{t}_p = 0 \ \textit{if} \ \frac{\partial U_p(t_p = 0, t_k^*)}{\partial t_p} < 0$$

given the concavity of $U_p(t_p, t_k)$ in $t_p$. On the other hand, if

$$\frac{\partial U_p(t_p = 0, t_k^*)}{\partial t_p} \geq 0$$

then the parent's best response satisfies the standard first-order condition

$$\frac{\partial U_p(\hat{t}_p, t_k)}{\partial t_p} = 0$$

Look at this first-order condition, we can learn

$$\frac{\partial U_p(\hat{t}_p, t_k)}{\partial t_p} = -u_p'(I_p - \hat{t}_p + t_k) + a_p \cdot u_k'(I_k - t_k + \hat{t}_p) = 0 \quad (15)$$

Therefore, we have

$$-u_p'(I_p-\hat{t}_p+t_k)=a_p\cdot u_k'(I_k-t_k+\hat{t}_p)$$

Differentiate to find

$$\frac{dt_p}{dt_k}=-\frac{\frac{\partial U_p(t_p,t_k)}{\partial t_k}}{\frac{\partial U_p(t_p,t_k)}{\partial t_p}}=-\frac{\frac{\partial V_p(t_p,t_k)}{\partial C_p}\cdot\frac{\partial C_p}{\partial t_k}+a_p\cdot\frac{\partial V_k(t_p,t_k)}{\partial C_k}\cdot\frac{\partial C_k}{\partial t_k}}{\frac{\partial V_p(t_p,t_k)}{\partial C_p}\cdot\frac{\partial C_p}{\partial t_p}+a_p\cdot\frac{\partial V_k(t_p,t_k)}{\partial C_k}\cdot\frac{\partial C_k}{\partial t_p}}$$

$$=-\frac{\frac{\partial V_p(t_p,t_k)}{\partial C_p}-a_p\cdot\frac{\partial V_k(t_p,t_k)}{\partial C_k}}{-\frac{\partial V_p(t_p,t_k)}{\partial C_p}+a_p\cdot\frac{\partial V_k(t_p,t_k)}{\partial C_k}}=1$$

as a general result. It is then clear that when $\hat{t}_p>0$, equation (15) can be written as

$$\hat{t}_p=t_k+f(a_p,I_p,I_k) \qquad (16)$$

where $f(a_p, I_p, I_k)$ is the function such as

$$\frac{\partial U_p(\hat{t}_p,t_k)}{\partial t_p}=-u_p'(I_p-f(a_p,I_p,I_k))+a_p\cdot u_k'(I_k+f(a_p,I_p,I_k))=0$$

Therefore, the parent's BRF is

$$\boxed{\begin{array}{l}\hat{t}_p=0,\ if\ t_k<-f(a_p,I_p,I_k);\\ \hat{t}_p=t_k+f(a_p,I_p,I_k),\ otherwise\end{array}}$$

Similarly, the kid's BRF is

$$\boxed{\begin{array}{l} \hat{t}_k = 0,\ if\ t_p < -g(a_k, I_p, I_k); \\ \hat{t}_k = t_p + g(a_k, I_p, I_k),\ otherwise \end{array}}$$

where $g(a_k, I_p, I_k)$ is the function

$$\frac{\partial U_k(t_p, \hat{t}_k)}{\partial t_p} = -u_k'(I_k - g(a_k, I_p, I_k)) + a_k \cdot u_p'(I_p + g(a_k, I_p, I_k)) = 0$$

Now, let us discuss how the model parameters $a_p$, $I_p$ and $I_k$ affect the N.E. transfers. When the value of $(\hat{t}_p\text{-}t_k)$ increases, the value of $[I_p\text{-}(\hat{t}_p\text{-}t_k)]$ decreases (i.e., the value of $C_p$ decreases) and the value of $[I_k\text{+}(\hat{t}_p\text{-}t_k)]$ increases (i.e., the value of $C_k$ increases). Since the strictly concave personal utility functions $u_p(C_p)$ and $u_k(C_k)$ have positive first derivatives and negative second derivatives, the value of $u_p'[I_p\text{-}(\hat{t}_p\text{-}t_k)]$ increases while the value of $u_k'[I_k\text{+}(\hat{t}_p\text{-}t_k)]$ decreases. Therefore, from equation (15), we can learn that the value $(\hat{t}_p\text{-}t_k)$ increases when the value of $a_p$ increases. From equation (16), we can learn the values of $f(a_p, I_p, I_k)$ and $(\hat{t}_p\text{-}t_k)$ change in the same direction. In addition, when the value of $I_p$ increases, the value of $[I_p\text{-}(\hat{t}_p\text{-}t_k)]$ increases and the value of $u_p'[I_p\text{-}(\hat{t}_p\text{-}t_k)]$ decreases because of the negative second derivative of the concave utility function. From equation (15), we can learn that $a_p$ increases when $I_p$ increases, so does the value of $f(a_p, I_p, I_k)$. In a word, the value of $(\hat{t}_p\text{-}t_k)$ changes in the same direction as $a_p$ and $I_p$ change. This result can be expressed algebraically as

$$\frac{\partial(t_p * - t_k)}{\partial a_p} = \frac{\partial f(a_p, I_p, I_k)}{\partial a_p} > 0\ and\ \frac{\partial(t_p * - t_k)}{\partial I_p} = \frac{\partial f(a_p, I_p, I_k)}{\partial I_p} > 0$$

Now, let us check how the value of $(\hat{t}_p\text{-}t_k)$ is affected by $I_k$. Taking the total derivative to equation (15), we have

$$da_p \cdot u_k'(I_k - t_k + \hat{t}_p) + d(\hat{t}_p - t_k) \cdot (u_p''(I_p - \hat{t}_p + t_k) + a_p \cdot u_k''(I_k - t_k + \hat{t}_p))$$
$$+ dI_k \cdot a_p \cdot u_k'(I_k - t_k + \hat{t}_p) + dI_p \cdot u_p'(I_p - \hat{t}_p + t_k) = 0 \qquad (17)$$

Because of the positive first derivatives and the negative second derivatives of the personal utility functions of the parent and the kid, with the zero values of the first term and the last term in equation (17), we can get

$$\frac{d(t_p * - t_k)}{dI_k} = \frac{\partial f(a_p, I_p, I_k)}{\partial I_k} < 0$$

which means that the parent's best response of transfer is negatively related to the kid's income. Similarly, we can derive the analogous results for the kid. Therefore, in general, we can conclude that the best response of a person's transfer is positively related to his own degree of altruism and his own income, and is negatively related to the other person's income.

Now set $t_p$=$t_k$=0. Are any transfers seen as desirable? If

$$-u_p'(I_p) + a_p \cdot u_k'(I_k) > 0$$

then the parent would like to make a positive transfer to the kid. This condition needs

$$a_p > \frac{u_p'(I_p)}{u_k'(I_k)}$$

Similarly, if

$$-u_k'(I_k) + a_k \cdot u_p'(I_p) > 0$$

then the kid would like to make a positive transfer to the parent. This condition needs

$$\frac{1}{a_k} < \frac{u_p'(I_p)}{u_k'(I_k)}$$

Then consider the following figure

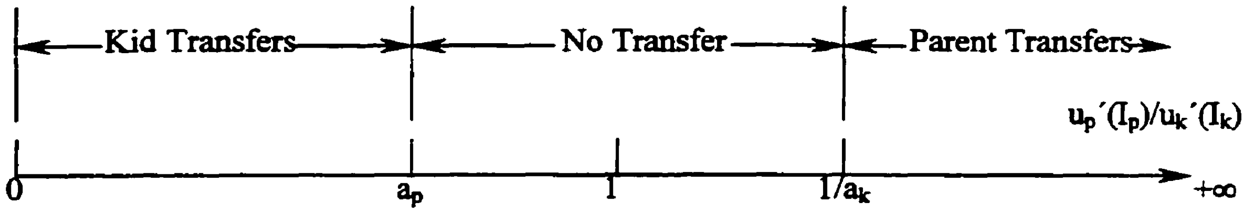


Figure 7

Therefore, when the parent's marginal utility of his own income is low enough relative to the kid's marginal utility of the kid's income, the parent will make a positive transfer to the kid. On the other hand, when the parent's marginal utility of his own income is high enough relative to the kid's marginal utility of the kid's income, the kid will make a positive transfer to the parent. Otherwise, neither of them will transfer to the other person. So far, we have discovered that the intuitive understanding and the main results that we found in the special case still hold in the more general situation.

## III. Summary of Problem 1:

Through the analysis of the special case and the general case in Problem 1 in which the model of reciprocal altruism involves fixed incomes, we can get the following results:

I. When $0 \leq a_i < 1$, ($i=p, k$), there are three kinds of Pareto-optimal N.E. transfers: (i) $t_p^* > 0$ and $t_k^* = 0$, (ii) $t_p^* = 0$ and $t_k^* > 0$, and (iii) $t_p^* = t_k^* = 0$.

II. When $0 \leq a_i < 1$, ($i=p, k$), the positive transfer is not two-sided but one-sided. In other words, in the family with reciprocal altruism, at most one person makes a positive transfer. When the parent's marginal utility of income is low enough relative to the kid's marginal utility of income, the parent will make a positive transfer to the kid but the kid will not transfer anything to the parent. On the contrary, when the parent's marginal utility of income is high enough relative to the kid's marginal utility of income, the parent will not transfer anything to the kid but the kid will make a positive transfer to the parent. Otherwise, neither of them will make a positive transfer to the other person.

III. A person's best response of transfer is related positively to his own income and degree of altruism, while it is related negatively to the other person's income.

IV. From the special case, we learn that the equilibrium personal utility of the person who makes a positive transfer to the other one in the case with reciprocal altruism is less than that in the case without altruism, while the equilibrium personal utility of the person who receives the transfer in the case with reciprocal altruism is greater than that in the case without altruism.

Figure 8 can show the above results from another view. Let us assume that point B is the parent's preferred point while point C is the kid's preferred point. The abscissa of point B expresses the parent's optimal consumption and the ordinate of point C expresses the kid's optimal consumption. Line DE is the endowment line in terms of the parent's and the kid's incomes. Point A, the point where $a_p=a_k$ and $I_p=I_k$, is the intersection between the 45-degree line and the endowment line. When $0\leq a_i<1$, (i=p, k),

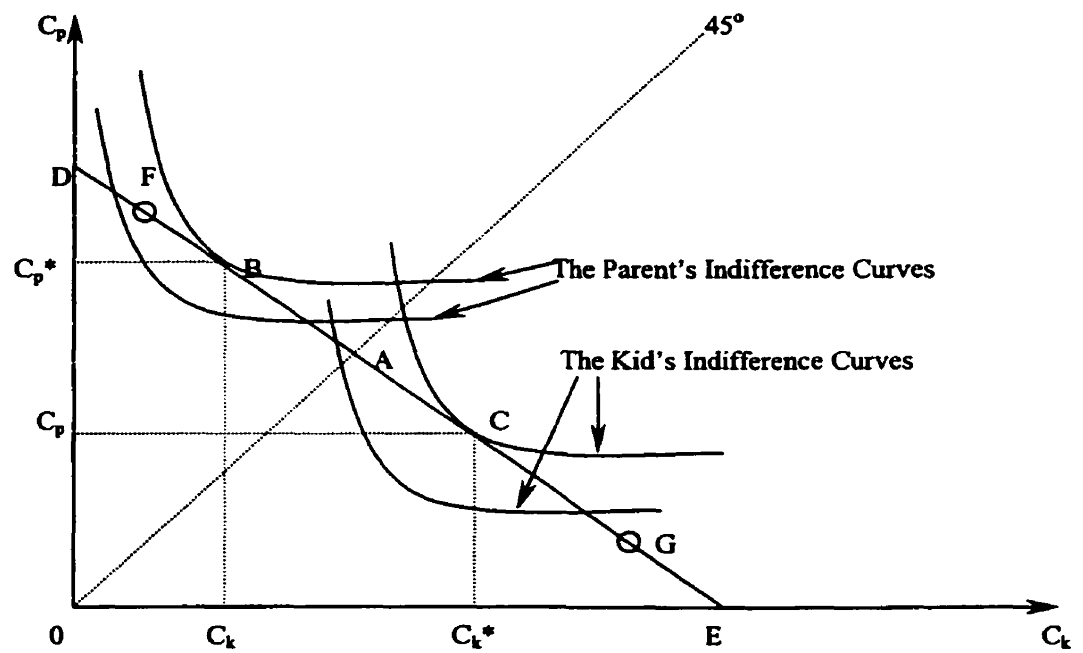

Figure 8 ($0\leq a_i<1$, (i=p, k) )

point B should be to the left of point C and point A, and point C should be to the right of point A. If the ratio of the parent's income to the kid's income is high enough so that the ratio point is on the segment DB such as point F, there will be a positive transfer from the parent to the kid but there will be zero transfer from the kid to the parent, since the parent's preferred point can be over-satisfied but the kid's preferred point cannot be satisfied. On the contrary, if the ratio of the parent's income to the kid's income is low enough so that the ratio point is on the segment CE such as point G, there will be a positive transfer from the kid to the parent, but there will be zero transfer from the parent to the kid. Otherwise, if the ratio point is on segment BC, neither the parent nor the kid will transfer to the other person.

