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Spectral Properties of Diagonally Dominant Infinite Matrices

BY

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Abstract

We establish Geršgorin-type theorems for diagonally dominant infinite matrices A acting as linear operators in the sequence spaces ℓ_p, $1 \leq p \leq \infty$. Two methods are used; in the first, the results are established using a sequence of infinite matrices A_n that converges to A in the generalized sense as $n \to \infty$. In the second method, the results are established using the continuity in the generalized sense of a family of closed operators $A(\mu)$, $\mu \in [0, 1]$. The first method allows us to approximate the eigenvalues and eigenvectors of A by those of A_n.

The dependence of the eigenvalues and the eigenvectors of a matrix operator upon its perturbation is also discussed. The results are established using the fixed-point principle for contraction maps.

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Contents

Introduction

The Gersgorin theory for the localization of eigenvalues of finite matrices is well established (see [22] and [2]). Eigenvalue problems for infinite matrices occur frequently in mathematics and engineering (see [4] and [31]). An often used method for finding eigenvalues is to truncate the system to finite $n \times n$ systems and to let n tend to infinity. However, it has been shown that an infinite system may have a nonzero eigenvalue although the truncated systems have only the zero eigenvalue (see [25]). Therefore it is natural to try and develop Gersgorin theorems for infinite matrices, where we consider them as linear operators, acting in the sequence spaces $\ell_p, 1 \leq p \leq \infty$ (see examples 3 and 4, page 195 of [9] for the definition of ℓ_p). An interesting paper establishing Gersgorin theorems for diagonally dominant infinite matrices was published recently by Shivakumar, Williams and Rudraiah [26], but their analysis is restricted to matrix operators acting in ℓ_1 and ℓ_∞.

In the paper [6], the authors develop an analogous theory for diagonally dominant matrix operators acting in ℓ_p, $1 \leq p \leq \infty$, but using a constructive approach involving a sequence $\{A_n\}$ of infinite matrix operators that converges to the matrix operator A in an appropriate sense (see Section 2 of [6]. Here we take advantage of the powerful theory of perturbations developed in the work of Kato [15]). There are some advantages in having results for the wider range of values of p; in particular, in the paper [6] the Hilbert space properties associated with $p = 2$ have admitted the application of strong results for selfadjoint operators (see Sections 5 and 6 of that paper). Also, the spectral properties of a given matrix operator may depend on the choice of the space ℓ_p. For example, Hanani, Netanyahu and

Reichaw [11] show that a matrix operator A can have zero as an eigenvalue with respect to ℓ_∞ while zero is not an eigenvalue of A with respect to ℓ_1. Although the analysis given in [6] succeeds in extending the range of p, the results when $p = 1, \infty$ are just weaker than those of the paper [26]. In the paper [7] we have shown how the results of [26] can be both strengthened and extended to more general values of p.

In this thesis, we cover the results of the papers [6] and [7] through Chapters Two and Five. In Chapter One, some preliminary results are introduced. In Chapter Two, we extend Gersgorin theory to infinite matrices with bounded perturbations where we show that for a matrix operator A, any set of r Gersgorin discs whose union is disjoint from all other Gersgorin discs intersects the spectrum of A in a finite set of eigenvalues of A with total algebraic multiplicity equal to r. Chapter Three generalizes and develops the Gersgorin theorems results of the paper [26], and Gersgorin theorems are established for diagonally dominant infinite matrix operators with relatively bounded perturbations acting in ℓ_p, $1 \leq p \leq \infty$. In Chapter Four, we use the spectral approximation theory described by Chatelin [3] and the results of Chapter Two to approximate the eigenvalues and the eigenvectors of the matrix operator A by those of the approximating sequence $\{A_n\}$ introduced in Chapter Two, and take advantage of the Hilbert space properties of the space ℓ_2 in getting stronger results. In Chapter Five, we study Mathieu's equation (see equation (5.1.1)) and prove that in the case $q = 1$, the eigenvalues corresponding to the eigenfunctions $\mathrm{ce}_{2n}(\theta, q)$ of equation (5.1.1) are the eigenvalues of a diagonally dominant matrix operator defined in ℓ_2. Then we use the results of Chapters Two and Four to approximate the eigenvalues of equation (5.1.1). In

Chapter Six, we discuss the dependence of the eigenvalues and eigenvectors of an operator upon its perturbation. We apply the fixed-point principle for contraction maps in obtaining the results. This problem has been discussed in [24], [13] and [33]. Our results generalize and improve the results in [24]. We also use these results to approximate the eigenvalues of Mathieu's equation, although it is recognized that sharper estimates for these eigenvalues can be found by other methods (see [18]).

Chapter 1

Preliminaries

In this chapter, we present some basic definitions and theorems that will be used throughout the thesis.

1.1 Linear Operators in Banach Spaces

Throughout this thesis, $\mathcal{X}$ and $\mathcal{Y}$ will denote Banach spaces over the field of complex numbers $\mathcal{C}$. The norm on $\mathcal{X}$ is denoted by $\|\cdot\|_{\mathcal{X}}$. A sequence $\{x_n\}$ of points in $\mathcal{X}$ is said to converge to a point x in $\mathcal{X}$ if $\lim_{n\to\infty} \| x_n - x \| = 0$. In this case we write $\lim_{n\to\infty} x_n = x$. If x and $x_n \in \mathcal{X}$ for all $n \in \mathcal{N}$, where $\mathcal{N}$ denotes the set of positive integers, and if $\lim_{n\to\infty} \sum_{i=1}^{n} x_i = x$, then we write $x = \sum_{i=1}^{\infty} x_i$.

A linear operator (or simply an operator) A from $\mathcal{X}$ into $\mathcal{Y}$ is a function which sends every vector u in a certain linear subspace $\mathcal{D}(A)$, called the domain of A, of $\mathcal{X}$ to a vector $v = Au = A(u) \in \mathcal{Y}$ and which satisfies the linearity condition:

$$A(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 A(u_1) + \alpha_2 A(u_2)$$

for all $u_1, u_2 \in \mathcal{D}(A)$ and $\alpha_1, \alpha_2 \in \mathcal{C}$. The range $\mathcal{R}(A)$ of the operator A from $\mathcal{X}$ into $\mathcal{Y}$ is defined as the set of all vectors of the form Au with $u \in \mathcal{D}(A)$. If $\mathcal{D}(A)$ is dense in $\mathcal{X}$, A is said to be densely defined. If $\mathcal{D}(A) = \mathcal{X}$, A is said to be defined on $\mathcal{X}$. If $\mathcal{X} = \mathcal{Y}$, we shall say that A is an operator in $\mathcal{X}$. If $\mathcal{Y} = \mathcal{C}$, A is called a linear functional.

If A and B are operators from $\mathcal{X}$ into $\mathcal{Y}$ and $\alpha, \beta \in \mathcal{C}$, the operator $\alpha A + \beta B$ is the operator from $\mathcal{X}$ into $\mathcal{Y}$ with domain $\mathcal{D}(\alpha A + \beta B) = \mathcal{D}(A) \bigcap \mathcal{D}(B)$ and which satisfies

$$(\alpha A + \beta B)(x) = \alpha A(x) + \beta B(x),$$

for all $x \in \mathcal{D}(A) \bigcap \mathcal{D}(B)$.

If A is an operator from $\mathcal{X}$ into $\mathcal{Y}$ and B is an operator from $\mathcal{Y}$ into $\mathcal{Z}$, where $\mathcal{Z}$ is a Banach space over $\mathcal{C}$, then the product BA is the operator from $\mathcal{X}$ into $\mathcal{Z}$ with domain $\mathcal{D}(BA) = \{x \in \mathcal{D}(A) : Ax \in \mathcal{D}(B)\}$ and $(BA)(x) = B(A(x))$ for all $x \in \mathcal{D}(BA)$.

An operator A from $\mathcal{X}$ into $\mathcal{Y}$ is said to be invertible if there exists an operator, denoted by A^{-1}, from $\mathcal{Y}$ into $\mathcal{X}$ such that

$$\mathcal{D}(A^{-1}) = \mathcal{R}(A), \ \mathcal{R}(A^{-1}) = \mathcal{D}(A),$$

and

$$A^{-1}(Au) = u, \ A(A^{-1}v) = v,$$

for all $u \in \mathcal{D}(A)$, $v \in \mathcal{R}(A)$.

The operator A^{-1} is called the inverse of A. It can be easily shown that if A is an invertible operator from $\mathcal{X}$ into $\mathcal{Y}$, then A^{-1} is unique. Also if A and B are invertible operators in $\mathcal{X}$, then AB is invertible in $\mathcal{X}$ and $(AB)^{-1} = B^{-1}A^{-1}$.

An operator A from $\mathcal{X}$ into $\mathcal{Y}$ is called bounded if

$$\| A \| = \sup\{\| Au \|_{\mathcal{Y}} : u \in \mathcal{D}(A), \| u \|_{\mathcal{X}} \leq 1\} < \infty. \tag{1.1.1}$$

In this case $\| A \|$ is called the norm (or the bound) of A. When there is no cause for confusion, we use the same notation for the norms on $\mathcal{X}$ and $\mathcal{Y}$. The space of all bounded operators A from $\mathcal{X}$ into $\mathcal{Y}$ with domains $\mathcal{D}(A) = \mathcal{X}$ is denoted by $\mathcal{L}(\mathcal{X},\mathcal{Y})$. With the norm $\| A \|$ defined in equation (1.1.1) for every $A \in \mathcal{L}(\mathcal{X},\mathcal{Y})$, $\mathcal{L}(\mathcal{X},\mathcal{Y})$ becomes a Banach space. When $\mathcal{X} = \mathcal{Y}$, we write $\mathcal{L}(\mathcal{X})$ instead of $\mathcal{L}(\mathcal{X},\mathcal{Y})$. It can be shown that if $A, B \in \mathcal{L}(\mathcal{X})$ then $AB \in \mathcal{L}(\mathcal{X})$ and $\| AB \| \leq \| A \| \| B \|$. The operator in $\mathcal{L}(\mathcal{X})$ which sends every $x \in \mathcal{X}$ into itself is called the identity operator and is denoted by I.

Throughout the thesis it will be required to find the inverse of operators of the form $I - A$, where $A \in \mathcal{L}(\mathcal{X})$ and $\| A \| < 1$. The formula for $(I - A)^{-1}$ is suggested by the geometric series :

$$1 + a + a^2 + \cdots = (1 - a)^{-1},\ 0 \leq a < 1.$$

THEOREM 1.1.1 Suppose $A \in \mathcal{L}(\mathcal{X})$ and $\| A \| < 1$. Then $(I - A)^{-1}$ exists and is in $\mathcal{L}(\mathcal{X})$. Moreover we have:

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$$

and

$$\| (I - A)^{-1} \| \leq (1 - \| A \|)^{-1},$$

where $A^0 = I$, $A^2 = AA$, $A^3 = AA^2$, etc.

See Theorem 8.1, page 70 of [9] (which holds true in Banach spaces) for the proof.

An operator which is not bounded is called unbounded.

An operator $A \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ is called compact if for every bounded sequence $\{u_n\}$ of points in $\mathcal{X}$,the sequence $\{Au_n\}$ in $\mathcal{Y}$ has a Cauchy subsequence. The following theorem shows that the product of a compact operator by a bounded operator is a compact operator.

THEOREM 1.1.2 Suppose $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is compact. If $B \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ and $C \in \mathcal{L}(\mathcal{Z}, \mathcal{X})$, where $\mathcal{Z}$ is a Banach space over $\mathcal{C}$, then BA and AC are compact.

See Theorem 4.8, page 158 of [15] for the proof.

The following theorem shows that the space of all compact operators in $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is a closed subspace of $\mathcal{L}(\mathcal{X}, \mathcal{Y})$.

THEOREM 1.1.3 Suppose $\{K_n\}$ is a sequence of compact operators in $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $\lim_{n\to\infty} \| K_n - K \| = 0$, where $K \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. Then K is compact.

See Theorem 4.7, page 158 of [15] for the proof.

REMARK 1.1.4 If $K \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is of finite rank, that is, $\dim \mathcal{R}(K)$ is finite, where $\dim \mathcal{R}(K)$ denotes the dimension of $\mathcal{R}(K)$ (in general if Z is a subspace of X, $\dim Z$ denotes the dimension of Z), then K is compact. For the proof, see [9], page 83.

The graph of an operator A defined in $\mathcal{X}$ is the set

$$\mathcal{G}(A) = \{(u, Au) : u \in \mathcal{D}(A)\}.$$

The set $\mathcal{G}(A)$ is a subspace of the product space $\mathcal{X} \times \mathcal{X}$, which is a Banach space with the norm: $\| (x, y) \| = \| x \| + \| y \|$ for all $x, y \in \mathcal{X}$ (other choices of the norm on $\mathcal{X} \times \mathcal{X}$ are possible; for example, $\| (x, y) \| = \sqrt{\| x \|^2 + \| y \|^2}$). An operator A in $\mathcal{X}$ is called closed if $\mathcal{G}(A)$ is a closed subspace of $\mathcal{X} \times \mathcal{X}$. It can be easily seen that an operator A in $\mathcal{X}$ is closed if and only if the conditions $\lim_{n\to\infty} u_n = u$, where $u_n \in \mathcal{D}(A)$ for all $n \in \mathcal{N}$ and $u \in \mathcal{X}$, and $\lim_{n\to\infty} Au_n = v$, where $v \in \mathcal{X}$, imply that $u \in \mathcal{D}(A)$ and $v = Au$. Also it can be shown that if $A \in \mathcal{L}(\mathcal{X})$ then A is closed if and only if $\mathcal{D}(A)$ is closed. The space of all closed operators in $\mathcal{X}$ is denoted by $\mathcal{C}(\mathcal{X})$. If A is an operator in $\mathcal{X}$, the inverse graph $\mathcal{G}'(A)$ is defined as:

$$\mathcal{G}'(A) = \{(Au, u) : u \in \mathcal{D}(A)\}.$$

It is clear that A is closed if and only if $\mathcal{G}'(A)$ is a closed subspace of the product space $\mathcal{X} \times \mathcal{X}$.

If A is an invertible operator in $\mathcal{X}$, then clearly

$$\mathcal{G}(A) = \mathcal{G}'(A^{-1}).$$

Thus A is closed if and only if A^{-1} is closed.

EXAMPLE 1.1.5 Let $\{\alpha_i\}$ be a sequence of complex numbers and suppose that either $\lim_{i\to\infty} | \alpha_i | = 0$ or $\lim_{i\to\infty} | \alpha_i | = \infty$. Let D be the infinite diagonal matrix with α_i, $i \in \mathcal{N}$, on the diagonal (we write $D = \operatorname{diag}(\alpha_1, \alpha_2, \ldots)$), and consider D as an operator (called a matrix operator) acting in ℓ_p, $1 \leq p \leq \infty$. Then D is closed in ℓ_p. From the previous paragraph it is sufficient to prove that D is closed in the case $\lim_{i\to\infty} | \alpha_i | = 0$. But this is clear since $D \in \mathcal{L}(\ell_p)$ (in this case $\| D \| = \max\{| \alpha_i | : i \in \mathcal{N}\}$).

The notion of the dual of an operator will be used in the thesis. The space $\mathcal{L}(\mathcal{X}, \mathcal{C})$ is denoted by $\mathcal{X}'$.

THEOREM 1.1.6 Let T be an operator in $\mathcal{X}$. Consider the points $(x', y') \in \mathcal{X}' \times \mathcal{Y}'$ satisfying the condition:

$$y'(Tx) = x'(x) \tag{1.1.2}$$

for all $x \in \mathcal{D}(T)$. Then x' is determined uniquely by y' if and only if $\mathcal{D}(T)$ is dense in $\mathcal{X}$.

See Theorem 1, page 193 of [34] for the proof.

DEFINITION 1.1.7 Let T be an operator in $\mathcal{X}$ with a dense domain $\mathcal{D}(T)$. The dual T' of T is defined as follows: the domain $\mathcal{D}(T')$ of T' is the totality of those $y' \in \mathcal{X}'$ such that there exists $x' \in \mathcal{X}'$ satisfying equation (1.1.2). If $y' \in \mathcal{D}(T')$, then $T'y' \in \mathcal{X}'$ such that $(T'y')(x) = y'(Tx)$ for all $x \in \mathcal{D}(T)$. From Theorem 1.1.6, T' is a linear operator in $\mathcal{X}'$.

The following is due to R.S. Phillips.

THEOREM 1.1.8 Let T be an operator in $\mathcal{X}$ with an inverse and suppose that $\mathcal{D}(T)$ and $\mathcal{R}(T)$ are both dense sets in $\mathcal{X}$. Then:

$$(T')^{-1} = (T^{-1})'.$$

See Theorem 1, page 224 of [34] for the proof.

1.2 Resolvents and Spectra

Let A be an operator in $\mathcal{X}$. A point $\varsigma \in \mathcal{C}$ is called a regular point of A if $A - \varsigma I$ is invertible and we write

$$R(\varsigma, A) = R(\varsigma) = (A - \varsigma I)^{-1}.$$

The set of all regular points of an operator A is called the resolvent set of A and is denoted by $\rho(A)$. The operator $R(\varsigma, A)$ is called the resolvent operator of A. Thus $R(\varsigma, A)$ has domain $\mathcal{X}$ and range $\mathcal{D}(A)$. The complement of $\rho(A)$, that is, the set $\{\lambda \in \mathcal{C} : \lambda \notin \rho(A)\}$ is called the spectrum of A and is denoted by $\sigma(A)$.

The following theorem shows that the spectrum of an operator is a closed set in $\mathcal{C}$.

THEOREM 1.2.1 Let A be an operator in $\mathcal{X}$. Then the resolvent set $\rho(A)$ is open in $\mathcal{C}$.

See Theorem 2, page 211 of [34] for the proof.

In the finite dimensional case, the spectrum of an operator A consists of a finite number of points (the eigenvalues of A), but the situation is much more complicated in the infinite dimensional case. In this case it is possible that the spectrum can be an uncountable set or the empty set as shown from the following examples.

EXAMPLE 1.2.2 Let $\mathcal{X} = \mathcal{C}[a,b]$ be the space of all complex valued continuous functions on the closed interval $[a,b]$. Define an operator T in $\mathcal{X}$ as : $\mathcal{D}(T)$ is the set of all continuously differentiable functions on $[a,b]$ and $Tu(x) = \frac{du(x)}{dx}$ (that is T is the derivative of u with respect to x), for all $u \in \mathcal{D}(T)$. Then $\sigma(T) = \mathcal{C}$. In fact for every $\lambda \in \mathcal{C}$, the equation $(T - \lambda I)u = \frac{du(x)}{dx} - \lambda u(x) = 0$ has a nontrivial solution $u(x) = e^{\lambda x}$, which belongs to $\mathcal{X}$.

Let $\mathcal{D}_1 = \{u \in \mathcal{D}(T) : u(a) = 0\}$. Define the operator T_1 in $\mathcal{X}$ as: $\mathcal{D}(T_1) = \mathcal{D}_1$ and $T_1 u(x) = \frac{du(x)}{dx}$. Then the spectrum $\sigma(T_1)$ of T_1 is the empty set. In fact the resolvent $R_1(\varsigma) = R(\varsigma, T_1)$ exists for every $\varsigma \in \mathcal{C}$ and is given by

$$R_1(\varsigma)v(y) = e^{\varsigma y} \int_a^y e^{-\varsigma x} v(x)dx.$$

EXAMPLE 1.2.3 Let S_l be the operator in ℓ_p, $1 \leq p \leq \infty$, defined as: $\mathcal{D}(S_l) = \ell_p$ and $S_l(x) = S_l(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$ for all $x = (x_1, x_2, \ldots) \in \ell_p$. Then $\sigma(S_l) = \{\lambda \in \mathcal{C} :\mid \lambda \mid \leq 1\}$ (see Example 1, page 283 of [28]). The operator S_l is called the left shift operator.

For a compact operator the spectrum is a countable set. More precisely, we have:

THEOREM 1.2.4 Let $K \in \mathcal{L}(\mathcal{X})$ be compact and $\mathcal{X}$ infinite dimensional. The spectrum of K is a countable set $\lambda_1, \lambda_2, \ldots$ which includes $\lambda = 0$. If $\lambda_i \neq 0$, then it is an eigenvalue of K. If $\{\lambda_i\}$ is an infinite set, then $\lim_{i\to\infty} \lambda_i = 0$.

See Theorem 6.1, page 248 of [9] for the proof.

A compact operator and its dual (see Definition 1.1.7) have the same nonzero eigenvalues.

THEOREM 1.2.5 Let K be a compact operator on $\mathcal{X}$. Then any nonzero number is an eigenvalue of K if and only if it is an eigenvalue of K'.

See Theorem 2, page 284 of [34] for the proof.

The relation between the spectrum of an invertible operator and the spectrum of its inverse is given in the following theorem.

THEOREM 1.2.6 Let A be an invertible operator in $\mathcal{X}$. Then

(a) Any nonzero complex number λ is an eigenvalue of A if and only if λ^{-1} is an eigenvalue of A^{-1}.

(b) The sets $\sigma(A) \setminus \{0\} = \{z \in \sigma(A) : z \neq 0\}$ and $\sigma(A^{-1}) \setminus \{0\}$ are mapped onto each other by the mapping $z \rightarrow z^{-1}$.

Proof. (a) Let λ be a nonzero complex number. The result follows from the equivalence of the following statements: $Ax = \lambda x$, $A^{-1}(Ax) = A^{-1}(\lambda x)$, $x = \lambda(A^{-1}x)$, $A^{-1}x = \lambda x$.

See Theorem 6.15, page 177 of [15] for the proof of part (b).

1.3 Stability Theorems

Let $T \in \mathcal{C}(\mathcal{X})$ and F be an operator in $\mathcal{X}$. The basic problem in perturbation theory for closed operators is to study the relation between the spectral properties of T and $T + F$. The operator F is called the perturbation.

In this section theorems investigating the stability, under small perturbations, of various spectral properties of closed operators in Banach spaces are stated. It is necessary to make precise what is meant by "small" perturbation. In this thesis we use the concept of the gap between the closed operators $S = T + F$ and T in measuring the smallness of the perturbation F.

DEFINITION 1.3.1 Let T and S be closed operators in $\mathcal{X}$, and let $\mathcal{G}(T)$ and $\mathcal{G}(S)$ be their graphs, respectively.

Define $\mathcal{G}_1(T) = \{(u, Tu) \in \mathcal{G}(T) :\| (u, Tu) \|= 1\}$.

Set $\delta(T, S) = \sup\{\text{dist}(w, \mathcal{G}(S)) : w \in \mathcal{G}_1(T)\}$, where $\text{dist}(w, G(S))$ denotes the distance between w and the closed set $\mathcal{G}(S)$ in the product space $\mathcal{X} \times \mathcal{X}$.

Let $\hat{\delta}(T, S) = \max\{\delta(T, S), \delta(S, T)\}$. Then $\hat{\delta}(T, S)$ is called the gap between T and S (or between S and T).

Some properties of the gap between two closed operators are listed in the following theorem.

THEOREM 1.3.2 Let T and S be closed operators in $\mathcal{X}$.

(1) If $T \in \mathcal{L}(\mathcal{X})$ and $\hat{\delta}(S, T) < \frac{1}{\sqrt{1+\|T\|^2}}$, then $S \in \mathcal{L}(\mathcal{X})$ and

$$\| S - T \| \leq \frac{(1 + \| T \|^2)\delta(S, T)}{1 - \sqrt{1 + \| T \|^2}\delta(S, T)}.$$

(2) If $S = T + A$ and $A \in \mathcal{L}(\mathcal{X})$, then

$$\hat{\delta}(T + A, T) \leq \| A \| .$$

(3) If $A \in \mathcal{L}(\mathcal{X})$, then

$$\hat{\delta}(S + A, T + A) \leq 2(1 + \| A \|^2)\hat{\delta}(S, T).$$

(4) If T and S are invertible,then

$$\delta(S^{-1}, T^{-1}) = \delta(S, T)$$

and

$$\hat{\delta}(S^{-1}, T^{-1}) = \hat{\delta}(S, T).$$

(5) If T is invertible with $T^{-1} \in \mathcal{L}(\mathcal{X})$ and $\hat{\delta}(S,T) < \left(1 + \| T^{-1} \|^2\right)^{-1/2}$, then S is invertible and $S^{-1} \in \mathcal{L}(\mathcal{X})$.

See Theorems 2.13,14,17,20 and 21 in Chapter IV of [15] for the proofs of (1),(2),(3),(4) and (5), respectively.

The first spectral property we will discuss is the upper semicontinuity of the spectrum. The following theorem shows that the spectrum of a bounded operator is upper semicontinuous.

THEOREM 1.3.3 (See Remark 3.3, page 208 of [15].) If $T \in \mathcal{L}(\mathcal{X})$, then for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\sup\{\mathrm{dist}(\lambda, \sigma(T)) : \lambda \in \sigma(S)\} < \epsilon,$$

if $\| S - T \| < \delta$.

Proof. Let $\epsilon > 0$ and $\Gamma = \{\varsigma \in C : \mathrm{dist}(\varsigma, \sigma(T)) \geq \epsilon\}$. Γ is nonempty since $\sigma(T)$ is a bounded set (if $| \varsigma | > \| T \|$ then $\varsigma \in \rho(T)$). Let $\varsigma \in \Gamma$ and $S \in \mathcal{L}(\mathcal{X})$. From the equality $S - \varsigma I = (T - \varsigma I)[I + R(\varsigma, T)(S - T)]$ and Theorem 1.1.1, it follows that if $\| S - T \| < \| R(\varsigma, T) \|^{-1}$ then

$$\varsigma \in \rho(S). \tag{1.3.1}$$

Now we show that the set $\{\| R(\varsigma, T) \|^{-1} : \varsigma \in \Gamma\}$ has a positive minimum. Since for $\varsigma \in \rho(T)$, $\| R(\varsigma, T) \| \to 0$ as $\varsigma \to \infty$ then we can find $\varsigma_1 \in \Gamma$ and a positive $M > \max\{| \varsigma_1 |, \| T \|\}$ such that

$$\| R(\varsigma, T) \|^{-1} > \| R(\varsigma_1, T) \|^{-1}, \tag{1.3.2}$$

if $| \varsigma | > M$. Since $\| R(\varsigma, T) \|$ is continuous in ς, then $\| R(\varsigma, T) \|^{-1}$ has a positive minimum δ on the compact set $\Gamma_1 = \{\varsigma \in \Gamma : | \varsigma | \leq M\}$. From Equation(1.3.2), we have $\delta = \min\{\| R(\varsigma, T) \|^{-1} : \varsigma \in \Gamma\}$. Therefore from Equation(1.3.1), it follows that $\Gamma \subset \rho(S)$ if $\| S - T \| < \delta = \min\{\| R(\varsigma, T) \|^{-1} : \varsigma \in \Gamma\}$. From the definition of Γ this means that there is a positive δ such that $\sup\{\mathrm{dist}(\lambda, \sigma(T)) : \lambda \in \sigma(S)\} < \epsilon$ if $\| S - T \| < \delta$, and this proves the requirement.

In the latter chapters we will be concerned with the case when the spectrum $\sigma(T)$ of a closed operator T contains a bounded part $\sigma_1(T)$ separated from the rest $\sigma_2(T)$ by a closed simple curve Γ consisting of regular points of T. The following theorem shows the stability of the separated part $\sigma_1(T)$ of the spectrum of T under small perturbations.

THEOREM 1.3.4 Let $T \in \mathcal{C}(\mathcal{X})$ and suppose that $\sigma(T)$ is separated into two parts $\sigma_1(T)$ and $\sigma_2(T)$ by a closed curve Γ consisting of regular points of T as stated above. Then there exists a δ depending on T and Γ such that for every $S \in \mathcal{C}(\mathcal{X})$ with $\hat{\delta}(S, T) < \delta$, $\sigma(S)$ is likewise separated by Γ into two parts $\sigma_1(S)$

and $\sigma_2(S)$ (Γ itself running in $\rho(S)$). Also we have:

$$\dim \mathcal{R}(P(\Gamma, T)) = \dim \mathcal{R}(P(\Gamma, S))$$

where

$$P(\Gamma, T) = \frac{-1}{2\pi\sqrt{-1}} \int_{\Gamma} R(\varsigma, T)\, d\varsigma$$

and

$$P(\Gamma, S) = \frac{-1}{2\pi\sqrt{-1}} \int_{\Gamma} R(\varsigma, S)\, d\varsigma.$$

See Theorem 3.16,page 212 of [15] for the proof.

