THE UNIVERSITY OF CALGARY

Exponentially Small Splittings and Solitary Water Waves

by

Kelly Charles Candy

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Exponentially Small Splittings and Solitary Water Waves" submitted by Kelly Candy in partial fulfillment of the requirements for the degree of Master of Science.

Supervisor, L. Bates
Department of Mathematics and Statistics

Sniatycki
Department of Mathematics and Statistics

D. Hobill
Department of Physics

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Abstract

In studying dynamical systems, mathematicians are often faced with what seems to be impossible integrations. Recently these problems have been attributed to the phenomenon of "Exponentially Small Splittings" of the orbits in the phase space. It seems that two problems in particular are studied to try and understand this phenomenon, that of solitary water waves and that of the rapidly forced pendulum. One such example of this occurrence with solitary water waves concerns the model equation

$$\varepsilon^2 \frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} - y + y^2 = 0$$

The analysis will involve the transformation

$$z(y) = \left(\frac{dy}{dx}\right)^2$$

in order to get a second order differential equation. Asymptotic expansions of $z(y)$ are then studied for ε small and we will try to get a hold of the "exponentially small splitting".

Acknowledgements

In the realization of the final draft of this thesis being written, I was suddenly aware of the number of people who had some contribution to its development, I hope to acknowledge, here as many of these people as possible without writing another manuscript of equal size. First and foremost I would like to thank my advisor Dr. Larry Bates, whose encouragement and infinite patience allowed me to complete the work at my own pace and not to the settings of a time clock. I would also like to thank Dr. J. Sniatycki and Dr. D. Hobill for agreeing to wade their way through my work for the purpose of my defense.

When I first started my post secondary career, I was attending the University of Lethbridge and it is here that I must put the credit for any interest sparked in me for mathematics. I would like to thank the University of Lethbridge Math Department for showing me that there is a little more to math then adding and subtracting. I would also like to thank my friend Dr. Lincoln Chew of the University of Lethbridge Psychology department for his guidance and words of wisdom throughout the search for my degree.

While at the University of Calgary, I met many people whose interests were very similar to mine, that is we all enjoyed doing our studying in the Den. I am sure the only reason that I did not go insane during the course of my studies was due to Peter Gibson, Halsey Boyd, Indy Lagu, Jim Stallard, Bonaventure Anthonio, Titus Misi, Len Bos, Chris Raham, Karen Slowinsky, Tom Morrison, and many others that are too numerous to mention. I raise a mug and give a toast to all of these.

Finally, I feel that much of my ambition and accomplishments can be attributed to my closest family and friends. I could not have gone as far as I have without the support and guidance of my dad Garry Candy and my mom Bev Candy. I also would like to thank my sisters Kerry and Dawn for making me leave the world of math and reenter the world of people. My brothers Jay and James were instrumental in helping me to relax while I was away from school. I really enjoyed the time we were able to spend in consultation over a few beer. I would like to thank my friends for all of their encouragement, of these I must make special mention of a few: Jeff Hill, Reece Dorigatti, Brian Veres, Mark Lenz, Blair Wilk, Bryan and Sandy Szumlas, Pat Spenrath, and last but definitely not least Treva Readman whose support for my scholastic achievement was never lacking.

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Chapter 1

Introduction

As we go about our everyday lives, we are constantly interacting with different systems; we depend upon the interaction of gears to be able to tell time, we depend upon the interaction of air molecules and other particles in the air so that the weather is bearable. The study of these systems has been going on since time began. In mankind's unquenchable thirst for knowledge, we have described as many of our "man-made" systems as we could. We are now moving into the realm of nature, and we are finding that nature's laws do not conform so easily to our own. The result is an area of study where we intentionally focus on the troublesome areas rather than ignoring them. In this thesis, I will try to explore one such area, that of exponentially small splittings of orbits. I will start with a brief overview of dynamical systems, then a quick annotated bibliography of some papers presented on exponentially small splittings. The main body of the thesis deals with a concrete example of exponentially small splittings in a homoclinic orbit of the equation for a solitary water wave. With this in mind, I will give an overview of some papers dealing with solitary water waves and then an in depth analysis of a paper by Wiktor Eckhaus, "Singular Perturbations of Homoclinic Orbits in $\mathbf{R}^4$" [5].

1.1 A Brief Introduction to Dynamical Systems.

A dynamical system is one whose state changes with time. In applications, we can deal with time continuously or discreetly. When time is continuous, the dynamics of the system are usually described by a differential equation whose solution or solutions are values in the phase space. Usually the phase space is Euclidean space or a subset thereof, but it can also be a non-Euclidean structure. We say the solution or solutions of the system of differential equations is the flow of the system, and specific solutions (i.e. solutions starting with particular initial conditions and then tracked as time changes), are called the orbits. We usually denote the flow by $\varphi_t : M \to M$ where M is the manifold or structure where the system is taking place. If $\varphi_t(x) = x^*$ for all $t \in \mathbf{R}$, then we say x^* is a fixed point of the flow. The sets of points $\{x \in M \mid \varphi_t(x) \to x^* \text{ as } t \to \infty(-\infty)\}$ are called the stable and unstable manifolds respectively. If a point lies in both the stable and the unstable manifold, then this point is referred to as a homoclinic point. If only one orbit contains both the unstable and stable manifold, then this orbit is referred to as a homoclinic orbit. Sometimes the unstable and stable manifolds intersect transversally and when this happens they intersect an infinite number of times. The result is a homoclinic tangle. The splitting between the unstable and stable manifolds of a homoclinic orbit is called exponentially small splitting due the fact that the splitting distance is exponentially small. This distance that we refer to is the actual measured distance between the stable manifold and the unstable manifold. When determining the splitting distance, it becomes interesting to also measure the splitting angle, this is the tangental angle at the one point where the stable and unstable manifolds intersect. Of course due to the fact that the stable and unstable manifolds will intersect an infinite number of times, it becomes necessary to look at the splitting angle and distance at one intersection point. One way to try and compute these measurements is called the Melnikov Function. Here a section of the unperturbed system is analyzed. In the perturbed system, the distance is measured along the Poincaré map between the stable and unstable manifolds. The distance changes for varying points and so an average is taken over the integral. However, this approach often fails due to the non-analyticity of the functions involved. In the case studied in this thesis, this integral is impossible to compute. So to try and work with these types of systems we use approximations to the functions involved. These approximations are said to be asymptotic if we have a series of increasingly accurate approximations to a function in a particular limit. An asymptotic limit does not necessarily have to converge and so can be used on a non analytic function. An important feature of an asymptotic series like $\sum \phi_n \varepsilon^n$ is that every term in the series is algebraic in ε. Transcendentally small terms like $\exp\left\{-\frac{1}{\varepsilon^2}\right\}$ are smaller than every term in the series as $\varepsilon \to 0$ and are not captured by it. These small terms are said to lie beyond all orders of the asymptotic expansion. In most applications, these tiny corrections are insignificant, but in some instances they play a very big part in the analysis. See figure 1.1 for an example of this phenomenon.

1.2 History of Exponentially Small Splittings.

The topic of exponentially small splittings was first suggested to me by my advisor in the paper "Exponentially Small Splittings of Separatrices with applications to

KAM Theory and Degenerate Bifurcations" [8]. In this paper, Philip Homes, Jerrold Marsden, and Jurgen Scheurle deal with the problem of rapidly forced systems with a homoclinic orbit. In particular, the rapidly forced pendulum described by

$$\ddot{\phi} + \sin(\phi) = \varepsilon f(t)$$

when $\varepsilon = 0$, we get homoclinic orbits given explicitly by the equations

$$\overline{\phi}(t) = \pm 2 \tan^{-1}(\sinh(t))$$

$$\overline{v}(t) = \pm 2 \operatorname{sech}(t)$$

For small values of ε the above orbits split and this is where the analysis takes place. The Melnikov function is then used to give some estimates for the splitting distance. The authors make the comment that the splitting distance should be of the order

$$d = 2\pi\varepsilon e^{-\frac{\pi}{2\varepsilon}} \tag{1.1}$$

however, they suggest that it is not easy to justify due to the fact that the errors are $O(\varepsilon^2)$,while 1.1 is already smaller than any power of ε. A Liapunov-Perron type iteration scheme is then used to locate the stable and unstable manifolds. In utilizing the iteration process itself, they keep track of all of the estimates made. A contraction mapping is then used to allow the existence of fixed points, and the stable and unstable manifolds determined by these fixed points are used for the analysis of the splitting distance. Estimates are made on the splitting distance by the use of

Sobolev estimates and some hard analysis. They then go on to extend their results to KAM theory and Bifurcation problems. The work carried out by the authors is extensive in its complexity, however they do mention that this paper was only meant to be an outline. It is interesting to note that in their introduction, they mention that Poincaré was aware of the problems that exponentially small splittings could cause in terms of integrability and the convergence of series expansions.

The challenge then of working on exponentially small splitting appealed to me, even though I had not really understood exactly what was going on in the paper. I then learned of a second paper by Marsden, Holmes, and Scheurle which was to expand upon the work carried out in their first paper. The title of this paper was "Exponentially Small Estimates for Separatrix Splittings" [13]. In this paper, the authors review their previous estimates and show that the assumption that the estimates of the splitting distance is given by $a\varepsilon^b e^{-\frac{c}{\varepsilon}}$ can sometimes be wrong. To do this, they used the following example of planar systems

$$\dot{x} = 1 - x^2$$

$$\dot{y} = \left[2x - (\alpha + 2\beta x)(1 - x^2)\right] y + \delta \cos(\frac{t}{\varepsilon})$$

where α, β, δ, and ε are constants. They assert that for $\delta = 0$ the system has the heteroclinic orbit

$$\Gamma : x = \tanh(t), y = 0$$

joining $(-1, 0)$ to $(1, 0)$. Their interest lies in the splitting of this orbit for small values of α, β, δ, and ε. Although many important remarks are made by the authors, one in particular seems to be of the most importance, namely that a singularity can occur in the resulting formula for the splitting distance even though there are no essential singularities in the given problem. Then an impressive amount of analysis is again used to show that

for any integer N, there are constants $c_N > 0$ and $\varepsilon_N > 0$ such that

for $\alpha = 0, \beta = -\varepsilon$ and $\delta = \varepsilon$, we have

$$d \geq \frac{c_N}{\varepsilon^N} e^{-\frac{\pi}{2\varepsilon}}$$

for all $0 < \varepsilon < \varepsilon_N$

which shows that no sharp upper estimate of the form $a\varepsilon^b e^{-\frac{\pi}{2\varepsilon}}$ can exist. This phenomenon occurs in this specific example with the splitting of a heteroclinic orbit, however the authors feel that this problem is generic and will occur as a result of the iteration schemes used. Thus making it very difficult indeed to actually get an actual upper estimate.

I then had the chance to talk with Dr. Martin Kummer of the University of Toledo, Ohio, and I was able to get a paper written by Dr. Kummer, James A. Ellison, and A.W. Sáenz [9]. In this paper, the rapidly forced pendulum is again used to demonstrate the splitting distance, however this paper uses much more geometry to try and illustrate the actual phenomenon taking place. The results of the work done in this paper are very neatly summed up in the conclusion and so I will quickly sum up these results. The authors make the point that much of the work done previously by Marsden, Holmes, and Scheurle was used, but there are some differences. The main point being that geometry was the main focus of this paper as opposed to only analysis, I found this to be a refreshing change from the hard-core analysis used previously. A second important point is that the paper by Kummer, Ellison, and Sáenz also proves a stable manifold theorem in the process. This paper is highly readable, and seems to explain much of the work done by Marsden, Holmes, and Scheurle.

1.3 History of the Problem of Solitary Water Waves.

In searching for a way to try and understand the problem of exponentially small splittings, I was referred to a paper by Wiktor Eckhaus concerning solitary water waves, this then brought upon a literature search for information concerning solitary water waves. One such paper is titled "A Theory of Solitary Water Waves in the Presence of Surface Tension", written by Charles J. Amick and Klaus Kirchgässner [1]. In this paper, the authors prove that solitary water waves exist on the surface of an inviscid fluid layer in the presence of small surface tension and gravity. They begin by making some assumptions concerning the characteristics of the fluid. They assume that the density of the fluid is constant, that the flow is irrotational, and that the fluid is at rest at infinity. Bernoulli's equation gives

$$p(x,y) + \frac{1}{2}\left|q(x,y)\right|^2 + gy = \text{constant} \tag{1.2}$$

in the flow domain S, where p denotes the pressure and $g > 0$ is the acceleration due to gravity. It is then assumed that the unknown free surface may be represented by a function Y and that the constant of proportionality is the surface tension T which is given. Then using this in 1.2 will give

$$\frac{1}{2}\left|q(x,Y(x)\right|^2 + gY(x) - TY''(x)/(1+Y'(x)^2)^{\frac{3}{2}} = \text{constant}$$

After some mathematical analysis the equation

$$\frac{\nu}{2}\exp(2\tau) - \int_0^{\phi}\sin(\theta)\exp(-\tau)ds + \gamma\partial_{\phi}\theta\exp(\tau) = \text{constant},\ \phi \in R$$

is derived, where $\nu = \frac{c^2}{gh}, \gamma = \frac{T}{gh^2}$ and c is the speed of light, h is the height of the wave. A reduction theorem is then used to bring the problem to a simpler conclusion. It turns out that when $\gamma > \frac{1}{3}$ the phase space is of dimension 3, however when $\gamma < \frac{1}{3}$ the dimension of the phase space jumps to 5. Some work is done in both cases, and the result is the model equation:

$$\varepsilon^2\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} - y + y^2 = 0 \tag{1.3}$$

A second paper written by J.M. Hammersley and G. Mazzarino, entitled "Computational Aspects of Some Autonomous Differential Equations"[7], begins with 1.3 and has the conditions

$$\frac{dy(0)}{dx} = 0, \frac{dy}{dx} < 0,\ \text{and}\ \lim_{x\to\infty} y(x) = 0$$

with the exception that the perturbation term is denoted as ε rather then ε^2. The main interest of their paper lies in the behavior of $\frac{d^3y}{dx^3}$ at the point $x = 0$. It will turn out that this is the exact same phenomenon studied by Eckhaus in the paper that

I will follow. They prove that this quantity is strictly positive and they investigate it numerically as a function of ε. It is then speculated that the process utilized by Hammersley and Mazzarino could possibly be applied to various autonomous differential equations. It will be noted in the following chapters where our results seem to be in agreement with those of Hammersley and Mazzarino.

The paper that I chose to follow was written by Wiktor Eckhaus of the Mathematical Institute in Rijksuniversiteit, Utrecht entitled "Singular Perturbations of Homoclinic Orbits in $\mathbf{R}^4$" [5]. This was an interesting paper due to the simplicity of the homoclinic orbit and the speed with which it splits. In his model equation, with the limit $\varepsilon = 0$, one finds an integral

$$\left(\frac{dy}{dx}\right)^2 = y^2 - \frac{2}{3}y^3 + c$$

and for $c = 0$ we then get a homoclinic orbit in the $\left(y, \frac{dy}{dx}\right)$ plane. See figure 1.2

The solution $y(x)$ of this limit problem tends to 0 as $x \to \pm\infty$, and the question is asked

Do there exist non-trivial solutions for the singularly perturbed model

ie. when $\varepsilon \neq 0$, which tend to zero for $x \to \pm\infty$?

This is the problem that Eckhaus addresses in his paper and he does so by examining the behavior of the homoclinic orbit and whether or not it exists after the perturbation term is introduced.