## Section II.2: Problem 2 –Reciprocal Altruism with Incomes Earned in an Impersonal Market

In Problem 2, we calculate and analyze the solutions of the N.E. transfers between the parent and the kid in a reciprocal altruism model with endogenous incomes, earned in an impersonal competitive market with wage rates $W_p$ and $W_k$. In this problem, I do not attempt to derive general results because the general problem is too complex. Define $Z_p$ and $Z_k$ as the work efforts of the parent and the kid respectively. Notice that in this problem, personal utility of person i, $u(C_i, Z_i)$, is related not only to consumption but also to work effort. Then define total utilities of the parent and the kid as

$$U_p(C_p, Z_p, C_k, Z_k) = u(C_p, Z_p) + a_p \cdot u(C_k, Z_k) \quad 0 \le a_p < 1 \quad (18)$$

$$U_k(C_p, Z_p, C_k, Z_k) = u(C_k, Z_k) + a_k \cdot u(C_p, Z_p) \quad 0 \le a_k < 1 \quad (19)$$

where $u(C_p,Z_p)$ and $u(C_k,Z_k)$ are the same weakly concave functions that ensure us to get interior solutions. Then observe that the parent's problem is

$$\max_{C_p, Z_p, t_p} u(C_p, Z_p) + a_p \cdot u(C_k, Z_k), \qquad s.t. \qquad C_p = W_p \cdot Z_p + t_k - t_p$$

Similarly, the kid's problem is

$$\max_{C_k, Z_k, t_k} u(C_k, Z_k) + a_k \cdot u(C_p, Z_p), \qquad s.t. \qquad C_k = W_k \cdot Z_k + t_p - t_k$$

Using the constraints to eliminate $C_p$ and $C_k$ as choice variables, we can simplify the problems. First, define

$$U_p(t_p, Z_p, t_k, Z_k) \equiv u(W_p \cdot Z_p + t_k - t_p) + a_p \cdot u(W_k \cdot Z_k + t_p - t_k)$$

$$U_k(t_p, Z_p, t_k, Z_k) \equiv u(W_k \cdot Z_k + t_p - t_k) + a_k \cdot u(W_p \cdot Z_p - t_p + t_k)$$

We now have a game in which the parent's strategies and payoff function are $t_p$, $Z_p$ and $U_p(t_p, t_k, Z_p, Z_k)$ and the kid's strategies and payoff function are $t_k$ , $Z_k$ and $U_k(t_p, t_k, Z_p, Z_k)$. Denote the Nash Equilibrium (N.E.) of this game by ($t_p$*, $t_k$*, $Z_p$*, $Z_k$*). To further simplify the problem, choose the following personal utility functions named as

$$u_p(C_p, Z_p) = C_p^\alpha - Z_p \;\; and \;\; u_k(C_k, Z_k) = C_k^\alpha - Z_k$$

where $0<\alpha<1$. Logically, there are 16 possibilities for ($t_p^*$, $t_k^*$, $Z_p^*$, $Z_k^*$). As it turns out, only 3 of them are possible: Case (i), $t_p^*>0$, $t_k^*=0$, $Z_p^*>0$ and $Z_k^*=0$; Case (ii), $t_p^*=0$, $t_k^*>0$, $Z_p^*=0$ and $Z_k^*>0$; Case (iii), $t_p^*=0$, $t_k^*=0$, $Z_p^*>0$ and $Z_k^*>0$.

Let us calculate the N.E. and the conditions for each of these possibilities by using the Kuhn-Tucker method. First, look at Case (i). Figure 9(a) and 9(b) depict transfers of the parent and the kid, and Figure 9(c) and 9(d) depict their work efforts respectively. Algebraically,

$$\frac{\partial U_p(Z_p{*} > 0, Z_k{*} = 0, t_p{*} > 0, t_k{*} = 0)}{\partial Z_p} = \alpha(W_p Z_p - t_p + t_k)^{\alpha-1} \cdot W_p - 1 = 0$$

$$\Rightarrow W_p \cdot Z_p^* - t_p^* = (\alpha \cdot W_p)^{\frac{1}{1-\alpha}} \qquad (20)$$

$$\frac{\partial U_p(Z_p{*}>0, Z_k{*}=0, t_p{*}>0, t_k{*}=0)}{\partial t_p} = -\alpha(W_p Z_p - t_p + t_k)^{\alpha-1} + \alpha \cdot a_p \cdot (W_k Z_k - t_k + t_p)^{\alpha-1} = 0$$

$$\Rightarrow \frac{W_p \cdot Z_p^{*} - t_p^{*}}{t_p^{*}} = A_p \qquad (21)$$

$$\frac{\partial U_k(Z_p{*}>0, Z_k{*}=0, t_p{*}>0, t_k{*}=0)}{\partial Z_k} = \alpha(W_k Z_k - t_k + t_p)^{\alpha-1} \cdot W_k - 1 < 0$$

$$\Rightarrow t_p^{*} < (\alpha \cdot W_k)^{\frac{1}{1-\alpha}} \qquad (22)$$

$$\frac{\partial U_k(Z_p{*}>0, Z_k{*}=0, t_p{*}>0, t_k{*}=0)}{\partial t_k} = -\alpha(W_k Z_k - t_k + t_p)^{\alpha-1} + \alpha \cdot a_k \cdot (W_p Z_p - t_p + t_k)^{\alpha-1} < 0$$

$$\Rightarrow \frac{W_p \cdot Z_p^{*} - t_p^{*}}{t_p^{*}} < 1/A_k \qquad (23)$$

$$\textit{where } A_p = a_p^{\frac{1}{\alpha-1}} \textit{ and } A_k = a_k^{\frac{1}{\alpha-1}}$$

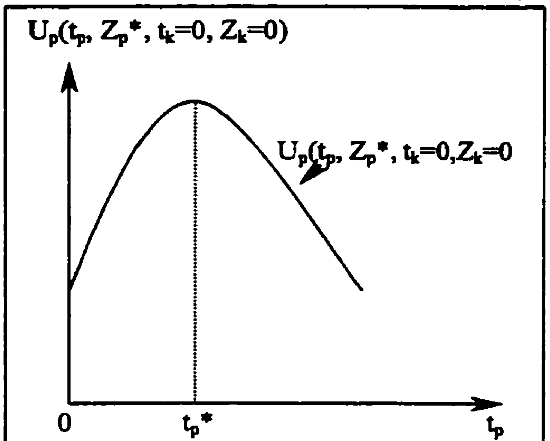

Figure 9(a)

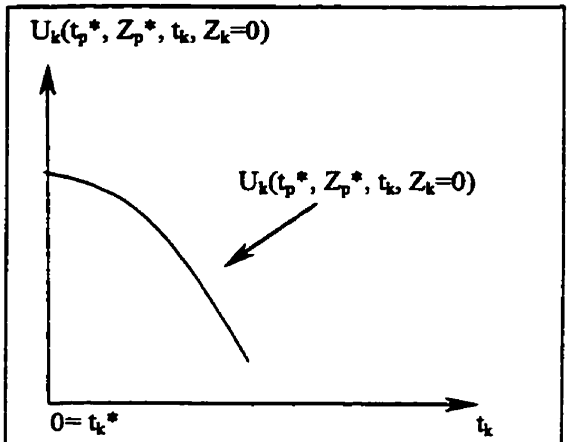

Figure 9(b)

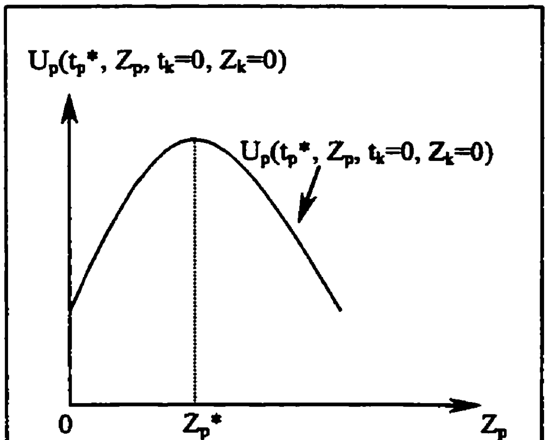

Figure 9(c)

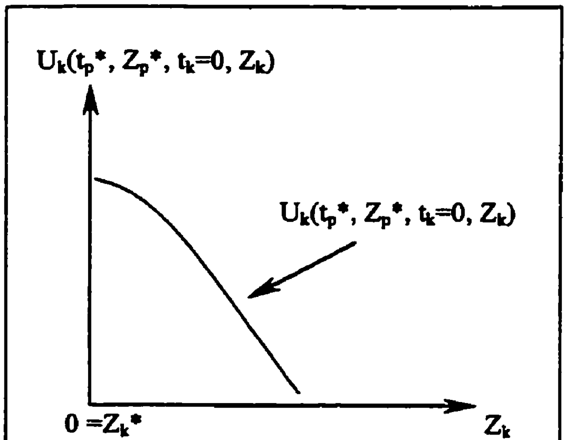

Figure 9(d)

Then deriving from the first-order conditions for $t_p^*$ and $Z_p^*$, and from parameter restrictions (22) and (23), we can get the result of case (i) as

$$when\ \frac{W_p}{W_k} > A_p,$$

$$t_p^* = \frac{(\alpha \cdot W_p)^{\frac{1}{1-\alpha}}}{A_p},\ t_k^* = 0,\ Z_p^* = \alpha^{\frac{1}{1-\alpha}} \cdot W_p^{\frac{\alpha}{1-\alpha}} \cdot \frac{1+A_p}{A_p}\ and\ Z_k^* = 0,$$

$$u_p^* = \alpha^{\frac{\alpha}{1-\alpha}} \cdot W_p^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}} \cdot W_p^{\frac{\alpha}{1-\alpha}} \cdot (1+\frac{1}{A_p}) \tag{24}$$

$$and\ u_k^* = \alpha^{\frac{\alpha}{1-\alpha}} \cdot W_p^{\frac{\alpha}{1-\alpha}} \cdot a_p^{\frac{\alpha}{1-\alpha}} \tag{25}$$

$$where\ A_p = a_p^{\frac{1}{\alpha-1}}$$

The result of Case (i) means that when the parent's wage rate is high enough relative to the kid's wage rate, the parent will make a positive transfer to the kid but kid will not work or transfer any more.

Case (ii) is analogous, but roles are reversed. The result of Case (ii) is

$$when\ \frac{W_p}{W_k} < \frac{1}{A_k},$$

$$t_p^* = 0,\ t_k^* = \frac{(\alpha \cdot W_k)^{\frac{1}{1-\alpha}}}{A_k},\ Z_p^* = 0\ and\ Z_k^* = \alpha^{\frac{1}{1-\alpha}} \cdot W_k^{\frac{\alpha}{1-\alpha}} \cdot \frac{1+A_k}{A_k},$$

$$u_p^* = \alpha^{\frac{\alpha}{1-\alpha}} \cdot W_k^{\frac{\alpha}{1-\alpha}} \cdot a_k^{\frac{\alpha}{1-\alpha}}\ and\ u_k^* = \alpha^{\frac{\alpha}{1-\alpha}} \cdot W_k^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}} \cdot W_k^{\frac{\alpha}{1-\alpha}} \cdot (1+\frac{1}{A_k}),\ where\ A_k = a_k^{\frac{1}{\alpha-1}}$$

Now, let us look at Case (iii). Figure 10(a) and 10(b) depict transfers of the parent and the kid, and Figure 10(c) and 10(d) depict their work efforts respectively. Algebraically,

$$\frac{\partial U_p(Z_p^* > 0, Z_k^* > 0, t_p^* = 0, t_k^* = 0)}{\partial Z_p} = \alpha(W_p Z_p - t_p + t_k)^{\alpha-1} \cdot W_p - 1 = 0$$

$$\Rightarrow Z_p^* = \alpha^{\frac{1}{1-\alpha}} \cdot W_p^{\frac{\alpha}{1-\alpha}} \quad (26)$$

$$\frac{\partial U_p(Z_p^* > 0, Z_k^* > 0, t_p^* = 0, t_k^* = 0)}{\partial t_p} = -\alpha(W_p Z_p - t_p + t_k)^{\alpha-1} + \alpha \cdot a_p \cdot (W_k Z_k - t_k + t_p)^{\alpha-1} < 0$$

$$\Rightarrow \frac{W_p \cdot Z_p^*}{W_k \cdot Z_k^*} \leq A_p \quad (27)$$

$$\frac{\partial U_k(Z_p^* > 0, Z_k^* > 0, t_p^* = 0, t_k^* = 0)}{\partial Z_k} = \alpha(W_k Z_k - t_k + t_p)^{\alpha-1} \cdot W_k - 1 = 0$$

$$\Rightarrow Z_k^* = \alpha^{\frac{1}{1-\alpha}} \cdot W_k^{\frac{\alpha}{1-\alpha}} \quad (28)$$

$$\frac{\partial U_k(Z_p^* > 0, Z_k^* > 0, t_p^* = 0, t_k^* = 0)}{\partial t_k} = -\alpha(W_k Z_k - t_k + t_p)^{\alpha-1} + \alpha \cdot a_k \cdot (W_p Z_p - t_p + t_k)^{\alpha-1} < 0$$

$$\Rightarrow \frac{W_p \cdot Z_p^*}{W_k \cdot Z_k^*} \leq A_k \quad (29)$$

$$\textit{where } A_p = a_p^{\frac{1}{\alpha-1}} \textit{ and } A_k = a_k^{\frac{1}{\alpha-1}}$$

Then deriving from these first-order conditions for $t_p$* and $t_k$*, and from the parameter restrictions (27) and (29), we can get the result of case (iii) as

$$\textit{When } \frac{1}{A_k} \leq \frac{W_p}{W_k} \leq A_p,$$

$$t_p^* = t_k^* = 0,\ Z_p^* = \alpha^{\frac{1}{1-\alpha}} \cdot W_p^{\frac{\alpha}{1-\alpha}}\ \ Z_k^* = \alpha^{\frac{1}{1-\alpha}} \cdot W_k^{\frac{\alpha}{1-\alpha}},$$

$$u_p^* = \alpha^{\frac{\alpha}{1-\alpha}} \cdot W_p^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}} \cdot W_p^{\frac{\alpha}{1-\alpha}} \textit{ and } u_k^* = \alpha^{\frac{\alpha}{1-\alpha}} \cdot W_k^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}} \cdot W_k^{\frac{\alpha}{1-\alpha}}$$

$$\textit{where } A_p = a_p^{\frac{1}{\alpha-1}} \textit{ and } A_k = a_k^{\frac{1}{\alpha-1}}$$

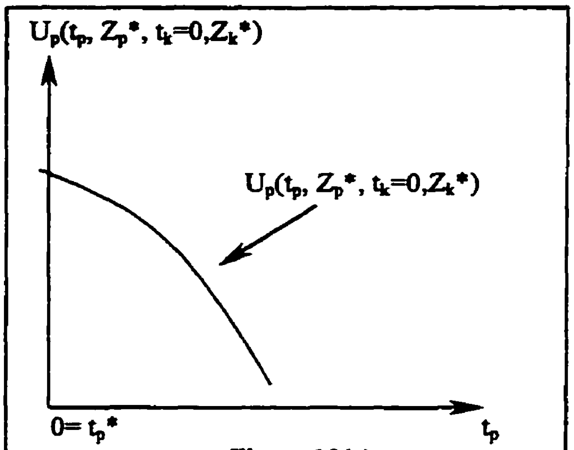


Figure 10(a)