DEFINITION 1.3.5 Let $T \in \mathcal{C}(\mathcal{X})$ and suppose the spectrum $\sigma(T)$ is separated into two parts as stated in Theorem 1.3.4 . If $\sigma_1(T) = \{\lambda_1, \ldots, \lambda_n\}$ is a finite set of distinct eigenvalues of T then around each λ_i construct a small closed curve Γ_i consisting of regular points of T and lying in $\operatorname{int}\Gamma$, where $\operatorname{int}\Gamma$ denotes the region enclosed by Γ. Also Γ_i satisfies $\lambda_j \notin \operatorname{int}\Gamma_i$ if $j \neq i$. If $P(\Gamma_i, T) = \frac{-1}{2\pi\sqrt{-1}} \int_{\Gamma_i} R(\varsigma, T)\, d\varsigma$ for all $i = 1, \ldots, n$, then $\dim \mathcal{R}(P(\Gamma_i, T))$ is called the algebraic multiplicity of λ_i. It is clear that if $\hat{\Gamma}_i$ is another closed curve around λ_i satisfying the above conditions, then $\dim \mathcal{R}(P(\hat{\Gamma}_i, T)) = \dim \mathcal{R}(P(\Gamma_i, T))$. So the algebraic multiplicity of λ_i does not depend on the choice of the closed curve Γ_i which satisfies the above properties. Also it can be proved that

$$P(\Gamma_i, T)P(\Gamma_j, T) = \begin{cases} P(\Gamma_i, T) & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

But since $P(\Gamma, T) = P(\Gamma_1, T) + \cdots + P(\Gamma_n, T)$, where $P(\Gamma, T)$ is defined in Theorem 1.3.4, then

$$\dim \mathcal{R}(P(\Gamma, T)) = \dim \mathcal{R}(P(\Gamma_1, T)) + \cdots + \dim \mathcal{R}(P(\Gamma_n, T)).$$

$\dim \mathcal{R}(P(\Gamma, T))$ is called the total algebraic multiplicity of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of T inside Γ.

REMARK 1.3.6 If in Theorem 1.3.4, $\dim \mathcal{R}(P(\Gamma, T)) = m < \infty$, then $\sigma_1(T)$ consists of a finite system of eigenvalues with the total algebraic multiplicity m.

1.4 Gersgorin Theorems for Finite Matrices

Let C^n be the set of all column vectors $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, where $x_i \in C$ for all $i = 1, \ldots, n$.

The set of all $n \times n$ matrices with complex entries is denoted by $C^{n \times n}$.

A description of regions of the complex plane containing the eigenvalues $\lambda_1, \ldots, \lambda_n$ of a matrix $A \in C^{n \times n}$ is presented in this section.

REMARK 1.4.1 If $A \in C^{n \times n}$ and $\lambda \in C$ then the polynomial $\det(A - \lambda I)$, where $\det(A - \lambda I)$ denotes the determinant of $A - \lambda I$, is called the characterstic polynomial. It is clear that the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A are the solutions of $\det(A - \lambda I) = 0$. Hence we have

$$(-1)^n \det(A - \lambda I) = \prod_{i=1}^{n} (\lambda - \lambda_i) = (\lambda - \lambda_1) \ldots (\lambda - \lambda_n). \qquad (1.4.1)$$

It can be proved that the algebraic multiplicity of the eigenvalue $\lambda_i, i \in \{1, \ldots, n\}$ of A defined in Definition 1.3.5 is equal to the number of times the factor $\lambda - \lambda_i$ appears in equation(1.4.1).

The following theorem shows that the zeros of a polynomial depend continuously on its coefficients.

THEOREM 1.4.2 Let $n \geq 1$ and let

$$p(x) = a_n x^n + \cdots + a_1 x + a_0,$$

$a_n \neq 0$, be a polynomial with complex coefficients. Then, for every $\epsilon > 0$, there is a $\delta > 0$ such that for any polynomial

$$q(x) = b_n x^n + \cdots + b_1 x + b_0,$$

satisfying $b_n \neq 0$ and

$$\max\{| a_i - b_i |: 0 \leq i \leq n\} < \delta,$$

there is a permutation τ of $1, \ldots, n$ with

$$\max\{| \lambda_i - \mu_{\tau(i)} |: 1 \leq i \leq n\} < \epsilon,$$

where $\lambda_1, \ldots, \lambda_n$ are the zeros of $p(x)$ and $\mu_1, \ldots, \mu_n$ are the zeros of $q(x)$ in some order, counting multiplicities.

See Appendix D of [12] for the proof.

We now prove one of the most useful and easily applied theorems that give bounds for the eigenvalues of finite matrices. This is known as the Gersgorin Theorem and was first published as recently as 1931.

THEOREM 1.4.3 (See Theorem 1, page 371 of [16].) If $A \in C^{n \times n}$ and a_{ij} denotes the elements of A, $i, j = 1, \ldots, n$ and

$$\rho_i = \sum_{j=1}^{n}{}' \mid a_{ij} \mid,$$

where $\sum_{j=1}^{n}{}'$ denotes the sum from $j = 1$ to $j = n$ excluding $j = i$, then every eigenvalue of A lies in at least one of the discs

$$\{z \in C : \mid z - a_{ii} \mid \leq \rho_i\},$$

$i = 1, \ldots, n$, in the complex plane.

Furthermore, a set of r discs whose union is disjoint from the remaining $n - r$ discs contains r and only r eigenvalues (counting their multiplicities) of A.

Proof. Let λ be an eigenvalue of A with the associated eigenvector x. Then $Ax = \lambda x$ or, writing this relation out as n scalar equations,

$$\sum_{j=1}^{n} a_{ij} x_j = \lambda x_i, \ i = 1, \ldots, n,$$

where $x_1, \ldots, x_n$ are the entries of the column vector x. Let $| x_p | = \max\{| x_i | : i = 1, \ldots, n\}$; then the pth equation in the above system gives:

$$| \lambda - a_{pp} || x_p | = | \sum_{j=1}^{n}{}' a_{pj} x_j | \leq \sum_{j}^{n}{}' | a_{pj} || x_j | \leq | x_p | \sum_{j=1}^{n}{}' | a_{pj} |.$$

Since $x \neq 0$ it must be that $| x_p | \neq 0$, and so we have

$$| \lambda - a_{pp} | \leq \rho_p = \sum_{j=1}^{n}{}' | a_{pj} |.$$

This proves the first part.

To prove the second part it is sufficient to prove if

$$\mathcal{G}_r = \bigcup_{i=1}^{r} \{ z \in C : | z - a_{N_i N_i} | \leq \rho_{N_i} \},$$

where $N_i \in \{1, \ldots, n\}$ for all $i = 1, \ldots, r$, is a connected set disjoint from the remaining $n - r$ discs, then $\mathcal{G}_r$ contains precisely r eigenvalues of A (counting their multiplicities).

Let $A = D + F$, where $D = \mathrm{diag}(a_{11}, \ldots, a_{nn})$, and define $A(t) = D + tF$ for all $t \in [0, 1]$. From the first part, the eigenvalues of $A(t)$, $t \in [0, 1]$, lie in the set

$$\bigcup_{i=1}^{n} \{ z \in C : | z - a_{ii} | \leq t\rho_i \},$$

which is a subset of $\bigcup_{i=1}^{n}\{z \in C : \mid z - a_{ii} \mid \leq \rho_i\}$. At $t = 0$, the eigenvalues of $A(0) = D$ are $a_{11}, \ldots, a_{nn}$. Since the eigenvalues of a matrix are continuous functions of the entries of that matrix (this follows from Theorem 1.4.2), then the eigenvalues of $A(t)$ are continuous functions of t, $t \in [0,1]$.Thus each $a_{ii}, i \in \{1, \ldots, n\}$, is joined to an eigenvalue of A, denoted by $\lambda_i = \lambda_i(1)$, by a continuous curve in the complex plane consisting of eigenvalues of $A(t)$. Denote such a curve by $\{\lambda_i(t) : 0 \leq t \leq 1\}$. Now we prove $\lambda_i \in \mathcal{G}_r$ for all $i \in \{N_1, \ldots, N_r\}$. If there were $k \in \{N_1, \ldots, N_r\}$ such that $\lambda_k \notin \mathcal{G}_r$, then from the first part λ_k is in one of the Gersgorin discs $\{z \in C : \mid z - a_{ii} \mid \leq \rho_i\}$ where $i \in \{1, \ldots, n\} \setminus \{N_1, \ldots, N_r\}$. Now the intermediate value theorem can be applied to the continuous curve $\{\lambda_k(t) : 0 \leq t \leq 1\}$ defined on the connected set $[0,1]$ to deduce that there is a $t_k \in (0,1]$ such that $\lambda_k(t_k) \notin \bigcup_{i=1}^{n}\{z \in C : \mid z - a_{ii} \mid \leq \rho_i\}$, which is impossible. Thus $\mathcal{G}_r$ contains at least r eigenvalues of A. Similarly the intermediate value theorem can be used to prove that $\lambda_i \notin \mathcal{G}_r$ for all $i \in \{1, \ldots, n\} \setminus \{N_1, \ldots, N_r\}$. Hence $\mathcal{G}_r$ contains precisely r eigenvalues of A (counting their multiplicities since we may have $\lambda_i(1) = \lambda_j(1)$ for $i, j \in \{N_1, \ldots, N_r\}$ and $i \neq j$).

Chapter 2

Gersgorin Theory for Diagonally Dominant Infinite Matrices with Bounded Perturbations

In this chapter we extend Gersgorin theory applied to the set of all finite square matrices (see Theorem 1.4.3) to a set of diagonally dominant infinite matrices with bounded perturbations.

Let $A = (a_{ij})$ be a matrix operator defined in ℓ_p, $1 \leq p \leq \infty$.

A is called row diagonally dominant if for all $i \in \mathcal{N}$,

$$| a_{ii} | > \sum_{j=1}^{\infty}{}' | a_{ij} |$$

where $\sum_{j=1}^{\infty}{}' | a_{ij} |$ denotes $(\sum_{j=1}^{\infty} | a_{ij} |) - | a_{ii} |$.

A is called colmun diagonally dominant if for all $j \in \mathcal{N}$,

$$| a_{jj} | > \sum_{i=1}^{\infty}{}' | a_{ij} |$$

where $\sum_{i=1}^{\infty}{}' | a_{ij} |$ denotes $(\sum_{i=1}^{\infty} | a_{ij} |) - | a_{ii} |$.

2.1 Notations and Preliminary Results

For a given matrix operator $A = (a_{ij})$ in ℓ_p, $1 \leq p \leq \infty$, and $x \in \mathcal{D}(A)$, the ith component of the vector Ax is denoted by $(Ax)_i$. We define the row and column sums of A :

$$P_i = \sum_{j=1}^{\infty}{}' | a_{ij} |, \; Q_i = \sum_{j=1}^{\infty}{}' | a_{ji} | \qquad (2.1.1)$$

and the corresponding Gersgorin discs (where they exist):

$$\mathcal{R}_i = \{z \in \mathcal{C} : | z - a_{ii} | \leq P_i\}, \; \mathcal{C}_i = \{z \in \mathcal{C} : | z - a_{ii} | \leq Q_i\}. \tag{2.1.2}$$

The following lemma gives the relation between the norm of a bounded matrix operator written in a block matrix form and the norms of its submatrices.

LEMMA 2.1.1 Let F be a matrix operator in ℓ_p, $1 \leq p \leq \infty$, and suppose that F is written in the block matrix form :

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$$

where F_{11} is the $(n \times n)$ leading submatrix of F.

Then $F \in \mathcal{L}(\ell_p)$ if and only if $F_{12} \in \mathcal{L}(\ell_p, \mathcal{C}^n)$, $F_{21} \in \mathcal{L}(\mathcal{C}^n, \ell_p)$ and $F_{22} \in \mathcal{L}(\ell_p)$.

Moreover if F is bounded, then

$$\| F \| \leq \sum_{i,j=1}^{2} \| F_{ij} \|, \; \| F_{ij} \| \leq \| F \|$$

for all $i, j = 1, 2$.

Proof. Suppose $F_{12} \in \mathcal{L}(\ell_p, \mathcal{C}^n)$, $F_{21} \in \mathcal{L}(\mathcal{C}^n, \ell_p)$ and $F_{22} \in \mathcal{L}(\ell_p)$. Let $x \in \ell_p$. Writing $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, where $x_1 \in \mathcal{C}^n$, we find from the assumption that $F_{11}x_1 + F_{12}x_2 \in \mathcal{C}^n$ and $F_{21}x_1 + F_{22}x_2 \in \ell_p$. Since $Fx = \begin{pmatrix} F_{11}x_1 + F_{12}x_2 \\ F_{21}x_1 + F_{22}x_2 \end{pmatrix}$ then $Fx \in \ell_p$. Thus $\mathcal{D}(F) = \ell_p$. Hence F can be written in the form:

$$F = \begin{pmatrix} F_{11} & O \\ O & O \end{pmatrix} + \begin{pmatrix} O & O \\ F_{21} & O \end{pmatrix} + \begin{pmatrix} O & F_{12} \\ O & O \end{pmatrix} + \begin{pmatrix} O & O \\ O & F_{22} \end{pmatrix} \tag{2.1.3}$$

where each block matrix in the right hand side of equation (2.1.3) has domain ℓ_p.(We did not distinguish between the zero operators written above though they do not have necessarily the same domains or ranges.) From the assumption and equation (2.1.3) , it follows that $F \in \mathcal{L}(\ell_p)$ and $\| F \| \leq \| F_{11} \| + \| F_{12} \| + \| F_{21} \| + \| F_{22} \|$.

Now suppose $F \in \mathcal{L}(\ell_p)$.The submatrices of F can be written as :

$$F_{11} = \begin{pmatrix} I_{n\times n} & 0_{n\times\infty} \end{pmatrix} \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \begin{pmatrix} I_{n\times n} \\ 0_{\infty\times n} \end{pmatrix} \tag{2.1.4}$$

$$F_{21} = \begin{pmatrix} 0_{\infty\times n} & I_{\infty\times\infty} \end{pmatrix} \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \begin{pmatrix} I_{n\times n} \\ 0_{\infty\times n} \end{pmatrix} \tag{2.1.5}$$

$$F_{12} = \begin{pmatrix} I_{n\times n} & 0_{n\times\infty} \end{pmatrix} \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \begin{pmatrix} 0_{n\times\infty} \\ I_{\infty\times\infty} \end{pmatrix} \tag{2.1.6}$$

$$F_{22} = \begin{pmatrix} 0_{\infty\times n} & I_{\infty\times\infty} \end{pmatrix} \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \begin{pmatrix} 0_{n\times\infty} \\ I_{\infty\times\infty} \end{pmatrix}. \tag{2.1.7}$$

The indices under the submatrices of the block matrices to the right and to the left of F in each equation of the above four equations indicate the size of these submatrices. For example, $0_{n\times\infty}$ is the zero matrix operator from ℓ_p into C^n. Since in each equation of the above four equations the block matrices to the right and to the left of F are bounded (the norm of each one is equal to one), then the boundedness of F implies the boundedness of each submatrix F_{ij} and that $\| F_{ij} \| \leq \| F \|$, $i, j = 1, 2$. This completes the proof of the lemma. □

DEFINITION 2.1.2 The Kronecker delta δ_{ij} is defined by:

$$\delta_{ij} = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases}$$

The following theorem will be used in the proof of Theorem 2.4.1.

THEOREM 2.1.3 Let $A = (a_{ij})$ be a matrix operator in ℓ_p, $1 \le p \le \infty$, $D = \operatorname{diag}(a_{11}, a_{22}, \ldots)$ and assume that

(1) $a_{ii} \ne 0$ for all $i \in \mathcal{N}$ and $| a_{ii} | \to \infty$ as $i \to \infty$.

(2) For every $i \in \mathcal{N}$ there exists $\sigma_i \in [0,1)$ such that

$$P_i = \sum_{j=1}^{\infty}{}' \mid a_{ij} \mid = \sigma_i \mid a_{ii} \mid.$$

(3) The matrix operator $F = ((1 - \delta_{ij})a_{ij})$ is in $\mathcal{L}(\ell_p)$.

(4) Either the matrix operator $I + FD^{-1}$ has a bounded inverse on ℓ_p or the matrix operator $I + D^{-1}F$ has a bounded inverse on ℓ_p.

Then if A is written in the block matrix form:

$$A = \begin{pmatrix} A_{11}^{(n)} & A_{12}^{(n)} \\ A_{21}^{(n)} & A_{22}^{(n)} \end{pmatrix} \tag{2.1.8}$$

where $A_{11}^{(n)}$ is the leading $n \times n$ submatrix of A, $A_{11}^{(n)}$ has a bounded inverse for every $n \in \mathcal{N}$, and there exists $M > 0$ such that $\| (A_{11}^{(n)})^{-1} \| \le M$ for all $n \in \mathcal{N}$ (that is, $(A_{11}^{(n)})^{-1}$ is uniformly bounded in n).

Proof. We consider the case $(I + D^{-1}F)^{-1} \in \mathcal{L}(\ell_p)$. The other case (see hypothesis (4)) has no new features. Suppose A is written in the matrix form given in equation (2.1.8). From Gersgorin's theorem for finite matrices (Theorem 1.4.3), the eigenvalues of $A_{11}^{(n)}$ lie in the Gersgorin discs $\{z \in \mathcal{C} : \mid z - a_{ii} \mid \le \sum_{j=1}^{n}{}' \mid a_{ij} \mid\}$, $i = 1, \ldots, n$. Since each one of these discs does not contain the origin (this follows from hypothesis(2)), then 0 is not an eigenvalue of $A_{11}^{(n)}$. But since every point of the spectrum of a finite matrix is an eigenvalue of this matrix, then 0 is a regular point of $A_{11}^{(n)}$. Thus $(A_{11}^{(n)})^{-1}$ exists. For every $n \in \mathcal{N}$, write $A_{11}^{(n)} = D_{11}^{(n)} + F_{11}^{(n)}$

where $D_{11}^{(n)} = \text{diag}(a_{11}, \ldots, a_{nn})$. For every $n \in \mathcal{N}$, let

$$E_n = \begin{pmatrix} (D_{11}^{(n)})^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

be a matrix operator in ℓ_p. Then by writing F in the block matrix form

$$F = \begin{pmatrix} F_{11}^{(n)} & F_{12}^{(n)} \\ F_{21}^{(n)} & F_{22}^{(n)} \end{pmatrix}$$

we get:

$$E_nF = \begin{pmatrix} (D_{11}^{(n)})^{-1}F_{11}^{(n)} & (D_{11}^{(n)})^{-1}F_{12}^{(n)} \\ 0 & 0 \end{pmatrix}.$$

Therefore,

$$I + E_nF = \begin{pmatrix} I_n + (D_{11}^{(n)})^{-1}F_{11}^{(n)} & (D_{11}^{(n)})^{-1}F_{12}^{(n)} \\ 0 & I \end{pmatrix}, \tag{2.1.9}$$

where I_n denotes the $n \times n$ identity matrix.

Now $I + E_nF = I + D^{-1}F - (D^{-1} - E_n)F$, or by hypothesis(4),

$$I + E_nF = [I - (D^{-1} - E_n)F(I + D^{-1}F)^{-1}]\,(I + D^{-1}F). \tag{2.1.10}$$

Since $D^{-1} - E_n = \begin{pmatrix} 0 & 0 \\ 0 & D_n^{-1} \end{pmatrix}$, where $D_n^{-1} = \text{diag}(a_{n+1,n+1}^{-1}, a_{n+2,n+2}^{-1}, \ldots)$, and $| a_{nn}^{-1} | \to 0$ as $n \to \infty$ (see hypothesis(1)), then $D^{-1} - E_n \in \mathcal{L}(\ell_p)$ and $\| D^{-1} - E_n \| = \max\{| a_{ii}^{-1} | : i \geq n+1\}$. Choose $n_1 \in \mathcal{N}$ such that for all $n \geq n_1$,

$$| a_{nn}^{-1} | < (1 + \| F \|)^{-1}(\| (I + D^{-1}F)^{-1} \|)^{-1}. \tag{2.1.11}$$

From equation (2.1.10) and inequality (2.1.11), it follows that

$$\| D^{-1} - E_n \| \, \| F \| \, \| (I + D^{-1}F)^{-1} \| < \frac{\| F \|}{1 + \| F \|},$$

for all $n \geq n_1$. Thus from Theorem 1.1.1, the operator $I-(D^{-1}-E_n)F(I+D^{-1}F)^{-1}$ has a bounded inverse on ℓ_p and

$$\begin{aligned} \| (I-(D^{-1}-E_n)F(I+D^{-1}F)^{-1})^{-1} \| &\leq \frac{1}{1- \| D^{-1}-E_n \| \| F \| \| (I+D^{-1}F)^{-1} \|} \\ &\leq 1+ \| F \| \end{aligned}$$

for all $n \geq n_1$. Hence $I+E_nF$ has a bounded inverse on ℓ_p and

$$\| (I+E_nF)^{-1} \| \leq (1+ \| F \|) \| (I+D^{-1}F)^{-1} \| \tag{2.1.12}$$

for all $n \geq n_1$. From equation (2.1.9), we have:

$$(I+E_nF)^{-1} = \begin{pmatrix} (I_n+(D_{11}^{(n)})^{-1}F_{11}^{(n)})^{-1} & Y \\ 0 & I \end{pmatrix}$$

where $Y = -(D_{11}^{(n)})^{-1}F_{12}^{(n)}(I_n+(D_{11}^{(n)})^{-1}F_{11}^{(n)})^{-1}$ (notice that for all $n \in \mathcal{N}$, $I_n + (D_{11}^{n})^{-1}F_{11}^{(n)}$ has an inverse on $\mathcal{C}^n$, since $A_{11}^{(n)} = D_{11}^{(n)}(I_n+(D_{11}^{(n)})^{-1}F_{11}^{(n)})$ and $(A_{11}^{(n)})^{-1}$ exists). Hence from Lemma 2.1.1 and inequality (2.1.12), we have

$$\| (I_n+(D_{11}^{(n)})^{-1}F_{11}^{(n)})^{-1} \| \leq (1+ \| F \|) \| (I+D^{-1}F)^{-1} \| \tag{2.1.13}$$

for all $n \geq n_1$. Thus from $A_{11}^{(n)} = D_{11}^{(n)}(I_n+(D_{11}^{(n)})^{-1}F_{11}^{(n)})$, $n \in \mathcal{N}$ and inequality (2.1.13), it follows that

$$\| (A_{11}^{(n)})^{-1} \| \leq (1+ \| F \|) \| (I+D^{-1}F)^{-1} \| \| D^{-1} \|$$

for all $n \geq n_1$. Now the proof is completed by taking

$$M = \max\{(1+ \| F \|) \| (I+D^{-1}F)^{-1} \| \| D^{-1} \|, \max\{\| (A_{11}^{(i)})^{-1} \| : 1 \leq i \leq n_1-1\}\}.$$

□

REMARK 2.1.4 If hypothesis (2) in Theorem 2.1.3 is replaced by column diagonal dominance, the same result follows.

2.2 Generalized Convergence

A sequence of closed operators $\{T_n\}$ in a Banach space $\mathcal{X}$ is said to converge to the operator $T \in \mathcal{C}(\mathcal{X})$ in the generalized sense if $\hat{\delta}(T_n, T) \to 0$ as $n \to \infty$ (see Definition 1.3.1 for the definition of the gap between two closed operators). In this case we write $T_n \xrightarrow{g} T$. Some of the properties that connect the convergence in the generalized sense with bounded operators and their norms are listed in the following theorem.

THEOREM 2.2.1 (See Theorem 2.23, page 206 of [15].) Let $T, T_n \in \mathcal{C}(\mathcal{X}), n = 1, 2, \ldots$.

(1) If T^{-1} exists and is in $\mathcal{L}(\mathcal{X})$, then $T_n \xrightarrow{g} T$ if and only if T_n^{-1} exists and is in $\mathcal{L}(\mathcal{X})$ for sufficiently large n and $\| T_n^{-1} - T^{-1} \| \to 0$ as $n \to \infty$.

(2) If $T_n \xrightarrow{g} T$ and if $F \in \mathcal{L}(\mathcal{X})$, then $T_n + F \xrightarrow{g} T + F$.

Proof. (1) Suppose T^{-1} exists and is in $\mathcal{L}(\mathcal{X})$. Assume that the sequence $\{T_n\}$ converges to T in the generalized sense. Then there exists $n_1 \in \mathcal{N}$ such that $\hat{\delta}(T_n, T) \leq (1 + \| T^{-1} \|^2)^{-1/2}$ for all $n \geq n_1$. Hence from Theorem 1.3.2 part(5), it follows that T_n^{-1} exists and is in $\mathcal{L}(\mathcal{X})$ for all $n \geq n_1$. To prove $\| T_n^{-1} - T^{-1} \| \to 0$ as $n \to \infty$ we first notice from Theorem 1.3.2 part(4) that $\hat{\delta}(T_n, T) = \hat{\delta}(T_n^{-1}, T^{-1}) \leq (1 + \| T^{-1} \|^2)^{-1/2}$ for all $n \geq n_1$. Hence from Theorem 1.3.2 part(1), we have for all $n \geq n_1$:

$$\| T_n^{-1} - T^{-1} \| \leq \frac{(1 + \| T^{-1} \|^2)\, \delta(T_n^{-1}, T^{-1})}{1 - (1 + \| T^{-1} \|^2)^{1/2}\, \delta(T_n^{-1}, T^{-1})}. \tag{2.2.1}$$

But since $\delta(T_n, T) = \delta(T_n^{-1}, T^{-1})$ (by Theorem 1.3.2 part(4)) and $\delta(T_n, T) \leq \hat{\delta}(T_n, T)$, then the right hand side of inequality (2.2.1) converges to zero as $n \to \infty$ and this proves the "if" part of (1). To prove the "only if" part assume that there exists $n_1 \in \mathcal{N}$ such that for all $n \geq n_1$, T_n^{-1} exists and is in $\mathcal{L}(\mathcal{X})$ and suppose $\| T_n^{-1} - T^{-1} \| \to 0$ as $n \to \infty$. Writing $T_n^{-1} = T^{-1} + (T_n^{-1} - T^{-1})$, $n \geq n_1$, we find from Theorem 1.3.2 part(2) that

$$\hat{\delta}(T_n^{-1}, T^{-1}) = \hat{\delta}(T^{-1} + (T_n^{-1} - T^{-1}),\ T^{-1}) \leq \| T_n^{-1} - T^{-1} \| .$$

But since $\| T_n^{-1} - T^{-1} \| \to 0$ as $n \to \infty$, then from Theorem 1.3.2 part(4) it follows that $\hat{\delta}(T_n, T) \to 0$ as $n \to \infty$. This completes the proof of part(1).

(2) Let $T_N \xrightarrow{g} T$ and $F \in \mathcal{L}(\ell_p)$. Then from Theorem 1.3.2 part(3) we have:

$$\hat{\delta}(T_n + F, T + F) \leq 2(1 + \| F \|^2)\hat{\delta}(T_n, T).$$

Now the result follows since $2(1 + \| F \|^2)\hat{\delta}(T_n, T) \to 0$ as $n \to \infty$. □

2.3 The Dual of a Matrix Operator

In Section 1.1 we have defined the dual of an operator with dense domain (see Definition 1.1.7). In this section we show that for a matrix operator T in ℓ_p, $1 \leq p < \infty$, with domain containing the unit coordinate vectors $e_1, e_2, \ldots$, the dual T' in ℓ_p' and the transpose T^{tr} in ℓ_q, where $1/p + 1/q = 1$, have the same eigenvalues.

First we need the following theorem.

THEOREM 2.3.1 Given $f \in \ell_p'$, $1 \leq p < \infty$, there exists a unique $\eta = (\eta_1, \eta_2, \ldots) \in \ell_q$, $1/p + 1/q = 1$, ($q = \infty$ when $p = 1$), such that for all $\varsigma = (\varsigma_1, \varsigma_2, \ldots) \in \ell_p$,

$$f(\varsigma) = \sum_{k=1}^{\infty} \varsigma_k \eta_k. \tag{2.3.1}$$

Moreover, $\| f \| = \| \eta \|$ and $\eta = (f(e_1), f(e_2), \ldots)$ (the ith component of e_i is one and all of its other components are zeros, for every $i \in \mathcal{N}$).

Conversely, given $(\eta_1, \eta_2, \ldots) \in \ell_q$, equation(2.3.1) defines an $f \in \ell_p'$.

See Theorem 5.2, page 143 of [28] for the proof.

REMARK 2.3.2 For every $f \in \ell_p'$, $1 \leq p < \infty$, set

$$J(f) = (f(e_1), f(e_2), \ldots).$$

Then from Theorem 2.3.1, it is clear that J is a linear operator on ℓ_p' mapping ℓ_p' onto ℓ_q, $1/p + 1/q = 1$, and $\| J(f) \| = \| f \|$ for all $f \in \ell_p'$.

THEOREM 2.3.3 Let $T = (t_{ij})$ be a matrix operator in ℓ_p, $1 \leq p < \infty$, with domain $\mathcal{D}(T)$ containing the unit coordinate vectors $e_1, e_2, \ldots$. Then T' exists and for every $y' \in \ell_p'$, $y' \in \mathcal{D}(T')$ if and only if $J(y') = y \in \mathcal{D}(T^{\mathrm{tr}})$ (T^{tr} is an operator in ℓ_q). In this case we have

$$J(T'y') = T^{\mathrm{tr}}y.$$

Proof. Since $e_i \in \mathcal{D}(T)$ for all $i \in \mathcal{N}$ and $\mathrm{span}\{e_1, e_2, \ldots\}$ is dense in ℓ_p, then $\mathcal{D}(T)$ is dense in ℓ_p. Thus by Theorem 1.1.6, T' exists.