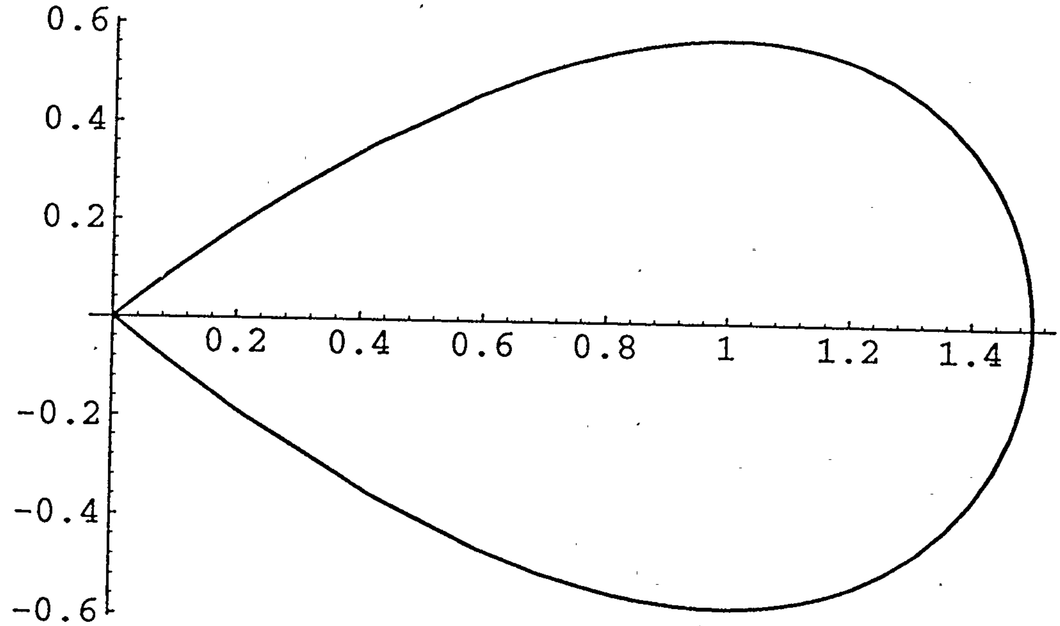

Figure 1.1: Homoclinic orbit before perturbation

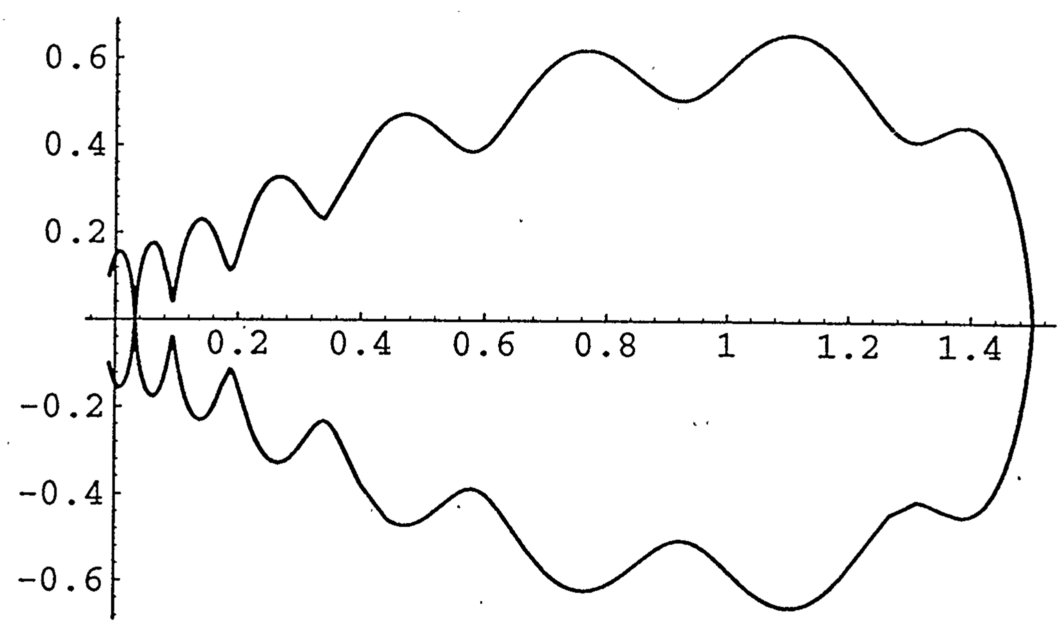

Figure 1.2: Homoclinic orbit after the perturbation term is introduced

Chapter 2

The Integral Part of the Thesis.

In this section, we will try and find an integral for 1.3.

2.1 An Equation for the Integral and Trajectories.

Let us multiply 1.3 by $\frac{dy}{dx}$ to get

$$\varepsilon^2 \frac{d^4y}{dx^4}\frac{dy}{dx} + \frac{d^2y}{dx^2}\frac{dy}{dx} - y\frac{dy}{dx} + y^2\frac{dy}{dx} = 0$$

Then notice that

$$\int y^2 \frac{dy}{dx} = \frac{1}{3}y^3 + c_1$$

$$\int y\frac{dy}{dx} = \frac{1}{2}y^2 + c_2$$

$$\int \frac{d^2y}{dx^2}\frac{dy}{dx} = \frac{1}{2}\left(\frac{dy}{dx}\right)^2 + c_3$$

and

$$\int \varepsilon^2 \frac{d^4y}{dx^4}\frac{dy}{dx} = \varepsilon^2\left[\frac{dy}{dx}\frac{d^3y}{dx^3} - \int \frac{d^3y}{dx^3}\frac{d^2y}{dx^2}\right] = \varepsilon^2\left[\frac{dy}{dx}\frac{d^3y}{dx^3} - \frac{1}{2}\left(\frac{d^2y}{dx^2}\right)^2 + c_4\right]$$

Thus we get

$$\varepsilon^2\left\{\frac{dy}{dx}\frac{d^3y}{dx^3} - \frac{1}{2}\left(\frac{d^2y}{dx^2}\right)^2\right\} = -\frac{1}{2}\left(\frac{dy}{dx}\right)^2 + \frac{1}{2}y^2 - \frac{1}{3}y^3 + c \tag{2.1}$$

Note that

$$\frac{d}{dy}\left(\frac{dy}{dx}\right)^2 = 2\frac{dy}{dx}\frac{d^2y}{dydx} = 2\frac{d^2y}{dx^2}$$

and

$$\frac{1}{2}\frac{dy}{dx}\frac{d^2}{dy^2}\left(\frac{dy}{dx}\right)^2 = \frac{1}{2}\frac{dy}{dx}\frac{d}{dy}\left(2\frac{dy}{dx}\frac{d^2y}{dydx}\right) = \frac{dy}{dx}\frac{d}{dy}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

so that if we now set

$$z = \left(\frac{dy}{dx}\right)^2 \tag{2.2}$$

then we will have

$$\frac{d^2y}{dx^2} = \frac{1}{2}\frac{dz}{dy}$$

and

$$\frac{d^3y}{dx^3} = \frac{1}{2}z^{\frac{1}{2}}\frac{d^2z}{dy^2}$$

So now we have:

$$\varepsilon^2\left\{z^{\frac{1}{2}}\left(\frac{1}{2}z^{\frac{1}{2}}\frac{d^2z}{dy^2}\right) - \frac{1}{2}\left(\frac{1}{2}\frac{dz}{dy}\right)^2\right\} = -\frac{1}{2}z + \frac{1}{2}y^2 - \frac{1}{3}y^3 + c$$

Which we write as:

$$\varepsilon^2\left\{z\frac{d^2z}{dy^2} - \frac{1}{4}\left(\frac{dz}{dy}\right)^2\right\} + z = y^2 - \frac{2}{3}y^3 + c \tag{2.3}$$

and this gives us an equation for the trajectories. We are searching for an equation for the trajectory where $c = 0$ in the hopes of finding a homoclinic orbit living in the $\left(y, \frac{dy}{dx}\right)$plane as in the limit case $\varepsilon = 0$. In 2.3 we have reduced the problem from 4th to 2nd order, and we have y as the independent variable. The problem is that the equation is highly non-linear and degenerates for $z = 0$. However, we will be using 2.3 in all the calculations that follow.

2.2 Formal Approximation by Straightforward Iteration

We are trying to find a homoclinic orbit, and so to do this we will put in 2.3 $c = 0$. We are searching for an expansion of the form

$$z = z_0 + \varepsilon^2 z_1 + \cdots$$

and we wish to find general formulae for the z_k. We will then try to find solutions by using these formal approximations. The equation we are trying to solve is

$$\varepsilon^2 \left\{ z\frac{d^2 z}{dy^2} - \frac{1}{4}\left(\frac{dz}{dy}\right)^2 \right\} + z = y^2\left(1 - \frac{2}{3}y\right) \tag{2.4}$$

with $\varepsilon = 0$ the above equation becomes

$$z_0 = y^2\left(1 - \frac{2}{3}y\right) \tag{2.5}$$

We introduce the transformation

$$z(y,\varepsilon) = z_0(y) + \varepsilon^2 \rho_1(y,\varepsilon)$$

to get the equation

$$\varepsilon^2 \left\{ (z_0 + \varepsilon^2\rho_1)\frac{d^2}{dy^2}(z_0 + \varepsilon^2\rho_1) - \frac{1}{4}\left(\frac{d}{dy}(z_0 + \varepsilon^2\rho_1)\right)^2 \right\} + z_0 + \varepsilon^2\rho_1 = y^2\left(1 - \frac{2}{3}y\right)$$

$$\Rightarrow \varepsilon^2 \left\{ (z_0 + \varepsilon^2 \rho_1) \left(\frac{d^2 z_0}{dy^2} + \varepsilon^2 \frac{d^2 \rho_1}{dy^2} \right) - \frac{1}{4} \left(\frac{dz_0}{dy} + \varepsilon^2 \frac{d\rho_1}{dy} \right)^2 \right\} + z_0 + \varepsilon^2 \rho_1 = z_0$$

$$\Rightarrow \varepsilon^2 \left\{ z_0 \frac{d^2 \rho_1}{dy^2} + \rho_1 \frac{d^2 z_0}{dy^2} + \varepsilon^2 \rho_1 \frac{d^2 \rho_1}{dy^2} - \frac{1}{2} \frac{dz_0}{dy} \frac{d\rho_1}{dy} - \frac{\varepsilon^2}{4} \left(\frac{d\rho_1}{dy} \right)^2 \right\} + \rho_1 = f_1(y)$$

where

$$f_1(y) = -z_0 \frac{d^2 z_0}{dy^2} + \frac{1}{4} \left(\frac{dz_o}{dy} \right)^2$$

so let us repeat this operation, performing the iteration again by putting

$$\rho_1(y, \varepsilon) = z_1(y) + \varepsilon^2 \rho_2(y, \varepsilon)$$
$$z_1(y) = f_1(y)$$

So substituting we obtain

$$\varepsilon^2 \left\{ \left(z_0 + \varepsilon^2 (z_1 + \varepsilon^2 \rho_2) \right) \frac{d^2}{dy^2} (z_1 + \varepsilon^2 \rho_2) - \frac{1}{2} \frac{dz_0}{dy} \frac{d}{dy} (z_1 + \varepsilon^2 \rho_2) \right.$$

$$\left. + (z_1 + \varepsilon^2 \rho_2) \frac{d^2 z_0}{dy^2} - \frac{\varepsilon^2}{4} \left(\frac{d}{dy} (z_1 + \varepsilon^2 \rho_2) \right)^2 \right\} + z_1 + \varepsilon^2 \rho_2 = z_1$$

$$\Rightarrow (z_0 + \varepsilon^2 z_1 + \varepsilon^4 \rho_2) \left(\frac{d^2 z_1}{dy^2} + \varepsilon^2 \frac{d^2 \rho_2}{dy^2} \right) - \frac{1}{2} \frac{dz_0}{dy} \left(\frac{dz_1}{dy} + \varepsilon^2 \frac{d\rho_2}{dy} \right) + z_1 \frac{d^2 z_0}{dy^2} + \varepsilon^2 \rho_2 \frac{d^2 z_0}{dy^2}$$

$$- \frac{\varepsilon^2}{4} \left(\frac{dz_1}{dy} + \varepsilon^2 \frac{d\rho_2}{dy} \right)^2 + \rho_2 = 0$$

$$\Rightarrow \varepsilon^2 \left\{ z_0 \frac{d^2\rho_2}{dy^2} + z_1 \frac{d^2 z_1}{dy^2} + \varepsilon^2 z_1 \frac{d^2\rho_2}{dy^2} + \varepsilon^2 \rho_2 \frac{d^2 z_1}{dy^2} + \varepsilon^4 \rho_2 \frac{d^2\rho_2}{dy^2} - \frac{1}{2}\frac{dz_0}{dy}\frac{d\rho_2}{dy} + \rho_2 \frac{d^2 z_0}{dy^2} \right.$$

$$\left. -\frac{1}{4}\left(\frac{dz_1}{dy} + \varepsilon^2 \frac{d\rho_2}{dy}\right)^2 \right\} + \rho_2 = -z_0 \frac{d^2 z_1}{dy^2} + \frac{1}{2}\frac{dz_0}{dy}\frac{dz_1}{dy} - z_1 \frac{d^2 z_0}{dy^2}$$

$$\Rightarrow \varepsilon^2 \left\{ (z_0 + \varepsilon^2 z_1 + \varepsilon^4 \rho_2)\frac{d^2\rho_2}{dy^2} + (z_1 + \varepsilon^2 \rho_2)\frac{d^2 z_1}{dy^2} - \frac{1}{2}\frac{dz_0}{dy}\frac{d\rho_2}{dy} - \frac{1}{2}\varepsilon^2 \frac{dz_1}{dy}\frac{d\rho_2}{dy} - \frac{1}{4}\left(\frac{dz_1}{dy}\right)^2 \right.$$

$$\left. -\frac{\varepsilon^4}{4}\left(\frac{d\rho_2}{dy}\right)^2 + \rho_2 \frac{d^2 z_0}{dy^2} \right\} + \rho_2 = -z_0 \frac{d^2 z_1}{dy^2} + \frac{1}{2}\frac{dz_0}{dy}\frac{dz_1}{dy} - z_1 \frac{d^2 z_0}{dy^2}$$

which can also be written as

$$\varepsilon^2 \left\{ (z_0 + \varepsilon^2 z_1 + \varepsilon^4 \rho_2)\frac{d^2\rho_2}{dy^2} - \frac{1}{2}\frac{dz_0}{dy}\frac{d\rho_2}{dy} - \frac{1}{2}\varepsilon^2 \frac{dz_1}{dy}\frac{d\rho_2}{dy} - \frac{\varepsilon^4}{4}\left(\frac{d\rho_2}{dy}\right)^2 \right.$$

$$\left. +\varepsilon^2 \rho_2 \frac{d^2 z_1}{dy^2} + \rho_2 \frac{d^2 z_0}{dy^2} \right\} + \rho_2 = -z_0 \frac{d^2 z_1}{dy^2} + \frac{1}{2}\frac{dz_0}{dy}\frac{dz_1}{dy} - z_1 \frac{d^2 z_0}{dy^2}$$

$$- \varepsilon^2 z_1 \frac{d^2 z_1}{dy^2} + \frac{\varepsilon^2}{4}\left(\frac{dz_1}{dy^2}\right)^2 \tag{2.6}$$

or if we notice that by doing this type of iteration repetitively we get

$$z(y,\varepsilon) = \sum_{n=0}^{m-1} \varepsilon^{2n} z_n(y) + \varepsilon^{2m} \rho_m(y,\varepsilon) \tag{2.7}$$