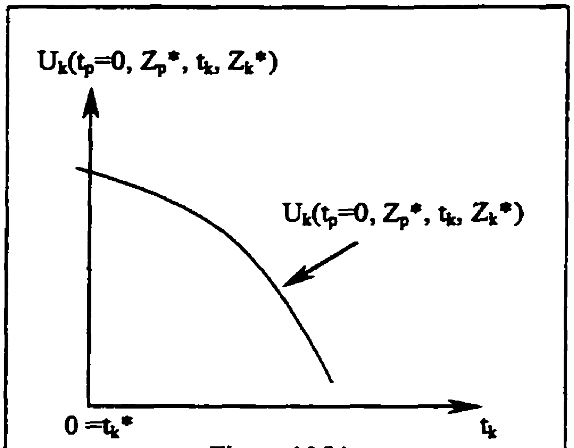


Figure 10(b)

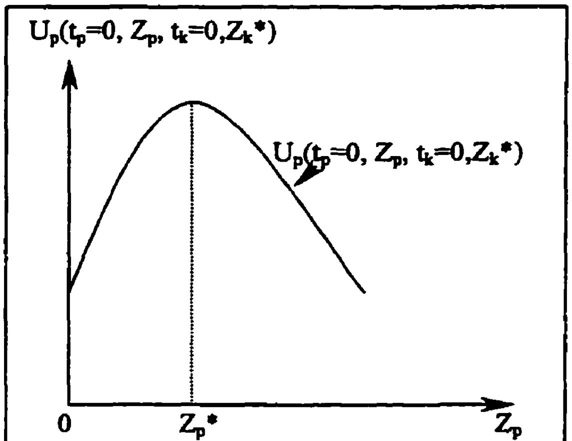


Figure 10(c)

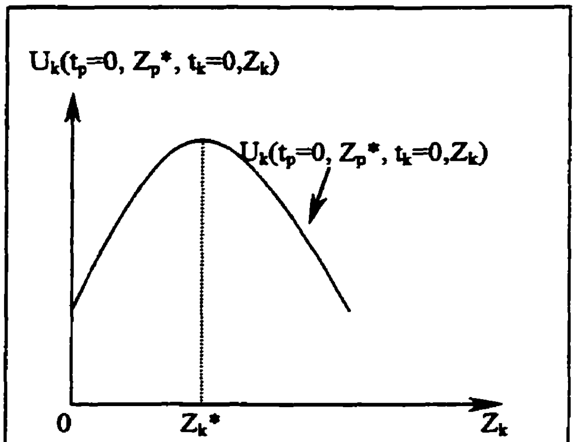


Figure 10(d)

The results of the above three cases can be depicted in $W_p$-$W_k$ space in Figure 11. The difference between the results in this special case and the results in Problem 1 is that the transfers in this case between the parent and the kid are related to their wage rates, while the transfers in Problem 1 are related to their incomes. From the results in the above three cases, we know that the parent's work effort in Case (i) is greater than his work effort in Case (iii), and that the kid's work effort in Case (ii) is greater than his work effort in Case (iii). Notice that when $a_p$ decreases, the value of $A_P$ increases, and when $a_k$ decreases, the value of $1/A_k$ decreases. Therefore, if the values of $a_p$ and $a_k$ decrease, the critical value line OA will rotate to axis $W_p$ and the critical value line OB will rotate to axis $W_k$. When

$a_p$=0 and $a_k$=0, OA will be coincident with axis $W_p$ and OB will be coincident with axis $W_k$. At that time, the whole $W_p$-$W_k$ space will illustrate Case (iii). Therefore, the work effort of the person who makes a positive transfer is greater than that in the case without transfers or without altruism.

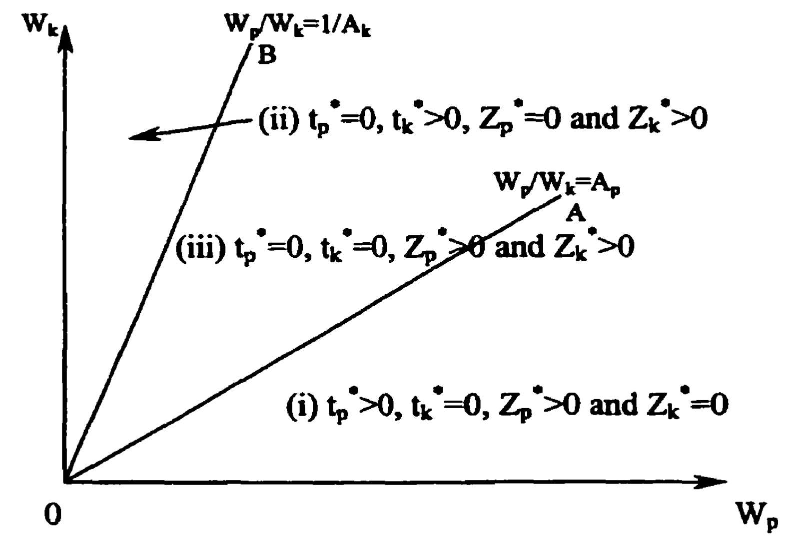


Figure 11

To see the implications of altruism, it is necessary to compare equilibrium values of the endogenous variables when there is altruism with corresponding equilibrium values with when there is none. Observe that when $a_i$=0, (i=p,k), and then $A_i$=0, (i=p,k). Denote $(u_p^*)'$ and $(u_k^*)'$ as the equilibrium personal utilities of the parent and the kid in the case without altruism. Setting $a_p$=$a_k$=0, we get the following equilibrium values of the endogenous variables for the case with no altruism:

$$(u_p^*)' = \alpha^{\frac{\alpha}{1-\alpha}} \cdot W_p^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}} \cdot W_p^{\frac{\alpha}{1-\alpha}} \quad and \quad (u_k^*)' = \alpha^{\frac{\alpha}{1-\alpha}} \cdot W_k^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}} \cdot W_k^{\frac{\alpha}{1-\alpha}}$$

Look at Case (i) for example. Comparing $u_p^*$ and $u_k^*$ in equations (24) and (25) with $(u_p^*)'$ and $(u_k^*)'$ respectively, it is easy to show $(u_p^*)'>u_p^*$, that is when the parent makes a positive transfer to the kid, the parent's personal utility is smaller than that if there is no transfer, although it is more difficult to show $(u_k^*)'<u_k^*$, that is when the parent transfers to the kid, the kid is better off. The detailed proof of this comparison is in Appendix I. Case (ii) is analogous. Therefore, the equilibrium personal utility of the person who makes a positive transfer is less than that in the case without altruism, while the equilibrium personal utility of the person who receives the transfer is greater than that in the case without altruism.

## Section II.3: Summary of Problem 1 and Problem 2:

In the analysis in Problem 1 and Problem 2, we have discussed the model of reciprocal altruism in the family that does not own a family firm. Problem 1 analyzes the model of reciprocal altruism with exogenous incomes, while Problem 2 analyzes the model of reciprocal altruism with endogenous incomes. Two main results can be summarized.

First, the positive transfer is not two-sided but one-sided. In other words, in the family with reciprocal altruism, at most one person makes a positive transfer. When $0 \leq a_i < 1$, ($i=p$, $k$), there are three kinds of N.E. transfers: (i) $t_p^* > 0$ and $t_k^* = 0$, (ii) $t_p^* = 0$ and $t_k^* > 0$ and (iii) $t_p^* = t_k^* = 0$, depending on the different values of model parameters and the different relationship among them. All of these three Nash Equilibriums are Pareto-optimal. The factors that determine the amount of the transfer and the person who does the positive transfer are the model parameters: the parameters related to personal utility functions, the marginal utilities of incomes (i.e., incomes or wage rates) and the degrees of altruism of the parent and the kid. According to the analysis, we have found that a person's best response of transfer is related positively to his own income (or wage rate) and his own degree of altruism, while it is related negatively to the other person's income (or wage rate).

Second, reciprocal altruism is not beneficial for the family that does not own a family firm. It only induces the party that makes a positive transfer to work harder, but makes the party that receives the transfer give up working. In fact, it could even lead the altruism among the family members to disappear completely. Moreover, in the family whose reciprocally altruistic members work in an impersonal market, the party that makes a positive transfer is worse off in term of personal utility, while the party that receives the transfer is better off.

# Chapter III: The Application of Reciprocal Altruism in Family Business

Recently, a lot of research on family business has been done by economists, since its importance is becoming increasingly obvious in the field of national economy of many countries, especially in the United States. Much of this research relates altruism to the family business in order to discuss the behavior of the family members. The Rotten-Kid Theorem proposed in Becker (1974) is the embryonic form of the analysis of altruism in the family business. Through the empirical research, Sharma, Chrisman and Chua (1996) estimate that the number of family firms in the United States may range from 4.1 million to 20.3 million firms, employ 19.8 million to 77.2 million individuals, and provide 12 percent to 49 percent of the GDP of the United.

After analyzing the model of reciprocal altruism in theory, what I am going to do in this chapter is to employ the results from Chapter II into the family business situation to analyze and solve the real problems in the economy. For convenient analysis, an introduction to the literature on the family business is necessary.

## Section III.1: Literature on the Family Business

There are more than 30 versions of the definition of family business in the history of papers in this area. According to Handler (1989), the definition of family business should involve four dimensions including the degree of ownership and management by family members, interdependent sub-systems, generational transfers and multiple conditions. However, most of them involve the first three dimensions above. Babicky (1987) and Leach, et. al. (1990) focused on the ownership and management by family members, Beckhard and Dyer (1983) and Davis (1983) focused on the interdependent sub-systems, Churchill and Hatten (1987) and Ward (1987) focused on the dimension of generational transfers, and Astrachman and Kolenko (1994) and Shanker and Astrachan (1995) focused on the last dimension in Handler (1989), the multiple conditions.

In the view of modern economists, the definition focusing on the ownership aspect is only a local definition. As a global definition of the family business, it should include more elements besides ownership, such as management, the influence among the family members or the influence coming from outside of family, and so on. At the same time, the intergenerational family business focuses on the influence of the management coming from different generations. On the basis of all of the above analysis, in my opinion, the family business can be defined as follows: A family business is an enterprise or an enterprise group which is started and developed by one or a few individuals and which is owned and controlled by a single family or a single family group that employs two or more family members, is an enterprise or an enterprise group which will be passed on to the next generations to own and manage, is an enterprise or an enterprise group which includes unique interdependent subsystems, and is an enterprise or an enterprise group which may be influenced by factors coming from outside of family.

Some empirical research has also been done in the area of family business. For instance, the empirical tests in Chua, Chrisman and Sharma (1999) show that the components of family involvement (categorized in terms of ownership and management) are very weak predictors of family firms' concerns over succession and professionalization (Chua, Chrisman and Sharma, 1999). The results of their empirical tests strengthen their contention that vision, intentions, and behavior are what should be used to distinguish family business from all others. Their definition of family business focuses on the vision held for the firm by a family or a small group of families and the intention of the dominant condition to shape and pursue this vision, potentially across generations of the same family or small group of families (Chua, Chrisman and Sharma, 1999).

In the view of the purpose of the family firm, actually, family and business somewhat clash. Family is a group whose members have clan or marriage relationships and is a group whose members are willing to live together or to hold closed relationships. The goal of the family is that the family members can care for each other and induce the development of the whole family, while the goal of the business is making profits. If you emphasize the altruistic behavior in the family, it may cause the failure of the business. If you, however, concentrate on the economic benefits, it may cause the disintegration of the family. In a word, altruism may not be consistent with profits. For example, although employing family members may lead to sub-optimal results or may be bad for the operation and development of the family firm, a number of firms are still willing to do this to maintain and develop good relationships among the members of the family. After asking 624 participants in Harvard's Smaller Company Management Program to rate the importance of seventy-four goals articulated by a hundred family firm managers for their firms, Tagiuri and Davis (1992) found that having a company where employees can be happy, productive and proud is the first of six most important purposes of the family business.

Succession is another important factor of the family business. It is the extension and the continuity of the family business. In reality however, only 20 percent of the family businesses in the United States can go through the process of the transition successfully. Thus, the transfer of a family firm has been an important topic fócused by modern economists. In Birley (1986), he estimated that only 30 percent of the family business in the United States would continue into the second generation and only 15 percent into the third. There are many reasons, especially internal reasons, that lead to the failure in succession. These internal reasons include the stress and strain created by combining family and firm (Rosenblatt, et. al., 1985), relationships between consecutive generations, educational situations of the family members, occupations of the family members, income distribution among the family members, relationships among the family members, culture in the family business, organizational structure of the family business, the type of family and the type of family business. It is no wonder that income distribution based on altruism is one of the most important factors. In family businesses throughout the world, the biggest scale of intergenerational replacement in the history of enterprise is coming. In just the United States, 43 per cent of family businesses will change the head before the end of 2002, since the entrepreneurs who have created the current wealth of the world are close to the age of retirement. At the cross of the road of development of the family businesses, succeeding in relief of the shift of the head will be very important to the development of the enterprises. Therefore, how reciprocal altruism in the family can affect the achievement of the family business looks to be a critical and realistic problem.

There are not only advantages but also some disadvantages in the family business. Let us begin by looking at the advantages. First, it's easier to appear some people who care about others' welfare in the family business. Second, it's relatively easy for all of the shareholders to take actions in unison in the family business relative to non-family business, and its administration is relatively easy because all of the important members of the company come from the same family. Third, relatives may have more than average commitment and interest in the long-term growth of the company, and employing relatives assures continuity and effective carrying out of important policies (Ewing, 1965).

Now, let us look at the disadvantages. First, due to altruism, punishment of the cheating actions of the selfish persons may not be effective, even if those actions hurt the welfare of the whole family business. Second, relatives may be difficult to fire if they prove incapable, and relatives may create doubt about the integrity and objectivity of top management (Ewing, 1965). Third, it may sow the seeds of future dissension as the "closeness" in family relationships dissipates over the generations. Fourth, should the business falter or fail and the shareholders lose money, personal and family relationships might be strained.

As stated above, some economists have done some research, especially the empirical research, in the area of family business. However, since theoretical research has not been done a lot in this area, there is very little theory in the family business. I am interested in doing some analytical research and trying to provide some theory by using the model of reciprocal altruism in the family business. In the literature on the family business, there seems to be a feeling that the family gets in the way, that is, family ties make it difficult to actually run a firm. Is this true? To see the answer clearly, let us construct a special model of family business along with team production in the next two sections. I am going to do two things. First, in Section III.2, I consider a model with a share contract but no transfer, and compare the N.E. work efforts and equilibrium personal utilities between the models with altruism and without altruism respectively. Second, in Section III.3, I add transfers to the model and make the same comparison.