Let $y' \in \mathcal{D}(T')$ and $z = (z_1, z_2, \ldots) \in \mathcal{D}(T)$. Define for all $n \in \mathcal{N}$, $z^{(n)} = (z_1, \ldots, z_n, 0, 0, \ldots)$. Since $z^{(n)} \in \mathrm{span}\{e_1, \ldots, e_n\}$, then $z^{(n)} \in \mathcal{D}(T)$ and

$$(T'y')(z^{(n)}) = y'(Tz^{(n)}). \tag{2.3.2}$$

If $J(y') = y = (y_1, y_2, \ldots)$ then from equation (2.3.2) and Theorem 2.3.1, we have

$$(T'y')(z^{(n)}) = y'(Tz^{(n)}) = \sum_{i=1}^{\infty} y_i (\sum_{j=1}^{n} t_{ij} z_j) = \sum_{j=1}^{n} z_j (\sum_{i=1}^{\infty} t_{ij} y_i). \tag{2.3.3}$$

Since $T'y'$ is a continuous linear functional on ℓ_p (that is, if $u_n \to u$ as $n \to \infty$ in ℓ_p, then $(T'y')u_n \to (T'y')u$ as $n \to \infty$) and $z^{(n)} \to z$ as $n \to \infty$, then from equation (2.3.3), we have:

$$(T'y')(z) = \lim_{n \to \infty} (T'y')(z^{(n)}) = \sum_{j=1}^{\infty} z_j (\sum_{i=1}^{\infty} t_{ij} y_i). \tag{2.3.4}$$

From Theorem 2.3.1, $J(T'y') = ((T'y')(e_1), (T'y')(e_2), \ldots) \in \ell_q$. But from equation(2.3.4), for all $k \in \mathcal{N}$ we have:

$$(T'y')(e_k) = \sum_{i=1}^{\infty} t_{ik} y_i.$$

Hence $J(T'y') = (\sum_{i=1}^{\infty} t_{i1} y_i, \sum_{i=1}^{\infty} t_{i2} y_i, \ldots) \in \ell_q$. This proves that $J(y') = y \in \mathcal{D}(T^{\mathrm{tr}})$ and $J(T'y') = T^{\mathrm{tr}} y$.

Now let $y = (y_1, y_2, \ldots) \in \mathcal{D}(T^{\mathrm{tr}})$. To complete the proof of the theorem, we should prove $y' = J^{-1}(y) \in \mathcal{D}(T')$. Let $x' = J^{-1}(T^{\mathrm{tr}} y)$. Then $x' \in \ell_p'$. For all $z = (z_1, z_2, \ldots) \in \mathcal{D}(T)$, define the linear functional

$$y_T'(z) = \sum_{i=1}^{\infty} y_i \, (Tz)_i.$$

Then from Cauchy-Schwartz inequality, it follows that for all $z = (z_1, z_2, \ldots) \in \mathcal{D}(T)$ and $n \in \mathcal{N}$ we have:

$$\mid y_T'(z) - y_T'(z^{(n)}) \mid \leq \| y \|_q \left(\sum_{i=1}^{\infty} \mid (Tz)_i - (Tz^{(n)})_i \mid^p \right)^{1/p}, \tag{2.3.5}$$

where $z^{(n)} = (z_1, \ldots, z_n, 0, 0, \ldots)$. Let T_m, $m \in \mathcal{N}$, be the matrix operator in ℓ_p whose first m rows coincide with the first m rows of T and all other elements of T_m are zeros. We have:

$$\| Tz - Tz^{(n)} \|_p \leq \| Tz - T_m z \|_p + \| T_m(z - z^{(n)}) \|_p + \| (T_m - T) z^{(n)} \|_p.$$

By choosing i and n large enough, the right hand side of the above inequality can be arbitrary small and hence from inequality (2.3.5) it follows that

$$\lim_{n \to \infty} y_T'(z^{(n)}) = y_T'(z).$$

Now for all $z = (z_1, z_2, \dots) \in \mathcal{D}(T)$, we have:

$$x'(z) = \sum_{j=1}^{\infty} z_j (T^{\mathrm{tr}} y)_j = \lim_{n\to\infty} \sum_{j=1}^{n} z_j (\sum_{i=1}^{\infty} t_{ij} y_i) = \lim_{n\to\infty} y'_T(z^{(n)}) = y'_T(z).$$

This proves that $y' \in \mathcal{D}(T')$ and $x' = T'y'$. This completes the proof of the theorem. □

Now we show that with the hypotheses of Theorem 2.3.3, the dual T' of an operator T and its transpose T^{tr} have the same eigenvalues.

COROLLARY 2.3.4 Let T be a matrix operator in ℓ_p, $1 \le p < \infty$, with domain $\mathcal{D}(T)$ containing the unit vectors $e_1, e_2, \dots$. Then T' and T^{tr} (as a matrix operator in ℓ_q, $1/p + 1/q = 1$) have the same eigenvalues.

Proof. Let $\lambda \in C$ be an eigenvalue of T'. Hence there exists a nonzero element y' in ℓ'_p such that $T'y' = \lambda y'$. Let $J(y') = y$. We have $y \neq 0$ since $y' \neq 0$ and J is one-to-one (this follows from $\| J(f) \| = \| f \|$ for all $f \in \ell'_p$). From Theorem 2.3.3, $y \in \mathcal{D}(T^{\mathrm{tr}})$ and $J(T'y') = T^{\mathrm{tr}} y$. But since $J(T'y') = J(\lambda y') = \lambda J(y') = \lambda y$ and J is one-to one, then $T^{\mathrm{tr}} y = \lambda y$ and this proves that λ is an eigenvalue of T^{tr}.

Similarly it can be proved using the inverse J^{-1} of J (J is invertible since it is one-to-one linear operator mapping ℓ'_p onto ℓ_q) that if λ is an eigenvalue of T^{tr}, then λ is an eigenvalue of T'. □

2.4 The Main Theorem

In this section we extend Gersgorin theory (see Theorem 1.4.3) to row diagonally dominant infinite matrices with bounded perturbations.

First we need the following lemma.

LEMMA 2.4.1 Let $A = (a_{ij})$ be a matrix operator in ℓ_p, $1 \leq p \leq \infty$ such that $a_{ii} \neq 0$ for all $i \in \mathcal{N}$, and let $D = \text{diag}(a_{11}, a_{22}, \ldots)$.

(1) If the domain of the matrix operator $F = ((1 - \delta_{ij})a_{ij})$ contains the domain of A, then A can be written in the form: $A = (I+F_1)D$, where $F_1 = ((1-\delta_{ij})a_{jj}^{-1}a_{ij})$.

(2) If $| a_{ii} | \geq 1$ for all but a finite number of indices i, then A can be written in the form: $A = D(I + F_2)$, where $F_2 = ((1 - \delta_{ij})a_{ii}^{-1}a_{ij})$.

Proof. (1) Suppose $\mathcal{D}(F) \supset \mathcal{D}(A)$. Let $x \in \mathcal{D}(A)$. Then $x \in \mathcal{D}(F)$. Since $a_{ii}x_i = (Ax)_i - (Fx)_i$, where x_i is the ith component of x, then for $p < \infty$,

$$(\sum_{i=1}^{\infty} | a_{ii}x_i |^p)^{1/p} \leq (\sum_{i=1}^{\infty} | (Ax)_i |^p)^{1/p} + (\sum_{i=1}^{\infty} | (Fx)_i |^p)^{1/p} < \infty$$

and for $p = \infty$, $i \in \mathcal{N}$ we have:

$$| a_{ii}x_i | \leq \sup\{| (Ax)_j | : j \in \mathcal{N}\} + \sup\{| (Fx)_j | : j \in \mathcal{N}\} < \infty.$$

This proves that $x \in \mathcal{D}(D)$. Also we have:

$$\begin{aligned} ((I + F_1)(Dx))_i &= \sum_{j=1}^{\infty}(\delta_{ij} + (1 - \delta_{ij})a_{jj}^{-1}a_{ij})a_{jj}x_j \\ &= \sum_{j=1}^{\infty} a_{ij}x_j \\ &= (Ax)_i. \end{aligned}$$

Hence $x \in \mathcal{D}((I + F_1)D)$ and

$$Ax = (I + F_1)Dx. \tag{2.4.1}$$

Conversely, if $x \in \mathcal{D}((I + F_1)D)$ then $((I + F_1)(Dx))_i = \sum_{j=1}^{\infty} a_{ij}x_j = (Ax)_i$ and hence $x \in \mathcal{D}(A)$. This proves $\mathcal{D}(A) = \mathcal{D}((I + F_1)D)$ and from equation (2.4.1), the result follows.

(2) Suppose $| a_{ii} | \geq 1$ for all but a finite number of indices i. If $x \in \mathcal{D}(D(I+F_2))$, then $((I+F_2)x)_i = x_i + \sum_{j=1}^{\infty}{}' a_{ii}^{-1} a_{ij} x_j$, and so $(D(I+F_2)x)_i = a_{ii}x_i + \sum_{j=1}^{\infty}{}' a_{ij} x_j = (Ax)_i$. This proves $x \in \mathcal{D}(A)$ and

$$D(I + F_2)x = Ax. \tag{2.4.2}$$

On the other hand, if $x \in \mathcal{D}(A)$ then $(Ax)_i = \sum_{j=1}^{\infty} a_{ij} x_j = a_{ii} \sum_{j=1}^{\infty} a_{ii}^{-1} a_{ij} x_j = a_{ii}((I + F_2)x)_i$. So $((I + F_2)x)_i = \frac{1}{a_{ii}}(Ax)_i$, and since $| a_{ii} | \geq 1$ for all $i \geq n_0$ for some $n_0 \in \mathcal{N}$, then $x \in \mathcal{D}(I+F_2)$. Now $(D(I+F_2)x)_i = a_{ii}((I+F_2)x)_i = (Ax)_i$ and this proves $x \in \mathcal{D}(D(I+F_2))$. Hence $\mathcal{D}(A) = \mathcal{D}(D(I+F_2)$ and from equation(2.4.2), the result follows. □

Now we state and prove the main theorem.

THEOREM 2.4.2 Let $A = (a_{ij})$ be a matrix operator in ℓ_p, $1 \leq p \leq \infty$, and assume that

(1) $a_{ii} \neq 0$ for all $i \in \mathcal{N}$ and $| a_{ii} | \to \infty$ as $i \to \infty$.

(2) There exists a $\sigma \in [0, 1)$ such that for all $i \in \mathcal{N}$:

$$P_i = \sum_{j=1}^{\infty}{}' \mid a_{ij} \mid = \sigma_i \mid a_{ii} \mid, \quad \sigma_i \in [0, \sigma].$$

(3) Either the matrix operator $I + F_1$, where $F_1 = ((1 - \delta_{ij})a_{jj}^{-1}a_{ij})$, has a bounded inverse on ℓ_p and the domain of the matrix operator $F = ((1 - \delta_{ij})a_{ij})$ contains the domain of A, or the matrix operator $I+F_2$, where $F_2 = ((1-\delta_{ij})a_{ii}^{-1}a_{ij})$, has a bounded inverse on ℓ_p.

Then A is a closed operator with a compact inverse and any point of the spectrum $\sigma(A)$ of A is an isolated eigenvalue of A that lies in the set $\bigcup_{i=1}^{\infty} \mathcal{R}_i$.

Furthermore, if $F = ((1-\delta_{ij})a_{ij}) \in \mathcal{L}(\ell_p)$ then any set of r Gersgorin discs whose union is disjoint from all other discs intersects $\sigma(A)$ in a finite set of eigenvalues of A with total algebraic multiplicity equal to r.

Proof. Let $D = \text{diag}(a_{11}, a_{22}, \ldots)$. We consider the case $(I + F_2)^{-1} \in \mathcal{L}(\ell_p)$. (From hypothesis (1), there is $n_0 \in \mathcal{N}$ such that $| a_{nn} | \geq 1$ for all $n \geq n_0$. Hence from Lemma 2.4.1, $A = D(I + F_2)$.) The other case (see hypothesis (3)) has no new features (the additional condition $\mathcal{D}(F) \supset \mathcal{D}(A)$ is only made to ensure that $A = (I + F_1)D$, see Lemma 2.4.1). The theorem will be established in five steps. The proof of step[3] can be found in Theorem 1 of [26].

Step [1] A is a closed operator with a compact inverse.

The operators $E_n = \text{diag}(a_{11}^{-1}, \ldots, a_{nn}^{-1}, 0, 0, \ldots)$, $n \in \mathcal{N}$, are compact on ℓ_p, since $\dim \mathcal{R}(E_n) < \infty$ for all $n \in \mathcal{N}$ (see Remark 1.1.4). But since $\lim_{n\to\infty} \| D^{-1} - E_n \| = 0$, then from Theorem 1.1.3, D^{-1} is compact on ℓ_p. But since $A = D(I + F_2)$, then from hypothesis (3) and Theorem 1.1.2, it follows that $A^{-1} = (I + F_2)^{-1}D^{-1}$ exists and is compact on ℓ_p. As A^{-1} is closed ($\mathcal{D}(A^{-1})$ is the whole space ℓ_p), so is A.

Step [2] If $\lambda \in \sigma(A)$ then λ is an isolated eigenvalue of A.

Since $A^{-1} \in \mathcal{L}(\ell_p)$ (see step [1]), then $0 \in \rho(A)$. Thus if λ is in $\sigma(A)$ then $\lambda \neq 0$ and, from Theorem 1.2.6 part(b), we have $\lambda^{-1} \in \sigma(A^{-1})$. Then from the compactness of A^{-1} and Theorem 1.2.4, λ^{-1} is an eigenvalue of A^{-1}. Thus from Theorem 1.2.6 part(a), λ is an eigenvalue of A. If λ were not isolated then there would be a sequence of distinct eigenvalues of A^{-1} converging to the nonzero eigenvalue λ^{-1} of A^{-1} which is impossible by the compactness of A^{-1}.

Step [3] If λ is an eigenvalue of A, then $\lambda \in \bigcup_{i=1}^{\infty} \mathcal{R}_i$ (the union of the discs defined by the row sums of A).

As λ is an eigenvalue of A, there is a nonzero vector $x \in \ell_p$ such that $Ax = \lambda x$.

Writing $x = (x_1, x_2, \ldots)$, we have $\sum_{j=1}^{\infty} a_{ij} = \lambda x_i$, for all $i \in \mathcal{N}$. This implies that, for all $i \in \mathcal{N}$,

$$| \lambda - a_{ii} || x_i | \leq \sum_{j=1}^{\infty}{}' | a_{ij} || x_j | . \tag{2.4.3}$$

Now consider the cases $p < \infty$ and $p = \infty$ separately.

Case (i) If $p < \infty$, let N be an integer for which $| x_N | = \max\{| x_i | : i \in \mathcal{N}\}$. We have $x_N \neq 0$, since $x \neq 0$. Setting $i = N$ in inequality (2.4.3) we get:

$$| \lambda - a_{NN} || x_N | \leq \sum_{j=1}^{\infty}{}' | a_{Nj} || x_j | \leq | x_N | (\sum_{j=1}^{\infty}{}' | a_{Nj} |)$$

and hence

$$| \lambda - a_{NN} | \leq \sum_{j=1}^{\infty}{}' | a_{Nj} | = P_N.$$

Thus $\lambda \in \mathcal{R}_N$ and so $\lambda \in \bigcup_{i=1}^{\infty} \mathcal{R}_i$.

Case (ii) If $p = \infty$, for any $\epsilon > 0$, define

$$\mathcal{R}_i(\epsilon) = \{z \in \mathcal{C} : | z - a_{ii} | \leq P_i(1 + \epsilon)\}.$$

We first prove that $\lambda \in \bigcup_{i=1}^{\infty} \mathcal{R}_i(\epsilon)$. Let $\epsilon \in (0, 1)$. Then there exists an $m = m(\epsilon) \in \mathcal{N}$, depending on ϵ, such that

$$| x_m | > \| x \|_{\infty}(1 - \frac{\epsilon}{2}).$$

Thus from inequality (2.4.3), with $i = m$, we get:

$$| \lambda - a_{mm} || x_m | < \frac{P_m | x_m |}{(1 - \frac{\epsilon}{2})}$$

and hence $| \lambda - a_{mm} | < P_m(1 + \epsilon)$. Thus $\lambda \in \mathcal{R}_m(\epsilon)$ and so $\lambda \in \bigcup_{i=1}^{\infty} \mathcal{R}_i(\epsilon)$ for $\epsilon \in (0, 1)$, and hence for all $\epsilon > 0$.

Now let $\sigma \neq 0$ be the constant of hypothesis (2), and let $\delta \in (0, (1-\sigma)/\sigma)$. Then $\sigma(1+\delta) < 1$, and if $z \in \mathcal{R}_k(\delta)$,

$$| z | > | a_{kk} | - P_k(1+\delta) \geq | a_{kk} | (1 - \sigma(1+\delta)).$$

But the right hand side diverges to infinity as $k \to \infty$. It follows that each disc of $\{\mathcal{R}_i(\delta)\}$ has nonempty intersection with only finitely many discs of the sequence. That is, there is a finite nonempty subset of positive integers, say I_0, such that $\lambda \in \mathcal{R}_i(\delta)$ if and only if $i \in I_0$. (For future reference, we note that this step of the argument holds for all $p, 1 \leq p \leq \infty$.) Hence for any $\epsilon \in (0, \delta)$, $\lambda \in \bigcup_{i \in I_0} \mathcal{R}_i(\epsilon)$ and in the limit as $\epsilon \to 0$, we obtain $\lambda \in \bigcup_{i \in I_0} \mathcal{R}_i \subset \bigcup_{i=1}^{\infty} \mathcal{R}_i$.

This establishes the first part of the theorem. Now suppose $F \in \mathcal{L}(\ell_p)$.

Step [4] There exists a sequence of compact operators converging in norm to A^{-1}.

Write

$$A = \begin{pmatrix} A_{11}^{(n)} & A_{12}^{(n)} \\ A_{21}^{(n)} & A_{22}^{(n)} \end{pmatrix}$$

where $A_{11}^{(n)}$ is the $n \times n$ leading submatrix of A. Let D_n be the diagonal of $A_{22}^{(n)}$ and define

$$A_n = \begin{pmatrix} A_{11}^{(n)} & 0 \\ A_{21}^{(n)} & D_n \end{pmatrix} \tag{2.4.4}$$

for all $n \in \mathcal{N}$. Let $\mathcal{R}_i^{(n)}$ and $P_i^{(n)}$ be the Gersgorin discs and radii of $A_{11}^{(n)}$, $i = 1, \ldots, n$. From Theorem 2.1.3, $A_{11}^{(n)}$ is invertible for each $n \in \mathcal{N}$. Hence A_n is invertible for each $n \in \mathcal{N}$. Hence A_n is invertible for every $n \in \mathcal{N}$ and

$$A_n^{-1} = \begin{pmatrix} (A_{11}^{(n)})^{-1} & 0 \\ -D_n^{-1} A_{21}^{(n)} (A_{11}^{(n)})^{-1} & D_n^{-1} \end{pmatrix}.$$

Since $A_{21}^{(n)}$ is the submatrix $\begin{pmatrix} 0_{\infty\times n} & I_{\infty\times\infty} \end{pmatrix} F \begin{pmatrix} I_{n\times n} \\ 0_{\infty\times n} \end{pmatrix}$ of F and $F \in \mathcal{L}(\ell_p)$, then from Lemma 2.1.1 it follows that $A_{21}^{(n)} \in \mathcal{L}(C^n, \ell_p)$ and $\| A_{21}^{(n)} \| \leq \| F \|$ for all $n \in \mathcal{N}$. Hence every submatrix of A_n^{-1} is bounded and, from Lemma 2.1.1, it follows that $A_n^{-1} \in \mathcal{L}(\ell_p)$. Now we have:

$$\begin{aligned} A^{-1} - A_n^{-1} &= A^{-1}(A_n - A)A_n^{-1} \\ &= A^{-1} \begin{pmatrix} 0 & -A_{12}^{(n)} \\ 0 & S_{nn} \end{pmatrix} \begin{pmatrix} (A_{11}^{(n)})^{-1} & 0 \\ -D_n^{-1} A_{21}^{(n)} (A_{11}^{(n)})^{-1} & D_n^{-1} \end{pmatrix} \end{aligned}$$

where $S_{nn} = D_n - A_{22}^{(n)}$. S_{nn} is bounded in ℓ_p and $\| S_{nn} \| \leq \| F \|$ for all $n \in \mathcal{N}$, since it is the restriction of the submatrix

$$F_{22}^{(n)} = \begin{pmatrix} 0_{\infty\times n} & I_{\infty\times\infty} \end{pmatrix} F \begin{pmatrix} 0_{n\times\infty} \\ I_{\infty\times\infty} \end{pmatrix}$$

of F (see Lemma 2.1.1) to the subspace $\mathcal{D}(A)$ and $F_{22}^{(n)} \in \mathcal{L}(\ell_p)$, $\| F_{22}^{(n)} \| \leq \| F \|$ (see Lemma 2.1.1). Also $A_{12}^{(n)} \in \mathcal{L}(\ell_p, C^n)$ and $\| A_{12}^{(n)} \| \leq \| F \|$ for all $n \in \mathcal{N}$ since it is the submatrix

$$\begin{pmatrix} I_{n\times n} & 0_{n\times\infty} \end{pmatrix} F \begin{pmatrix} 0_{n\times\infty} \\ I_{\infty\times\infty} \end{pmatrix}$$

of the bounded operator $F \in \mathcal{L}(\ell_p)$ (see Lemma 2.1.1). We obtain

$$\| A^{-1} - A_n^{-1} \| \leq \| A^{-1} \| \left\| \begin{pmatrix} A_{12}^{(n)} D_n^{-1} A_{21}^{(n)} (A_{11}^{(n)})^{-1} & -A_{12}^{(n)} D_n^{-1} \\ -S_{nn} D_n^{-1} A_{21}^{(n)} (A_{11}^{(n)})^{-1} & S_{nn} D_n^{-1} \end{pmatrix} \right\|.$$

But since $(A_{11}^{(n)})^{-1}$ is uniformly bounded in n (Theorem 2.1.3) and $\| D_n^{-1} \| \to 0$ as $n \to \infty$, then each of the matrices in the block matrix operator in the right hand side of the above inequality converges to zero in norm as $n \to \infty$. Since $A^{-1} \in \mathcal{L}(\ell_p)$, it follows that $A_n^{-1} \to A^{-1}$ as $n \to \infty$. To complete the proof of this step it remains to show that A_n^{-1} is compact for every $n \in \mathcal{N}$. Since $\dim \mathcal{R}((A_{11}^{(n)})^{-1}) < \infty$ then $(A_{11}^{(n)})^{-1}$ is compact (see Remark 1.1.4), and from Theorem 1.1.2, $D_n^{-1}A_{21}^{(n)}(A_{11}^{(n)})^{-1}$ is compact. The sequence

$$\{\mathrm{diag}(a_{n+1,n+1}^{-1}, \ldots, a_{n+m,n+m}^{-1}, 0, 0, \ldots)\}_{m=1}^{\infty}$$

is a sequence of compact operators, since the range of every operator in the sequence is of finite dimension. Hence from Theorem 1.1.3, D_n^{-1} is compact since $\mathrm{diag}(a_{n+1,n+1}^{-1}, \ldots, a_{n+m,n+m}^{-1}, 0, 0, \ldots)$ converges to D_n^{-1} in norm as $m \to \infty$. Thus every matrix in the block matrix operator A_n^{-1} is compact. Let $K_1 = (A_{11}^{(n)})^{-1}$, $K_2 = -D_n^{-1}A_{21}^{(n)}(A_{11}^{(n)})^{-1}$ and $K_3 = D_n^{-1}$, and suppose $\{x^{(m)}\}$ is a sequence of bounded vectors in ℓ_p. Write

$$x^{(m)} = \begin{pmatrix} x_1^{(m)} \\ x_2^{(m)} \end{pmatrix}$$

where $x_1^{(m)}$ is a vector in $\mathcal{C}^n$ for all $m \in \mathcal{N}$. We have

$$A_n^{-1}x^{(m)} = \begin{pmatrix} K_1 x_1^{(m)} \\ K_2 x_1^{(m)} + K_3 x_2^{(m)} \end{pmatrix}.$$

From the compactness of K_1, there is a subsequence $\{m'\}$ of $\{m\}$ such that $\{K_1 x_1^{(m')}\}$ converges. From the compactness of K_2, there is a subsequence $\{m''\}$ of $\{m'\}$ such that $\{K_2 x_1^{(m'')}\}$ converges. Finally from the compactness of K_3 there a subsquence $\{m'''\}$ of $\{m''\}$ such that $\{K_3 x_2^{(m''')}\}$ converges. It is clear that the sequence $\{A_n^{-1}x^{(m''')}\}$ converges and this proves the compactness of A_n^{-1}.

Step [5] Let $\mathcal{S} = \bigcup_{i=1}^{r} \mathcal{R}_{k_i}$, $k_1 < \cdots < k_r$, be disjoint from the other Gersgorin discs. Then the spectrum of A in $\mathcal{S}$ is a finite set of eigenvalues of A with total algebraic multiplicity equal to r.

From step [3], there exists a $\delta > 0$ such that each disc in the sequence $\{\mathcal{R}_i(\delta)\}$ has a nonempty intersection with only finitely many discs of the sequence. Hence a Jordan closed curve Γ can be drawn so that $\mathcal{S} = \bigcup_{i=1}^{r} \mathcal{R}_{k_i}$ is a proper subset of $\mathcal{U}$, where $\mathcal{U}$ is the interior of the set bounded by Γ, and $\bar{\mathcal{U}} \cap \mathcal{R}_j = \emptyset$ (= the empty set) if $j \notin \{k_1, \ldots, k_r\}$ ($\bar{\mathcal{U}}$ denotes the closure of $\mathcal{U}$). Let $\mathcal{R}_i^{(n)}$ be the Gersgorin discs of A_n for all $i, n \in \mathcal{N}$, where A_n are given by equation (2.4.4). Since $\mathcal{R}_i^{(n)} \subset \mathcal{R}_i$, then the set $\mathcal{R}^{(n)} = \bigcup_{i=1}^{r} \mathcal{R}_{k_i}^{(n)}$ is a proper subset of $\mathcal{U}$ and $\bar{\mathcal{U}} \cap \mathcal{R}_j^{(n)} = \emptyset$ if $j \notin \{k_1, \ldots, k_r\}$. If m is an integer in $[k_r, \infty)$, then the Gersgorin theorem for finite matrices applied to $A_{11}^{(m)}$ implies that the eigenvalues of $A_{11}^{(m)}$ in $\bar{\mathcal{U}}$ lie in $\mathcal{R}^{(m)} = \bigcup_{i=1}^{r} \mathcal{R}_{k_i}^{(m)}$ and their total algebraic multiplicity is equal to r. Hence $\dim \mathcal{R}(P(\Gamma, A_{11}^{(m)})) = r$, where

$$P(\Gamma, A_{11}^{(m)}) = \frac{-1}{2\pi\sqrt{-1}} \int_{\Gamma} (A_{11}^{(m)} - \varsigma I_m)^{-1}\, d\varsigma \qquad (2.4.5)$$

and I_m is the $m \times m$ identity matrix operator. Since the eigenvalues of A_m are the eigenvalues of $A_{11}^{(m)}$ together with a_{ii}, $i \geq m+1$, and from the compactness of A_m^{-1} every point in the spectrum of A_m is an eigenvalue of A_m, then Γ consists of regular points of A_m. Similarly Γ consists of regular points of $\hat{A}_m$, where

$$\hat{A}_m = \begin{pmatrix} A_{11}^{(m)} & 0 \\ 0 & D_m \end{pmatrix}.$$

Since $\sigma(D_m) \cap \bar{\mathcal{U}} = \emptyset$, then for every $z \in \ell_p$ it is clear that $(D_m - \varsigma I)^{-1} z$ is differentiable in ς on $\mathcal{U}$ and continuous in ς on Γ. Then from Theorem 19.2 of [1], we have

$$\int_\Gamma (D_m - \varsigma I)^{-1} z \, d\varsigma = 0. \tag{2.4.6}$$

From the inequality

$$\| A^{-1} - \hat{A}_n^{-1} \| \leq \| A^{-1} - A_n^{-1} \| + \| A_n^{-1} - \hat{A}_n^{-1} \|$$

and step [4], it follows that

$$\lim_{n\to\infty} \| A^{-1} - \hat{A}_n^{-1} \| = 0.$$

Thus from Theorem 2.2.1 part (1), $\hat{A}_n \xrightarrow{g} A$. Hence from Theorem 1.3.4, there exists a positive integer $m_0 \geq k_r$ such that for all $m \geq m_0$,

$$\dim \mathcal{R}(P(\Gamma, A)) = \dim \mathcal{R}(P(\Gamma, \hat{A}_m)), \tag{2.4.7}$$

where

$$P(\Gamma, \hat{A}_m) = \frac{-1}{2\pi\sqrt{-1}} \int_\Gamma R(\varsigma, \hat{A}_m) \, d\varsigma$$

and

$$P(\Gamma, A) = \frac{-1}{2\pi\sqrt{-1}} \int_\Gamma R(\varsigma, A) \, d\varsigma.$$

But since for all $x = \begin{pmatrix} x^{(1)} \\ x^{(2)} \end{pmatrix} \in \ell_p$, where $x^{(1)} \in C^m$,

$$\begin{aligned} P(\Gamma, \hat{A}_m)x &= \frac{-1}{2\pi\sqrt{-1}} \int_\Gamma \begin{pmatrix} (A_{11}^{(m)} - \varsigma I)^{-1} & 0 \\ 0 & (D_m - \varsigma I)^{-1} \end{pmatrix} \begin{pmatrix} x^{(1)} \\ x^{(2)} \end{pmatrix} d\varsigma \\ &= \frac{-1}{2\pi\sqrt{-1}} \int_\Gamma \begin{pmatrix} (A_{11}^{(m)} - \varsigma I)^{-1} x^{(1)} \\ (D_m - \varsigma I)^{-1} x^{(2)} \end{pmatrix} d\varsigma, \end{aligned}$$

hence from equation (2.4.6) it follows that for all $m \geq k_r$,

$$\dim \mathcal{R}(P(\Gamma, \hat{A_m})) = \dim \mathcal{R}(P(\Gamma, A_{11}^{(m)})). \tag{2.4.8}$$

Therefore from equations (2.4.7) and (2.4.8), we have

$$\dim \mathcal{R}(P(\Gamma, A)) = \dim \mathcal{R}(P(\Gamma, A_{11}^{(m)})) = r$$

for all $m \geq m_0$. So the spectrum of A in $\mathcal{U}$ is a finite set of eigenvalues of A with total algebraic multiplicity equal to r. Now the result follows since every eigenvalue of A in $\mathcal{U}$ is in $\mathcal{S}$. □

REMARK 2.4.3 It is clear that we have used hypotheses (1) and (2) only in the proof of step [3] of Theorem 2.4.2 .