With this, we may now introduce the definition

$$\Phi_m = \sum_{n=0}^{m-1} \varepsilon^{2n} z_n \tag{2.8}$$

so that we now have

$$z = \Phi_m + \varepsilon^{2m} \rho_m \tag{2.9}$$

Now we can write equation 2.6 as

$$\varepsilon^2 \left\{ (\Phi_2 + \varepsilon^{2\cdot 2}\rho_2)\frac{d^2\rho_2}{dy^2} - \frac{1}{2}\frac{d\Phi_2}{dy}\frac{d\rho_2}{dy} + \frac{d^2\Phi_2}{dy^2}\rho_2 - \frac{1}{4}\varepsilon^{2\cdot 2}\left(\frac{d\rho_2}{dy}\right)^2 \right\} + \rho_2 = f_2 \tag{2.10}$$

where

$$f_2 = -\left\{ (z_0 + \varepsilon^2 z_1)\frac{d^2 z_1}{dy^2} - \frac{1}{2}\frac{dz_0}{dy}\frac{dz_1}{dy} + \frac{d^2 z_0}{dy^2} z_1 - \frac{1}{4}\varepsilon^2 \left(\frac{dz_1}{dy}\right)^2 \right\}$$

and we are hoping for a general recursion formula, so we do the integration one more time in hopes of finding a pattern. So we set

$$\rho_2 = z_2 + \varepsilon^2 \rho_3$$

$$z_2 = f_2$$

$$\varepsilon^2 \left\{ (z_0 + \varepsilon^2 z_1 + \varepsilon^4 (z_2 + \varepsilon^2 \rho_3))\frac{d^2}{dy^2}(z_2 + \varepsilon^2\rho_3) - \frac{1}{2}\frac{dz_0}{dy}\frac{d}{dy}(z_2 + \varepsilon^2\rho_3) \right.$$

$$-\frac{1}{2}\varepsilon^2 \frac{dz_1}{dy}\frac{d}{dy}(z_2 + \varepsilon^2\rho_3) - \frac{\varepsilon^4}{4}\left(\frac{d}{dy}(z_2 + \varepsilon^2\rho_3)\right)^2 + \varepsilon^2 (z_2 + \varepsilon^2\rho_3)\frac{d^2 z_1}{dy^2}$$

$$\left. +(z_2 + \varepsilon^2\rho_3)\frac{d^2 z_0}{dy^2} \right\} + (z_2 + \varepsilon^2\rho_3) = z_2$$

$$\Rightarrow (z_0 + \varepsilon^2 z_1 + \varepsilon^4 z_2 + \varepsilon^6 \rho_3)\left(\frac{d^2 z_2}{dy^2} + \varepsilon^2 \frac{d^2\rho_3}{dy^2}\right) - \frac{1}{2}\frac{dz_0}{dy}\left(\frac{dz_0}{dy} + \varepsilon^2 \frac{d\rho_3}{dy}\right)$$

$$-\frac{1}{2}\varepsilon^2\frac{dz_1}{dy}\left(\frac{d^2z_2}{dy^2}+\varepsilon^2\frac{d^2\rho_3}{dy^2}\right)-\frac{\varepsilon^4}{4}\left(\frac{d^2z_2}{dy^2}+\varepsilon^2\frac{d^2\rho_3}{dy^2}\right)^2+\varepsilon^2(z_2+\varepsilon^2\rho_3)\frac{d^2z_1}{dy^2}$$

$$+(z_2+\varepsilon^2\rho_3)\frac{d^2z_0}{dy^2}+\rho_3=0$$

$$\Rightarrow \varepsilon^2\left\{(z_0+\varepsilon^2z_1+\varepsilon^4z_2+\varepsilon^6\rho_3)\frac{d^2\rho_3}{dy^2}+\varepsilon^4\rho_3\frac{d^2z_2}{dy^2}-\frac{1}{2}\frac{dz_0}{dy}\frac{d\rho_3}{dy}\right.$$

$$\left.-\frac{\varepsilon^2}{2}\frac{dz_1}{dy}\frac{d\rho_3}{dy}-\frac{\varepsilon^4}{2}\frac{dz_2}{dy}\frac{d\rho_3}{dy}-\frac{\varepsilon^6}{4}\left(\frac{d\rho_3}{dy}\right)^2+\varepsilon^2\rho_3\frac{d^2z_1}{dy^2}+\rho_3\frac{d^2z_0}{dy^2}\right\}+\rho_3$$

$$=-(z_0+\varepsilon^2z_1+\varepsilon^4z_2)\frac{d^2z_2}{dy^2}+\frac{1}{2}\frac{dz_0}{dy}\frac{dz_2}{dy}+\frac{\varepsilon^2}{2}\frac{dz_1}{dy}\frac{dz_2}{dy}+\frac{\varepsilon^4}{4}\left(\frac{dz_2}{dy}\right)^2$$

$$-\varepsilon^2z_2\frac{d^2z_1}{dy^2}-z_2\frac{d^2z_0}{dy^2}$$

$$\Rightarrow \varepsilon^2\left\{(\Phi_3+\varepsilon^{2\cdot 3}\rho_3)\frac{d^2\rho_3}{dy^2}-\frac{1}{2}\left(\frac{dz_0}{dy}+\varepsilon^2\frac{dz_1}{dy}+\varepsilon^4\frac{dz_2}{dy}\right)\frac{d\rho_3}{dy}\right.$$

$$\left.+\left(\frac{d^2z_0}{dy^2}+\varepsilon^2\frac{d^2z_1}{dy^2}+\varepsilon^4\frac{d^2z_2}{dy^2}\right)\rho_3-\varepsilon^{2\cdot 3}\left(\frac{d\rho_3}{dy}\right)^2\right\}+\rho_3=-\left\{\Phi_3\frac{d^2z_2}{dy^2}\right.$$

$$\left.-\frac{1}{2}\left(\frac{dz_0}{dy}+\varepsilon^2\frac{dz_1}{dy}\right)\frac{dz_2}{dy}+\left(\frac{d^2z_0}{dy^2}+\varepsilon^2\frac{d^2z_1}{dy^2}\right)z_2-\frac{\varepsilon^{2\cdot 2}}{4}\left(\frac{dz_2}{dy}\right)^2\right\}$$

which can be expressed as

$$\varepsilon^2\left\{(\Phi_3+\varepsilon^{2\cdot 3}\rho_3)\frac{d^2\rho_3}{dy^2}-\frac{1}{2}\frac{d\Phi_3}{dy}\frac{d\rho_3}{dy}+\frac{d^2\Phi_3}{dy^2}\rho_3-\frac{\varepsilon^{2\cdot 3}}{4}\left(\frac{d\rho_3}{dy}\right)^2\right\}+\rho_3$$

$$=-\left\{\Phi_3\frac{d^2z_2}{dy^2}-\frac{1}{2}\frac{d\Phi_2}{dy}\frac{dz_2}{dy}+\frac{d^2\Phi_2}{dy^2}z_2-\frac{\varepsilon^{2\cdot 2}}{4}\left(\frac{dz_2}{dy}\right)^2\right\}$$

So now comparing this with 2.10 allows us to speculate on a general recursion formula for ρ_n and f_n, namely

$$\varepsilon^2 \left\{ (\Phi_n + \varepsilon^{2n}\rho_n)\frac{d^2\rho_n}{dy^2} - \frac{1}{2}\frac{d\Phi_n}{dy}\frac{d\rho_n}{dy} + \frac{d^2\Phi_n}{dy^2}\rho_n - \frac{\varepsilon^{2n}}{4}\left(\frac{d\rho_n}{dy}\right)^2 \right\} + \rho_n$$

$$= -\left\{ \Phi_n \frac{d^2 z_{n-1}}{dy^2} - \frac{1}{2}\frac{d\Phi_{n-1}}{dy}\frac{dz_{n-1}}{dy} + \frac{d^2\Phi_{n-1}}{dy^2} z_{n-1} - \frac{\varepsilon^{2(n-1)}}{4}\left(\frac{dz_{n-1}}{dy}\right)^2 \right\} \tag{2.11}$$

we will refer to the right side of this equation as f_n which by our substitution is just z_n. So to be thorough a proof by induction is needed so that we are sure the formula holds for all postitive integers.

Proof: The base case has been done above. So we assume 2.11 and use the substitution $\rho_n = z_n + \varepsilon^2\rho_{n+1}$.

$$\varepsilon^2 \left\{ \left(\Phi_n + \varepsilon^{2n}(z_n + \varepsilon^2\rho_{n+1})\right) \frac{d^2}{dy^2}(z_n + \varepsilon^2\rho_{n+1}) - \frac{1}{2}\frac{d\Phi_n}{dy}\frac{d}{dy}(z_n + \varepsilon^2\rho_{n+1}) \right.$$

$$\left. + \frac{d^2\Phi_n}{dy^2}(z_n + \varepsilon^2\rho_{n+1}) - \frac{1}{4}\varepsilon^{2n}\left(\frac{d}{dy}(z_n + \varepsilon^2\rho_{n+1})\right)^2 \right\} + z_n + \varepsilon^2\rho_{n+1} = z_n$$

which gives

$$(\Phi_n + \varepsilon^{2n}z_n + \varepsilon^{2(n+1)}\rho_{n+1})\left(\frac{d^2 z_n}{dy^2} + \varepsilon^2\frac{d^2\rho_{n+1}}{dy^2}\right) - \frac{1}{2}\frac{d\Phi_n}{dy}\left(\frac{dz_n}{dy} + \varepsilon^2\frac{d\rho_{n+1}}{dy}\right)$$

$$+ \frac{d^2\Phi_n}{dy^2}(z_n + \varepsilon^2\rho_{n+1}) - \frac{1}{4}\varepsilon^{2n}\left(\frac{dz_n}{dy} + \varepsilon^2\frac{d\rho_{n+1}}{dy}\right)^2 + \rho_n = 0$$

so we get

$$\varepsilon^2 \left\{ (\Phi_{n+1} + \varepsilon^{2(n+1)}\rho_{n+1}) \frac{d^2\rho_{n+1}}{dy^2} - \frac{1}{2}\left(\frac{d\Phi_n}{dy} + \varepsilon^{2n}\frac{dz_n}{dy}\right)\frac{d\rho_{n+1}}{dy} \right.$$

$$\left. + \left(\frac{d^2\Phi_n}{dy^2} + \varepsilon^{2n}\frac{d^2 z_n}{dy^2}\right)\rho_{n+1} - \frac{\varepsilon^{2(n+1)}}{4}\left(\frac{d\rho_{n+1}}{dy}\right)^2 \right\} + \rho_{n+1} = -\left\{ \Phi_{n+1}\frac{d^2 z_n}{dy^2} - \frac{1}{2}\frac{d\Phi_n}{dy}\frac{dz_n}{dy} \right.$$

$$\left. + \frac{d^2\Phi_n}{dy^2} z_n - \frac{1}{4}\varepsilon^{2n}\left(\frac{dz_n}{dy}\right)^2 \right\}$$

and this is just

$$\varepsilon^2 \left\{ (\Phi_{n+1} + \varepsilon^{2(n+1)}\rho_{n+1}) \frac{d^2\rho_{n+1}}{dy^2} - \frac{1}{2}\frac{d\Phi_n}{dy}\frac{d\rho_{n+1}}{dy} + \frac{d^2\Phi_{n+1}}{dy^2}\rho_{n+1} - \frac{\varepsilon^{2(n+1)}}{4}\left(\frac{d\rho_{n+1}}{dy}\right)^2 \right\}$$

$$+\rho_{n+1} = -\left\{ \Phi_{n+1}\frac{d^2 z_n}{dy^2} - \frac{1}{2}\frac{d\Phi_n}{dy}\frac{dz_n}{dy} + \frac{d^2\Phi_n}{dy^2} z_n - \frac{1}{4}\varepsilon^{2n}\left(\frac{dz_n}{dy}\right)^2 \right\}$$

which completes the proof by induction.

2.3 Formal Approximation by Power Series

Now we will compute a formal power series expansion for $z(y)$ which at the same time is an asymptotic expansion in ε. So define

$$Lz := \varepsilon^2 \left\{ z\frac{d^2 z}{dy^2} - \frac{1}{4}\left(\frac{dz}{dy}\right)^2 \right\} + z - y^2 + \frac{2}{3}y^3 \tag{2.12}$$

and we introduce

$$\widetilde{\Phi}_m(y) = \alpha y^2 - \beta y^3 - \sum_{n=1}^{m} a_n y^{n+3} \tag{2.13}$$

Which is just a truncated power series. Now we substitute $\widetilde{\Phi}_m(y)$ in for $z(y)$ to produce

$$L\widetilde{\Phi}_m = \varepsilon^2 \left\{ \widetilde{\Phi}_m \frac{d^2\widetilde{\Phi}_m}{dy^2} - \frac{1}{4}\left(\frac{d\widetilde{\Phi}_m}{dy}\right)^2 \right\} + \widetilde{\Phi}_m - y^2 + \frac{2}{3}y^3$$

$$= \varepsilon^2 \left\{ (\alpha y^2 - \beta y^3 - \sum_{n=1}^{m} a_n y^{n+3}) \frac{d^2}{dy^2}(\alpha y^2 - \beta y^3 - \sum_{n=1}^{m} a_n y^{n+3}) \right.$$

$$\left. -\frac{1}{4}\left(\frac{d}{dy}(\alpha y^2 - \beta y^3 - \sum_{n=1}^{m} a_n y^{n+3})\right)^2 \right\} + \alpha y^2 - \beta y^3 - \sum_{n=1}^{m} a_n y^{n+3} - y^2 + \frac{2}{3}y^3$$

$$= \varepsilon^2 \left\{ \left(\alpha y^2 - \beta y^3 - \sum_{n=1}^{m} a_n y^{n+3}\right)\left(2\alpha - 6\beta y - \sum_{n=1}^{m} a_n (n+3)(n+2) y^{n+1}\right) \right.$$

$$\left. -\frac{1}{4}\left(2\alpha y - 3\beta y^2 - \sum_{n=1}^{m} a_n (n+3) y^{n+2}\right)^2 \right\} + (\alpha - 1)y^2 + \left(\frac{2}{3} - \beta\right) y^3 - \sum_{n=1}^{m} a_n y^{n+3}$$

$$= \varepsilon^2 \left\{ \alpha^2 y^2 - 5\alpha\beta y^3 - \alpha y^2 \sum_{n=1}^{m} a_n (n+3)(n+2) y^{n+1} + \frac{15}{4}\beta^2 y^4 \right.$$

$$+\beta y^3 \sum_{n=1}^{m} a_n (n+3)(n+2) y^{n+1} - 2\alpha \sum_{n=1}^{m} a_n y^{n+3} + 6\beta y \sum_{n=1}^{m} a_n y^{n+3}$$

$$+\sum_{n=1}^{m} a_n y^{n+3} \sum_{n=1}^{m} a_n (n+3)(n+2) y^{n+1} - \frac{3}{2}\beta y^2 \sum_{n=1}^{m} a_n (n+3) y^{n+2} + \alpha y \sum_{n=1}^{m} a_n (n+3) y^{n+2}$$

$$\left. -\frac{1}{4}\sum_{n=1}^{m} a_n (n+3) y^{n+2} \sum_{n=1}^{m} a_n (n+3) y^{n+2} \right\} + (\alpha - 1)y^2 + \left(\frac{2}{3} - \beta\right) y^3 - \sum_{n=1}^{m} a_n y^{n+3}$$

$$= (\varepsilon^2 \alpha^2 + \alpha - 1)y^2 + \left(\frac{2}{3} - \beta(1 + 5\alpha\varepsilon^2)\right) y^3$$

$$+y^4\sum_{n=1}^{m}a_n\left[-\varepsilon^2\alpha(n+3)(n+2)-2\varepsilon^2\alpha+\varepsilon^2\alpha(n+3)-1\right]y^{n-1}+y^4\varepsilon^2\left\{\frac{15}{4}\beta^2\right.$$

$$+y\beta\sum_{n=1}^{m}a_n\left[(n+3)(n+2)+6-\frac{3}{2}(n+3)\right]y^{n-1}+y^2\left[\sum_{n=1}^{m}a_ny^{n-1}\sum_{n=1}^{m}a_n(n+3)(n+2)y^{n-1}\right.$$

$$\left.\left.-\frac{1}{4}\sum_{n=1}^{m}a_n(n+3)y^{n-1}\sum_{n=1}^{m}a_n(n+3)y^{n-1}\right]\right\}$$

and so collecting like terms and reorganizing

$$L\widetilde{\Phi}_m=(\varepsilon^2\alpha^2+\alpha-1)y^2+\left(\frac{2}{3}-\beta(1+5\alpha\varepsilon^2)\right)y^3$$

$$-y^4\sum_{n=1}^{m}a_n\left[1+\varepsilon^2\alpha\left[(n+3)(n+1)+2\right]\right]y^{n-1}+\varepsilon^2y^4\left\{\frac{15}{4}\beta^2\right.$$

$$+y\beta\sum_{n=1}^{m}a_n\left[(n+3)(n+\frac{1}{2})+6\right]y^{n-1}+y^2\left[\sum_{n=1}^{m}a_ny^{n-1}\sum_{n=1}^{m}a_n(n+3)(n+2)y^{n-1}\right.$$

$$\left.\left.-\frac{1}{4}\left(\sum_{n=1}^{m}a_n(n+3)y^{n-1}\right)^2\right]\right\} \tag{2.14}$$