## Section III.2: The Case with a Share Contract but No Transfers

Now, let us still consider the special case in Section II.2 in which the parent and the kid have the same weakly concave personal utility functions named as

$$u_p(C_p, Z_p) = C_p^{\alpha} - Z_p \ \ and \ \ u_k(C_k, Z_k) = C_k^{\alpha} - Z_k$$

where $0<\alpha<1$. However, what happens if the family owns a family firm? As per the example used in Alchian and Demsetz (1972), the firm is described by an income production function as:

$$I(Z_p, Z_k) = Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}}, \ if \ Z_k > 0 \ and \ Z_p > 0$$

$$I(Z_p, Z_k) = 0, \ otherwise$$

In addition, let the parent's share of the firm's income be $s_p$ and the kid's share be $s_k$. Notice that sum of $s_p$ and $s_k$ is equal to 1. When there is no transfer, we observe that the parent's problem is

$$\max_{C_p, Z_p} u(C_p, Z_p) + a_p \cdot u(C_k, Z_k), \qquad s.t. \ \ C_p = s_p \cdot I(Z_p, Z_k) = s_p \cdot Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}}$$

Similarly, the kid's problem is

$$\max_{C_k, Z_k} u(C_k, Z_k) + a_k \cdot u(C_p, Z_p), \qquad s.t. \ \ C_k = s_k \cdot I(Z_p, Z_k) = s_k \cdot Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}}$$

Using the constraints to eliminate $C_p$ and $C_k$ as choice variables, we can simplify the problems to be

$$\max_{Z_p} U_p(Z_p, Z_k) = [(s_p \cdot Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}})^{\alpha} - Z_p] + a_p \cdot [(s_k \cdot Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}})^{\alpha} - Z_k] \quad (30)$$

$$\max_{Z_k} U_k(Z_p, Z_k) = [(s_k \cdot Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}})^{\alpha} - Z_k] + a_k \cdot [(s_p \cdot Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}})^{\alpha} - Z_p] \quad (31)$$

We now have a game in which the parent's strategy and payoff function are $Z_p$ and $U_p(Z_p, Z_k)$ and the kid's strategy and payoff function are $Z_k$ and $U_k(Z_p, Z_k)$. Denote the Nash Equilibrium (N.E.) of this game by ($Z_p$*, $Z_k$*). As it turns out, there are only two possibilities for ($Z_p$*, $Z_k$*) calling: (i) $Z_p^*>0$ and $Z_k^*>0$ and (ii) $Z_p^*=0$ and $Z_k^*=0$. Let us find the best response functions (BRFs) for this special case first. Look at the first derivative of the parent's total utility with respect to his transfer

$$\frac{\partial U_p(\hat{Z}_p, Z_k)}{\partial t_p} = \frac{\alpha}{2} \cdot (\hat{Z}_p)^{\frac{\alpha-2}{2}} \cdot Z_k^{\frac{\alpha}{2}} \cdot (s_p^{\alpha} + a_p \cdot s_k^{\alpha}) - 1 = 0 \quad (32)$$

We imagine that when $Z_k=0$, the derivative is equal to –1 which is negative. This leads to $\hat{Z}_p=0$. Therefore, the parent's best response is

$$\hat{Z}_p = 0 \; if \; Z_k = 0$$

On the other hand, if $Z_k>0$ and then $\hat{Z}_p>0$, then the parent's best response satisfies equation (32). These observations yield the parent's BRF to be

$$\boxed{\begin{array}{l} \hat{Z}_p = (\frac{\alpha}{2})^{\frac{2}{2-\alpha}} \cdot Z_k^{\frac{\alpha}{2-\alpha}} \cdot (s_p^{\alpha} + a_p \cdot s_k^{\alpha})^{\frac{2}{2-\alpha}} \quad (33), \; if \; Z_k > 0 \\ \hat{Z}_p = 0, \; otherwise \; (i.e., \; Z_k = 0) \end{array}}$$

Similarly, the kid's BRF is

$$\boxed{\begin{array}{l}\hat{Z}_k = (\frac{\alpha}{2})^{\frac{2}{2-\alpha}} \cdot Z_p^{\frac{\alpha}{2-\alpha}} \cdot (s_k^\alpha + a_k \cdot s_p^\alpha)^{\frac{2}{2-\alpha}} \quad (34),\ if\ Z_p > 0 \\ \hat{Z}_k = 0,\ otherwise\ (i.e.,\ Z_p = 0)\end{array}}$$

Solving equations (33) and (34), we can derive their N.E. work efforts in Case (i) as

$$\boxed{\begin{array}{l}Z_p^* = (\frac{\alpha}{2})^{\frac{1}{1-\alpha}} \cdot (s_k^\alpha + a_k \cdot s_p^\alpha)^{\frac{\alpha}{2-2\alpha}} \cdot (s_p^\alpha + a_p \cdot s_k^\alpha)^{\frac{2-\alpha}{2-2\alpha}} \quad (35) \\ Z_k^* = (\frac{\alpha}{2})^{\frac{1}{1-\alpha}} \cdot (s_p^\alpha + a_p \cdot s_k^\alpha)^{\frac{\alpha}{2-2\alpha}} \cdot (s_k^\alpha + a_k \cdot s_p^\alpha)^{\frac{2-\alpha}{2-2\alpha}} \quad (36)\end{array}}$$

Their equilibrium personal utilities are

$$\boxed{\begin{array}{l}u_p^* = (\frac{\alpha}{2})^{\frac{\alpha}{1-\alpha}} \cdot (s_k^\alpha + a_k \cdot s_p^\alpha)^{\frac{\alpha}{2-2\alpha}} \cdot (s_p^\alpha + a_p \cdot s_k^\alpha)^{\frac{2-\alpha}{2-2\alpha}} \cdot (\frac{s_p^\alpha}{s_p^\alpha + a_p \cdot s_k^\alpha} - \frac{\alpha}{2}) \quad (37) \\ u_k^* = (\frac{\alpha}{2})^{\frac{\alpha}{1-\alpha}} \cdot (s_k^\alpha + a_k \cdot s_p^\alpha)^{\frac{2-\alpha}{2-2\alpha}} \cdot (s_p^\alpha + a_p \cdot s_k^\alpha)^{\frac{\alpha}{2-2\alpha}} \cdot (\frac{s_k^\alpha}{s_k^\alpha + a_k \cdot s_p^\alpha} - \frac{\alpha}{2}) \quad (38)\end{array}}$$

Their N.E. work efforts in Case (ii) are

$$\boxed{Z_p^* = 0\ and\ Z_k^* = 0}$$

Actually, this N.E. is irrelevant since it gives zero utilities to both players. It is dominated by the N.E. in Case (i) because the N.E. in Case (i) gives both players positive utilities when

$$\frac{s_p^\alpha}{s_p^\alpha + a_p \cdot s_k^\alpha} > \frac{\alpha}{2}\ and\ \frac{s_k^\alpha}{s_k^\alpha + a_k \cdot s_p^\alpha} > \frac{\alpha}{2}$$

Hence, the N.E. $Z_p^* = Z_k^* = 0$ should be ignored.

Denote $(Z_p^*)'$ and $(Z_k^*)'$ as the N.E. work efforts of the parent and the kid in the case without altruism. By setting $a_p=a_k=0$, we get the work efforts and equilibrium personal utilities of the parent and the kid in the case without altruism from equations (35) – (38) as

$$(Z_p^*)' = (\frac{\alpha}{2})^{\frac{1}{1-\alpha}} \cdot s_k^{\frac{\alpha^2}{2-2\alpha}} \cdot s_p^{\frac{\alpha(2-\alpha)}{2-2\alpha}} \qquad (39)$$

$$(Z_k^*)' = (\frac{\alpha}{2})^{\frac{1}{1-\alpha}} \cdot s_p^{\frac{\alpha^2}{2-2\alpha}} \cdot s_k^{\frac{\alpha(2-\alpha)}{2-2\alpha}} \qquad (40)$$

$$(u_p^*)' = (\frac{\alpha}{2})^{\frac{\alpha}{1-\alpha}} \cdot s_k^{\frac{\alpha^2}{2-2\alpha}} \cdot s_p^{\frac{\alpha(2-\alpha)}{2-2\alpha}} \cdot (1-\frac{\alpha}{2}) \qquad (41)$$

$$(u_k^*)' = (\frac{\alpha}{2})^{\frac{\alpha}{1-\alpha}} \cdot s_p^{\frac{\alpha^2}{2-2\alpha}} \cdot s_k^{\frac{\alpha(2-\alpha)}{2-2\alpha}} \cdot (1-\frac{\alpha}{2}) \qquad (42)$$

Comparing equation (35) with equation (39) and comparing equation (36) with equation (40), we observe that both the parent and the kid in the family firm work harder if they are altruistic than they would if they are not. Therefore, reciprocal altruism can induce all the family members in the family owning a family firm to work harder. However, do they necessarily both get larger personal utilities when they are altruistic? From the expressions for their equilibrium personal utilities, we observe that the comparison is not straight forward, and therefore I use numerical methods instead of mathematical means. Intuitively, we might expect that both parties will be better off when they are altruistic if their altruism is balanced (that is, when $a_p$ and $a_k$ are approximately equal), and that the less altruistic party will be better off and more altruistic party will be worse off when their altruism is unbalanced. I use numerical methods to examine this intuition. In particular, I make use of Mathematica 4. Let us look at the example in which $s_p$=0.5 and $\alpha$=0.2 are fixed. With different $a_p$, the boundaries of $a_k$ that ensures both of them to be better off are different, and vice versa. These sets of ($a_p$, $a_k$) can be illustrated by Figure 12. After connecting the points with curves, we observe that in the blank area, both parties are better off, and that in the shaded area, less altruistic party is better off and more altruistic party is worse off. Since this model is symmetric, the situation of the model is analogous to this example. The details of the numerical calculation and the results are attached in Appendix II.

Recall that in last chapter, reciprocal altruism never makes both better off in terms of personal utilities. In the context of family business, however, it makes both better off and makes the firm more efficient if their altruism is balanced.

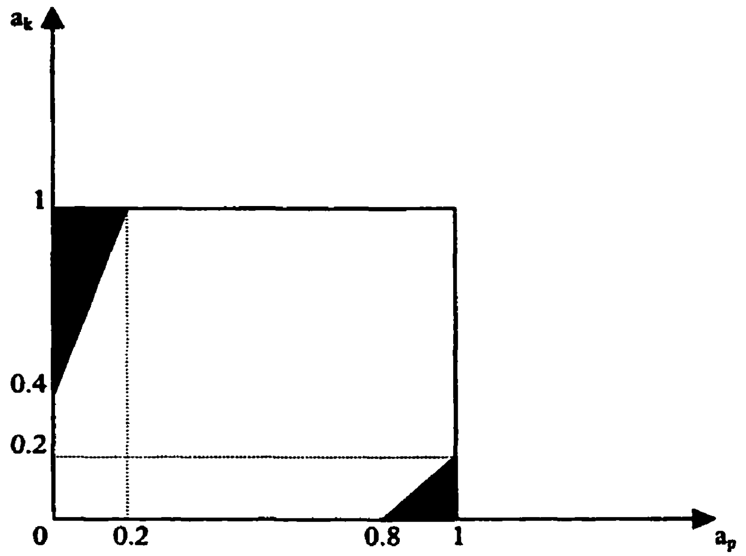


Figure 12

## Section III.3: The Case with Share Contract and Transfers

Now, let us add transfer into the model of the previous section. Then observe that the parent's problem is

$$\max_{C_p, Z_p, t_p} \quad u(C_p, Z_p) + a_p \cdot u(C_k, Z_k), \quad s.t.\ C_p = s_p \cdot I(Z_p, Z_k) + t_k - t_p = s_p \cdot Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}} + t_k - t_p$$

Similarly, the kid's problem is

$$\max_{C_k, Z_k, t_k} \quad u(C_k, Z_k) + a_k \cdot u(C_p, Z_p), \quad s.t.\ C_k = s \ \cdot I(Z_p, Z_k) - t_k + t_p = s_k \cdot Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}} - t_k + t_p$$

Using the constraints to eliminate $C_p$ and $C_k$ as choice variables, we can simplify the problems to be

$$\max_{t_p, Z_p} U_p(Z_p, Z_k, t_p, t_k) = [(s_p \cdot Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}} + t_k - t_p)^{\alpha} - Z_p] + a_p \cdot [(s_k \cdot Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}} + t_p - t_k)^{\alpha} - Z_k] \quad (43)$$

$$\max_{t_k, Z_k} U_k(Z_p, Z_k, t_p, t_k) = [(s_k \cdot Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}} + t_p - t_k)^{\alpha} - Z_k] + a_k \cdot [(s_p \cdot Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}} + t_k - t_p)^{\alpha} - Z_p] \quad (44)$$

We now have a game in which the parent's strategies and payoff function are $t_p$, $Z_p$ and $U_p(t_p, t_k, Z_p, Z_k)$ and the kid's strategies and payoff function are $t_k$ , $Z_k$ and $U_k(t_p, t_k, Z_p, Z_k)$. Denote the Nash Equilibrium of this game by $(t_p^*, t_k^*, Z_p^*, Z_k^*)$. Logically, there are 16 possibilities for ($t_p^*$, $t_k^*$, $Z_p^*$ and $Z_k^*$). As it turns out, only 4 of them are possible calling: Case (i), $t_p^*>0$, $t_k^*=0$, $Z_p^*>0$ and $Z_k^*>0$; Case (ii), $t_p^*=0$, $t_k^*>0$, $Z_p^*>0$ and $Z_k^*>0$;

Case (iii), $t_p^*=0$, $t_k^*=0$, $Z_p^*>0$ and $Z_k^*>0$; Case (iv), $t_p^*=t_k^*=0$ and $Z_p^*=Z_k^*=0$. As before, however, zero-effort equilibrium is Pareto-dominated and should be ignored.