REMARK 2.4.4 At $p = \infty$, hypotheses (1) and (2) of Theorem 2.4.2 imply hypothesis (3). In this case the matrix operator F_2 is bounded on ℓ_∞ and $\| F_2 \| < 1$, and so $I + F_2$ has a bounded inverse on ℓ_∞ (see Theorem 1.1.1). The boundedness of F_2 follows from the fact that a matrix operator $T = (t_{ij})$ defined in ℓ_∞ is bounded on ℓ_∞ if and only if

$$M = \sup\{\sum_{j=1}^{\infty} \mid t_{ij} \mid : i \in \mathcal{N}\} < \infty.$$

In this case $M = \| T \|_\infty$ (see [8]).

REMARK 2.4.5 From step [5], it is clear that for a matrix operator A satisfying all the hypotheses of Theorem 2.4.2, if r Gersgorin discs of A are disjoint from the remaining Gersgorin discs then the spectrum of A is nonempty.

When all the Gersgorin discs are disjoint we obtain:

COROLLARY 2.4.6 Let $A = (a_{ij})$ be a matrix operator in ℓ_p, $1 \leq p \leq \infty$, and assume that

(1) $a_{ii} \neq 0$ for all $i \in \mathcal{N}$ and $| a_{ii} | \to \infty$ as $i \to \infty$.

(2) There exists a $\sigma \in [0,1)$ such that for all $i \in \mathcal{N}$:

$$P_i = \sum_{j=1}^{\infty}{}' \mid a_{ij} \mid = \sigma_i \mid a_{ii} \mid, \ \sigma_i \in [0,\sigma].$$

(3) The matrix operator $F = ((1-\delta_{ij})a_{ij}) \in \mathcal{L}(\ell_p)$ and either $(I + FD^{-1})^{-1}$ exists with $(I+FD^{-1})^{-1} \in \mathcal{L}(\ell_p)$ or $(I+D^{-1}F)^{-1}$ exists with $(I+D^{-1}F)^{-1} \in \mathcal{L}(\ell_p)$.

(4) $| a_{ii} - a_{kk} | > P_i + P_k$ for all $i, k \in \mathcal{N}$, $i \neq k$.

Then the spectrum $\sigma(A)$ of the closed operator A is nonempty and consists of a countable set of eigenvalues $\{\lambda_i\}$. For every $i \in \mathcal{N}$, $\lambda_i \in \mathcal{R}_i$ and λ_i is a simple eigenvalue of A. Moreover if A is real (that is, a_{ij} are real numbers for all $i, j \in \mathcal{N}$), so are the λ_i.

Proof. It is clear that, with the exception of the last statement, all the conclusions of the corollary follow from Theorem 2.4.2. This follows from the fact that, if A is real then any eigenvalues arise in conjugate pairs, or they are real. Since the discs are symmetric with respect to the real line and each contains precisely one eigenvalue, conjugate pairs can not arise. □

2.5 The Dual Theorem

In this section, we consider the case of column diagonally dominant infinite matrices with bounded perturbations, and develop Gersgorin theory for such matrices.

If $p \geq 1$ in Theorem 2.4.2, then the following theorem can be considered as a dual of Theorem 2.4.2 where the row diagonal dominance (hypothesis (2)) is replaced by column diagonal dominance. Recall the definitions of equations (2.1.1) and (2.1.2). (There is a corresponding dual statement for Corollary 2.4.6.)

THEOREM 2.5.1 Let $A = (a_{ij})$ be a matrix operator in ℓ_p, $1 \le p < \infty$, and assume that

(1) $a_{jj} \neq 0$ for all $j \in \mathcal{N}$, $| a_{jj} | \to \infty$ as $j \to \infty$.

(2) There exists a $\sigma \in [0, 1)$ such that

$$Q_j = \sum_{i=1}^{\infty}{}' \mid a_{ij} \mid = \sigma_j \mid a_{jj} \mid, \ \sigma_j \in [0, \sigma].$$

(3) Either the matrix operator $I + F_1$, where $F_1 = ((1 - \delta_{ij})a_{jj}^{-1}a_{ij})$, has a bounded inverse on ℓ_p and the domain of the matrix operator $F = ((1 - \delta_{ij})a_{ij})$ contains the domain of A, or the matrix operator $I+F_2$, where $F_2 = ((1-\delta_{ij})a_{ii}^{-1}a_{ij})$, has a bounded inverse on ℓ_p.

Then A is a closed operator with a compact inverse and any point in the spectrum $\sigma(A)$ of A is an isolated eigenvalue of A that lies in the set $\bigcup_{j=1}^{\infty} C_j$.

Furthermore, if $F = ((1-\delta_{ij})a_{ij}) \in \mathcal{L}(\ell_p)$ then any set of r Gersgorin discs whose union is disjoint from all other discs intersects $\sigma(A)$ in a finite set of eigenvalues of A with total algebraic multiplicity equal to r.

The proof of Theorem 2.5.1 is the same as that of Theorem 2.4.2, except in step [3], so we confine our discussion to that step. Namely:

If λ is an eigenvalue of A then $\lambda \in \bigcup_{j=1}^{\infty} C_j$ (the union of the discs defined by the colmun sums of A).

Since $Q_j < \infty$ for all $j \in \mathcal{N}$, then the unit coordinate vectors $e_j \in \mathcal{D}(A)$ for all $j \in \mathcal{N}$. So $\mathcal{D}(A)$ is dense in ℓ_p (as $p < \infty$) and using Theorem 1.1.6, it follows that the dual A' of A exists. As in step [1] of Theorem 2.4.2, A^{-1} is compact on ℓ_p. The boundedness of A^{-1} implies $\lambda \neq 0$ and, from Theorem 1.2.6 part (a), λ^{-1} is an eigenvalue of A^{-1}. Now we may use Theorem 1.2.5 to deduce from the compactness of A^{-1} that λ^{-1} is an eigenvalue of $(A^{-1})'$ ($(A^{-1})'$ exists since $\mathcal{D}(A^{-1}) = \ell_p$). But since $\mathcal{R}(A) = \mathcal{D}(A^{-1}) = \ell_p$ and $\mathcal{D}(A)$ is dense in ℓ_p, then from Theorem 1.1.8, $(A')^{-1}$ exists and $(A')^{-1} = (A^{-1})'$. Hence λ^{-1} is an eigenvalue of $(A')^{-1}$. So from Theorem 1.2.6 part(a), it follows that λ is an eigenvalue of A'. Thus from Corollary 2.3.4, λ is an eigenvalue of the transpose A^{tr} of A, where A^{tr} is an operator in ℓ_q, $\frac{1}{p} + \frac{1}{q} = 1$.

It is easily seen that A^{tr} satisfies hypotheses (1) and (2) of Theorem 2.4.2 on ℓ_q. Using the same proof given in step [3] of Theorem 2.4.2 one can prove that λ belongs to the Gersgorin discs defined by the rows of A^{tr} and the result follows. (As mentioned in Remark 2.4.3, we used hypotheses (1) and (2) only of Theorem 2.4.2 in proving step [3] of that theorem.) □

A similar result on ℓ_∞ to that given in Theorem 2.5.1 is introduced in the following theorem.

THEOREM 2.5.2 Let $A = (a_{ij})$ be a matrix operator in ℓ_∞ and assume that

(1) $a_{jj} \neq 0$ for all $j \in \mathcal{N}$ and $|\, a_{jj} \,| \to \infty$ as $j \to \infty$.

(2) There exists a $\sigma \in [0, 1)$ such that for all $j \in \mathcal{N}$:

$$Q_j = \sum_{i=1}^{\infty}{}' \mid a_{ij} \mid = \sigma_j \mid a_{jj} \mid, \; \sigma_j \in [0, \sigma].$$

(3) Either the matrix operator $I + F_1$, where $F_1 = ((1 - \delta_{ij}) a_{jj}^{-1} a_{ij})$, has a bounded inverse on ℓ_∞ and the domain of the matrix operator $F = ((1 - \delta_{ij}) a_{ij})$ contains the domain of A, or the matrix operator $I + F_2$, where $F_2 = ((1 - \delta_{ij}) a_{ii}^{-1} a_{ij})$, has a bounded inverse on ℓ_∞.

(4) Every row of F is in ℓ_1.

Then A is a closed operator with a compact inverse and every point of the spectrum $\sigma(A)$ of A is an isolated eigenvalue of A that lies in the set $\bigcup_{j=1}^{\infty} C_j$.

Furthermore, if $F = ((1-\delta_{ij})a_{ij}) \in \mathcal{L}(\ell_\infty)$ then any set of r Gersgorin discs whose union is disjoint from the other Gersgorin discs intersects $\sigma(A)$ in a finite set of eigenvalues of A with total algebraic multiplicity equal to r.

The proof of Theorem 2.5.2 is the same as that of Theorem 2.4.2, except in step [3], so we confine our discussion to that step. Namely:

If λ is an eigenvalue of A then $\lambda \in \bigcup_{j=1}^{\infty} C_j$.

Let $S = A^{\text{tr}}$ acting in ℓ_1. From hypothesis (4), every column of S is in ℓ_1 and so $e_i \in \mathcal{D}(S)$ for all $i \in \mathcal{N}$. Thus $\mathcal{D}(S)$ is dense in ℓ_1, and from Theorem 1.1.6 the dual S' of S exists. Now since $A^{-1} \in \mathcal{L}(\ell_\infty)$ then from Theorem 2.3.1, $(S')^{-1}$ exists and is in $\mathcal{L}(\ell_1')$; in fact for all $y' \in \ell_1'$,

$$(S')^{-1}y' = J^{-1}(A^{-1}(J(y'))),$$

where J is the bijective map defined in Remark 2.3.2. Hence from Lemma 1, page 224 of [34], $\mathcal{R}(S)$ is dense in ℓ_1. But since S^{-1} exists (this follows from Theorem 2, page 225 of [34]), then from Theorem 1.1.8 we have $(S^{-1})' = (S')^{-1}$. Also from Theorem 2.3.1, the compactness of A^{-1} implies the compactness of $(S')^{-1}$ (if $\{y_n'\}_{n=1}^{\infty}$ is a bounded sequence of points in ℓ_1' then $\{J(y_n')\}_{n=1}^{\infty}$ is a bounded sequence of points in ℓ_∞ and so there exists a subsequence $\{y_{n_k}'\}_{k=1}^{\infty}$ of $\{y_n'\}_{n=1}^{\infty}$ and a vector $y_0 \in \ell_\infty$ such that $A^{-1}(J(y_{n_k}')) \to y_0$ as $k \to \infty$. Thus $J^{-1}(A^{-1}(J(y_{n_k}'))) \to J^{-1}(y_0)$ as $k \to \infty$. Hence $(S')^{-1}$ is also compact). Then from $(S^{-1})' = (S')^{-1}$, we have $(S^{-1})'$ is compact on ℓ_1'. Thus from Schauder's Theorem, page 282 of [34], it follows that S^{-1} is compact.

Now let λ be an eigenvalue of A. Then from Corollary 2.3.4, λ is an eigenvalue of S'. But since $(S')^{-1} \in \mathcal{L}(\ell_1')$ then $\lambda \neq 0$, and from Theorem 1.2.6 part (a), λ^{-1} is an eigenvalue of $(S')^{-1} = (S^{-1})'$. Now we may use Theorem 1.2.5 to deduce from the compactness of S^{-1} and the fact that λ^{-1} is an eigenvalue of $(S^{-1})'$ that λ^{-1} is an eigenvalue of S^{-1}. So from Theorem 1.2.6 part (a), it follows that λ is an eigenvalue of $S = A^{\mathrm{tr}}$. It is easily seen that A^{tr} satisfies hypotheses (1) and (2) of Theorem 2.4.2 on ℓ_1. Using the same proof given in step [3] of Theorem 2.4.2 one can prove that λ belongs to the Gersgorin discs defined by the row sums of A^{tr} and the result follows. $\square$

Chapter 3

Gersgorin Theory for Diagonally Dominant Infinite Matrices with Relatively Bounded Perturbations

In this chapter, it is shown how the Gersgorin theorem results introduced in the paper [26], which are restricted to the spaces ℓ_1 and ℓ_∞, can be both strengthened and extended to the sequence spaces ℓ_p, $1 \leq p \leq \infty$.

3.1 Definitions and Remarks

A matrix operator $A = (a_{ij})$ in ℓ_p, $1 \leq p \leq \infty$, is said to have relatively bounded perturbation if either the matrix operator $F_1 = ((1-\delta_{ij})a_{jj}^{-1}a_{ij}) \in \mathcal{L}(\ell_p)$ or the matrix operator $F_2 = ((1-\delta_{ij})a_{ii}^{-1}a_{ij}) \in \mathcal{L}(\ell_p)$.

As stated in Section 2.1, the symbol $\sum_{j=1}^{\infty}{}'$ denotes the sum from one to infinity excluding the index $j = i$. Recall the definitions of equations (2.1.1) and (2.1.2).

DEFINITION 3.1.1 Let $A(\mu)$ be an operator valued function (in μ) from [0,1] into $\mathcal{C}(\mathcal{X})$. $A(\mu)$ is said to be continuous in the generalized sense at $\mu_0 \in [0,1]$ if given an $\epsilon \in (0,\infty)$, there is a $\delta \in (0,\infty)$ such that if $\mu \in [0,1]$ and $|\mu - \mu_0| < \delta$ then the gap $\hat{\delta}(A(\mu), A(\mu_0))$ between $A(\mu)$ and $A(\mu_0)$ satisfies $\hat{\delta}(A(\mu), A(\mu_0)) < \epsilon$.

REMARK 3.1.2 Let $T(\mu) \in \mathcal{C}(\mathcal{X})$ for all $\mu \in [0,1]$ and let $\mu_0 \in [0,1]$. In a similar way to the proof of Theorem 2.2.1 part (1), we can prove that if $T^{-1}(\mu_0)$

exists and is in $\mathcal{L}(\mathcal{X})$, then $T(\mu)$ is continuous in the generalized sense at μ_0 if and only if $T^{-1}(\mu)$ exists and is in $\mathcal{L}(\mathcal{X})$ for μ in a small neighbourhood of μ_0 and $\lim_{\mu\to\mu_0} \| T^{-1}(\mu) - T^{-1}(\mu_0) \| = 0$.

3.2 Row Diagonally Dominant Matrices

In this section we extend Gersgorin theory (see Theorem 1.4.3) to row diagonally dominant infinite matrices with relatively bounded perturbations.

THEOREM 3.2.1 Let $A = (a_{ij})$ be a matrix operator in ℓ_p, $1 \leq p \leq \infty$, and assume that

(1) $a_{ii} \neq 0$ for all $i \in \mathcal{N}$ and $| a_{ii} | \to \infty$ as $i \to \infty$.

(2) There exists a $\sigma \in [0,1)$ such that for all $i \in \mathcal{N}$:

$$P_i = \sum_{j=1}^{\infty}{}' \, | a_{ij} | = \sigma_i \, | a_{ii} |, \; \sigma_i \in [0,\sigma].$$

(3) Either $F_1 = ((1-\delta_{ij})a_{jj}^{-1}a_{ij}) \in \mathcal{L}(\ell_p)$, $I + \mu F_1$ has a bounded inverse on ℓ_p for all $\mu \in (0,1]$ and the domain of $F = ((1-\delta_{ij})a_{ij})$ contains the domain of A, or $F_2 = ((1-\delta_{ij})a_{ii}^{-1}a_{ij}) \in \mathcal{L}(\ell_p)$ and $I + \mu F_2$ has a bounded inverse on ℓ_p for all $\mu \in (0,1]$.

Then A is a closed operator with a compact inverse and any point of the spectrum $\sigma(A)$ of A is an isolated eigenvalue of A that lies in the set $\bigcup_{i=1}^{\infty} \mathcal{R}_i$.

Furthermore, any set of r Gersgorin discs whose union is disjoint from all other Gersgorin discs intersects $\sigma(A)$ in a finite set of eigenvalues of A with total algebraic multiplicity equal to r.

Proof. Let $D = \mathrm{diag}(a_{11}, a_{22}, \ldots)$. We consider the case in which $F_2 \in \mathcal{L}(\ell_p)$ and $(I + \mu F_2)^{-1} \in \mathcal{L}(\ell_p)$ for all $\mu \in [0,1]$. (From hypothesis (1) there is $n_0 \in \mathcal{N}$

such that $| a_{nn} | \geq 1$ for all $n \geq n_0$. Hence from Lemma 2.4.1, $A(\mu) = D(I + \mu F_2)$, where $A(\mu) = (\delta_{ij}a_{ij} + (1 - \delta_{ij})\mu a_{ij})$.) The other case (see hypothesis (3)) has no new features (the additional condition $\mathcal{D}(F) \supset \mathcal{D}(A)$ is only made to ensure that $A(\mu) = (I + \mu F_1)D$, see Lemma 2.4.1). The theorem will be established in five steps.

Step [1] For every $\mu \in [0, 1]$, $A(\mu)$ is a closed operator with a compact inverse.

Let $\mu \in [0, 1]$. Since $A(\mu) = D(I + \mu F_2)$ and D^{-1} is compact (see step [1] of Theorem 2.4.2), then from hypothesis (3) and Theorem 1.1.2, it follows that $A^{-1}(\mu) = (I + \mu F_2)^{-1}D^{-1}$ exists and is compact on ℓ_p for every $\mu \in [0, 1]$. As $A^{-1}(\mu)$ is closed $(\mathcal{D}(A^{-1}(\mu)) = \ell_p)$, so is $A(\mu)$. Since $\lambda = \| A^{-1} \|$ is in the spectrum of A^{-1}, then $\sigma(A)$ is nonempty.

Step [2] For every $\mu \in [0, 1]$, the spectrum $\sigma(A(\mu))$ of $A(\mu)$ consists of isolated eigenvalues.

Since $A^{-1}(\mu) \in \mathcal{L}(\ell_p)$ (see step [1]), then $0 \in \rho(A(\mu))$. Thus if $\lambda \in \sigma(A(\mu))$ then $\lambda \neq 0$ and from Theorem 1.2.6 part (b), we have $\lambda^{-1} \in \sigma(A^{-1}(\mu))$. Then from the compactness of $A^{-1}(\mu)$ and Theorem 1.2.4, λ^{-1} is an eigenvalue of $A^{-1}(\mu)$. Thus from Theorem 1.2.6 part (a), λ is an eigenvalue of $A(\mu)$. If λ were not isolated then there would be a sequence of distinct eigenvalues of $A^{-1}(\mu)$ converging to the nonzero eigenvalue λ^{-1} of $A^{-1}(\mu)$ which is impossible by the compactness of $A^{-1}(\mu)$.

Step [3] If λ is an eigenvalue of $A(\mu)$ and $\mu \in [0, 1]$, then $\lambda \in \{z \in C : | z - a_{ii} | \leq \mu P_i\}$.

As in step [3] of Theorem 2.4.2, this result follows from hypotheses (1) and (2).

Step [4] The operator valued function $A(\mu)$ is a continuous function in the generalized sense at every $\mu_0 \in [0,1]$.

Let $\mu_0 \in [0,1]$. From Remark 3.1.2, it is sufficient to prove that

$$\lim_{\mu \to \mu_0} \| A^{-1}(\mu) - A^{-1}(\mu_0) \| = 0.$$

We have:

$$A^{-1}(\mu) - A^{-1}(\mu_0) = ((I + \mu F_2)^{-1} - (I + \mu_0 F_2)^{-1})D^{-1}.$$

But since $F_2 \in \mathcal{L}(\ell_p)$ (see hypothesis (3)), then

$$\begin{aligned} A^{-1}(\mu) - A^{-1}(\mu_0) &= (I + \mu F_2)^{-1}((I + \mu_0 F_2) - (I + \mu F_2))(I + \mu_0 F_2)^{-1} D^{-1} \\ &= -(\mu - \mu_0)(I + \mu F_2)^{-1} F_2 (I + \mu_0 F_2)^{-1} D^{-1} \end{aligned}$$

and so

$$A^{-1}(\mu) - A^{-1}(\mu_0) = -(\mu - \mu_0)((I + \mu_0 F_2) + (\mu - \mu_0) F_2)^{-1} F_2 (I + \mu_0 F_2)^{-1} D^{-1}.$$

If $| \mu - \mu_0 | < \delta_1 = \frac{1}{2}(1 + \| F_2 \|)^{-1} \| (I + \mu_0 F_2)^{-1} \|^{-1}$, then the operator $I + (\mu - \mu_0)\,(I + \mu_0 F_2)^{-1} F_2$ has a bounded inverse on ℓ_p and from Theorem 1.1.1 we have:

$$\| (I + (\mu - \mu_0)\,(I + \mu_0 F_2)^{-1} F_2)^{-1} \| \leq 2.$$

Therefore if $| \mu - \mu_0 | < \delta_1$,

$$\| A^{-1}(\mu) - A^{-1}(\mu_0) \| \leq 2\, | \mu - \mu_0 | \| F_2 \| \, \| (I + \mu_0 F_2)^{-1} \|^2 \, \| D^{-1} \| . \qquad (3.2.1)$$

Let ϵ be a positive number. If δ is a positive number such that

$$\delta = \min\left\{ \frac{\epsilon}{2 \| D^{-1} \| (1 + \| F_2 \|) \| (I + \mu_0 F_2)^{-1} \|^2}, \frac{1}{2(1 + \| F_2 \|) \| (I + \mu_0 F_2)^{-1} \|} \right\}$$

then from equation (3.2.1), we have

$$\| A^{-1}(\mu) - A^{-1}(\mu_0) \| < \epsilon$$

for all $\mu \in [0,1]$ such that $\mid \mu - \mu_0 \mid < \delta$ and this proves $\lim_{\mu \to \mu_0} \parallel A^{-1}(\mu) - A^{-1}(\mu_0) \parallel = 0$.

Step [5] Let $\mathcal{S} = \bigcup_{i=1}^{\infty} \mathcal{R}_{k_i}$, $k_1 < \cdots < k_r$, be disjoint from all other Gersgorin discs. Then the spectrum of A in $\mathcal{S}$ is a finite set of eigenvalues of A with total algebraic multiplicity equal to r.

From step [3] (see the proof of step [3] of Theorem 2.4.2), there exists a real $\delta > 0$ such that each disc of the sequence $\{\mathcal{R}_i(\delta)\}$ has a nonempty intersection with only finitely many discs of the sequence. Since $\mathcal{S}$ is disjoint from the other discs, then a closed Jordan curve Γ can be drawn so that (a) $\mathcal{S}$ is a proper subset of $\mathcal{U}$, where $\mathcal{U}$ is the interior of the set bounded by Γ, and (b) $\bar{\mathcal{U}} \bigcap \mathcal{R}_j = \emptyset$ if $j \notin \{k_1, \ldots, k_r\}$. Thus from steps [2] and [3], it is clear that Γ consists of regular points of $A(\mu)$ for all $\mu \in [0,1]$. Let $P(\mu)$ be the Riesz projector for $A(\mu)$ and Γ for every $\mu \in [0,1]$, that is,

$$P(\mu) = \frac{-1}{2\pi\sqrt{-1}} \int_{\Gamma} (A(\mu) - \varsigma I)^{-1} \, d\varsigma.$$

At $\mu = 0$, $\sigma(A(\mu))$ consists of the eigenvalues a_{ii} with corresponding eigenvectors e_i, $i \in \mathcal{N}$, where e_i are the unit coordinate vectors in ℓ_p. Hence from properties (a) and (b), it is easily seen that $\dim \mathcal{R}(P(0)) = r$. From step [4] and Theorem 1.3.4, there exists a $\delta_1 \in (0,1]$ such that $\dim \mathcal{R}(P(\mu) = r$ for all $\mu \in [0, \delta_1]$.

Define δ_0 to be:

$$\delta_0 = \sup\{\delta \in (0,1] : \dim \mathcal{R}(P(\mu)) = r, \, \mu \in [0, \delta]\}.$$

Then $\delta_0 \in (0,1]$ and,in fact, we are to prove $\delta_0 = 1$. Since Γ consists of regular points of $A(\delta_0)$, then we may use Theorem 1.3.4 and step [4] to show that if $\dim \mathcal{R}(P(\delta_0)) \neq r$, we can find a $\delta \in (0, \delta_0)$ close enough to δ_0 such that $\dim \mathcal{R}(P(\delta)) \neq r$, which is impossible by the definition of δ_0. Hence $\dim \mathcal{R}(P(\delta_0)) = r$.

If $\delta_0 < 1$, then from the fact that $\dim \mathcal{R}(P(\delta_0)) = r$ and the definition of δ_0, it follows that for any $\delta' \in (\delta_0, 1]$ there is a $\mu' \in (\delta_0, \delta']$ such that $\dim \mathcal{R}(P(\mu')) \neq r$. But from the fact that $\dim \mathcal{R}(P(\delta_0)) = r$ and step [4], we may use Theorem 1.3.4 to find a $\delta_0' \in (\delta_0, 1]$such that $\dim \mathcal{R}(P(\mu)) = r$ for all $\mu \in [\delta_0, \delta_0']$, a contradiction. Therefore we must have $\delta_0 = 1$, and from $\dim \mathcal{R}(P(\delta_0)) = r$, the result follows. This completes the proof of the theorem. □

3.3 Column Diagonally Dominant Matrices

In this section we extend Gersgorin theory (see Theorem 1.4.3) to column diagonally dominant infinite matrices with relatively bounded perturbations.

If $p > 1$ in Theorem 3.2.1, the following theorem can be considered as the dual of that theorem.

THEOREM 3.3.1 Let $A = (a_{ij})$ be a matrix operator in ℓ_p, $1 \leq p < \infty$, and assume that

(1) $a_{jj} \neq 0$ for all $j \in \mathcal{N}$ and $| a_{jj} | \to \infty$ as $j \to \infty$.

(2) There exists a $\sigma \in [0, 1)$ such that for all $j \in \mathcal{N}$

$$Q_j = \sum_{i=1}^{\infty}{}' \mid a_{ij} \mid = \sigma_j \mid a_{jj} \mid, \; \sigma_j \in [0, \sigma].$$

(3) Either $F_1 = ((1 - \delta_{ij}) a_{jj}^{-1} a_{ij}) \in \mathcal{L}(\ell_p)$, $I + \mu F_1$ has a bounded inverse on ℓ_p for all $\mu \in (0, 1]$ and the domain of $F = ((1 - \delta_{ij}) a_{ij})$ contains the domain of A,

or $F_2 = ((1-\delta_{ij})a_{ii}^{-1}a_{ij}) \in \mathcal{L}(\ell_p)$ and $I + \mu F_2$ has a bounded inverse on ℓ_p for all $\mu \in (0,1]$.

Then A is a closed operator with a compact inverse and any point of the spectrum $\sigma(A)$ of A is an isolated eigenvalue of A that lies in the set $\bigcup_{j=1}^{\infty} C_j$.

Furthermore, any set of r Gersgorin discs whose union is disjoint from all other Gersgorin discs intersects $\sigma(A)$ in a finite set of eigenvalues of A with total algebraic multiplicity equal to r.

The proof of Theorem 3.3.1 is the same as that of Theorem 3.2.1, except in step [3]. The proof of step [3] in Theorem 3.3.1 is exactly the same as that of step [3] in Theorem 2.5.1.

A result on ℓ_∞ similar to, and complementing that given in Theorem 3.3.1 is introduced in the following theorem.

THEOREM 3.3.2 Let $A = (a_{ij})$ be a matrix operator in ℓ_∞ and assume that

(1) $a_{jj} \neq 0$ for all $j \in \mathcal{N}$ and $| a_{jj} | \to \infty$ as $j \to \infty$.

