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Combinatorial Games of Ramsey and Conway Types

by

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Abstract

Ramsey and Conway type games are two well developed branches of the Theory of Combinatorial Games. Chapters 2 and 4 introduce the mathematical theories along with the main relevant results in these two areas. Chapter 3 is devoted to a sub-class of Ramsey-type games — the path-forming games. It focuses on new results about the outcomes of two-person Hamiltonian path and Hamiltonian cycle games. In Chapter 5, a new theoretical tool for evaluating Conway-type games within infinitesimals is introduced — the Reduced Canonical Form. Chapter 6 is about the quintessential partizan game: Domineering. An original solution within infinitesimals to the game of $2 \times n$ Domineering is presented. It comes as an application of the theoretical construct in Chapter 5.

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Chapter 1

Introduction

There is no general, universally accepted definition for a combinatorial game. The attribute "combinatorial" refers to the discrete nature of the playing pattern. As opposed to the classical von Neumann and Morgenstern games [VM53] or to differential games, the players of a combinatorial game take turns making moves at discrete moments in time. The other essential restrictions are that no randomness component is allowed and, at every moment, the players must have complete knowledge of all the previous moves made in the game.

Two types of combinatorial game have generated significant interest among mathematicians and rich supporting theories have developed through their study.

One type branched out from classical Ramsey Theory and generated the class of Ramsey-type games. A Ramsey-type game is essentially described by a set of "options" that two players take turns in picking. There is also a collection of "objectives" – an objective is simply a set of options. From a player's perspective, the purpose of the game is to gather all the options that belong to an objective. The objective (!) in Chapter 2 is to give a formal and consistent introduction to the subject of Ramsey-type games. As there is no standardized terminology in the literature on this subject, many of the definitions here are original. The focus of Chapter 3 is on path-forming games. They are games where the options and the objectives are edges and respectively Hamiltonian paths (cycles.) The main two results, Theorems 32 and 34, are based on the new collapsing technique introduced in Section 3.2.

The second part of this manuscript concentrates on the class of combinatorial games where the outcome is decided in the favor of the player who makes the last move. As discovered by J. H. Conway [Con77], these games have a rich mathematical structure. They form a partially ordered infinitely generated abelian group Γ. Only games with finitely many options are considered. In Chapter 4, the introduction to the subject is based mostly on the two important texts in the area, "*On Numbers and Games*" by J. H. Conway and "*Winning Ways for Your Mathematical Plays*" by E. R. Berlekamp, J. H. Conway and R. K. Guy. The author's theory of the *reduced canonical forms* is developed in Chapter 5. The central result here is a direct sum decomposition theorem for Γ. This technique may become especially useful when trying to estimate game values to within infinitesimals. We use it in Chapter 6 to find the complete solution of the game of Domineering on $2 \times n$ boards to within infinitesimals. The outcome confirms Berlekamp's conjectured values in [Ber88]. Also, D. Wolfe coded the reduced canonical form operator in his combinatorial games software package [Wol96].

Unless otherwise specified, the theorems and the proofs presented in Chapters 3, 5 and 6 are original.

Chapter 2

Ramsey-type games

2.1 From TIC-TAC-TOE to The Generalized Continuum Hypothesis

The study of Ramsey-type games as mathematical objects has its beginnings as late as 1963 in the pioneering paper of A. W. Hales and R. I. Jewett on regularity and positional games [HJ63]. However, such games have been played by humans (and maybe by other creatures too) since the primitive times before *Mathematics*. Rather than a formal definition, the following oldest and best known positional game will do the introduction. It is **Tic-Tac-Toe** alias **Noughts-and-Crosses**. We cite Dudeney [Dud58] for the rules:

"Every child knows how to play this game. You make a square of nine cells, and each of the players, playing alternatively, puts his mark (a nought or a cross, as the case may be) in a cell with the object of getting three in a line. Whichever player first gets three in a line, wins with the exulting cry[1]:

"Tit, tat, toe,
My last go;
Three jolly butcher boys
All in a row"."

It doesn't take very long to see that, played correctly, Tic-Tac-Toe ends in a tie. A tie-strategy for the second player (the one that traditionally plays the noughts) is shown in Figure 1. This strategy should be used in conjunction with the following obvious rules, the first having precedence over the second:

- Place a nought and instantly make three in a row whenever you can.
- Block any of your opponent's two in a row.

In a situation not covered by Figure 1, and not subject to the two rules, any move

[1]Mother Goose Nursery Rhymes

ensures a tie.

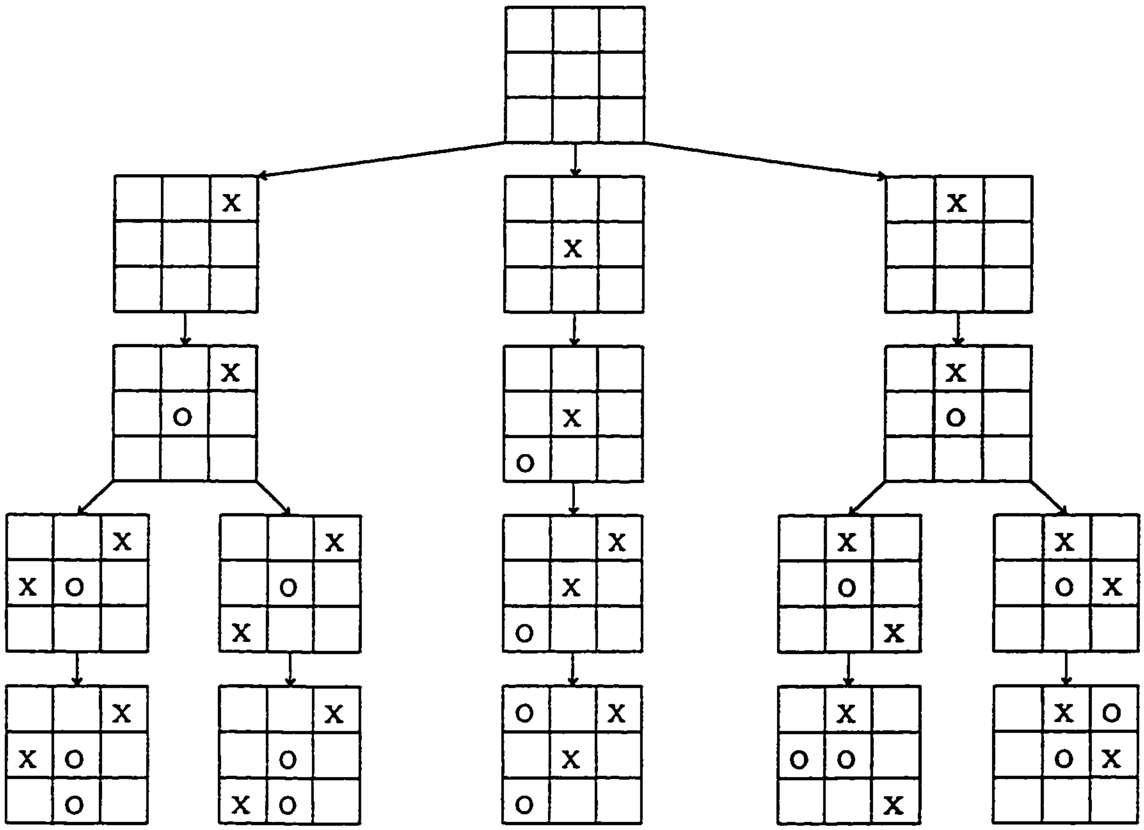


Figure 1.

It is unnecessary to provide a tie-strategy for the first player because the second player can never win unless the first player makes a mistake like in the diagram below.

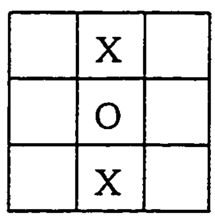


Figure 2

The following argument, known in the Game Theory folklore as **the strategy-stealing principle**, shows why. Suppose that the second player has a winning strategy, and let

T_{12} be the position reached after a move of the first player and a correct reply of the second. Hence, T_{12} is a win for the second player. But now, the first can "steal" the second's strategy like this: let T_2 be the position obtained by omitting the first player's move in T_{12}; firstly, T_2 is obviously at least as good for the second player as T_{12} was, and he will win it going second in the same way he would win T_{12}. However, the first player wins T_2 going first by using the second player's "would be" winning reply to the existing move in T_2. This is a contradiction and, indeed, the second player can never win.

Tic-Tac-Toe can be reformulated as a game on the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, where the players take turns at colouring the elements (one by one) in their own colour. The first player to colour three numbers that add up to the "magic" 15, wins. The reason why the two games really are the same becomes apparent when looking at the magic square

2	9	4
7	5	3
6	1	8

The reader may recognize that Tic-Tac-Toe being a tie implies, in particular, a Ramsey-type statement, namely that the chromatic number of the hypergraph with the digits one to nine as vertices, and triplets (sets of three elements) adding to 15 as edges, is at most two (hence, exactly two).

As the previous paragraph implies, it is time to move to the second objective of this section — the construction of the mathematical object Ramsey-type game. Games of this type have been also referred to as *positional*, *amoeba* or *achievement* by different authors. First, a few basic combinatorial notions need to be defined.

Definition 1 *A* ***hypergraph*** *is a collection of subsets of a set V. These subsets are called* ***hyperedges*** *(shortly* ***edges****), and the elements of V are the* ***vertices*** *of the hypergraph. Any function defined on the set of vertices of a hypergraph with values in* $\mathbf{n} = \{0, ..., n-1\}$ *is called an* ***n-colouring****. A colouring is called* ***good*** *if no restriction to a hyperedge is constant. In other words, a colouring is good if every edge is* *assigned more than one colour. In this setting, the* **chromatic number** *of a hypergraph is the least positive integer n (if there exists one) such that the hypergraph admits a good n-colouring. Otherwise, the chromatic number is ∞. Any restriction of a colouring is called a* **partial colouring.** *A hypergraph is called* **finite** *if it has finitely many vertices, hence finitely many edges.*

Based on Conway's idea of formalizing two person (normal winning convention) combinatorial games (see Chapter 4 of this manuscript), it is convenient to identify the actual positions during the play as being games themselves. Formally:

Definition 2 *A* ***Ramsey-type game*** *is a partially 2-coloured finite hypergraph.*

We will refer to an empty-coloured hypergraph as an 'initial position'. Although it is not formally necessary, defining how a game is played and lost or won from the perspective of two (human) players gives a better intuitive basis for the theory. Some game-related terminology will be introduced next.

Definition 3 *The vertices of a Ramsey-type game are called* **options** *and the set of all vertices is called the* **pick-set.** *The edges are the* **objectives.** *The two colours are M and B (rather than 0 and 1) and they are also called* **marks.** *There are two players, named the* **maker** *and the* **breaker,** *who take turns at picking (not previously picked) options and marking them M, respectively B. For quick reference, the maker is a female (she) and the breaker is a male (he). Unless otherwise specified, the maker will have the first move and the game stops when all the options are marked one way or the other.*

The aim of the maker is to mark all the elements of some objective M, while the aim of the breaker is, depending on the initial convention, either to prevent the maker from achieving her aim, or to mark all the elements of some objective B before she does. As with most other types of game, we are only interested in games played 'perfectly'. Therefore, the concept of 'existence of a winning strategy' is essential. The following are recursive definitions; the base cases are the already-played games.

Definition 4 *A totally-coloured game is said to be a **weak WIN** if it contains an objective whose options are all marked M. A (strictly) partially coloured game is a **weak WIN** if there is an unmarked option such that after the maker marks it M, the new game is either totally-coloured and a weak WIN or, if it is not totally coloured, after any reply by the breaker the new game is a weak WIN. We will refer to such a game as being a WIN in the **weak convention**.*

In other words, the notion of weak WIN simulates the existence of a strategy for the maker to mark all the elements of some objective no matter what the breaker does. Note that we have defined the notion of weak WIN from the perspective of the maker having the first move.

It is useful to observe here that once the maker completes marking every option M in an objective, the game will end up being a weak WIN no matter what the subsequent moves will be. Therefore, the players will agree to stop playing as soon as this situation occurs. If the maker first completely marks an objective on her nth move we will say that she wins the weak–convention game in n moves. Later in this manuscript we will consider the problem of finding the minimum number of moves necessary for the maker to win certain games.

Definition 5 *A game that is not a weak WIN is called a **weak LOSS**.*

Definition 6 *In the **strong convention** the game stops as soon as a monochromatic objective is achieved by either player (note that we only consider partially coloured games where there is no objective already monochromatically coloured). Then, the game is called a **strong WIN**, respectively a **strong LOSS**, depending on whether the common mark of the objective is M or B . If it doesn't contain a monochromatic objective, the game is a **strong WIN** if the maker can mark an option in it so that the game is over and is a strong WIN or, after any reply by the breaker to her move the outcome is a strong WIN. Similarly, the game is a **strong LOSS** if, after any move by the maker, there is a reply by the breaker so that the outcome is a strong LOSS.*

The notions of strong WIN and strong LOSS simulate the existence of strategies for the maker, respectively the breaker, to mark all the elements of some objective by M, respectively B, before the opponent does it.

Definition 7 *A game that is neither a strong WIN nor a strong LOSS is called a TIE.*

In the new terms, Tic-Tac-Toe is a TIE and it is an easy verification that it is also a weak WIN (indeed, the maker can easily win if she doesn't have to block any of her opponent's two-in-a-row's — she can make two open-ended two-in-a-row's through her first three moves, and the breaker cannot block both at the same time). The basic properties in the next proposition are straightforward consequences of the definitions and of a general strategy-stealing principle:

Proposition 8 *The following are true for any initial position G:*

1). If G is a strong WIN then G is a weak WIN.

2). If G is a weak LOSS then G is a TIE.

3). If the chromatic number $\chi(G) \geq 3$ then G is a strong WIN.

4). G is not a strong LOSS.

It is difficult to say which convention (that is, the weak or the strong one) is more interesting and a better or more natural object of study. Logically, it can be argued that it is pointless to play a game in the strong convention where one of the players can never win against a perfect opponent (see the previous proposition). However, games that people actually play (herein referred to as **real games**) are hard enough so that nobody knows how to play them perfectly. Notable exceptions include, of course, Tic-Tac-Toe; also, *Nine Men's Morris* and *Go-Moku* have been recently solved by extensively using computers (Nine Men's Morris is a tie and Go-Moku is a strong WIN – see [Gas96] and [AVH93] respectively). The game of *Checkers* seems to be the next one likely to fall [Sch96]. Nevertheless, it is the symmetric nature of the strong convention (that is, both players follow the same type of goal as opposed to the weak convention where the breaker's sole goal is to prevent the maker from achieving hers) that makes it more appealing in real games.

Most of the interesting mathematical problems arise from the study of the weak convention. Beside the already-mentioned logical pointlessness of the game in the strong convention at perfect play, there is another reason why the study of the weak one is the favored choice: there are no general methods or results concerning strong games. While playing a strong game, both players have their own threats and either of them, fending off the other's, may build his own. Therefore, a play is a delicate balancing between threats and counter-threats; this intricate structure has resisted all attempts to be modelled mathematically. In consequence, unless otherwise specified, games will be assumed to be in the weak convention from now on.

For the rest of this section, two main possible variations to the rules will be briefly discussed.

In [CE73], Chvátal introduced the "biased games" with the following explanation: "When a game is overwhelmingly in favor of one of the players, one can make up this handicap by allowing the other to pick many vertices in a move." Precisely, given two positive integers m and b, the definitions of 'WIN' and 'LOSS' can be modified by allowing the maker and the breaker to pick m, respectively b, options in each move. The outcomes will be called (m, b)-*biased WIN* and (m, b)-*biased LOSS*, respectively.

The other major way to break the rules is to allow infinitely many options while the objectives are still finite. The game ends when the maker picks all the options of some objective (in the case of the weak convention) or if either of the players picks all the options of some objective (in the case of the strong convention) and, if none of these happen, the game ends if the players have taken their nth turns for every natural number n. To say when such a game is a (weak) WIN or LOSS, the notion of *strategy* is required.

Definition 9 (*J. Beck*) *A strategy for the maker is a function S with domain the set of finite sequences of options in the game (including the empty sequence) such that $S(\langle w_0, w_1, ..., w_{n-1}\rangle)$ is always an option different from the options $v_i = S(\langle w_0, w_1, ..., w_{i-1}\rangle)$*

and w_i for $i = 0, 1, ..., n-1$ (if there is such an option). Here, the w_j denote moves by the breaker, and the v_j denote moves by the maker.

In a play according to this strategy, the maker determines all her moves by S as follows. Suppose the options $v_0, w_0, v_1, w_1, ..., v_{n-1}, w_{n-1}$ have already been picked in this order (the v_i by the maker and the w_i by the breaker). Then, the maker's nth move is $v_n = S(\langle w_0, w_1, ..., w_{n-1}\rangle)$. A strategy S is a winning strategy for the maker if every play according to S leads to a situation where the maker has picked all the options of some objective. The notions of a strategy for the breaker and a play according to a strategy for the breaker are similarly defined. A game is a WIN if the maker has a winning strategy; otherwise it is a LOSS (this also implies the existence of a winning strategy for the breaker). The study of infinite games (in the weak convention) can be reduced to the finite case as the following compactness result shows. The proof is due to Fred Galvin [BGLM73].

Theorem 10 *An infinite game H is a WIN if and only if some finite subhypergraph G of H is a WIN.*

Proof: The *if* part is obvious. To show the *only if* part, assume that the breaker wins on every finite subhypergraph $G \subset H$, and fix a winning strategy S_G for him in each G. We combine these S_G to get a winning strategy for the breaker in the big game. To do this let A be the family of finite subhypergraphs of H, and choose an ultrafilter[2] U on A such that for every $G \in A$ we have $\{G' \in A : G' \supset G\} \in U$. Now the breaker plays as follows: if there is some option v of H such that almost all (in the sense of U) of the strategies agree on v as the next move of the breaker, then he chooses v; if there is no such option, then he makes an arbitrary move. Now suppose that the maker wins the play in which the options $v_0, w_0, v_1, w_1, ..., v_{n-1}, w_{n-1}, v_n$ were chosen in this order, and the breaker picked all the options w_i by the strategy described above. Then there is

[2]An ultrafilter on A is a maximal proper family of non-empty subsets of A that contains any superset of a member and such that the family is closed under finite intersection.

some finite hypergraph $G \in A$ such that v_i and w_i are options in G, the set $\{v_0, v_1, ..., v_n\}$ contains an objective of G, and for every $0 \leq i < n$, $S_G(\langle v_0, v_1, ..., v_i \rangle) \notin \{v_{i+1}, ..., v_n\}$. Thus picking the options $v_0, v_1, ..., v_n$ in this order, the maker wins against the strategy S_G, a contradiction. □

We will close this section with an exotic game of P. Erdős which is both biased and infinite. Let E_n be the $(1, \omega)$-biased game (where the breaker picks countably many options in one move) having as options the real numbers and as objectives the arithmetic progressions of length $n+1$. Also, the players are allowed to make infinitely many moves — it is easy to see that for $n \geq 2$ the maker can never win in finite time. J. Beck and L. Csirmaz [BC82C] cite the following striking result of F. Galvin and Zs. Nagy [Gal78]:

Theorem 11 *The game E_n is a WIN if and only if $2^{\chi_0} \geq \chi_n$. Here χ_n denotes the nth infinite cardinal and 2^{χ_0} the cardinality of the power set of a countable set.*

2.2 Results for Special Classes of Ramsey-type Games

The object of this section is to overview the more prominent breeds of Ramsey-type games. Excepted is the class of path-forming games which are treated separately in the next section.

2.2.1 Generalized Tic-Tac-Toe

A. W. Hales and R. I. Jewett [HJ63] suggested the following natural generalization of Tic-Tac-Toe: The pick-set is the d-dimensional cube of size n

$$HJ(d,n) := \{\langle a_1, a_2, ..., a_d \rangle : a_j \in \mathbf{Z} \text{ and } 0 \leq a_j < n \text{ for each } 1 \leq j \leq d\}.$$

The objectives in $HJ(d,n)$ are those n-element subsets $\{\langle a_1^i, a_2^i, ..., a_d^i \rangle : 0 \leq i < n\}$ of $HJ(d,n)$ so that, for each j, the sequence $\langle a_j^0, a_j^1, ..., a_j^{n-1} \rangle$ composed of the jth coordinates is either strictly increasing (from 0 to $n-1$), or strictly decreasing (from $n-1$ to 0), or constant.

In particular, $HJ(2,3)$ is the same game as 3x3 Tic-Tac-Toe and it is a WIN (and, as we saw in the previous section, a strong tie). Oren Patashnik [Pat80] showed that $HJ(3,4)$ — commercially known as *Qubic* — is a strong WIN by using 1500 hours of 1980-computer time. Also, in [HJ63], Hales and Jewett showed:

Theorem 12 *If n is odd and $n \geq 3^d - 1$ or if n is even and $n \geq 2^{d+1} - 2$ then $HJ(d,n)$ is a LOSS, but for each n there exists $d = d(n)$ so that $HJ(d,n)$ is a WIN.*

P. Erdős and J. Selfridge obtained a far better estimate for the minimum n such that $HJ(d,n)$ is a LOSS by considering a general game played on an n-uniform hypergraph (that is, a hypergraph where every edge has size n). Such a game is called n-uniform. We will present next their main result in [ES73]. The valuation method used in the original proof is at the core of several subsequent results so the proof will be given too.

Theorem 13 *Given a positive integer n, let $m(n)$ be the smallest integer for which there exists an n-uniform game with $m(n)$ objectives that is a WIN. Then, $m(n) = 2^{n-1}$.*

Proof: Let $S = \{a_1, ..., a_n, b_1, ..., b_n\}$ be a set with $2n$ elements. First, we exhibit 2^{n-1} sets of n elements each so that the corresponding game is a WIN. Let G_n be such a collection. Put $G_1 = \{a_1\}$ and $G_1' = \{b_1\}$. By induction define G_{i+1} to be the collection of the sets G_i and G_i' with a_{i+1} adjoined to each and define G_{i+1}' to be the collection of sets G_i and G_i' with b_{i+1} adjoined to each. The maker's strategy to win G_n is clear. First pick a_n. If the breaker picks a_i then she picks b_i and vice versa. On each move the breaker can only block half of the remaining objectives. On the nth move, the maker will complete an objective.

Now we must prove that for any smaller collection of objectives of size n the breaker can keep the maker from winning. Here is how: Give each option a value which is the sum of the values of those objectives it belongs to and which are not already blocked by the breaker. The value of such an objective having i options picked is 2^i. The breaker picks an option of largest value. To prove that the maker cannot win, we show that the sum C of the values of all the objectives remaining after her ith turn is less than 2^n. So before her next move this sum C is less than $2^n - V$, where V is the sum of the values of the objectives just blocked by the breaker, that is, the value of the option picked by him. Now on the maker's next move she doubles the value of each objective containing the option picked, that is, she adds the sum of the previous values V' to C. But clearly $V' \leq V$ since V was a maximum. □

The same method gives a similar result in the general case when the game is not n-uniform:

Theorem 14 *Let $\{A_k\}$ be a set of objectives, $|A_i| = n_i$, for which*

$$\sum_i \frac{1}{2^{n_i}} < \frac{1}{2}.$$

Then the corresponding game is a LOSS. On the other hand, if integers n_i are given for which

$$\sum_i \frac{1}{2^{n_i}} \geq \frac{1}{2},$$

we can find sets of options A_i, $|A_i| = n_i$ so that the corresponding game is a WIN.

It can be shown that in the d-dimensional cube of size n there are

$$\frac{1}{2}\left\{(n+2)^d - n^d\right\}$$

objectives. Thus, applying Theorem 13 to $HJ\,(d,n)$ gives the sharper condition

$$n > cd\ln d \text{ for some constant } c$$

for the game being a LOSS. This still falls short of the Hales–Jewett conjecture that

$HJ(d,n)$ is a LOSS if

$$n > 2\left(2^{1/d}-1\right)^{-1} \approx \frac{2d}{\ln 2} - 1.$$

J. Beck proves this conjecture for $d \geq 100$ in his 1981 paper on positional games [Bec81A] by considering almost disjoint games:

Definition 15 *A game is called* ***almost disjoint*** *if the intersection of any two objectives contains at most one option.*

He first establishes the following:

Theorem 16 *If an almost disjoint n-uniform game with m objectives is a WIN then*

$$m \geq \frac{4^{n-2\sqrt{3n}}}{64n^2}.$$

Then, he shows:

Theorem 17 *If $n \geq (\ln 3/\ln 2)\, d + 4\,(d \ln d)^{1/2} + 4$ and $d \geq 100$, then $HJ(d,n)$ is a LOSS.*

2.2.2 The Ramsey and van der Waerden Games

The study of these games was obviously suggested by the following two well-known combinatorial problems:

Problem 1. Let K_M^k be the family of all k-tuples of elements of an m-element set M. Then, what is the least integer m (denoted by $R_k(n)$ and called the Ramsey number) such that if we partition K_M^k into two classes then at least one of them contains all the k-tuples of some n-element subset of M ? (Note that even the proof of the existence of the Ramsey numbers is not a trivial problem, being the subject of a famous theorem of Ramsey.)

Problem 2. For each positive integer n, what is the smallest integer m (denoted by $W(n)$ — the van der Waerden number) with the property that if the integers from

1 to m are partitioned into two classes, then at least one class contains an arithmetic progression of n terms? (The existence of $W(n)$ for every n is the object of van der Waerden's well-known theorem.)

Two game-outcome problems can be associated naturally with problems 1 and 2:

*Problem 1**. The Ramsey game $R(k,n,m)$ is played on the pick-set formed with all the k-element subsets of a set M with m elements and for any n-element subset N of M, the collection of all k-element subsets of N form an objective. Then, what is the smallest value of m, denoted by $R_k^*(n)$, such that $R(k,n,m)$ is a WIN?

*Problem 2**. The van der Waerden game $W(n,m)$ is played on the pick-set $\{1,2,...,m\}$ and the objectives are the arithmetic progressions of n terms. Then, what is the smallest value of m, denoted by $W^*(n)$, such that $W(n,m)$ is a WIN?

It seems to be hopelessly difficult to find precise expressions for $R_k(n)$ and $W(n)$, in the sense that the best known upper and lower bounds are far apart; even the existing estimates are very poor. The best lower bound on $W(n)$ known, due to E. R. Berlekamp [Ber68], asserts that $W(p) > p2^p$ if p is a prime and $W(n) > c2^n$ for all n; as for the upper bound, even the following question is unanswered: is there an integer k such that $W(n) < \exp_k n$ for each n, where $\exp_k n$ denotes the k-fold iterated exponent of n? Also, the true order of magnitude of the Ramsey numbers is not known either; for instance, the best estimates known for $R_3(n)$ are the following [ER52]:

$$2^{n^2/6} < R_3(n) < 2^{2^{4n-10}}.$$

In light of this, it was rather surprising when J. Beck [Bec81B] found asymptotically tight estimates for the "game-numbers" $R_k^*(n)$ and $W^*(n)$:

Theorem 18 *For every positive integer n*

$$2^{\frac{n}{2}} < R_2^*(n) < (2+\epsilon)^n$$

and

$$2^{\frac{n^{k-1}}{k!}} < R_k^*(n) < n^{k+1}2^{\binom{n}{k}} \textit{ for } 3 \le k < n.$$

Theorem 19 *For every integer $n > 2$*

$$2^{n-7n^{7/8}} < W^*(n) < n^3 2^{n-4}.$$

2.2.3 Games of A Hybrid Type

This type of games was suggested by F. Harary and others (see [Sim69], [MRH74] and [RY74]) and borrows features from the Ramsey-type games as well as from combinatorial games in the sense of Berlekamp-Conway-Guy (see Chapter 4 for the latter ones). As in Ramsey-type games, there are two players, a set of options and a set of objectives; however, the players use the same markings for the options they pick and the winning convention is different too: the first player to complete an objective is the winner (loser) when the game is played in the achievement (avoidance) convention. Unlike in Ramsey-type games where one player tries to completely mark an objective while an opponent tries to deter her, the players have the same goal in a hybrid game: namely, in the *achievement* convention, to pick the last unpicked (by either of the players) option of some objective before the opponent does it; and, in the *avoidance* convention, to avoid picking the last option of any objective. Because of the symmetry of the goals, it is more reasonable to call the players *Alpha* and *Beta*. Alpha has the first move and, as for strong Ramsey-type games, a game is called a WIN if Alpha has a winning strategy, a LOSS if Beta has a winning strategy and a DRAW if neither of the players has a winning strategy. The possibility of a DRAW only exists in infinite games.

The following classes of hybrid-type games have been studied (however results have only been obtained for certain particular cases):

- The pick-set is the edge-set of a finite graph G. The objectives are diameter-2 graphs (Buckley and Harary [BH84]).

- The same pick-set and the objectives are connected graphs (Harary and Robinson [HR84]).
- The same pick-set and the objectives are graphs with a node of fixed degree (Harary and Plochinsky [HP87]).
- The same pick-set and the objectives are graphs with fixed minimum degree (Gordon, Robinson and Harary [GRH94]).
- The pick-set is the vertex set of a finite graph G. The objectives are maximal independent sets (Harary and Tuza [HT93]).

There seems to be no general method to tackle hybrid-type games. However, in situations where the game tends to split into disjunctive components (see Chapter 4), the Nim theory of impartial games ([Spr35], [Gru39] or [BCG82]) could be very effective. A notable example is the last of the above mentioned problems for the case when the graph is a path P_n of length $n-1$. It was proposed as an open problem by Harary and Tuza [HT93]. However, it is equivalent to "Dawson's Kayles" (and to a few other games) which had been solved 44 years earlier (see [GS56]), by using the Sprague-Grundy theory of impartial games. Dawson's Kayles is played like this: there are one or more heaps of beans and the two players take turns at making moves consisting of choosing a heap, followed by removing exactly two beans from it and splitting the rest of that heap into at most two heaps; the player unable to move loses. The equivalence between Harary and Tuza's game and Dawson's Kayles is obvious.

Chapter 3

Path-Forming Games

3.1 Preliminaries

The scope of this chapter is a collection of positional games generically called *path-forming games*. The following definition describes a general form of such a game.

Definition 20 *Given a positive integer m and a finite graph G with vertex set $V(G)$ and edge set $E(G)$, the m-path game on G is the positional game where the pick-set is $E(G)$ and the objectives are the paths of length m in G (a path of length k is generated by $k+1$ distinct vertices $\{v_1, v_2, ..., v_{k+1}\}$ with the property that, for all $1 \leq i \leq k$, v_i and v_{i+1} are connected through the edge e_i; the subset $\{e_1, e_2, ..., e_k\}$ of $E(G)$ forms the path.) The maker has the first move.*

The main variations to this definition are the following:

- Games with different types of handicaps that generally favor the breaker. The reason why such handicaps are considered is that path-forming games have a tendency to bias in favor of the maker as the size n of the vertex set of G increases. One such handicap consists of allowing the breaker to make a number of initial moves in G. If G is the complete graph K_n say, a typical choice for the number of initial breaker moves is $an + b$ where a and b are non-negative integers. Another type of handicap is to allow the breaker to pick more than one edge at a time — the number of edges he can pick can again be linear in n.

- Games where the maker has to achieve more than one path — often there is the extra condition that the paths she forms must be edge-disjoint.

Typically, the main problem associated with a path-forming game is to find out the winner. If the maker wins then it is also of interest to find the minimum (or at least good estimates for the minimum) number of moves that are necessary for her to win.

Over the past decades there has been significant interest in studying path-forming games but, in the absence of a comprehensive monograph in the area of Ramsey-type games, many results have been independently duplicated. In this context we will give an account of the main known results and explain how the results in the next section relate to them.

In [Pap82], Alexis Papaioannou first proved that the Hamiltonian path[1] game on K_n is a WIN for $n \geq 5$. This problem was first proposed by B. Bollobás as a variant of an Achievement Game of Harary [Har82]. Papaioannou also gave an informal argument for the Hamiltonian *cycle* game on K_n (for very large values of n) being a WIN as well (the objectives here are Hamiltonian cycles.)

Independently in 1992, Bill Sands suggested the Hamiltonian path game on K_n. During the same year D. Duffus, O. Gelbord, B. Sands and R. Woodrow gave a different proof that it is a WIN for $n \geq 5$ (personal communication of B. Sands). Neither of these two approaches considered the problem of winning in a minimum number of moves.