Let us calculate the first three N.E.s and their conditions by again using the Kuhn-Tucker method. First, look at Case (i). Figure 13(a) and 13(b) depict transfers of the parent and the kid, and Figure 13(c) and 13(d) depict their work efforts respectively. Algebraically,

$$\frac{\partial U_p(Z_p^*>0, Z_k^*>0, t_p^*>0, t_k^*=0)}{\partial Z_p} = \frac{1}{2}\alpha \cdot s_p \cdot Z_p^{-\frac{1}{2}} \cdot Z_k^{\frac{1}{2}} \cdot (s_p \cdot Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}} - t_p + t_k)^{\alpha-1} - 1$$
$$+\frac{1}{2}\alpha \cdot a_p \cdot s_k \cdot Z_p^{-\frac{1}{2}} \cdot Z_k^{\frac{1}{2}} \cdot (s_k \cdot Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}} + t_p - t_k)^{\alpha-1} = 0$$
$$\Rightarrow s_p \cdot (s_p \cdot Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}} - t_p)^{\alpha-1} + a_p \cdot s_k \cdot (s_k \cdot Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}} + t_p)^{\alpha-1} = \frac{2}{\alpha} \cdot (\frac{Z_p}{Z_k})^{\frac{1}{2}} \quad (45)$$

$$\frac{\partial U_p(Z_p^*>0, Z_k^*>0, t_p^*>0, t_k^*=0)}{\partial t_p} = -\alpha(s_p \cdot Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}} - t_p + t_k)^{\alpha-1}$$
$$+\alpha \cdot a_p \cdot (s_k \cdot Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}} - t_k + t_p)^{\alpha-1} = 0$$
$$\Rightarrow \frac{s_p \cdot Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}} - t_p}{s_k \cdot Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}} + t_p} = A_p \quad (46)$$

$$\frac{\partial U_k(Z_p^*>0, Z_k^*>0, t_p^*>0, t_k^*=0)}{\partial Z_k} = \frac{1}{2}\alpha \cdot s_k \cdot Z_k^{-\frac{1}{2}} \cdot Z_p^{\frac{1}{2}} \cdot (s_k \cdot Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}} + t_p - t_k)^{\alpha-1} - 1$$
$$+\frac{1}{2}\alpha \cdot a_k \cdot s_p \cdot Z_k^{-\frac{1}{2}} \cdot Z_p^{\frac{1}{2}} \cdot (s_p \cdot Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}} - t_p + t_k)^{\alpha-1} = 0$$
$$\Rightarrow s_k \cdot (s_k \cdot Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}} + t_p)^{\alpha-1} + a_k \cdot s_p \cdot (s_p \cdot Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}} - t_p)^{\alpha-1} = \frac{2}{\alpha} \cdot (\frac{Z_p}{Z_k})^{-\frac{1}{2}} \quad (47)$$

$$\frac{\partial U_k(Z_p^* > 0, Z_k^* > 0, t_p^* > 0, t_k^* = 0)}{\partial t_k} = -\alpha(s_k \cdot Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}} - t_k + t_p)^{\alpha-1}$$

$$+\alpha \cdot a_k \cdot (s_p \cdot Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}} - t_p + t_k)^{\alpha-1} < 0$$

$$\Rightarrow \frac{s_p \cdot Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}} - t_p}{s_k \cdot Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}} + t_p} < 1/A_k \qquad (48)$$

$$where\ s_p + s_k = 1,\ A_p = a_p^{\frac{1}{\alpha-1}}\ and\ A_k = a_k^{\frac{1}{\alpha-1}}$$

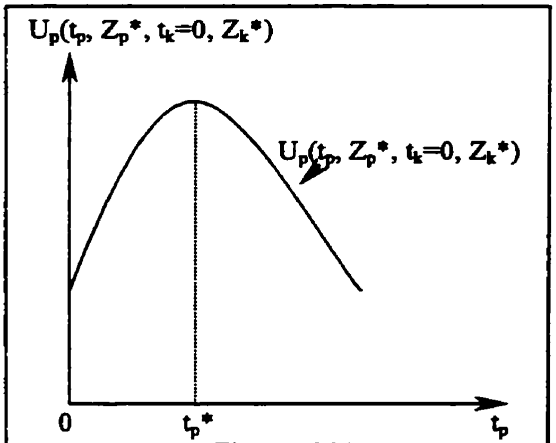

Figure 13(a)

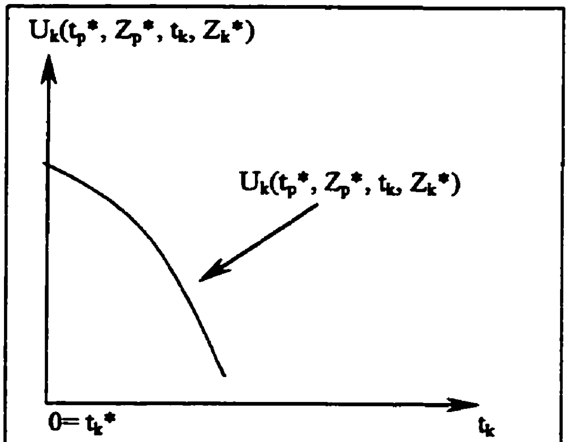

Figure 13(b)

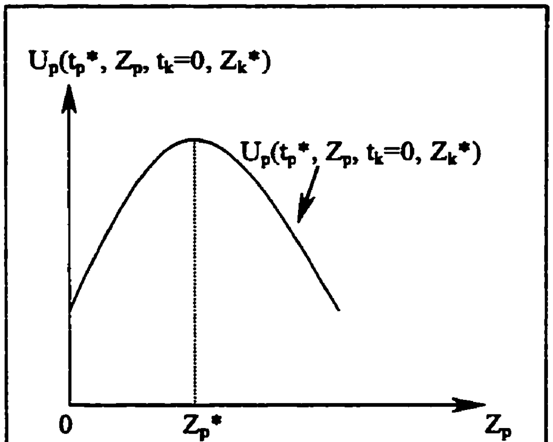

Figure 13(c)

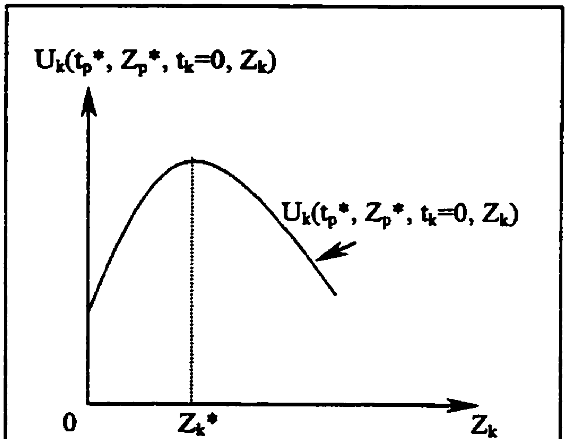

Figure 13(d)

Then deriving $t_p^*$, $Z_p^*$ and $Z_k^*$ from these first-order conditions, and from parameter restriction (48), we can get the result of case (i) as

$$when\ \frac{s_p}{s_k} > A_p,$$

$$t_p^* = (\frac{\alpha}{2})^{\frac{1}{1-\alpha}} \cdot (\frac{s_k + a_p \cdot a_k \cdot s_p}{a_p})^{\frac{1}{2\alpha-2}} \cdot (\frac{s_p}{A_p} - s_k),\ t_k^* = 0,$$

$$Z_p^* = (\frac{\alpha}{2})^{\frac{1}{1-\alpha}} \cdot (1 + \frac{1}{A_p}) \cdot (\frac{s_k + a_p \cdot a_k \cdot s_p}{a_p})^{\frac{2-\alpha}{2\alpha-2}}$$

$$and\ Z_k^* = (\frac{\alpha}{2})^{\frac{1}{1-\alpha}} \cdot (1 + \frac{1}{A_p}) \cdot (\frac{s_k + a_p \cdot a_k \cdot s_p}{a_p})^{\frac{\alpha}{2\alpha-2}}$$

$$where\ A_p = a_p^{\frac{1}{\alpha-1}}$$

The result of Case (i) tells us that when the parent's share of the firm's income is high enough relative to the kid's share, the parent will make a positive transfer to the kid but the kid will not transfer anything to the parent, although both of them are going to work. Then their equilibrium personal utilities are

$$u_p^* = (\frac{\alpha}{2})^{\frac{\alpha}{1-\alpha}} \cdot (\frac{s_k + a_p \cdot a_k \cdot s_p}{a_p})^{\frac{\alpha}{2\alpha-2}} \cdot [1 - \frac{\alpha}{2} \cdot (1 + \frac{1}{A_p}) \cdot (\frac{a_p}{s_k + a_p \cdot a_k \cdot s_p})]$$

$$u_k^* = (\frac{\alpha}{2})^{\frac{\alpha}{1-\alpha}} \cdot (\frac{s_k + a_p \cdot a_k \cdot s_p}{a_p})^{\frac{\alpha}{2\alpha-2}} \cdot [(a_p)^{\frac{\alpha}{1-\alpha}} - \frac{\alpha}{2} \cdot (1 + \frac{1}{A_p})]$$

Case (ii) is analogous, but the roles are reversed. The result of Case (ii) is

$$when \frac{s_p}{s_k} < \frac{1}{A_k},$$

$$t_p^* = 0,\ t_k^* = (\frac{\alpha}{2})^{\frac{1}{1-\alpha}} \cdot (\frac{s_p + a_p \cdot a_k \cdot s_k}{a_k})^{\frac{1}{2\alpha-2}} \cdot (\frac{s_k}{A_k} - s_p),$$

$$Z_p^* = (\frac{\alpha}{2})^{\frac{1}{1-\alpha}} \cdot (1 + \frac{1}{A_k}) \cdot (\frac{s_p + a_p \cdot a_k \cdot s_k}{a_k})^{\frac{2-\alpha}{2\alpha-2}}$$

$$and\ Z_k^* = (\frac{\alpha}{2})^{\frac{1}{1-\alpha}} \cdot (1 + \frac{1}{A_k}) \cdot (\frac{s_p + a_p \cdot a_k \cdot s_k}{a_k})^{\frac{\alpha}{2\alpha-2}}$$

$$whereA_k = a_k^{\frac{1}{\alpha-1}}$$

and their equilibrium personal utilities are

$$u_p^* = (\frac{\alpha}{2})^{\frac{\alpha}{1-\alpha}} \cdot (\frac{s_p + a_p \cdot a_k \cdot s_k}{a_k})^{\frac{\alpha}{2\alpha-2}} \cdot [(a_k)^{\frac{\alpha}{1-\alpha}} - \frac{\alpha}{2} \cdot (1 + \frac{1}{A_k})]$$

$$u_k^* = (\frac{\alpha}{2})^{\frac{\alpha}{1-\alpha}} \cdot (\frac{s_p + a_p \cdot a_k \cdot s_k}{a_k})^{\frac{\alpha}{2\alpha-2}} \cdot [1 - \frac{\alpha}{2} \cdot (1 + \frac{1}{A_k}) \cdot (\frac{a_k}{s_p + a_p \cdot a_k \cdot s_k})]$$

Now, let us look at Case (iii). Figure 14(a) and 14(b) depict the transfers of the parent and the kid, and Figure 14(c) and 14(d) depict their work efforts respectively. We observe that Case (iii) is exactly the same as the case without cash transfers in Section III.2. Therefore, the result of Case (iii) is

$$When\ \frac{1}{A_k} \le \frac{s_p}{s_k} \le A_p,$$

$$t_p^* = 0,\ t_k^* = 0,\ Z_p^* = (\frac{\alpha}{2})^{\frac{1}{1-\alpha}} \cdot (s_k^\alpha + a_k \cdot s_p^\alpha)^{\frac{\alpha}{2-2\alpha}} \cdot (s_p^\alpha + a_p \cdot s_k^\alpha)^{\frac{2-\alpha}{2-2\alpha}}$$

$$and\ Z_k^* = (\frac{\alpha}{2})^{\frac{1}{1-\alpha}} \cdot (s_p^\alpha + a_p \cdot s_k^\alpha)^{\frac{\alpha}{2-2\alpha}} \cdot (s_k^\alpha + a_k \cdot s_p^\alpha)^{\frac{2-\alpha}{2-2\alpha}},$$

$$where\ A_p = a_p^{\frac{1}{\alpha-1}}\ and\ A_k = a_k^{\frac{1}{\alpha-1}}$$

The result of Case (iii) tells us that when the parent's share of the firm's income is neither high enough nor low enough relative to the kid's share, neither of them will make a

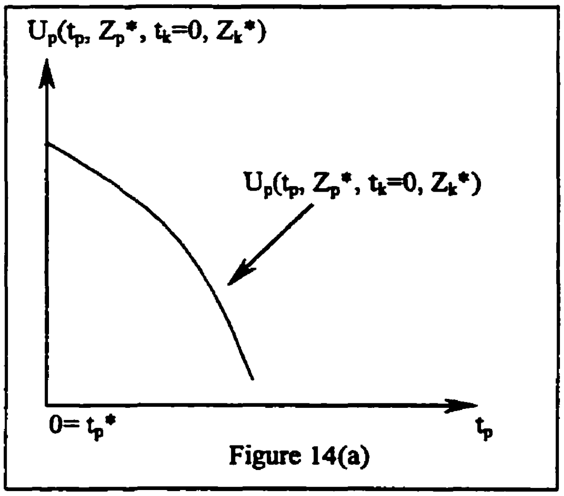

Figure 14(a)

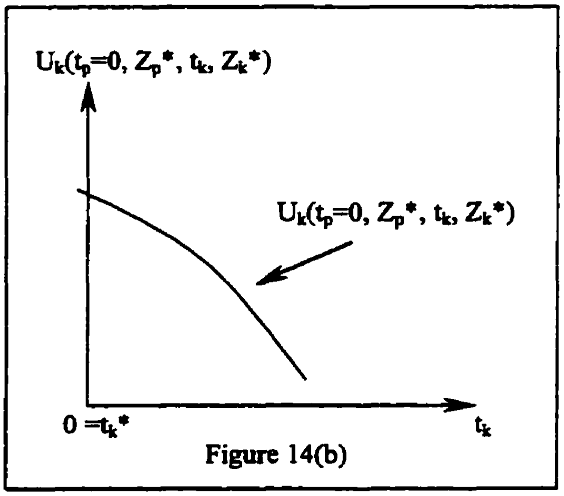

Figure 14(b)

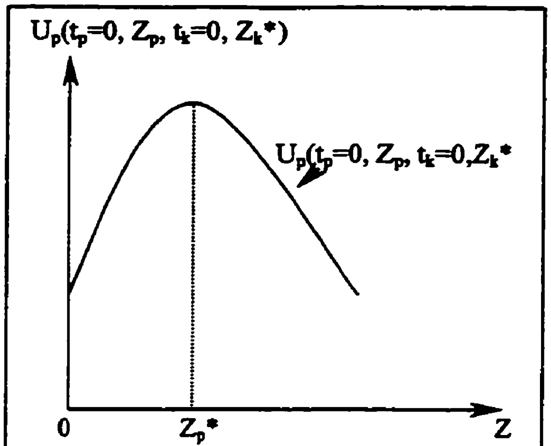


Figure 14(c)

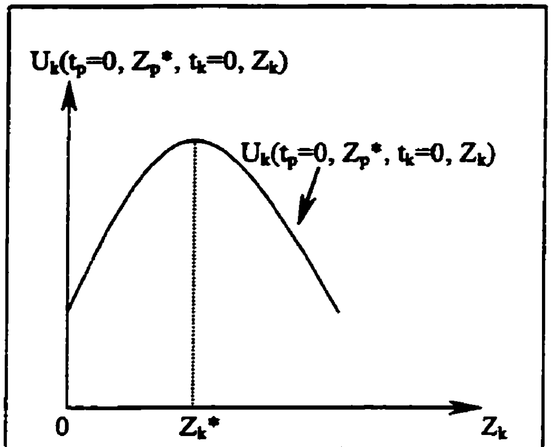


Figure 14(d)

positive transfer to the other one. Moreover, the equilibrium personal utilities of the parent and the kid are the same as those in the case without transfers in Section III.2.