(2) There exists a $\sigma \in [0,1)$ such that for all $j \in \mathcal{N}$

$$Q_j = \sum_{i=1}^{\infty}{}' \mid a_{ij} \mid = \sigma_j \mid a_{jj} \mid, \ \sigma_j \in [0,\sigma].$$

(3) Either $F_1 = ((1-\delta_{ij})a_{jj}^{-1}a_{ij}) \in \mathcal{L}(\ell_\infty)$, $I + \mu F_1$ has a bounded inverse on ℓ_∞ for all $\mu \in (0,1]$ and the domain of $F = ((1-\delta_{ij})a_{ij})$ contains the domain of A, or $F_2 = ((1-\delta_{ij})a_{ii}^{-1}a_{ij}) \in \mathcal{L}(\ell_\infty)$ and $I + \mu F_2$ has a bounded inverse on ℓ_∞ for all $\mu \in (0,1]$.

(4) Every row of $F = ((1-\delta_{ij})a_{ij})$ is in ℓ_1.

Then A is a closed operator with a compact inverse and any point of the spectrum $\sigma(A)$ of A is an isolated eigenvalue of A that lies in the set $\bigcup_{j=1}^{\infty} C_j$.

Furthermore, any set of r Gersgorin discs whose union is disjoint from all other Gersgorin discs intersects $\sigma(A)$ in a finite set of eigenvalues of A with total algebraic multiplicity equal to r.

NOTE. If $F_2 \in \mathcal{L}(\ell_\infty)$, then from Remark 2.4.4 and the condition $a_{ii} \neq 0$ for all $i \in \mathcal{N}$, it is clear that

$$| a_{ii}^{-1} | \sum_{j=1}^{\infty}{}' | a_{ij} | \leq \| F_2 \| < \infty.$$

Hence for every $i \in \mathcal{N}$, $\sum_{j=1}^{\infty}{}' | a_{ij} | < \infty$. Thus every row of F is in ℓ_1.

Proof. As in Theorem 2.5.2, we prove that if λ is an eigenvalue of A then $\lambda \in \bigcup_{j=1}^{\infty} C_j$. The rest of the proof is the same as that of Theorem 3.2.1. □

3.4 Almost Disjoint Discs

In this section we make a hypothesis on the geometry of the Gersgorin discs (see hypothesis (4) of Theorem 3.4.2 below) that can be loosely described as almost disjoint discs. At the same time we are to weaken the condition of diagonal dominance somewhat (see hypothesis (2) of Theorem 3.4.2 below). In this respect (and in the action on ℓ_p, of course) this theorem generalizes Theorem 5 of [26]. First we need the following lemma.

LEMMA 3.4.1 Suppose the matrix operator $A = (a_{ij})$ in ℓ_p, $1 \leq p \leq \infty$, satisfies the following hypotheses:

(1) $| a_{ii} | \to \infty$ as $i \to \infty$.

(2) $| a_{ii} - a_{jj} | \geq P_i + P_j$ for all $i, j \in \mathcal{N}$, $i \neq j$.

If the disc $\mathcal{S} = \{z \in \mathcal{C} : | z - a | \leq r\}$, $r \in [0, \infty)$ and $a \in \mathcal{C}$, is disjoint from $\bigcup_{i=1}^{\infty} \mathcal{R}_i$, then there is a real $\delta > 0$ such that the disc $\{z \in \mathcal{C} : | z - a | \leq r + \delta\}$ is also disjoint from $\bigcup_{i=1}^{\infty} \mathcal{R}_i$.

NOTE. It is clear that for every $n \in \mathcal{N}$,

$$\begin{aligned}
\operatorname{dist}(\mathcal{S}, \mathcal{R}_n) &= \min\{\operatorname{dist}(u, \mathcal{R}_n) : u \in \mathcal{S}\} \\
&= \min\{\min\{| u - v |: v \in \mathcal{R}_n\} : u \in \mathcal{S}\} \\
&> 0.
\end{aligned}$$

This is clear since the function $\psi_u : \mathcal{R}_n \to \mathcal{R}$ defined by $\psi_u(v) =| u - v |, u \in \mathcal{S}$ is continuous on the compact set $\mathcal{R}_n$ and attains positive values ($u \notin \mathcal{R}_n$). So it has a positive minimum. Hence the function $\phi : \mathcal{S} \to \mathcal{R}$ defined by

$$\phi(u) = \min\{| u - v |: v \in \mathcal{R}_n\} = \operatorname{dist}(u, \mathcal{R}_n)$$

attains positive values. Also ϕ is continuous on the compact set $\mathcal{S}$, and hence it must have a positive minimum.

Proof. Suppose the contrary. Then for every $n \in \mathcal{N}$, there is a positive integer k_n such that

$$0 < \operatorname{dist}(\mathcal{S}, \mathcal{R}_{k_n}) < \frac{1}{n}. \tag{3.4.1}$$

Since $| a_{k_n k_n} | \to \infty$ as $n \to \infty$, then from inequality (3.4.1) it follows that $P_{k_n} \to \infty$ as $n \to \infty$. Now it follows from the compactness of $\mathcal{S}$ that there are two different positive integers k_m and k_r such that $0 <| a_{k_m k_m} - a_{k_r k_r} |< P_{k_m} + P_{k_r}$, which is impossible by hypothesis (2). □

THEOREM 3.4.2 Let $A = (a_{ij})$ be a matrix operator in ℓ_p, $1 \le p \le \infty$, and assume that

(1) $a_{ii} \neq 0$ for all $i \in \mathcal{N}$ and $| a_{ii} | \to \infty$ as $i \to \infty$.

(2) For every $i \in \mathcal{N}$,

$$P_i = \sum_{j=1}^{\infty}{}' \mid a_{ij} \mid = \sigma_i \mid a_{ii} \mid, \; \sigma_i \in [0,1].$$

(3) Either $F_1 = ((1-\delta_{ij})a_{jj}^{-1}a_{ij}) \in \mathcal{L}(\ell_p)$, $I + \mu F_1$ has a bounded inverse on ℓ_p for all $\mu \in (0,1]$ and the domain of $F = ((1-\delta_{ij})a_{ij})$ contains the domain of A, or $F_2 = ((1-\delta_{ij})a_{ii}^{-1}a_{ij}) \in \mathcal{L}(\ell_p)$ and $I + \mu F_2$ has a bounded inverse on ℓ_p for all $\mu \in (0,1]$.

(4) $\mid a_{ii} - a_{jj} \mid \geq P_i + P_j$ for all $i, j \in \mathcal{N}$, $i \neq j$.

Then the spectrum $\sigma(A)$ of A consists of a discrete, countable set of nonzero eigenvalues $\{\lambda_i : i \in \mathcal{N}\}$, and for every $i \in \mathcal{N}$, $\lambda_i \in \mathcal{R}_i$.

Furthermore, if strict inequality obtains in (4) for a fixed i and all $j \neq i$, then λ_i is a simple eigenvalue.

NOTE. From hypotheses (2) and (4), it is clear that there are two different positive integers n_1 and n_2 such that $0 \notin \mathcal{R}_i$ if $i \in \mathcal{N} \setminus \{n_1, n_2\}$.

Proof. Let $D = \mathrm{diag}(a_{11}, a_{22}, \ldots)$. We consider the case in which $F_2 \in \mathcal{L}(\ell_p)$ and $(I + \mu F_2)^{-1} \in \mathcal{L}(\ell_p)$ for all $\mu \in [0,1]$. From hypothesis (1) there is $n_0 \in \mathcal{N}$ such that $\mid a_{nn} \mid \geq 1$ for all $n \geq n_0$. Hence from Lemma 2.4.1, the matrix operator $A(\mu) = (\delta_{ij}a_{ij} + (1-\delta_{ij})\mu a_{ij})$ can be written in the form $A(\mu) = D(I + \mu F_2)$. The other case (see hypothesis (3)) has no new features (the additional condition $\mathcal{D}(F) \supset \mathcal{D}(A)$ is only made to ensure that $A(\mu)$ can be written in the form $A(\mu) = (I + \mu F_1)D$, see Lemma 2.4.1). The theorem will be established in eight steps. The proofs of steps [1],[2],[3] and [4] are the same as those of steps [1],[2],[3] and [4] of Theorem 3.2.1, respectively.

Step [1] For every $\mu \in [0,1]$, $A(\mu)$ is a closed operator with a compact inverse.

Step [2] For every $\mu \in [0,1]$, the spectrum $\sigma(A(\mu))$ of $A(\mu)$ consists of isolated eigenvalues.

Step [3] If λ is an eigenvalue of $A(\mu)$ and $\mu \in [0,1)$, then

$$\lambda \in \bigcup_{i=1}^{\infty} \{z \in \mathcal{C} : | z - a_{ii} | \leq \mu P_i\}.$$

Step [4] The operator $A(\mu)$ is a continuous function in the generalized sense at every $\mu \in [0,1]$.

Step [5] If λ is an eigenvalue of A, then $\lambda \in \bigcup_{i=1}^{\infty} \mathcal{R}_i$.

Suppose the contrary. Then there is an eigenvalue λ of A such that $\lambda \notin \bigcup_{i=1}^{\infty} \mathcal{R}_i$. From step [2] ($\lambda$ is an isolated eigenvalue of A) and Lemma 3.4.1 there is a disc $\mathcal{D}$ with centre λ and a positive radius such that $\mathcal{D} \bigcap \sigma(A) = \{\lambda\}$ and $\mathcal{D} \bigcap (\bigcup_{i=1}^{\infty} \mathcal{R}_i) = \emptyset$ (= the empty set). Let $P(\mu)$ be the Riesz projector for $A(\mu)$, $\mu \in [0,1]$, and the boundary of $\mathcal{D}$, that is,

$$P(\mu) = \frac{-1}{2\pi\sqrt{-1}} \int_{\partial \mathcal{D}} (A - \varsigma I)^{-1} \, d\varsigma$$

where $\partial \mathcal{D}$ denotes the boundary of $\mathcal{D}$. Since $\dim \mathcal{R}(P(1)) \neq 0$, then from step [4] and Theorem 1.3.4 there is a $\mu_0 \in [0,1)$ such that $\dim \mathcal{R}(P(\mu_0)) = \dim \mathcal{R}(P(1)) \neq 0$, which is impossible since by steps [2] and [3], the spectrum of $A(\mu_0)$ is a subset of the set $\bigcup_{i=1}^{\infty} \mathcal{R}_i$.

Step [6] Let $i \in \mathcal{N}$ and $\mu \in [0,1)$, then we have:

(a) The set $\mathcal{R}_i$ contains one and only one eigenvalue of $A(\mu)$; if $P_i > 0$ then this eigenvalue of $A(\mu)$ is simple and is in the interior of $\mathcal{R}_i$.

(b) The set $\mathcal{R}_i$ contains at least one eigenvalue of A.

Let $i \in \mathcal{N}$ and $\mu \in [0,1)$. If $P_i = 0$, then $\mathcal{R}_i = \{a_{ii}\}$ and a_{ii} is an eigenvalue of $A(\mu)$ with a corresponding eigenvector e_i.

Now consider the case $P_i > 0$. Since $A(\mu)$ satisfies the hypotheses of Theorem 3.2.1 and the Gersgorin disc $\{z \in \mathcal{C} : \mid z - a_{ii} \mid \leq \mu P_i\}$ of $A(\mu)$ is disjoint from the other Gersgorin discs of $A(\mu)$, then from Theorem 3.2.1, it follows that the disc $\{z \in \mathcal{C} : \mid z - a_{ii} \mid \leq \mu P_i\}$ contains one and only one simple eigenvalue of $A(\mu)$. Hence from hypothesis (4) and the fact that the eigenvalues of $A(\mu)$ lie in the set $\bigcup_{j=1}^{\infty}\{z \in \mathcal{C} : \mid z - a_{jj} \mid \leq \mu P_j\}$ (see Theorem 3.2.1), the disc $\mathcal{R}_i = \{z \in \mathcal{C} : \mid z - a_{ii} \mid \leq P_i\}$ contains in its interior one and only one simple eigenvalue of $A(\mu)$. Denote this eigenvalue by $\lambda_i(\mu)$. This proves part (a).

If $P_i > 0$ then from Theorem 1.3.4, step [4] and part (a), it follows that $\mathcal{R}_i$ must contain at least one eigenvalue of A. If $P_i = 0$, then $\mathcal{R}_i = \{a_{ii}\}$ and a_{ii} is an eigenvalue of A with a corresponding eigenvector e_i. This proves part (b).

Step [7] The set of the eigenvalues of A is a countable set $\{\lambda_i : i \in \mathcal{N}\}$, and for every $i \in \mathcal{N}$, $\lambda_i \in \mathcal{R}_i$.

Let $j \in \mathcal{N}$. From step [4] and Remark 3.1.2, it follows that $\lim_{\mu \to 1^-} \| A^{-1}(\mu) - A^{-1} \| = 0$. Hence from Theorem 1.3.3, we have $\lim_{\mu \to 1^-} \text{dist}(\lambda_j^{-1}(\mu), \sigma(A^{-1})) = 0$ (see step [6] for the definition of $\lambda_j(\mu)$). This implies that if $z \in \sigma(A) \cap \mathcal{R}_j$, then

$$\mid \lambda_j(\mu) - z \mid = \mid \lambda_j(\mu) \mid \mid z^{-1} - \lambda_j^{-1}(\mu) \mid \mid z \mid \to 0$$

as $\mu \to 1^-$. Hence $\lim_{\mu \to 1^-} \text{dist}(\lambda_j(\mu),\ \sigma(A) \cap \mathcal{R}_j) = 0$. Hence by part (b) of step (6), there is an eigenvalue λ_j of A in $\mathcal{R}_j$ such that $\lim_{\mu \to 1^-} \lambda_j(\mu) = \lambda_j$. This proves that $\sigma(A)$ contains the countable set $\{\lambda_i : i \in \mathcal{N}\}$ of eigenvalues of A, where $\lambda_i \in \mathcal{R}_i$ for every $i \in \mathcal{N}$. Now we prove $\sigma(A) = \{\lambda_i : i \in \mathcal{N}\}$.

Now let $\lambda \in \sigma(A)$. From the condition $\mid a_{ii} \mid \to \infty$ as $i \to \infty$ (each diagonal element a_{ii} of A, $i \in \mathcal{N}$, is repeated finitely many times along the diagonal), step [5] and hypothesis (4) there is a finite nonempty subset $\mathcal{N}_\lambda$ of $\mathcal{N}$ such that $\lambda \in \mathcal{R}_n$ if and only if $n \in \mathcal{N}_\lambda$. Hence from step [2] (λ is isolated in $\sigma(A)$) and Lemma 3.4.1, it follows that there is a positive real δ'_λ such that the disc $\mathcal{D}(\lambda, \delta'_\lambda) = \{z \in \mathcal{C} :| z - \lambda |\leq \delta'_\lambda\}$ satisfies

$$\mathcal{D}(\lambda, \delta'_\lambda) \bigcap \mathcal{R}_i = \emptyset,$$

if $i \notin \mathcal{N}_\lambda$, and

$$\mathcal{D}(\lambda, \delta'_\lambda) \bigcap \sigma(A) = \{\lambda\}.$$

If $\lambda \notin \{\lambda_i : i \in \mathcal{N}\}$, then there is no $i \in \mathcal{N}$ with $\lim_{\mu \to 1^-} \lambda_i(\mu) = \lambda$. Hence there is a positive $\delta_\lambda < \delta'_\lambda$ and a strictly increasing sequence $\{\mu_n\}_{n=1}^{\infty}$ of points in $(0,1)$ such that $\mu_n \to 1^-$ and the disc $\mathcal{D}(\lambda, \delta_\lambda) = \{z \in \mathcal{C} :| z - \lambda |\leq \delta_\lambda\}$ consists of regular points of $A(\mu_n)$ for all $n \in \mathcal{N}$ (here we use the fact that $\mathcal{R}_i \bigcap \sigma(A(\mu_n)) = \{\lambda_i(\mu_n)\}$ for all $i, n \in \mathcal{N}$, see step [6] part (a)). Let $P(\mu)$ be the Riesz projector for $A(\mu)$, $\mu \in [0,1]$, and the boundary $\partial\mathcal{D}$ of $\mathcal{D}(\lambda, \delta_\lambda)$, that is,

$$P(\mu) = \frac{-1}{2\pi\sqrt{-1}} \int_{\partial\mathcal{D}} (A(\mu) - \varsigma I)^{-1}\, d\varsigma.$$

Since $\dim \mathcal{R}(P(1)) \neq 0$, then from Theorem 1.3.4 and step [4] there is an $n_0 \in \mathcal{N}$ such that $\dim \mathcal{R}(P(\mu_n)) = \dim \mathcal{R}(P(1)) \neq 0$ for all $n \geq n_0$, which is impossible since $\dim \mathcal{R}(P(\mu_n)) = 0$ for all $n \in \mathcal{N}$. This proves the requirement.

Step [8] If $k \in \mathcal{N}$ is such that $| a_{kk} - a_{jj} |> P_k + P_j$ for all $j \neq k$, then λ_k is a simple eigenvalue of A.

From the assumption and step [7], it follows that $\sigma(A) \bigcap \mathcal{R}_k = \{\lambda_k\}$, and by Lemma 3.4.1, a closed Jordan curve Γ_k can be drawn so that $\mathcal{R}_k$ is a proper subset of $\mathcal{U}_k$, where $\mathcal{U}_k$ is the interior of the set bounded by Γ_k, and $\bar{\mathcal{U}}_k \bigcap \mathcal{R}_j = \emptyset$ if $j \neq k$.

From steps [2], [3] and [5], it is clear that Γ_k consists of regular points of $A(\mu)$ for every $\mu \in [0,1]$. From step [4], step [6] part (a) and Theorem 1.3.4 it is easily seen that λ_k is a simple eigenvalue. This completes the proof of the theorem. □

REMARK 3.4.3 If $p > 1$ in Theorem 3.4.2, then there is a dual theorem on ℓ_q, $\frac{1}{p} + \frac{1}{q} = 1$, where the row diagonal dominance (see hypothesis (2) of Theorem 3.4.2) is replaced by column diagonal dominance.

3.5 General Remarks

In this section we show the advantages of Theorem 3.4.2 over Theorem 5 of [26].

(1) When $p = \infty$ and $\sup\{\sigma_i : i \in \mathcal{N}\} < 1$, then the condition $a_{ii} \neq 0$ for all $i \in \mathcal{N}$ and hypothesis (2) imply hypothesis (3). In this case, $F_2 \in \mathcal{L}(\ell_\infty)$ and $\| F_2 \| < 1$ since

$$\| F_2 \| = \sup\{| a_{ii}^{-1} | \sum_{j=1}^{\infty}{}' | a_{ij} | : i \in \mathcal{N}\}$$

(see Remark 2.4.4). Hence Theorem 3.4.2 includes Theorem 5 of [26].

(2) In Theorem 5 of [26] it is assumed that every column of the matrix operator A is in ℓ_∞, while in Theorem 3.4.2 we do not require such a hypothesis. We give an example to illustrate this point. Define a matrix operator $A = (a_{ij})$ on ℓ_∞ by:

$$a_{11} = 1, a_{12} = \frac{1}{2}$$

and

$$a_{ij} = \begin{cases} i(i+1) & \text{if } j = i \geq 2 \\ i-1 & \text{if } j = 1, i \geq 2 \\ 1 & \text{if } j = i+1 \geq 3 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 5 of [26] does not apply to this matrix operator, since the first column of A is not in ℓ_∞. However, it is clear that A satisfies hypotheses (1) and (4) of Theorem 3.4.2. Also A satisfies hypothesis (2) of this theorem, since $P_i \leq \frac{1}{2} \mid a_{ii} \mid$ for all $i \in \mathcal{N}$. Since $\| F_2 \| = \sup\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} = \frac{1}{2} < 1$, then A satisfies hypothesis (3) of Theorem 3.4.2. So $\sigma(A) = \{\lambda_i : i \in \mathcal{N}\}$ where $\mid \lambda_1 - 1 \mid \leq \frac{1}{2}$ and $\mid \lambda_i - i(i+1) \mid \leq i-1$ for all $i \geq 2$.

(3) Matrix operators that would satisfy the hypotheses of Theorem 3.4.2 (and specially hypothesis (2)) and would not be included in Theorem 5 of [26] can be constructed as follows:

Let the matrix operator $A = (a_{ij})$ have diagonal entries defined by $a_{11} = 2$, and

$$a_{ii} = \frac{(2i-1)(i+1)}{i} a_{i-1,i-1}, \; i = 2, 3, \ldots$$

and it is assumed that -1 is not an eigenvalue of F_2. The off-diagonal elements may then be chosen in any way consistent with the condition that

$$P_i = \frac{i}{i+1} a_{ii}, \; i \in \mathcal{N}.$$

Chapter 4

Spectral Approximation of Diagonally Dominant Infinite Matrices

Our proofs of the Gersgorin-type theorems in Chapter Two are constructive in the sense that we have an explicit sequence of matrix operators $\{A_n\}$ with the property that $A_n \xrightarrow{g} A$. Furthermore, because of the block-triangular form of A_n (see equation (2.4.4)), the spectral properties of A_n are tractable. Our next step is therefore to draw conclusions concerning the relationship of eigenvalues and eigenvectors of A to those of A_n. For this purpose we apply some results on the "stable approximation" of closed operators.

4.1 Stable Approximation of Closed Operators

Let $\mathcal{X}$ be a separable Banach space, $A \in \mathcal{C}(\mathcal{X})$ and let $\{A_n\}$ be a sequence of operators in $\mathcal{C}(\mathcal{X})$. We suppose that $\mathcal{D}(A_n) \supset \mathcal{D}(A)$ for all $n \in \mathcal{N}$.

DEFINITION 4.1.1 We say that $\{A_n\}$ is an approximation of A if $A_n x \to Ax$ as $n \to \infty$ for all $x \in \mathcal{D}(A)$, and we write $A_n \xrightarrow{p} A$ on $\mathcal{D}(A)$.

Now let λ be an isolated eigenvalue of A, of finite algebraic multiplicity m. The point λ is isolated by the closed Jordan curve Γ, the interior of which defines the domain Δ.

DEFINITION 4.1.2 The spectrum of $\{A_n\}$ in Δ converges to λ if

$$\lim_{n\to\infty} (\sigma(A_n) \bigcap \Delta) = \{\lambda\}.$$

DEFINITION 4.1.3 The approximation $\{A_n\}$ of A is called stable at $z \in \rho(A)$ if there exists a positive integer $N(z)$ (depending on z) and a positive real $M(z)$ (depending on z) such that, for all $n \geq N(z)$, $z \in \rho(A_n)$ and

$$\| R_n(z) \| = \| (A_n - zI)^{-1} \| \leq M(z).$$

The following proposition gives the relation between the stability and the convergence of the spectrum for an approximating sequence $\{A_n\}$ ($A_n \in \mathcal{C}(\mathcal{X})$ for all $n \in \mathcal{N}$) of the closed operator A.

PROPOSITION 4.1.4 If the approximation $\{A_n\}$ of A is stable in $\Delta \setminus \{\lambda\}$ (that is, stable at every $z \in \Delta \setminus \{\lambda\}$), then $\lim_{n\to\infty}(\sigma(A_n) \bigcap \Delta) = \{\lambda\}$.

See Proposition 2.2 of [3] for the proof.

We now introduce the notion of strong stability.

DEFINITION 4.1.5 An approximation $\{A_n\}$ of A, stable in $\Delta \setminus \{\lambda\}$, is said to be strongly stable in $\Delta \setminus \{\lambda\}$ if $\dim P\mathcal{X} = \dim P_n\mathcal{X}$ for all n large enough where

$$P = \frac{-1}{2\pi\sqrt{-1}} \int_\Gamma (A - zI)^{-1}\, dz, \quad P_n = \frac{-1}{2\pi\sqrt{-1}} \int_\Gamma (A_n - zI)^{-1}\, dz.$$

We shall apply results using strong stability in the following form. Recall that λ is an isolated eigenvalue of A, of finite algebraic multiplicity m, and Γ and Δ are as defined above.

LEMMA 4.1.6 Let $0 \notin \bar{\Delta}$, the closure of Δ. Suppose A^{-1} exists and is compact. Let $\{A_n\}$ be an approximation of A for which A_n^{-1} exists and is in $\mathcal{L}(\mathcal{X})$ for all $n \in$

$\mathcal{N}$. If $\| A_n^{-1} - A^{-1} \| \to 0$ as $n \to \infty$, then $\{A_n\}$ is a strongly stable approximation of A in $\Delta \setminus \{\lambda\}$.

We need the following lemma (see page 4 of [10]) for the proof. We note that Theorem 1.3.3 can be deduced from this lemma.

LEMMA 4.1.7 Let $T \in \mathcal{L}(\mathcal{X})$ and $\mathcal{M}$ be a closed set in $\mathcal{C}$. If $\mathcal{M} \subset \rho(T)$, the resolvent set of T, then there exists a $\delta > 0$ such that $\mathcal{M} \subset \rho(S)$ for all $S \in \mathcal{L}(\mathcal{X})$ with $\| S - T \| < \delta$.

Proof of Lemma 4.1.6.

Since A^{-1} is compact, then any point $z \in \bar{\Delta} \setminus \{\lambda\}$ is a regular point of A. If $z \in \Delta \setminus \{\lambda\}$, then $z \neq 0, z \in \rho(A)$ and from Theorem 1.2.6 part(b), $z^{-1} \in \rho(A^{-1})$. So from Lemma 4.1.7 it follows that there exists an $N_1(z) \in \mathcal{N}$ such that $z^{-1} \in \rho(A_n^{-1})$ for all $n \geq N_1(z)$. Since $A_n^{-1} \in \mathcal{L}(\mathcal{X})$ and $z \neq 0$, we have $z \in \rho(A_n)$ for all $n \geq N_1(z)$. Also, for all $n \geq N_1(z)$

$$\| (A_n - zI)^{-1} \| \leq \| (A - zI)^{-1} \| + \| (A_n - zI)^{-1} - (A - zI)^{-1} \| . \tag{4.1.1}$$

Since $\| A_n^{-1} - A^{-1} \| \to 0$ as $n \to \infty$, then from Theorem 2.2.1 part(1) we have $A_n \xrightarrow{g} A$ and so by using Theorem 2.2.1 part(2), $A_n - zI \xrightarrow{g} A - zI$. Then we may use Theorem 2.2.1 part(1) again to deduce $\lim_{n\to\infty} \| (A_n - zI)^{-1} - (A - zI)^{-1} \| = 0$. Using this conclusion in inequality (4.1.1) and the fact that $\| (A - zI)^{-1} \| < \infty$, we find that there exists an $N(z) \in \mathcal{N}$ such that $N(z) > N_1(z)$ and there exists a positive real $M(z)$ such that, for all $n \geq N(z)$,

$$\| R_n(z) \| = \| (A_n - zI)^{-1} \| \leq M(z) < \infty.$$

This proves that the approximation $\{A_n\}$ of A is stable at every $z \in \Delta \setminus \{\lambda\}$. Using

Theorem 1.3.4 and the fact that $A_n \xrightarrow{g} A$, we find $\dim P_n\mathcal{X} = \dim P\mathcal{X}$ for n large enough and this completes the proof of the lemma. □

Now suppose $\lim_{n\to\infty}(\sigma(A_n)\bigcap\Delta) = \{\lambda\}$. Let λ_n be an eigenvalue of A_n for all $n \in \mathcal{N}$ such that $\lim_{n\to\infty}\lambda_n = \lambda$. For each λ_n there exists a normalized eigenvector, that is, there exists a $y_n \in \mathcal{D}(A_n)$ such that $A_n y_n = \lambda_n y_n$ and $\| y_n \| = 1$.

DEFINITION 4.1.8 The eigenvectors of A_n associated with $\sigma(A_n)\bigcap\Delta$ are said to be convergent if any sequence of normalized eigenvectors $\{y_n\}_{n=1}^{\infty}$ associated with $\{\lambda_n\}_{n=1}^{\infty}$ contains a subsequence $\{y_{n_k}\}_{k=1}^{\infty}$ converging to some normalized eigenvector of A associated with λ.

PROPOSITION 4.1.9 If $\{A_n\}$ is a strongly stable approximation of A in $\Delta\setminus\{\lambda\}$, then the eigenvectors of $\{A_n\}$ associated with $\sigma(A_n)\bigcap\Delta$ are convergent.

See Proposition 2.3 part(iii) of [3] for the proof.

4.2 Approximation of Eigenvalues and Eigenvectors

The results of Chapter Two and Section 4.1 are now combined to show that the eigenvalues and the eigenvectors of infinite diagonally dominant matrices can be approximated by those of approximating matrices of the form (2.4.4). We state the results for row diagonal dominance only. There are exactly parallel statements in the case of column diagonal dominance. The matrices are assumed to act in ℓ_p with $1 \leq p < \infty$. The case $p = \infty$ is excluded because our proofs require separability of the underlying space, and ℓ_∞ does not enjoy this property (see Example 3, page 200 of [9]).

COROLLARY 4.2.1 Suppose the matrix operator A satisfies the hypotheses of Corollary 2.4.6 and assume $1 \leq p < \infty$, then the sequence of closed matrix operators $\{A_n\}$ defined by equation (2.4.4) is a strongly stable approximation of A in $\mathcal{R}_i \setminus \{\lambda_i\}$ where λ_i is the only eigenvalue of A in $\mathcal{R}_i$ (see equations (2.1.1) and (2.1.2) for the definition of $\mathcal{R}_i$).