In 1985, József Beck [Bec85] proved the following negative (for the maker) result using random graph process techniques:

[1]a Hamiltonian path is a path which meets every vertex of the graph

Theorem 21 *If we denote by Hamilt $(n; b)$ the Hamiltonian path game on K_n where the breaker is allowed b moves at a time, then the breaker wins Hamilt $(n; b)$ if*

$$b > \frac{(\log 2 - \varepsilon)\, n}{\log n}$$

for n large enough.

He also conjectured that, asymptotically, this is the true breaking point in the sense that there is a positive constant k so that for

$$b < \frac{kn}{\log n}$$

the maker wins. He quoted a heuristic argument of P. Erdős to support it.

In 1992, Xiaoyun Lu [Lu92] proved the following:

Theorem 22 *In the Hamiltonian cycle game on K_n, the maker can achieve at least $\left(\frac{1}{16} - o(1)\right) n$ edge-disjoint Hamiltonian cycles for sufficiently large n ($o(1)$ denotes a sequence which converges to zero as n goes to infinity.)*

Three years later [Lu95], he showed:

Theorem 23 *In the Hamiltonian cycle game on $K_{n,n}$ (the complete bipartite graph of order $2n$), the maker can achieve at least $\frac{1}{37}n$ edge-disjoint Hamiltonian cycles for sufficiently large n.*

In Lu's version of the game the play ends when all the edges of the graph are taken.

In 1996, Martin Knor [Kno96] used an approach different from Papaioannou's and Lu's to show the following results:

Theorem 24 *The Hamiltonian path game on K_n with a handicap of at most $\left\lfloor \frac{n-5}{2} \right\rfloor$ initial breaker edges is a WIN for $n \geq 5$.*

Theorem 25 *The Hamiltonian cycle game on K_n with a handicap of at most $\left\lfloor \frac{n-15}{2} \right\rfloor$ initial breaker edges is a WIN for $n \geq 15$.*

The author's results in the rest of this chapter were obtained independently from the above by using the new method of collapsing described in the next section. The two main results (Theorems 32 and 34) complement the theorems mentioned so far in several ways: they give tight bounds on the minimum number of moves required by the maker to win; Theorem 32 improves the known bound on the handicap of initial breaker edges from $\left\lfloor \frac{n-5}{2} \right\rfloor$ to $n-4$ so that the Hamiltonian path game is still a win. None of the above researchers have dealt with the problem of finding strategies that minimize the number of moves required for the maker to win. Theorem 32 gives the best possible bound of $n-1$ for the number of moves required by the maker to win the Hamiltonian path game. Theorem 34 shows that the maker can win the Hamiltonian cycle game in $n+4$ moves — this is close to the best possible because, clearly, the maker cannot win in only n moves.

In general, any position in a path-forming game (reached after some moves have been made by the players) will be either a WIN or a LOSS, depending on whether the maker has or has not a winning strategy when moving first from there. For any position Γ, it is convenient to denote the set of edges already picked (marked) by the maker and the breaker by $M(\Gamma)$ and $B(\Gamma)$ respectively and we shall do so. We will use the notation $V = \{0, 1, ..., n-1\}$ for the vertex set of a game G on n vertices. For $i, j \in V$, $i \neq j$, each of ij and ji denote the edge between i and j. For clarity, an extra space will be inserted between the names of the vertices forming the edge whenever necessary. For instance, the edge between 0 and $n-1$ will be denoted $0\ n-1$. The terms "mark an edge" and "join two vertices" will be used interchangeably for the action of either the maker, or the breaker, of picking an edge and marking it M, respectively B.

It is convenient to regard the non-edges of G as edges marked B at the beginning of the game played on a complete[2] graph with the same number of vertices as G. We also call an M-marked edge an *M-edge,* and a path made entirely of M-edges an *M-path.*

[2]A complete graph is a graph where any two vertices are joined by an edge.

3.2 The Collapsing Technique

The proofs in the rest of this chapter will be of an inductive nature and will rely on the construction in Definition 26 and on Lemma 27. Before giving the formal definition, the notion of collapse will be briefly described using Figure 1.

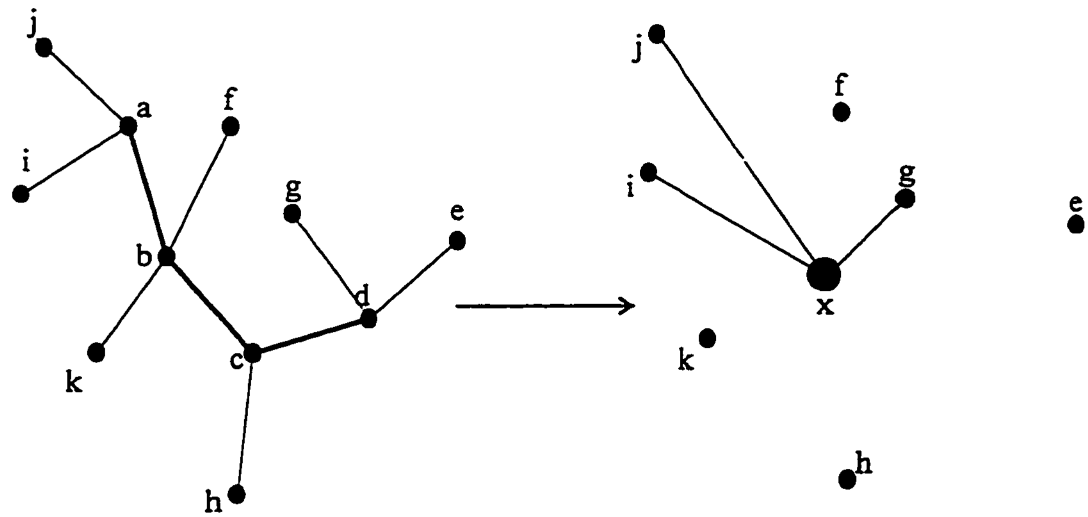


Figure 1. The M-path (a, b, c, d) with specified B-edge de attached at the end collapses to one vertex x (thick lines are used for M-edges and thin lines for B-edges.)

Given a position containing an M-path (determined by the sequence (a, b, c, d) of vertices in Figure 1) and some B-edges (shown by thin lines) so that at least one B-edge (de in this case) is connected to one end of the M-path, we obtain the collapse of this position by replacing the vertices a, b, c, d in the M-path by a single vertex x. Also, we discard the B-edges containing interior vertices of the path (bf, bk, ch in Figure 1) as well as the specified B-edge at the end of the path (de). All the other edges are inherited from the original position; for example, ai and aj become xi and xj respectively.

Definition 26 *Consider the position Γ, reached after some moves have been made in a* *game on n vertices. Also, assume that*

$$\{i_0i_1, i_1i_2, ..., i_{s-2}i_{s-1}\} \subseteq M(\Gamma) \text{ and } i_{s-1}i_s \in B(\Gamma),$$

where $i_0, i_1, ..., i_s$ are all distinct, and $2 \leq s \leq n-1$. Finally, assume that the M-path determined by $(i_0, ..., i_{s-1})$ is maximal, in the sense that for any $j \notin \{i_0, i_1, ..., i_{s-1}\}$, ji_0 and $ji_{s-1} \notin M(\Gamma)$. We define the **collapse** *of Γ, denoted by $co(\Gamma)$, to be the following position on the vertices $\{0, 1, ..., n-1\} \setminus \{i_0, i_1, ..., i_{s-2}\}$. Here are the conditions when a pair of vertices i, j from this set determines an M (respectively B) edge of $co(\Gamma)$:*

- *$ij \in M(co(\Gamma))$ whenever $ij \in M(\Gamma)$;*
- *$ij \in B(co(\Gamma))$ whenever:*

$$i \neq i_{s-1} \text{ and } j \neq i_{s-1} \text{ and } ij \in B(\Gamma), \text{ or}$$

$$i = i_{s-1} \text{ and } j \neq i_s \text{ and } \{i_0j, i_{s-1}j\} \cap B(\Gamma) \neq \emptyset, \text{ or}$$

$$i = i_{s-1} \text{ and } j = i_s \text{ and } i_0j \in B(\Gamma).$$

In other words, given a maximal M-path with a specified B-edge hanging at one end, $co(\Gamma)$ can be obtained from Γ by collapsing all the vertices of the path to one vertex, and erasing the specified B-edge at the end of the path as well as all the M-edges and B-edges that are connected to interior vertices of the path. Any B-edge connecting either end of the path to some other vertex, becomes an edge connecting that vertex to i_{s-1}. All the rest stays unchanged.

Note that the specified B-edge at the end of the path can be regarded as an optional feature of Definition 26 in the sense that if the maker can win in $co(\Gamma)$ without erasing the specified B-edge, then she can certainly win when the specified B-edge is erased. Therefore the following lemma is still true if the specified B-edge is not erased.

We make the observation that the minimum number of moves required by the maker to form a Hamiltonian path is $n-1$ — one less than the number of vertices in the graph. In the case of Γ this number contains the $s-1$ M-edges already present.

Lemma 27 *The position Γ is a WIN in a minimal number of moves if $co(\Gamma)$ is a WIN in a minimal number of moves.*

Proof: It will be shown that, playing according to the following algorithm, the maker can "copy" the winning strategy from $co(\Gamma)$ to Γ:

- *step 1.*

  The maker makes a (valid) winning move in $co(\Gamma)$, say ij. If this move completes a Hamiltonian path in $co(\Gamma)$, then *STOP* after the next step.

- *step 2.*

  if ($i \neq i_{s-1}$ and $j \neq i_{s-1}$), then the maker marks ij in Γ.

  if $ij = i_{s-1}i_s$, then the maker marks i_0i_s in Γ.

  if ($i = i_{s-1}$ and $j \neq i_s$ and i_{s-1} has not been previously joined by an M-edge to some other vertex in $co(\Gamma)$) then the maker marks $i_{s-1}j$ in Γ.

  if ($i = i_{s-1}$ and $j \neq i_s$ and i_{s-1} has been previously joined by an M-edge to some other vertex in $co(\Gamma)$) then the maker marks i_0j in Γ.

- *step 3.*

  The breaker replies by making some move in Γ, say kl.

- *step 4.*

  The corresponding move of the breaker in $co(\Gamma)$ will be:

  $i_{s-1}l$ if ($k = i_0$ and $l \notin \{i_1, i_2, ..., i_{s-1}\}$ and $i_{s-1}l$ is not marked yet),

  kl if ($\{k, l\} \cap \{i_0, i_1, ..., i_{s-2}\} = \emptyset$ and kl is not marked yet),

  any other move, otherwise.

- *step 5.*

  Go back to *step 1.*

In order to prove that this algorithm is correct, we have to show that at every stage, *step 2* is possible, i.e. the edge that the maker has to mark in Γ must not be already marked. It is easy to see that, because of the way $co(\Gamma)$ was constructed, if the edge in Γ required at *step 2* had already been marked when the algorithm started, then the edge ij in $co(\Gamma)$ must have been marked before the previous step (*step 1*). This is in contradiction with the legality of the M-mark on ij at *step 1.* Assume, now, that the edge required at *step 2* had been marked at a previous iteration of *step 2* or *step 3.* But this is not possible because the transformations of edges of $co(\Gamma)$ into edges of Γ defined by moving from *step 1* to *step 2* and also from *step 3* to *step 4* are one-to-one.

Finally, we will prove that when the play stops, an M-Hamiltonian path (that is, a Hamiltonian path whose edges are all marked M) has been created in Γ. Denote by

$$l_0,\ l_1,\ \ldots,\ l_m = i_{s-1},\ l_{m+1},\ \ldots,\ l_{n-s}$$

the consecutive vertices of the M-Hamiltonian path in $co(\Gamma)$. If $1 \leq m \leq n-s-1$, then $(l_0, l_1, \ldots, l_{m-1})$ and $(l_{m+1}, l_{m+2}, \ldots, l_{n-s})$ are two disjoint M-paths in Γ by virtue of the first case of *step 2.* Then, because of the alternate way in which the M-marks in Γ are made at *step 2* in the last three cases (when one of the endpoints of the input edge from *step 1* is i_{s-1}), we either have

$$l_{m-1}i_0 \text{ and } l_{m+1}i_{s-1} \in M(\Gamma) \text{ or}$$

$$l_{m-1}i_{s-1} \text{ and } l_{m+1}i_0 \in M(\Gamma).$$

If the former is true, then

$$(l_0, l_1, \ldots, l_{m-1}, i_0, i_1, \ldots, i_{s-1}, l_{m+1}, \ldots, l_{n-s})$$

is an M-Hamiltonian path in Γ, and if the latter is true, then

$$(l_0, l_1, ..., l_{m-1}, i_{s-1}, i_{s-2}, ..., i_0, l_{m+1}, ..., l_{n-s})$$

is an M-Hamiltonian path in Γ.

If i_{s-1} is one of the endpoints in $(l_0, l_1, ..., l_{n-s})$, say if $m = 0$, then either

$$(i_0, i_1, ..., i_{s-1}, l_1, ..., l_{n-s}) \text{ or}$$

$$(i_{s-1}, i_{s-2}, ..., i_0, l_1, ..., l_{n-s})$$

is an M-Hamiltonian path in Γ.

Note that the assumption that the maker can win $co(\Gamma)$ in a minimum number of moves (which is $n-s$) implies that, including the already existing $s-1$ M-edges in Γ, the maker has made $(n-s)+(s-1) = n-1$ moves in Γ and this is a win in a minimum number of moves. □

The following string of four lemmas are direct applications of the collapsing technique defined above. They will be used to establish the induction step in the proof of Theorem 32. For the rest of this section, unless otherwise stated, we will call any Hamiltonian path game simply a game. Also, we define the M-degree, respectively B-degree of a vertex to be the number of M-edges, respectively B-edges, adjacent to that vertex. We will call any game played on K_n having k initial B-marked edges (or simply B-edges), a game of type $\langle n, k \rangle$.

Lemma 28 *Let $n \geq 8$. If any game of type $\langle n-2, n-6 \rangle$ is a WIN in a minimal number of moves, then so is any game of type $\langle n, n-4 \rangle$ that has a vertex of B-degree at least four.*

Proof: Let a be a vertex in a game G of type $\langle n, n-4 \rangle$ so that ab, ac, ad and ae are marked by the breaker (the relevant part of the graph, including four other vertices where the breaker may reply, is shown in Figure 2 below.)

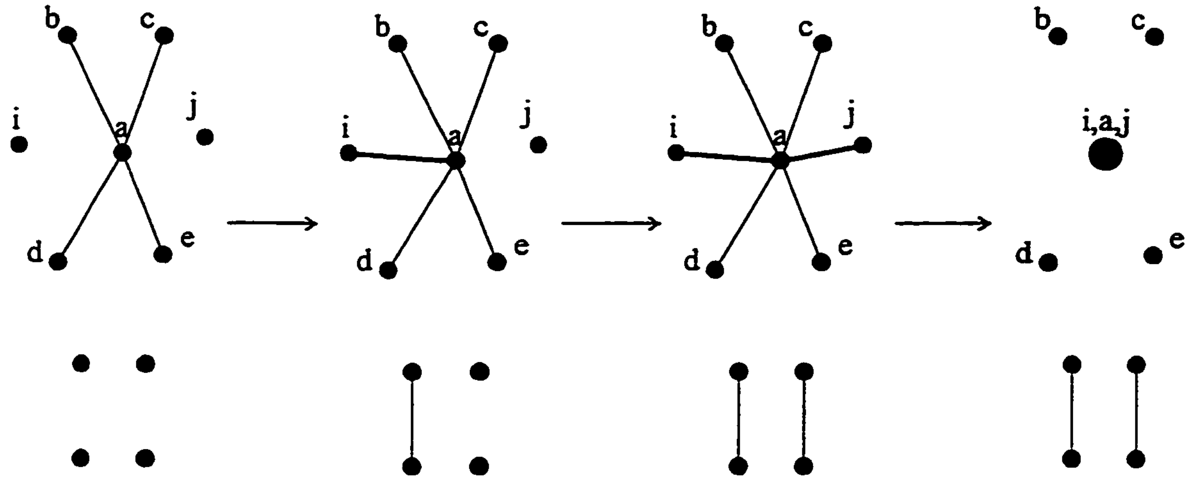


Figure 2

Then, the maker's first move will be to mark ai for some vertex i. This is possible because at least $n-1-(n-4)=3$ edges from a are not marked yet. After the breaker's reply, the maker will mark some other edge aj. This is again possible because there is at least one more edge from a that is not marked yet. After the breaker's reply, the collapse of G relative to the M-path determined by i, a and j is a game of type $\langle n-2, m\rangle$ with $m \leq n-6$ and so it is a WIN because $\langle n-2, n-6\rangle$ is a WIN and any fewer B-edges can only help the maker. Now, applying Lemma 27, it follows that G is a WIN as well. Note that we have applied Lemma 27 in the weaker form because by collapsing along i, a and j, we did not need to delete any potential B-edge connected to i or j (see the remark after Definition 26.) $\square$

Lemma 29 *Let $n \geq 8$ and assume that any game of type $\langle n-2, n-6\rangle$ is a WIN in a minimal number of moves. Then, if a game G of type $\langle n, n-4\rangle$ has a B-edge ab and a vertex c ($c \neq a$) of B-degree at least three such that $ac \notin B(G)$, this game is a WIN and moreover in a minimal number of moves.*

Proof: At the first move in the game G, the maker will mark the edge ac (see Figure 3).

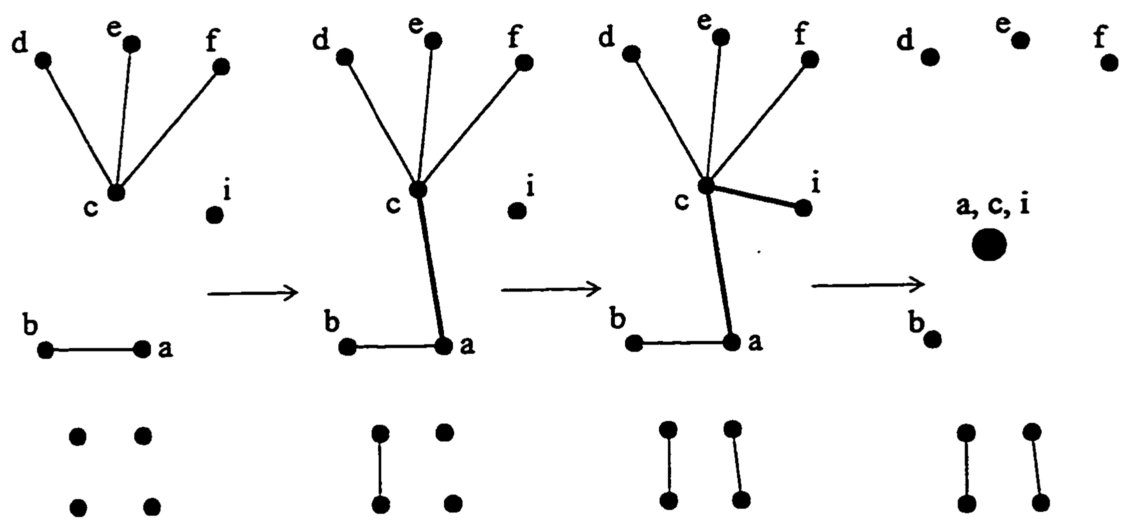


Figure 3

After the breaker's reply, the maker will mark ci for some vertex i. After the breaker replies again, the collapse of G relative to the M-path determined by a, c and i, and with specified B-edge ab, is a game of type $\langle n-2, m\rangle$ with $m \leq n-6$ and so it is a WIN. Applying Lemma 27 again, G must be a WIN as well. □

Lemma 30 *Let $n \geq 9$ and assume that any game of type $\langle n-2, n-6\rangle$ is a WIN in a minimal number of moves. Then, if a game G of type $\langle n, n-4\rangle$ contains distinct vertices a, b, c, d, e and f such that $\{ab, ac, de, ef, fd\} \subseteq B(G)$ and $\{ad, ae, af\} \cap B(G) = \emptyset$, G is a WIN in a minimal number of moves.*

Proof: The maker first marks ad (see Figure 4).

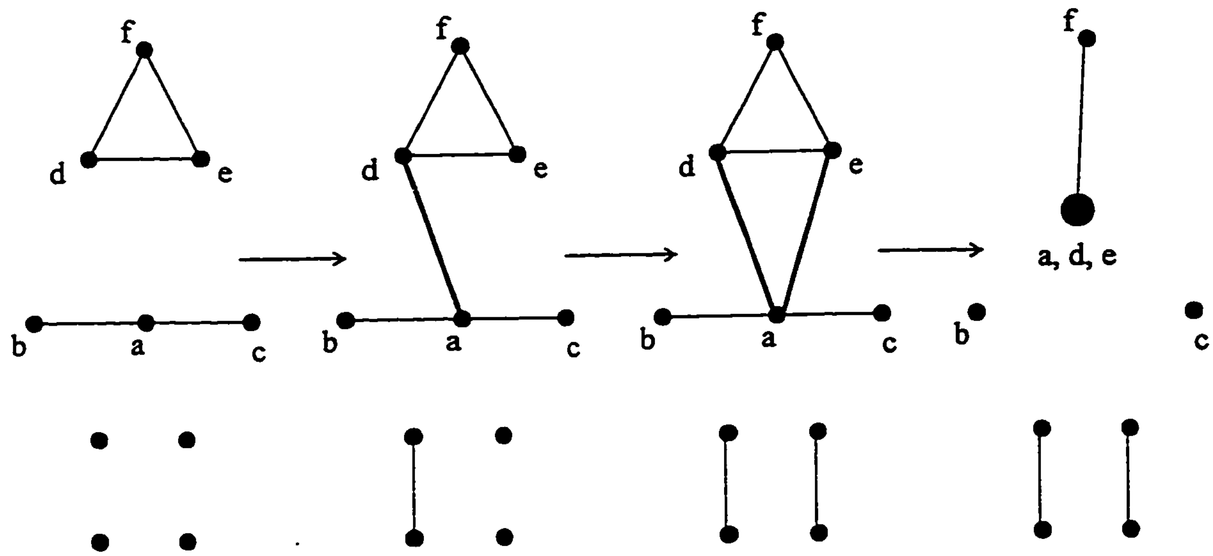


Figure 4

After the breaker's reply, at least one of ae and af is available so the maker can mark one of them, say she marks ae. After the breaker replies again, the collapse of G relative to the M-path determined by d, a and e, and with specified B-edge df, is a game of type $\langle n-2, m\rangle$ with $m \leq n-6$ and so it is a WIN. Applying Lemma 27 again, G must be a WIN as well. □

Lemma 31 *Let $n \geq 7$ and assume that any game of type $\langle n-1, n-5\rangle$ is a WIN in a minimal number of moves. Then, if a game G of type $\langle n, n-4\rangle$ contains distinct vertices a, b, c and d such that $\{ab, bc, cd\} \subseteq B(G)$ and $ac \notin B(G)$, G is a WIN in a minimal number of moves.*

Proof: The maker first marks ac (see Figure 5).

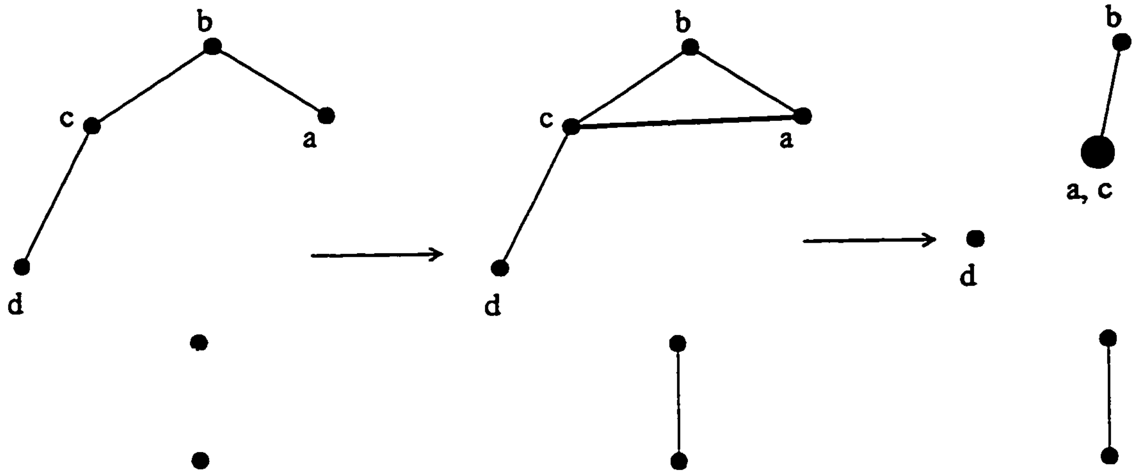


Figure 5

After the breaker's reply, the collapse of G relative to the M-path determined by a and c, and with specified B-edge cd, is a game of type $\langle n-1, m \rangle$ with $m \leq n-5$ and so it is a WIN. Applying Lemma 27 again, G must be a WIN as well. □

3.3 The Hamiltonian Path Game

Theorem 32 *For any graph G such that $|V(G)| = n \geq 5$ and $|E(G)| \geq \binom{n}{2} - n + 4$, the Hamiltonian path game on G is a WIN. In other words, every game of type $\langle n, k \rangle$ is a WIN for $k \leq n-4$. Furthermore, the maker can win in $n-1$ moves.*

Proof: Consider a game G such that $|V(G)| = n \geq 5$ and $|E(G)| \geq \binom{n}{2} - n + 4$. We proceed by induction on n and split the proof into two parts. The first part is the induction step. The second part is much longer and establishes the initial cases required by the induction argument. It is enough to show the theorem for $k = n - 4$ — fewer initial B-edges can only help the maker.

Part 1.

Let $n \geq 13$ and assume that every game of type $\langle n-2, n-6\rangle$ and $\langle n-1, n-5\rangle$ is a WIN. We will show that any game of type $\langle n, n-4\rangle$ is a win as well. Let G be a game of this type. We will show that G has to satisfy the hypothesis of one of the Lemmas 28, 29, 30, 31, and therefore G is a WIN. Suppose, for a contradiction, that this is not the case. Then, as Lemmas 28 and 29 don't apply, G contains no vertices of B-degree three or more. Therefore, if we denote by G^* the graph on the same vertices as G but only having the initial B-edges, the connected components of G^* are paths and cycles. As Lemma 31 doesn't apply, the only connected components are isolated edges, paths of length two and triangles. As Lemma 30 doesn't apply, a triangle and a path of length two cannot coexist as connected components and neither do two triangles. We will be in one of the following two situations:

1. The connected components of G^* are isolated edges and, possibly, one triangle.

   In this case, the sum of the degrees of the vertices of G^* is at most $(n-3)+(2\times 3) = n+3$. Hence, $|E(G^*)| \leq \left\lfloor \frac{n+3}{2} \right\rfloor$. On the other hand, we know that $|E(G^*)| = n-4$. This means that $n-4 \leq \left\lfloor \frac{n+3}{2} \right\rfloor \leq \frac{n+3}{2}$ so $n \leq 11$, contradiction.

2. The connected components of G^* are isolated edges and paths of length two.

   In this case, there are at least twice as many vertices of degree one in G^* than vertices of degree two. Then, the sum of the degrees of the vertices of G^* is at most $\frac{4}{3}|V(G^*)| = \frac{4}{3}n$. This leads to the inequality $n-4 \leq \frac{2}{3}n$ and then $n \leq 12$, a contradiction again.

This completes the proof of **Part 1**.

Part 2.

Let $5 \leq n \leq 12$ and assume that G is of type $\langle n, n-4\rangle$. We will show that G is a WIN by describing a winning strategy in two phases for the maker. In Phase 1, the maker will create a number k $(2 \leq k \leq 5)$ of independent[3] paths in her colour that cover all the

[3]paths that, pairwise, have no vertices in common

n vertices. Furthermore, this has to be done in such a way that, when the k paths are completed, the number of *active* B-edges is at most $3(k-1)$. By **active B-edges** we mean B-edges that connect endpoints of distinct M-paths. It is intuitive that a B-edge connecting an interior vertex of an M-path to some other vertex or a B-edge connecting the two endpoints of the same M-path cannot be of any use for the breaker because the maker will never need to mark those edges anyway — otherwise, she would not be able to win G in a minimal number of moves. We may use the term active B-edge, *before* the maker has completed her k paths, for any B-edge that has the potential to still be active when the M-paths are completed. A B-edge that becomes non-active will be called **dead** and when the maker turns a B-edge dead we say that she **kills** it.

In Phase 2, the maker will join (in $k-1$ moves) the k M-paths to form a Hamiltonian path. This is done in the presence of at most $3(k-1)$ active B-edges.

We will proceed by proving first that Phase 1 can be completed for every $5 \leq n \leq 12$ and then that Phase 2 can be completed. However, when showing Phase 1 for $n = 8$, 10 and 12 we need to assume that both phases have been completed for smaller values of n. This is not a problem because the proof that Phase 2 can be completed for every k does not use the fact that Phase 1 can be completed. More precisely, if we denote by $P1(n)$ and $P(n)$ the statements that Phase 1 can be completed for n vertices and respectively that the game is a WIN for n vertices, then the logical sequence of our proofs will be as follows:

- Show that Phase 2 can be completed for all $2 \leq k \leq 5$.
- $P1(5)$
- $P1(6)$
- $P1(7)$
- $(P(6) \text{ and } P1(7)) \Longrightarrow (P1(8) \text{ or } P(8))$
- $P1(7) \Longrightarrow P1(9)$

- $(P\,(8)$ and $P1\,(9)) \Longrightarrow (P1\,(10)$ or $P\,(10))$
- $P1\,(9) \Longrightarrow P1\,(11)$
- $(P\,(10)$ and $P1\,(11)) \Longrightarrow (P1\,(12)$ or $P\,(12))$

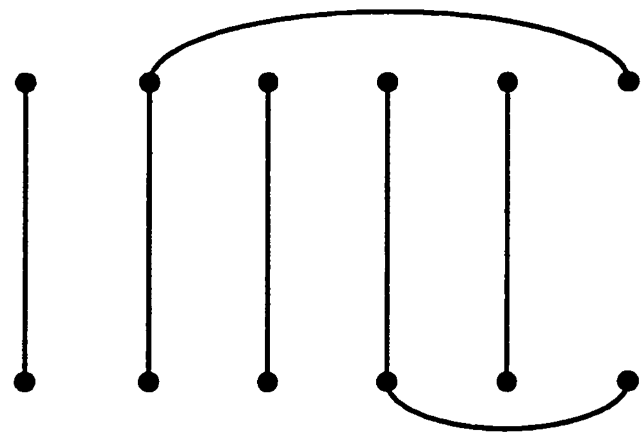


Figure 6: Phase 1 completed.

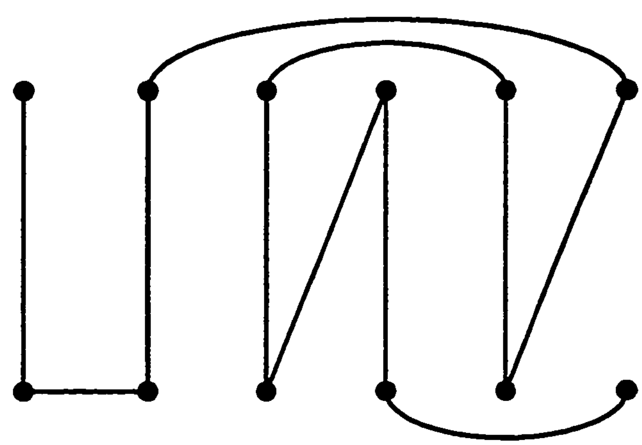


Figure 7 : Phase 2 completed

- Phase 1.

When $n = 5$, the maker can afford to make the first two moves anywhere as long as they are independent edges. After the breaker's reply, there are $(5-4)+1+1=3$

B-edges. On her third move, the maker will join, if possible, the M-isolated[4] vertex x to a vertex which is adjacent to a B-edge. This will kill the B-edge and, after the breaker's reply, there will be at most three active B-edges and the goal of Phase 1 will be achieved — there will be $k = 2$ paths covering all the vertices and at most $3 = 3\,(k-1)$ active B-edges. If it is not possible for the maker to join x as specified, then all the three B-edges are adjacent to x and she can join x to the only available vertex a, say. Now, the B-edge bx becomes dead, where b is the other endpoint of the M-edge from a (see Figure 8), and again, after the breaker's reply, there are at most three active B-edges, as required.