Let us consider the N.E. transfers in these three cases first. Observe that $A_p>1$ and $1/A_k<1$. Then consider the following figure:

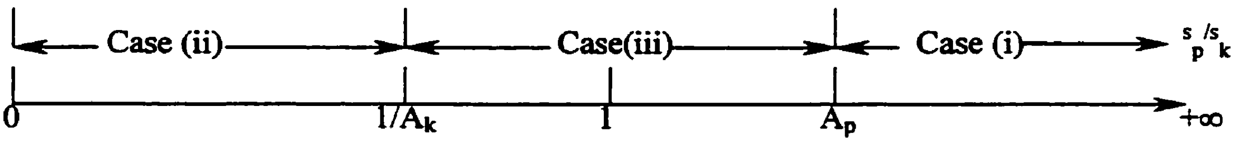


Figure 15

Now, let us look at the indifference curves of $U_p(t_p, t_k)$ and $U_k(t_p, t_k)$. The parent's MRS is

$$\frac{dt_p}{dt_k} = -\frac{\frac{\partial U_p(t_p,t_k)}{\partial t_k}}{\frac{\partial U_p(t_p,t_k)}{\partial t_p}} = -\frac{\alpha(s_p \cdot Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}} + t_k - t_p)^{\alpha-1} - a_p \cdot \alpha \cdot (s_k \cdot Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}} + t_p - t_k)^{\alpha-1}}{-\alpha(s_p \cdot Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}} + t_k - t_p)^{\alpha-1} + a_p \cdot \alpha \cdot (s_k \cdot Z_p^{\frac{1}{2}} \cdot Z_k^{\frac{1}{2}} + t_p - t_k)^{\alpha-1}} = 1$$

Similarly, the kid's MRS is 1. As before, we can learn that all of their indifference curves are straight lines with slope 1. Using the same method as in Chapter II, we can testify that the above three N.E. transfers are Pareto-optimal, given the N.E. work efforts.

We observe that Case (iii) in which there is no transfer is exactly the same as the case in Section III.2 so that we can make the same statements as we do in the previous section. What we need to do is to consider either Case (i) or Case (ii) since they are expected for reversed roles. Now, let us consider the N.E. work efforts in these two cases. The results of these cases can be depicted by Figure 16. One of the differences between Figure 16 and Figure 11 is that the work efforts in all three cases in Figure 16 are greater than zero. It is also difficult for me to compare the N.E. work efforts in the case with altruism and without altruism algebraically, so I am going to make use of similar simulation method to compare them as before in Section III.2. The N.E. work efforts of the parent and the kid in the case without altruism are expressed by equations (39) and (40). Thus, in contrast to the case in which family members earn incomes in an impersonal market and one member transfers to the other, when they work in the family firm, reciprocal altruism does not completely destroy the incentive of one party to work. Balanced reciprocal altruism induces both of them to work harder. The detailed program and results of this comparison are attached in Appendix III.

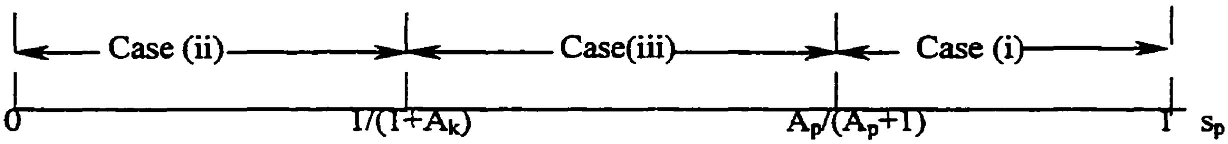


Figure 16

Now, let's compare their equilibrium personal utilities in the case with altruism and without altruism. Look at Case (i) for example again. The similar intuition can be proposed as in Section III.2: both parties will be better off when they are altruistic if their altruism is balanced (that is, when $a_p$ and $a_k$ are approximately equal), and that the less altruistic party will be better off and more altruistic party will be worse off when their altruism is unbalanced. I make use of numerical methods by means of Mathematica 4 again to examine this intuition. The details of the program and results of this simulation method are attached in Appendix IV. One of the differences between this program and the program in Section III.2 is that the relationship between $s_p$ and $s_k$ is involved in this

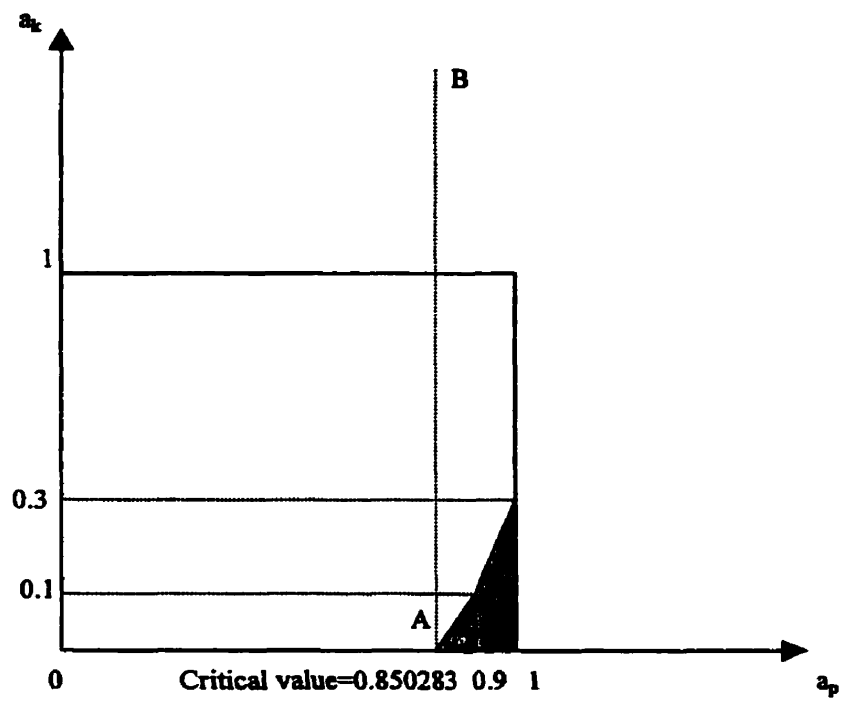


Figure 17

case so that their ranges are different from those in the program of Section III.2. Let's look at the example in which $s_p$=0.6 and $\alpha$=0.6 are fixed. As per the restriction $s_p/s_k > A_p$, we have a critical value for $a_p$ as 0.850283 (that is, the value of ap has to be greater than 0.850283). From the results of the program, we learn that the sets of ($a_p$, $a_k$) that ensure both of them to be better off can be illustrated by Figure 17. After connecting the points with curves, we observe that in the blank area to the right of the dashed line AB, both parties are better off, and that in the shaded area, less altruistic party is better off while more altruistic party is worse off. Therefore, in terms of personal utilities, balanced altruism can make both parties in the family firm better off and can make the firm more efficient.

# Chapter IV: Conclusion

What is the main result of this thesis? It is a tentative explanation of the fact that the family firm plays such a large role in the real economy. What is the explanation? It is based on three hypotheses: first, that members of the same family are reciprocally altruistic; second, that the altruism is balanced (that is, $a_p$ and $a_k$ are not too different); third, that the technology of the firm is a team production technology. The explanation itself is that reciprocal altruism is a partial solution to the shirking problems that ordinarily arise in a team production context. In particular, when altruism is balanced, both members of the family firm experience larger equilibrium personal utilities than they would if they were not altruistic. In short, balanced reciprocal altruism allows family members to more effectively exploit opportunities in a team production context more effectively than individuals who are not altruistic, and it makes the firm more efficient in term of personal utility.

Moreover, when reciprocal altruism exists, there is a difference between the family members' work efforts in the family that does not own a family firm, as opposed to that owns a family firm. When they work in an impersonal market, the party that makes a positive transfer works harder and the party that receives the transfer works less. In contrast, balanced reciprocal altruism in family firm induces both to work harder.

Therefore, a family firm with a share contract and balanced reciprocal altruism can do better than a firm with the same share contract but without altruism can. This seems contrary to the available literature in the area of family business. In the sense of the introduction and analysis in the literature on family business, there seems to be a feeling that the family gets in the way that family ties make it difficult to actually run a firm, and that the altruism in the family can, in the least, make the family firm more inefficient. However, the conclusions made from Chapter III tell us that this may not be true. Through the special cases with cash transfers and without cash transfers in the family owning a family firm, we have reached the same conclusion that balanced reciprocal altruism can make both party better off and can make the family firm more efficient in terms of personal utilities.

# Appendix

## Appendix I

As per equations (24) and (25), the equilibrium personal utilities of the parent and the kid in Case (i) are

$$when\ \frac{W_p}{W_k} > A_p,$$

$$u_p^* = \alpha^{\frac{\alpha}{1-\alpha}} \cdot W_p^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}} \cdot W_p^{\frac{\alpha}{1-\alpha}} \cdot (1+\frac{1}{A_p})\ and\ u_k^* = \alpha^{\frac{\alpha}{1-\alpha}} \cdot W_p^{\frac{\alpha}{1-\alpha}} \cdot a_p^{\frac{\alpha}{1-\alpha}},$$

$$where\ A_p = a_p^{\frac{1}{\alpha-1}}$$

and their equilibrium personal utilities in the case without altruism are

$$(u_p^*)' = \alpha^{\frac{\alpha}{1-\alpha}} \cdot W_p^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}} \cdot W_p^{\frac{\alpha}{1-\alpha}}\ and\ (u_k^*)' = \alpha^{\frac{\alpha}{1-\alpha}} \cdot W_k^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}} \cdot W_k^{\frac{\alpha}{1-\alpha}}$$

$$where\ A_p = a_p^{\frac{1}{\alpha-1}}\ and\ A_k = a_k^{\frac{1}{\alpha-1}}$$

Since

$$0 \le a_p < 1\ and\ then\ \frac{1}{A_p} > 0$$

we have

$$u_p^* = \alpha^{\frac{\alpha}{1-\alpha}} \cdot W_p^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}} \cdot W_p^{\frac{\alpha}{1-\alpha}} \cdot (1+\frac{1}{A_p}) < (u_p^*)' = \alpha^{\frac{\alpha}{1-\alpha}} \cdot W_p^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}} \cdot W_p^{\frac{\alpha}{1-\alpha}}$$

Now, we need to compare $u_k^*$ and $(u_k^*)'$. Assume that $u_k^* > (u_k^*)'$, we need

$$\alpha^{\frac{\alpha}{1-\alpha}} \cdot W_p^{\frac{\alpha}{1-\alpha}} \cdot a_p^{\frac{\alpha}{1-\alpha}} > \alpha^{\frac{\alpha}{1-\alpha}} \cdot W_k^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}} \cdot W_k^{\frac{\alpha}{1-\alpha}}$$

and then

$$W_p^{\frac{\alpha}{1-\alpha}} \cdot a_p^{\frac{\alpha}{1-\alpha}} > W_k^{\frac{\alpha}{1-\alpha}} \cdot (1-\alpha)$$

So we need

$$\frac{W_p}{W_k} > \frac{1}{a_p} \cdot (1-\alpha)^{\frac{1-\alpha}{\alpha}}$$

Since $W_p/W_k > A_p$ in Case (i), we need

$$\frac{1}{a_p} \cdot (1-\alpha) < A_p = a_p^{\frac{1}{\alpha-1}} \textit{ and then } a_p < (1-\alpha)^{\frac{(\alpha-1)^2}{\alpha^2}} \quad (A1)$$

Since $0<\alpha<1$, $0<a_p<1$ and then

$$(1-\alpha)^{\frac{(\alpha-1)^2}{\alpha^2}} > 1$$

inequation (A1) is right. Hence, when $W_p/W_k > A_p$, $u_k^* > (u_k^*)'$. The above proof tells us that when the parent makes a positive transfer to the kid, his equilibrium personal utility will be damaged while the kid's equilibrium personal utility will be advanced. Similarly, in Case (ii), we can prove that the parent's equilibrium personal utility will be advanced and the kid's equilibrium personal utility will be damaged when the kid makes a positive transfer to the parent. In Case (iii), when neither of them makes a transfer to the other one, their equilibrium personal utilities are the same as those in the case without altruism.

# Appendix II

I examine the intuition on the relationship between the equilibrium personal utilities of the parent and the kid in the case with altruism and without altruism respectively by Mathematica 4. Since we find that when $a_p=a_k=0$, the equilibrium personal utilities in the case with altruism are the same as those in the case without altruism, we only need to compare the values of $u_p^*$ and $u_k^*$ with those when $a_p=a_k=0$ in each group involving fixed $s_p$ and $\alpha$. Since the values of $s_p$ and $\alpha$ cannot be 0 or 1 on the basis of the expressions of their equilibrium personal utilities, I set the ranges of $s_p$ and $\alpha$ from 0.2 to 0.8 and set the ranges of $a_p$ and $a_k$ from 0 to 1. In addition, since the result matrix would be very huge if I used a small interval, I set the intervals of $s_p$ and $\alpha$ to be 0.3 and set the intervals of $a_p$ and $a_k$ to be 0.2. The result matrix is constructed as $\{s_p, \alpha, a_p, a_k, u_p^*, u_k^*\}$. The program of this comparison and the result matrix are attached behind.