Proof. From Lemma 3.4.1, given $i \in \mathcal{N}$ there is a positive real δ_i such that the disc $\bar{\Delta}_i = \{z \in C : | z - a_{ii} | \leq P_i + \delta_i\}$ satisfies $\bar{\Delta}_i \cap \mathcal{R}_j = \emptyset$ if $j \neq i$. Also the disc $\bar{\Delta}_i$ is chosen so that $0 \notin \bar{\Delta}_i$ (notice that from hypothesis (2) of Corollary 2.4.6, $0 \notin \mathcal{R}_i$). Hence for every $i \in \mathcal{N}$, λ_i is isolated by the boundary Γ_i of $\bar{\Delta}_i$, where $\Gamma_i \subset \rho(A)$. We see from Corollary 2.4.6 that A^{-1} exists and is compact, A_n^{-1} exists and is in $\mathcal{L}(\ell_p)$ for every $n \in \mathcal{N}$, and $\| A_n^{-1} - A^{-1} \| \to 0$ as $n \to \infty$. Therefore the hypotheses of Lemma 4.1.6 are satisfied if we can show $A_n x \to Ax$ as $n \to \infty$ for every $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} \in \mathcal{D}(A)$ (we have $\mathcal{D}(A) = \mathcal{D}(A_n)$ for all $n \in \mathcal{N}$, since the perturbation operator $F = ((1-\delta_{ij})a_{ij}) \in \mathcal{L}(\ell_p)$). To see this, write $x = \begin{pmatrix} x_1^{(n)} \\ x_2^{(n)} \end{pmatrix}$, where $x_1^{(n)} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $x_2^{(n)} = \begin{pmatrix} x_{n+1} \\ x_{n+2} \\ \vdots \end{pmatrix}$. Then

$$Ax - A_n x = \begin{pmatrix} 0 & A_{12}^{(n)} \\ 0 & -S_{nn} \end{pmatrix} \begin{pmatrix} x_1^{(n)} \\ x_2^{(n)} \end{pmatrix} = \begin{pmatrix} A_{12}^{(n)} x_2^{(n)} \\ -S_{nn} x_2^{(n)} \end{pmatrix}$$

where $A = \begin{pmatrix} A_{11}^{(n)} & A_{12}^{(n)} \\ A_{21}^{(n)} & A_{22}^{(n)} \end{pmatrix}$ and $S_{nn} = D_n - A_{22}^{(n)}$ (see equation (2.4.4)). Since $F \in \mathcal{L}(\ell_p)$, then $A_{12}^{(n)}$ and S_{nn} are bounded uniformly with respect to n (see step [4] of Theorem 2.4.2). Also since $x \in \ell_p$, then $\lim_{n\to\infty} x_2^{(n)} = 0$. So $\lim_{n\to\infty} A_{12}^{(n)} x_2^{(n)} = 0$ and $\lim_{n\to\infty} S_{nn} x_2^{(n)} = 0$. This proves that $\{A_n\}$ is an approximation of A. Now the result follows by applying Lemma 4.1.6. □

Combining this corollary with propositions 4.1.4 and 4.1.9 we obtain:

THEOREM 4.2.2 Suppose that the matrix operator A satisfies the hypotheses of Corollary 2.4.6, and suppose $1 \leq p < \infty$. Define matrix operators A_n in ℓ_p as in equation (2.4.4). Denote the simple eigenvalue of A (of A_n) in $\mathcal{R}_i$ by λ_i (by $\lambda_i^{(n)}$). Then for $1 \leq i < \infty$, $\lim_{n\to\infty} \lambda_i^{(n)} = \lambda_i$ and the associated sequence of normalized eigenvectors $\{y_i^{(n)}\}_{i=1}^{\infty}$ contains a subsequence converging to the normalized eigenvector of A associated with λ_i.

We remark that, when A is a banded matrix, the eigenvectors of A_n (see equation (2.4.4)) have only finitely many nonzero terms. If A_n has an eigenvalue $\lambda_n \neq a_{n+i,n+i}$ for some $i \in \mathcal{N}$ then with an obvious partition of the eigenvector,

$$\begin{pmatrix} A_{11}^{(n)} & 0 \\ A_{21}^{(n)} & D_n \end{pmatrix} \begin{pmatrix} x_1^{(n)} \\ x_2^{(n)} \end{pmatrix} = \lambda_n \begin{pmatrix} x_1^{(n)} \\ x_2^{(n)} \end{pmatrix}.$$

Thus $x_1^{(n)}$ is the corresponding eigenvector of $A_{11}^{(n)}$ and

$$x_2^{(n)} = (\lambda_n I - D_n)^{-1} A_{21}^{(n)} x_1^{(n)}.$$

When A is banded, $A_{21}^{(n)}$ is strictly upper triangular and the number of nonzero entries in $x_2^{(n)}$ does not exceed the number of sub-diagonal bands of A.

Now consider bounded matrix operators $F^{(p)} \in \mathcal{L}(\ell_p), F^{(q)} \in \mathcal{L}(\ell_q)$ which are defined by the same infinite matrix F ($A = (a_{ij}), F = ((1-\delta_{ij})a_{ij})$ as usual). It is clear from our construction that the eigenvalues of the approximating sequence $\{A_n\}$ will not depend on the choice of the space ℓ_p or ℓ_q. Thus their limits are the same, and we can conclude that if $1 \leq p, q < \infty$, then $A^{(p)} = D(I + D^{-1}F^{(p)})$ and $A^{(q)} = D(I + D^{-1}F^{(q)})$, where $D = \operatorname{diag}(a_{11}, a_{22}, \ldots)$, have the same spectrum. A similar remark applies for the eigenvectors. So we have:

THEOREM 4.2.3 Let the matrix operator $A = (a_{ij})$ satisfy the hypotheses of Corollary 2.4.6 and assume that the matrix operator $F = ((1-\delta_{ij})a_{ij})$ is in $\mathcal{L}(\ell_p)$ and in $\mathcal{L}(\ell_q)$, $p \neq q, 1 \leq p, q < \infty$. Then A defines closed operators $A^{(p)}, A^{(q)}$ with domains in ℓ_p, ℓ_q, respectively, whose spectra and eigenvectors coincide.

4.3 Selfadjoint Matrix Operators

For a matrix operator A acting in ℓ_2 there is a possibility of obtaining stronger conclusions for the spectral properties using the Hilbert space structure of ℓ_2. To illustrate this point, we know, for example, that in Theorem 4.2.2 the convergence of the sequence $\{\lambda_i^{(n)}\}_{n=1}^{\infty}$ to λ_i for each $i \in \mathcal{N}$ depends on estimates for the norm of the resolvent of the matrix operator A, which are not necessarily available. However for a matrix operator A acting in ℓ_2, such estimates are available if A is selfadjoint (see equation (3.16), page 272 of [15]).

The following theorem shows that, in principle, the ith eigenvalue of a selfadjoint matrix operator $A = (a_{ij})$ acting in ℓ_2 and satisfying the hypotheses of Corollary 2.4.6 can be approximated to any prescribed accuracy $\epsilon > 0$ by truncation to the leading $N \times N$ submatrix of A, where the dependence of N on ϵ is given below (see equation (4.3.2)). Before introducing the theorem we notice that for such a matrix operator A the Gersgorin discs $\mathcal{R}_i$, $i \in \mathcal{N}$, are disjoint (see hypothesis (4) of Corollary 2.4.6) and hence from Lemma 3.4.1, it follows that for every $i \in \mathcal{N}$ the number

$$\delta_i = \inf\{\mid a_{ii} - a_{jj} \mid -(P_i + P_j) : j \neq i\}$$

is a positive real number. For brevity, we introduce the following symbols:

$$\epsilon_i = \frac{\delta_i}{2(\mid a_{ii} \mid + P_i)(\mid a_{ii} \mid + P_i + \delta_i)}, \; i \in \mathcal{N}$$

and

$$K = \| D^{-1} \| (1 + \| F \|) \| (I + D^{-1}F)^{-1} \|$$

where $D = \mathrm{diag}(a_{11}, a_{22}, \ldots)$ and $F = ((1 - \delta_{ij})a_{ij})$ (we consider the case $(I + D^{-1}F)^{-1} \in \mathcal{L}(\ell_2)$; see hypothesis (3) of Corollary 2.4.6). If $\mathcal{F}$ is a subset of $\mathcal{C}$ not including zero then we define the set $\mathcal{F}^{-1} = \{z \in \mathcal{C} : z^{-1} \in \mathcal{F}\}$.

THEOREM 4.3.1 Assume the hypotheses of Corollary 2.4.6, let $p = 2$ (the case $(I + D^{-1}F)^{-1} \in \mathcal{L}(\ell_2)$ is considered here; the case $(I + FD^{-1})^{-1} \in \mathcal{L}(\ell_2)$ has a similar statement) and assume that A is selfadjoint. Let $i \in \mathcal{N}$, $\epsilon \in (0, \epsilon_i)$ and also $\epsilon < \| D^{-1} \| \| A^{-1} \|$. If

$$k = 4 \| A^{-1} \| (1 + K)(1 + \| F \|)^2 \tag{4.3.1}$$

is a constant depending only on A, and N is an integer for which

$$\mid a_{nn} \mid > k\epsilon^{-1}(1 + \mid a_{ii} \mid + P_i)^2 \tag{4.3.2}$$

whenever $n \geq N$, then $| \lambda_i^{(N)} - \lambda_i | < \epsilon$ ($\lambda_i^{(N)}$ is the unique eigenvalue of A_N in the Gersgorin disc $\mathcal{R}_i$, see equation (2.4.4) for the definition of A_N).

Proof. Let $T = A^{-1}$ and $T_n = A_n^{-1}$ for all $n \in \mathcal{N}$. Fix $i \in \mathcal{N}$ and let Γ_{i_1} be the boundary of the set $\mathcal{R}_i$ and Γ_{i_2} be the boundary of the set $\tilde{\mathcal{R}}_i = \{z \in \mathcal{C} : | z - a_{ii} | \leq P_i + \delta_i\}$. Since $\inf\{\text{dist}(z, \Gamma_{i_1}) : z \in \Gamma_{i_2}\} = \delta_i$ then for all $z \in \Gamma_{i_1}$ and $z' \in \Gamma_{i_2}$ we have:

$$| \frac{1}{z} - \frac{1}{z'} | = \frac{| z' - z |}{| z || z' |} \geq \frac{\delta_i}{(| a_{ii} | + P_i)(| a_{ii} | + P_i + \delta_i)}.$$

If $a_{ii} = | a_{ii} | e^{\sqrt{-1}\phi_0}$, then

$$| \frac{1}{z_0} - \frac{1}{z_0'} | = \frac{| z_0' - z_0 |}{| z_0 || z_0' |} = 2\epsilon_i$$

where $z_0 = (| a_{ii} | + P_i)e^{\sqrt{-1}\phi_0}$ and $z_0' = (| a_{ii} | + P_i + \delta_i)e^{\sqrt{-1}\phi_0}$. Thus we have

$$\inf\{\text{dist}(\varsigma, \Gamma_{i_1}^{-1}) : \varsigma \in \Gamma_{i_2}^{-1}\} = 2\epsilon_i.$$

Choose $\epsilon \in (0, \epsilon_i)$ and also $\epsilon < \| D^{-1} \| \| A^{-1} \|$. Then from equation (3.16), page 272 of [15], we have

$$\max\{\| R(\varsigma, T) \| : \varsigma \in \Delta_i\} = (1 + | a_{ii} | + P_i)^2 / \epsilon,$$

where

$$\Delta_i = \{\varsigma \in \mathcal{R}_i^{-1} : | \varsigma - \varsigma_i | \geq \epsilon(1 + | a_{ii} | + P_i)^{-2}\}$$

and $\varsigma_i = \lambda_i^{-1}$. From Theorem 2.1.3 we know that if we choose $N_1 \in \mathcal{N}$ such that $| a_{nn}^{-1} | < \frac{\|D^{-1}\|}{K}$ for all $n \geq N_1$, then $\| (A_{11}^{(n)})^{-1} \| \leq K$ for all $n \geq N_1$. Now we choose $N \in \mathcal{N}$ such that for all $n \geq N$,

$$| a_{nn}^{-1} | < \min \left\{ \frac{\epsilon}{k(1 + | a_{ii} | + P_i)^2}, \frac{\| D^{-1} \|}{K} \right\}.$$

Since $\epsilon <\| D^{-1} \| \| A^{-1} \|$, then from equation (4.3.1) the above inequality is equivalent to equation (4.3.2). Hence from step [4] of Theorem 2.4.2 and Lemma 2.1.1, it follows that $\| T_n - T \| < \epsilon(1+ | a_{ii} | + P_i)^{-2}$ for all $n \geq N$ (the matrix operator $D_n^{-1} = \text{diag}(a_{n+1,n+1}^{-1}, a_{n+2,n+2}^{-1}, \ldots)$ satisfies $\| D_n^{-1} \| < \frac{\epsilon}{k(1+|a_{ii}|+P_i)^2}$ for all $n \geq N$). So for all $\varsigma \in \Delta_i$ and $n \geq N$, $\| R(\varsigma, T) \| \| T_n - T \| < 1$ and hence $\varsigma \in \rho(T_n)$. But since $\mathcal{R}_i^{-1}$ contains a unique eigenvalue of T_n, namely $\varsigma_i^{(n)} = (\lambda_i^{(n)})^{-1}$, then $| \varsigma_i^{(n)} - \varsigma_i | < \epsilon(1+ | a_{ii} | + P_i)^{-2}$ for all $n \geq N$. Thus we get

$$| \lambda_i^{(n)} - \lambda_i | = | \lambda_i^{(n)} (\varsigma_i - \varsigma_i^{(n)}) \lambda_i | \leq \frac{(| a_{ii} | + P_i)^2 \epsilon}{(1+ | a_{ii} | + P_i)^2} < \epsilon$$

for all $n \geq N$ and this proves the requirement. □

Chapter 5

Mathieu's Equation

The main result of this chapter is Theorem 5.4.1 which shows that the eigenvalues of the Mathieu differential operator and of a certain infinite matrix operator are the same. This is a result that is frequently used implicitly in the literature but, to the author's knowledge, is proved here for the first time. To establish this result some facts on the solution of infinite systems of linear equations are introduced in Section 5.3.

5.1 Introduction

We will be concerned with Mathieu's differential equation

$$\frac{d^2y}{d\theta^2} + (a - 2q\cos 2\theta)y = 0 \tag{5.1.1}$$

and we will concentrate on solutions which are even with period π, denoted usually by $\mathrm{ce}_{2n}(\theta, q)$. For a given q, the problem will be to find the eigenvalues a.

Mathieu's equation is a special case of Hill's equation

$$\frac{d^2y}{d\theta^2} + (a - g(\theta))y = 0,$$

where $g(\theta)$ is a periodic function with periodicity π. Titchmarsh in [29] and [30] discussed the dependence of the eigenvalues and the eigenvectors of the perturbed equation

$$\frac{d^2y}{d\theta^2} + (a - g(\theta) - \epsilon\sigma(\theta))y = 0$$

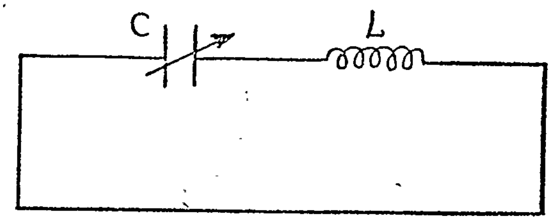

Figure 5.1: $L - C$ circuit.

upon ϵ, assuming that the spectrum of the unperturbed equation is discrete. He considered two cases ; the first when the spectrum of the perturbed system is discrete, and the second when the spectrum of the perturbed system is continuous.

Mathieu's equation appears in many applications. One example is the "Direct Capacitance Modulation" shown below (for more examples see Chapter XV of [18]).

EXAMPLE 5.1.1 Consider the electric circuit shown schematically in Figure 1, where an inductance L is in series with a capacitance C, which varies with time t, so we can write $C = C(t)$. If Q denotes the quantity of electricity in the capacitance then from Kirchhoff's second law (see page 23 of [19]), the circuital differential equation is

$$\frac{d^2Q}{dt^2} + \frac{Q}{LC(t)} = 0. \tag{5.1.2}$$

We shall assume that $C(t) = C_0(1 + \epsilon \cos 2\omega_1 t)$, C_0 being constant and $0 < \epsilon \ll 1$ (that is, ϵ is a very small positive number compared with one). Using the binomial expansion of $(1 + \epsilon \cos 2\omega_1 t)^{-1}$, equation (5.1.2) becomes:

$$\frac{d^2Q}{dt^2} + \frac{1}{LC_0}(1 - \epsilon \cos 2\omega_1 t + \epsilon^2 \cos^2 2\omega_1 t - \epsilon^3 \cos^3 2\omega_1 t + \ldots)Q = 0.$$

Omitting the terms in $\epsilon^2, \epsilon^3, \ldots$ in the above equation as a first approximation, we obtain

$$\frac{d^2Q}{dt^2} + \frac{1}{LC_0}(1 - \epsilon \cos 2\omega_1 t)Q = 0. \tag{5.1.3}$$

Writing $\theta = \omega_1 t$ and $y = Q$ in equation (5.1.3) we are led to Mathieu's equation

$$\frac{d^2y}{d\theta^2} + (a - 2q\cos 2\theta)y = 0$$

where $a = \frac{1}{\omega_1^2 LC_0}$ and $q = \frac{\epsilon a}{2}$.

5.2 The Space $\mathcal{L}_2[-\frac{\pi}{2}, \frac{\pi}{2}]$

This space of functions will play an important role in finding the eigenvalues of Mathieu's equation.

DEFINITION 5.2.1 Let $\mathcal{L}_2[-\frac{\pi}{2}, \frac{\pi}{2}]$ be the vector space of all complex valued Lebesgue measurable functions f defined on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ with the property that $|f|^2$ is Lebesgue integrable. If we identify functions which are equal almost everywhere, then

$$\langle f, g\rangle = \int_{-\pi/2}^{\pi/2} f(\theta)\bar{g}(\theta)\, d\theta,$$

where $\bar{g}(\theta)$ denotes the complex conjugate of $g(\theta)$, defines an inner product on $\mathcal{L}_2[-\frac{\pi}{2}, \frac{\pi}{2}]$.

REMARK 5.2.2 Since for $k, j \in \mathcal{N}$,

$$\int_{-\pi/2}^{\pi/2} \cos 2(k-1)\theta \cos 2(j-1)\theta\, d\theta = \left\{ \begin{array}{ll} 0 & \text{if } k \neq j \\ \pi/2 & \text{if } k = j \neq 1 \\ \pi & \text{if } k = j = 1 \end{array} \right\}, \tag{5.2.1}$$

$$\int_{-\pi/2}^{\pi/2} \sin 2k\theta \sin 2j\theta \, d\theta = \left\{ \begin{array}{ll} 0 & \text{if } k \neq j \\ \pi/2 & \text{if } k = j \end{array} \right\} \tag{5.2.2}$$

and

$$\int_{-\pi/2}^{\pi/2} \cos 2(k-1)\theta \sin 2j\theta \, d\theta = 0, \tag{5.2.3}$$

then by giving a similar argument to that given in page 29 of [9], one can prove that the countable collection

$$\left\{ \frac{1}{\sqrt{\pi}} \right\} \bigcup \left\{ \sqrt{\frac{2}{\pi}} \cos 2n\theta : n \in \mathcal{N} \right\} \bigcup \left\{ \sqrt{\frac{2}{\pi}} \sin 2n\theta : n \in \mathcal{N} \right\}$$

is an orthonormal basis for $\mathcal{L}_2[-\frac{\pi}{2}, \frac{\pi}{2}]$ (see page 26 of [9] for the definition of an orthonormal basis for a Hilbert space).

One of the properties that connect the Hilbert spaces ℓ_2 and $\mathcal{L}_2[-\frac{\pi}{2}, \frac{\pi}{2}]$ is stated in the following theorem.

THEOREM 5.2.3 (See Theorem 11.2 part(iii), page 25 of [9].) If $x_k \in \mathcal{C}$ for all $k \in \mathcal{N}$ then the vector $x = (x_1, x_2, \ldots)$ is in ℓ_2 if and only if the series

$$\sum_{k=1}^{\infty} x_k \cos 2(k-1)\theta$$

converges in $\mathcal{L}_2[-\frac{\pi}{2}, \frac{\pi}{2}]$.

NOTE. The series $\sum_{k=1}^{\infty} x_k \cos 2(k-1)\theta$ converges in $\mathcal{L}_2[-\frac{\pi}{2}, \frac{\pi}{2}]$ if there exists a function $y \in \mathcal{L}_2[-\frac{\pi}{2}, \frac{\pi}{2}]$ such that the sequence $\{y_n\}_{n=1}^{\infty}$ defined by $y_n = \sum_{k=1}^{n} x_k \cos 2(k-1)\theta$, $n \in \mathcal{N}$, converges to y as $n \to \infty$. The function y is written as $\sum_{k=1}^{\infty} x_k \cos 2(k-1)\theta$.

Proof. For all $n \in \mathcal{N}$, let $y_n = \sum_{k=1}^{n} x_k \cos 2(k-1)\theta$ and $s_n = \sum_{k=1}^{n} | x_k |^2$. Then from equation (5.2.1) it follows that for $n > m$,

$$\| y_n - y_m \|^2 = \langle \sum_{k=m+1}^{n} x_k \cos 2(k-1)\theta, \sum_{j=m+1}^{n} x_j \cos 2(j-1)\theta \rangle$$

$$= \sum_{k=m+1}^{n} \int_{-\pi/2}^{\pi/2} \cos 2(k-1)\theta \sum_{j=m+1}^{n} \bar{x}_j \cos 2(j-1)\theta \, d\theta$$
$$= \frac{\pi}{2} \sum_{k=m+1}^{n} | x_k |^2$$
$$= s_n - s_m.$$

Thus $\{y_n\}$ is a Cauchy sequence if and only if $\{s_n\}$ is a Cauchy sequence. Therefore $\{y_n\}$ converges if and only if $\{s_n\}$ converges, and this proves the requirement. □

In the next sections we will use the following lemma which shows the continuity of the inner product on a Hilbert space.

LEMMA 5.2.4 Let $\mathcal{H}$ be a Hilbert space over $\mathcal{C}$. If $\{x_n\}$ is a sequence of points in $\mathcal{H}$ converging to $x \in \mathcal{H}$, and if $y \in \mathcal{H}$ then $\langle x_n, y\rangle \to \langle x, y\rangle$ as $n \to \infty$, where $\langle\ ,\ \rangle$ denotes the inner product on $\mathcal{H}$.

See Lemma 11.1, page 25 of [9] for the proof.

5.3 Linear Systems and Truncation

In this section some basic results about the solution of infinite linear systems, which will be used in the next section, are introduced. Linear equations with infinite matrices occur in various topics of mathematics, for example, interpolation [5], sequence spaces [4], and summability [32].

Let $A = (a_{ij})$ be a matrix operator in ℓ_p, $1 \le p \le \infty$, and $y = (y_1, y_2, \ldots) \in \ell_p$. We define the truncations $A(n)$ and $y(n)$ by

$$A(n) = \begin{pmatrix} A_{11}^{(n)} & 0 \\ 0 & 0 \end{pmatrix},$$

where $A_{11}^{(n)}$ is the leading $n \times n$ submatrix of A (see Section 2.1), and the ith component $(y(n))_i$ of the vector $y(n) \in \ell_p$ is

$$(y(n))_i = \begin{cases} y_i & \text{if } 1 \leq i \leq n \\ 0 & \text{if } i > n. \end{cases}$$

For $1 \leq i, j \leq n$, the cofactor of the element a_{ij} in $A_{11}^{(n)}$ is denoted by $\langle a_{ij}^{(n)} \rangle$. We define $\frac{\langle a_{ij} \rangle}{\det A} = \lim_{n \to \infty} \frac{\langle a_{ij}^{(n)} \rangle}{\det A_{11}^{(n)}}$, provided the limit exists. If $\left(A_{11}^{(n)}\right)^{-1}$ exists then we define the matrix $\hat{A}^{-1}(n)$ to be the infinite matrix for which

$$(\hat{A}^{-1}(n))_{ij} = \begin{cases} ((A_{11}^{(n)})^{-1})_{ij} & \text{if } 1 \leq i, j \leq n \\ 0 & \text{if } i, j \geq n+1. \end{cases}$$

Now we consider the linear systems

$$Ax = y \tag{5.3.1}$$

and

$$A(n)\eta(n) = y(n), \tag{5.3.2}$$

where the ith component $(\eta(n))_i$ of $\eta(n)$ is zero for $i > n$.

THEOREM 5.3.1 Let $A = (a_{ij})$ be a matrix operator in ℓ_p, $1 \leq p \leq \infty$, $D = \operatorname{diag}(a_{11}, a_{22}, \ldots)$ and assume that

(1) The diagonal elements a_{ii} satisfy $a_{ii} \neq 0$ for all $i \in \mathcal{N}$ and $\mid a_{ii} \mid \to \infty$ as $n \to \infty$.

(2) For every $i \in \mathcal{N}$ there exists a $\sigma_i \in [0, 1)$ such that

$$P_i = \sum_{j=1}^{\infty}{}' \mid a_{ij} \mid = \sigma_i \mid a_{ii} \mid .$$

(3) The matrix operator $F = ((1 - \delta_{ij})a_{ij}) \in \mathcal{L}(\ell_p)$.

(4) Either the matrix operator $I + FD^{-1}$ has a bounded inverse on ℓ_p or the matrix operator $I + D^{-1}F$ has a bounded inverse on ℓ_p.

Then for every $y = (y_1, y_2, \ldots) \in \ell_p$, the linear systems (5.3.1) and (5.3.2) have unique solutions $x = (x_1, x_2, \ldots) \in \ell_p$ and $\eta(n) = (\eta_1, \eta_2, \ldots) \in \ell_p$, respectively, and $\lim_{n\to\infty} \| x - \eta(n) \|_p = 0$.

Furthermore, the ith component of x is given by

$$x_i = \sum_{j=1}^{\infty} \left(\frac{\langle a_{ji} \rangle}{\det A}\right) y_j .$$

Proof. We consider the case $(I + D^{-1}F)^{-1} \in \mathcal{L}(\ell_p)$. The other case (see hypothesis (4)) has no new features. Let $y = (y_1, y_2, \ldots) \in \ell_p$. Then from Theorems 2.4.2 and 2.1.3, the matrix operators A and $A_{11}^{(n)}$, $n \in \mathcal{N}$, have bounded inverses on ℓ_p and C^n, respectively. Hence the systems (5.3.1) and (5.3.2) have unique solutions $x = A^{-1}y$ and $\eta(n) = \hat{A}^{-1}(n)y(n)$. To prove $\lim_{n\to\infty} \| x - \eta(n) \|_p = 0$, we consider the following two cases:

(i) The case $p < \infty$. Since $\| x - \eta(n) \| = \| A^{-1}y - \hat{A}^{-1}(n)y(n) \|$, it follows that

$$\| x - \eta(n) \| \leq \| A^{-1} - \hat{A}^{-1}(n) \| \| y \| + \| \hat{A}^{-1}(n) \| \| y - y(n) \| . \tag{5.3.3}$$

Also from step [4] of Theorem 2.4.2, we have

$$\| A^{-1} - \hat{A}^{-1}(n) \| \leq \| A^{-1} - A_n^{-1} \| + \left\| \begin{pmatrix} 0 & 0 \\ -D_n^{-1} A_{21}^{(n)} \left(A_{11}^{(n)}\right)^{-1} & D_n^{-1} \end{pmatrix} \right\| . \tag{5.3.4}$$

Then from inequalities (5.3.3),(5.3.4) and the fact $\| \hat{A}^{-1}(n) \| = \| \left(A_{11}^{(n)}\right)^{-1} \|$ (see the definition of $\hat{A}^{-1}(n)$), it follows that

$$\| x - \eta(n) \| \leq \| A^{-1} - A_n^{-1} \| \| y \| + \left\| \begin{pmatrix} 0 & 0 \\ -D_n^{-1} A_{21}^{(n)} \left(A_{11}^{(n)}\right)^{-1} & D_n^{-1} \end{pmatrix} \right\| \| y \|$$

$$+ \| (A_{11}^{(n)})^{-1} \| \| y - y(n) \| . \tag{5.3.5}$$

From step [4] of Theorem 2.4.2, $\lim_{n\to\infty} \| A^{-1} - A_n^{-1} \| = 0$ and from Theorem 2.1.3, there is a positive real M such that $\| (A_{11}^{(n)})^{-1} \| \leq M$ for all $n \in \mathcal{N}$. Also since $\| D_n^{-1} \| \to 0$ as $n \to \infty$ then from Lemma 2.1.1, the norm of the block matrix in the right hand side of inequality (5.3.5) tends to zero as $n \to \infty$. Thus the right hand side of inequality (5.3.5) tends to zero as $n \to \infty$ and this proves $\| x - \eta(n) \| \to 0$ as $n \to \infty$.