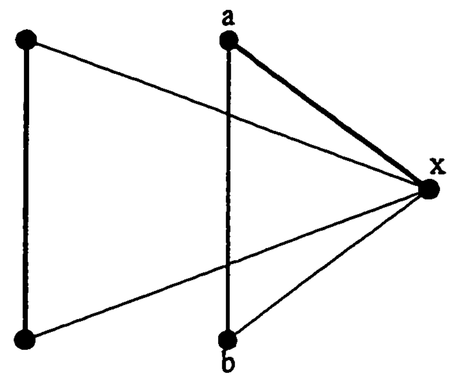


Figure 8

For $n = 6$, there are $6 - 4 = 2$ initial B-edges and, on her first two moves, the maker can create an M-path of length two and kill both initial B-edges (see Figure 9(a) for what M does if the initial B-edges have a common vertex, and Figure 9(b) for what to

[4]by M-isolated vertex in this context we mean a vertex that does not belong to any M-edge

do if they don't.)

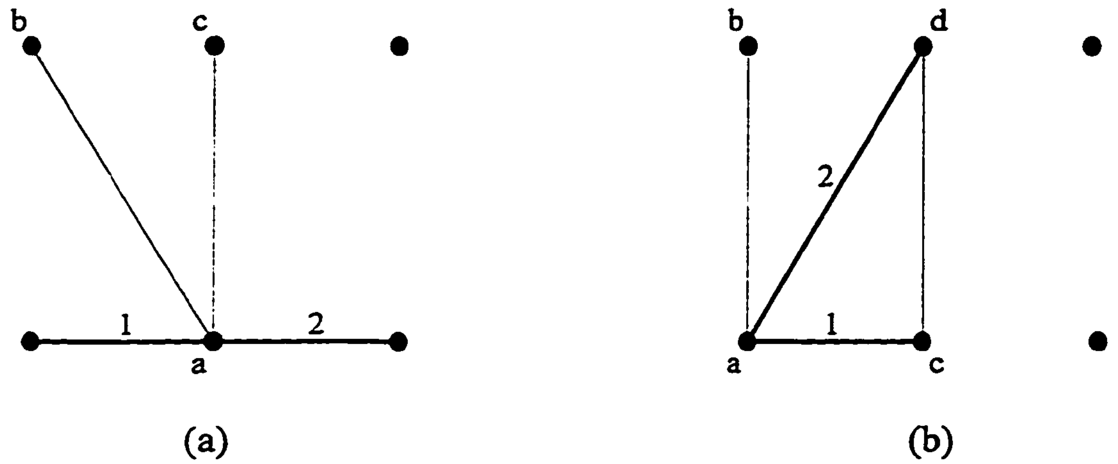


Figure 9

The maker can now regard the M-path as an M-edge (by eliminating the inner vertex) and, as there are at most two active B-edges (the two replies by the breaker), the situation is at least as good for the maker as the one for $n = 5$ after one move by each player has been made. Indeed, if the M-path in Figure 9 kills more than two B-edges then it is not necessary for her to kill a B-edge through her last move in Phase 1. As established before, the maker can achieve the goal of Phase 1 when $n = 5$ no matter what her first move is, so she can achieve the goal of Phase 1 when $n = 6$ as well.

For $n = 7$, we will show that the maker can make any first move as long as it is adjacent to some B-edge, and still win. There are $7 - 4 = 3$ initial B-edges so after such a move by the maker and a breaker's reply, there are at most three B-edges between the remaining five vertices (those not connected by the maker's first move). On the second move, the maker will mark an edge that is independent of the first one she marked using the following strategy. If there are exactly three B-edges among the five vertices not used by the maker yet, then the maker will do the following: if one of the five vertices is incident to all three B-edges then she will connect it to some other vertex; if not, then there must be a vertex connected to two of the B-edges (three edges among five vertices guarantee this) and the maker will connect it to something else. If there are two or fewer B-edges among these five vertices, then the maker's choice is easy: she will join the vertex which is part of both B-edges (if they are two and adjacent) to some other vertex, or she will take an edge that is adjacent to at least one B-edge otherwise. In each of these cases, after the maker's second move and the subsequent reply by the breaker, there are at most two B-edges connecting the three vertices not used by the maker. Therefore, the maker can join two of these vertices — the only restriction on this third move is that it be adjacent to a B-edge (note that this is clearly possible as, among the five B-edges present, at least one has to be connected to one of these three vertices.) After the breaker's reply, the situation is as in Figure 10 where the six B-edges are not displayed.

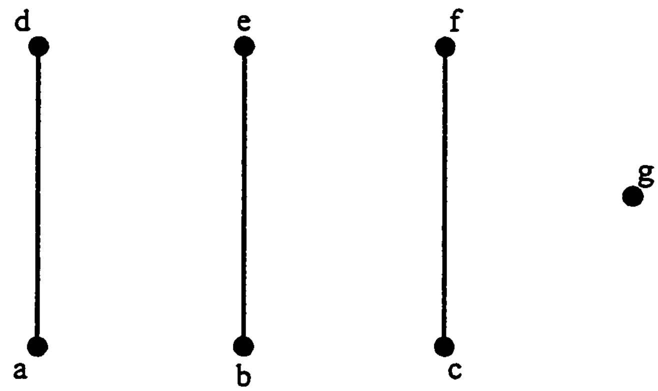


Figure 10

All the maker has to do now is join g to some other vertex and kill a B-edge at the same time. This would imply that, after the breaker's reply, there would be at most $6-1+1=6$ active B-edges and three M-paths hence the maker's Phase 1 goal would be achieved. We will show next how the maker can join g as described. We first discuss the possible B-degrees of g. First, g cannot have B-degree six because this would mean that, after the first move (for each player), g had already B-degree at least four and hence, it would have been joined by the maker on her second move. If g has B-degree five then, without loss of generality, g is connected to f by a B-edge but not to c. The maker will then connect g to c thus killing gf. We are left with the situation when g has B-degree at most four. Without loss of generality, let ab be a B-edge. Following the notation in Figure 10, the maker can kill ab by marking either ag or bg if they are not marked B yet. Suppose that ag and bg are already marked B. Then, if possible, the maker will kill ag (bg) by marking dg (respectively eg). If this is not possible, then dg and eg are already marked B too. In this situation, the sixth remaining B-edge is not joined to g (g has B-degree at most four) but it has to be joined to either c or f — otherwise the M-edge fc would not be adjacent to any B-edge which is a contradiction with the way the maker marked her edges. There is then a B-edge connecting c or f to one of a, b, d or e. Then the maker can mark either cg or fg to kill this B-edge.

For $n = 8$, we will reduce the problem to fewer vertices by using an argument similar to the one for $n = 6$. If all four initial B-edges are connected to the same vertex, then Lemma 28 applies and the game is a WIN as we already showed that the maker wins when $n = 6$. If there is no vertex of B-degree four but there is a vertex a of B-degree at least two then all the maker has to do is mark an edge that is adjacent to at least two B-edges at one end, and adjacent to at least one B-edge at the other end; on her second move, she connects the first mentioned end to some other vertex. She achieves a configuration as in Figure 11 by marking ad and af thus killing ab and ac. It is possible that dc is a B-edge instead of de but this will not change the argument. Only the relevant part of the graph is displayed in Figure 11 — the other B-edges can be anywhere.

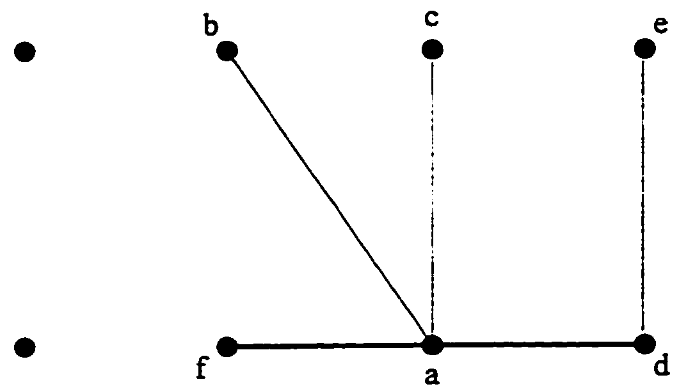


Figure 11

The maker can regard fd as her first move in a game on seven vertices (all but a) so that it is adjacent to at least one B-edge: de. Furthermore, the number of active B-edges is at most four (this accounts for the two replies of the breaker to ad and af). When the number of active B-edges is exactly four the situation is the same as what we had for $n = 7$ after one move by each player. As has already been established, the objective of Phase 1 is achieved here. If there are fewer than four B-edges or if the breaker uses the vertex a again, thus reducing the maximum number of active B-edges, then the maker's task is simpler as she doesn't have to kill a B-edge by her last move in Phase 1 anymore.

Finally, if there is no vertex of B-degree at least two, the initial position looks like that in Figure 12.

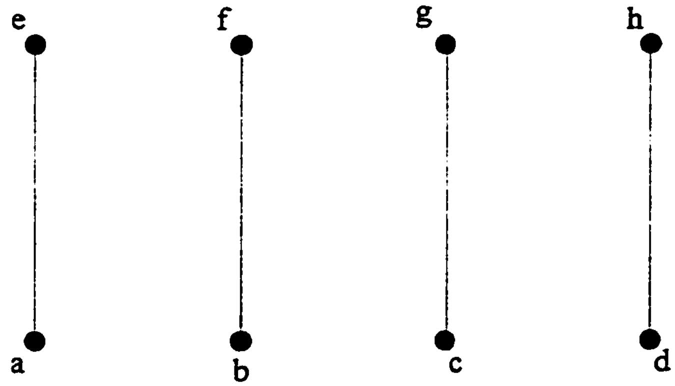


Figure 12

The maker's first move is ab. If, in reply, the breaker joins a or b to some other vertex, then the situation is equivalent with the case when the initial B-degree is at least two, so the maker wins. If the breaker replies by joining e or f to some other vertex different from a and b, for instance fg, then the maker joins af. Again, she can regard the path baf as a first move bf which is adjacent to the B-edge fg in a game on seven vertices (with a removed). This reduction to $n = 7$ is correct because there are two fewer B-edges (the B-edges ae and bf from the original graph got killed and the maker's first move in the reduced graph satisfies the requirement from case $n = 7$ to be adjacent to at least one B-edge.) The last possible situation occurs when the breaker replies by joining two of c, d, g and h. Let us assume his reply is cd. Then, the maker can mark ch on her second move and, applying Lemma 27 to the path ch and with specified B-edge cg, the problem reduces to $n = 7$ again (by collapsing c and h to x and deleting cg, there are two fewer B-edges and the situation is equivalent to the one occurring in a game on seven vertices with three initial B-edges after a move by the maker has been made). We have thus showed that, when $n = 8$, either Phase 1 can be completed, or the game is a WIN.

For $n = 9$ or 11 the argument is similar to the one for $n = 7$. We will show that the maker can make any first move as long as it is adjacent to a B-edge and still win. The strategy of the maker for the subsequent three (when $n = 9$), respectively four (when $n = 11$) moves, is to mark independent edges. She will make sure that, at every moment, her move is adjacent to at least one B-edge, if possible. By this, we simply mean that, if none of the vertices that the maker could use to mark a new independent edge are connected by at least a B-edge, then the maker can make any move as long as it is independent of her previous moves. More precisely, for $n = 9$ ($n = 11$), after the first M and B move, the seven (nine) vertices not yet used by M contain at most five (seven) B-edges.

Let us consider first the case when all these seven (nine) vertices contain five (seven) B-edges. Among these seven (nine) vertices either

- there is a vertex connected to at least three of the five (seven) B-edges, or
- there are two vertices not connected by a B-edge but connected through two B-edges (the first vertex) and through at least one B-edge (the second vertex) to other vertices.

In the first case the maker connects any such vertex to some other vertex (and such a vertex must be available). If the first case doesn't apply she connects the two vertices specified in the second case. The purpose is to ensure that after the breaker's second move, the five (seven) vertices the maker hasn't used contain at most three (five) B-edges, which is a situation that we saw before when $n = 7$ (and, respectively, just settled for $n = 9$). In conclusion, after four, respectively five (when $n = 11$) moves by each player, there is exactly one vertex x that has not been used by the maker yet, and there are 9 (respectively 12) B-edges.

In the case when, after the first M and B move, there are strictly less than five (seven) B-edges among the vertices not used by the maker, the task of connecting all but one vertex by independent edges is definitely easier — the maker has more choices for placing her marks in addition to all the choices she had before. Again, she will play adjacent to a B-edge whenever possible.

The maker will complete the goal of Phase 1 by joining x to some other vertex and killing some B-edge at the same time (four, respectively five M-paths would be formed this way and, after accounting for the breaker's reply, the number of active B-edges is at most 9, respectively 12, as required at the end of Phase 1). We will show next that this (join x and kill a B-edge) is possible. First, x cannot be joined by B-edges to every other vertex because, if that was the case, x must have had B-degree at least five (respectively six when $n = 11$) after the breaker's first move in the game, so the maker would have joined it on her second move. Let d be a vertex that has not been joined to x by the breaker (see Figure 13 for the case $n = 9$; the case $n = 11$ is similar).

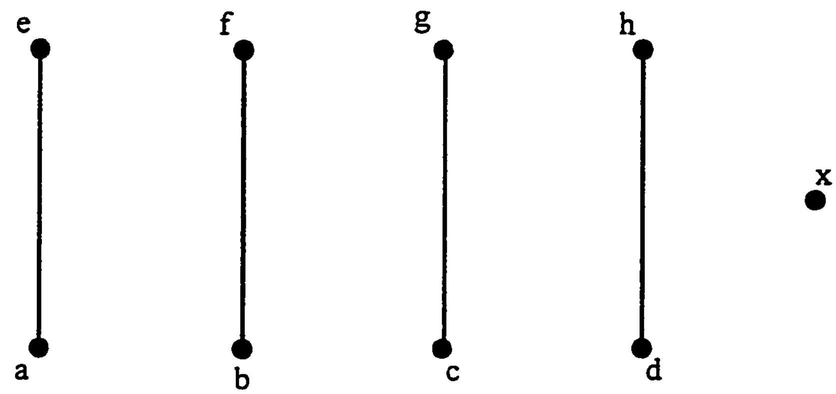


Figure 13

Then, if xh is marked by the breaker, the maker can kill it by marking xd so let us assume that xh is not marked by the breaker either. If h or d is joined by a B-edge to some other vertex then the maker can kill this edge by joining h or d to x. We are left with the case when h and d are not joined by a B-edge to anything else. The same argument can be made about each of the pairs a and e, b and f, c and g. In such a pair either both vertices are connected through B-edges to x or they are not part of any B-edge at all.

There are however nine B-edges in the graph which means that x has to be connected through B-edges to at least three of these pairs of vertices. This will imply that x has B-degree at least six and also that, among a, b, c, d, e, f, g and h, there are at least two vertices of B-degree zero (so, actually x will have B-degree exactly six). But then, the maker would have joined x on her third or fourth move instead of joining the pair of vertices of zero B-degree — this is because her strategy was to play adjacent to a B-edge whenever possible.

For $n = 10$ or 12, the reduction to the cases $n = 8$ or 9, respectively $n = 10$ or 11 is straightforward. Indeed, there are 6, respectively 8 initial B-edges whence, if there is a vertex of B-degree at least four, Lemma 28 reduces the problem to $n = 8$, respectively $n = 10$ as before. In the case when there is no vertex of B-degree at least four the maker's task is easier than when $n = 8$ because there always is a vertex of B-degree at least two — this means that a similar situation to the one in Figure 12, where all the vertices had B-degree one, cannot occur. In this case, if the maker can join a vertex of B-degree at least two to a vertex of B-degree at least one, the problem reduces to $n = 9$, respectively $n = 11$, by using the same construct as in Figure 11. There is only one situation when the maker cannot join a vertex of B-degree at least two to a vertex of B-degree at least one. This situation only occurs when $n = 10$ and the six initial B-edges form a complete graph on four vertices. Then, if a, b, c, d, e, f, g, h, i and j are the vertices, and ab, ac, ad, bc, bd and cd are the initial B-edges, the maker will mark ae on her first move. If, in reply, the breaker connects a to some other vertex then the B-degree of a becomes four and the problem reduces to $n = 8$ as before — on her second move, the maker will connect a to some other vertex. Otherwise, if the breaker doesn't connect a to some other vertex, then the maker can connect a to some other vertex x such that either e or x is one of the ends of B's reply. Here, the problem reduces again to $n = 9$ by using the same argument as the one for Figure 11.

- Phase 2.

We have to show that the maker can join the ends of the M-paths formed after Phase

1 and, hence, form an M-Hamiltonian path in G. The case-discussion will be upon the number k of M-paths at the beginning of Phase 2. In each case, we can assume the maximal number $3(k-1)$ of active B-edges (fewer active B-edges can only favor the maker). As the maker only joins endpoints of different paths, we can simplify the problem by replacing every M-path by an M-edge. We will only count active B-edges.

For $k = 2$, there are three B-edges, so the maker can join the two M-edges using the fourth (available) edge.

For $k = 3$, there are six B-edges hence the sum of the B-degrees of the vertices is twelve. We will show that the maker can join two vertices and kill four B-edges at the same time. After the breaker's reply, there will be two M-paths and three active B-edges, so the problem would reduce to the case when $k = 2$. We will start with the observation that if one of the vertices has B-degree three and all the others have non-zero B-degree, then the maker can join the vertex of degree three to the only available vertex, killing at least $3 + 1 = 4$ B-edges. Assume that there is a vertex of B-degree four and one of B-degree zero which must be the vertex joined by an M-edge to the vertex of B-degree four. If there is also a vertex of B-degree three, then the only possibility for the decreasing sequence of the B-degrees of the vertices is $(4, 3, 2, 2, 1, 0)$. In this case the game looks like that in Figure 14 and the maker can mark ce which kills bc, be, de and df.

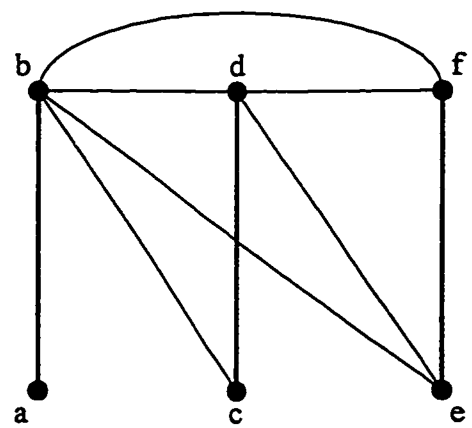


Figure 14

If there is a vertex of B-degree three, a vertex of B-degree zero and no vertex of B-degree four, then the possibilities for the B-degrees of the vertices are:

— $(3, 3, 2, 2, 2, 0)$; in this case, one of the vertices of degree two can be joined to one of the vertices of non-zero B-degree.

— $(3, 3, 3, 2, 1, 0)$; in this case, the vertex of B-degree one can be joined to one of the vertices of degree three — this is always possible because the vertex of B-degree one is incident to at most two edges: a B-edge and an M-edge.

— $(3, 3, 3, 3, 0, 0)$; this case is actually impossible because the four vertices of degree four would form a complete subgraph with all the edges marked B, which is a contradiction with the fact that there are three independent M-edges.

Now, if there is a vertex of B-degree four and no vertex of B-degree three, the possibilities for the B-degrees are:

— $(4, 2, 2, 2, 2, 0)$; in this case, two of the B-degree two vertices can be joined to each other.

— $(4, 2, 2, 2, 1, 1)$; in this case the game either looks like in Figure 15

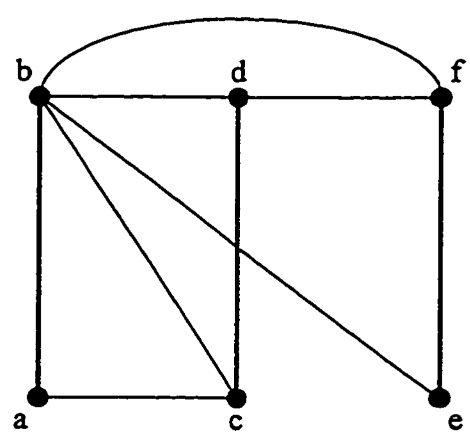


Figure 15

and the maker can mark af which kills ac, be, df and bf, or like in Figure 16

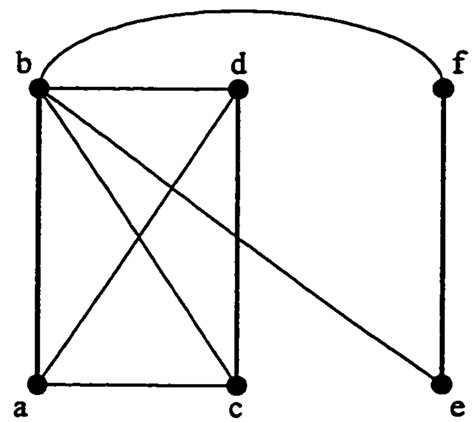


Figure 16

and the maker can mark ae which kills ac, ad, be and bf, or like in Figure 17

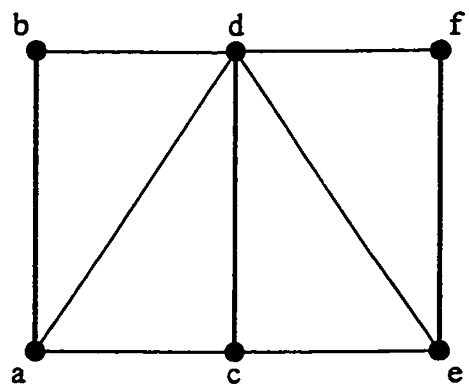


Figure 17

and the maker can mark bc which kills ac, ce, ad and bd.

Finally, if there are no vertices of B-degree three or four, then each vertex must have B-degree two so, clearly, the maker can join two of them killing four B-edges.

For $k = 4$, there are nine B-edges hence the sum of the B-degrees of the vertices is 18. Again, we will show that the maker can join two vertices and kill four B-edges at the same time. After the breaker's reply, there will be three M-paths and six B-edges, so the problem would reduce to the case when $k = 3$. If one of the vertices has B-degree four or five, we are done because the maker can join this vertex to some other and this will kill at least four B-edges; so assume there are no such vertices. Similarly, assume that if a vertex has B-degree three, then at least three other vertices have B-degree zero (otherwise, the maker can always join the vertex of degree three to a vertex of degree at least one). But this case is obviously impossible. It must be then the case that there is a vertex of B-degree six. On the other hand, there cannot be more than one vertex of degree six (otherwise two vertices of B-degree six would account for at least eleven B-edges, and we only have nine). Therefore, the only possibilities for the B-degrees are $(6, 2, 2, 2, 2, 2, 2, 0)$ and $(6, 2, 2, 2, 2, 2, 1, 1)$. In both cases, the maker can join two vertices of degree two, killing four B-edges.

For $k = 5$, there are twelve B-edges and the sum of the B-degrees is 24. Again, we will show that the maker can join two vertices and kill four B-edges at the same time. After the breaker's reply, there will be four M-paths and nine (active) B-edges, so the problem would reduce to the case when $k = 4$. As before, when there is a vertex of B-degree 7, 6, 5 or 4, the maker can easily kill at least four B-edges. Assume then that there are no vertices of B-degree 7, 6, 5 or 4. If there is a vertex of B-degree three, then, unless there are at least five vertices of B-degree zero, the maker can again join the vertex of degree three to a vertex of non-zero B-degree and kill at least four edges. As it is impossible to have five vertices of B-degree zero and at the same time no vertices of B-degree 7, 6, 5 or 4, the only possible values for the B-degrees left to consider are 8, 2, 1 or 0. The only possible configurations are $(8, 2, 2, 2, 2, 2, 2, 2, 2, 0)$ and $(8, 2, 2, 2, 2, 2, 2, 2, 1, 1)$. In both cases, the maker can join two vertices of degree two, killing again four B-edges. This completes the proof of Theorem 32. □

Theorem 32 only leaves open the games of type $\langle n, n-3 \rangle$ and $\langle n, n-2 \rangle$. This is the case because a game of type $\langle n, k \rangle$ with $k \geq n-1$ is a LOSS when one vertex is connected (by initial B-edges) to everything else. All we can tell is that for $n = 5$ and $n = 6$, any game of type $\langle n, n-3 \rangle$ or $\langle n, n-2 \rangle$ is a LOSS so the theorem is sharp here. More precisely, the following problem is still open:

Problem *Show that there exist n_1 and n_2 such that every game of type $\langle n, n-3 \rangle$ is a WIN for $n \geq n_1$, and every game of type $\langle n, n-2 \rangle$ is a WIN for $n \geq n_2$. Furthermore, what are the smallest n_1 and n_2 with this property?*

3.4 The Hamiltonian Cycle Game

This section is devoted to the Hamiltonian cycle game. A. Papaioannou [Pap82] proposed a conjecture that can be reformulated as follows:

Conjecture 33 *The Hamiltonian cycle game on K_n is a WIN for $n > 7$.*

The next theorem shows that this is the case when $n > 13$. The problem of finding the minimal value of n for which the Hamiltonian cycle game is a WIN seems to be significantly harder than the similar problem for Hamiltonian paths, and the answer to it is not known yet. The author of this manuscript failed in the attempt to analyze it by computer — an exhaustive search is impossible due to the large number of possibilities; also, no computationally efficient algorithm for deciding if two partially-coloured graphs are isomorphic is known and therefore no significant simplifications seem to be possible. All we can tell is that the minimum value of n for which the game is a WIN is at least six — the breaker can win a Hamiltonian game on five vertices. Indeed, if $n = 5$ then the maker's only chance is to win in five moves (because five is half the total number of edges in K_5.) But this is impossible because the breaker can close the path that the maker created with her fourth move.

Theorem 34 *The Hamiltonian cycle game on K_n is a WIN for $n \geq 14$. Furthermore, the maker can win in at most $n + 4$ moves.*

Note that making a Hamiltonian cycle requires n M-edges. However, a win is not possible in n moves because, on his $(n-1)$st move, the breaker can join the endpoints of the Hamiltonian path formed by the maker after her $(n-1)$st move.

Proof: We will start with an outline of the maker's winning strategy.

In a first phase the maker will construct a Hamiltonian path in a minimal number of moves ($n-1$ moves). This is possible because of Theorem 32. Furthermore, we will show that the maker can form this path so that neither of its ends has B-degree $n-2$ (this requirement is clearly necessary because a vertex of B-degree $n-2$ cannot be part of a maker's Hamiltonian cycle). Then, if $(0, 1, ..., n-1)$ is the path, the maker will mark an edge $0i$ (where $2 \leq i \leq n-2$) such that $i-1$ has "very small" B-degree and, on her next move, the edge $j\,(n-1)$ (where $1 \leq j \leq n-3$), such that $j+1$ has "very small" B-degree too (see Figure 18).

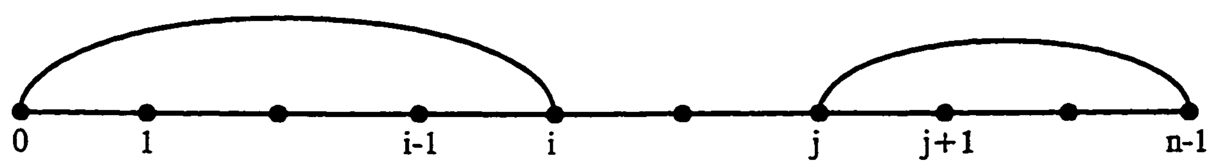


Figure 18

Then, if say $i < j$, the new Hamiltonian path

$$(i-1, i-2, ..., 0, i, i+1, ..., j, n-1, n-2, ..., j+1)$$

has endpoints of small B−degree. Then, using this path, the maker can construct a Hamiltonian cycle in at most three moves. Next, we will specify how small "very small" is, and we will prove that the maker can indeed follow this strategy to win. For the purposes of this proof, we will denote the $B-$degree of a vertex a by $b(a)$. We assume without loss of generality that $b(0) \geq b(n-1)$. We will show that the maker can join the vertex 0 to some interior vertex i such that $b(i-1) \leq 3$.

First, we will show how the maker will make sure that $b(0)$ is different from $n-2$ and $n-1$. The maker can prevent a vertex of B-degree at least $n-2$ by constructing the Hamiltonian path in the following way. Throughout her first four moves she constructs a path of length four. Corresponding to these four moves by the maker, there are four replies by the breaker. If the highest B-degree of any vertex is at most two at this moment, then all the maker has to do is to complete the path of length four to a Hamiltonian path; this way she limits the highest B-degree of any vertex to at most $2+(n-5)=n-3$ and so $b(0)$ cannot be $n-2$ or $n-1$. The maker can indeed complete the path of length four to a Hamiltonian path because, according to Theorem 32, the game obtained by collapsing the path of length four (with no specified B-edge) is a (Hamiltonian-path) WIN, being a game on $n-4$ vertices with 4 initial B-edges (recall that $n \geq 14$ but we only need $n-4 \geq 8$ here.) This means that the non-collapsed game is a (Hamiltonian-path) WIN by Lemma 27. If the highest B-degree after the M-path of length four is formed is at least three, then the maker will do the following. In the more general case when the vertex x of B-degree at least three is not part of the M-path, the maker creates through her next two moves a path of length two which is independent of the path of length four. This will be done such that x is the middle of the new path. By collapsing the game twice (first, corresponding to the path of length four and, next, corresponding to the path of length two), the resulting game has $n-4-2=n-6$ vertices and at most $4+2-3=3$ B-edges. As $n \geq 14$ (we only need $n \geq 13$ here), the collapsed game is a (Hamiltonian path) WIN, and therefore, the maker can achieve a Hamiltonian path of length $n-1$ in the (original) non-collapsed game. Furthermore, when the Hamiltonian path is completed, neither of its endpoints can have B-degree $n-2$ or more because an interior vertex of the path (x) has degree at least three. The task of the maker is simpler if x is either an endpoint of the path of length four, or an interior vertex of the path of length four. If x is an endpoint, then the maker connects x to some other vertex to form a path of length five which, again, collapses to a (Hamiltonian path) WIN. If x is an interior vertex then the path of length four can be immediately collapsed to get the desired result. In fact, it is easy to show that the maker can force a lower B-degree for the ends of the path but we do not need it.

If $b(0) = n-3$, then the maker can join 0 to the only available vertex i and, after the breaker's reply,

$$b(i-1) \leq (n-1) - (n-3) + 1 = 3.$$

Note that, if $i = n-1$ then the maker instantly wins the game by closing the Hamiltonian path into a cycle.

If $b(0) = n-4$, then there are two vertices i_1 and i_2 such that $0i_1$ and $0i_2$ are not marked yet. The sum of the B-degrees of $i_1 - 1$ and $i_2 - 1$ is at most six (the maximum value for the sum of their B-degrees is reached when they are both connected to 0 by B-edges, there is a B-edge between them and the remaining two B-edges are also connected to $i_1 - 1$ or $i_2 - 1$). By the pigeon hole principle, at least one of them has B-degree at most three.

If $b(0) = n-5$, then there are three vertices i_1, i_2 and i_3 such that $0i_1$, $0i_2$ and $0i_3$ are not marked yet. By a similar argument, the sum of the B-degrees of $i_1 - 1$, $i_2 - 1$ and $i_3 - 1$ is at most ten so, again, at least one of them has B-degree at most three.

If $b(0) = n-6$, then there are four vertices i_1, i_2, i_3 and i_4 such that $0i_1$, $0i_2$, $0i_3$ and $0i_4$ are not marked yet. The sum of the B-degrees of $i_1 - 1$, $i_2 - 1$, $i_3 - 1$ and $i_4 - 1$ is at most fourteen so at least one of them has B-degree at most three.