**Program and Results of Comparison between the Equilibrium Personal utilities in Section III.2**

```
ap =.; ak =.; sp =.; α =.;
Upstar = ((α / 2) ^ (α / (1 - α))) * (((sp) ^α + ap * (1 - sp) ^α) ^ ((2 - α) / (2 - 2 α))) *
   (((1 - sp) ^α + ak * (sp) ^α) ^ (α / (2 - 2 α))) *
   ((((sp) ^α) / ((sp) ^α + ap * (1 - sp) ^α)) - (α / 2));
Ukstar =  ((α / 2) ^ (α / (1 - α))) * (((sp) ^α + ap * (1 - sp) ^α) ^ (α / (2 - 2 α))) *
   (((1 - sp) ^α + ak * (sp) ^α) ^ ((2 - α) / (2 - 2 α))) *
   ((((1 - sp) ^α) / ((1 - sp) ^α + ak * (sp) ^α)) - (α / 2));

W = Table[Print[{sp, α, ap, ak, Upstar, Ukstar}],
   {sp, 0.2, 0.8, 0.3}, {α, 0.2, 0.8, 0.3}, {ap, 0, 1, 0.2}, {ak, 0, 1, 0.2}];
Dimensions[
 W]

{0.2, 0.2, 0, 0, 0.35039, 0.462342}

{0.2, 0.2, 0, 0.2, 0.356626, 0.462646}

{0.2, 0.2, 0, 0.4, 0.362181, 0.461804}

{0.2, 0.2, 0, 0.6, 0.367197, 0.460039}

{0.2, 0.2, 0, 0.8, 0.371774, 0.457513}

{0.2, 0.2, 0, 1., 0.375988, 0.454343}

{0.2, 0.2, 0.2, 0, 0.35022, 0.476078}

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{0.2, 0.2, 0.2, 0.6, 0.367018, 0.473706}

{0.2, 0.2, 0.2, 0.8, 0.371594, 0.471104}

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```

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{0.2, 0.5, 0.2, 0.2, 0.0570373, 0.127237}

{0.2, 0.5, 0.2, 0.4, 0.0595735, 0.128312}

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{3, 3, 6, 6}
```

## Appendix III

I examine the intuition on the relationship between the N.E. work efforts of the parent and the kid in the case with altruism and without altruism respectively by Mathematica 4. Look at Case (i) for example. Since there is a restriction $s_p/s_k>A_p$ and since the value of $\alpha$ cannot be 0 or 1 on the basis of the expressions of their N.E. work efforts, I set the ranges of $s_p$ and $a_p$ from 0.6 to 1, set the range of $\alpha$ from 0.2 to 0.8 and set the range of $a_k$ from 0 to 1. In addition, I set the interval to be 0.2. $(Z_p^*)'$ and $(Z_k^*)'$ in equations (35) and (36) are the N.E. work efforts of the parent and the kid in the case without altruism. The result matrix is constructed as $\{s_p, \alpha, a_p, a_k, Z_p^*, (Z_p^*)', Z_k^*, (Z_k^*)'\}$. The program of this comparison and the result matrix are attached behind.

**Program and Results of Comparison between N.E. Work Efforts in Section III .3**

```
ap =.; ak =.; sp =.; α =.;
Zpstar = ((α / 2) ^ (1 / (1 - α))) *
   ((((ap * ak * sp + (1 - sp)) / ap) ^ ((2 - α) / (2 - 2 α)))) * (1 + (ap) ^ (1 / (1 - α)));
Zkstar = ((α / 2) ^ (1 / (1 - α))) * ((((ap * ak * sp + (1 - sp)) / ap) ^ ((α) / (2 - 2 α)))) *
   (1 + (ap) ^ (1 / (1 - α)));
Zp = ((α / 2) ^ (1 / (1 - α))) * (((sp) ^α) ^ ((2 - α) / (2 - 2 α))) * (((1 - sp) ^α) ^ ((α) / (2 - 2 α)));
Zk = ((α / 2) ^ (1 / (1 - α))) * (((sp) ^α) ^ ((α) / (2 - 2 α))) * (((1 - sp) ^α) ^ ((2 - α) / (2 - 2 α)));

W = Table[Print[{sp, α, ap, ak, Zpstar, Zp, Zkstar, Zk}],
   {sp, 0.6, 1, 0.2}, {α, 0.2, 0.8, 0.2}, {ap, 0.2, 1, 0.2}, {ak, 0, 1, 0.2}];
Dimensions[
 W]
```

```
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{0.8, 0.6, 0.2, 0, 0.0501768, 0.0189021, 0.0501768, 0.00822764}

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{0.8, 0.6, 0.2, 0.6, 0.0996463, 0.0189021, 0.0673286, 0.00822764}

{0.8, 0.6, 0.2, 0.8, 0.119256, 0.0189021, 0.072717, 0.00822764}

{0.8, 0.6, 0.2, 1., 0.140356, 0.0189021, 0.0779754, 0.00822764}

{0.8, 0.6, 0.4, 0, 0.0161385, 0.0189021, 0.0322771, 0.00822764}

{0.8, 0.6, 0.4, 0.2, 0.0262342, 0.0189021, 0.0397488, 0.00822764}

{0.8, 0.6, 0.4, 0.4, 0.0383567, 0.0189021, 0.0467764, 0.00822764}

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{0.8, 0.6, 0.8, 0.8, 0.0632132, 0.0189021, 0.0710261, 0.00822764}

{0.8, 0.6, 0.8, 1., 0.0844222, 0.0189021, 0.0804021, 0.00822764}

{0.8, 0.6, 1., 0, 0.00589706, 0.0189021, 0.0294853, 0.00822764}

{0.8, 0.6, 1., 0.2, 0.0164954, 0.0189021, 0.0458205, 0.00822764}

{0.8, 0.6, 1., 0.4, 0.0313934, 0.0189021, 0.060372, 0.00822764}

{0.8, 0.6, 1., 0.6, 0.0502023, 0.0189021, 0.0738269, 0.00822764}

{0.8, 0.6, 1., 0.8, 0.0726644, 0.0189021, 0.0865053, 0.00822764}

{0.8, 0.6, 1., 1., 0.0985901, 0.0189021, 0.0985901, 0.00822764}

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{0.8, 0.8, 0.2, 0.4, 0.0235592, 0.000456419, 0.0178479, 0.000150562}

{0.8, 0.8, 0.2, 0.6, 0.0332066, 0.000456419, 0.0224369, 0.000150562}

{0.8, 0.8, 0.2, 0.8, 0.0451825, 0.000456419, 0.0275503, 0.000150562}

{0.8, 0.8, 0.2, 1., 0.0597388, 0.000456419, 0.0331882, 0.000150562}

{0.8, 0.8, 0.4, 0, 0.00129311, 0.000456419, 0.00258621, 0.000150562}

{0.8, 0.8, 0.4, 0.2, 0.00297411, 0.000456419, 0.00450622, 0.000150562}

{0.8, 0.8, 0.4, 0.4, 0.00570382, 0.000456419, 0.00695588, 0.000150562}

{0.8, 0.8, 0.4, 0.6, 0.0097365, 0.000456419, 0.0099352, 0.000150562}

{0.8, 0.8, 0.4, 0.8, 0.0153264, 0.000456419, 0.0134442, 0.000150562}

{0.8, 0.8, 0.4, 1., 0.0227277, 0.000456419, 0.0174828, 0.000150562}

{0.8, 0.8, 0.6, 0, 0.00040875, 0.000456419, 0.00122625, 0.000150562}

{0.8, 0.8, 0.6, 0.2, 0.00132508, 0.000456419, 0.00268598, 0.000150562}

{0.8, 0.8, 0.6, 0.4, 0.0030777, 0.000456419, 0.00471077, 0.000150562}

{0.8, 0.8, 0.6, 0.6, 0.00593783, 0.000456419, 0.00730061, 0.000150562}

{0.8, 0.8, 0.6, 0.8, 0.0101767, 0.000456419, 0.0104555, 0.000150562}

{0.8, 0.8, 0.6, 1., 0.0160655, 0.000456419, 0.0141755, 0.000150562}

{0.8, 0.8, 0.8, 0, 0.000212429, 0.000456419, 0.000849715, 0.000150562}

{0.8, 0.8, 0.8, 0.2, 0.000937012, 0.000456419, 0.00228539, 0.000150562}

{0.8, 0.8, 0.8, 0.4, 0.00251778, 0.000456419, 0.00441716, 0.000150562}

{0.8, 0.8, 0.8, 0.6, 0.00528886, 0.000456419, 0.00724501, 0.000150562}

{0.8, 0.8, 0.8, 0.8, 0.00958437, 0.000456419, 0.010769, 0.000150562}

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{0.8, 0.8, 1., 0, 0.00016384, 0.000456419, 0.0008192, 0.000150562}

{0.8, 0.8, 1., 0.2, 0.000955515, 0.000456419, 0.00265421, 0.000150562}

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{0.8, 0.8, 1., 1., 0.02048, 0.000456419, 0.02048, 0.000150562}

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{1., 0.2, 0.2, 0.8, 0.0496013, 0., 0.0620016, 0.}

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{1., 0.2, 1., 0.8, 0.0874996, 0., 0.109375, 0.}

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{1., 0.4, 0.2, 0.4, 0.0215376, 0., 0.0538439, 0.}

{1., 0.4, 0.2, 0.6, 0.0369816, 0., 0.0616359, 0.}

{1., 0.4, 0.2, 0.8, 0.0542713, 0., 0.0678391, 0.}

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{1., 0.4, 0.4, 0.2, 0.00973723, 0., 0.0486861, 0.}

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{1., 0.4, 0.4, 0.6, 0.0421305, 0., 0.0702176, 0.}

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{1., 0.4, 0.6, 1., 0.0975936, 0., 0.0975936, 0.}

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{1., 0.6, 0.6, 1., 0.0630412, 0., 0.0630412, 0.}

{1., 0.6, 0.8, 0, 0., 0., 0., 0.}

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{1., 0.6, 0.8, 0.6, 0.0317059, 0., 0.0528432, 0.}

{1., 0.6, 0.8, 0.8, 0.0524545, 0., 0.0655681, 0.}

{1., 0.6, 0.8, 1., 0.0775132, 0., 0.0775132, 0.}

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{1., 0.6, 1., 0.6, 0.0403272, 0., 0.067212, 0.}

{1., 0.6, 1., 0.8, 0.0667176, 0., 0.083397, 0.}

{1., 0.6, 1., 1., 0.0985901, 0., 0.0985901, 0.}

{1., 0.8, 0.2, 0, 0., 0., 0., 0.}

{1., 0.8, 0.2, 0.2, 0.0000819462, 0., 0.000409731, 0.}

{1., 0.8, 0.2, 0.4, 0.00065557, 0., 0.00163892, 0.}

{1., 0.8, 0.2, 0.6, 0.00221255, 0., 0.00368758, 0.}

{1., 0.8, 0.2, 0.8, 0.00524456, 0., 0.0065557, 0.}

{1., 0.8, 0.2, 1., 0.0102433, 0., 0.0102433, 0.}

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{1., 0.8, 0.4, 0.4, 0.000662071, 0., 0.00165518, 0.}

{1., 0.8, 0.4, 0.6, 0.00223449, 0., 0.00372415, 0.}

{1., 0.8, 0.4, 0.8, 0.00529657, 0., 0.00662071, 0.}

{1., 0.8, 0.4, 1., 0.0103449, 0., 0.0103449, 0.}

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{1., 0.8, 0.6, 0, 0., 0., 0., 0.}

{1., 0.8, 0.6, 0.2, 0.0000882901, 0., 0.00044145, 0.}

{1., 0.8, 0.6, 0.4, 0.000706321, 0., 0.0017658, 0.}

{1., 0.8, 0.6, 0.6, 0.00238383, 0., 0.00397305, 0.}

{1., 0.8, 0.6, 0.8, 0.00565057, 0., 0.00706321, 0.}

{1., 0.8, 0.6, 1., 0.0110363, 0., 0.0110363, 0.}

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{1., 0.8, 0.8, 0.2, 0.000108764, 0., 0.000543818, 0.}

{1., 0.8, 0.8, 0.4, 0.000870108, 0., 0.00217527, 0.}

{1., 0.8, 0.8, 0.6, 0.00293662, 0., 0.00489436, 0.}

{1., 0.8, 0.8, 0.8, 0.00696087, 0., 0.00870108, 0.}

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{1., 0.8, 1., 0.4, 0.00131072, 0., 0.0032768, 0.}

{1., 0.8, 1., 0.6, 0.00442368, 0., 0.0073728, 0.}

{1., 0.8, 1., 0.8, 0.0104858, 0., 0.0131072, 0.}

{1., 0.8, 1., 1., 0.02048, 0., 0.02048, 0.}

Out[7]= {3, 4, 5, 6}
```

# Appendix IV

I examine the intuition on the relationship between the equilibrium personal utilities of the parent and the kid in the case with altruism and without altruism respectively by Mathematica 4. Since there is a restriction $s_p/s_k > A_p$ in Case (i) of Section III.3, I set the range of $s_p$ from 0.6 to 1, set the range of $a_p$ from 0.8 to 1, set the range of $\alpha$ from 0.2 to 0.8 and set the range of $a_k$ from 0 to 1. In addition, the intervals of $s_p$ and $\alpha$ are 0.2 and the intervals of $a_p$ and $a_k$ are 0.1. $(U_p^*)'$ and $(U_k^*)'$ in equations (41) and (42) are the equilibrium personal utilities of the parent and the kid in the case without altruism. The result matrix is constructed as $\{s_p, \alpha, a_p, a_k, U_p^*, (U_p^*)', U_k^*, (U_k^*)'\}$. The program of this comparison and the result matrix are attached behind.