We can now determine α, β, and $a_n, n=1,...,m$ by putting all the coefficients of $y^p, p=2,...,m+3$ on the right hand side of equation 2.14 equal to zero. So we have that

$$\varepsilon^2\alpha^2+\alpha-1=0$$

$$\Rightarrow \alpha=\frac{-1\pm\sqrt{1+4\varepsilon^2}}{2\varepsilon^2}$$

and we take α to be the positive solution to the above so that α is the solution the linearized model equation. We also find that

$$\frac{2}{3}-\beta(1+5\alpha\varepsilon^2)=0\Rightarrow\beta=\frac{2}{3}\frac{1}{1+5\alpha\varepsilon^2}$$

Now set the coefficient of y^4 equal to zero to get

$$-a_1\left[1+\varepsilon^2\alpha\,(4\cdot 2+2)\right]+\varepsilon^2\frac{15}{4}\beta^2=0$$

$$\Rightarrow a_1=\varepsilon^2\frac{15\beta^2}{4}\frac{1}{1+10\varepsilon^2\alpha}$$

we will also look at the coefficients of y^5 in order to get an equation for a_2.

$$-a_2\left[1+\varepsilon^2\alpha\,(5\cdot 3+2)\right]+\varepsilon^2\beta a_1\left[4(1+\frac{1}{2})+6\right]=0$$

$$\Rightarrow a_2=\varepsilon^4\beta^3\frac{45}{1+10\varepsilon^2\alpha}\frac{1}{1+17\varepsilon^2\alpha}$$

So we see that without the condition that α was positive, we would have a_1 possibly being negative, and we want positive values for all the a_n. Let us now determine a_3 to look for a pattern in their structure.

$$-a_3\left[1+\varepsilon^2\alpha\,(6\cdot 4+2)\right]+\varepsilon^2\beta a_2\left[5\cdot\frac{11}{2}+6\right]+a_1a_1 4\cdot 3-\frac{1}{4}a_1a_1 4\cdot 4=0$$

$$a_3\left[1+26\varepsilon^2\alpha\right]=\frac{67}{2}\varepsilon^2\beta\varepsilon^4\beta^3\frac{45}{1+10\varepsilon^2\alpha}\frac{1}{1+17\varepsilon^2\alpha}-8\left(\varepsilon^2\frac{15\beta^2}{4}\frac{1}{1+10\varepsilon^2\alpha}\right)^2$$

$$\Rightarrow a_3=\left[\frac{67}{2}\varepsilon^6\beta^4\frac{45}{1+10\varepsilon^2\alpha}\frac{1}{1+17\varepsilon^2\alpha}-8\left(\varepsilon^2\frac{15\beta^2}{4}\frac{1}{1+10\varepsilon^2\alpha}\right)^2\right]\frac{1}{1+26\varepsilon^2\alpha}$$

So by inspection of the formulas, we are able to determine that

$$a_n=O(\varepsilon^{2n}),\, a_n>0$$

We summarize these results as

$$\widetilde{\Phi}_m(y) = \alpha y^2 - \beta y^3 - \sum_{n=0}^{m} \varepsilon^{2n}\overline{a}_n y^{n+3}$$

$$\overline{a}_n = O(1)$$

$$\overline{a}_n > 0 \text{ for } \varepsilon > 0$$

$$L\widetilde{\Phi}_m = \varepsilon^{2(m+1)} y^{m+4} \sum_{p=0}^{m} \sigma_p y^p$$

$$\sigma_p = O(1), \sigma_p > 0$$

This will then give us

$$z = \widetilde{\Phi}_m + \varepsilon^{2(m+2)}\overline{\rho}_m$$

where $\overline{\rho}_m$ is a remainder term. We will be using the power series expansion when we deal with the problem of splitting the homoclinic orbit.

In the work done by Hammersley and Mazzarino [7], a formal power series similar to 2.13 is used, in particular

$$f = \lambda^2 z^2 \left(1 - \sum_{n=1}^{\infty} c_n z^n\right)$$

as a basic step in their analysis. The coefficients are determined by a recursion formula and convergence results are given. However, for ε small, the authors do not do the analysis. The results gained by Hammersley and Mazzarino [7] are encouraging as they do support the results obtained here. Their numerical computations suggest the same results that will be shown in the following chapters.

Chapter 3

What Does it All Mean?

3.1 The Solution for z(y) and the Result

Our basic equation is

$$\varepsilon^2 \left\{ z\frac{d^2z}{dy^2} - \frac{1}{4}\left(\frac{dz}{dy}\right)^2 \right\} + z = y^2(1 - \frac{2}{3}y) \tag{3.1}$$

We are trying to find a homoclinic orbit, so in order to do this we look for solutions $z(y)$ which for $y \to 0$ behave as y^2. We look for this type of solution due to the way $z(y) = y^2 - \frac{2}{3}y^3$ approximates y^2 as $y = 0$. To achieve this we introduce the transformation

$$z = y^2\overline{z}$$

to get the equation

$$\varepsilon^2 \left\{ y^2 \left[\overline{z}\frac{d^2\overline{z}}{dy^2} - \frac{1}{4}\left(\frac{d\overline{z}}{dy}\right)^2 \right] + 3y\overline{z}\frac{d\overline{z}}{dy} \right\} + \overline{z}(1 + \varepsilon^2\overline{z}) = 1 - \frac{2}{3}y \tag{3.2}$$

then write

$$\overline{z} = \varphi_m + \varepsilon^{2m}\psi_m$$

where φ_m is the result of the formal iteration of section 2.2 or the truncated power series of section 2.3, in both cases with y^2 factored out. So φ_m is really just a polynomial in y. Substituting $\overline{z}$ into 3.2 we derive

$$\varepsilon^2 \left\{ y^2 \left[(\varphi_m + \varepsilon^{2m}\psi_m) \frac{d^2}{dy^2}(\varphi_m + \varepsilon^{2m}\psi_m) - \frac{1}{4}\left(\frac{d}{dy}(\varphi_m + \varepsilon^{2m}\psi_m)\right)^2 \right] \right.$$

$$\left. +3y(\varphi_m + \varepsilon^{2m}\psi_m)\frac{d}{dy}(\varphi_m + \varepsilon^{2m}\psi_m) \right\} + (\varphi_m + \varepsilon^{2m}\psi_m)(1 + \varepsilon^2(\varphi_m + \varepsilon^{2m}\psi_m))$$

$$= 1 - \frac{2}{3}y$$

$$\Rightarrow \varepsilon^2 y^2 (\varphi_m + \varepsilon^{2m}\psi_m)\left(\varphi_m'' + \varepsilon^{2m}\frac{d^2\psi_m}{dy^2}\right) - \frac{\varepsilon^2 y^2}{4}\left(\varphi_m' + \varepsilon^{2m}\frac{d}{dy}\psi_m\right)^2$$

$$+3\varepsilon^2 y(\varphi_m + \varepsilon^{2m}\psi_m)\left(\varphi_m' + \varepsilon^{2m}\frac{d}{dy}\psi_m\right) + (\varphi_m + \varepsilon^{2m}\psi_m)(1 + (\varphi_m + \varepsilon^{2m}\psi_m))$$

$$= 1 - \frac{2}{3}y$$

which simplifies to

$$\left[\varepsilon^{2m} y^2 (\varphi_m + \varepsilon^{2m}\psi_m)\right] \frac{d^2\psi_m}{dy^2} + \varepsilon^2 \left[3y(\varphi_m + \varepsilon^{2m}\psi_m) - \frac{1}{2}y^2(\varphi_m' + \frac{1}{2}\varepsilon^{2m}\psi_m')\right] \frac{d\psi_m}{dy}$$

$$+ \psi_m = \overline{f}_m(y) - \varepsilon^2 \left\{ y^2\varphi_m'' + 3y\varphi_m' + (2\varphi_m + \varepsilon^{2m}\psi_m) \right\} \psi_m \qquad (3.3)$$

where

$$\overline{f}_m(y) = \frac{1 - \frac{2}{3}y - \varepsilon^2 y^2 \varphi_m \varphi_m'' + \frac{\varepsilon^2 y^2}{4}(\varphi_m') - \varepsilon^2 \varphi_m^2}{\varepsilon^{2m}}$$

which is just a polynomial in y since φ_m is a polynomial in y. We use both primes and the $\frac{d}{dy}$ notation to denote the derivative so that we can try to produce general formulas. In Chapter 4 we will prove the following main result.

Result

There exists a unique solution $\psi_m(y;\varepsilon)$*, which on an interval* $y \in [0, y_0]$ *is bounded for* $\varepsilon \downarrow 0$ *and the same is true for the derivative* $\psi'_m(y;\varepsilon)$*.* y_0 *satisfies the estimate*

$$|\varphi_m(y;\varepsilon)| \geq c\varepsilon^m$$

with c *a constant and* $m \geq 2$*.*

Remarks

Note that m is an arbitrary integer and so we have the existence of a solution which starts out as y^2 and goes to zero faster than any power of ε. We also have that $y_0 > \frac{3}{2}$. We wonder what happens to the continuation of the solution. $z(y)$ can not stop at some positive value, it must continue, so suppose that the continuation escapes to some large value. This is only possible at $y > y_0$. The problem arises because at some $z > 0$ one would have $\frac{dz}{dy} = 0$ and $\frac{d^2z}{dy^2} \geq 0$ and this leads to a contradiction in the equation 3.1. Now suppose that $z \to 0$ as $y \to y_1 > y_0$,with z' also tending to zero. Since z' comes from negative values and did not pass through zero, we must still have $z'' \geq 0$ and this again gives a contradiction in 3.1. This contradiction remains if we take $y_1 = +\infty$ or if we assume that z tends to a non-zero positive value as $y \to \infty$. Thus the continuation of the solution $z(y)$ must reach $z = 0$ with a non-zero slope.

3.2 Splitting of the Homoclinic Orbit

We start by defining two half orbits $y_+(x)$, $x \in (-\infty, 0]$, $y_-(x)$, $x \in [0, \infty)$ as solutions of

$$\begin{aligned} \left(\frac{dy}{dx}\right)_{\pm} &= \pm\sqrt{z(y)} \\ y_+(0) = y_-(0) &= y_1(\varepsilon) \\ z(y_1) &= 0 \end{aligned} \tag{3.4}$$

What happens to $y_+(x)$ and $y_-(x)$ at $x = 0$? We are interested in the regularity of the union of these two half orbits. Since the model equation is of fourth order, we need continuity up to and including the fourth derivative. So let us compute

$$\left(\frac{d^2y}{dx^2}\right)_{\pm} = \frac{d}{dx}\left(\frac{dy}{dx}\right)_{\pm} = \frac{d}{dx}\left(\pm\sqrt{z(y)}\right) = \pm\frac{1}{2}\frac{1}{\sqrt{z}}\left(\frac{dz}{dx}\right)_{\pm}$$

$$= \pm\frac{1}{2}\frac{1}{\sqrt{z}}\frac{dz}{dy}\left(\frac{dy}{dx}\right)_{\pm} = \pm\frac{1}{2}\frac{1}{\sqrt{z}}\frac{dz}{dy}(\pm)\sqrt{z} = \frac{1}{2}\frac{dz}{dy}$$

thus

$$\left(\frac{d^2y}{dx^2}\right)_{+} = \left(\frac{d^2y}{dx^2}\right)_{-} \quad \text{at } x = 0$$

From 1.3, the above, and the fact that $y_+(0) = y_-(0)$ we get that

$$\left(\frac{d^4y}{dx^4}\right)_{+} = \left(\frac{d^4y}{dx^4}\right)_{-} \quad \text{at } x = 0$$

Now from previous results we have

$$\left(\frac{d^3y}{dx^3}\right)_{\pm} = \frac{1}{2}\left(\frac{dy}{dx}\right)_{\pm}\frac{d^2z}{dy^2} = \pm\frac{1}{2}\sqrt{z}\frac{d^2z}{dy^2}$$

remember that we are trying to establish continuity of the third derivative. We compute the third derivative by the use of the Laplace transform. We use

$$L\{f\}(s) = \int_{-\infty}^{0} f(-t)e^{st}dt$$

Now we make the substitution $u = \frac{d^2y}{dx^2}$ and so the equation that we are trying to solve is $\varepsilon^2 u'' + u = (y - y^2)$. So consider

$$L\left\{\varepsilon^2 u'' + u\right\} = L\left\{y - y^2\right\}$$

$$\Rightarrow L\left\{\varepsilon^2 u''\right\} + L\{u\} = L\left\{y - y^2\right\}$$

$$\Rightarrow \varepsilon^2 s^2 L\{u\} + L\{u\} = L\left\{y - y^2\right\}$$

$$\Rightarrow L\{u\} = \frac{1}{\varepsilon^2 s^2 + 1} L\left\{y - y^2\right\}$$

$$\Rightarrow L\{u\} = \frac{1}{\varepsilon} \frac{\frac{1}{\varepsilon}}{s^2 + \frac{1}{\varepsilon^2}} L\left\{y - y^2\right\}$$

$$\Rightarrow L\{u\} = \frac{1}{\varepsilon} L\left\{\sin\left(\frac{1}{\varepsilon}x\right)\right\} L\left\{y - y^2\right\}$$

and so by the convolution theorem

$$\Rightarrow u = \frac{1}{\varepsilon}\int_{-\infty}^{0} \sin\left(\frac{1}{\varepsilon}(x-\xi)\right)\left[y(\xi) - y^2(\xi)\right] d\xi$$

thus we have

$$\left(\frac{d^2y}{dx^2}\right)_+ = \frac{1}{\varepsilon}\int_{-\infty}^{0} \sin\left(\frac{1}{\varepsilon}(x-\xi)\right)\left[y(\xi) - y^2(\xi)\right] d\xi$$

So now we differentiate to get

$$\frac{d^3y_+}{dx^3}(0) = I(\varepsilon)$$

where

$$I(\varepsilon) = \frac{1}{\varepsilon^2}\int_{-\infty}^{0} \cos(\frac{1}{\varepsilon}\xi)\left[y(\xi) - y^2(\xi)\right] d\xi$$

We wish to compute the integral along approximate trajectories, so we replace $z(y)$ in 3.4 by its asymptotic expansions. So we define $y^{(m)}(x), x < 0$, as

$$\frac{dy^{(m)}}{dx} = \sqrt{\Phi_m(y;\varepsilon)}$$

with Φ_m the asymptotic expansion defined in section 2.2. Now, define

$$I^{(m)}(\varepsilon) = \frac{1}{\varepsilon^2}\int_{-\infty}^{0} \cos\left(\frac{1}{\varepsilon}\xi\right)\left[y^{(m)}(\xi) - (y^{(m)}(\xi))^2\right] d\xi$$

In the Appendix, we have the integrals computed for $m = 1$ and $m = 2$. These integrals are due to dr. N. Temme of C.W.I. Amsterdam who did the analysis involved. These results can be summarized as follows:

$$\Phi_m = \sum_{n=0}^{m-1} \varepsilon^{2n} z_n(y), m = 1, 2$$

$$I^{(m)}(\varepsilon) = -\frac{\pi}{\varepsilon^5} c_m e^{-\frac{\pi}{\varepsilon}}\left[1 + o(1)\right]$$

$$c_1 = 6, c_2 \cong 3.6c_1$$

So a small correction in the trajectories does not result in a small change in the constant c_m. Hammersley and Mazzarino [7] came up with a very similar result for the integral I_m, their result was the reward for complicated and extensive numerical calculations. For large values of m, we can expect that the constants c_m will settle to a definite value but analytical determination seems beyond possibility. The complicated evaluation for $m = 1$ and $m = 2$ gives this impression. However, we can show that the integral $I^{(m)}$ is negative for all m as was shown in [7].