If $b(0) \leq n-7$, there are $k = n - b(0) - 2 \geq 5$ vertices different from 0, 1 and $n-1$ that haven't been joined to 0 yet (we assumed here that $0\, n-1$ is marked B — otherwise the maker wins instantly). Denote these vertices by $i_1, i_2, ..., i_k$. Then,

$$b(i_1 - 1) + b(i_2 - 1) + \cdots + b(i_k - 1) \leq k + 2(n - 1 - b(0)) \tag{1}$$

(the term k in relation (1) accounts for the situation when $i_1 - 1, i_2 - 1, ..., i_k - 1$ are all connected to 0, while $n - 1 - b(0)$ accounts for the situation when all the B-edges that are not adjacent to 0 are determined by vertices from $\{i_1 - 1, i_2 - 1, ..., i_k - 1\}$). From (1) it follows that there must exist some i_s, $1 \leq s \leq k$ such that

$$b(i_s - 1) \leq \frac{k + 2(n - 1 - b(0))}{k} = 3 + \frac{2}{k} \tag{2}$$

As $k \geq 5$, it follows that $b(i_s - 1) \leq 3$ so, in any case, the maker can join 0 to i so that $b(i - 1) \leq 3$. So, the maker will join 0 to i and after the breaker's reply, the new path

$$(i - 1, i - 2, ..., 0, i, i + 1, i + 2, ..., n - 1)$$

will have the left-most vertex $i - 1$ of degree at most four.

Now, the maker can repeat the same procedure for the right-most vertex $n - 1$. More precisely, she will join $n - 1$ to a vertex j such that $j + 1$ has B-degree at most three. With one exception, we have that $b(n - 1) \leq n - 7$ — this follows from the assumption that $b(0) \geq b(n - 1)$ and the fact that there are n B-edges present with $n \geq 14$. The exception occurs for $n = 14$ when, at the moment the first Hamiltonian path is completed by the maker (and after the 13th reply by the breaker), $b(0) = b(13) = 7$, which means that, after the new path is created by the maker, we could have $b(13) = 8$. We will consider the exceptional case separately later. If $b(n - 1) \leq n - 7$, an argument similar to the one for $b(0) \leq n - 7$ applies. The only difference is that there is an extra B-edge — the breaker's reply to the last move $(0i)$ of the maker. The analogues of (1) and (2) are

$$b(j_1 + 1) + b(j_2 + 1) + \cdots + b(j_k + 1) \leq k + 2(n - b(n - 1)) \tag{1'}$$

and

$$b(j_s + 1) \leq \frac{k + 2(n - b(n - 1))}{k} = 3 + \frac{4}{k}, \tag{2'}$$

where $j_1, ..., j_k$ are the vertices different from $n-1$, $n-2$ or $i-1$ which are not yet joined to $n-1$ and $k = n - b(n-1) - 2 \geq 5$. Thus, $b(j_s + 1) \leq 3$. In the exceptional case we have that $b(0) = 7$ or 8, and $b(13) = 8$ at the time the B-degree of 13 has to be reduced. As $k = 4$, the analogue of (1') is

$$b(j_1 + 1) + b(j_2 + 1) + b(j_3 + 1) + b(j_4 + 1) \leq 16. \tag{1"}$$

This means that, unless

$$b(j_1 + 1) = b(j_2 + 1) = b(j_3 + 1) = b(j_4 + 1) = 4,$$

there is $1 \leq s \leq 4$ such that $b(j_s + 1) \leq 3$. But it is impossible to have all these four B-degrees equal to four as this implies that there is a vertex of B-degree at least seven, a vertex of B-degree eight and four vertices of B-degree four, which would bring the total number of B-edges to at least 16, a contradiction. In conclusion, the B-degree of $j_s + 1$ was reduced to at most three.

After the breaker's $(n+1)$th move, and after a convenient permutation of the names of the vertices, a new $M-$path $(0, 1, ..., n-1)$ is formed such that

$$0\ n-1 \text{ is a } B\text{-edge, } b(0) \leq 5, \text{ and } b(n-1) \leq 4. \tag{3}$$

Note that if $0\ n-1$ is not a B-edge then the maker can instantly make a Hamiltonian cycle by marking it herself.

Let C be the set of pairs of consecutive vertices $(j, j+1)$ such that $0 < j < n-2$ and neither $0\ j+1$ nor $j\ n-1$ has been marked by the breaker. We will call such a pair a *couple*. Each couple gives a choice of two forcing moves for the maker: if the maker marks $0\ j+1$ or $j\ n-1$ then the breaker will have to mark the other one (otherwise the maker will mark it next and form a Hamiltonian cycle). Note that the forcing moves corresponding to one couple are distinct from the forcing moves corresponding to any other couple. This means that the total number m of couples in C is at least

$$m \geq n - 3 - (b(0) + b(n-1) - 2). \tag{4}$$

Here, $n - 3$ is the total number of potential couples and $b(0) + b(n-1) - 2$ is the maximum possible number of pairs which are not couples due to B-edges from 0 or $n-1$; we subtracted two from the sum of the B-degrees of 0 and $n-1$ in order to account for the B-edge $0\ n-1$ which cannot be required by a couple and it was counted twice in $b(0) + b(n-1)$. Combining (3) and (4), we obtain the inequality

$$m \geq n - 10. \tag{5}$$

Consider, now, the following set of edges:

$$Q = \{i-1\ j : (i-1, i) \in C, (j-1, j) \in C, i < j\}.$$

The number of elements of Q is $|Q| = \binom{m}{2}$. It follows from (4) that

$$|Q| \geq \binom{n-1-b(0)-b(n-1)}{2}. \tag{6}$$

Our aim is to show that the set $Q \setminus B(G)$ is not empty. An upper bound for the size of $Q \cap B(G)$ is

$$|Q \cap B(G)| \leq (n+1) - (b(0) + b(n-1) - 1) = n + 2 - b(0) - b(n-1) \tag{7}$$

(here $n+1$ is the total number of B-edges and $b(0) + b(n-1) - 1$ is the total number of B-edges which are adjacent to 0 or $n-1$ and therefore not in Q). Combining (6) and (7) gives the following sufficient condition for $Q \setminus B(G)$ to be nonempty:

$$\binom{n-1-b(0)-b(n-1)}{2} > n + 2 - b(0) - b(n-1). \tag{8}$$

By rearranging the terms, we find that (8) is equivalent to the following:

$$(n - b(0) - b(n-1))(n - b(0) - b(n-1) - 5) > 2. \tag{9}$$

For $n > 14$ this condition is satisfied (it follows from (3)). This means that, for $n > 14$, there is an edge $i-1\ j$ in Q which is not marked by the breaker. Then, the maker can construct the Hamiltonian cycle in three more moves as follows (see Figure 19):

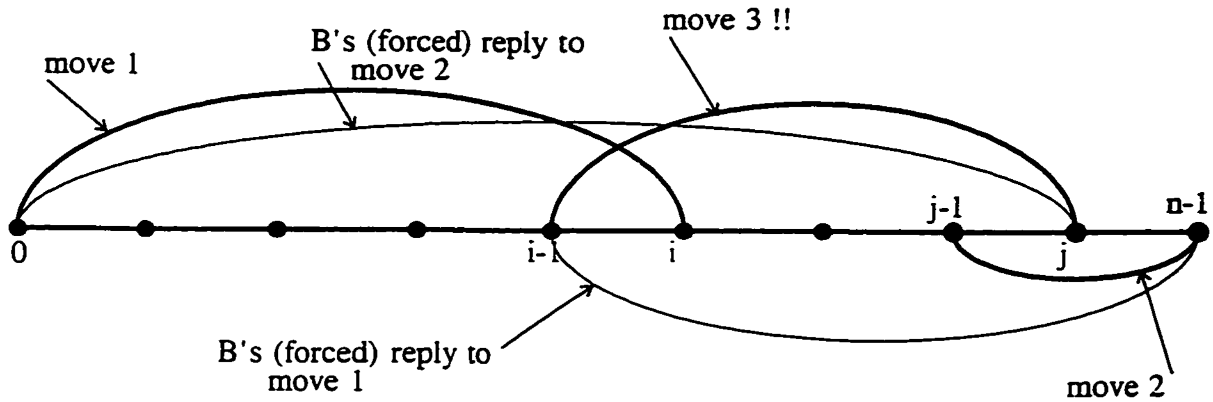


Figure 19

— first, mark the edge $0i$. As $(i-1, i)$ is a couple, the breaker is forced to reply by marking $i-1\ n-1$ because, otherwise the maker will mark it and create the cycle

$$(i-1, i-2, \ldots, 0, i, i+1, i+2, \ldots, n-1, i-1).$$

— next, the maker will mark the edge $j-1\ n-1$, and this time, the breaker will have to respond with $0j$ because, otherwise, the maker will create the cycle

$$(j-1, j-2, \ldots, 0, j, j+1, j+2, \ldots, n-1, j-1).$$

— finally, the maker's last move will be $i-1\ j$ which will complete the cycle

$$(i-1, i-2, ..., 0, i, i+1, i+2, ..., j-1, n-1, n-2, ..., j, i-1)$$

and the maker wins.

We have to consider now separately the case when $n = 14$. It follows from (5) that there are at least four couples. We denote them by $(a, a+1)$, $(b, b+1)$, $(c, c+1)$ and $(d, d+1)$ and we have $1 \leq a < b < c < d \leq 11$. We will show that the maker can win in three moves. Her first two moves will induce forced replies by the breaker and the third move will complete a Hamiltonian cycle. Here is an example of a potential three-move winning sequence for the maker: first, she marks $0\ a+1$; the breaker must mark $a\ 13$ in reply; next, she marks $b\ 13$; the breaker must reply with $0\ b+1$; finally, if the edge $a\ b+1$ is not taken, the maker will mark it and form a Hamiltonian path. The only problem here is that the edge $a\ b+1$ may be already marked by the breaker. There are 14 B-edges that connect to at least one interior node of the M-path (all except 0 13). The idea of the proof is to describe sufficiently many three-move sequences for the maker, so that the set containing the third edge of every sequence has at least 15 elements. This will ensure that at least one of them has not been taken by the breaker and, therefore, the maker can win using the corresponding three-move sequence. The fact that the first two moves in each sequence are not yet marked by the breaker follows from the fact that $(a, a+1)$, $(b, b+1)$, $(c, c+1)$ and $(d, d+1)$ are couples. We can easily identify the following 18 potential winning sequences by considering the three (similarly constructed) sequences for each pair of couples:

1. $0\ a+1 \longrightarrow b\ 13 \longrightarrow a\ b+1$

2. $0\ b+1 \longrightarrow a\ 13 \longrightarrow a-1\ b$

3. $0\ b+1 \longrightarrow a\ 13 \longrightarrow a+1\ b+2$

4. $0\ a+1 \longrightarrow c\ 13 \longrightarrow a\ c+1$

5. $0\ c+1 \longrightarrow a\ 13 \longrightarrow a-1\ c$

6. $0\ c+1 \longrightarrow a\ 13 \longrightarrow a+1\ c+2$

7. $0\ a+1 \longrightarrow d\ 13 \longrightarrow a\ d+1$

8. $0\ d+1 \longrightarrow a\ 13 \longrightarrow a-1\ d$

9. $0\ d+1 \longrightarrow a\ 13 \longrightarrow a+1\ d+2$

10. $0\ b+1 \longrightarrow c\ 13 \longrightarrow b\ c+1$

11. $0\ c+1 \longrightarrow b\ 13 \longrightarrow b-1\ c$

12. $0\ c+1 \longrightarrow b\ 13 \longrightarrow b+1\ c+2$

13. $0\ b+1 \longrightarrow d\ 13 \longrightarrow b\ d+1$

14. $0\ d+1 \longrightarrow b\ 13 \longrightarrow b-1\ d$

15. $0\ d+1 \longrightarrow b\ 13 \longrightarrow b+1\ d+2$

16. $0\ c+1 \longrightarrow d\ 13 \longrightarrow c\ d+1$

17. $0\ d+1 \longrightarrow c\ 13 \longrightarrow c-1\ d$

18. $0\ d+1 \longrightarrow c\ 13 \longrightarrow c+1\ d+2$

It is easy to see that the only situation when the number of mutually distinct final edges in these sequences is less than 15 occurs when a, b, c and d are consecutive integers. For example, if $a < b < c$ are consecutive integers and $c < d$ are not consecutive, then the only possible duplications are $(a, b+1) = (b-1, c)$, $(a+1, b+2) = (b, c+1)$ and $(a+1, c+2) = (c-1, d)$. We will consider next the situation when a, b, c and d are consecutive integers. We can also assume without loss of generality that there are exactly four couples (a higher number of couples can only help the maker to achieve her goal). If $a = 1$, then the following is a three-move winning sequence for the maker

$$04 \longrightarrow 03 \longrightarrow 1\ 13$$

(the edges 04, 03 and 1 13 are not marked because $(3,4)$, $(2,3)$ and respectively $(1,2)$ are couples). Similarly, if $d = 11$, then the following is a three-move winning sequence for the maker

$$9\ 13 \longrightarrow 10\ 13 \longrightarrow 0\ 12.$$

Now, if $a > 1$ and $d < 11$, then none of the final edges in the above 18 sequences connects to one of the ends (0 or 13) of the Hamiltonian M-path. But there are only $15 - b(0) - b(13) + 1$ interior B-edges and 12 mutually distinct final edges in the 18 sequences. As there are exactly four couples, we get from (4) that the number of interior B-edges is at most 7 hence there are at least $12 - 7 = 5$ winning sequences for the maker. □

We will conclude this chapter by mentioning three problems related to the Hamiltonian cycle game that are still left open.

Problem 1. What is the smallest value n_0 of n for which the Hamiltonian cycle game on n vertices is a WIN? (All we can tell at this moment is that $6 \leq n_0 \leq 14$.)

Problem 2. How many initial B-edges can we allow so that the maker still wins? It is expected that the maximum number of initial B-edges will increase with n but the author could not establish a similar result to the one for Hamiltonian paths where we could allow $n - 4$ initial B-edges. For $n = 14$ for instance, the proof of Theorem 34 could not be modified to account for even one initial B-edge. The situation may be different for higher values of n but for each value of n a separate estimation would be necessary. Note that this the best estimate one could hope for is $n - 4$ because otherwise, if there are at least $n - 3$ initial B-edges all adjacent to the same vertex, then the maker has no time to connect that vertex twice by edges in her colour.

Problem 3. What is the minimal number $m(n)$ of moves necessary for the maker in order to win the Hamiltonian cycle game on n vertices? (all we can tell in this case is that $n+1 \leq m(n) \leq n+4$.)

Chapter 4

Games In The Normal Play Convention

4.1 Preliminaries

The term *combinatorial game* usually refers to the special breed of perfect information games introduced by J. H. Conway in [Con72] and extensively developed later in [Con76] and [BCG82]. It came as an extension of the Nim theory of impartial games which was discovered independently by R. P. Sprague [Spr35] and P. M. Grundy [Gru39]. It is the object of the next three sections to present the main body of this theory for combinatorial games with finitely many options. The two books referenced above, "*Winning Ways*" by E. R. Berlekamp, J. H. Conway and R. K. Guy and "*On Numbers and Games*" by J. H. Conway are the standard texts in the field and will be referred to as [WW] and [ONAG] respectively throughout the rest of this manuscript.

The mathematical object of a combinatorial game has been found to be an effective model for solving games satisfying the following requirements [WW, page 16]:

1. There are two players, called Left and Right.

2. There are finitely[1] many positions, and often a particular starting position.

3. There are clearly defined rules that specify the moves that either player can make from a given position to its options.

4. Left and Right move alternately.

5. In the normal play convention a player unable to move loses.

6. The rules are such that play will always come to an end because some player will be unable to move.

7. Both players know what is going on, i.e. there is complete information.

8. There are no chance moves such as rolling dice or shuffling cards.

At first sight, Rule 5 seems to be the most restrictive one. For instance, the Ramsey-type games in the second chapter fail here. Certain games, however, can easily be reformulated in order to comply with Rule 5. Games that satisfy these conditions can be shown to have a value. These values are sometimes numbers but not necessarily. We adopt the convention that Left is the "positive" player and Right is the "negative" player. Any game whose end result is a non-zero integer score (the winner is Left if the score is positive, respectively Right if negative) can be artificially continued by allowing abstract moves consisting of, decreasing (by Left if it is Left's turn to move) and respectively increasing (by Right if it is Right's turn to move) the score by one. The only restriction is that the result of such a move is non-negative after a Left move, and non-positive after a Right move. If the normal play convention (Rule 5) is now applied, the outcome is clearly the same as for the original game. A good example of a game where the winner is decided by an integer score is *Go* (see [BW94] for a comprehensive treatment of Go from a mathematical perspective.) There are two main scoring methods in Go: the

[1]In [Con72] Conway considers the general case when there are infinitely many options; many of the results in this chapter (including the uniqueness of canonical forms) are not true in this general setting.

Japanese and the Chinese. Essentially, the Chinese scoring method is similar to Rule 5 — the players have to fill their own territories until one player runs out of space and loses. Some exceptional cases aside, the Japanese scoring method (where players stop playing once their territories are established and count an integer score representing the difference between the sizes of their territories minus the difference between the captives on each side) yields the same outcome as the Chinese scoring method.

The next game characteristic is not a formal requirement for the theory to work — it is rather an ideal situation where Conway's theory of combinatorial games is in a certain sense optimal (see Section 4.5).

9. There is a general tendency for any position to split into components with the property that any subsequent move will not affect more than one component.

An extreme example when Rule 9 is satisfied, is the game of Nim ([Bou02]; also, see Section 4.5) where all the possible positions (including the starting position) are unions of disjoint components which are single nim-heaps.

Göran Andersson's game of *Domineering* [WW, page 115] a.k.a. Dominoes [ONAG, page 74] or Cross-Cram [Gar74] will be used as a test game to exemplify and apply the theoretical results in this chapter. Domineering is played by two players on a board that can be any subset of the unit squares of a sheet of graph paper. On each turn, Left places a vertical domino on the board (a 2×1 piece that occupies exactly two squares); on his turn, Right places a horizontal domino (a 1×2 piece that occupies exactly two squares.) Dominoes are not allowed to overlap. The game ends when someone is unable to move; the player who makes the last move wins. Figure 1 is an example of how a Domineering game is played.

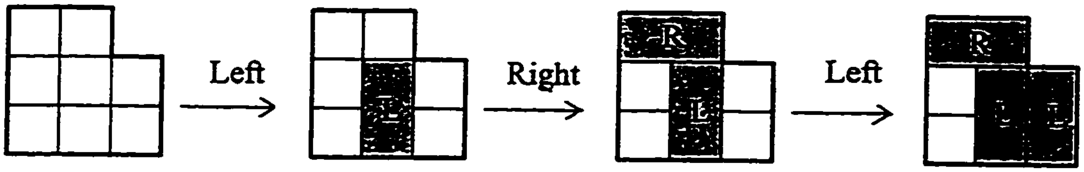


Figure 1. Left wins because Right cannot move.

4.2 Games In General

Definition 35 *A game G is defined as any ordered pair of finite sets L and R (possibly empty) whose elements are games. A member of L is called a **Left option**, and a member of R is called a **Right option**. The collection of all such games is denoted by Γ.*

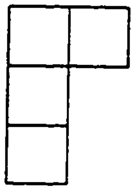
We will usually informally use the generic notation $G = \left\{G^L | G^R\right\}$ where G^L is a typical member of L and G^R is a typical member of R.

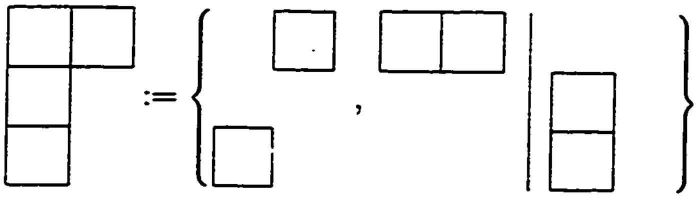
Note that this recursive definition doesn't require a base. Most of the definitions to follow are of the same recursive nature. There is precisely one game whose definition doesn't depend on any other game. This is the game with empty sets of Left and Right options. We denote this game by **zero**:

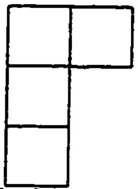
$$0 := \{ \mid \} .$$

In the language of Domineering, the following is an incarnation of zero:

$$0 = \square .$$

Neither player can place a domino on a 1×1 board so the sets of Left and Right options are empty. According to Definition 35, the Domineering game on the board  can be described as follows:

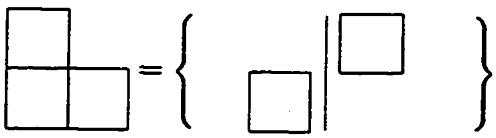
$G =$  $:= \{0, \{\ |0\}|\{0|\ \}\}$

A Left option in  is generated by the action of Left placing a vertical domino on the board and thus reducing the playable zone to a smaller board. Formally, these smaller boards *are* the Left options. Similarly, the results of Right's moves are Right options. In general, any game G (in the sense of Definition 35) is played like this: after deciding on who starts, say if Right has the honor, Right will pick an option G^R which will be the current position. Then, Left will pick an option G^{RL} of G^R and this will be the new current position; then Right follows again, and so on. Eventually, the player who will have no option to pick when his or her turn comes loses.

Definition 36 *The following game is called Star:*

$$* := \{0|0\}$$

For example, the following Domineering position is star:

 $= \{0|0\} = *$

Also, the nim-heap of size one is $*$.

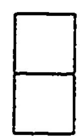
Most of the inductive definitions to follow are based on a ranking function on Γ, called the **birthday**. The birthday of a game G is a non-negative integer $b(G)$, defined recursively as

$$b(G) = \min\{n \in \mathbf{Z} : n \geq 0 \text{ and } n > b\left(G^L\right) \text{ and } n > b\left(G^R\right)$$
$$\text{for all Left and Right options } G^L, G^R\}$$

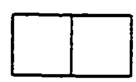
For example, $b(0) = 0$ and $b(*) = 1$. The only other two games with birthday one are

$$1 := \{0|\ \} \qquad \text{and} \qquad -1 := \{\ |0\}.$$

The following Domineering boards have values 1 and −1 respectively:

 $= 1$

and

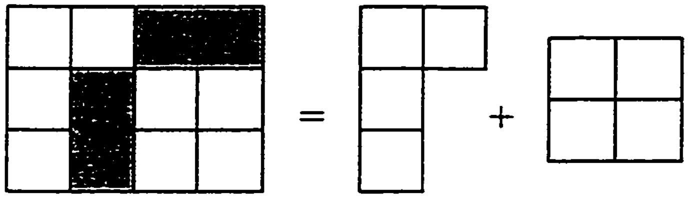
 $= -1.$

At the end of Section 4.5 we present a collection of domineering boards whose values are games of birthday two.

Definition 37 *The sum of two games* $G = \left\{G^L|G^R\right\}$ *and* $H = \left\{H^L|H^R\right\}$ *is the game*

$$G + H := \{G^L + H, G + H^L | G^R + H, G + H^R\}.$$

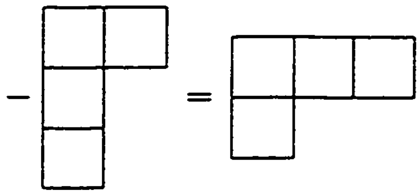
We stress that G and H may have multiple Left and/or Right options and $\left\{G^L|G^R\right\}$ is a generic notation for $\left\{G^{L_1}, G^{L_2}, ...|G^{R_1}, G^{R_2}, ...\right\}$. Similarly, $G^L + H$ is short for "$G^{L_1} + H, G^{L_2} + H, ...$". In other words, Left can either choose to make a move G^L in G and leave H unchanged, and the resulting position is $G^L + H$, or choose to make a move H^L in H and leave G unchanged (and the resulting position is $G + H^L$). Right has similar options. According to Definition 37, if the players play two games at once, a legal move being to choose a component and make a move in it until no moves are possible in either of the components, we say that they play the game $G + H$. For example, a disconnected Domineering board as the one below equals the sum of its components.



Definition 38 *The negative of a game* $G = \left\{G^L | G^R\right\}$ *is defined recursively as:*

$$-G := \left\{-G^R | -G^L\right\}.$$

For example

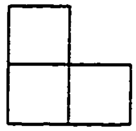


The significance of the negative of a game is that of "turning the boards around" between the players — the moves available for Left (Right) in the negative game are the same as the ones available for Right (Left) in the initial game. We remind the reader that the notation $\left\{G^L | G^R\right\}$ is informal in the sense that G^L and G^R are generic Left and respectively Right options.

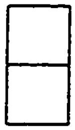
The following relation will express Left's preferences between games. As it will be shown later, if $G \leq H$, then Left will always rather play H than G.

Definition 39 *Given two games* G *and* H*, we say that* $G \leq H$ *(we will also write* $H \geq G$*) if there is no Left option* G^L *such that* $H \leq G^L$ *and there is no Right option* H^R *such that* $H^R \leq G$.

This is again a recursive definition that does not require a base. For an example let us consider two Domineering positions that we have seen already:

$G =$  $= *$

and

$H =$  $= 1.$

We will show that $G \leq H$. There is no Right option from H so all we need to show is that for every Left option in G, it is not true that $H \leq G^L$. But there is only one option G^L which is the 1×1 board of value 0. By looking at the definition again, in order to show that $H \leq 0$ is not true, it is enough to show that there is a Left option H^L in H so that $H^L \geq 0$. The only Left option in H is the empty board which has value 0. We have thus to show that $0 \leq 0$. But this is vacuously true because there are no Left and no Right options from 0.

Definition 40 *If $G \leq H$ is not true and $H \leq G$ is not true, we say that G and H are* ***incomparable*** *and write $G||H$.*

Definition 41 *A binary relation on a set A is called a* ***quasi-order*** *if it is reflexive and transitive. An antisymmetric quasi-order is called a* ***partial order****. If "$\leq$" is a partial order on A, and A has an abelian group structure $(A, +, 0)$ such that, for any g, h, g', h' in A, $g \geq h$ and $g' \geq h'$ implies $g + g' \geq h + h'$ (in other words, the order and the group structures are compatible), then A is called a* ***partially ordered group****.*

Theorem 42 *[ONAG, pages 16,78,79] The relation "$\leq$" is a quasi-order on Γ.*

Definition 43 *We say that the games G and H are* ***equal*** *if $G \leq H$ and $H \leq G$. We write $G = H$. When $G = H$ we will also say that G and H have the* ***same value****.*

We are using here Conway's non-orthodox but consecrated notion of 'equality as a defined relation'. In fact, "=" is an equivalence relation on Γ and this follows immediately from Theorem 42. Indeed, a class in this equivalence relation is formed by all games H with the property that $H \leq G$ and $G \leq H$ for a fixed G. The order relation induced by "$\leq$" on the set of equivalence classes $\Gamma/=$ has the properties of a partial order, that is, antisymmetry holds. We will not commonly need to distinguish between games in the same equivalence class. In the situations when this need arises however, we will say that two games which are equal but not identical in the sense of Definition 35, have different *forms*. If they are identical in the sense of Definition 35, we will say that they have the same form (and not merely the 'same value') and write $G \equiv H$.

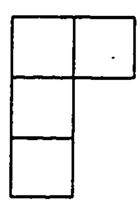
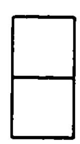
Theorem 44 *[ONAG, pages 16-18,78-79] The structure* $\left(\Gamma/=, \leq, +, \widehat{0}\right)$ *is a partially ordered abelian group* ($\widehat{0}$ *denotes the equivalence class containing* 0.)

Note that, although the addition operation is defined on Γ, it only induces a group structure on the equivalence classes in $\Gamma/=$. Theorem 44 justifies the statement 'G is always at least as advantageous for Left as H is'. '$G \geq H$' means that for any test game K, $G + K$ is at least as advantageous for Left as $H + K$ is. Intuitively, by "G is at least as advantageous for Left as H is" we mean that Left would prefer to play the game G against Right rather than the game H. This can be reformulated mathematically as follows:

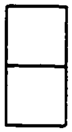
$$G \geq H \Longleftrightarrow (\text{For every } K, \text{ if Left wins } H + K \text{ then Left wins } G + K).$$

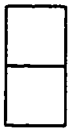
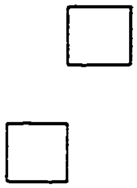
An informal but rigorous proof of Theorem 44 can be found in Chapter 7 of ONAG.

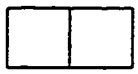
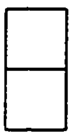
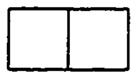
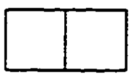
Example To see how the order relation works, let us prove that

$H =$  $\leq$  $= G.$

By Definition 39, we have to show that neither of the following is true:

1. $G =$  $\leq$  $= H^{L_1}$

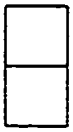
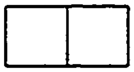
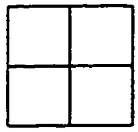
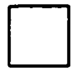
2.  $\leq$  $= H^{L_2}$

Note that the clause 'no $G^R \leq H$' holds vacuously since there are no Right options in G. Assume, for a contradiction, that Statement 1 is true. Then, using Definition 39 again, we obtain that  is not equal or less than 0 (the only Left option in  is the game $0 = \{ \mid \}$.) But  ≤ 0 vacuously because 0 has no Right options and  has no Left options. A contradiction can be obtained in a similar way if we assume Statement 2 to be true. Thus, we proved that $H \leq G$. We will see later that this inequality is actually strict, that is, $H \neq G$. In general, we will write $H < G$ when $H \leq G$ and $H \neq G$.

We will describe next the notion of the *outcome* of a game. Essentially, the outcome of a game G points to the player who wins G under the assumption that the players "play perfectly". In a perfect play, each player will always pick a winning move whenever one is available. A winning move can be defined inductively like this: after any reply of the opponent (if there is any possibility of a reply) there is a winning move from the new position. This is a well-defined notion since every game ends after a finite number of moves. Given a game G, and depending on who has the honor (horror) of the first move, exactly one of the following outcomes is possible at best play:

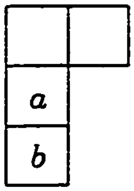
- Left wins whoever starts — G is called **positive**.
- Right wins whoever starts — G is called **negative**.
- The player who goes next wins — G is called **fuzzy**.

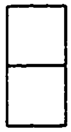
- The player who goes second (or the player who went previously) wins — G is called **null** (or **zero game**.)

For example,  is positive;  is negative;  is fuzzy and  is null.

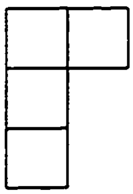
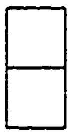
Theorem 45 *[ONAG, pages 73-76] If $G = H$ then G and H have the same outcome.*

Note that the converse is not true. For instance,



and 

are clearly both positive, and hence, they have the same outcome (as long as Left doesn't play ab above). However, as we will see later,

 $<$ .

The following theorem shows how the outcomes relate to the previously introduced order relation.

Theorem 46 *[ONAG, pages 73-76] For any game G,*

- *$G > 0$ if and only if G is positive;*
- *$G < 0$ if and only if G is negative;*
- *$G \,||\, 0$ if and only if G is fuzzy;*
- *$G = 0$ if and only if G is null.*

It is an important observation that the outcome of a sum $G+H$ cannot be determined solely based on the individual outcomes of G and H. All we can say, in general, about the outcome of $G+H$ is summarized in the next table ($+$, $-$, $||$ and 0 are short-hand for positive, negative, fuzzy and null respectively; ? stands for 'any outcome is possible'.)

	$+$	$-$	$\|\|$	0
$+$	$+$	?	$+$ or $\|\|$	$+$
$-$	?	$-$	$-$ or $\|\|$	$-$
$\|\|$	$+$ or $\|\|$	$-$ or $\|\|$	?	$\|\|$
0	$+$	$-$	$\|\|$	0

4.3 Cold Games

We will show how the important subgroup of $\Gamma/=$ formed by the *cold* games is constructed. Intuitively, a cold game is such that neither player would like to have the first move in it.