**Program and Results of Comparison between the Equilibrium Personal Utilities in Section III .3**

```
ap =.; ak =.; sp =.; α =.;
Upstar = ((α / 2) ^ (α / (1 - α))) * (((ap * ak * sp + (1 - sp)) / ap) ^ (α / (2 α - 2))) *
   (1 - (α / 2) * (ap / (ap * ak * sp + (1 - sp))) * (1 + (ap) ^ (1 / (1 - α))));
Ukstar = ((α / 2) ^ (α / (1 - α))) * (((ap * ak * sp + (1 - sp)) / ap) ^ (α / (2 α - 2))) *
   ((ap) ^ (α / (1 - α)) - (α / 2) * (1 + (ap) ^ (1 / (1 - α))));
Up = ((α / 2) ^ (1 / (1 - α))) * ((sp) ^ (α * (2 - α) / (2 - 2 α))) *
   ((1 - sp) ^ (α * α / (2 - 2 α))) * (1 - α / 2);
Uk = ((α / 2) ^ (1 / (1 - α))) * ((1 - sp) ^ (α * (2 - α) / (2 - 2 α))) *
   ((sp) ^ (α * α / (2 - 2 α))) * (1 - α / 2);
Upstar
Ukstar
Up
Uk
W = Table[Print[{sp, α, ap, ak, Upstar, Up, Ukstar, Uk}],
   {sp, 0.6, 1, 0.2}, {α, 0.2, 0.8, 0.2}, {ap, 0.8, 1, 0.1}, {ak, 0, 1, 0.1}];
Dimensions[
 W]
```

Out[6]= $2^{-\frac{\alpha}{1-\alpha}} \left(\frac{1 - sp + ak\, ap\, sp}{ap}\right)^{\frac{\alpha}{-2+2\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} \left(1 - \frac{ap\left(1 + ap^{\frac{1}{1-\alpha}}\right)\alpha}{2\,(1 - sp + ak\, ap\, sp)}\right)$

Out[7]= $2^{-\frac{\alpha}{1-\alpha}} \left(\frac{1 - sp + ak\, ap\, sp}{ap}\right)^{\frac{\alpha}{-2+2\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} \left(ap^{\frac{\alpha}{1-\alpha}} - \frac{1}{2}\left(1 + ap^{\frac{1}{1-\alpha}}\right)\alpha\right)$

Out[8]= $2^{-\frac{1}{1-\alpha}} (1 - sp)^{\frac{\alpha^2}{2-2\alpha}} sp^{\frac{(2-\alpha)\alpha}{2-2\alpha}} \left(1 - \frac{\alpha}{2}\right) \alpha^{\frac{1}{1-\alpha}}$

Out[9]= $2^{-\frac{1}{1-\alpha}} (1 - sp)^{\frac{(2-\alpha)\alpha}{2-2\alpha}} sp^{\frac{\alpha^2}{2-2\alpha}} \left(1 - \frac{\alpha}{2}\right) \alpha^{\frac{1}{1-\alpha}}$

```
{0.6, 0.2, 0.8, 0, 0.397796, 0.0440939, 0.472243, 0.0406593}

{0.6, 0.2, 0.8, 0.1, 0.414959, 0.0440939, 0.465601, 0.0406593}

{0.6, 0.2, 0.8, 0.2, 0.427834, 0.0440939, 0.459714, 0.0406593}

{0.6, 0.2, 0.8, 0.3, 0.437675, 0.0440939, 0.454437, 0.0406593}

{0.6, 0.2, 0.8, 0.4, 0.445303, 0.0440939, 0.449659, 0.0406593}

{0.6, 0.2, 0.8, 0.5, 0.451279, 0.0440939, 0.445298, 0.0406593}

{0.6, 0.2, 0.8, 0.6, 0.455997, 0.0440939, 0.441291, 0.0406593}

{0.6, 0.2, 0.8, 0.7, 0.459738, 0.0440939, 0.437586, 0.0406593}

{0.6, 0.2, 0.8, 0.8, 0.462712, 0.0440939, 0.434144, 0.0406593}

{0.6, 0.2, 0.8, 0.9, 0.465074, 0.0440939, 0.430931, 0.0406593}

{0.6, 0.2, 0.8, 1., 0.466945, 0.0440939, 0.42792, 0.0406593}
```

{0.6, 0.2, 0.9, 0, 0.359562, 0.0440939, 0.489367, 0.0406593}

{0.6, 0.2, 0.9, 0.1, 0.384679, 0.0440939, 0.481682, 0.0406593}

{0.6, 0.2, 0.9, 0.2, 0.403198, 0.0440939, 0.474963, 0.0406593}

{0.6, 0.2, 0.9, 0.3, 0.417192, 0.0440939, 0.469003, 0.0406593}

{0.6, 0.2, 0.9, 0.4, 0.427969, 0.0440939, 0.463655, 0.0406593}

{0.6, 0.2, 0.9, 0.5, 0.436391, 0.0440939, 0.45881, 0.0406593}

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# Reference

1. A Fresh Look at the Rotten Kid Theorem--and Other Household Mysteries

Theodore C. Bergstrom

*The Journal of Political Economy*, Vol. 97, No. 5. (Oct., 1989), pp. 1138-1159.

2. The Rotten-Kid Theorem Meets the Samaritan's Dilemma

Neil Bruce, Michael Waldman

*Quarterly Journal of Economics*, Vol. 105, No. 1. (Feb., 1990), pp. 155-165.

3. The Strategic Bequest Motive

B. Douglas Bernheim, Andrei Shleifer, Lawrence H. Summers

*The Journal of Political Economy*, Vol. 93, No. 6. (Dec., 1985), pp. 1045-1076.

4. The Expanding Domain of Economics

Jack Hirshleifer

*The American Economic Review*, Vol. 75, No. 6. (Dec., 1985), pp. 53-68.

5. Altruism, Egoism, and Genetic Fitness: Economics and Sociobiology

Gary S. Becker

*Journal of Economic Literature*, Vol. 14, No. 3. (Sep., 1976), pp. 817-826.

6. Giving with Impure Altruism: Applications to Charity and Ricardian Equivalence

James Andreoni

*The Journal of Political Economy*, Vol. 97, No. 6. (Dec., 1989), pp. 1447-1458.

9. Altruism within the Family Reconsidered: Do Nice Guys Finish Last?

B. Douglas Bernheim, Oded Stark

*The American Economic Review*, Vol. 78, No. 5. (Dec., 1988), pp. 1034-1045.

10. Income and Outcomes: A Structural Model of Intrahousehold Allocation

Martin Browning, Francois Bourguignon, Pierre-Andre Chiappori, Valerie Lechene

*The Journal of Political Economy*, Vol. 102, No. 6. (Dec., 1994), pp. 1067-1096.

11. Child Endowments and the Quantity and Quality of Children

Gary S. Becker, Nigel Tomes

*The Journal of Political Economy*, Vol. 84, No. 4, Part 2: Essays in Labor Economics in Honor of H. Gregg Lewis. (Aug., 1976), pp. S143-S162.

12. Nobel Lecture: The Economic Way of Looking at Behavior

Gary S. Becker

*The Journal of Political Economy*, Vol. 101, No. 3. (Jun., 1993), pp. 385-409.

13. Parental Altruism and Inter Vivos Transfers: Theory and Evidence

Joseph G. Altonji, Fumio Hayashi, Laurence J. Kotlikoff

*The Journal of Political Economy*, Vol. 105, No. 6. (Dec., 1997), pp. 1121-1166.

14. Bequest Behavior and the Effect of Heirs' Earnings: Testing the Altruistic Model of Bequests

Mark O. Wilhelm

*The American Economic Review*, Vol. 86, No. 4. (Sep., 1996), pp. 874-892.

15. Bequest as a Public Good within Marriage: A Note

Junsen Zhang

*The Journal of Political Economy*, Vol. 102, No. 1. (Feb., 1994), pp. 187-193.

16. Uncertain Altruism and Investment in Children

Subir Chakrabarti, William Lord, Peter Rangazas

*The American Economic Review*, Vol. 83, No. 4. (Sep., 1993), pp. 994-1002.

17. Transfers in Kind: Why They Can be Efficient and Nonpaternalistic

Neil Bruce, Michael Waldman

*The American Economic Review*, Vol. 81, No. 5. (Dec., 1991), pp. 1345-1351.

18. Intergenerational Transfers and Liquidity Constraints

Donald Cox

*Quarterly Journal of Economics*, Vol. 105, No. 1. (Feb., 1990), pp. 187-217.

19. A Reexamination of the Strategic Bequest Motive (in Comment)

Maria G. Perozek

*The Journal of Political Economy*, Vol. 106, No. 2. (Apr., 1998), pp. 423-445.

20. How Strong Are Bequest Motives? Evidence Based on Estimates of the Demand for Life Insurance and Annuities

B. Douglas Bernheim

*The Journal of Political Economy*, Vol. 99, No. 5. (Oct., 1991), pp. 899-927.

21. Bequest and Wealth Accumulation: Are Some Pieces of the Puzzle Missing?

Denis Kessler, Andre Masson

*The Journal of Economic Perspectives*, Vol. 3, No. 3. (Summer, 1989), pp. 141-152.

22. Household Bequest Behaviour and the National Distribution of Wealth

John Laitner

*The Review of Economic Studies*, Vol. 46, No. 3. (Jul., 1979), pp. 467-483.

23. Non-market based transfers of wealth and power: a research framework for family business

Churchill, N.C. and K.J. Hatten (1987)

*American Jounal of Small Business Management,* 11 (3): pp. 51-64

24. Keeping the family business healthy: How to plan for continuing growth, profitability, and family leadership

Ward, J. L. (1987)

San Francisco: Jossey-Bass

25. The special role of strategic planning for family businesses

Ward, J. L. (1988)

*Family Business Review,* 1(2); pp. 105-117

26. Managing change in the family firm – Issues and Strategies

Beckhard, R. and W.G. Dyer, Jr. (1983a)

*Sloan Management Review,* 24 (3): pp. 59-65

27. Managing continuity in the family owned business

Beckhard, R. and W.G. Dyer, Jr. (1983b)

*Organizational Dynamics, Summer*: pp. 47-56

28. Realizing the Potential of the family Business

Davis, P. (1983)

*Organizational Dynamics*, Summer: pp. 47-56

29. Transfer of the Family-owned Business

Ambrose, D.M (1983)

*Journal of Small Business Management*, 21 (1): 49-56

30. Family Partnerships

Anonymous (1983)

*Estate Planning, CPA Journal,* May: pp. 71-78

31. Transferring Power in the Family Business

Barnes. L.B. and S.A. Hershon (1976)

*Harvard Business Review*, 53 (4): 105-114

32. Succession in Family Firm: The Inheritors View

Birley, S. (1986)

*Journal of Small Business Management*, 2493): pp. 36-43

33. The Influence of Life-stage on Father-son Work Relationships in Family Companies

Davis, J.A. and R. Tagiuri (1989)

*Family Business Review*, 2 (1): pp. 47-74

34. Is Nepotism So Bad?

Ewing, D.W. (1965)

*Harvard Business Review*, 43 (1): pp. 22

35. A Comparison of Successor Development in Family and Non-family Businesses

Fiegener, M.K., B.M. Brown, R.A. Prince and K.M. File (1994)

*Family Business Review*, 7(4): pp. 313-329

36. Executive Succession In Small Companies

Trow, D.B. (1961)

*Administrative Science Quarterly*, 6: pp. 228-239

37. The New Nepotism

Toy, S. (1988)

*Business Week*, April 4

38. Intergenerational Trade, Longevity, and Economic Growth.

Ehrlich, Isaac and Lui, Francis T. (1991)

*Journal of Political Economy* 99: pp. 1029-59

39. Altruism within the family reconsidered: Do nice guys finish last? Bernheim,

B. Douglas and Stark, Oded(1988)

*American Economic Review* 78: pp. 1034-45

40. Motives for Private Income transfers

Cox, Donald (1987)

*Journal of Political Economy* 95: pp. 508-46

41. Transfer Behavior within the Family: Results from the Asset and Health Dynamics Survey

Kathleen McGarry & Robert F. Schoeni (Apr. 1995)

NBER Working Paper, W5099

42. Transfer Behavior: Measurement and the Redistribution of Resources within the Family

Kathleen McGarry & Robert F. Schoeni

NBER Working Paper, W4607

43. Intergenerational Transfers and Savings

Laurence J. Kotlikoff (Nov 1989)

NBER Working Paper, W2237

44. Intergenerational Transfers, Aging, and Uncertainty

David N. Weil (Sep 1993)

NBER Working Paper, W4477

45. The Distribution of Family Income: Measuring and Explaining Changes in the 1980s for Canada and the United States

McKinley L. Blackburn and David E. Bloom (Mar 1991)

NBER Working Paper, W3659

46. The Contribution of Intergenerational Transfers to Total Wealth: A Reply

Laurence J. Kotlikoff and Lawrence H. Summers (Dec 1988)

NBER Working Paper, W1827

47. The International Transfer of Semi-Conductor Technology Through U.S. Based Firms

William F. Finan (Dec 1975)

NBER Working Paper, W0118

48. Intergenerational Transfers and Inequality

D.L. Bevan and Joseph E. Stiglitz (Aug 1980)

NBER Working Paper, R0076

49. Intergenerational Altruism and Social Welfare: A Critique of the Dynastic Model

B. Douglas Bernheim (Jun 1987)

NBER Working Paper, W2288

50. Defining the Family Business by Behavior

Jess H. Chua, James J. Chrisman and Pramodita Sharma

*Entrepreneurship Theory and Practice, Volume 23*, No. 4 (Summer, 199), P19-39

51. Alturism and Beyond: An Economic Analysis of Transfers and Exchanges within Families and Groups

John Hoddinott

52. A Theory of Social Interactions

Gary S. Becker

*Journal of Political Economy*, 1974, Vol. 82, No. 6

53. How Altruism Can Prevail in an Evolutionary Environment

Theodore C. Bergstrom, Oded Stark

*The American Economic Review*, Volume 83, Issue 2.

54. Altruism, the Samaritan's Dilemma, and Government Transfer Policy

Stephen Coate

*The American Economic Review*, Volume 85, Issue 1 (Mar., 1995), pp. 46-57.

55. Altruism and Economics

Herbert A. Simon

*The American Economic Review*, Volume 83, Issue 2

56. Fiscal Policy with Impure Integeneational Altruism

Andrew B. Abel, B. Douglas Bernheim

*Econometrica*, Volume 59, Issue 6 (Nov. 1991), pp. 1687-1711

57. Economic Growth with Intergenerational Altruism

B Douglas Bernheim, Debraj Ray

*The Review of Economic Studies*, Volume 54, Issue 2 (Apr., 1987), pp. 227-241

58. Intergenerational Altruism, Dynastic Equilibrua and Social Welfare

B. Douglas Bernheim

*The Review of Economic Studies*, Voulume 56, Issue 1 (Jan., 1989), pp. 119-128