We recall some relevant formulas

$$Lz = \varepsilon^2 \left(z\frac{d^2z}{dy^2} - \frac{1}{4}\left(\frac{dz}{dy}\right)^2 \right) + z - y^2 + \frac{2}{3}y^3$$

$$\tilde{\Phi}_m(y) = \alpha y^2 + \beta y^3 - \sum_{n=1}^{m} \varepsilon^{2n} \overline{a}_n y^{n+3}$$

$$L\tilde{\Phi}_m = \varepsilon^{2(m+1)} f_m(y)$$

$$f_m(y) = y^{m+4} \sum_{p=0}^{m} \sigma_p y^p$$

$$\sigma_p = o(1), \sigma_p > 0$$

Now we want to look at $\tilde{y}^{(m)}(x)$ which is a solution for $x < 0$ of the equation

$$\frac{d\tilde{y}^{(m)}}{dx} = \sqrt{\tilde{\Phi}_m(y^{(m)})}$$

Now consider the original equation

$$\varepsilon^2 \frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} = y - y^2$$

and using 2.3 we get

$$\frac{d}{dy}\left[\varepsilon^2\left\{z\frac{d^2z}{dy^2}-\frac{1}{4}\left(\frac{dz}{dy}\right)^2\right\}+z\right]=2\left[\varepsilon^2\frac{d^4y}{dx^4}+\frac{d^2y}{dx^2}\right]$$

and so

$$\frac{d}{dy}\left[\varepsilon^2\left\{z\frac{d^2z}{dy^2}-\frac{1}{4}\left(\frac{dz}{dy}\right)^2\right\}+z-y^2+\frac{2}{3}y^3\right]$$

$$=2\left[\varepsilon^2\frac{d^4y}{dx^4}+\frac{d^2y}{dx^2}\right]-2y+2y^2$$

So $\tilde{y}^{(m)}$ satisfies

$$\varepsilon^2\frac{d^4\tilde{y}^{(m)}}{dx^4}+\frac{d^2\tilde{y}^{(m)}}{dx^2}=\frac{1}{2}\varepsilon^{2(m+1)}f_m'(\tilde{y}^{(m)})+\tilde{y}^{(m)}-\left(\tilde{y}^{(m)}\right)^2$$

with

$$f_m'(\tilde{y}^{(m)})=\frac{d}{dy}f_m(\tilde{y}^{(m)})$$

so by using the Laplace transform once again, we get

$$\left(\frac{d^2\tilde{y}^{(m)}}{dx^2}\right)_+=\frac{1}{\varepsilon}\int_{-\infty}^{0}\sin\left(\frac{1}{\varepsilon}(x-\xi)\right)\left[\tilde{y}^{(m)}(\xi)-\left(\tilde{y}^{(m)}\right)^2(\xi)+\frac{1}{2}\varepsilon^{2(m+1)}f_m'(\tilde{y}^{(m)})\right]d\xi$$

and so

$$\frac{d^3\tilde{y}_+^{(m)}}{dx^3}(0)=\frac{1}{\varepsilon^2}\int_{-\infty}^{0}\cos\left(-\frac{1}{\varepsilon}\xi\right)\left[\tilde{y}^{(m)}-\left(\tilde{y}^{(m)}\right)^2+\frac{1}{2}\varepsilon^{2(m+1)}f_m'(\tilde{y}^{(m)})\right]$$

$$=\frac{1}{\varepsilon^2}\int_{-\infty}^{0}\cos\left(\frac{1}{\varepsilon}\xi\right)\left[\tilde{y}^{(m)}-\left(\tilde{y}^{(m)}\right)^2+\frac{1}{2}\varepsilon^{2(m+1)}f_m'(\tilde{y}^{(m)})\right]$$

So we have that

$$\frac{d^3\tilde{y}_+^{(m)}}{dx^3} = \frac{1}{2}\tilde{\Phi}_m^{\frac{1}{2}}\frac{d^2\tilde{\Phi}_m}{(d\tilde{y}^{(m)})^2}$$

since the quantity on the right hand side of the above equation is just a polynomial, we have $\frac{d^3\tilde{y}_+^{(m)}}{dx^3}(0) = 0$ since this is how we determined the coefficients for $\tilde{\Phi}_m$. This then allows us to get

$$0 = \frac{1}{\varepsilon^2}\int_{-\infty}^{0}\cos\left(\frac{1}{\varepsilon}\xi\right)\left[\tilde{y}^{(m)}(\xi) - \left(\tilde{y}^{(m)}(\xi)\right)^2 + \frac{1}{2}\varepsilon^{2(m+1)}f'(\tilde{y}^{(m)})\right]d\xi$$

$$\Rightarrow 0 = \frac{1}{\varepsilon^2}\int_{-\infty}^{0}\cos\left(\frac{1}{\varepsilon}\xi\right)\left[\tilde{y}^{(m)}(\xi) - \left(\tilde{y}^{(m)}(\xi)\right)^2\right]d\xi$$

$$+\frac{1}{2}\frac{1}{\varepsilon^2}\int_{-\infty}^{0}\varepsilon^{2(m+1)}\cos\left(\frac{1}{\varepsilon}\xi\right)f'(\tilde{y}^{(m)})d\xi$$

$$\Rightarrow I^{(m)}(\varepsilon) = -\frac{1}{2}\varepsilon^{2m}\int_{-\infty}^{0}\cos\left(\frac{1}{\varepsilon}\xi\right)f'(\tilde{y}^{(m)})d\xi$$

Now remember that

$$f'(\tilde{y}^{(m)}) = \frac{d}{dy}f(\tilde{y}^{(m)}) = \frac{d}{dy}\left(y^{m+4}\sum_{p=0}^{m}\sigma_p y^p\right)$$

$$= \frac{d}{dy}\sum_{p=0}^{m}\sigma_p y^{p+m+4}$$

$$= \sum_{p=0}^{m}(p+m+4)\sigma_p y^{p+m+3}$$

So

$$I^{(m)}(\varepsilon) = -\frac{1}{2}\varepsilon^{2m}\int_{-\infty}^{0}\cos\left(\frac{1}{\varepsilon}\xi\right)\sum_{p=0}^{m}(p+m+4)\sigma_p\left[\tilde{y}^{(m)}\right]^{p+m+3}d\xi$$

$$= -\frac{1}{2}\varepsilon^{2m}\sum_{p=0}^{m}\sigma_p(p+m+4)\int_{-\infty}^{0}\cos\left(\frac{1}{\varepsilon}\xi\right)\left[\tilde{y}^{(m)}\right]^{p+m+3}d\xi$$

Now in order for $I^{(m)}(\varepsilon)$ to be negative, the integrals in the summation must all be positive. $\tilde{y}^{(m)}$ is a monotonic function and for all ξ ranging from $-\infty$ to 0 is positive. So if we consider the function on intervals like $\left[-\frac{2\pi}{\varepsilon}, 0\right], \left[-\frac{4\pi}{\varepsilon}, -\frac{2\pi}{\varepsilon}\right]$, etc. we see that we get a net positive contribution. We also have that $\sigma_p > 0$ thus

$$I^{(m)}(\varepsilon) < 0, \forall m.$$

Chapter 4

The Proof of the Result

In this section we will prove the result given in Chapter 3 and we will simplify notations by dropping the subscript m.

4.1 The General Idea

The equation 3.3 is of the following structure

$$\Psi(\psi) = \overline{f}(y) + \varepsilon^2 g_1(y, \psi) \tag{4.1}$$

$$\Psi \doteq \varepsilon^2 A(y, \psi) \frac{d^2}{dy^2} + \varepsilon^2 B(y, \psi, \psi') \frac{d}{dy} + 1 \tag{4.2}$$

$$A(y, \psi) = y^2(\varphi + \varepsilon^{2m}\psi) \tag{4.3}$$

$$B(y, \psi, \psi') = -\frac{1}{2}y^2(\varphi' + \frac{1}{2}\varepsilon^{2m}\psi') + 3y(\varphi + \varepsilon^{2m}\psi) \tag{4.4}$$

$$g_1(y, \psi) = -\left\{y^2\varphi'' + 3y\varphi' + (2\varphi + \varepsilon^{2m}\psi)\right\}\psi \tag{4.5}$$

We want to prove the existence of a bounded solution. We would like to decompose $\Psi = \Psi_1 + \Psi_2$ so that

$$\Psi_1(\psi) + \Psi_2(\psi) = \overline{f} + \varepsilon^2 g_1$$

or

$$\Psi_1(\psi) = -\Psi_2(\psi) + \overline{f} + \varepsilon^2 g_1$$

and if Ψ_1^{-1} existed then we could find

$$\psi = \Psi_1^{-1}\left(-\Psi_2(\psi) + \overline{f} + \varepsilon^2 g_1\right)$$

To make things work out nicely, we will try to use a linear operator for Ψ_1. The problem is that we get nonlinear terms on the right hand side of the equation. It will be possible to get rid of occurrences of the second derivative, but the first derivative can not be avoided and so we must study the problem for ψ'. So consider

$$\Psi(\psi) = \overline{f}(y) + \varepsilon^2 g_1(y, \psi) \tag{4.6}$$

Now use 4.2 and differentiate with respect to y

$$\frac{d}{dy}\Psi(\psi) = \left[\frac{d}{dy}\varepsilon^2 y^2(\varphi + \varepsilon^{2m}\psi)\frac{d^2\psi}{dy^2} + \varepsilon^2\left(-\frac{1}{2}y^2(\varphi' + \frac{1}{2}\varepsilon^{2m}\psi') + 3y(\varphi + \varepsilon^{2m}\psi\right)\frac{d\psi}{dy}\right.$$

$$\left. +\psi\cdot\right]$$

$$= \varepsilon^2\left(y^2(\varphi + \varepsilon^{2m}\psi)\right)\frac{d^2\psi'}{dy^2} + \varepsilon^2\frac{d\psi'}{dy}\left[2y(\varphi + \varepsilon^{2m}\psi) + y^2(\varphi' + \varepsilon^{2m}\psi')\right.$$

$$\left. -\frac{1}{2}y^2\frac{1}{2}\varepsilon^{2m}\psi' - \frac{1}{2}y^2(\varphi' + \frac{1}{2}\varepsilon^{2m}\psi') + 3y(\varphi + \varepsilon^{2m}\psi)\right] + \psi'$$

$$+\psi'\varepsilon^2\left[-y(\varphi' + \frac{1}{2}\varepsilon^{2m}\psi') - \frac{1}{2}y^2\varphi'' + 3y(\varphi' + \varepsilon^{2m}\psi') + 3(\varphi + \varepsilon^{2m}\psi)\right]$$

which gives

$$\varepsilon^2 y^2(\varphi + \varepsilon^{2m}\psi)\frac{d^2\psi'}{dy^2} + \varepsilon^2\frac{d\psi'}{dy}\left[5y(\varphi + \varepsilon^{2m}\psi) + \frac{1}{2}y^2(\varphi' + \varepsilon^{2m}\psi')\right] + \psi'$$

$$+\psi'\varepsilon^2\left[2y\varphi' + \frac{5}{2}y\varepsilon^{2m}\psi' - \frac{1}{2}y^2\varphi'' + 3(\varphi + \varepsilon^{2m}\psi)\right]$$

and

$$\frac{d}{dy}\left(\overline{f}(y) + \varepsilon^2 g_1(y,\psi)\right)$$

$$= \overline{f}(y) - \frac{d}{dy}\varepsilon^2\left\{y^2\varphi'' + 3y\varphi' + (2\varphi + \varepsilon^{2m}\psi)\right\}\psi$$

$$= \overline{f}(y) - \varepsilon^2\left[(2y\varphi'' + y^2\varphi''' + 3\varphi' + 3y\varphi'' + 2\varphi' + \varepsilon^{2m}\psi')\psi\right.$$

$$\left. + \left(y^2\varphi'' + 3y\varphi' + (2\varphi + \varepsilon^{2m}\psi)\right)\psi'\right]$$

$$= \overline{f}(y) - \varepsilon^2\left[(y^2\varphi''' + 5y\varphi'' + 5\varphi')\psi + \left(y^2\varphi'' + 3y\varphi' + (2\varphi + \varepsilon^{2m}\psi) + \varepsilon^{2m}\psi\right)\psi'\right]$$

So we now have

$$\widetilde{\Psi} = \varepsilon^2 A(y,\psi)\frac{d^2}{dy^2} + \varepsilon^2\overline{B}(y,\psi,\psi')\frac{d}{dy} + 1 \tag{4.7}$$

with A as in 4.3 and

$$\overline{B}(y,\psi,\psi') = 5y(\varphi + \varepsilon^{2m}\psi) + \frac{1}{2}y^2(\varphi' + \varepsilon^{2m}\psi') \tag{4.8}$$

$$\widetilde{\Psi}(\psi') = \overline{f}'(y) + \varepsilon^2 g_2(y,\psi,\psi')$$

where

$$g_2 = -(y^2\varphi''' + 5y\varphi'' + 5\varphi')\psi - \left(y^2\varphi'' + 3y\varphi' + (2\varphi + 2\varepsilon^{2m}\psi)\right)\psi'$$

$$-\psi'\left[2y\varphi' + \frac{5}{2}y\varepsilon^{2m}\psi' - \frac{1}{2}y^2\varphi'' + 3(\varphi + \varepsilon^{2m}\psi)\right]$$

$$= -(y^2\varphi''' + 5y\varphi'' + 5\varphi')\psi - \left\{5(\varphi + \varepsilon^{2m}\psi) + y(5\varphi' + \frac{5}{2}\varepsilon^{2m}\psi')\right.$$

$$+y^2(\varphi'' - \frac{1}{2}\varphi'')\Big\}\,\psi'$$

$$= -(y^2\varphi''' + 5y\varphi'' + 5\varphi')\psi - \Big\{5(\varphi + \varepsilon^{2m}\psi) + 5y(\varphi' + \varepsilon^{2m}\psi')$$

$$+\frac{1}{2}y^2\varphi'' - \frac{5}{2}\varepsilon^{2m}y\psi'\Big\}\,\psi' \tag{4.9}$$

Now we put our attention to finding a suitable operator Ψ_1 and for this we take a small recess into WKB functions.