Definition 47 *For any non-negative integer n, define recursively $n+1 := \{n|\ \}$ and $-n-1 := \{\ |-n\}$. Such games are called **integers**. Zero is also an integer.*

Note that the definition of negative integers is consistent with the $G \longmapsto -G$ operation introduced earlier. This notation for integer games was prompted by the fact that, algebraically, they behave like integer numbers:

Theorem 48 *The function $\Phi : \mathbf{Z} \to \Gamma/=$, where $\Phi(n) = n$, is a one-to-one, order-preserving group homomorphism.*

This ensures that the notation introduced by Definition 47 is consistent with game-addition. Also, the order structure of the integer games is the same as the (natural) order structure of $\mathbf{Z}$. Once the integer games are defined, we can use them to construct fractional numbers:

Definition 49 *For any dyadic rational $n/2^k$ with n odd and $k \geq 1$, we define (inductively on k) the game with the same name:*

$$\frac{n}{2^k} := \left\{ \frac{n-1}{2^k} \middle| \frac{n+1}{2^k} \right\}.$$

We will call such games simply ***numbers*** *from now on.*

The definition is correct because, by assumption, n is odd and so $n-1$ and $n+1$ are even, which means that, when in lowest terms, the fractions $\frac{n-1}{2^k}$ and $\frac{n+1}{2^k}$ have the power of two in the denominator strictly less than k and, therefore, they have already been defined. We mention that, if we allow games with infinitely many options, then other numbers are possible (see ONAG for a general treatment of numbers as games with infinitely many options.)

Theorem 50 *Numbers form a subgroup of $\Gamma/=$ which is isomorphic to the subgroup of dyadic rationals in $\mathbb{Q}$ and order is preserved.*

The connection between numbers and those games in which nobody wants to move first is expressed by the following result, called the *simplicity rule*. First we will need the following definition

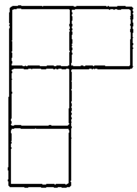
Definition 51 *Given two numbers $n_1/2^{k_1}$ and $n_2/2^{k_2}$ (where n_i is an odd integer whenever $k_i \geq 1$), we say that $n_1/2^{k_1}$ is* ***simpler*** *than $n_2/2^{k_2}$ if $k_1 < k_2$ or ($k_1 = k_2$ and $|n_1| < |n_2|$).*

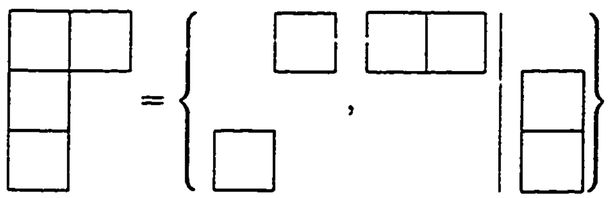
The "simpler than" relation can be regarded as a lexicographic order on pairs of the form $(k, |n|)$ from $\mathbf{Z}_+ \times \mathbf{Z}_+$.

Theorem 52 *[ONAG, page 23] A game $G = \left\{G^L | G^R\right\}$ is equal to a number if and only if there exists a number x such that*

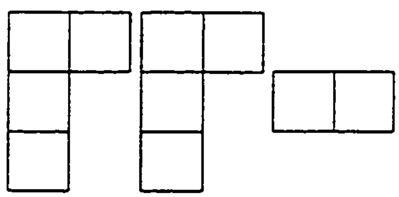
$$G^L \lhd\!\!| \;\; x \quad \textit{and} \quad x \;\; \lhd\!\!| \; G^R$$

for every G^L and G^R(where '$\lhd\!\mid$' stands for 'less than or incomparable to'.) Then, G equals the simplest such x.

Example We will show that  $= 1/2$. Indeed,

 $= \{0, -1|1\}$,

since $\frac{1}{2}$ is the simplest number that is greater or incomparable to 0 and -1, and smaller or incomparable to 1. In particular, this implies that the game

 $= \frac{1}{2} + \frac{1}{2} + (-1) = 0.$

The reader may also check that this is the case by actually playing the sum of the three games above — the result will be a win for the second player to move.

When playing a sum of games, both Left and Right will never make a move in a number component unless there are no Left, respectively Right, options available in the other components. Once the play reaches the moment when all components are numbers, the players can stop playing, and just add up the (number-) values of the components, as people do when playing Go. If this sum is positive Left wins, if it is negative Right wins, and if it is zero, the player who is next to move loses. The following theorem, called the *translation principle*, explains why good players should play like this.

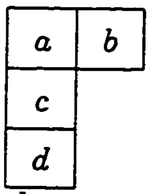
Theorem 53 *[ONAG, page 112] If $G = \left\{G^L|G^R\right\}$ is not a number, then, for any number x*

$$G + x = \left\{G^L + x \mid G^R + x\right\}.$$

4.4 Simplifying Games

There are infinitely many games in every equivalence class of Γ. This is a direct consequence of the fact that there are infinitely many null games. In order to play a sum of games $G = G_1 + \ldots + G_n$ perfectly, all that the players need to know are the values of the components G_i. The value of the *compound* game G will be the sum of these values. Therefore, it is highly desirable to find the simplest possible games which are equal to $G_1, \ldots, G_n$. J. H. Conway proved [ONAG, pages 109-112] that it is always possible to find the simplest form for a game, and also gave the procedure for finding it. In this context, 'simplest' is used with the meaning of 'smallest birthday' — the birthday is a good measure of how complicated the game is.

Definition 54 *If G^{L_1} and G^{L_2} are Left options in G such that $G^{L_1} \leq G^{L_2}$, then G^{L_1} is called a* ***dominated*** *option. We say that G^{L_2}* ***dominates*** *G^{L_1}. Similarly, if G^{R_1} and G^{R_2} are Right options in G such that $G^{R_1} \geq G^{R_2}$, then G^{R_1} is a dominated option.*

For example, in 

, Left's option to play (cd) is dominated by her option of playing (ac). This is because the resulting position after playing (cd) has value -1 which is less than the value of the position after playing (ac) which is 0.

Theorem 55 (*deleting dominated options — ONAG, pages 110-111*) *If G^{L_1} and G^{L_2} are Left options in G such that G^{L_2} dominates G^{L_1}, then G^{L_1} can be deleted without changing the value of G. In a similar way, if G^{R_2} dominates G^{R_1}, then G^{R_1} can be deleted.*

Definition 56 *If a Left option G^L of G has a Right option G^{LR} such that $G^{LR} \leq G$ then we say that G^L is* ***reversible through*** *G^{LR}. Similarly, if a Right option G^R of G has a Left option G^{RL} such that $G^{RL} \geq G$ then we say that G^R is reversible through G^{RL}.*

Theorem 57 (*bypassing reversible moves — ONAG, pages 110-111*) *If a Left option G^L is reversible through G^{LR} then it can be replaced by all the Left options $G^{LRL_1}, G^{LRL_2}, \ldots$*

of G^{LR} *without affecting the value of* G. *Similarly, if* G^R *is reversible through* G^{RL} *then it can be replaced by all the Right options* $G^{RLR_1}, G^{RLR_2}, \ldots$ *of* G^{RL} *without affecting the value of* G.

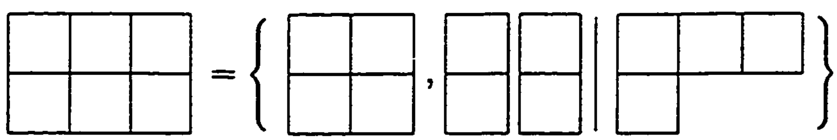
Definition 58 *If a game* G *has no dominated options and no reversible moves, and all the options of* G *are in canonical form, then* G *is said to be in* ***canonical form***.

Note that this is again a recursive definition — it is assumed that it is known what the canonical forms of the options of G are at the moment when the canonical form of G is defined. It is an easy check that the games we have defined so far — zero, integers, numbers in general, star — are all in canonical form.

Theorem 59 *[ONAG, pages 111-112] For every game* G *the canonical form exists, is unique and has minimal birthday among the games that are equal to* G.

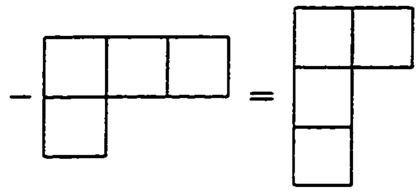
In order to make the reading easier for games with more complicated forms we will use a multiple vertical bar notation to make the distinction between the separator of the Left and Right options of a game (which will consist of the largest number of vertical bars) and the separators within the Left or Right options of the game (which will have fewer vertical bars). For instance, we will denote $G = \{\{2|1\}\,|0\}$ by $\{2|1\|\,0\}$ — the double separator is used to identify $\{2|1\}$ as the Left option of G, and the inner brackets are dropped as they are not necessary anymore. Also, given a positive number a, we will use the notation $\pm a$ for the game $\{a|{-a}\}$.

Example We will find the canonical form of the following Domineering position:

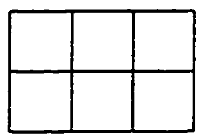


$$= \left\{\pm 1, 2 \,\middle|\, -\frac{1}{2}\right\}.$$

Note that the omitted Right options are equal to the listed one — they are just rotations or reflections of the same shape. We have used the fact that



which was proved before to equal 1/2. We claim that the option $G^{L_1} = \{1|-1\}$ is dominated by $G^{L_2} = 2$. Indeed, $G^{L_1} - G^{L_2} = \{-1|-3\}$ by the translation principle. As -1 and -3 are negative games, Left loses whoever starts in $\{-1|-3\}$. This means that $G^{L_1} - G^{L_2}$ is negative so $G^{L_1} < G^{L_2}$. We can therefore delete the dominated option G^{L_1} and

 $= \left\{2 \,\middle|\, -\frac{1}{2}\right\}.$

This is the canonical form because the options 2 and $-\frac{1}{2}$ are not reversible. Indeed, there is no Right option from $2 = \{1| \}$ and the Left option in $-\frac{1}{2} = \{-1|0\}$ is not greater than or equal to G (it is, in fact, strictly less than G).

It is also useful to note here the effect of replacing an option in general. The following theorem has great practical importance.

Theorem 60 *[ONAG, page 110] The value of G is unaltered or increased if we*

- *increase any Left or Right options G^L or G^R,*
- *remove some G^R or add a new G^L.*
- *replace the set of Right options of G by the set of Right options of K, for any game $K \geq G$.*

4.5 Special Subclasses of Γ

Definition 61 *A game G is called* ***infinitesimal*** *if $-x < G < x$ for every positive number x. If two games G and H differ by an infinitesimal we say that G is* ***infinitesimally shifted*** *from H, and write $G = H$-ish.*

There is a convenient way to decide if a game is an infinitesimal or not. It involves the notion of *stops*:

Definition 62 *The Left and Right stops of a game are numbers defined recursively as follows:*

$$L(G) = \begin{cases} G & \text{if } G = x \text{ for some number } x, \\ \max_{G^L} R\left(G^L\right) & \text{otherwise;} \end{cases}$$

$$R(G) = \begin{cases} G & \text{if } G = x \text{ for some number } x, \\ \max_{G^R} L\left(G^R\right) & \text{otherwise.} \end{cases}$$

Example Let us consider the following game where each player has two options:

$$G = \left\{ \{2|1\}, \frac{3}{2} \,\middle\|\, \{0|-5\}, -1 \right\}.$$

We have used the double slash as the separator between Left and Right options to stress that it is the main separator as opposed to the single slash separator in $\{2|1\}$ which only separates Left and Right options of a Left option. In order to calculate the Left stop $L(G)$ we need to see first if G is a number or not. The Left option $\frac{3}{2}$ is greater than the Right option -1 and therefore the simplicity rule (Theorem 52) does not apply for any x and G is not a number. We use then the second clause in the definition of $L(G)$ and we look at the Right stops of all Left options in G. The Right stop of $\{2|1\}$ is 1 because, by the definition of the Right stop, it is the Left stop of 1 which is a number so the first clause in the definition of Right stops applies. The Right stop of the other Left option is $\frac{3}{2}$ because it is a number. Thus the maximum of $L(\{2|1\})$ and $L\left(\frac{3}{2}\right)$ is $\frac{3}{2}$ and so $L(G) = \frac{3}{2}$. Similarly, one can find that $R(G) = -1$.

Theorem 63 *[ONAG, page 101] A game G is an infinitesimal if and only if $L(G) = R(G) = 0$.*

In general, the stops of a game G determine the interval, usually called the **confusion interval**, of numbers that are incomparable with G:

Theorem 64 *[ONAG, pages 98-100] For any game G, $R(G) \leq L(G)$ and*

- *G is incomparable with any number x such that $R(G) < x < L(G)$.*
- *G is greater than any number $x < R(G)$.*
- *G is smaller than any number $x > L(G)$.*

In the previous example, the endpoints of the confusion interval of

$$G = \left\{\{2|1\}, \frac{3}{2} \,\Big\|\, \{0|-5\}, -1\right\}$$

are -1 and $\frac{3}{2}$. The confusion interval is closed in this case because -1 is incomparable with G and so is $\frac{3}{2}$. The typical way to check that, say, $-1||G$ is to prove that the difference $G - (-1) = G + 1 = \left\{\{3|2\}, \frac{5}{2} \,\Big\|\, \{1|-4\}, 0\right\}$ is incomparable with 0 by showing that $G + 1$ is a fuzzy game. Indeed, Left wins when going first by moving to $\frac{5}{2}$ (the other option does it too) which is positive. Right wins when going first by moving to 0 which is a losing position for Left as she is the next to move in it.

We already have two examples of infinitesimals: zero and star. These are the simplest examples of an important class of infinitesimals that will be introduced next — the impartial games.

Definition 65 *A game G is called **impartial** if the set of Left options is the same as the set of Right options and all the options are impartial games. If these options are $G^1, ..., G^n$, we will use the notation $G = \{G^1, ..., G^n\}$.*

Note that this is again a recursive definition. An impartial game can be informally described as a game where, at any moment of any possible play, the two players have the same options to choose from no matter whose turn it is to play. Unlike chess, in an impartial game the possible moves do not depend on who, Left or Right, makes them, but only on what the current position is. Because of this symmetry, only two outcomes are possible — null or fuzzy. The traditional names for impartial null and impartial fuzzy games are P-positions, respectively, N-positions. The term N-position is short for "Next-player-win position" (which is the definition of fuzzy game). A P-position is a "Previous-player-win position" where the player that has just played wins, meaning that the player next to move loses.

It is an easy inductive check that the stops of any impartial game are both zero, hence impartial games are infinitesimals.

The game of Nim [Bou02] is the classical example of an impartial game. Nim can be played with heaps of matches. A move is to choose a heap and remove any positive number of matches from it.

All the possible values of impartial games can be completely characterized. This result, together with the actual rules for recursively computing and adding impartial values, was the object of the first theoretical development in the area of combinatorial games — it was discovered by R. P. Sprague [Spr35] and P. M. Grundy [Gru39].

Theorem 66 *Let* $*0 := 0$ *and* $*1 := *$ *and, inductively, for* $n \geq 1$ *:*

$$*(n+1) = \{*0, *1, ..., *n\}.$$

Then, every impartial game has a value $*n$ *for some non-negative integer* n*. Furthermore, the value of a game* $G = \{*n_1, ..., *n_k\}$ *can be computed by the so called* ***mex rule*** *(mex comes from 'minimum–excluded'):*

$$G = *m \text{ where}$$

$$m = \min\{n \geq 0 : n \neq n_i \textit{ for every } 1 \leq i \leq k\} \quad := mex\{n_1, ..., n_k\}.$$

Theorem 67 *Let m and n be two non-negative integers having base two expansions $m = \overline{a_k...a_1}$ and $n = \overline{b_k...b_1}$ (any two integers can be written in this form if we fill zeros in front of the smaller one). Then,*

$$*m + *n = *s$$

where

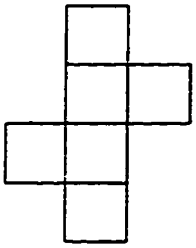
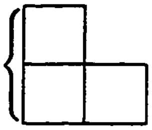
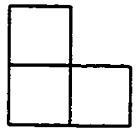
$$s = \overline{s_k...s_1} \textit{ and } s_i = a_i + b_i \pmod 2.$$

Corollary 68 *The following are general properties of impartial games:*

1. *For any non-negative n,*
$$*n + *n = *0\,(= 0).$$
2. *For any non-negative n,*
$$- *n = *n$$
and therefore nim-subtraction is the same as nim-addition.
3. *Every impartial value except zero is fuzzy.*

The solution to Nim follows immediately from Theorems 66 and 67. The value of the Nim game on one heap of n matches is $*n$ — the formula for calculating it is the same as the formula for $*n$ in Theorem 66 (this is the reason why impartial values are often called **nimbers**.) The value of a Nim position with heaps of sizes $n_1, ..., n_k$ is just the sum $*n_1 + \cdots + *n_k$ and it can be computed using the rule given in Theorem 67. If this value is not zero, then the next player to move wins. A winning move has to leave a position of value zero for the opponent to move in.

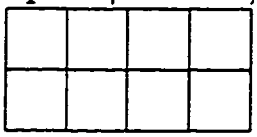
There are many kinds of infinitesimals which are not nimbers:

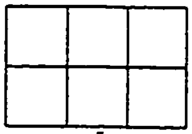
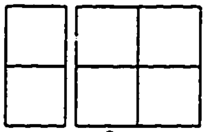
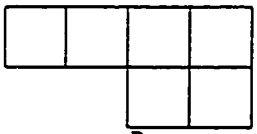
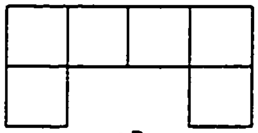
Example Let $G =$  $= \Big\{$  $, 0 \;\Big|\Big|$  $\Big\} = \{*, 0 | *\}$.

One can quickly check that Left wins in G no matter who starts. This means that $G > 0$. But now, $*$ is a reversible Left option because the Right option 0 in $*$ is smaller than G itself. We can apply Theorem 57 and replace the Right option $*$ by all the Left options in zero. There are no Left options in zero hence $*$ just gets deleted. We have thus

$$G = \{0|*\}.$$

This game is called Up and we write $G = \uparrow$. The negative of $\uparrow$ is Down, and it is denoted by $\downarrow = \{*|0\}$. As $* = \{0|0\}$, both stops of $\uparrow$ are zero, so $\uparrow$ is an infinitesimal, indeed.

Example Let us calculate $G =$  $=$

$\Bigg\{$  ,  $\Bigg|$  ,  $\Bigg\}$.

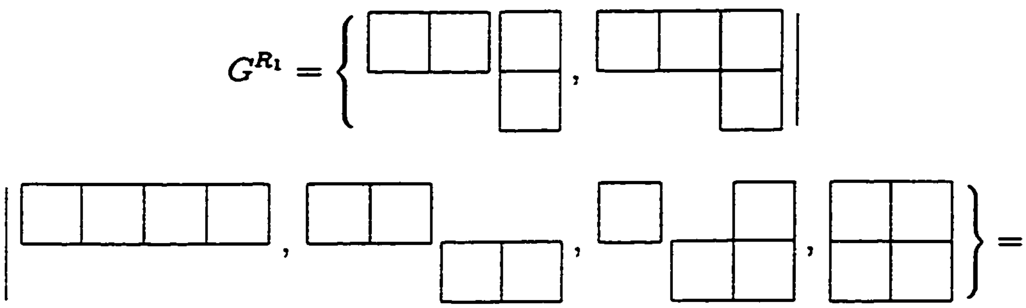
G^{L_1} $\quad$ G^{L_2} $\quad$ G^{R_1} $\quad$ G^{R_2}

$G^{L_1} = \left\{2 \middle| -\frac{1}{2}\right\}$ as seen before.

$G^{L_2} = 1 + \{1|-1\} = \{2|0\}$.

By playing $H = G^{L_2} - G^{L_1}$ we see that Left wins H (so long as she doesn't play in the G^{L_2} component) no matter who starts, so the option G^{L_1} is dominated by G^{L_2}.

$G^{R_1} =$



$$= \left\{0, -\frac{1}{2} \,\middle|\, -2, *, \pm 1\right\} = \{0|-2\}.$$

$G^{R_2} = 0$ (this is because the first player to move cannot win in G^{R_2}).

We obtain, then,

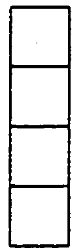
$$G = \{2|0 \,||\, \{0|-2\}, 0\}.$$

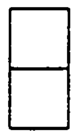
The option G^{R_1} is reversible since its Left option $0 \geq G$ (Right can win in G whoever starts) and it is replaced by the Right option of 0, i.e. the empty set. It follows that $G = \{2|0||\,0\}$, which is the canonical form ($\{2|0\}$ is not reversible). We denote this game by **Miny-two** and write $-_2 = \{2|0||\,0\}$. The negative of $-_2$ is **Tiny-two**: $+_2 = \{0\,||0|-2\}$. In general, for any positive game G, we define $-_G := \{G|0||\,0\}$ and $+_G = \{0\,||0|-G\}$. The fact that $-_2$ is negative although all the positions that could occur in $-_2$ are non-negative numbers, may seem puzzling. At a closer inspection however, we see that the position of value 2 can never be reached if Right is playing well and, no matter who starts, Right has a 'timing advantage' in that Left will always be the first to play from zero (which is a losing situation for Left).

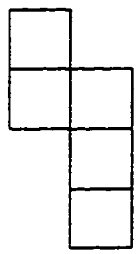
The following relations describe the relative magnitude of the infinitesimals introduced so far:

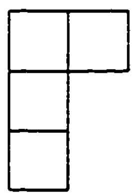
- $\downarrow < -_2 < 0 < +_2 < \uparrow$.
- $*$ is incomparable to all of the above, but $\downarrow + \downarrow < * < \uparrow + \uparrow$.
- $\downarrow < *n < \uparrow$, but $*n \,||\, +_2$ and $*n \,||\, -_2$ for all $n \geq 2$.

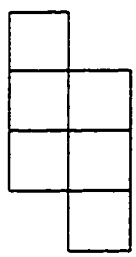
We close this section with a collection of Domineering examples. As mentioned before, there are four games with birthday zero or one: 0, $*$, 1 and -1. Therefore, there are $4^4 = 256$ games with birthday at most two, corresponding to all possible pairs of subsets of $\{0, *, 1, -1\}$. However, there are only 22 distinct values among these. With one exception (the game $*2$), we found Domineering boards whose canonical forms are games with birthday at most two:

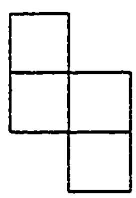
$2 =$ 

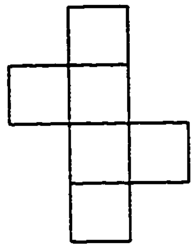
$1 =$ 

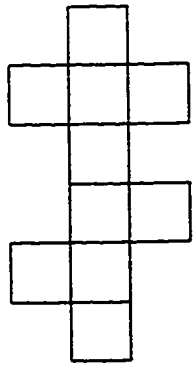
$1* := \{1|1\} =$ 

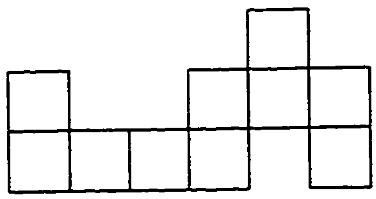
$\frac{1}{2} =$ 

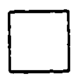
$\{1|*\} =$ 

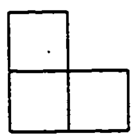
$\{1|\,0\} =$ 

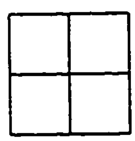
$\uparrow =$ 

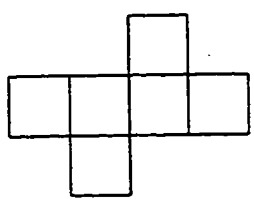
$\uparrow * := \uparrow + * =$ 

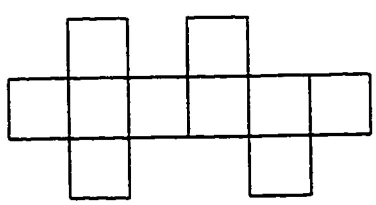
$\{1 | 0, *\} =$ 

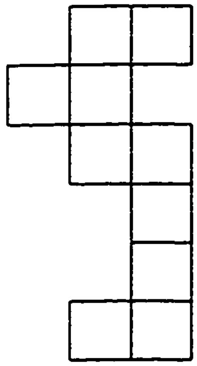
$0 =$ 

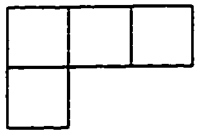
$* =$ 

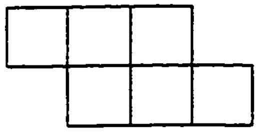
$\pm 1 := \{1 | -1\} =$ 

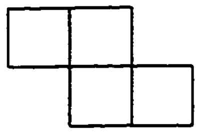
$\downarrow =$ 

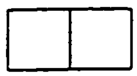
$\downarrow * := \downarrow + * =$ 

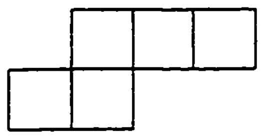
$\{0, * \mid -1\} =$ 

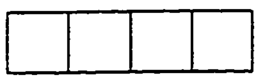
$-\frac{1}{2} =$ 

$\{* \mid -1\} =$ 

$\{0 \mid -1\} =$ 

$-1 =$ 

$-1* := \{-1 \mid -1\} =$ 

$-2 =$ 

The partial order structure of the 22 games with birthday at most two is shown in Figure 2. It is interesting to note that the 22 games form a distributive lattice. Furthermore, if we remove the non-zero integers (the games -2, -1, 1, and 2) the remaining 18 games form the free distributive lattice with three generators. We remark that this lattice corrects figures appearing on page 15 of [Guy91], on page 2128 of [Guy95] and on page 55 of [Guy96].

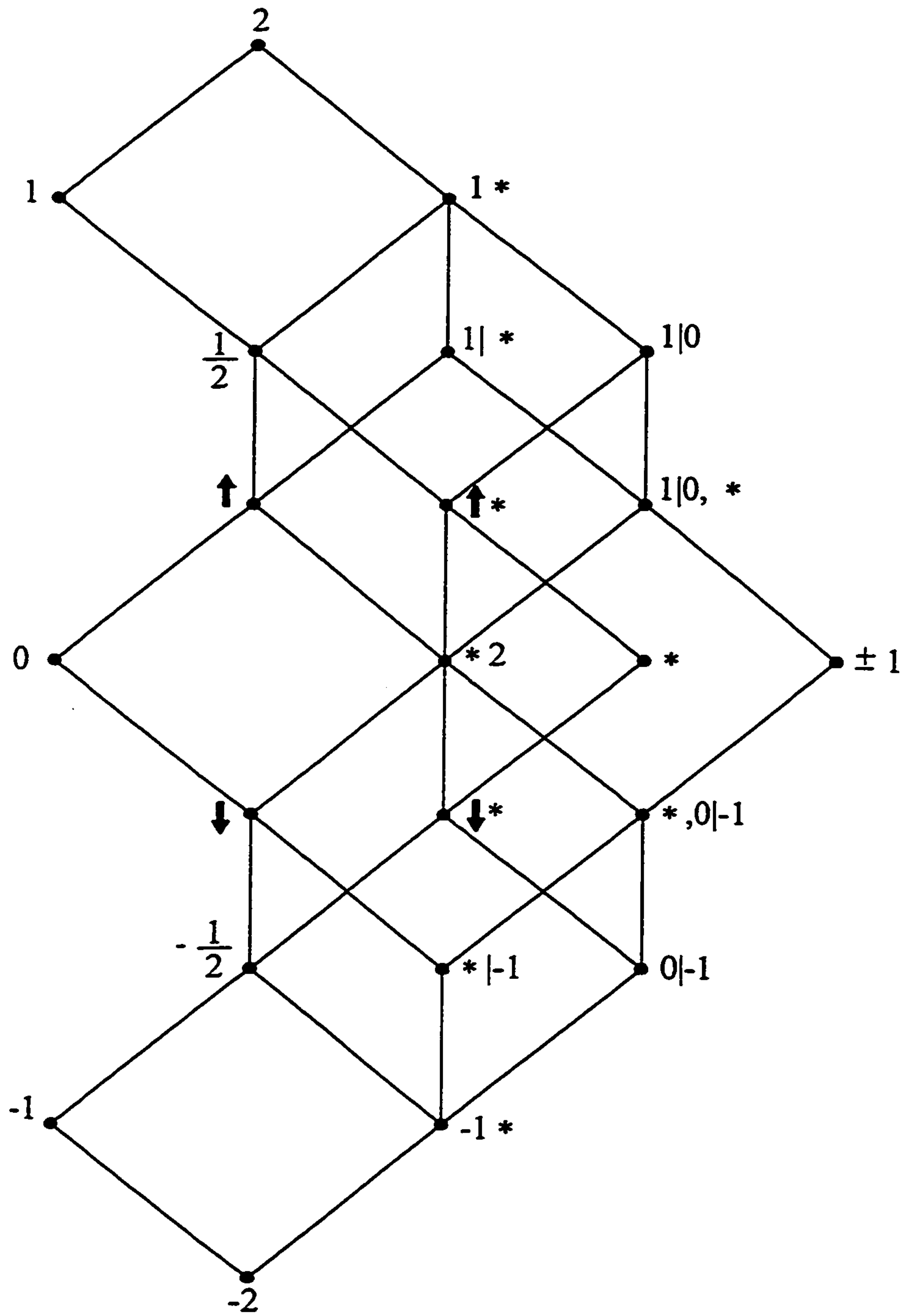


Figure 2. The lattice of games of birthday at most two

4.6 Mean Value and Temperature

Games which are not numbers may have very complicated values (even the canonical forms), and the task of adding them up is often not computationally tractable. A possible approach is to use some kind of approximation for games with complicated values. As we have already seen, an approximate measure for the magnitude of a game G relative to the number scale is the confusion interval $(R(G), L(G))$. The problem with this is that we cannot estimate the stops of a sum of games, based solely on the stops of the summands. J. H. Conway and S. Norton [ONAG, pages 102-104] and also E. R. Berlekamp [WW] introduced the notions of 'mean value' and 'temperature' of a game such that they have the desired additivity property. The following result is called the 'Mean Value Theorem'.

Theorem 69 *For every game G there is a number $m(G)$, called the mean value of G, and a number t such that*

$$n \cdot m(G) - t \leq n \cdot G \leq n \cdot m(G) + t$$

for all integers n, where $n \cdot G$ denotes the sum of n copies of G. Also, for any games G and H,

$$m(G+H) = m(G) + m(H).$$

Intuitively, the mean value of a game can be obtained by reducing the incentives of the players to move first; if we go far enough, none of the players would want to move first, which is the characteristic of number games. The number reached this way is the mean value. Formally, we use the following 'cooling operator'.

Definition 70 *If G is a game and $t \geq 0$ is a number, then we define the cooled game G_t by the recursive formula*

$$G_t = \left\{G_t^L - t | G_t^R + t\right\},$$

unless possibly this formula defines a number (which it will for all sufficiently large t). For the smallest values of t for which this happens, the number turns out to be constant *(that is, independent of t), and we define G_t to be this constant number for all larger t.*

Definition 71 *Let $t \geq 0$ be the smallest number such that G_t is a number. Then, t is called the* ***temperature*** *of G.*

Theorem 72 *If t is the temperature of G then, for any number $\tau > t$, $m(G) = G_\tau$.*

Example Let $a > b$ be two numbers. Then, using the previous results, we find that

$$m(\{a|b\}) = \frac{a+b}{2}$$

and

$$\text{temperature}(\{a|b\}) = \frac{a-b}{2}.$$

It is not true, in general, that the mean value is the arithmetic mean of the stops. For instance, the game $G = \{2|1||0\}$ has stops $L(G) = 1$ and $R(G) = 0$, while $m(G) = 3/4$ which does not equal the arithmetic mean of the stops (the temperature of G is also $3/4$).