(ii) The case $p = \infty$. From hypothesis (3) and Remark 2.4.4, there exists a positive real M such that

$$\sup\{P_i : i \in \mathcal{N}\} < M.$$

Hence from hypothesis (1), $\sigma_i = \frac{P_i}{|a_{ii}|} \to 0$ as $i \to \infty$. Thus the matrix operator A satisfies hypotheses (5.1) and (5.2) (with $\lim_{i\to\infty} \sigma_i = 0$) of Theorem 7 in [26], and using the proof of that theorem it follows that $\lim_{n\to\infty} \| x - \eta(n) \|_\infty = 0$. (Note that their hypothesis (5.4) is not required in this argument.)

Now fix $j \in \mathcal{N}$ and let $n \in \mathcal{N}$, $n \geq j$. Then, for $1 \leq i \leq n$ and $y = e_j$, we have

$$(\eta(n))_i = (\hat{A}^{-1}(n)y(n))_i = \sum_{k=1}^{\infty} (\hat{A}^{-1}(n))_{ik}(y(n))_k = (\hat{A}^{-1}(n))_{ij}$$

and from the definition of $\hat{A}^{-1}(n)$, we get

$$(\eta(n))_i = ((A_{11}^{(n)})^{-1})_{ij} = \frac{\langle a_{ji}^{(n)} \rangle}{\det A_{11}^{(n)}}.$$

Similarly,

$$x_i = (A^{-1}y)_i = (A^{-1})_{ij}.$$

Thus,

$$\lim_{n\to\infty} \frac{\langle a_{ji}^{(n)} \rangle}{\det A_{11}^{(n)}} = \lim_{n\to\infty} (\eta(n))_i = x_i = (A^{-1})_{ij}$$

exists and hence, $(A^{-1})_{ij} = \frac{\langle a_{ji} \rangle}{\det A}$.

For a general $y = (y_1, y_2, \ldots) \in \ell_p$ and $i \in \mathcal{N}$, we have

$$x_i = (A^{-1}y)_i = \sum_{j=1}^{\infty} (A^{-1})_{ij} y_j = \sum_{j=1}^{\infty} \frac{\langle a_{ji} \rangle}{\det A} y_j$$

and this completes the proof of the theorem. □

REMARK 5.3.2 Let $p = \infty$ in Theorem 5.3.1. From the proof of this theorem, there is a positive real M such that

$$\sup\{P_i : i \in \mathcal{N}\} < M.$$

Also from hypothesis (1) there exists a positive integer i_0 such that $| a_{ii} |^{-1} < \frac{1}{M+1}$ for all $i > i_0$. Thus from hypothesis (2), we get

$$\sup\{\frac{P_i}{| a_{ii} |} : i \in \mathcal{N}\} < 1.$$

So $D^{-1}F \in \mathcal{L}(\ell_\infty)$ and $\| D^{-1}F \| < 1$. Thus hypotheses (1),(2) and (3) of Theorem 5.3.1 imply hypothesis (4) in the case $p = \infty$.

REMARK 5.3.3 Let $p = \infty$ in Theorem 5.3.1. It is clear that hypotheses (1),(2) and (3) of Theorem 5.3.1 are equivalent to conditions (H.1),(1.2) and (H.2) in [27]. Thus from Remark 5.3.2, Theorem 5.3.1 generalizes Theorem 2 of [27]. It also develops it since we do not assume in Theorem 5.3.1 the condition (H.3) of Theorem 2 in [27]. On the other hand, Theorem 7 in [26] is more general than Theorem 5.3.1 at the case $p = \infty$. This can be seen from the fact that the boundedness of F together with hypothesis (1) of Theorem 5.3.1 imply the condition $\sigma_i \to 0$ as $i \to \infty$, while this condition does not necessarily imply the boundedness of F.

LEMMA 5.3.4 (See Lemma 5 of [26].) Suppose $A = (a_{ij})$ is a matrix operator in ℓ_p, $1 \leq p \leq \infty$, that satisfies hypotheses (1),(2),(3) and (4) of Theorem 5.3.1.

Then, for $i, j \in \mathcal{N}$

$$| \frac{\langle a_{ji} \rangle}{\det A} | \leq \sigma_i | \frac{\langle a_{jj} \rangle}{\det A} |, \; j \neq i$$

and

$$(| a_{ii} | + P_i)^{-1} \leq | \frac{\langle a_{ii} \rangle}{\det A} | \leq (| a_{ii} | - P_i)^{-1}.$$

Proof. Fix i, j and $n \in \mathcal{N}$ with $n > i$ and $n > j$. It follows from inequalities (13) and (10) in [21] that the above inequalities are valid for $A_{11}^{(n)}$. From Theorem 5.3.1, $\lim_{n\to\infty} \frac{\langle a_{ji}^{(n)} \rangle}{\det A_{11}^{(n)}} = \frac{\langle a_{ji} \rangle}{\det A}$ exists and the lemma is proved. □

For tridiagonal matrices we have the following interesting result.

THEOREM 5.3.5 Let $A = (a_{ij})$ be a matrix operator in ℓ_p, $1 \leq p \leq \infty$, that satisfies hypotheses (1),(2),(3) and (4) of Corollary 2.4.6 and assume that:

(5) The diagonal elements a_{ii} satisfy $a_{ii} < a_{i+1,i+1}$ for all $i \in \mathcal{N}$.

(6) $a_{i,i+1}, a_{i+1,i}$ are nonzero real numbers for all $i \in \mathcal{N}$.

(7) $a_{ij} = 0$ if $| i - j | \geq 2$.

Fix $N \in \mathcal{N}$. If λ is the eigenvalue of A in $\mathcal{R}_N$ (see Corollary 2.4.6) and $x = (x_1, x_2, \ldots)$ is an eigenvector of A corresponding to λ, then there exist $n_0 \in \mathcal{N}$ and a positive real r such that for all $n \geq n_0$,

$$| x_n | < r\sigma_n.$$

Proof. We consider the case $(I + D^{-1}F)^{-1} \in \mathcal{L}(\ell_p)$. The other case (see hypothesis (3) of Corollary **2.4.6**) has no new features. Because of hypothesis (5), the hypothesis

$$| a_{ii} - a_{kk} | > P_i + P_k$$

for all $i, k \in \mathcal{N}$, $i \neq k$ (see hypothesis (4) of Corollary (2.4.6)) is equivalent to

$$a_{ii} + P_i < a_{jj} - P_j, \; i < j. \tag{5.3.6}$$

Also since $| \lambda - a_{NN} | \leq P_N$ and λ is real, then

$$a_{NN} - P_N \leq \lambda \leq a_{NN} + P_N. \tag{5.3.7}$$

(This means, by hypothesis (2), that λ is a positive real.) Choose $n_1 \in \mathcal{N}$ such that $n_1 \geq N$ and

$$\| D_{n_1}^{-1} \| \leq \frac{1}{(1+\lambda)(1+ \| F \|)}, \tag{5.3.8}$$

where $D_{n_1} = \operatorname{diag}(a_{n_1+1,n_1+1}, a_{n_1+2,n_1+2}, \ldots)$ and $F = ((1-\delta_{ij})a_{ij})$. Define the matrix operator $E = (e_{ij})$ by:

$$e_{ij} = \begin{cases} a_{n_1+i,n_1+i} - \lambda & \text{if } j = i, i \geq 1 \\ a_{n_1+i,n_1+j} & \text{if } j = i+1, i \geq 1 \\ a_{n_1+i,n_1+j} & \text{if } j = i-1, i \geq 2 \\ 0 & \text{otherwise.} \end{cases} \tag{5.3.9}$$

Thus the system $(A - \lambda I)x = 0$ implies

$$E \begin{pmatrix} x_{n_1+1} \\ x_{n_1+2} \\ x_{n_1+3} \\ \vdots \end{pmatrix} = \begin{pmatrix} -a_{n_1+1,n_1} x_{n_1} \\ 0 \\ 0 \\ \vdots \end{pmatrix}. \tag{5.3.10}$$

Let $\hat{F} = ((1-\delta_{ij})e_{ij})$ and $\hat{D} = D_{n_1} - \lambda I$. We show that E satisfies the hypotheses of Theorem 5.3.1.

Since $a_{ii} \to \infty$ as $i \to \infty$, then $| e_{ii} | = | a_{i+n_1,i+n_1} - \lambda | \to \infty$ as $i \to \infty$. Also from inequalities (5.3.7),(5.3.6) and hypothesis (2), we have

$$\lambda \leq a_{NN} + P_N < a_{jj} - P_j < a_{jj},$$

for $j \geq N+1$. Therefore, since $n_1 \geq N$, we have $a_{jj} - \lambda > 0$ for all $j \geq n_1 + 1$. Hence E satisfies hypothesis (1) of Theorem 5.3.1.

For $i \geq n_1 + 2$,

$$\begin{aligned}
\mid e_{i-n_1,i-n_1} \mid - \sum_{j=n_1+1}^{\infty}{}' \mid e_{i-n_1,j-n_1} \mid &= (a_{ii} - \lambda) - P_i \\
&\geq (a_{ii} - P_i) - (a_{NN} + P_N) \\
&\geq (a_{n_1+1,n_1+1} + P_{n_1+1}) - (a_{n_1+1,n_1+1} - P_{n_1+1}) \\
&= 2P_{n_1+1} \\
&> 0.
\end{aligned}$$

(Notice that we used inequalities (5.3.7) and (5.3.6) in the first and second inequalities of the above system, respectively, and we used hypothesis (6) in the last inequality.)

For $i = n_1 + 1$,

$$\begin{aligned}
\mid e_{11} \mid - \sum_{j=n_1+1}^{\infty}{}' \mid e_{1,j-n_1} \mid &= (a_{n_1+1,n_1+1} - \lambda) - \mid a_{n_1+1,n_1+2} \mid \\
&\geq a_{n_1+1,n_1+1} - (a_{NN} + P_N) - \mid a_{n_1+1,n_1+2} \mid \\
&\geq a_{n_1+1,n_1+1} - (a_{n_1+1,n_1+1} - P_{n_1+1}) - \mid a_{n_1+1,n_1+2} \mid \\
&= \mid a_{n_1+1,n_1} \mid \\
&> 0.
\end{aligned} \tag{5.3.11}$$

Therefore, E satisfies hypothesis (2) of Theorem 5.3.1. (Notice that we used $P_{n_1+1} = \mid a_{n_1+1,n_1} \mid + \mid a_{n_1+1,n_1+2} \mid$ in the second equality of equation (5.3.11) and used hypothesis (6) in the last inequality of the equation.)

Since $F \in \mathcal{L}(\ell_p)$ we see from Lemma 2.1.1 that $\hat{F} \in \mathcal{L}(\ell_p)$ and $\parallel \hat{F} \parallel \leq \parallel F \parallel$. Thus E satisfies hypothesis (3) of Theorem 5.3.1.

Finally, from $\hat{D} = D_{n_1}^{-1}(I - \lambda D_{n_1})$, inequality (5.3.8) and Theorem 1.1.1, it follows that $\hat{D}$ has a bounded inverse on ℓ_p and

$$\begin{aligned} \| \hat{D}^{-1} \| &\leq \| (I - \lambda D_{n_1}^{-1})^{-1} \| \| D_{n_1}^{-1} \| \\ &\leq \frac{\| D_{n_1}^{-1} \|}{1 - \lambda \| D_{n_1}^{-1} \|} \\ &\leq \frac{\| D_{n_1}^{-1} \|}{1 - \frac{1}{1+\|F\|}}. \end{aligned}$$

Hence from $\| \hat{F} \| \leq \| F \|$, we have

$$\| \hat{F} \| \| \hat{D}^{-1} \| \leq (1 + \| F \|) \| D_{n_1}^{-1} \| .$$

But since $\| D_{n_1}^{-1} \| < \frac{1}{1+\|F\|}$, we obtain $\| \hat{F} \| \| \hat{D}^{-1} \| < 1$. Thus from Theorem 1.1.1, $(I + \hat{D}^{-1}\hat{F})$ has a bounded inverse on ℓ_p. Therefore, E satisfies hypothesis (4) of Theorem 5.3.1. Then from the system (5.3.10) and Theorem 5.3.1, we have for $i \geq n_1 + 1$,

$$x_i = -\langle e_{1i} \rangle (\det E)^{-1} a_{n_1+1,n_1} x_{n_1}. \tag{5.3.12}$$

If $\rho_i = (\sum_{j=1}^{\infty}{}' \mid e_{ij} \mid) / \mid e_{ii} \mid$, $i \in \mathcal{N}$, then from equation (5.3.12) and Lemma 5.3.4 we have for $i \geq n_1 + 1$,

$$\begin{aligned} \mid x_i \mid &\leq \rho_{i-n_1} \mid \langle e_{11} \rangle \mid \mid \det E \mid^{-1} \mid a_{n_1+1,n_1} \mid \mid x_{n_1} \mid \\ &\leq \rho_{i-n_1} (\mid e_{11} \mid - \mid a_{n_1+1,n_1+2} \mid)^{-1} \mid a_{n_1+1,n_1} \mid \mid x_{n_1} \mid \\ &= \rho_{i-n_1} (a_{n_1+1,n_1+1} - \lambda - \mid a_{n_1+1,n_1+2} \mid)^{-1} \mid a_{n_1+1,n_1} \mid \mid x_{n_1} \mid \\ &\leq \rho_{i-n_1} \mid a_{n_1+1,n_1} \mid^{-1} \mid a_{n_1+1,n_1} \mid \mid x_{n_1} \mid . \end{aligned}$$

(Notice that we used Lemma 5.3.4 to get the second inequality of the above system and used equation (5.3.11) to get the last one.) Thus for $i \geq n_1 + 1$,

$$\mid x_i \mid \leq \rho_{i-n_1} \mid x_{n_1} \mid . \tag{5.3.13}$$

Since $\rho_{i-n_1} = P_i(a_{ii} - \lambda)^{-1}$ for all $i \geq n_1 + 2$, $P_i = \sigma_i \mid a_{ii} \mid$ (see hypothesis (2) of Corollary 2.4.6) and $\lim_{i\to\infty} \frac{a_{ii}}{a_{ii}-\lambda} = 1$, we have $\lim_{i\to\infty} \frac{\rho_{i-n_1}}{\sigma_i} = 1$. Let $r_0 \in (1, \infty)$ and $r = r_0 \mid x_{n_1} \mid$. Then there exists $n_0 \in \mathcal{N}$ such that $\rho_{i-n_1}\sigma_i^{-1} < r_0$ for all $i \geq n_0$. Choose $n_0 \geq n_1 + 2$. Hence from inequality (5.3.13) we have for all $i \geq n_0$,

$$\mid x_i \mid \leq \rho_{i-n_1} \mid x_{n_1} \mid < \sigma_i r_0 \mid x_{n_1} \mid = \sigma_i r,$$

and this proves the requirement. □

5.4 The Eigenvalues of Mathieu's Equation

In this section we show that, in the case $q = 1$, the eigenvalues corresponding to the eigenfunctions ce_{2n} of Mathieu's equation (equation (5.1.1)) are the eigenvalues of a matrix operator defined in ℓ_2.

First we rewrite equation (5.1.1) in the form

$$\left(-\frac{d^2}{d\theta^2} + 2q\cos 2\theta\right)y = ay.$$

Thus $(-\frac{d^2}{d\theta^2} + 2q\cos 2\theta)$ will define an operator if a (suitable) space of functions is chosen.

Let $\mathcal{L}$ be the space of all even complex valued functions on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ such that the second derivative $y''(\theta)$ of $y(\theta) \in \mathcal{L}$ exists and is piecewise continuous on $(-\frac{\pi}{2}, \frac{\pi}{2})$. Then $\mathcal{L}$ is a subspace of $\mathcal{L}_2[-\frac{\pi}{2}, \frac{\pi}{2}]$. Let $T = -\frac{d^2}{d\theta^2} + 2\cos 2\theta$ be acting on $\mathcal{L}$. Then T is a linear operator from $\mathcal{L}$ into $\mathcal{L}_2[-\frac{\pi}{2}, \frac{\pi}{2}]$. The following theorem relates the eigenvalues of T to those of a matrix operator in ℓ_2.

THEOREM 5.4.1 Let $B = (b_{ij})$ be a matrix operator in ℓ_2 defined by

$$b_{12} = b_{23} = 1,\ b_{21} = 2$$

and

$$b_{ij} = \begin{cases} 1 & \text{if } j = i \pm 1, i \geq 3 \\ 4(i-1)^2 & \text{if } j = i, i \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then the matrix operator B and the differential operator T have the same eigenvalues.

Proof. Let a be an eigenvalue of T with a corresponding eigenvector $y(\theta)$. Since $y(\theta)$ is an even function in $\mathcal{L}_2[-\frac{\pi}{2}, \frac{\pi}{2}]$, we have

$$y(\theta) = \sum_{k=1}^{\infty} x_k \cos 2(k-1)\theta, \tag{5.4.1}$$

where the convergence is in $\mathcal{L}_2[-\frac{\pi}{2}, \frac{\pi}{2}]$. Multiplying both sides of equation (5.4.1) by $\cos 2(j-1)\theta$, $j \in \mathcal{N}$, integrating both sides from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ and using Lemma 5.2.4, we get

$$\begin{aligned} \int_{-\pi/2}^{\pi/2} y(\theta) \cos 2(j-1)\theta \, d\theta &= \int_{-\pi/2}^{\pi/2} \cos 2(j-1)\theta \Big(\sum_{k=1}^{\infty} x_k \cos 2(k-1)\theta\Big) \, d\theta \\ &= \sum_{k=1}^{\infty} x_k \int_{-\pi/2}^{\pi/2} \cos 2(j-1)\theta \cos 2(k-1)\theta \, d\theta. \end{aligned}$$

Thus from equation (5.2.1), the coefficients x_k, $k \in \mathcal{N}$, in equation (5.4.1) are given by

$$x_1 = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} y(\theta) \, d\theta, \; x_k = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} y(\theta) \cos 2(k-1)\theta \, d\theta \, , \; k \geq 2. \tag{5.4.2}$$

Also since $y'(\theta) \in \mathcal{L}_2[-\frac{\pi}{2}, \frac{\pi}{2}]$, we have

$$y'(\theta) = \sum_{k=1}^{\infty} \alpha_k \cos 2(k-1)\theta + \sum_{k=2}^{\infty} \beta_k \sin 2(k-1)\theta, \tag{5.4.3}$$

where $\alpha_k, \beta_k \in \mathcal{C}$ for all $k \in \mathcal{N}$. (The convergence in equation (5.4.3) is in $\mathcal{L}_2[-\frac{\pi}{2}, \frac{\pi}{2}]$.) Multiplying both sides of equation (5.4.3) by $\cos 2(j-1)\theta$, $j \in \mathcal{N}$, integrating both sides from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ and using Lemma 5.2.4, we get

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} y'(\theta)\cos 2(j-1)\theta\, d\theta &= \sum_{k=1}^{\infty}\alpha_k \int_{-\pi/2}^{\pi/2}\cos 2(j-1)\theta\cos 2(k-1)\theta\, d\theta + \\ &\quad \sum_{k=1}^{\infty}\beta_k \int_{-\pi/2}^{\pi/2}\cos 2(j-1)\theta\sin 2(k-1)\theta\, d\theta. \quad (5.4.4)\end{aligned}$$

From equations (5.2.1) and (5.2.3), equation (5.4.4) gives

$$\alpha_1 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} y'(\theta)\, d\theta = 0 \tag{5.4.5}$$

($y(\theta)$ is an even function) and for all $k \geq 2$,

$$\alpha_k = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} y'\cos 2(k-1)\theta\, d\theta.$$

Integrating the above integral by parts and using the fact that $y(\theta)$ is an even function, we get for all $k \geq 2$,

$$\alpha_k = 0. \tag{5.4.6}$$

To determine the coefficients β_k in equation (5.4.3), we multiply both sides of equation (5.4.3) by $\sin 2(j-1)\theta$, $j \in \mathcal{N}\setminus\{1\}$, then integrate from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ and use Lemma 5.2.4 to get for all $k \geq 2$,

$$\beta_k = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} y'(\theta)\sin 2(k-1)\theta\, d\theta.$$

Integrating the above integral by parts and using equation (5.4.2), we get for all $k \geq 2$,

$$\beta_k = -\frac{4(k-1)}{\pi}\int_{-\pi/2}^{\pi/2} y(\theta)\cos 2(k-1)\theta\, d\theta = -2(k-1)x_k. \tag{5.4.7}$$

From equations (5.4.5),(5.4.6) and (5.4.7), $y'(\theta)$ in equation (5.4.3) takes the form

$$y' = -2\sum_{k=2}^{\infty} x_k \sin 2(k-1)\theta. \tag{5.4.8}$$

From $Ty(\theta) = ay(\theta)$, we have $y''(\theta) = (2\cos 2\theta - a)y(\theta)$ and hence $y''(\theta) \in \mathcal{L}_2[-\frac{\pi}{2}, \frac{\pi}{2}]$. Using a similar argument to that given for determining the coefficients α_k and β_k in equation (5.4.3), we can show that

$$y''(\theta) = -\sum_{k=2}^{\infty} 4(k-1)^2 x_k \cos 2(k-1)\theta, \tag{5.4.9}$$

where the convergence is in $\mathcal{L}_2[-\frac{\pi}{2}, \frac{\pi}{2}]$. Since the series representation of $y(\theta)$ in equation (5.4.1) holds in the space $\mathcal{L}_2[-\frac{\pi}{2}, \frac{\pi}{2}]$, from Theorem 5.2.3 $x = (x_1, x_2, \ldots) \in \ell_2$. Thus from Theorem 5.2.3, the series $\sum_{k=1}^{\infty} x_k \cos 2k\theta$ and the series $\sum_{k=1}^{\infty} x_k \cos 2(k-2)\theta$ both converge in $\mathcal{L}_2[-\frac{\pi}{2}, \frac{\pi}{2}]$. So from the equality

$$2(\cos 2\theta)\ \cos 2(k-1)\theta = \cos 2k\theta + \cos 2(k-2)\theta,$$

where $k \in \mathcal{N}$, and from equation (5.4.1), it follows that

$$2(\cos 2\theta)\ y(\theta) = \sum_{k=1}^{\infty} x_k \cos 2k\theta + \sum_{k=1}^{\infty} x_k \cos 2(k-2)\theta, \tag{5.4.10}$$

where the convergence holds in $\mathcal{L}_2[-\frac{\pi}{2}, \frac{\pi}{2}]$. Therefore from equations (5.4.9) and (5.4.10), the equation $Ty(\theta) = ay(\theta)$ gives

$$\sum_{k=2}^{\infty} 4(k-1)^2 x_k \cos 2(k-1)\theta + \sum_{k=1}^{\infty} x_k \cos 2k\theta$$
$$= a\sum_{k=1}^{\infty} x_k \cos 2(k-1)\theta - \sum_{k=1}^{\infty} x_k \cos 2(k-2)\theta.$$

Thus,

$$x_2 + (2x_1 + 4(2-1)^2 x_2 + x_3)\cos 2\theta + \sum_{k=3}^{\infty}(x_{k-1} + 4(k-1)^2 x_k + x_{k+1})\cos 2(k-1)\theta.$$
$$= a\sum_{k=1}^{\infty} x_k \cos 2(k-1)\theta \tag{5.4.11}$$

Integrating both sides of equation (5.4.11) from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ and using Lemma 5.2.4, we get

$$\begin{aligned}\pi x_2 + \left(2x_1 + 4(2-1)^2 x_2 + x_3\right) \int_{-\pi/2}^{\pi/2} \cos 2\theta \, d\theta = \\ a \sum_{k=1}^{\infty} x_k \int_{-\pi/2}^{\pi/2} \cos 2(k-1)\theta \, d\theta - \\ \sum_{k=3}^{\infty} \left(x_{k-1} + 4(k-1)^2 x_k + x_{k+1}\right) \int_{-\pi/2}^{\pi/2} \cos 2(k-1)\theta \, d\theta.\end{aligned}$$

Using equation (5.2.1), the above equation gives

$$x_2 = a x_1. \tag{5.4.12}$$

If we multiply both sides of equation (5.4.11) by $\cos 2\theta$, then integrate both sides from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ and use Lemma 5.2.4, we get

$$\begin{aligned}x_2 \int_{-\pi/2}^{\pi/2} \cos 2\theta \, d\theta + \left(2x_1 + 4(2-1)^2 x_2 + x_3\right) \int_{-\pi/2}^{\pi/2} \cos^2 2\theta \, d\theta = \\ a \sum_{k=1}^{\infty} x_k \int_{-\pi/2}^{\pi/2} (\cos 2\theta) \cos 2(k-1)\theta \, d\theta - \\ \sum_{k=3}^{\infty} \left(x_{k-1} + 4(k-1)^2 + x_{k+1}\right) \int_{-\pi/2}^{\pi/2} (\cos 2\theta) \cos 2(k-1)\theta \, d\theta.\end{aligned}$$

Using equation (5.2.1), we get

$$2x_1 + 4(2-1)^2 x_2 + x_3 = a x_2. \tag{5.4.13}$$

Similarly if we multiply both sides of equation (5.4.11) by $\cos 2(j-1)\theta$, j is an integer greater than 2, then integrate both sides of the equation from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ and use Lemma 5.2.4 and equation (5.2.1), we get for all $k \geq 3$,

$$x_{k-1} + 4(k-1)^2 x_k + x_{k+1} = a x_k. \tag{5.4.14}$$

Equations (5.4.12),(5.4.13) and (5.4.14) give $Bx = ax$, where $x = (x_1, x_2, \ldots) \in \ell_2$. Hence a is an eigenvalue of B with a corresponding eigenvector x ($x \neq 0$; for if $x = 0$ then from equation (5.4.2), $\int_{-\pi/2}^{\pi/2} y(\theta) \cos 2(k-1)\theta \, d\theta = 0$ for all $k \in \mathcal{N}$. But since $\int_{-\pi/2}^{\pi/2} y(\theta) \sin 2k\theta \, d\theta = 0$ for all $k \in \mathcal{N}$ and $\{\frac{1}{\sqrt{\pi}}\} \bigcup \{\sqrt{\frac{2}{\pi}} \cos 2n\theta : n \in \mathcal{N}\} \bigcup \{\sqrt{\frac{2}{\pi}} \sin 2n\theta : n \in \mathcal{N}\}$ is an orthonormal basis for $\mathcal{L}_2[-\frac{\pi}{2}, \frac{\pi}{2}]$, then by Lemma 5.2.4 it follows that $\int_{-\pi/2}^{\pi/2} y(\theta) f(\theta) \, d\theta = 0$ for any $f \in \mathcal{L}_2[-\frac{\pi}{2}, \frac{\pi}{2}]$. In particular we have $\int_{-\pi/2}^{\pi/2} | y(\theta) |^2 \, d\theta = 0$. So $y = 0$ almost everywhere, which is impossible since $y \neq 0$ and y is continuous).

Now suppose that a is an eigenvalue of $B = (b_{ij})$ with a corresponding eigenvector $x = (x_1, x_2, \ldots) \in \ell_2$. If $S = \text{diag}(\sqrt{2}, 1, 1, \ldots)$, then $SBS^{-1} = \tilde{B}$, where the entries $\tilde{b}_{ij}$ of $\tilde{B}$ are given by $\tilde{b}_{12} = \tilde{b}_{21} = \sqrt{2}$ and $\tilde{b}_{ij} = b_{ij}$ for all other indices i and j. Hence a is an eigenvalue of $\tilde{B}$ with the corresponding eigenvector x. If $A = \tilde{B} + 2I$, that is, $A = (a_{ij})$ is the matrix operator defined by

$$A = \begin{pmatrix} 2 & \sqrt{2} & 0 & 0 & 0 & 0 & \ldots \\ \sqrt{2} & 6 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 18 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 38 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{5.4.15}$$

then $a + 2$ is an eigenvalue of A with the corresponding eigenvector x. We show that A satisfies the hypotheses of Theorem 5.3.5 at $p = 2$. Define

$$D = \text{diag}(a_{11}, a_{22}, \ldots), \; F = ((1 - \delta_{ij}) a_{ij}) \tag{5.4.16}$$

as usual. It is clear that $F \in \mathcal{L}(\ell_2)$. Also since

$$D^{-1}F = \begin{pmatrix} 0 & \sqrt{2}/2 & 0 & 0 & 0 & \dots \\ \sqrt{2}/6 & 0 & 1/6 & 0 & 0 & \dots \\ 0 & 1/18 & 0 & 1/18 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

then $\| D^{-1}F \|_1 = \frac{1+9\sqrt{2}}{18}$ and $\| D^{-1}F \|_\infty = \frac{\sqrt{2}}{2}$ (see [8]). But since $\| D^{-1}F \|_2^2 \leq \| D^{-1}F \|_1 \| D^{-1}F \|_\infty$ (see page 366 in [16]), then

$$\| D^{-1}F \|_2 \leq \frac{(18+\sqrt{2})^{1/2}}{6} < 1. \tag{5.4.17}$$

Thus from Theorem 1.1.1, $(I + D^{-1}F)^{-1} \in \mathcal{L}(\ell_2)$, and hence A satisfies hypothesis (3) of Theorem 5.3.5. It is clear that A satisfies all other hypotheses of this theorem. Hence there exists $n_0 \in \mathcal{N}$ and a positive real r such that for all $n \geq n_0$,

$$| x_n | < r\sigma_n. \tag{5.4.18}$$

Since $\sigma_n = | a_{nn} |^{-1} \sum_{k=1}^{\infty}{}' | a_{nk} | = \frac{2}{4(n-1)^2+2}$ for all $n \geq 3$, then if we choose $n_0 \geq 3$, inequality (5.4.18) gives

$$\sum_{n=n_0}^{\infty} | x_n | \leq r \sum_{n=n_0}^{\infty} \frac{2}{4(n-1)^2+2} < \infty,$$

where the convergence of the series follows from the fact $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Hence $\sum_{n=1}^{\infty} | x_n | < \infty$ and this means that $x = (x_1, x_2, \dots) \in \ell_1$. Then the series $\sum_{k=1}^{\infty} x_k \cos 2(k-1)\theta$ converges for every $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. So we can define a function $y(\theta)$ by

$$y(\theta) = \sum_{k=1}^{\infty} x_k \cos 2(k-1)\theta, \tag{5.4.19}$$

where the convergence is pointwise in C.