4.2 Excerpt on "exact WKB functions".

WKB (Wentzel, Kramers, Brillouin) approximations have been used by physicists for many applications. Here, we are only going to use a special case and the procedure we need is as follows. Consider the homogeneous equation

$$\varepsilon^2 a\frac{d^2\theta}{dy^2} + \varepsilon^2 b\frac{d\theta}{dy} + \theta = 0 \tag{4.10}$$

where a and b are given functions. We introduce

$$\theta = e^{\frac{1}{\varepsilon}Q}$$

So that

$$\frac{d\theta}{dy} = \frac{1}{\varepsilon}Q'e^{\frac{1}{\varepsilon}Q}$$

$$\frac{d^2\theta}{dy^2} = \frac{1}{\varepsilon^2}\left(Q'\right)^2 e^{\frac{1}{\varepsilon}Q} + \frac{1}{\varepsilon}Q''e^{\frac{1}{\varepsilon}Q}$$

To get

$$\varepsilon^2 a\left(\frac{1}{\varepsilon^2}\left(Q'\right)^2 e^{\frac{1}{\varepsilon}Q}+\frac{1}{\varepsilon}Q''e^{\frac{1}{\varepsilon}Q}\right)+\varepsilon^2 b\frac{1}{\varepsilon}Q'e^{\frac{1}{\varepsilon}Q}+e^{\frac{1}{\varepsilon}Q}=0$$

which yields

$$a\left(Q'\right)^2 e^{\frac{1}{\varepsilon}Q}+\varepsilon aQ''e^{\frac{1}{\varepsilon}Q}+\varepsilon bQ'e^{\frac{1}{\varepsilon}Q}+e^{\frac{1}{\varepsilon}Q}=0$$

or

$$a\left(Q'\right)^2+\varepsilon aQ''+\varepsilon bQ'+1=0$$

Now introduce a formal expression for Q.

$$Q=q_0+\varepsilon q_1+\varepsilon^2 q_2+\cdots$$

$$Q'=q_0'+\varepsilon q_1'+\varepsilon^2 q_2'+\cdots$$

$$Q''=q_0''+\varepsilon q_1''+\varepsilon^2 q_2''+\cdots$$

So we have

$$a(q_0'+\varepsilon q_1'+\varepsilon^2 q_2'+\cdots)^2+\varepsilon a(q_0''+\varepsilon q_1''+\varepsilon^2 q_2''+\cdots)$$

$$+\varepsilon b(q_0'+\varepsilon q_1'+\varepsilon^2 q_2'+\cdots)+1=0$$

and equating coefficients of ε^0 and ε^1 will give

$$\varepsilon^0: a(q_0')^2+1=0$$

$$\varepsilon^1: aq_1'q_0'+aq_0'q_1'+aq_0'+bq_0'=0$$

So

$$(q_0')^2 = -\frac{1}{a} \text{ giving } q_0' = \pm i \frac{1}{\sqrt{a}}$$

and

$$\begin{aligned} q_1' &= \frac{-aq_0'' - bq_0'}{2aq_0'} \\ &= \frac{-a'}{4a^2(q_0')^2} - \frac{1}{2}\frac{b}{a} \\ &= \frac{a'}{4a} - \frac{1}{2}\frac{a}{b} \end{aligned}$$

So

$$\begin{aligned} q_0 &= \pm i \int^y \frac{1}{\sqrt{a(t)}} dt \\ q_1 &= \frac{1}{4}\int \frac{a'}{a} - \int^y \frac{1}{2}\frac{b(t)}{a(t)} dt \\ &= \frac{1}{4}\ln a - \frac{1}{2}\int^y \frac{b}{a} dt \end{aligned}$$

Hence as a formal WKB approximation we get

$$\theta \sim \exp\left\{ \frac{1}{\varepsilon}\left(\pm i \int^y \frac{1}{\sqrt{a(t)}} dt + \varepsilon \ln a^{\frac{1}{4}} - \frac{\varepsilon}{2}\int^y \frac{b}{a} dt \right) \right\}$$

So call

$$\theta_0 = \exp\left\{ -\frac{1}{2}\int^y \frac{b(t)}{a(t)} dt \right\} a^{\frac{1}{4}} \exp\left\{ \pm \frac{i}{\varepsilon} \int^y \frac{1}{\sqrt{a(t)}} dt \right\}$$

In our case the equations 4.3, 4.4, 4.8 vanish at $y = 0$ and the proof of validity appears difficult. Luckily we will not need to prove validity in the approach we will follow. We will consider what differential equation is satisfied exactly by the WKB approximation. In the case of $\widetilde{\Psi}$ with $m = 0$ we have

$$\widetilde{\Psi} = \varepsilon^2 y^2 \varphi \frac{d^2}{dy^2} + \varepsilon^2 (5y\varphi + \frac{1}{2} y^2 \varphi') \frac{d}{dy} + 1$$

In regards to the computation of the WKB approximation,

$$a = y^2 \varphi$$

$$b = 5y\varphi + \frac{1}{2} y^2 \varphi'$$

so

$$\begin{aligned}
\theta_0 &= \exp\left\{ -\frac{1}{2} \int^y \frac{5y\varphi + \frac{1}{2} y^2 \varphi'}{y^2 \varphi} dy \right\} \left(y^2 \varphi \right)^{\frac{1}{4}} \exp\left\{ \pm \frac{i}{\varepsilon} \int^y \frac{1}{\sqrt{t^2 \varphi}} dt \right\} \\
&= y^{\frac{-5}{2}} \varphi^{\frac{-1}{4}} y^{\frac{1}{2}} \varphi^{\frac{1}{4}} \exp\left\{ \pm \frac{i}{\varepsilon} \int^y \frac{1}{t\sqrt{\varphi}} dt \right\} \\
&= \frac{1}{y^2} \exp\left\{ \pm \frac{i}{\varepsilon} \int^y \frac{1}{t\sqrt{\varphi}} dt \right\}
\end{aligned}$$

Now

$$\frac{d\theta_0}{dy} = \frac{1}{y^2} \exp\left\{ \pm \frac{i}{\varepsilon} \int^y \frac{1}{t\sqrt{\varphi}} dt \right\} \left[\frac{1}{\varepsilon} \left(\frac{-1}{y^2 \varphi} \right)^{\frac{1}{2}} + \frac{-2}{y^3} \right]$$

$$\varepsilon^2 b \frac{d\theta_0}{dy} = \left[\frac{5\varepsilon\varphi}{y} \left(\frac{-1}{y^2 \varphi} \right)^{\frac{1}{2}} - \frac{10\varepsilon^2 \varphi}{y^2} + \frac{\varepsilon\varphi'}{2} \left(\frac{-1}{y^2 \varphi} \right)^{\frac{1}{2}} - \frac{\varepsilon^2 \varphi'}{y} \right] \exp\left\{ \pm \frac{i}{\varepsilon} \int^y \frac{1}{t\sqrt{\varphi}} dt \right\}$$

and

$$\begin{aligned}
\frac{d^2\theta_0}{dy^2} &= \frac{1}{y^2} \exp\left\{ \pm \frac{i}{\varepsilon} \int^y \frac{1}{t\sqrt{\varphi}} dt \right\} \frac{1}{2\varepsilon} \left(\frac{-1}{y^2 \varphi} \right)^{\frac{1}{2}} \left(\frac{2y\varphi + y^2 \varphi'}{(y^2 \varphi)^2} \right) \\
&+ \frac{1}{\varepsilon} \left(\frac{-1}{y^2 \varphi} \right)^{\frac{1}{2}} \exp\left\{ \pm \frac{i}{\varepsilon} \int^y \frac{1}{t\sqrt{\varphi}} dt \right\} \left[\frac{1}{\varepsilon y^2} \left(\frac{-1}{y^2 \varphi} \right)^{\frac{1}{2}} - \frac{2}{y^3} \right]
\end{aligned}$$

$$-\exp\left\{\pm\frac{i}{\varepsilon}\int^{y}\frac{1}{t\sqrt{\varphi}}dt\right\}\left[\frac{2}{y^3}\frac{1}{\varepsilon}\left(\frac{-1}{y^2\varphi}\right)^{\frac{1}{2}}-\frac{6}{y^4}\right]$$

so

$$\varepsilon^2 a\frac{d^2\theta_0}{dy^2}=\left[\frac{\varepsilon\varphi}{2}\left(\frac{-1}{y^2\varphi}\right)^{-\frac{1}{2}}\left(\frac{2\varphi+y\varphi'}{y^3\varphi^2}\right)+\varphi\left(\frac{-1}{y^2\varphi}\right)-\frac{4\varepsilon\varphi}{y}\left(\frac{-1}{y^2\varphi}\right)^{\frac{1}{2}}\right.$$
$$\left.+\frac{6\varepsilon^2\varphi}{y^2}\right]\exp\left\{\pm\frac{i}{\varepsilon}\int^{y}\frac{1}{t\sqrt{\varphi}}dt\right\}$$

Now we can evaluate

$$\varepsilon^2\left\{a\frac{d^2\theta_0}{dy^2}+b\frac{d\theta_0}{dy}\right\}+\theta_0$$

$$=\exp\left\{\pm\frac{i}{\varepsilon}\int^{y}\frac{1}{t\sqrt{\varphi}}dt\right\}\left[\frac{\varepsilon}{2}\left(\frac{-1}{y^2\varphi}\right)^{-\frac{1}{2}}\left(\frac{2\varphi+y\varphi'}{y^3\varphi}\right)-\frac{1}{y^2}-\frac{4\varepsilon\varphi}{y}\left(\frac{-1}{y^2\varphi}\right)^{\frac{1}{2}}+\frac{6\varepsilon^2\varphi}{y^2}\right.$$
$$\left.+\frac{5\varepsilon\varphi}{y}\left(\frac{-1}{y^2\varphi}\right)^{\frac{1}{2}}-\frac{10\varepsilon^2\varphi}{y^2}+\frac{\varepsilon\varphi'}{2}\left(\frac{-1}{y^2\varphi}\right)^{\frac{1}{2}}-\frac{\varepsilon^2\varphi'}{y}+\frac{1}{y^2}\right]$$
$$=\exp\left\{\pm\frac{i}{\varepsilon}\int^{y}\frac{1}{t\sqrt{\varphi}}dt\right\}\left(\frac{-1}{y^2\varphi}\right)^{-\frac{1}{2}}\left[\frac{\varepsilon}{2}\left(\frac{2\varphi+y\varphi'}{y^3\varphi}\right)-\frac{\varepsilon}{y^3}-\frac{\varepsilon\varphi'}{2y^2\varphi}\right.$$
$$\left.-\left(\frac{-1}{y^2\varphi}\right)^{\frac{1}{2}}\left(\frac{4\varepsilon^2\varphi}{y^2}+\frac{\varepsilon^2 y\varphi'}{y^2}\right)\right]$$
$$=-\exp\left\{\pm\frac{i}{\varepsilon}\int^{y}\frac{1}{t\sqrt{\varphi}}dt\right\}\left(\frac{4\varepsilon^2\varphi+\varepsilon^2 y\varphi'}{y^2}\right)$$
$$=-\varepsilon^2(4\varphi+y\varphi')\theta_0$$

and so

$$\varepsilon^2\left\{a\frac{d^2\theta_0}{dy^2}+b\frac{d\theta_0}{dy}\right\}+\theta_0=-\varepsilon^2(4\varphi+y\varphi')\theta_0$$

and so θ_0 exactly satisfies a mildly perturbed original equation. Note that the above equation holds for φ replaced by $\varphi+\varepsilon^{2m}\psi$. This is very elegant yet at the same time, Eckhaus discards it since difficulties arise in trying to use it in the construction of a contraction mapping. We will now define explicitly the operator Ψ_1.

$$\Psi_1 = \varepsilon^2 y^2 \varphi \frac{d^2}{dy^2} + \varepsilon^2 \left[5y\varphi + \frac{1}{2}y^2\varphi'\right] \frac{d}{dy} + \left[1 + \varepsilon^2(4\varphi + y\varphi')\right] \cdot 1 \tag{4.11}$$

Then

$$\theta_0^{(1)}, \theta_0^{(2)} = \frac{1}{y^2} \exp\left\{\pm\frac{i}{\varepsilon}\int^y \frac{dt}{t\sqrt{\varphi}}\right\}$$

gives two linearly independent solutions to the homogeneous equation $\Psi_1\theta_0 = 0$.

Eckhaus remarks that the simple approach of adding in an additional perturbation term to allow us to use the WKB approximation as an exact solution is not found in any of the literature.