A useful tool for visualizing the cooling process is the **thermograph**. The thermograph of a game is obtained by plotting the Left and Right stops of the game cooled by

t as functions of t. For an example, let us look at the thermograph of $\{2|-1\}$:

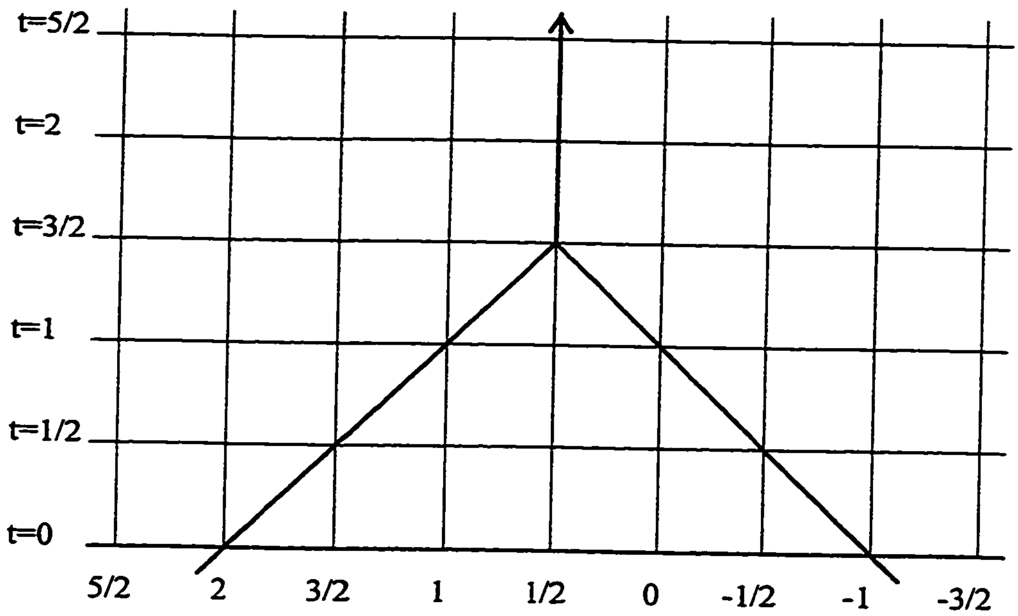


Figure 3. The thermograph of $\{2|-1\}$.

The sloping line to the left represents the Left stops as t varies from 0 to $\frac{3}{2}$, and the sloping line to the right represents the Right stops as t varies from 0 to $\frac{3}{2}$. Note that the Left and Right stops become equal when $t = \frac{3}{2}$, which means that the temperature of the game is $\frac{3}{2}$. Any further increase in t will leave G_t unchanged because, once it "freezes" down to a number (which is the mean value and equals $\frac{1}{2}$ in this case), any further cooling will have no effect at all.

We will give next a simple example of how to compute the thermograph of $G = \{2|1||0\}$ using the thermographs of its Left option $\{2|1\}$ and Right option 0.

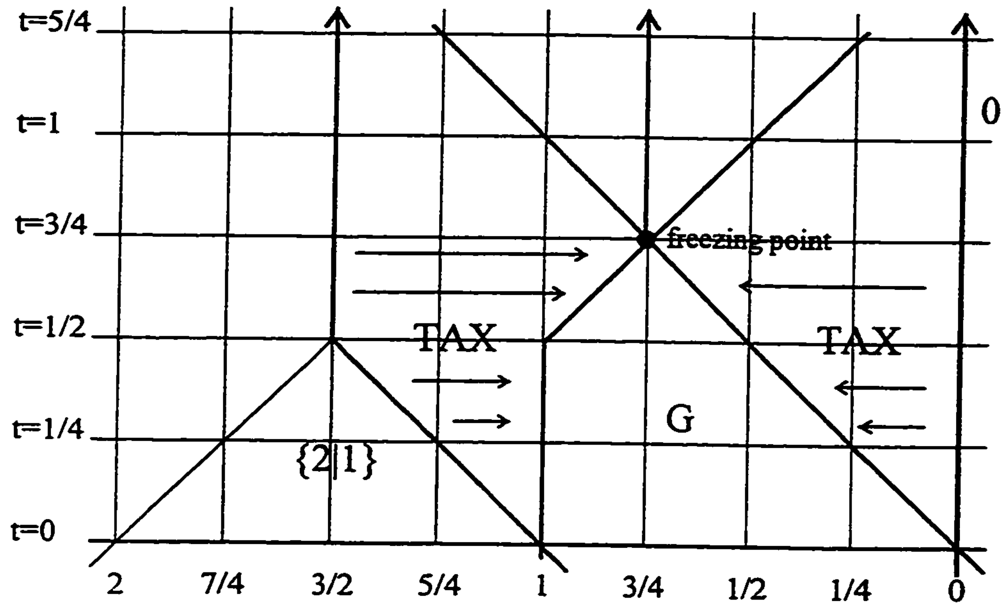


Figure 4. The thermographs of $\{2|1\}$, G and 0.

Because of the way the stops of a game are defined, the effect of cooling on the right boundary of the thermograph of the Left option $\{2|1\}$ is like that of a tax that moves it rightwards by the amount of the tax. Similarly, the Left boundary of the thermograph of 0 (which is just an infinite mast) moves leftwards by the amount of the tax. The freezing point is at the intersection of these taxed boundaries — in this case at height $\frac{3}{4}$ above the point $\frac{3}{4}$, showing that $G_t = \frac{3}{4}$ for all $t > \frac{3}{4}$. The thermograph of G is therefore as in Figure 4, both boundaries coinciding with the vertical mast above this point.

By definition, all numbers have zero temperature. There are games with zero temperature which are not numbers. Such games are called **tepid** — they lie at the boundary between cold games (the numbers), and the games of positive temperature which are called hot games. For instance, all infinitesimals other than zero are tepid (it can be verified inductively that the result of cooling an infinitesimal by any positive number is zero.)

E. R. Berlekamp [WW, Chapter 6] suggested the possibility to recover (at least partially) the information that is lost through the cooling process by applying certain operators which in ideal cases would invert cooling. Let us assume for the moment that such an operator $\int^t$ exists such that $\int^t (G_t) = G$ and $\int^t (H_t) = H$ (where, say, G and H are complicated games) and $\int^t$ is additive. Then, one can compute G_t and H_t, which are simple games, and then compute $G_t + H_t$, and finally $\int^t (G_t + H_t)$ which, as $\int^t$ is additive, equals $G + H$. The following family of operators was discovered by Conway & Norton in Cambridge, and Berlekamp & Hickerson in Berkeley. They are additive, but do not have the property of reversing cooling in general.

Definition 73 *Let G be a game whose canonical form is $G = \left\{G^L | G^R\right\}$. Then we define G* ***heated by*** *t as*

$$\int^t G = \begin{cases} G & \text{if } G \text{ is a number,} \\ \left\{t + \int^t G^L \,\middle|\, -t + \int^t G^R\right\} & \text{otherwise.} \end{cases}$$

For any positive game s, we define G ***overheated from*** *s* ***to*** *t as*

$$\int_s^t G = \begin{cases} G \cdot s & \text{if } G \text{ is an integer,} \\ \left\{t + \int_s^t G^L \,\middle|\, -t + \int_s^t G^R\right\} & \text{otherwise.} \end{cases}$$

For specific values of s and t, the overheating operator inverts cooling within infinitesimal errors for certain classes of games such as Blockbusting [Ber88] and Go [BW94]. There is no known general method to decide whether a given overheating operator inverts cooling on a given class of games.

Chapter 5

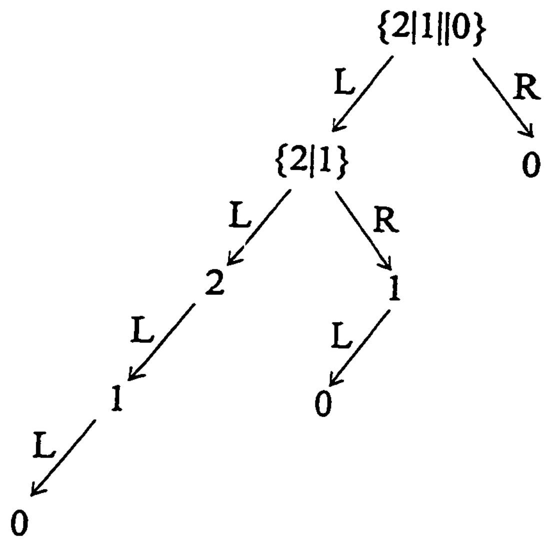
The Reduced Canonical Form

Before introducing the new concept of the *reduced* canonical form we give a brief discussion of the canonical form. Recall from Theorem 59 that any game G admits a unique canonical form, i.e. a form with no dominated options and no reversible moves. The canonical form is the simplest among all the games that have the same value as G, where "simplest" is used with the meaning of least or earliest birthday. Actually, the canonical form of a game G has an additional property which will be defined in terms of the game-tree. The game-tree of G is a graph where the nodes are all the options and sub-options of G and there is an edge between each sub-option and the option or sub-option it belongs to. A game G will be called sparser than a game H if the size of the edge-set of the game-tree of G is less than the size of the edge-set of the tree corresponding to H. The canonical form of a game G is the sparsest game among all games having the same value as G.

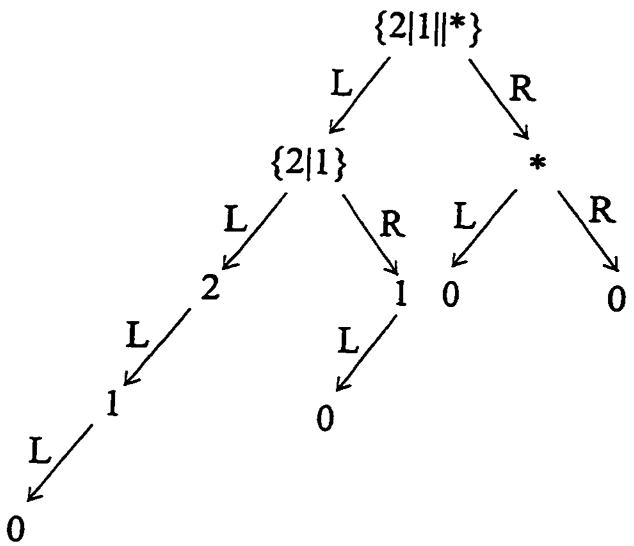
For an example, let us consider the following two games:

$$G = \{2|1||0\} \text{ and } H = \{2|1||*\}.$$

These games have the following game-trees:



The game-tree of G



The game-tree of H

The sizes of the edge-sets of G and H are 7 and 9, respectively. This means that, although they both have the same birthday (four), G is sparser than H.

The aim of this chapter is to introduce a yet simpler form denoted by $\overline{G}$, called the

reduced canonical form, by relaxing the condition that it should have the same value as the initial game to the condition that it should be at most infinitesimally shifted from the initial game. This new form should be the sparsest possible game subject to this condition, and the transformation $G \to \overline{G}$ should be linear. Algebraically, we will show that the reduced canonical forms form a subgroup $\mathbf{Rcf}$, and the group of games $\Gamma/=$ is the direct sum $\mathbf{I} \oplus \mathbf{Rcf}$, where $\mathbf{I}$ is the subgroup of infinitesimals. Often, the information provided by the $\mathbf{Rcf}$-component of a game G is enough to decide the outcome class of G. For games of this type, it is important to know when knowledge of the reduced canonical forms of the options of G would imply knowledge of the reduced canonical form of G. This will be answered by Theorem 84. As we will see, the games G and H in the above example are in canonical form, and G is the reduced canonical form of H.

5.1 Construction of $G \to \overline{G}$

Definition 74 *Given $G = \{G^L | G^R\}$, G^L and G^R being generic Left, respectively Right options of G, then G* **cooled by** $*$, *written G_*, is defined recursively by*

$$G_* = \begin{cases} G & \text{if } G \text{ is a number,} \\ \{G^L_* + * | G^R_* + *\} & \text{otherwise.} \end{cases}$$

where ${G^L}_*$ is an abbreviation for $\left(G^L\right)_*$.

It is an easy inductive check that, if $G - H$ is a zero game, then $G_* - H_*$ is a zero game too. Indeed, if one of G and H is a number then we are done by applying the first clause of Definition 74. Otherwise, any move of Left or Right in $G - H$ together with the winning reply of Right, respectively Left, can be mirrored in $G_* - H_*$ by using the second clause of Definition 74. This means that the definition of G_* is independent of the form of G, so G_* is well-defined.

This definition of cooling closely resembles the notion of cooling by numbers (see Definition 70.) The difference here is that games are cooled by $*$ instead of by positive numbers.

Definition 75 *If $H_0 \equiv \{H_0^L | H_0^R\}$ is the canonical form of a game H (which means that H_0^L and H_0^R are in canonical form, too), then define $p(H)$, the* ***$*$-projection** of H recursively:*

$$p(H) = \begin{cases} x & \text{if } H = x \text{ or } x + *, \text{ where } x \text{ is a number,} \\ \{p(H_0^L) | p(H_0^R)\} & \text{otherwise.} \end{cases}$$

Again, H_0^L and H_0^R are generic Left, respectively Right options of H_0.

Because of the uniqueness of the canonical form, the definition of $p(H)$ is independent of the form of H.

Example Let us consider again the game Up: $\uparrow = \{0|*\}$. Up is in canonical form and it is not a number. Then, $p(\uparrow) = \{p(0) | p(*)\}$. But 0 and $*$ satisfy the first condition in the definition of the $*$-projection, so $p(0) = p(*) = 0$. Therefore, $p(\uparrow) = \{0|0\} = *$.

Given a number x, we will use the notation $x*$ for the game $x + *$.

Definition 76 *The* ***reduced canonical form*** *of G, denoted by $\overline{G}$, is $p(G_*)$.*

Observe that $p(G_*)$ is, in particular, a canonical form, because p is defined in terms of the canonical form of G_*, and it follows by induction that $p(G_*)$ is in canonical form as well.

Example The game of Col is a map-colouring game invented by Colin Vout [WW, page 39 and ONAG, page 96]. Each player, when it is her, respectively his, turn to move, paints one region of the map, Left using the colour blue and Right using red. No two regions having a common frontier edge may be painted the same colour. Whoever is unable to paint a region loses.

Conway & Guy [ONAG, pages 91-96] discovered that the values of Col positions are surprisingly simple. They proved that every Col position is either a number, or a number plus star. For Col positions, the reduced canonical forms are all numbers. Indeed, if $G = x*$, where x is a number, then $G_* = (x*)_* = \{x|x\}_*$ (we used the translation principle for numbers here). Then, by the definition of cooling by star,

$$G_* = \{x_*|x_*\} = \{x|x\} = x* = G.$$

We have then $p(G_*) = p(x*) = x$ after using the first clause in the definition of the $*$-projection.

We will use the following property of the game $*$, called the 'translation principle for stars'.

Theorem 77 *[WW, page 123] For any numbers x and y, $\{x|y\}+* = \{x*|\,y*\}$ if $x \geq y$, and $\{x\,|y*\} + * = \{x*|\,y\}$ if $x > y$.*

Example Let $G = \{\{2|0\}, 1 \,\|\, 0\}$. Note that G is in canonical form. Then

$$G_* = \{\{2|0\}_* + *, 1* \,\|\, *\} = \{\{2*|\,*\} + *, 1* \,\|\, *\}$$

and, using the translation principle for stars,

$$G_* = \{\{2|0\}, 1*\|\, *\}.$$

Since this is a win for Left whoever starts (provided that Left does not move to $\{2|0\}$), the Left option $\{2|0\}$ is reversible so it can be replaced by all the Left options from 0, and as there are no options from 0, $G_* = \{1*|*\}$. It is easy to check that $\{1*|*\}$ has no reversible moves, so it is the canonical form of G_*, hence $\overline{G} = p(G_*) = \{p(1*)|p(*)\} = \{1|0\}$. So, in this example, the reduced canonical form of G is sparser than the canonical form of G.

5.2 Properties of $\overline{G}$

Theorem 78 *The transformation $G \to \overline{G}$ is a homomorphism.*

Proof: We will show that $G \to G_*$ and $H \to p(H)$ are homomorphisms, hence their composition is a homomorphism. We first show inductively that neither of the players can win going first in $(G-H)_* - G_* + H_*$, and therefore $(G-H)_* = G_* - H_*$. If we consider first the more general case when G, H and $G-H$ are not numbers, we have:

$$(G-H)_* - G_* + H_* = \{(G^L - H)_* + *, (G - H^R)_* + * \parallel (G^R - H)_* + *, (G - H^L)_* + *\}$$

$$+\{(-G^R)_* + * \parallel (-G^L)_* + *\} + \{H_*^L + * \parallel H_*^R + *\}.$$

We can then see that, assuming the property true for pairs such as (G^L, H), (G^R, H), (G, H^L), (G, H^R), every move in $(G-H)_* - G_* + H_*$ has an exact counter, i.e. for any move of one player in it, there is a reply by the other player that brings the position to a value of 0.

If at least two of G, H, and $G-H$ are numbers, then all of them are numbers and the equality to be proved is trivial since we are in the first case of Definition 74.

If precisely one of G, H is a number x, say $H = x$, then we need to show that $(G-x)_* = G_* - x$. As we are in the second case of Definition 74, this is equivalent to

$$\{(G^L - x)_* + * | (G^R - x)_* + *\} = \{G_*^L + * | G_*^R + *\} - x.$$

Applying the translation principle (with x) one more time, we are done — because $(G^L - x)_* = G_*^L - x$ and $(G^R - x)_* = G_*^R - x$ by the induction hypothesis.

The proof that $H \to p(H)$ is a homomorphism is very similar: it is enough to show that

$$p(G) + p(H) + p(K) = 0$$

if $G + H + K = 0$ and G, H and K are in canonical form.

Suppose that none of G, H, K is of the form x or $x*$ for some number x. It is enough to prove the inequality $p(G) + p(H) + p(K) \leq 0$ because, by symmetry, the opposite inequality can be obtained in the same way. If Left moves first in $p(G) + p(H) + p(K)$, she will leave for Right a position like $p(G^L) + p(H) + p(K)$. As G, H and K were in canonical form and $G + H + K = 0$, there is a Right reply, H^R say, in a different component, so that $G^L + H^R + K \le 0$. The Left reply cannot be in the same component because G^{LR} is not reversible. Applying the induction hypothesis to this, we obtain $p(G^L) + p(H^R) + p(K) \le 0$, which means that, going second in $p(G) + p(H) + p(K)$, Right wins, so $p(G) + p(H) + p(K) \le 0$.

Finally, if exactly one of G, H, and K, say H, is of the form x or $x*$ for some number x, then the implication

$$G + H + K = 0 \Rightarrow p(G) + p(H) + p(K) = 0$$

is immediate modulo the observation that if $\{G^L|G^R\}$ is a game in canonical form, then the game formed with the canonical forms of G^L+x* and G^R+x* as Left and, respectively, Right options is also in canonical form.

□

The following lemma shows one sense in which $\overline{G}$ approximates G.

Lemma 79 *For any game G, $L(\overline{G}) = L(G_*) = L(G)$ and $R(\overline{G}) = R(G_*) = R(G)$, i.e. G, G_* and $\overline{G}$ have the same stops.*

Proof: We need to show separately that $L(G_*) = L(G)$ and $L(p(H)) = L(H)$. The latter relation can be obtained inductively: if $H \neq x$ or $x*$, and H is in canonical form,

$$L(p(H)) = L(\{p(H^L)|p(H^R)\}) = \max R(p(H^L)) = \max R(H^L) = L(H),$$

and similarly for the Right stops.

For the former relation, we observe first that $R(G + *) = R(G)$ for any game G, and hence, if G is not a number,

$$L(G_*) = \max R(G_*^L + *) = \max R(G_*^L) = \max R(G^L) = L(G),$$

which completes the proof. □

Theorem 80 *A game G is an infinitesimal if and only if $\overline{G} = 0$.*

Proof: The "if" direction follows from the lemma. Next, we will prove that, if G is an infinitesimal, then $G_* = 0$ (and hence $\overline{G} = 0$). We will do so by showing inductively

$$(L(G) \leq 0 \text{ and } R(G) \leq 0) \Longrightarrow G_* \leq 0 \tag{1}$$

and

$$(L(G) \geq 0 \text{ and } R(G) \geq 0) \Longrightarrow G_* \geq 0. \tag{2}$$

Because of the symmetry of the definition of G_*, it is enough to prove (1), so let G satisfy the hypothesis of (1). Suppose G_* is a number. This is an easy case because, from the lemma, G_* has the same stops as G, so $G_* \leq 0$ and we are done. Suppose G_* is not a number. Then, Left's move in G_* will lead to a position $G_*^L + *$. Now, if G_*^L is a number, then, applying the lemma again, we find that $G_*^L \leq 0$, hence Right's move from $G_*^L + *$ to G_*^L will force a loss for Left, so $G_* \leq 0$. Suppose now that G_*^L is not a number. Then,

$$G_*^L + * = \{G_*^{LL} + * | G_*^{LR} + *\} + *.$$

As $L(G) \leq 0$, there must exist a Right-option G^{LR_0} in G^L so that $L(G^{LR_0}) \leq 0$. Therefore, Right can move from $G_*^L + *$ to $G_*^{LR_0} + * + * = G_*^{LR_0}$ and, applying the lemma one more time, $L(G_*^{LR_0}) \leq 0$. Now, if $G_*^{LR_0}$ is a number, then it cannot be strictly positive, so Left will lose going first in $G_*^{LR_0}$. Finally, if $G_*^{LR_0}$ is not a number, then $R(G_*^{LR_0}) \leq 0$ (because we already know that $L(G_*^{LR_0}) \leq 0$), so $G_*^{LR_0}$ satisfies the conditions of the induction hypothesis, so $G_*^{LR_0} \leq 0$, so Left will lose going first in G_* in any case, so $G_* \leq 0$ and the proof is completed. □

Theorem 81 *The reduced canonical form $\overline{G}$ is infinitesimally close to G.*

Proof: We will show first that G_* is infinitesimally close to G and then that $p(H)$ is infinitesimally close to H. We will establish inductively that:

$$G_* - G - x \leq 0 \text{ for every positive number } x. \tag{3}$$

This will be enough to ensure that $\overline{G}$ is infinitesimally close to G, because applying (3) to $-G$ yields $(-G)_* + G - x \leq 0$, so $-(-G)_* - G + x \geq 0$. As $-(-G)_* = G_*$ (G_* is a homomorphism), $G_* - G + x \geq 0$, so $G_* - G$ will be greater than all negative numbers and smaller than all positive numbers.

We only have to consider the case when G is not a number. In this case,

$$G_* - G - x = \{G_*^L + * | G_*^R + *\} + \{-G^R - x | -G^L - x\}.$$

After Left makes his first move in this game, Right can reply to either of the following:

$$G_*^L + * - G^L - x = (G_*^L - G^L - x/2) + (* - x/2)$$

$$-G^R - x + G_*^R + * = (G_*^R - G^R - x/2) + (* - x/2).$$

The induction hypothesis applies to $(G_*^L - G^L - x/2)$ and $(G_*^R - G^R - x/2)$, so they are both negative, and as $(* - x/2)$ is also negative, Left loses, hence $G_* - G - x \leq 0$, as desired.

□

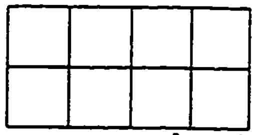
The proof of $p(G) - G - x \leq 0$ follows precisely the same steps if we choose G to be in canonical form.

Theorem 82 *$\overline{G}$ is the sparsest game infinitesimally close to G.*

Proof: Let H be infinitesimally close to G. We need to show that $\overline{G}$ is at least as sparse as H. As $G - H$ is an infinitesimal, $\overline{G - H} = 0$ so $\overline{G} = \overline{H}$. As $\overline{K}$ is in canonical form for any game K, we have $\overline{G} \equiv \overline{H}$, so all we need to show is that $\overline{H}$ is at least as sparse as H for any game H. For this purpose, we can relax the definition of $\overline{G}$ in the sense that p is not applied to the canonical form of G_*, but directly to G_* (that is to the form obtained after cooling G by $*$, without deleting any dominated options or bypassing any reversible moves). If we denote the result by $\tilde{G}$, then $\tilde{G}$ will be at least as sparse as G — it can be seen inductively that the only thing achieved in the process of forming $\tilde{G}$ is to replace $x*$ by x everywhere in G. Yet, $\overline{G}$ is at least as sparse as $\tilde{G}$ because when p is applied to the canonical form K^c of a game K, the outcome will be at least as sparse as when p is applied directly to K ($p(K^c)$ is at least as sparse as $p(K)$ because, considering the sequence $K, K_1, K_2, ..., K_n = K^c$ where for every i, K_{i+1} is obtained from K_i by deleting a dominated option or by bypassing a reversible move, $p(K_{i+1})$ will be at least as sparse as $p(K_i)$). We have thus proved that $\tilde{G}$ is at least as sparse as G, and $\overline{G}$ is at least as sparse as $\tilde{G}$, which implies that $\overline{G}$ is at least as sparse as G. □

We needed this kind of argument because it can occur that G is sparser than G_*. For example, let $G = \{1|*\}$. Then $G_* = \{1*|*\}$, whereas $\overline{G} = p(G_*) = \{1|0\}$.

Definition 83 *Let $G = \{G^L|G^R\}$. A number x is* ***permitted*** *by G if $G^L \not> x$ and $x \not> G^R$ for every G^L, G^R.*

Example Let us consider the Domineering position $G =$  . We have seen in Section 4.5 that $G = -_2 = \{2|0||0\}$. Although $-_2$ is not a number, it does permit 0 because $\{2|0\} \not> 0$ (they are incomparable) and $0 \not> 0$. There are no other permitted numbers for G. Indeed, the set of numbers that are not strictly less than G^L is $[0, \infty)$, and the set of numbers that are not strictly greater than G^R is $(-\infty, 0]$ and their intersection is $\{0\}$.

Theorem 84 *Let $G = \{G^L|G^R\}$ be such that $\{\overline{G^L}|\overline{G^R}\}$ permits at most one number. Then, $\overline{G} = \overline{\{\overline{G^L}|\overline{G^R}\}}$.*

Proof: Suppose that at least one of G and $\{\overline{G^L}|\overline{G^R}\}$ is not a number. Then, for any positive number x, the translation principle gives

$$H = \{G^L|G^R\} + \{-\overline{G^R}| - \overline{G^L}\} - x =$$

$$= \{G^L - x|G^R - x\} + \{-\overline{G^R}| - \overline{G^L}\}$$

or

$$H = \{G^L|G^R\} + \{-\overline{G^R} - x| - \overline{G^L} - x\},$$

and therefore, for any Left option in H, there is a Right response in the other component that leaves a negative game (applying Theorem 81), so Left, going first in H, loses, hence $H \leq 0$. Similarly, we show that

$$H' := \{G^L|G^R\} + \{-\overline{G^R}| - \overline{G^L}\} + x \geq 0,$$

so

$$-x \leq G - \{\overline{G^L}|\overline{G^R}\} \leq x.$$

This means that $G - \{\overline{G^L}|\overline{G^R}\}$ is an infinitesimal, so

$$0 = \overline{G - \{\overline{G^L}|\overline{G^R}\}} = \overline{G} - \overline{\{\overline{G^L}|\overline{G^R}\}}$$

and the result is proved in this case.

Suppose, now, that G and $\{\overline{G^L}|\overline{G^R}\}$ are both numbers. As $\{\overline{G^L}|\overline{G^R}\}$ permits at most one number, and it is a number itself, we must have $R(\overline{G^L}) = L(\overline{G^R}) = \{\overline{G^L}|\overline{G^R}\}$. Applying Lemma 79 we find that G^L and G^R have the same stops as $\overline{G^L}$ and, respectively $\overline{G^R}$ and, since G is a number as well, we obtain $G = \{\overline{G^L}|\overline{G^R}\}$, hence $\overline{G} = \overline{\{\overline{G^L}|\overline{G^R}\}}$. $\square$

David Wolfe has implemented this approximation operator in his Games Kit [Wol96]. The Games Kit is essentially a (freely available) combinatorial-games calculator — all the operations introduced in Chapters 4 and 5 (and others) can be directly tested by the user on concrete games of her choice. Also, the program contains routines for evaluating several well-known combinatorial games, notably Domineering. The reduced canonical form routine can be accessed by typing $G[e]$, the program will return the reduced canonical form of G.

Chapter 6

Applications to Domineering of the Reduced Canonical Form

6.1 Known Values for Certain types of Boards

We denote by $D_{n \times m}$ the value of the Domineering game played on a rectangular board of size n by m. Note that, for any square Domineering board, G^L is a Left option if and only if $-G^L$ is a Right option. Therefore, any such game can be represented in the form

$$\left\{G^{L_1}, G^{L_2}, \ldots \middle| -G^{L_1}, -G^{L_2}, \ldots\right\} := \pm\left\{G^{L_1}, G^{L_2}, \ldots\right\}.$$

Clearly, $D_{1 \times 1} = 0$ and $D_{2 \times 2} = \pm 1$. The following are the other known values for square boards.

$$D_{3 \times 3} = \pm\left\{1, \left\{\frac{1}{2}\middle| -2\right\}\right\} \qquad \text{([Con76])}$$

$$D_{4 \times 4} = \pm\left\{0, \left\{\{2|0\}, 2 + {+_2} \,|\{2|0\}, {-_2}\right\} \| \, 0\right\} \qquad \text{([Wol96])}$$

$$D_{5 \times 5} = 0. \qquad \text{([Con76])}$$

To this list we can add one other:

$$D_{6\times 6} = \pm 1\text{-ish}.$$

The proof for this last one is based on the following inequalities:

$$D_{2\times 6} + D_{2\times 6} + D_{2\times 6} \le D_{6\times 6} \le D_{6\times 2} + D_{6\times 2} + D_{6\times 2}.$$

This inequality is based on the fact that, by splitting the 6×6 board into three horizontal 2×6 strips, we only delete Left options from $D_{6\times 6}$, and leave the Right options untouched. By Theorem 60, this operation yields a position which is less than or equal to $D_{6\times 6}$. The value of a 2×6 strip is relatively easy to calculate: $D_{2\times 6} = \{1 + -_2 | -1\} = \pm 1\text{-ish}$. By symmetry, $D_{6\times 2} = \{1 \ | -1 + +_2\} = \pm 1\text{-ish}$. This implies that

$$D_{6\times 6} = \pm 1\text{-ish} + \pm 1\text{-ish} + \pm 1\text{-ish} = \pm 1\text{-ish}.$$

Based on this idea we can find bounds for $D_{8\times 8}$:

$$-\frac{1}{2}\text{-ish} \le D_{8\times 8} \le \frac{1}{2}\text{-ish}.$$

We used the value

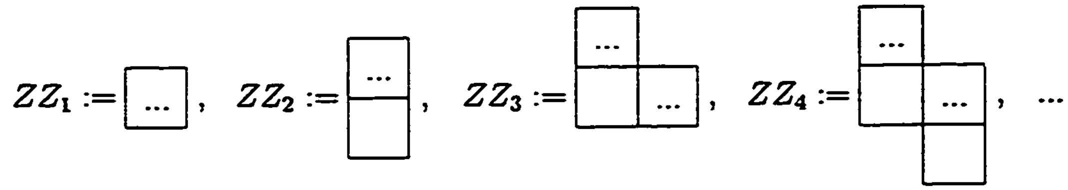
$$D_{2\times 8} = \left\{2 + -_2 | 0 \,\middle\|\, -\frac{1}{2} \middle| -2\right\} = \left\{2 | 0 \,\middle\|\, -\frac{1}{2} \middle| -2\right\}\text{-ish},$$

and the fact that the sum of the four copies

$$\left\{2|0 \,\middle\|\, -\frac{1}{2}\middle| -2\right\} + \left\{2|0 \,\middle\|\, -\frac{1}{2}\middle| -2\right\} + \left\{2|0 \,\middle\|\, -\frac{1}{2}\middle| -2\right\} + \left\{2|0 \,\middle\|\, -\frac{1}{2}\middle| -2\right\} = -\frac{1}{2}.$$

The conjecture is that $D_{8\times 8}$ is an infinitesimal.

In [ONAG, pages 117-118] Conway gives the values for the following Domineering positions, called 'zig-zags':



He shows that

$$ZZ_{8n+1 \text{ or } 8n+3} = 0\text{-ish}$$

$$ZZ_{8n-1 \text{ or } 8n-3} = \pm 1\text{-ish} \ .$$

$$ZZ_{8n+2} = 1\text{-ish}$$

$$ZZ_{8n-2} = \{2|0\}\text{-ish}$$

$$ZZ_{4n} = \{n|n-1||n-2|||n-3|||| \ \ldots \ 2|||||\ldots||1|||||\ldots|||0\}\text{-ish}$$

A misprint in ONAG is corrected in the expression for $ZZ_{8n-1 \text{ or } 8n-3}$ above.

In [Wol93], D. Wolfe finds the values within -ish for the so-called snakes — generalizations of the zig-zags. By using similar snake patterns, Yonghoan Kim [Kim96] shows that every number (as in Definition 49) is the value of some Domineering position. For instance, the board in Figure 1 has values

$$\frac{1}{2^9}, \frac{1}{2^8}, \ldots, \frac{1}{2^2},$$

and smaller values are obtained in an obvious way.