Also from $x \in \ell_1$, it follows that the power series $\sum_{k=1}^{\infty} \frac{1}{2} x_k z^{k-1}$ has a radius of convergence equal to ∞. For $z \in C$, define the function

$$f(z) = \sum_{k=1}^{\infty} \frac{1}{2} x_k z^{k-1}. \tag{5.4.20}$$

From the equality

$$\cos 2(k-1)\theta = \frac{1}{2}(e^{2(k-1)\sqrt{-1}\theta} + e^{-2(k-1)\sqrt{-1}\theta}),$$

it follows that for all $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$,

$$y(\theta) = f(z_1) + f(z_2), \tag{5.4.21}$$

where $z_1 = e^{2\sqrt{-1}\theta}$ and $z_2 = e^{-2\sqrt{-1}\theta}$. From Theorem 7.1, page 76 of [17], the first and second derivatives of $f(z)$ exist for all $z \in C$ and

$$f'(z) = \sum_{k=2}^{\infty} \frac{1}{2}(k-1) x_k z^{k-2}, \; f''(z) = \sum_{k=3}^{\infty} \frac{1}{2}(k-1)(k-2) x_k z^{k-3}. \tag{5.4.22}$$

From equations (5.4.21) and (5.4.22), it follows that $y'(\theta)$ exists for every $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and

$$\begin{aligned} y'(\theta) &= f'(z_1)\frac{dz_1}{d\theta} + f'(z_2)\frac{dz_2}{d\theta} \\ &= \sum_{k=2}^{\infty} -2(k-1) x_k \sin 2(k-1)\theta. \end{aligned} \tag{5.4.23}$$

Similarly from equations (5.4.22) and (5.4.23), it follows that $y''(\theta)$ exists for every $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and

$$y''(\theta) = \sum_{k=2}^{\infty} -4(k-1)^2 x_k \cos 2(k-1)\theta. \tag{5.4.24}$$

Since the third derivative of $f(z)$, $f'''(z)$, exists (this implies that $y''(\theta)$ is continuous on $(-\frac{\pi}{2}, \frac{\pi}{2})$), from equation (5.4.19) ($y(\theta)$ is even) and equation (5.4.24), it follows that $y = y(\theta) \in \mathcal{L}$.

The equation $Bx = ax$ gives equations (5.4.12),(5.4.13) and (5.4.14). Hence from the convergence of the series $\sum_{k=1}^{\infty} x_k \cos 2(k-1)\theta$, we have

$$x_2 + (2x_1 + 4(2-1)^2 x_2 + x_3)\cos 2\theta + \sum_{k=3}^{\infty}(x_{k-1} + 4(k-1)^2 x_k + x_{k+1})\cos 2(k-1)\theta$$
$$= a\sum_{k=1}^{\infty} x_k \cos 2(k-1)\theta,$$

where the convergence is in C and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Thus from the convergence of the series $\sum_{k=3}^{\infty} x_{k-1}\cos 2(k-1)\theta$, $\sum_{k=3}^{\infty} x_{k+1}\cos 2(k-1)\theta$ and $\sum_{k=3}^{\infty} 4(k-1)^2 x_k \cos 2(k-1)\theta$ (see equation (5.4.24)), we get the equality

$$\sum_{k=2}^{\infty} 4(k-1)^2 x_k \cos 2(k-1)\theta + \sum_{k=1}^{\infty} x_k \cos 2k\theta + \sum_{k=1}^{\infty} x_k \cos 2(k-2)\theta$$
$$= a\sum_{k=1}^{\infty} x_k \cos 2(k-1)\theta = ay(\theta),$$

where the convergence is in C and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Therefore, from the equality

$$2(\cos 2\theta)\cos 2(k-1)\theta = \cos 2k\theta + \cos 2(k-2)\theta$$

and equation (5.4.24), we have for all $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$,

$$-y''(\theta) + 2(\cos 2\theta)y(\theta) = ay(\theta)$$

or

$$Ty(\theta) = ay(\theta).$$

Hence a is an eigenvalue of T with the corresponding eigenvector $y = y(\theta)$ (since $y(0) = \sum_{k=1}^{\infty} x_k \neq 0$ and y is continuous, there is an interval $\mathcal{M}$ containing zero such that $y(\theta) \neq 0$ for all $\theta \in \mathcal{M}$. Thus $y \neq 0$). This completes the proof of the theorem. □

5.5 A Numerical Estimation

In this section we use the results of Section 4.3 to approximate the eigenvalues of Mathieu's equation in the case of $q = 1$.

We have shown in the previous section that in the case $q = 1$, a is an eigenvalue of Mathieu's equation with a corresponding eigenvector ce_{2n}, $n \in \mathcal{N}$, if and only if $a + 2$ is an eigenvalue of the matrix operator A defined by equation (5.4.15). Therefore, in order to approximate the eigenvalues corresponding to the eigenfunctions ce_{2n} of Mathieu's equation it is sufficient to approximate the eigenvalues of A. Since A is selfadjoint, Theorem 5.4.1 shows that A satisfies the hypotheses of Theorem 4.3.1. So the eigenvalues of A can be approximated to any degree of accuracy provided the constant $K = \| D^{-1} \| (1+ \| F \|) \| (I + D^{-1}F)^{-1} \|$, where D and F are defined by equation (5.4.16), and the constant k defined by equation (4.3.1) are estimated.

First we evaluate the norm of F. For any $y = (y_1, y_2, \ldots) \in \ell_2$ with $\| y \| = 1$, we have

$$\begin{aligned}
\| Fy \|^2 &= 2| y_2 |^2 + | \sqrt{2}y_1 + y_3 |^2 + | y_2 + y_4 |^2 + | y_3 + y_5 |^2 + \cdots \\
&\leq 2| y_2 |^2 + (2| y_1 |^2 + | y_3 |^2 + | y_2 |^2 + | y_4 |^2 + | y_3 |^2 + | y_5 |^2 + \cdots) \\
&\quad + 2(\sqrt{2} | y_1 || y_3 | + | y_2 || y_4 | + | y_3 || y_5 | + \cdots) \\
&= | y_2 |^2 + 2(| y_1 |^2 + | y_2 |^2 + | y_3 |^2 + \cdots) \\
&\quad + 2(\sqrt{2} | y_1 || y_3 | + | y_2 || y_4 | + | y_3 || y_5 | + \cdots)
\end{aligned}$$

$$\begin{aligned}
&\leq \; |y_2|^2 + 2 + 2 \, \| (\sqrt{2}\,|y_1|, |y_2|, |y_3|, \ldots) \| \| (|y_3|, |y_4|, \ldots) \| \\
&= \; 2 + |y_2|^2 + 2[(2|y_1|^2 + |y_2|^2 + |y_3|^2 + \cdots)(|y_3|^2 + |y_4|^2 + \cdots)]^{1/2} \\
&= \; 2 + |y_2|^2 + 2[(1 + |y_1|^2)(1 - |y_1|^2 - |y_2|^2)]^{1/2} \\
&= \; 2 + |y_2|^2 + 2[(1 - |y_1|^4) - |y_2|^2(1 + |y_1|^2)]^{1/2} \\
&\leq \; 2 + |y_2|^2 + 2[1 - |y_2|^2]^{1/2},
\end{aligned}$$

but since the function $\psi(\eta) = 2+\eta^2+2(1-\eta^2)^{1/2}$ is a continuous function on $[0,1]$, has a negative derivative on $(0,1)$ and $\psi(0) = 4$, $2 + |y_2|^2 + 2[1 - |y_2|^2]^{1/2} \leq 4$. Thus for any $y = (y_1, y_2, \ldots) \in \ell_2$ with $\| y \| = 1$, we have $\| Fy \| \leq 2$, and hence

$$\| F \| \leq 2. \tag{5.5.1}$$

On the other hand, the sequence $\{x^{(k)}\}_{k=0}^{\infty}$ of vectors in ℓ_2, where the ith component $(x^{(k)})_i$ of $x^{(k)}$ is given by

$$(x^{(k)})_i = \begin{cases} 1 & \text{if } 1 \leq i \leq k+3 \\ 0 & \text{if } i \geq k+4 \end{cases}$$

satisfies

$$\| x^{(k)} \|^2 = k+3\,,\; \| Fx^{(k)} \|^2 = 4k + \| Fx^{(0)} \|^2$$

for all $k = 0, 1, 2, \ldots$. Hence,

$$\sup\left\{ \frac{\| Fx^{(k)} \|}{\| x^{(k)} \|} : k = 0, 1, 2, 3, \ldots \right\} = 2. \tag{5.5.2}$$

From equations (5.5.1) and (5.5.2), we get

$$\| F \| = 2. \tag{5.5.3}$$

From equation (5.4.17) and Theorem 1.1.1, we have

$$\| (I + D^{-1}F)^{-1} \| \leq 3.765. \tag{5.5.4}$$

Hence from equations (5.5.3) and (5.5.4), we get

$$K \leq (\frac{1}{2})(3)(3.765) < 5.648. \tag{5.5.5}$$

Also from inquality (5.5.4), we have

$$\| A^{-1} \| \leq \| D^{-1} \| \| (I + D^{-1}F)^{-1} \| \leq (\frac{1}{2})(3.765) < 1.883. \tag{5.5.6}$$

Thus from equations (5.5.3),(5.5.5) and (5.5.6), we have

$$k < (4)(1.883)(6.648)(9) < 451. \tag{5.5.7}$$

Using the notations introduced before Theorem 4.3.1 in Section 4.3, we have at $i = 1$,

$$\delta_1 = 3 - \sqrt{2},\ \epsilon_1 = \frac{3 - \sqrt{2}}{2(2 + \sqrt{2})(2 + \sqrt{2} + 3 - \sqrt{2})} < 0.008$$

and $(1 + |\, a_{11} \,| + P_1)^2 = (3 + \sqrt{2})^2$. Then equations (4.3.2) and (5.5.7) show, for example, that to guarantee an error of not more than 0.005 in the first eigenvalue (which is the eigenvalue of A in the Gersgorin disc $\mathcal{R}_1 = \{z \in \mathcal{C} :|\, z - 2 \,| \leq \sqrt{2}\}$), we can truncate A to the leading submatrix of size 664.

REMARK 5.5.1 Determining the eigenvalues in ℓ_2 of the matrix operator $A = (a_{ij})$ defined by equation (5.4.15) is equivalent to solving

$$Kx = 0,$$

where x is a nonzero vector in ℓ_2 and $K = (k_{ij})$ is defined by

$$k_{ij} = \begin{cases} (a_{ij} - \lambda)/a_{ii} & \text{if } j = i \\ a_{ij}/a_{ii} & \text{otherwise} \end{cases}$$

where $\lambda \in \mathcal{C}$. In contrast with our method of approximating the eigenvalues of A, Mennicken and Schmidt [20] established similar results based on the concept of the vanishing of $\det K$. The determinant $\det K$ is defined as

$$\det K = \lim_{n \to \infty} \det K_n,$$

where K_n, $n \in \mathcal{N}$, are the leading $n \times n$ submatrices of K. The existence of the above limit has been proved by Poincaré [23].

Chapter 6

Perturbation of Linear Operators in Banach Spaces

In this chapter we discuss the dependence of the eigenvalues and the eigenvectors of the perturbed system upon the perturbation provided it is small enough, and use this result to approximate the eigenvalues of Mathieu's equation (see Chapter Five). First we state the fixed-point theorem.

6.1 The Fixed-Point Principle

Consider a complete metric space $\mathcal{X}$ with a metric ρ and a closed subset Ω of $\mathcal{X}$. Assume that there is a mapping P defined on Ω that maps Ω into itself.

DEFINITIONS 6.1.1

(1) A point $x^* \in \Omega$ is called a fixed point of P if $x^* = P(x^*)$.

Thus the fixed points of P are the solutions of the equation

$$x = P(x). \tag{6.1.1}$$

(2) P is a contraction map if there exists an $\alpha \in [0, 1)$ such that

$$\rho(P(x), P(x')) \leq \alpha\rho(x, x')$$

for all $x, x' \in \Omega$.

If P is a contraction map, one can guarantee the existence, and even uniqueness, of a fixed point.

THEOREM 6.1.2 If P is a contraction map on Ω, then there is a unique solution x^* in Ω for equation (6.1.1).

Moreover, x^* can be obtained as the limit of the sequence $\{x_n\}$, where

$$x_{n+1} = P(x_n)$$

for all $n = 0, 1, 2, \ldots$, and where x_0 is an arbitrary point in Ω.

See Theorem 1, page 474 of [14] for the proof.

6.2 The Eigenvalues and Eigenvectors of the Perturbed System

In this section we show under certain conditions (see Theorem 6.2.1 below) that if the unperturbed system has an eigenvalue λ_0 with a corresponding eigenvector x_0, then an upper bound for the perturbation (in norm) can be given to guarantee that the perturbed system has an eigenvalue with a corresponding eigenvector that are both close to λ_0 and x_0, respectively. Such a problem has been discussed in [24] and [33].

THEOREM 6.2.1 Let D be a closed operator defined in a Banach space $\mathcal{X}$ such that $\mathcal{D}(D)$ is dense in $\mathcal{X}$ and assume that

(1) The point λ_0 is an eigenvalue of both D and D' (the dual of D) with corresponding eigenvectors x_0 and x_0', respectively. The vector x_0 satisfies $\| x_0 \| = 1$.

(2) The restriction of the operator $(D - \lambda_0 I)$ to the space $\mathcal{X}_1 = \{x \in \mathcal{D}(D) : x_0'(x) = 0\}$ has a bounded inverse R mapping $\mathcal{X}_1$ into itself.

(3) The vector x_0' satisfies $x_0'(x_0) = 1$.

Then for any $r \in (0, (\frac{\|S\|}{\|x_0'\|\|R\|})^{1/2})$ and $F \in \mathcal{G}_r = \{U \in \mathcal{L}(\mathcal{X}) : \| U \| \leq \delta\}$, where

$$\delta = \frac{r}{\| x_0' \|\| R \| r^2 + (\| x_0' \|\| R \| + \| S \|)r + \| S \|}$$

and

$$S = R(I - x_0 x_0'),$$

the system of equations

$$Ax = (D + F)x = \lambda x,\ x_0'(x) = 1 \tag{6.2.1}$$

has a unique solution (x, λ) in the set $\{z \in \mathcal{D}(D) :\| z - x_0 \|\leq r\} \times \mathcal{C}$. (Concerning the location of λ in $\mathcal{C}$ see Remark 6.2.2.)

Proof. Define $P = I - x_0 x_0'$. Hence $S = RP$.

Fix $r \in (0, (\frac{\|S\|}{\|x_0'\|\|R\|})^{1/2})$ and $F \in \mathcal{G}_r$. Writing $x = x_0 + y$ and $\lambda = \lambda_0 + \eta$, and substituting into equation (6.2.1), we find it is required to solve

$$(D - \lambda_0 I)y = \eta(x_0 + y) - F(x_0 + y), \tag{6.2.2}$$

where $y \in \mathcal{Y} = \mathcal{X}_1 \bigcap \{z \in \mathcal{X} :\| z \|\leq r\}$. From the second hypothesis, the left hand side of equation (6.2.2) is in $\mathcal{X}_1$, and so acting on both sides of the equation by x_0' we get

$$\eta = \eta x_0'(x_0 + y) = x_0'(F(x_0 + y)). \tag{6.2.3}$$

Therefore it is required to solve in $\mathcal{Y}$ the equation

$$(D - \lambda_0 I)y = x_0'(F(x_0 + y))(x_0 + y) - F(x_0 + y). \tag{6.2.4}$$

From the definition of P, it is easy to see that solving equation (6.2.4) in $\mathcal{Y}$ is equivalent to solving

$$(D - \lambda_0 I)y = x_0'(F(x_0 + y))y - PF(x_0 + y) \tag{6.2.5}$$

in $\mathcal{Y}$. Now we may use hypothesis (2) and act on both sides of equation (6.2.5) by the operator R to deduce that the equation

$$y = x_0'(F(x_0 + y))Ry - SF(x_0 + y) \tag{6.2.6}$$

has the same solution set in $\mathcal{Y}$ as equation (6.2.5) has.

Define the map Q_F on the closed set $\mathcal{Y}$ by

$$Q_F(y) = x_0'(F(x_0 + y))Ry - SF(x_0 + y)$$

for all $y \in \mathcal{Y}$. We prove that Q_F is a contraction map mapping $\mathcal{Y}$ into itself.

Let $y \in \mathcal{Y}$. Since P maps $\mathcal{X}$ onto $\mathcal{X}_1$ (for $x \in \mathcal{X}$, we have $x_0'(Px) = x_0'(x) - x_0'(x)x_0'(x_0) = x_0'(x) - x_0'(x) = 0$, by the third hypothesis), we have $PF(x_0+y) \in \mathcal{X}_1$. Hence from the second hypothesis both of $SF(x_0 + y)$ and Ry are in $\mathcal{X}_1$. Thus $Q_F(y) \in \mathcal{X}_1$. Also using the triangular inequality and the fact that $\| x_0 \| = 1$ and $\| y \| \leq r$, we get

$$\| Q_F(y) \| \leq (\| x_0' \| \| F \| \| R \| r + \| S \| \| F \|)(1 + r).$$

But since $\| F \| \leq \dfrac{r}{\| x_0' \| \| R \| r^2 + (\| x_0' \| \| R \| + \| S \|)r + \| S \|}$, we have $\| Q_F(y) \| \leq r$, and this proves that Q_F maps $\mathcal{Y}$ into itself.

Now let y_1 and y_2 be in $\mathcal{Y}$. We have

$$\begin{aligned} \| Q_F(y_1) - Q_F(y_2) \| &\leq \| F \| (2 \| x_0' \| \| R \| r + \| x_0' \| \| R \| + \| S \|) \| y_1 - y_2 \| \\ &\leq \frac{2 \| x_0' \| \| R \| r + \| x_0' \| \| R \| + \| S \|}{\| x_0' \| \| R \| r + \| x_0' \| \| R \| + \| S \| + \frac{\| S \|}{r}} \| y_1 - y_2 \| . \end{aligned}$$

Hence we have

$$\| Q_F(y_1) - Q_F(y_2) \| \leq \alpha \| y_1 - y_2 \|,$$

where $\alpha = \dfrac{2 \| x_0' \| \| R \| r + \| x_0' \| \| R \| + \| S \|}{\| x_0' \| \| R \| r + \| x_0' \| \| R \| + \| S \| + \frac{\| S \|}{r}}$. But since

$$0 < r < \left(\frac{\| S \|}{\| x_0' \| \| R \|} \right)^{1/2}$$

then $\alpha \in (0,1)$ and this completes the proof that Q_F is a contraction map mapping $\mathcal{Y}$ into itself. Thus from Theorem 6.1.2, Q_F has a unique fixed point $y^* = Q_F(y^*) \in \mathcal{Y}$. Hence from the definition of Q_F it follows that equation (6.2.6) has the unique solution y^* in $\mathcal{Y}$. But since the two equations (6.2.4) and (6.2.6) have the same solution set in $\mathcal{Y}$, then equation (6.2.2) has a unique solution (y^*, η^*) in $\mathcal{Y} \times \mathcal{C}$ where y^* is the unique solution of equation (6.2.4) and η^* is given by

$$\eta^* = x_0'(F(x_0 + y^*)).$$

Thus the system (6.2.1) has the unique solution $(x, \lambda) \in \{z \in \mathcal{C} : \| z - x_0 \| \leq r\} \times \mathcal{C}$, where $x = x_0 + y^*$ and $\lambda = \lambda_0 + \eta^*$. This completes the proof of the theorem. □

REMARK 6.2.2 In Theorem 6.2.1, by choosing r and δ small enough, the solution (x, λ) of the system (6.2.1) can come close to x_0, λ_0 to any degree of accuracy. To explain this point, let $\epsilon > 0$. If we let r vary in the interval $(0, (\frac{\|S\|}{\|x_0'\|\|R\|})^{1/2})$, then

$$\delta = \delta(r) = \frac{r}{\| x_0' \| \| R \| r^2 + (\| x_0' \| \| R \| + \| S \|) r + \| S \|}$$

is a function of r. This function has a greatest lower bound equal to zero, and is increasing since the derivative

$$\delta'(r) = \frac{\| S \| - \| x_0' \| \| R \| r^2}{(\| x_0' \| \| R \| r^2 + (\| x_0' \| \| R \| + \| S \|) r + \| S \|)^2} > 0$$

for all $r \in (0, (\frac{\|S\|}{\|x_0'\|\|R\|})^{1/2})$. So if we choose $r < \epsilon$ such that

$$\delta = \delta(r) < \frac{\epsilon}{\| x_0' \| (1+r)},$$

the unique solution (x, λ) of the system (6.2.1) in the product space $\{z \in \mathcal{D}(D) : \| z - x_0 \| \leq r\} \times C$ satisfies

$$\| x - x_0 \| < \epsilon$$

and

$$| \lambda - \lambda_0 | \leq \| x_0' \| (1 + \| y \|) \| F \| < \frac{\epsilon \| x_0' \| (1 + \| y \|)}{\| x_0' \| (1+r)} \leq \epsilon.$$

REMARK 6.2.3 In Theorem 6.2.1, we have improved the upper bound for the norm of the perturbation given in [24] which is

$$\frac{r}{2 \| x_0' \| \| R \| r^2 + (\| x_0' \| \| R \| + \| S \|) r + \| S \|}.$$

6.3 Approximation of the Eigenvalues of Mathieu's Equation

In Theorem 5.4.1, we proved that the eigenvalues of Mathieu's equation (5.1.1), with $q = 1$, are the eigenvalues of the infinite matrix operator $\tilde{B} = (\tilde{b}_{ij})$ defined in ℓ_2 by

$$\tilde{b}_{12} = \tilde{b}_{21} = \sqrt{2},\ \tilde{b}_{23} = 1$$

and

$$\tilde{b}_{ij} = \begin{cases} 1 & \text{if } j = i \pm 1, i \geq 3 \\ 4(i-1)^2 & \text{if } j = i, i \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

and that the eigenvalues of $\tilde{B}$ lie in the Gersgorin discs (defined by the row sums) of $\tilde{B}$. Each Gersgorin disc contains one and only one simple eigenvalue of $\tilde{B}$.

In Section 5.5, we have shown that the matrix operator $F = ((1 - \delta_{ij})\tilde{b}_{ij})$ is in $\mathcal{L}(\ell_2)$ and $\| F \| = 2$. Hence we can write $\tilde{B} = D + F$, where $D = \text{diag}(0, 4, 16, 36, \ldots)$. Now we show that D satisfies the hypotheses of Theorem 6.2.1. It is clear that $d_i = 4(i-1)^2$, $i \in \mathcal{N}$, are the eigenvalues of D with corresponding eigenvectors $x_i = e_i$, where e_i are the unit coordinate vectors. Also since $\mathcal{D}(D)$ is dense in ℓ_2, the dual D' of D exists, and from Corollary 2.3.4, it follows that for every $i \in \mathcal{N}$, d_i is an eigenvalue of D' with a corresponding eigenvector $x'_i = J^{-1}(e_i)$, where J is the map defined in Remark 2.3.2.

Now fix $i \in \mathcal{N}$. Let $\mathcal{X}_i = \{x \in \ell_2 : x'_i(x) = 0\}$. From Theorem 2.3.1, it is clear that $e_j \in \mathcal{X}_i$ for all $j \neq i$ and $x'_i(e_i) = 1$. Hence from the continuity of x'_i, it follows that $\mathcal{X}_i$ is the closure of the set $\text{span}\{e_j : j \in \mathcal{N} \setminus \{i\}\}$. Therefore the restriction of the operator $(D - d_i I)$ to the space $\mathcal{X}_i$ has an inverse $R_i = (r_{jk}^{(i)})$ mapping $\mathcal{X}_i$ into itself and is given by

$$r_{jk}^{(i)} = \begin{cases} c & \text{if } k = j = i \\ (1/4)\left((j-1)^2 - (i-1)^2\right)^{-1} & \text{if } k = j \neq i \\ 0 & \text{otherwise} \end{cases}$$

and c is a complex number. Thus $\| R_1 \| = \frac{1}{4}$ and $\| R_i \| = \frac{1}{4(2i-3)}$ for all $i = 2, 3, 4, \ldots$. Also the operator $S_i = R_i(I - e_i x'_i)$, where I is the identity operator, has the same norm as R_i has. To prove this; let $z = \sum_{j=1}^{\infty}{}' \alpha_j e_j \in \mathcal{X}_i$, then since $(I - e_i x'_i)(z) = z$, we have

$$\| R_i \| \leq \| S_i \| . \tag{6.3.1}$$

On the other hand let $x = \sum_{j=1}^{\infty} \alpha_j e_j \in \ell_2$ and $\| x \| \leq 1$. We have

$$
\begin{aligned}
(I - e_i x_i')(x) &= x - x_i'(\sum_{j=1}^{\infty} \alpha_j e_j) e_i \\
&= x - x_i'(\sum_{j=1}^{\infty} \alpha_j e_j) e_i \\
&= x - (\sum_{j=1}^{\infty} \alpha_j x_i'(e_j)) e_i \\
&= x - \alpha_i e_i \\
&= \sum_{j=1}^{\infty} \alpha_j e_j - \alpha_i e_i,
\end{aligned}
$$

which is an element in $\mathcal{X}_i$; denote it by $\tilde{x}$. From Theorem 4.1, page 10 of [9], we have $\| \tilde{x} \| \leq 1$. Therefore $\| S_i x \| = \| R_i \tilde{x} \| \leq \| R_i \|$. Hence we have

$$\| S_i \| \leq \| R_i \| . \tag{6.3.2}$$

Equations (6.3.1) and (6.3.2) prove $\| S_i \| = \| R_i \|$. Thus D satisfies the hypotheses of Theorem 6.2.1.

Now for every integer $i \geq 3$, we solve the equation

$$2 = \frac{r_i}{(r_i^2 + 2r_i + 1) \| R_i \|}. \tag{6.3.3}$$

Substituting from $\| R_i \| = \frac{1}{4(2i-3)}$ into equation (6.3.3), we get

$$r_i^2 + 4(2 - i) r_i + 1 = 0, \tag{6.3.4}$$

where $i \geq 3$ (for $i = 1, 2$, equation (6.3.3) does not have a real solution). Now for all $i \geq 3$ choose r_i to be the smaller root of equation (6.3.4),that is, for all $i \geq 3$ choose r_i to be

$$r_i = 2(i-2) - \sqrt{4(i-2)^2 - 1}. \tag{6.3.5}$$

It is clear that $r_i \in (0,1)$ for all $i \geq 3$. Since $\| F \|= 2$, $\| x_i' \|=\| e_i \|= 1$ (see Theorem 2.3.1) and $\| S_i \|=\| R_i \|$ for all $i \in \mathcal{N}$, then from Theorem 6.2.1 (where we take $\delta =\| F \|= 2$), the system of equations

$$(D + F)(x_i + y_i) = \lambda_i(x_i + y_i),\ x_i'(x_i + y_i) = 1$$

where $i \geq 3$, has a unique solution $(x_i + y_i, \lambda_i)$ in the set $\{z \in \mathcal{D}(D) :\| z - x_i \|\leq r_i\} \times \mathcal{C}$, where r_i is given by equation (6.3.5). Also we have for all $i \geq 3$,

$$\lambda_i - d_i = x_i'(F(e_i + y_i)). \tag{6.3.6}$$

Since $Fe_i \in \text{span}\{e_{i-1}, e_{i+1}\}$ and $x_i'(e_j) = \delta_{ij}$ for all $i \geq 3$ and $j \in \mathcal{N}$, then $\lambda_i - d_i = x_i'(F(y_i))$. Thus we have

$$| \lambda_i - d_i |\leq\| F \|\| y_i \|\leq 2r_i < 1$$

for all $i \geq 3$. Thus for all $i \geq 3$, λ_i is the simple eigenvalue of A in the Gersgorin disc $\mathcal{R}_i$ of A. Since $\lim_{i\to\infty} r_i = 0$, then $\lim_{i\to\infty} | \lambda_i - 4(i-1)^2 |= 0$. For example,

for $i = 3$:

$$r_3 = 2 - \sqrt{3} < 0.268$$

hence $| \lambda_3 - 16 |< 0.536$

for $i = 4$:

$$r_4 = 4 - \sqrt{15} < 0.1271$$

hence $| \lambda_4 - 36 |< 0.2542$

for $i = 10$:

$$r_{10} = 16 - \sqrt{255} < 0.032$$

hence $| \lambda_{10} - 324 |< 0.064$.

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