4.3 Transformation to Integral Equation

Let us consider an inhomogeneous problem

$$\Psi_1(\widehat{\psi}) = R \tag{4.12}$$

where Ψ_1 is the operator defined in the previous section. We would like to find bounded solutions on some non-empty interval $y \in [0, y_0]$ where $y_0 > 0$. So we suppose a solution for 4.12 of the form

$$\widehat{\psi} = u_1\theta_0^{(1)} + u_2\theta_0^{(2)}$$

then using variation of parameters we have

$$u_1' = -\frac{\theta_0^{(2)}}{\theta_0^{(1)}\left(\theta_0^{(2)}\right)' - \theta_0^{(2)}\left(\theta_0^{(1)}\right)'}R(y)$$

$$u_2' = -\frac{\theta_0^{(1)}}{\theta_0^{(1)}\left(\theta_0^{(2)}\right)' - \theta_0^{(2)}\left(\theta_0^{(1)}\right)'}R(y)$$

and

$$\theta_0^{(1)}\left(\theta_0^{(2)}\right)' - \theta_0^{(2)}\left(\theta_0^{(1)}\right)'$$

$$= \theta_0^{(1)}\left[\frac{1}{y^2}\exp\left\{-\frac{i}{\varepsilon}\int^y \frac{dt}{y\sqrt{\varphi}}\right\}\frac{-1}{\varepsilon}\left(\frac{-1}{y^2\varphi}\right)^{\frac{1}{2}} + \frac{-2}{y^3}\exp\left\{-\frac{i}{\varepsilon}\int^y \frac{dt}{y\sqrt{\varphi}}\right\}\right]$$

$$-\theta_0^{(2)}\left[\frac{1}{y^2}\exp\left\{\frac{i}{\varepsilon}\int^y \frac{dt}{y\sqrt{\varphi}}\right\}\frac{-1}{\varepsilon}\left(\frac{-1}{y^2\varphi}\right)^{\frac{1}{2}} + \frac{-2}{y^3}\exp\left\{\frac{i}{\varepsilon}\int^y \frac{dt}{y\sqrt{\varphi}}\right\}\right]$$

$$= \theta_0^{(1)}\left(\theta_0^{(2)}\frac{1}{\varepsilon}(-i)\frac{1}{y\sqrt{\varphi}} + \theta_0^{(2)}\frac{-2}{y}\right) - \theta_0^{(2)}\left(\theta_0^{(1)}\frac{1}{\varepsilon}i\frac{1}{y\sqrt{\varphi}} + \theta_0^{(1)}\frac{-2}{y}\right)$$

$$= \frac{-2i}{\varepsilon}\theta_0^{(1)}\theta_0^{(2)}\left(\frac{1}{y\sqrt{\varphi}}\right)$$

So

$$u_1' = \frac{R\varepsilon}{2i\theta_0^{(1)}y\sqrt{\varphi}}$$

$$u_2' = \frac{-R\varepsilon}{2i\theta_0^{(2)}y\sqrt{\varphi}}$$

Thus

$$u_1 = \frac{\varepsilon}{2i}\int_0^y \frac{R(t)dt}{\theta_0^{(1)}t\sqrt{\varphi(t)}}$$

$$u_2 = \frac{-\varepsilon}{2i}\int_0^y \frac{R(t)dt}{\theta_0^{(2)}t\sqrt{\varphi(t)}}$$

Now we can get

$$\widehat{\psi} = \left(\frac{\varepsilon}{2i}\int_0^y \frac{R(t)}{\theta_0^{(1)}(t)t\sqrt{\varphi(t)}}dt\right)\theta_0^{(1)}(y)$$

$$+\left(\frac{-\varepsilon}{2i}\int_0^y \frac{R(t)}{\theta_0^{(2)}(t)t\sqrt{\varphi(t)}}dt\right)\theta_0^{(2)}(y)$$

$$= \frac{\varepsilon}{2i}\int_0^y \left(\frac{\theta_0^{(1)}(y)}{\theta_0^{(1)}(t)} - \frac{\theta_0^{(2)}(y)}{\theta_0^{(2)}(t)}\right)\frac{R(t)}{t\sqrt{\varphi(t)}}dt$$

$$= \frac{\varepsilon}{2i}\int_0^y \frac{\theta_0^{(1)}(y)\theta_0^{(2)}(t) - \theta_0^{(2)}(y)\theta_0^{(1)}(t)}{\theta_0^{(1)}(t)\theta_0^{(2)}(t)}\frac{R(t)}{t\sqrt{\varphi(t)}}dt$$

and

$$\theta_0^{(1)}(t)\theta_0^{(2)}(t) = \frac{1}{t^4}$$

$$\widehat{\psi} = \frac{\varepsilon}{2i}\int_0^y \left\{\theta_0^{(1)}(y)\theta_0^{(2)}(t) - \theta_0^{(2)}(y)\theta_0^{(1)}(t)\right\} t^3\frac{R(t)}{\sqrt{\varphi(t)}}dt$$

However, substituting back into 4.12 leaves us with $\Psi_1(\widehat{\psi}) = \varepsilon^2 R$ and so we must adjust $\widehat{\psi}$ by a factor of $\frac{1}{\varepsilon^2}$. So now we define

$$\Omega(t,y) = \int_y^t \frac{d\xi}{\xi\sqrt{\varphi(\xi)}}$$

and

$$\theta_0^{(1)}(y)\theta_0^{(2)}(t) = \frac{1}{y^2}\exp\left\{\frac{i}{\varepsilon}\int_0^y \frac{dt}{t\sqrt{\varphi}}\right\}\frac{1}{t^2}\exp\left\{-\frac{i}{\varepsilon}\int_0^t \frac{dt}{t\sqrt{\varphi}}\right\}$$

$$= \frac{1}{y^2t^2}\exp\left\{\frac{-i}{\varepsilon}\int_y^t \frac{dt}{t\sqrt{\varphi}}\right\}$$

$$= \frac{1}{y^2t^2}\exp\left\{\frac{-i}{\varepsilon}\Omega\right\}$$

and by the same reasoning

$$\theta_0^{(2)}(y)\theta_0^{(1)}(t) = \frac{1}{y^2t^2}\exp\left\{\frac{i}{\varepsilon}\Omega\right\}$$

Now putting it all together yields

$$\widehat{\psi}(y) = \frac{1}{\varepsilon^2}\frac{\varepsilon}{2i}\int_0^y \frac{1}{y^2}\left(e^{-\frac{1}{\varepsilon}\Omega} - e^{\frac{1}{\varepsilon}\Omega}\right) t\frac{1}{\sqrt{\varphi(t)}}R(t)dt$$

and so

$$\widehat{\psi}(y) = \frac{-1}{\varepsilon y^2}\int_0^y \frac{e^{-\frac{1}{\varepsilon}\Omega} - e^{\frac{1}{\varepsilon}\Omega}}{2i} t\frac{1}{\sqrt{\varphi(t)}}R(t)dt$$

$$= \frac{-1}{\varepsilon y^2}\int_0^y \sin\left[\frac{1}{\varepsilon}\Omega\right] t\frac{1}{\sqrt{\varphi(t)}}R(t)dt$$

Now we show that $\widehat{\psi}(y)$ is bounded as $y \to 0$. First we note that since$|\sin x| \leq 1$ for all x hence

$$\left|\widehat{\psi}(y)\right| \leq \frac{1}{\varepsilon}\frac{1}{y^2}\left(\int_0^y \frac{t}{\sqrt{\varphi(t)}}dt\right) \sup_{y\in[0,y_0]} |R(y)|$$

This estimate is valid on all intervals $y \in [0, y_0]$ with $\varphi(y) \geq 0$. Of course $R(y)$ must be such that $\sup |R(y)|$ exists on the interval $[0, y_0]$. Now

$$\frac{1}{y^2}\int_0^y \frac{t}{\sqrt{\varphi(t)}}dt \leq c \text{ for some constant } c$$

and so

$$\left|\widehat{\psi}(y)\right| \leq \frac{c}{\varepsilon}\sup |R|$$

We now suppose that R is differentiable, and we will use the fact that

$$\frac{d}{dt}\cos\left[\frac{1}{\varepsilon}\Omega(t,y)\right] = -\sin\left[\frac{1}{\varepsilon}\Omega(t,y)\right]\frac{1}{\varepsilon}\frac{d}{dt}\Omega(t,y)$$

$$= -\frac{1}{\varepsilon}\sin\left[\frac{1}{\varepsilon}\Omega(t,y)\right]\frac{1}{t\sqrt{\varphi(t)}}$$

to show that

$$\widehat{\psi}(y) = \frac{1}{y^2}\int_0^y\left(\frac{d}{dt}\cos\left[\frac{1}{\varepsilon}\Omega(t,y)\right]\right)t^2R(t)dt$$

We now integrate by parts to get

$$\widehat{\psi}(y) = \frac{1}{y^2}\left(t^2R(t)\cos\left[\frac{1}{\varepsilon}\Omega(t,y)\right]_0^y - \int_0^y\cos\left[\frac{1}{\varepsilon}\Omega(t,y)\right]\frac{d}{dt}t^2R(t)dt\right)$$

$$= R(y) - \frac{1}{y^2}\int_0^y\cos\left[\frac{1}{\varepsilon}\Omega(t,y)\right]\frac{d}{dt}\left(t^2R(t)\right)dt$$

Now let $\varepsilon\downarrow 0$

$$\left|\frac{1}{y^2}\int_0^y\cos\left[\frac{1}{\varepsilon}\Omega(t,y)\right]\frac{d}{dt}\left(t^2R(t)\right)dt\right| \leq \frac{1}{y^2}y^2R(y) = R(y)$$

and so $\widehat{\psi}(y)$ is bounded as $\varepsilon\downarrow 0$. Thus $\widehat{\psi}(y)$ is $O(1)$. Now suppose $R(t)$ is twice differentiable so that we may integrate

$$-\frac{1}{y^2}\int_0^y\cos\left[\frac{1}{\varepsilon}\Omega(t,y)\right]\frac{d}{dt}\left(t^2R(t)\right)dt$$

by parts. First notice that

$$\frac{d}{dt}\sin\left[\frac{1}{\varepsilon}\Omega(t,y)\right] = \frac{1}{\varepsilon}\cos\left[\frac{1}{\varepsilon}\Omega(t,y)\right]\frac{1}{t\sqrt{\varphi(t)}}$$

and so

$$\cos\left[\frac{1}{\varepsilon}\Omega(t,y)\right] = \varepsilon t\sqrt{\varphi(t)}\frac{d}{dt}\sin\left[\frac{1}{\varepsilon}\Omega(t,y)\right]$$

Thus

$$-\frac{1}{y^2}\int_0^y \cos\left[\frac{1}{\varepsilon}\Omega(t,y)\right]\frac{d}{dt}\left(t^2R(t)\right)dt$$

becomes

$$-\frac{\varepsilon}{y^2}\int_0^y t\sqrt{\varphi(t)}\frac{d}{dt}\sin\left[\frac{1}{\varepsilon}\Omega(t,y)\right]\frac{d}{dt}\left(t^2R(t)\right)dt$$

so now doing the integration will give

$$-\frac{\varepsilon}{y^2}\left(t\sqrt{\varphi(t)}\frac{d}{dt}\left(t^2R(t)\right)\sin\left[\frac{1}{\varepsilon}\Omega(t,y)\right]_0^y\right.$$

$$\left.-\int_0^y \sin\left[\frac{1}{\varepsilon}\Omega(t,y)\right]\frac{d}{dt}\left(t\sqrt{\varphi(t)}\frac{d}{dt}(t^2R(t))\right)dt\right)$$

$$=\frac{\varepsilon}{y^2}\int_0^y \sin\left[\frac{1}{\varepsilon}\Omega(t,y)\right]\frac{d}{dt}\left[t\sqrt{\varphi}\frac{d}{dt}\left(t^2R\right)\right]dt$$

and so

$$\widehat{\psi}(y) = R(y) + \frac{\varepsilon}{y^2}\int_0^y \sin\left[\frac{1}{\varepsilon}\Omega(t,y)\right]\frac{d}{dt}\left[t\sqrt{\varphi}\frac{d}{dt}\left(t^2R\right)\right]dt$$

which then tells us that $\widehat{\psi}(y) = R(y) + O(\varepsilon)$. Now we turn our attention to the problem for ψ and ψ'. In this section of the analysis, it will be convenient to define

$$S_1[R](y) = -\frac{1}{y^2}\int_0^y \sin\left[\frac{1}{\varepsilon}\Omega(t,y)\right]\frac{t}{\sqrt{\varphi(t)}}R(t)dt$$

$$S_2[R](y) = -\frac{1}{y^2}\int_0^y \cos\left[\frac{1}{\varepsilon}\Omega(t,y)\right]\frac{t}{\sqrt{\varphi(t)}}R(t)dt$$

so that on intervals $y \in [0, y_0]$ with $\varphi(y) \geq 0$ we have

$$|S_{1,2}[R]| \leq c \sup |R|$$

and if $R(t)$ is twice differentiable then

$$\frac{1}{\varepsilon}S_1[R](y) = R(y) + O(\varepsilon)$$

4.3.1 The problem for ψ.

We start with 4.1 to 4.5 and 4.11 so that we get

$$\begin{aligned}
\Psi_1(\psi) = \varepsilon^2 y^2 \varphi \frac{d^2\psi}{dy^2} + \varepsilon^2\left[5y\varphi + \frac{1}{2}y^2\varphi'\right]\frac{d\psi}{dy} + \left[1 + \varepsilon^2(4\varphi + y\varphi')\right]\psi \\
= \varepsilon^2 y^2(\varphi + \varepsilon^{2m}\psi)\frac{d^2\psi}{dy^2} - \varepsilon^2 y^2 \varepsilon^{2m}\psi\frac{d^2\psi}{dy^2} \\
+\varepsilon^2\left(\frac{-1}{2}y^2(\varphi' + \frac{1}{2}\varepsilon^{2m}\psi') + 3y(\varphi + \varepsilon^{2m}\psi)\right)\frac{d\psi}{dy} \\
+\varepsilon^2\left[2y\varphi + y^2\varphi' + \frac{1}{4}y^2\varepsilon^{2m}\psi' - 3y\varepsilon^{2m}\psi\right]\frac{d\psi}{dy} + \psi + \varepsilon^2(4\varphi + y\varphi')\psi
\end{aligned}$$

$$= \Psi(\psi) - \varepsilon^2 y^2 \varphi \frac{d^2\psi}{dy^2} + \varepsilon^2 \left[2y\varphi + y^2\varphi' + \frac{1}{4}y^2\varepsilon^{2m}\psi' - 3y\varepsilon^{2m}\psi\right] \frac{d\psi}{dy}$$

$$+\varepsilon^2(4\varphi + y\varphi')\psi$$

$$= \overline{f}(y) + \varepsilon^2 g_1(y,\psi) + \varepsilon^2(2y\varphi + y^2\psi)\psi' + \varepsilon^2\left(\frac{1}{4}y^2\varepsilon^{2m}\psi' - 3y\varepsilon^{2m}\psi\right)\psi'$$

$$+\varepsilon^2(4\varphi + y\varphi')\psi - \varepsilon^{2(m+1)}y^2\psi\frac{d^2\psi}{dy^2}$$

$$= \overline{f}(y) + \varepsilon^2\left[-y^2\varphi'' - 3y\varphi' - (2\varphi + \varepsilon^{2m}\psi)\right]\psi + \varepsilon^2(2y\varphi + y^2\varphi')\psi'$$

$$+\varepsilon^{2(m+1)}\left(\frac{1}{4}y^2\psi' - 3y\psi\right)\psi' + \varepsilon^2\left(4\varphi + y\varphi'\right)\psi - \varepsilon^{2(m+1)}y^2\psi\frac{d^2\psi}{dy^2}$$

$$= \overline{f}(y) + \varepsilon^2\left[-y^2\varphi'' - 2y\varphi' + 2\varphi - \varepsilon^{2m}\psi\right]\psi + \varepsilon^2\left[2y\varphi + y^2\varphi'\right]\psi'$$

$$+\left[\varepsilon^{2(m+1)}\frac{1}{4}y^2\psi' - 3y\varepsilon^{2(m+1)}\psi\right]\psi' - \varepsilon^{2(m+1)}y^2\psi\frac{d^2\psi}{dy^2}$$

$$= \overline{f}(y) + \varepsilon^2\left[-y^2\varphi'' - 2y\varphi' + 2\varphi\right]\psi - (\varepsilon^{2m}\psi)\psi + \varepsilon^2\left[2y\varphi + y^2\varphi'\right]\psi'$$

$$+\varepsilon^{2m}\left[\frac{1}{4}y^2\psi' - 3y\psi\right]\psi' - \varepsilon^{2(m+1)}y^2\psi\frac{d^2\psi}{dy^2}$$

$$= \overline{f}(y) + \varepsilon^2 G(y,\psi,\psi') - \varepsilon^{2(m+1)}y^2\psi\frac{d^2\psi}{dy^2}$$

where $G(y,\psi,\psi')$ is of the form

$$\beta_0(y)\psi + \gamma_0(y)\psi' + \varepsilon^{2m}\left[\beta_1(y)\psi + \gamma_1(y)\psi'\right]\psi' - \left(\varepsilon^{2m}\psi\right)\psi$$