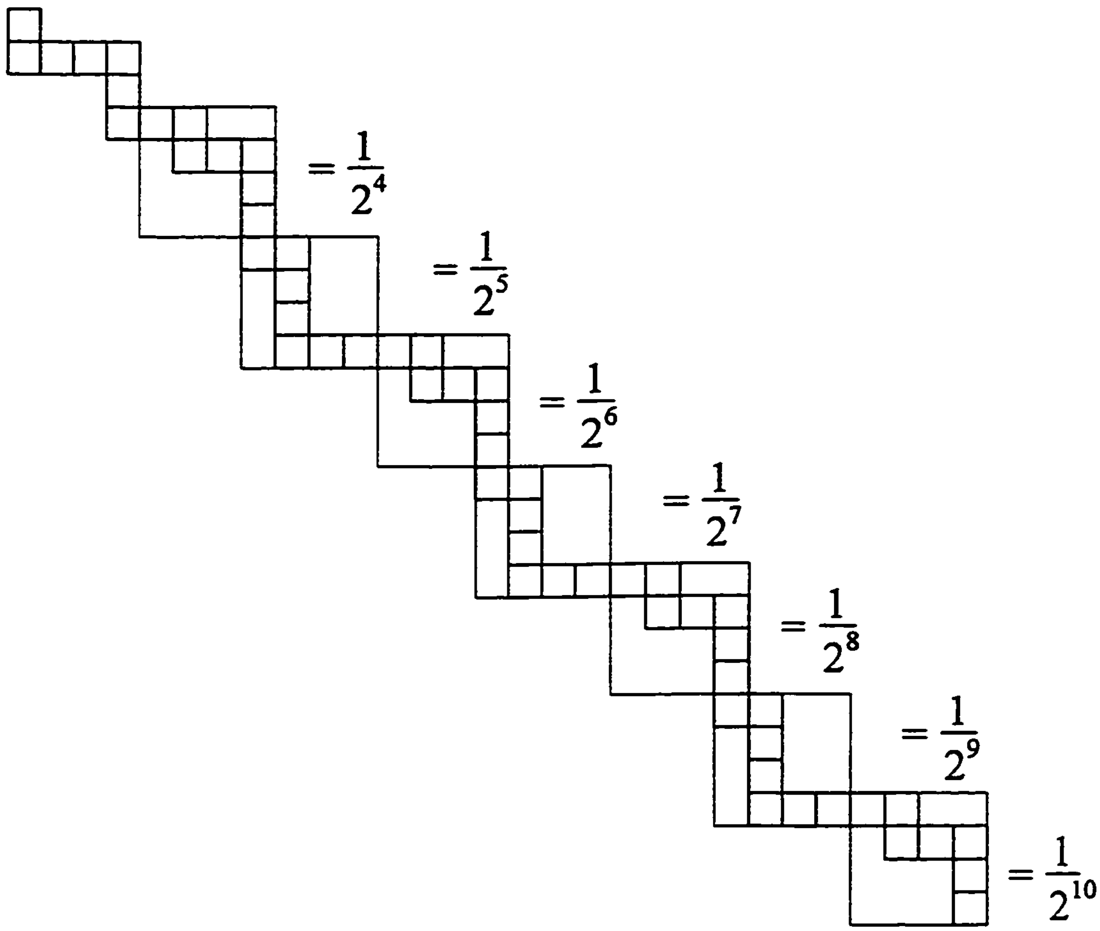


Figure 1.

By successively replicating each of the two bottom–right blocks of seven squares we obtain boards of values

$$\frac{1}{2^{11}}, \frac{1}{2^{12}}, \dots .$$

E. R. Berlekamp [Ber88] extensively studied Domineering on rectangular boards of sizes $2 \times n$ and $3 \times n$. Except for special sub-classes, he found their values within -ish. He has also shown (personal notes) that

$$D_{4\times n} = 2 \cdot D_{2\times n} + g,$$

where g is a game of zero mean and of temperature less than or equal to that of $D_{2\times n}$. He conjectured that g is zero for n odd. The method used is essentially strategic — it is based on outlining an optimal line of play for both Left and Right that reduces the problem (within infinitesimals) to a less complex, one-dimensional game, called 'Blockbusting'. We will present in the next section a different approach to the study of $2\times n$ Domineering that completely characterizes these games within infinitesimals and confirms Berlekamp's conjectured values.

6.2 A Recursive Approach to $2 \times n$ Domineering

The values D_n of $2 \times n$ Domineering display a pattern that is common to several other classes of games like, for instance 'Snort' [ONAG, page 91 and WW, page 96] played on chains of length n, or 'Toads and Frogs' (WW, page 14 and [EH83]). As n increases, the values become very complicated very fast. However, at least in the case of $2 \times n$ Domineering, the complications are only due to infinitesimals. In other words, the reduced canonical forms are simple and relatively easy to compute. Moreover, as we will show, they display the following striking periodic behavior:

$$\overline{D}_{n+10} = \overline{D}_n + \left\{0 \,\middle|\, -\frac{3}{2}\right\} \text{ for all } n \geq 6. \tag{1}$$

Note that (1) essentially solves the problem in general and proving it will be the principal objective of this chapter. All we need to do is compute the first 15 values of $\overline{D}_n$. This was done using David Wolfe's Games Kit [Wol96] and the following results were obtained:

n	$\overline{D}_n$
1	1
2	± 1
3	$\left\{2 \mid -\frac{1}{2}\right\}$
4	0
5	$\frac{1}{2}$
6	± 1
7	$\left\{\frac{3}{2} \mid -\frac{1}{2}\right\}$
8	$\left\{2 \mid 0 \parallel -\frac{1}{2} \mid -2\right\}$
9	$\left\{\frac{5}{2} \mid \frac{1}{2} \parallel 0 \mid -\frac{3}{2}\right\}$
10	$\left\{3 \mid 1 \parallel \frac{1}{2} \mid -1 \mid\mid\mid -\frac{3}{2}\right\}$
11	$\left\{\frac{7}{2} \mid \frac{3}{2} \parallel 1 \mid -\frac{1}{2} \mid\mid\mid -1\right\}$
12	$-\frac{1}{2}$
13	0
14	$\left\{\frac{1}{2} \mid\mid\mid 0 \mid -\frac{3}{2} \parallel -2 \mid -\frac{7}{2}\right\}$
15	$\left\{1 \mid\mid\mid \frac{1}{2} \mid -1 \parallel -\frac{3}{2} \mid -3\right\}$

which confirms Berlekamp's results.

From the definition of addition for games we get

$$\left\{0 \middle| -\frac{3}{2}\right\} + \left\{0 \middle| -\frac{3}{2}\right\} + \left\{0 \middle| -\frac{3}{2}\right\} + \left\{0 \middle| -\frac{3}{2}\right\} = -3,$$

and we can re-write (1) in the following form:

$$\overline{D}_{n+40} = \overline{D}_n - 3 \text{ for all } n \geq 6. \tag{2}$$

In other words, for n large enough, any additional 2×40 piece in a game of $2 \times n$ Domineering has a net value of 3 for Right.

Typically, in order to prove a periodicity result like (1), one would first derive a recursive formula which has a general form like this:

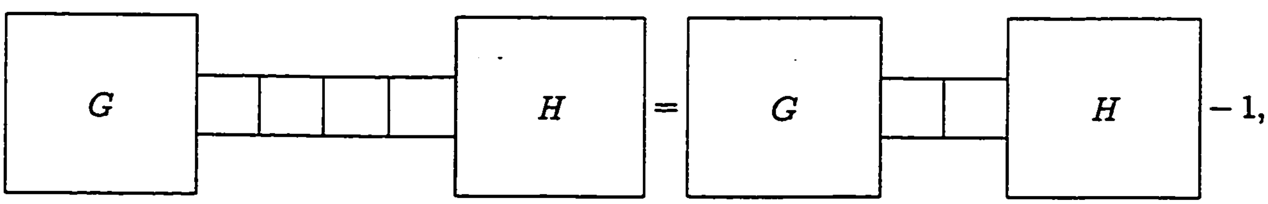
$$D_{n+1} = \{f_L(D_0, D_1, ..., D_n) \mid f_R(D_0, D_1, ..., D_n)\}, \tag{3}$$

where f_L and f_R define sets of options based on simpler positions. The difficulty in obtaining such a formula for $2 \times n$ Domineering lies in that, although every Left move splits the board into simpler positions, this is not the case for the Right moves which only loosely disconnect the board. The essence of Theorem 87 is to show that, at best play, a Right domino will effectively split the board in the sense that any potential subsequent move by Right that affects both components must be a bad move. In the case of $2 \times n$ Domineering the recursive formula will involve the reduced canonical forms of the positions rather than the values themselves. Also, the recursive formula will be expressed in terms of two other sequences of games (beside D_n itself) related to D_n and corresponding to certain "boundary conditions" on the $2 \times n$ Domineering strip. Once the recursive formula is established, Theorem 89 will be used to confirm (1). Theorem 89 is tailored for the specific purposes of our recursion. For a version that applies to impartial combinatorial games please refer to [WW, page 100] or to [GS56].

6.3 The Separation Theorem

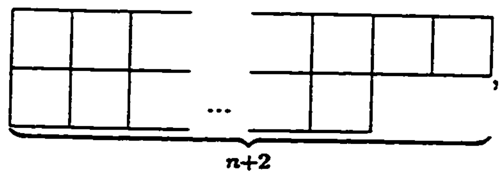
The object of this section is to establish a rather technical result (Theorem 86) which is essential in the process of deriving a recursive formulation for $2 \times n$ Domineering. We will also use the following result of David Wolfe [Wol93]:

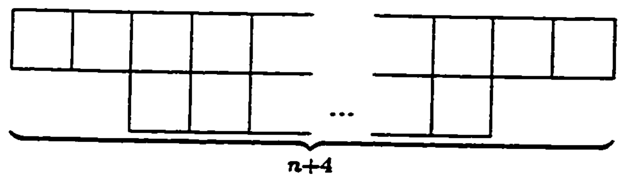
Lemma 85



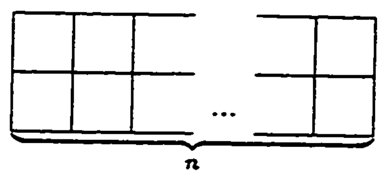
where G and H are domineering positions that only attach to each other through a 1×4 and respectively 1×2 Domineering strip.

We will also introduce the following notation for certain positions arising in $2 \times n$ Domineering. For all non-negative integers n consider the following games and their values:

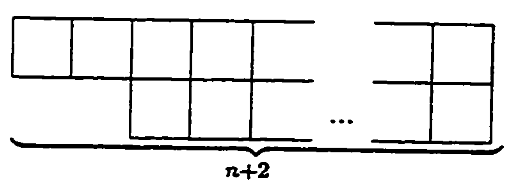
$B_n =$ 

$C_n =$ 

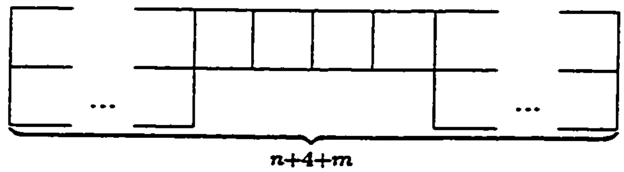
As mentioned before, D_n is the $2 \times n$ Domineering game:

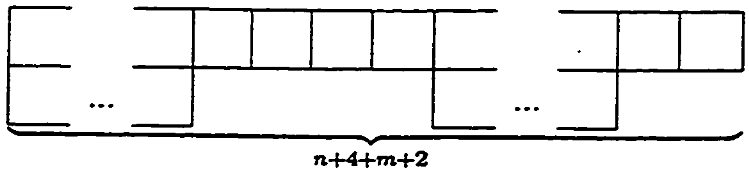
$D_n =$ 

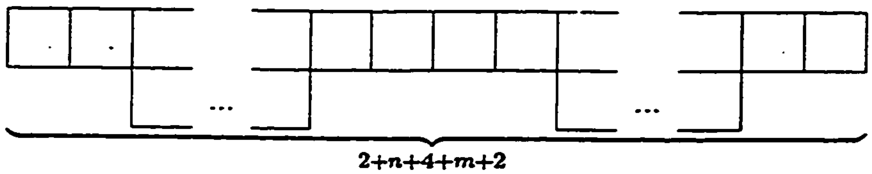
Note that the board

$B'_n =$ 

has the same value as B_n because of symmetry. By linking together boards of the above types we obtain the following:

$B_nB'_m =$ 

$B_nC_m =$ 

$C_nC_m =$  .

Theorem 86 *(Separation) For any $n, m \geq 0$,*

$$B_nB'_m = B_n + B_m \text{ and}$$
$$B_nC_m = B_n + C_m \text{ and}$$
$$C_nC_m = C_n + C_m.$$

Intuitively, Theorem 86 says that, given a board made of two $2 \times n$ Domineering boards linked to each other by a 1×4 piece, one can cut it along the middle of the 1×4 piece into two boards which sum up to the same value as the initial board. Note that the games $B_nB'_m$, B_nC_m and C_nC_m are always less than or equal to $B_n + B_m$, $B_n + C_m$ and $C_n + C_m$ respectively. This is true because Right has the extra option to place a domino that overlaps with both the B_n and B'_m components in $B_nB'_m$ while all the other options of Left and Right in $B_nB'_m$ have directly corresponding options in either B_n or B'_m. Therefore, the real problem is to show that the extra option of Right that we have just mentioned is never useful for him.

Proof: We will show that $x_n y_m = x_n + y_m$ for all n, $m \geq 0$, where x_n is either B_n or C_n, and y_m is B'_m or C_m. We use this compressed notation because the arguments in the proof are identical for each of these cases.

It is enough to prove the relation $x_n y_m = x_n + y_m$ for n, $m \geq 1$ because when n or m are zero we have a particular instance of Lemma 85. As noted before, it is always true that $x_n y_m \leq x_n + y_m$ so we will concentrate on showing that $x_n y_m - x_n - y_m \geq 0$, which is equivalent to Left winning by going second in $G = x_n y_m + (-x_n) + (-y_m)$. We stripe the horizontal component and box the vertical components as shown in Figure 2 following the idea of Berlekamp in [Ber88].**Stripes** are alternate 2×1 domino positions. The first and last available columns are not used as stripes. **Boxes** are 2×2 contiguous squares and the boxing starts at the lower ends of the vertical components with the first available spaces (by convention, the lower ends of $(-x_n)$ and $(-y_m)$ are those that correspond to the left and, respectively right end of $x_n y_m$.) When the distinction is not necessary, we will refer to both boxes and stripes as **parcels**.

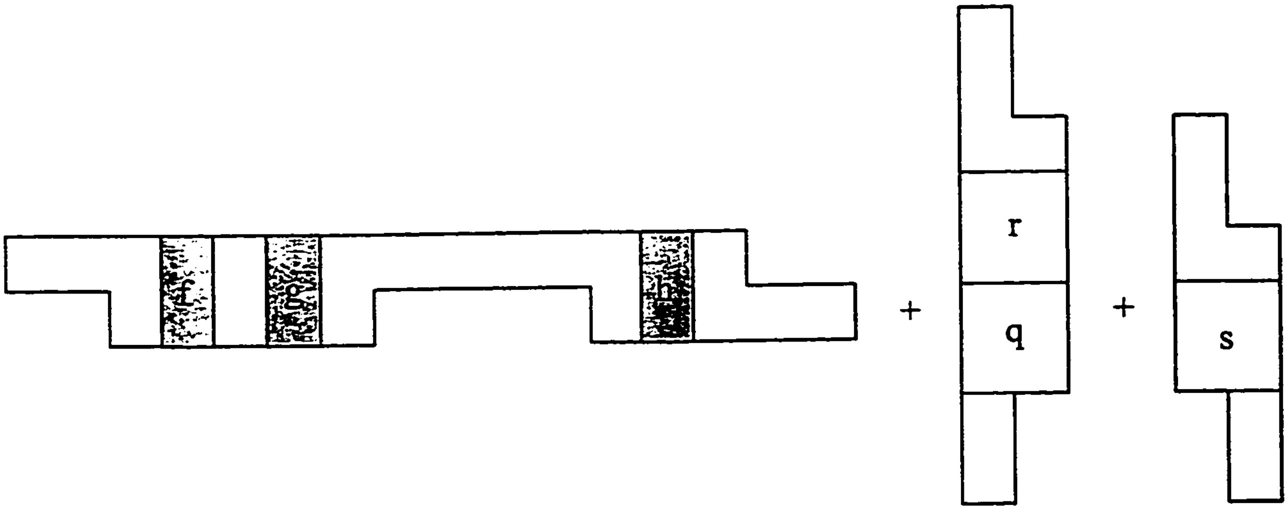


Figure 2. The stripes are f, g and h; the boxes are q, r and s.

We will now define four types of Left moves and four types of Right moves:

- A move $Lv1$ is realized by Left when placing a (vertical) domino into an unoccupied box of a vertical component. We adopt the convention to place it on the right of the box.
- A move $Lv2$ is realized by Left when placing a domino into a box where a move $Lv1$ has previously been made, i.e. it will be on the left of the box.
- A move $Lh1$ is realized by Left when placing a domino over a stripe of the horizontal component.
- A move $Lh2$ is realized by Left when placing a domino beside a stripe of the horizontal component.
- A move $Rh1$ is realized by Right when placing a (horizontal) domino on the horizontal component such that this is the first domino that overlaps the (unique) stripe which it intersects.
- A move $Rh2$ is realized by Right when placing a domino on the horizontal component such that this is the second domino that overlaps the stripe it intersects.
- A move $Rv1$ is realized by Right when placing a domino on a vertical component such that it is the first domino to be placed in that box.
- A move $Rv2$ is realized by Right when placing a domino on a vertical component such that it is the second Right domino to be placed in that box.

Note that so far we have not considered moves that do not correspond to any parcel. These are moves located at the ends of the boards $(-x_n)$, $(-y_m)$ and $x_n y_m$ and also over the portion where x_n and y_m link to form $x_n y_m$. It is easy to deal with the moves located at the ends of the boards $(-x_n)$, $(-y_m)$ or $x_n y_m$ — on $x_n y_m$ we can introduce 0, 1 or 2 stripes of shape 1×1 at the ends of the board, depending on whether x_n and y_m are

B_n or C_m, which creates the possibility for 0, 1 or 2 extra *Rh*2 moves respectively (the stripes are located as shown in Figure 3.)

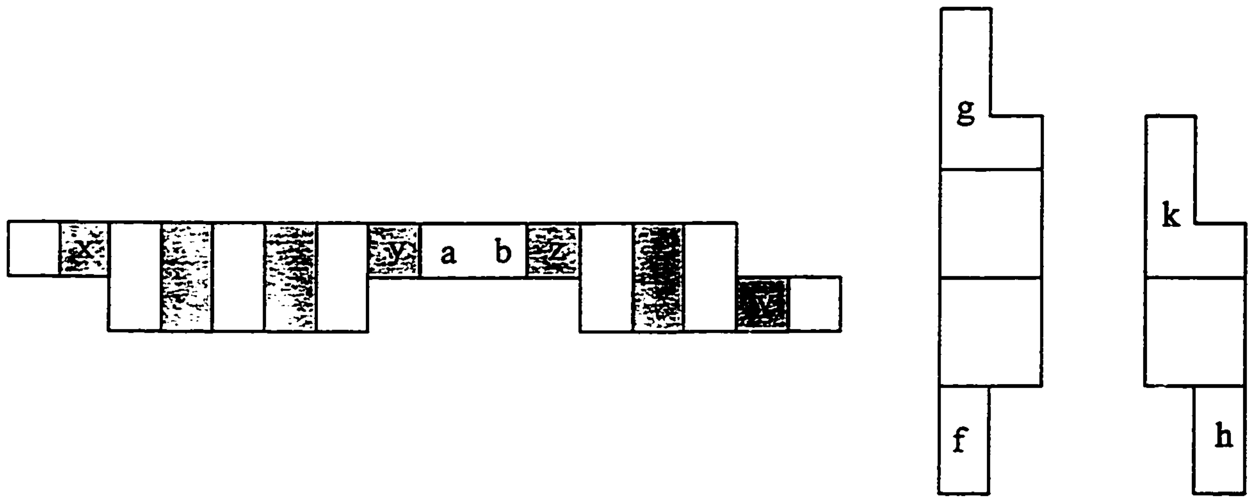


Figure 3. The new stripes are x, y, z and v; the new boxes are f, g, h and k.

Similarly, depending on whether x_n and y_m are B_n or C_m, we introduce 2×1 boxes at the ends of $(-x_n)$ and $(-y_m)$ which creates the possibility for more *Lv*2 moves (see Figure 3 again.) Finally, we introduce two more 1×1 stripes on the portion of $x_n y_m$ where x_n and y_m link such that they lie two squares away from the nearest stripes — this will create the possibility for two more *Rh*2 moves for Right. In the unique situation when n and m are both odd integers there is one more possible move for Right on the horizontal board (ab in Figure 3) and we will call it "the special move".

An immediate consequence of the definition of parcels is that the number α of boxes is the same as the number of stripes and

$$\alpha = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor + \beta + 2$$

where $\beta = 0$ if $x_n = B_n$ and $y_m = B'_m$; $\beta = 1$ if $x_n = B_n$ and $y_m = C_m$; $\beta = 2$ if $x_n = C_n$ and $y_m = C_m$. Furthermore, if for each *type* we denote by $f\,(type)$ the maximum number of possible moves (of that type) which can occur during any one actual play of G, we obtain the following values for f:

$$
\begin{aligned}
f\,(Lv1) &= \alpha - \beta - 2 = f\,(Rh1) \\
f\,(Lh1) &= \alpha - \beta - 2 = f\,(Rv1) \\
f\,(Lv2) &= \alpha = f\,(Rh2) \\
f\,(Lh2) &= \alpha = f\,(Rv2)\,.
\end{aligned}
$$

Therefore the maximum overall number of possible moves for each player is 4α except in the case when n and m are odd integers and hence Right has $4\alpha + 1$ moves after accounting for the special move. We will extend the notation $f\,(type)$ to the maximum number of moves that are still possible at a given moment during the play.

We can show now that using the following "move-replication" strategy, Left can win going second in $G = x_n y_m + (-x_n) + (-y_m)$. Every time Right makes a move of type $Rh1$ or $Rv1$, Left will reply by a move of type $Lv1$ or respectively $Lh1$ in the corresponding parcel of the component of opposite orientation (the two orientations are vertical and horizontal.) If the corresponding parcel already contains a Left move of type $Lv1$ or $Lh1$, then Left will make any other move of type 1 (that is $Lv1$ or $Lh1$) if there are any left and, if there are none left, she will make a move of type 2 ($Lv2$ or $Lh2$.) Note that the parcel corresponding to (in the component of opposite orientation) the Right move of type $Rh1$ or $Rv1$ cannot be occupied by anything but a domino of type $Lv1$ or, respectively $Lh1$ because otherwise the parcel where Right moved would have been occupied by a previous reply of Left. Every time Right makes a move of type $Rh2$ or $Rv2$, Left will reply by a move of type $Lv1$ or $Lh1$ if there are any available; otherwise she will make a move of type $Lv2$ or respectively $Lh2$. In order to prove that this is a winning strategy for Left we need to investigate the effects of every type of move on the values of $f\,(type)$ for each of the eight types. The results are summarized in the table below, where the entries along a row represent the changes in the values of $f\,(type)$ for all eight types after a move of the type given at the beginning of the row is made:

	$f\,(Lv1)$	$f\,(Lh1)$	$f\,(Lv2)$	$f\,(Lh2)$	$f\,(Rv1)$	$f\,(Rh1)$	$f\,(Rv2)$	$f\,(Rh2)$
$Lv1$	-1	0	0	0	-1	0	-1	0
$Lh1$	0	-1	0	0	0	-1	0	-1
$Lv2$	0	0	-1	0	0	0	0 or -1	0
$Lh2$	0	0	0	-1	0	0	0	0
$Rv1$	-1	0	-1	0	-1	0	0	0
$Rh1$	0	-1	0	-1	0	-1	0	0
$Rv2$	0	0	0	0	0	0	-1	0
$Rh2$	0	0	0 or -1	0	0	0	0	-1

If at a given moment during the play we denote by l the maximum possible remaining number of Left moves and by r the maximum possible remaining number of Right moves, then if it is Left's turn to move and she makes a move of type 1 ($Lv1$ or $Lh1$) she will increase the value of $l - r$ for the resulting position by one. Similarly, if it is Right's turn to move and he makes a move of type 1 ($Rh1$ or $Rv1$) then he decreases the value of $l - r$ by one. Any other moves by the two players will either keep $l - r$ constant or change it in favor of the other player — of course, Left favors higher values of $l - r$ while Right favors lower values for $l - r$. As the purpose of the game is to have the last move, and $l - r$ equals zero or -1 at the beginning (recall that $l - r = -1$ when n and m are odd because of Right's special move) Right cannot afford to make fewer than $\alpha - \beta - 3$ moves of type 1 throughout the play (otherwise Left can make at least $\alpha - \beta - 1$ type 1 moves and, even accounting for Right's special move, Right will run out of moves before Left and lose). But Left replies to every move of type 1 by a move of type 1 of her own which means that, throughout the play, Right can make at most one move of type 2 before all his moves of type 1 are consumed. We will examine all positions which can occur immediately after all type 1 moves are exhausted and at most one type 2 move had been made by Right; we will show that all these are winning positions for Left, and this will complete the proof. More specifically, if no move of type 2 had been made by Right, then each player has made $\alpha - \beta - 2$ moves of type 1 and, as Right started, we need to show that, from such a position, Left wins going second. If Right had made one move of type 2, then the position after the exhaustion of type 1 moves does not depend on when (during the play) Right made his type 2 move. We can assume that he made it immediately after all the type 1 moves were exhausted and this brings us to the previous case, when no move of type 2 had been made at all. We will call such positions (obtained just after the exhaustion of the type 1 moves) **saturated** and show that they have values greater than or equal to zero.

For a saturated position, consider the connected-board components of $x_n y_m$ in which at least one Right domino has been placed. Also consider the connected-board components of $(-x_n)$ and $(-y_m)$ in which at least one Left domino has been placed. The claim is that the value of a horizontal board connected component equals half the number of empty 2×1 columns in the component minus the number of Right dominoes adjacent to the component (we make the convention that the 2×1 and 1×2 spaces at the boundaries of $(-x_n)$, $(-y_m)$ and $x_n y_m$ as well as the 1×4 portion of the $x_n y_m$ board where x_n and y_m link are counted as adjacent to Left and Right dominoes.) Similarly, the value of a vertical board connected component equals the number of adjacent Left dominoes minus half the number of empty 1×2 rows in the component. We will prove the formula for horizontal board components — the proof of the formula for vertical boards is analogous.

The first observation we make on a horizontal board component is that it contains at most two 2×1 empty columns. This follows from the fact that every stripe must have been overlapped by either a horizontal or a vertical domino as a consequence of the fact that all possible moves of type 1 have been made in $x_n y_m$.

The second observation on a horizontal board component goes like this: if two horizontal dominoes are adjacent then one can remove the 2×2 portion of the board containing one of them and link back the two amputated sides (see Figure 4 below)

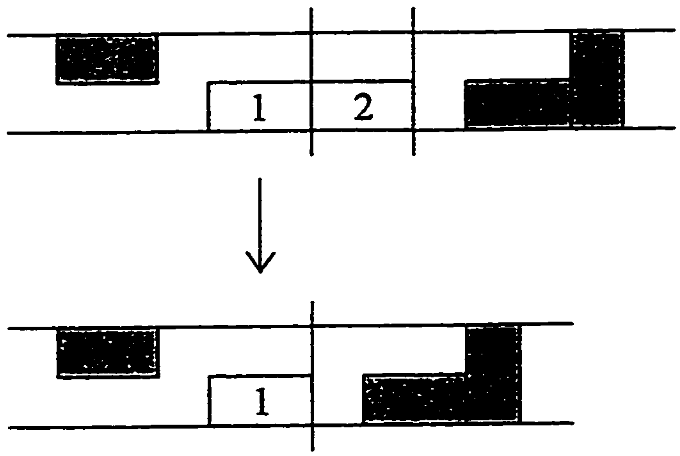
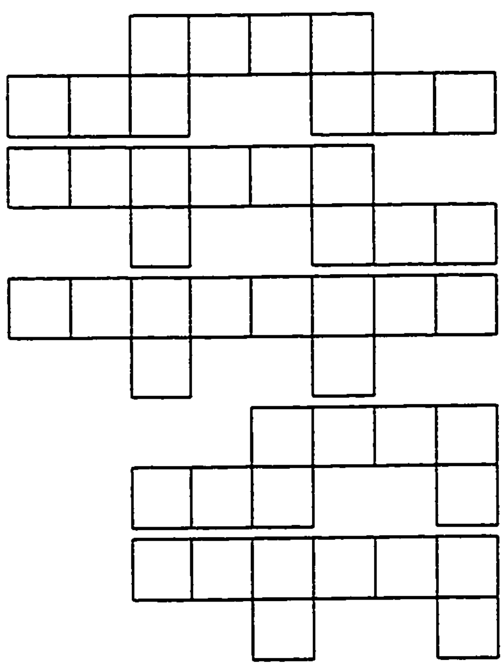


Figure 4. The size 2 square containing domino 2 is removed.

thus increasing the value of the component by one — this is a direct consequence of Lemma 85. Therefore, we only need to prove the formula for connected horizontal components which do not contain adjacent Right dominoes. The following are the only possible such components and their values verify the claimed formula:



$$= -2\left(= \frac{1}{2} \times 2 - 3\right)$$

$$= -2$$

$$= -2$$

$$= -1\left(= \frac{1}{2} \times 2 - 2\right)$$

$$= -1$$

Therefore, these two regions contribute a value of at least $1 + \frac{1}{2} = \frac{3}{2}$ to the total value. In conclusion, the net change to the total value induced by this situation is

$$\pm 1 + \frac{3}{2} = \left\{ \frac{5}{2} \middle| \frac{1}{2} \right\} > 0.$$

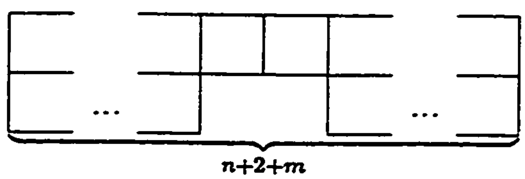
This proves that the total value is greater than or equal to zero in any case and the proof is completed. $\square$

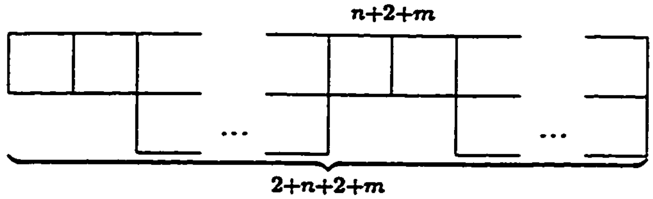
6.4 The Solution of $2\times n$ Domineering to within -ish

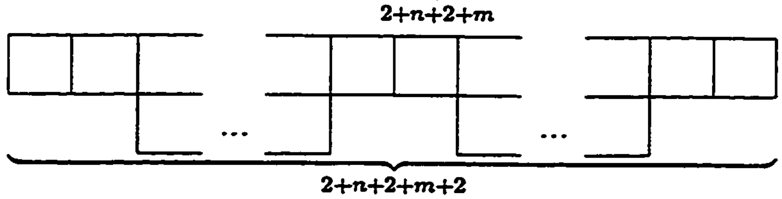
Theorem 87 *For all n, $m > 0$,*

$$B_nD_m = B_n + B_m + 1 \textit{ and}$$
$$C_nD_m = C_n + B_m + 1 \textit{ and}$$
$$C_nB_m = C_n + C_m + 1,$$

where

$B_nD_m =$ 

$C_nD_m =$ 

$C_nB_m =$ 

Proof: It follows directly by applying Theorem 86 and Lemma 85 for $G = D_n$ and $H = D_m$ for the first equation; $G = B'_n$ and $H = D_m$ for the second equation; $G = B'_n$

and $H = B_m$ for the third. $\square$

Based on Theorem 87 we can now establish the following recursive formulae for D_n, B_n and C_n:

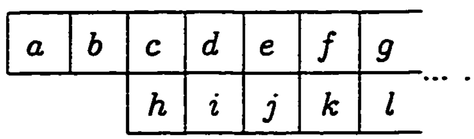
$$D_0 = 0,\ B_0 = -1,\ C_0 = -2;$$
$$D_1 = 1,\ B_1 = -\tfrac{1}{2},\ C_1 = -\tfrac{3}{2};$$

and for $n \geq 2$:

$$D_n = \Big\{\{D_i + D_j\}_{i+j=n-1} \,|\, \{B_i + B_j + 1\}_{i+j=n-2}\Big\} \tag{6}$$
$$B_n = \Big\{\{D_i + B_j\}_{i+j=n-1} \,|\, \{B_i + C_j + 1\}_{i+j=n-2}\Big\} \tag{7}$$
$$C_n = \Big\{\{B_i + B_j\}_{i+j=n-1} \,|\, \{C_i + C_j + 1\}_{i+j=n-2}\Big\} \tag{8}$$

In order to be able to apply Theorem 87 to confirm (7) and (8) ((6) follows directly), we need to make sure that those Right moves that overlap with the "kinks" at the ends of B_n and C_n are dominated and therefore only (interior) moves by Right that lead to positions of type C_iD_j and respectively C_iB_j need to be considered. This will be ensured by the following lemma:

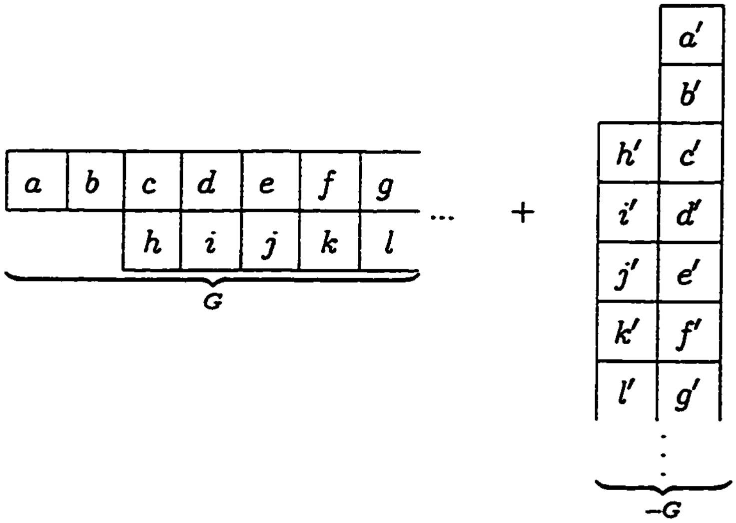
Lemma 88 *Consider a domineering board of shape B'_n or C_n where $n \geq 2$:*



Then the Right moves $[ab]$ and $[bc]$ are dominated by $[cd]$.

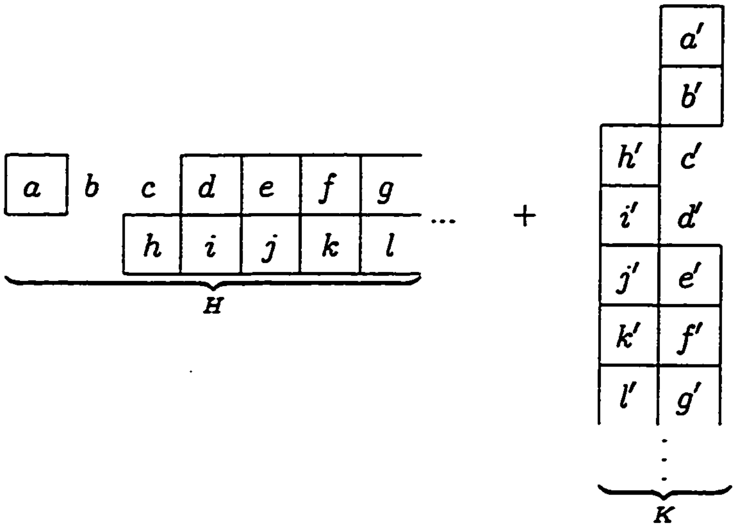
Proof: We will show that $[cd]$ dominates $[bc]$ (the proof for $[cd]$ dominating $[ab]$ goes similarly and it is in fact simpler.) The proof follows the same steps whether the board is B'_n or C_n so we will denote the board by G. We will show that if in playing $G + (-G)$ Right starts by placing a domino at $[bc]$ in G and Left replies by placing a domino at

$[c'd']$ in $(-G)$, then Left wins.



$G + (-G)$ before any moves are made

We denote the position reached after these first two moves are made by $H + K$:



We will show more, namely that Left wins in all *symmetric partial board positions* of $H + K$. By a symmetric partial board position we mean the position obtained after any number of pairs ($[xy]$ and $[x'y']$) of dominoes have been placed on $H + K$ (if $[xy]$ is a vertical domino in H then $[x'y']$ is the corresponding horizontal domino in K, and if $[xy]$ is a horizontal domino in H then $[x'y']$ is the corresponding vertical domino in K.) We will proceed by induction on the number of 1×1 boxes in a symmetric partial board position. The initial cases for the induction are naturally trivial because Right has no place to play a domino and hence he loses because he is the first to play. Let $H' + K'$ be a symmetric partial board position and let $[xy]$ be a first move by Right in $H' + K'$. If $[xy]$ is played in K, then x' and y' are not taken in H and Left will place a domino $[x'y']$ and win by applying the induction hypothesis that symmetric partial board positions of lesser size are wins for Left. Similarly, if $[xy]$ is played in H and $[xy] \neq [de]$ then Left will replicate this move in K and win. The last possibility is when Right plays $[xy] = [de]$. We have two cases here. If $[e'f']$ is available in K then Left plays a domino there. Now we can use Lemma 85 on the resulting position and remove $[hi]$ and $[h'i']$ from the board without altering the value. The resulting position is a symmetric partial board position of lesser size so Left wins again. The second case occurs when $[e'f']$ is not available in K after Right played at $[de]$. Left's response in this case is at $[i'j']$. In this case we apply Lemma 85 to remove $[hi]$ and $[a'b']$ from the board without changing the value of the position and we notice that for the resulting board $H^* + K^*$ we have $H^* = -K^*$ (this follows from the observation that a domino $[\alpha\beta]$ can be placed on H^* if and only if $[\alpha'\beta']$ can be placed on K^*). This means that Left can win going second and the proof is complete. □

The next theorem is the central result of this section — it establishes the periodicity behaviour of $2 \times n$ Domineering within -ish.

Theorem 89 *For all $n \geq 6$ the reduced canonical forms of $2 \times n$ Domineering positions satisfy the following arithmetic-periodicity formulae:*

$$\overline{D_{n+10}} = \overline{D_n} + \left\{0 \middle| -\frac{3}{2}\right\} \tag{9}$$

$$\overline{B_{n+10}} = \overline{B_n} + \left\{0 \middle| -\frac{3}{2}\right\} \tag{10}$$

$$\overline{C_{n+10}} = \overline{C_n} + \left\{0 \middle| -\frac{3}{2}\right\} \tag{11}$$

Proof: We will proceed by induction on n to prove the following three relations

$$\overline{D_{n+20}} = \overline{D_n} - \frac{3}{2} \tag{12}$$
$$\overline{B_{n+20}} = \overline{B_n} - \frac{3}{2} \tag{13}$$
$$\overline{C_{n+20}} = \overline{C_n} - \frac{3}{2} \tag{14}$$

Once (12), (13) and (14) are proved, it is a straightforward check over one period length that (9), (10) and (11) are true too. At the same time, we will prove by induction on n the following statements

$$\left\{\left\{\overline{D_i}+\overline{D_j}\right\}_{i+j=n-1}\middle|\left\{\overline{B_i}+\overline{B_j}+1\right\}_{i+j=n-2}\right\} \text{ permits at most one number} \tag{15}$$
$$\left\{\left\{\overline{D_i}+\overline{B_j}\right\}_{i+j=n-1}\middle|\left\{\overline{B_i}+\overline{C_j}+1\right\}_{i+j=n-2}\right\} \text{ permits at most one number} \tag{16}$$
$$\left\{\left\{\overline{B_i}+\overline{B_j}\right\}_{i+j=n-1}\middle|\left\{\overline{C_i}+\overline{C_j}+1\right\}_{i+j=n-2}\right\} \text{ permits at most one number} \tag{17}$$

The initial cases for the induction were verified using Wolfe's Game Kit [Wol96] for all $6 \le n \le 32$. The actual values are summarized later in this section. Let $k \ge 33$ and assume that (12), (13), (14), (15), (16) and (17) are true for every $6 \le n < k$. We will show that (12) and (15) hold for $n = k$ (the proof of (13), (14), (16) and (17) is almost identical.) Applying the reduced canonical form on both sides of (6) yields

$$\overline{D_{k+20}} = \overline{\left\{\{D_i + D_j\}_{i+j=k+19}\middle|\{B_i + B_j + 1\}_{i+j=k+18}\right\}}. \tag{18}$$

There exist infinitesimals ε_{ij} and δ_{ij} such that

$$\overline{D_{k+20}} = \overline{\left\{\left\{\overline{D_i}+\overline{D_j}+\varepsilon_{ij}\right\}_{i+j=k+19}\middle|\left\{\overline{B_i}+\overline{B_j}+\delta_{ij}+1\right\}_{i+j=k+18}\right\}}. \tag{19}$$

We can apply now the induction hypothesis on (13) and (14) because for every i and j such that $i+j=k+19$, either $i \geq 26$ or $j \geq 26$ and hence either $\overline{D_i} = \overline{D_{i-20}} - \frac{3}{2}$ or $\overline{D_j} = \overline{D_{j-20}} - \frac{3}{2}$, and similarly for $\overline{B_i}$ and $\overline{B_j}$. We obtain

$$\overline{D_{k+20}} = \overline{\left\{ \left\{ \overline{D_i} + \overline{D_j} - \frac{3}{2} + \varepsilon_{ij} \right\}_{i+j=k-1} \middle| \left\{ \overline{B_i} + \overline{B_j} - \frac{3}{2} + \delta_{ij} + 1 \right\}_{i+j=k-2} \right\}}. \tag{20}$$

Using the induction hypothesis on (15) we find that

$$\left\{ \left\{ \overline{D_i} + \overline{D_j} \right\}_{i+j=k-1} \middle| \left\{ \overline{B_i} + \overline{B_j} + 1 \right\}_{i+j=k-2} \right\}$$

permits at most one number. Denote the permitted number (if it exists) by x. Then

$$\left\{ \left\{ \overline{\overline{D_i} + \overline{D_j} - \frac{3}{2} + \varepsilon_{ij}} \right\}_{i+j=k-1} \middle| \left\{ \overline{\overline{B_i} + \overline{B_j} - \frac{3}{2} + \delta_{ij} + 1} \right\}_{i+j=k-2} \right\}$$

also permits at most one number which (if it exists) has to be $x - \frac{3}{2}$. We have used the fact that $\overline{\varepsilon_{ij}} = \overline{\delta_{ij}} = 0$ (being infinitesimals) and that the reduced canonical form is a linear operator. This in turn implies that

$$\left\{ \left\{ \overline{D_i + D_j} \right\}_{i+j=k+19} \middle| \left\{ \overline{B_i + B_j + 1} \right\}_{i+j=k+18} \right\}$$

permits at most one number and thus we proved (15) for $n = k$. We use this fact (and also the Translation Principle) to apply Theorem 84 to the right-hand side of (20). We obtain

$$\overline{D_{k+20}} = \overline{\left\{ \{D_i + D_j\}_{i+j=k-1} \middle| \{B_i + B_j + 1\}_{i+j=k-2} \right\}} - \frac{3}{2} = \overline{D_k} - \frac{3}{2}$$

which is precisely (12) for $n = k$. □

We will show next the results of the calculation of D_n, B_n and C_n over the length of the pre-period ($0 \leq n \leq 5$) and over the length of a full principal period ($6 \leq n \leq 15$.)

These values will be expressed in terms of the following three hot games:

Let

$$m = \pm 1,$$
$$n = \left\{\tfrac{5}{4}\middle|-\tfrac{5}{4}\right\},$$
$$q = \left\{\left\{\tfrac{53}{16}\middle|\tfrac{21}{16}\middle\|\tfrac{13}{16}\middle|-\tfrac{11}{16}\right\}\middle\|\middle|-\tfrac{19}{16}\right\}.$$

An easy calculation shows that m, n and q all have mean zero and their temperatures are 1, $\frac{5}{4}$ and $\frac{19}{16}$ respectively. The saltus of the arithmetic periodicity (the value of $D_{n+10} - D_n$ for $n \geq 6$) equals $\left\{0\middle|-\frac{3}{2}\right\}$. Moreover, the saltus equals $-\frac{3}{4} + m + 4q$.

$D_0 = 0$	$B_0 = -1$	$C_0 = -2$
$D_1 = 1$	$B_1 = -\frac{1}{2}$	$C_1 = -\frac{3}{2}$
$D_2 = m$	$B_2 = -1 + m$	$C_2 = -\frac{9}{4} + m + 4q$
$D_3 = \frac{3}{4} + n$	$B_3 = -\frac{1}{2} + m$	$C_3 = -\frac{7}{4} + m + 4q$
$D_4 = 0$	$B_4 = -\frac{9}{8} + 2q$	$C_4 = -\frac{5}{2}$
$D_5 = \frac{1}{2}$	$B_5 = -\frac{5}{8} + 2q$	$C_5 = -2$

$D_6 = m$	$B_6 = -\frac{21}{16} + q$	$C_6 = -\frac{5}{2} + m$
$D_7 = \frac{1}{2} + m$	$B_7 = -\frac{13}{16} + q$	$C_7 = -2 + m$
$D_8 = -\frac{1}{8} + 2q$	$B_8 = -\frac{3}{2}$	$C_8 = -\frac{11}{4} + 4q$
$D_9 = \frac{3}{8} + 2q$	$B_9 = -1$	$C_9 = -\frac{9}{4} + 4q$
$D_{10} = -\frac{5}{16} + q$	$B_{10} = -\frac{13}{8} + 2q + m$	$C_{10} = -\frac{23}{8} + m + 6q$
$D_{11} = \frac{3}{16} + q$	$B_{11} = -\frac{9}{8} + 2q + m$	$C_{11} = -\frac{19}{8} + m + 6q$
$D_{12} = -\frac{1}{2}$	$B_{12} = -\frac{7}{4} + 4q$	$C_{12} = -3$
$D_{13} = 0$	$B_{13} = -\frac{5}{4} + 4q$	$C_{13} = -\frac{5}{2}$
$D_{14} = -\frac{5}{8} + m + 2q$	$B_{14} = -\frac{15}{8} + m + 6q$	$C_{14} = -\frac{13}{4} + m + 4q$
$D_{15} = -\frac{1}{8} + m + 2q$	$B_{15} = -\frac{11}{8} + m + 6q$	$C_{15} = -\frac{11}{4} + m + 4q$

We will conclude this section with some algebraic considerations. First it is important to observe that, as elements in the group Γ of game values, m, n and q have finite orders.

More precisely, *order* $(m) = 2$, *order* $(n) = 2$ and *order* $(q) = 8$.

As mentioned in the preliminaries to this chapter, the algebraic formulation of Conway's mean value and temperature analysis is the following direct sum decomposition of Γ:

$$\Gamma \simeq \mathbf{H_0} \oplus \mathbf{N}$$

where $\mathbf{H_0}$ is the subgroup of (hot) games of mean zero and $\mathbf{N}$ is the subgroup of numbers. The cooling operator is the instrument used to compute the components in the direct sum for any given game.

In the same context, the algebraic formulation of the reduced canonical form theory presented in this thesis is the following further decomposition of Γ:

$$\Gamma \simeq \mathbf{R_0} \oplus \mathbf{I} \oplus \mathbf{N}$$

where $\mathbf{R_0}$ is the subgroup of reduced canonical forms of mean zero which we call *the hot kernel* of Γ, and $\mathbf{I}$ is the subgroup of infinitesimals. If we call the group of all $2 \times n$ and $n \times 2$ domineering positions (not necessarily connected) Dom_2 then an algebraic interpretation of the results in this section would be that the projection of Dom_2 on $\mathbf{R_0}$ (the hot kernel of $2 \times n$ domineering) is isomorphic to $\mathbf{Z_2} \times \mathbf{Z_2} \times \mathbf{Z_8}$, where $\mathbf{Z}_n$ is the cyclic group with n elements. The generators are m, n and q.

6.5 Directions for Further Research

The following problems are all related to the notion of reduced canonical form. The author strongly believes that they have a good chance to be solved, at least partially.

- Solve the game of **Snort** (see [BCG82], page 141) on chains of length n within infinitesimals. It is conjectured that it will ultimately display a periodicity pattern.

- Study partizan **Take-and-break** and **subtraction** games (see [BCG82] and [FK87]) in reduced canonical form. At present, the only known results are about the outcomes (rather than the values) of certain subclasses of such games. As for Snort, it seems that the main complications are due to a rapid increase in the complexity of the infinitesimals involved.

- The reduced canonical form approximates general games within infinitesimals. Devise a similar operator for the "small world" — it should approximate infinitesimals with errors bounded by $\uparrow$. For example, $\{5.\!\uparrow \mid \uparrow \| 2.\!\downarrow\}$ would be a typical reduced form for a "relatively hot" infinitesimal (here, $n.\!\uparrow$ denotes $\underbrace{\uparrow + \uparrow + \cdots + \uparrow}_{n \text{ terms}}$ and $n.\!\downarrow$ denotes $\underbrace{\downarrow + \downarrow + \cdots + \downarrow}_{n \text{ terms}}$.)

Bibliography

[ACN79] M. Ajtai, L. Csirmaz and Zs. Nagy [1979], On a generalization of the game Go-Moku I, Studia Scientiarum Math. Hungar. 14, 209-226.

[AS92] L. V. Allis, P. N. A. Schoo [1992], Qubic solved again, Heuristic Programming and Artificial Intelligence, 3: The Third Computer Olympiad (H. J. van den Herik and L. V. Allis, eds.), pp. 192-204, Ellis Horwood, Chichester, England.

[AVH93] L. V. Allis, H. J. van den Herik and M. P. H. Huntjens [1993], Go-Moku solved by new search techniques, Proc. 1993 AAAI Fall Symp. on Games: Planning and Learning, AAAI Press Tech. Report FS(#-02, pp. 1-9, Menlo Park, CA.

[AH87] M. Anderson and F.Harary [1987], Achievement and avoidance games for generating abelian groups, Internat. J. Game Theory 16, 321-325.

[BGLM73] J. Baumgartner, F. Galvin, R. Laver, and R. McKenzie [1973], Game theoretic versions of partition relations, in: Infinite and Finite Sets, Vol. I, pp. 131-135, Colloq. Keszthely.

[Bec81A] J. Beck [1981], On positional games, J. Combin. Theory (Ser. A) 30, 117-133.

[Bec81B] J. Beck [1981], Van der Waerden and Ramsey type games, Combinatorica 1, 103-116.

[Bec82A] J. Beck [1982], On a generalization of Kaplansky's game, Discrete Math. 42, 27-35.

[Bec82B] J. Beck [1982], Remarks on positional games, I, Acta Math. Acad. Sci. Hungar. 40(1-2), 65-71.

[Bec85] J. Beck [1985], Random graphs and positional games on the complete graph, Ann. Discrete Math. 28, 7-13.

[BC82C] J. Beck and L. Csirmaz [1982], Variations on a game, J. Combin. Theory (Ser. A) 33, 297-315.

[Ber68] E. R. Berlekamp [1968], A construction for partitions which avoid long arithmetic progressions, Canad. Math. Bull. 11, 409-414.

[Ber88] E. R. Berlekamp [1991], Blockbusting and Domineering, J. Combin. Theory (Ser. A) 49, 67-116. An earlier version, entitled Introduction to Blockbusting and Domineering, appeared in: The Lighter Side of Mathematics, Proc. E. Strens Memorial Conf. on Recr. Math. and its History (R. K. Guy and R. E. Woodrow, eds.), Calgary, 1986, Spectrum Series, Math. Assoc. of America, Washington, DC, 1994, pp. 134-148.

[BCG82] E. R. Berlekamp, J. H. Conway and R. K. Guy [1982], Winning Ways (two volumes), Academic Press, London.

[BW94] E. R. Berlekamp and D. Wolfe [1994], Mathematical Go — Chilling Gets the Last Point, AK Peters, Wellesley, MA 02181.

[Bou02] C. L. Bouton [1902], Nim, a game with a complete mathematical theory, Ann. of Math. 3(2), 35-39.

[BH84] F. Buckley and F. Harary [1984], Diameter avoidance games for graphs, Bull. Malaysian Math. Soc. 7, 29-33.

[Cal96] D. Calistrate [1996], The reduced canonical form of a game, in: Games of No Chance, Proc. MSRI Workshop on Combinatorial Games, July, 1994, Berkeley, CA (R. J. Nowakowski, ed.), MSRI Publ. Vol. 29, Cambridge University Press, Cambridge, pp. 409-416.

[CE73] V. Chvátal and P. Erdös [1978], Biased positional games, Ann. Discrete Math. 2: Algorithmic Aspects of Combinatorics, (B. Alspach, P. Hell and D. J. Miller, eds.), Qualicum Beach, B.C., Canada, 1976, North-Holland, pp. 221-229.

[Con72] J. H. Conway [1972], All numbers great and small, Univ. of Calgary Math. Dept. Res. Paper No. 149.

[Con76] J. H. Conway [1976], On Numbers and Games, Academic Press, London.

[Con77] J. H. Conway [1977], All games bright and beautiful, Amer. Math. Monthly 84, 417-434.

[Csi80] L. Csirmaz [1980], On a combinatorial game with applications to Go-Moku, Discrete Math. 29 , 19-23.

[CN79] L. Csirmaz and Zs. Nagy [1979], On a generalization of the game Go-Moku, II, Studia Scientarium Math. Hung. 14, 461-469.

[Dud58] H. E. Dudeney [1958], The Canterbury Puzzles, 4th ed., Mineola, NY.

[ER52] P. Erdős and R. Rado [1952], Combinatorial theorems on classifications of subsets of a given set, Proc. of the London Math. Soc. 2, 417-439.

[ES73] P. Erdős and J. L. Selfridge [1973], On a combinatorial game, J. Combin. Theory (Ser. A) 14, 298-301.

[EH83] M. Erickson and F. Harary [1983], Picasso animal achievement games, Bull. Malaysian Math. Soc. 6, 37-44.

[Fer74] T. S. Ferguson [1974], On sums of graph games with last player losing, Internat. J. Game Theory 3, 159-167.

[FK87] A. S. Fraenkel and A. Kotzig [1987], Partizan octal games: partizan subtraction games, Internat. J. Game Theory 16, 145-154.

[Gal78] F. Galvin [1978], Indeterminacy of point-open games. Bull. de l'Académie Polonaise des Sciences (Math., Astr. et Phys.) 26, 445-449.

[Gar74] M. Gardner [1974], Cram, crosscram and quadraphage: new games having elusive winning strategies. Scientific Amer., 230(2), 106-108.

[Gas96] R. Gasser [1996], Solving Nine Men's Morris, in : Games of No Chance, Proc. MSRI Workshop on Combinatorial Games, July, 1994, Berkeley, CA (R. J. Nowakowski, ed.), MSRI Publ. Vol. 29, Cambridge University Press, Cambridge, pp. 101-114.

[GRH94] D. M. Gordon, R. W. Robinson and F. Harary [1994], Minimum degree games for graphs, Discrete Math. 128, 151-163.

[Gru39] P. M. Grundy [1939], Mathematics and Games, Eureka 2, 6-8; reprinted in Eureka 27 (1964), 9-11.

[GSS44] P. M. Grundy, R. S. Scorer and C. A. B. Smith [1944], Some binary games, Math. Gaz. 28, 96-103.

[GS56] R. K. Guy and C. A. B. Smith [1956], The G-values of various games, Proc. Cambridge Phil. Soc. 52, 514-526.

[Guy91] R. K. Guy [1991], What is a game?, in Richard K. Guy (editor) Combinatorial Games, Proc. Symp. Appl. Math., 43(1991) Amer. Math. Soc., 1-21.

[Guy95] R. K. Guy [1995], Combinatorial Games, cha. 43 in R. Graham, M. Grötschel, L. Lovász (editors), Handbook of Combinatorics, North-Holland, 1995, vol. II, 2117-2162.

[Guy96] R. K. Guy [1996], What is a game?, Games of No Chance, Proc. MSRI Workshop on Combinatorial Games, July, 1994, Berkeley, CA (R. J. Nowakowski, ed.), MSRI Publ. Vol. 29, Cambridge University Press, Cambridge, pp. 43-60.

[HJ63] A. W. Hales and R. I. Jewett [1963], Regularity and positional games, Trans. Amer. Math. Soc. 106, 222-229.

[Har82] F. Harary [1982], Achievement and avoidance games for graphs, Ann. Discrete Math. 13, 111-120.

[HP87] F. Harary and K. Plochinski [1987], On degree achievement and avoidance games for graphs, Math. Mag. 60, 316-321.

[HR84] F. Harary and R. Robinson [1984], Connect-it games, College Math. J. 15, 411-419.

[HT93] F. Harary and Z. Tuza [1993], Two graph-coloring games, Bull. Austral. Math. Soc. 48, 141-149.

[Isb92] J. Isbell [1992], The Gordon game of a finite group, Amer. Math. Monthly 99, 567-569.

[Kim96] Y. Kim [1996], New values in Domineering, Theoret. Comput. Sci. (Math Games) 156, 263-280.

[Kno96] M. Knor [1996], On Ramsey-Type Games for Graphs, Australasian J. of Combinatorics 14, 199-206.

[Knu74] D. E. Knuth [1974], Surreal Numbers, Addison-Wesley, Reading, MA.

[Leh64] A. Lehman [1964], A solution to the Shannon switching game , SIAM J. Appl. Math. 12, 687-725.

[Lu91] X. Lu [1992], A matching game, Discrete Math. 94, 199-207.

[Lu92] X. Lu [1992], Hamiltonian games, J. Combin. Theory (Ser. B) 55, 18-32.

[Lu95] X. Lu [1995], A Hamiltonian game on $K_{n,n}$, Discrete Math. 142, 185-191.

[MRH74] E. Mead, A. Rosa & C. Huang [1974] The game of SIM: A winning strategy for the second player, Math. Mag. 47, 243-247.

[Pap82] A. Papaioannou [1982], A Hamiltonian game, Ann. Discrete Math. 13, 171-178.

[Pat80] O. Patashnik [1980], Qubic: $4 \times 4 \times 4$ Tic-Tac-Toe, Math. Mag. 53, 202-216.

[Ple91] V. Pless [1991], Games and codes, in: Combinatorial Games, Proc. Symp. Appl. Math. 43 (R. K. Guy, ed.), Amer. Math. Soc., Providence, RI, pp. 101-110.

[RY74] E. M. Rounds & S. S. Yau [1974], A winning strategy for SIM, J. Recr. Math. 7, 193-202.

[Sch96] J. Schaeffer [1996], Solving the game of checkers, in: Games of No Chance, Proc. MSRI Workshop on Combinatorial Games, July, 1994, Berkeley, CA (R. J. Nowakowski, ed.), MSRI Publ. Vol. 29, Cambridge University Press, Cambridge, pp 119-134.

[Sim69] G. J. Simmons [1969] The Game of SIM, J. Recr. Math. 2, 193-202.

[Spr35] R. Sprague [1935-36], Über mathematische Kampfspiele, Tôhoku Math. J. 41, 438-444.

[Spr37] R. Sprague [1937], Über zwei Abarten von Nim, Tôhoku Math. J. 43, 351-359.

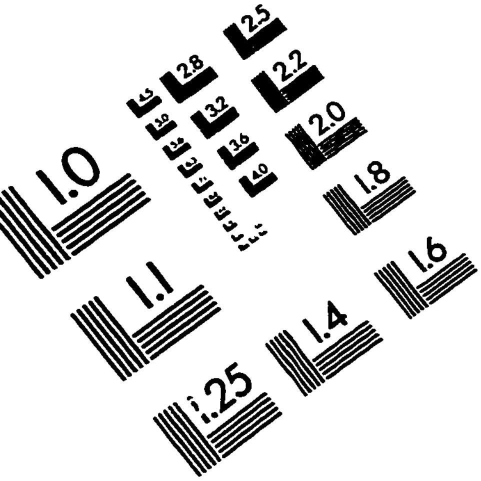
[VM53] J. von Neumann and O. Morgenstern [1953], Theory of Games and Economic Behaviour, 3rd ed., Princeton University Press, Princeton, NJ.

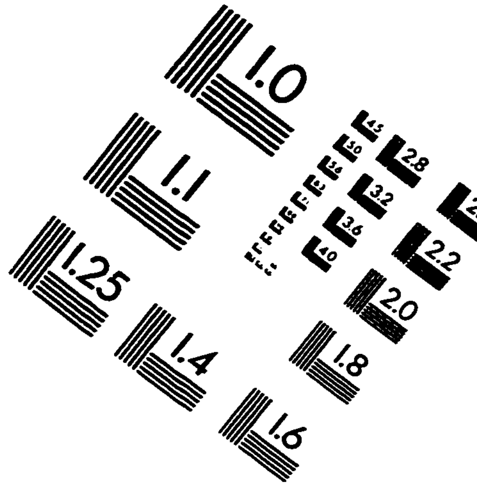
[Wes96] J. West [1996], Champion-level play of domineering, in: Combinatorial Games, Proc. MSRI Workshop on Combinatorial Games, July, 1994, Berkley, CA (R. J. Nowakowski, ed.), MSRI Publ. Vol. 29, Cambridge University Press, Cambridge, pp. 345-350.

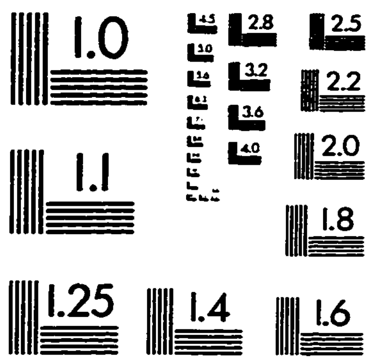
[Wol93] D. Wolfe [1993], Snakes in domineering games, Theoret. Comput. Sci. (Math Games) 119, 323-329.

[Wol96] D. Wolfe [1996], The gamesman's toolkit, in: Games of No Chance, Proc. MSRI Workshop on Combinatorial Games, July, 1994, Berkeley, California (R. J. Nowakowski, ed.) MSRI Publ. Vol. 29, Cambridge University Press, Cambridge, pp. 93-98 (the software can be downloaded from *http://www.gac.edu/~wolfe/papers-games/*)

IMAGE EVALUATION TEST TARGET (QA-3)







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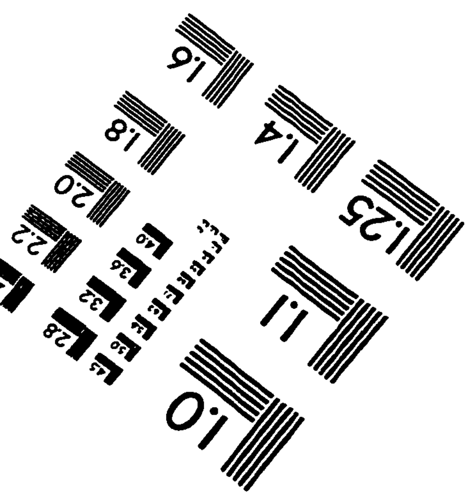

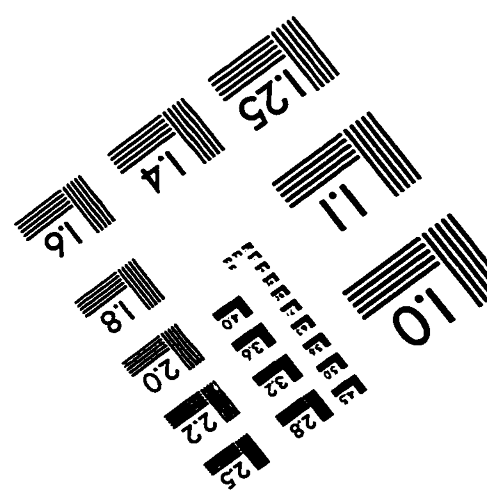

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