The β's and the γ's are polynomials in y. In the following computations, it will be convenient to use G to denote anything of the above form. We have

$$\Psi_1(\psi) = \overline{f}(y) + \varepsilon^2 G(y, \psi, \psi') - \varepsilon^{2(m+1)} y^2 \psi \frac{d^2\psi}{dy^2}$$

and so

$$\psi = \frac{1}{\varepsilon} S_1 \left[\Psi_1(\psi)\right](y)$$

$$= \frac{1}{\varepsilon} S_1 \left[\overline{f}(y) + \varepsilon^2 G(y, \psi, \psi') - \varepsilon^{2(m+1)} y^2 \psi \frac{d^2\psi}{dy^2}\right](y)$$

and

$$S_1 \left[\overline{f}(y) + \varepsilon^2 G(y, \psi, \psi') - \varepsilon^{2(m+1)} y^2 \psi \frac{d^2\psi}{dy^2}\right](y)$$

$$= -\frac{1}{y^2} \int_0^y \sin\left[\frac{1}{\varepsilon}\Omega(t, y)\right] \frac{t}{\sqrt{\varphi(t)}} \left[\overline{f}(t) + \varepsilon^2 G(t, \psi, \psi') - \varepsilon^{2(m+1)} t^2 \psi \frac{d^2\psi}{dt^2}\right] dt$$

$$= -\frac{1}{y^2} \int_0^y \sin\left[\frac{1}{\varepsilon}\Omega(t, y)\right] \frac{t}{\sqrt{\varphi(t)}} \overline{f}(t) dt$$

$$+\varepsilon^2 \left(\frac{-1}{y^2}\right) \int_0^y \sin\left[\frac{1}{\varepsilon}\Omega(t, y)\right] \frac{t}{\sqrt{\varphi(t)}} G(t, \psi, \psi') dt$$

$$+\varepsilon^{2(m+1)} \frac{1}{y^2} \int_0^y \sin\left[\frac{1}{\varepsilon}\Omega(t, y)\right] \frac{t^3}{\sqrt{\varphi(t)}} \psi \frac{d^2\psi}{dy^2} dt$$

$$= S_1\left[\overline{f}(t)\right] + \varepsilon^2 S_1\left[G(t, \psi, \psi')\right] + \varepsilon^{2(m+1)} \frac{1}{y^2} \int_0^y \sin\left[\frac{1}{\varepsilon}\Omega\right] \frac{t^3}{\sqrt{\varphi(t)}} \psi \frac{d^2\psi}{dt^2} dt$$

So

$$\psi = \frac{1}{\varepsilon} S_1\left[\overline{f}(t)\right] + \varepsilon S_1\left[G(t, \psi, \psi')\right] + \varepsilon^{2m+1} \frac{1}{y^2} \int_0^y \sin\left[\frac{1}{\varepsilon}\Omega\right] \frac{t^3}{\sqrt{\varphi(t)}} \psi \frac{d^2\psi}{dt^2} dt$$

and by integrating the last term by parts we get

$$\frac{\varepsilon^{2m+1}}{y^2}\left[\left[\sin[\frac{1}{\varepsilon}\Omega]\frac{t^3}{\sqrt{\varphi(t)}}\psi\frac{d\psi}{dt}\right]_0^y - \int_0^y \frac{d\psi}{dt}\left(\cos[\frac{1}{\varepsilon}\Omega]\frac{1}{\varepsilon}\frac{t^2}{\varphi}\psi\right.\right.$$

$$\left.\left.+\sin[\frac{1}{\varepsilon}\Omega]\frac{d}{dt}\left[\frac{t^3\psi}{\sqrt{\varphi}}\right]\right)dt\right]$$

for the last term and simplifying yields

$$-\frac{\varepsilon^{2m+1}}{y^2}\int_0^y \sin[\frac{1}{\varepsilon}\Omega(t,y)]\frac{d}{dt}\left[\frac{t^3\psi}{\sqrt{\varphi}}\right]\frac{d\psi}{dt}dt - \frac{\varepsilon^{2m}}{y^2}\int_0^y \cos[\frac{1}{\varepsilon}\Omega(t,y)]\frac{t^2}{\varphi}\psi\frac{d\psi}{dt}dt$$

and we conclude

$$\psi = \frac{1}{\varepsilon}S_1[\overline{f}] + \varepsilon S_1[G] - -\frac{\varepsilon^{2m+1}}{y^2}\int_0^y \sin[\frac{1}{\varepsilon}\Omega(t,y)]\frac{d}{dt}\left[\frac{t^3\psi}{\sqrt{\varphi}}\right]\frac{d\psi}{dt}dt$$

$$-\frac{\varepsilon^{2m}}{y^2}\int_0^y \cos[\frac{1}{\varepsilon}\Omega(t,y)]\frac{t^2}{\varphi}\psi\frac{d\psi}{dt}dt$$

Notice that

$$S_2\left[\frac{t}{\sqrt{\varphi}}\psi\psi'\right] = -\frac{1}{y^2}\int_0^y \cos[\frac{1}{\varepsilon}\Omega]\frac{t}{\sqrt{\varphi}}\frac{t}{\sqrt{\varphi}}\psi\psi' dt$$

$$= -\frac{1}{y^2}\int_0^y \cos[\frac{1}{\varepsilon}\Omega]\frac{t^2}{\varphi}\psi\psi' dt$$

and

$$S_1\left[\frac{t^2}{\varphi}\varphi'\psi\psi'\right] = -\frac{1}{y^2}\int_0^y \sin[\frac{1}{\varepsilon}\Omega]\frac{t}{\sqrt{\varphi}}\frac{t^2}{\varphi}\varphi'\psi\psi' dt$$

$$= -\frac{1}{y^2}\int_0^y \sin[\frac{1}{\varepsilon}\Omega]\frac{t^3}{\varphi\sqrt{\varphi}}\varphi'\psi\psi' dt$$

We also have that

$$\frac{d}{dt}\left[\frac{t^3}{\sqrt{\varphi}}\psi\right] = \frac{d}{dt}\left(\frac{t^3}{\sqrt{\varphi}}\right)\psi + \frac{t^3}{\sqrt{\varphi}}\frac{d\psi}{dt}$$

$$= \frac{\sqrt{\varphi}3t^2 - t^3\frac{1}{2}\varphi^{-\frac{1}{2}}\varphi'}{\varphi}\psi + \frac{t^3}{\sqrt{\varphi}}\frac{d\psi}{dt}$$

$$= \left(\frac{3t^2}{\sqrt{\varphi}} - \frac{t^3\varphi'}{2\varphi\sqrt{\varphi}}\right)\psi + \frac{t^3}{\sqrt{\varphi}}\frac{d\psi}{dt}$$

So the result for ψ is now

$$\psi = \frac{1}{\varepsilon}S_1[\overline{f}] + \varepsilon S_1[G] - \frac{\varepsilon^{2m+1}}{y^2}\int_0^y \sin[\frac{1}{\varepsilon}\Omega]\frac{3t^2}{\sqrt{\varphi}}\psi\psi' dt$$

$$+\frac{\varepsilon^{2m+1}}{y^2}\int_0^y \sin[\frac{1}{\varepsilon}\Omega]\frac{t^3\varphi'}{2\varphi\sqrt{\varphi}}\psi\psi' dt - \frac{\varepsilon^{2m+1}}{y^2}\int_0^y \sin[\frac{1}{\varepsilon}\Omega]\frac{t^3}{\sqrt{\varphi}}\psi'\psi' dt$$

$$+\varepsilon^{2m}S_2[\frac{t}{\sqrt{\varphi}}\psi\psi']$$

which means that

$$\psi = \frac{1}{\varepsilon}S_1[\overline{f}] + \varepsilon S_1[G] - \frac{\varepsilon^{2m+1}}{2}S_1\left[\frac{t^2}{\varphi}\varphi'\psi\psi'\right] + \varepsilon^{2m}S_2\left[\frac{t}{\sqrt{\varphi}}\psi\psi'\right]$$

$$-\frac{\varepsilon^{2m+1}}{y}\int_0^y \sin[\frac{1}{\varepsilon}\Omega]\frac{3t^2}{\sqrt{\varphi}}\psi\psi' dt - \frac{\varepsilon^{2m+1}}{y}\int_0^y \sin[\frac{1}{\varepsilon}\Omega]\frac{t^3}{\sqrt{\varphi}}\psi'\psi' dt$$

$$= \frac{1}{\varepsilon}S_1[\overline{f}] + \varepsilon S_1[G] - \frac{\varepsilon^{2m+1}}{2}S_1\left[\frac{t^2}{\varphi}\varphi'\psi\psi'\right] + \varepsilon^{2m}S_2\left[\frac{t}{\sqrt{\varphi}}\psi\psi'\right]$$

$$-\frac{\varepsilon^{2m+1}}{y}\int_0^y \sin[\frac{1}{\varepsilon}\Omega]\frac{t}{\sqrt{\varphi}}\left[3t\psi\psi' + t^2\psi'\psi'\right] dt$$

$$= \frac{1}{\varepsilon}S_1[\overline{f}] + \varepsilon S_1[G] - \frac{\varepsilon^{2m+1}}{2}S_1\left[\frac{t^2}{\varphi}\varphi'\psi\psi'\right] + \varepsilon^{2m}S_2\left[\frac{t}{\sqrt{\varphi}}\psi\psi'\right]$$

since the term $-\frac{\varepsilon^{2m+1}}{y}\int_0^y \sin[\frac{1}{\varepsilon}\Omega]\frac{t}{\sqrt{\varphi}}[3t\psi\psi' + t^2\psi'\psi']\,dt$ is of the form $S_1[G]$, of course with different polynomials $\beta_1(y)$ and $\gamma_1(y)$ then in the original G. Thus we have that

$$\psi = \psi_0 + \varepsilon S_1[G(t,\psi,\psi')] + \varepsilon^{2m} S_2[\frac{t}{\sqrt{\varphi}}\psi\psi'] - \frac{1}{2}\varepsilon^{2m+1} S_1[\frac{t^2}{\varphi}\varphi'\psi\psi'] \qquad (4.13)$$

where

$$\psi_0 = \frac{1}{\varepsilon}S_1[\overline{f}]$$

Since $\overline{f}$ is twice differentiable, we can use previous results to state that

$$\frac{1}{\varepsilon}S_1[\overline{f}] = \overline{f} + O(\varepsilon)$$

We need to keep the terms separate so that we may pursue the analysis as far as possible into values of y where $\varphi(y)$ become very small.

4.3.2 The problem for ψ'

We now turn our attention to equations 4.6 to 4.9. We want to remove the nonlinearity in the term with the second derivative of ψ'. We do this by multiplying the equations by the factor

$$\frac{\varphi}{\varphi + \varepsilon^{2m}\psi}$$

and using the identity

$$\frac{\varphi}{\varphi + \varepsilon^{2m}\psi} = 1 - \frac{\varepsilon^{2m}\psi}{\varphi + \varepsilon^{2m}\psi}$$

So

$$\frac{\varphi}{\varphi+\varepsilon^{2m}\psi}\left[\varepsilon^2 y^2(\varphi+\varepsilon^{2m}\psi)\frac{d^2\psi'}{dy^2}+\varepsilon^2\left(5y(\varphi+\varepsilon^{2m}\psi)+\frac{1}{2}y^2(\varphi'+\varepsilon^{2m}\psi')\right)\frac{d\psi'}{dy}+\psi'\right]$$

$$=\frac{\varphi}{\varphi+\varepsilon^{2m}\psi}\left[\overline{f}'(y)+\varepsilon^2 g_2(y,\psi,\psi')\right]$$

$$\Rightarrow \varepsilon^2 y^2\varphi\frac{d^2\psi'}{dy^2}+\left[\varepsilon^2 5y\varphi+\frac{\varepsilon^2}{2}y^2\varphi\frac{\varphi'+\varepsilon^{2m}\psi'}{\varphi+\varepsilon^{2m}\psi}\right]\frac{d\psi'}{dy}+\frac{\psi'\varphi}{\varphi+\varepsilon^{2m}\psi}$$

$$=\left(1-\frac{\varepsilon^{2m}\psi}{\varphi+\varepsilon^{2m}\psi}\right)(\overline{f}'(y)+\varepsilon^2 g_2)$$

which yields

$$\varepsilon^2 y^2\varphi\frac{d^2\psi'}{dy^2}+\varepsilon^2(5y\varphi+\frac{1}{2}y^2\varphi')\frac{d\psi'}{dy}+\psi'=\left(1-\frac{\varepsilon^{2m}\psi}{\varphi+\varepsilon^{2m}\psi}\right)(\overline{f}'(y)+\varepsilon^2 g_2)$$

$$+\left[\varepsilon^2\frac{y^2}{2}\varphi'\left(\frac{\varepsilon^{2m}\psi}{(\varphi+\varepsilon^{2m}\psi)}\right)-\varepsilon^2\frac{y^2}{2}\varepsilon^{2m}\psi'+\frac{\varepsilon^2 y^2}{2}\frac{\varepsilon^{4m}\psi'\psi}{(\varphi+\varepsilon^{2m}\psi)}\right]\frac{d\psi'}{dy}+\frac{\varepsilon^{2m}\psi}{(\varphi+\varepsilon^{2m}\psi)}\psi'$$

and simplifying results in

$$\varepsilon^2 y^2\varphi\frac{d^2\psi'}{dy^2}+\varepsilon^2(5y\varphi+\frac{1}{2}y^2\varphi')\frac{d\psi'}{dy}+\psi'=\left(1-\frac{\varepsilon^{2m}\psi}{\varphi+\varepsilon^{2m}\psi}\right)(\overline{f}'(y)+\varepsilon^2 g_2)$$

$$+\varepsilon^{2m+2}\frac{1}{2}y^2(\varphi+\varepsilon^{2m}\psi)(\psi\varphi'-\psi'\varphi)\frac{d\psi'}{dy}+\frac{\varepsilon^{2m}\psi}{(\varphi+\varepsilon^{2m}\psi)}\psi'$$

So now we introduce Ψ_1 as defined by 4.11 and use the generic notation for G as done earlier. This then produces

$$\Psi_1(\psi')=-\varepsilon^2(4\varphi+y\varphi')\psi'+\left(1-\frac{\varepsilon^{2m}\psi}{\varphi+\varepsilon^{2m}\psi}\right)(\overline{f}'(y)+\varepsilon^2 g_2)$$

$$+\varepsilon^{2m+2}\frac{1}{2}y^2(\varphi+\varepsilon^{2m}\psi)(\psi\varphi'-\psi'\varphi)\frac{d\psi'}{dy}+\frac{\varepsilon^{2m}\psi}{\varphi+\varepsilon^{2m}\psi}\psi'$$

$$=-4\varepsilon^2\varphi\psi'-\varepsilon^2 y\varphi'\psi'+\overline{f}'(y)+\varepsilon^2 g_2-\frac{\varepsilon^{2m}\psi}{(\varphi+\varepsilon^{2m}\psi)}\overline{f}'(y)-\frac{\varepsilon^{2m+2}\psi}{(\varphi+\varepsilon^{2m}\psi)}g_2$$

$$+\varepsilon^{2m+2}\frac{1}{2}y^2(\varphi+\varepsilon^{2m}\psi)^{-1}(\psi\varphi'-\psi'\varphi)\frac{d\psi'}{dy}+\frac{\varepsilon^{2m}\psi}{\varphi+\varepsilon^{2m}\psi}\psi'$$

$$=\overline{f}'(y)+\varepsilon^2\left[-4\varphi\psi'-y\varphi'\psi'+g_2\right]+\varepsilon^{2m}(\varphi+\varepsilon^{2m}\psi)\psi\left[-\overline{f}'(y)-\varepsilon^2 g_2+\psi'\right]$$

$$+\varepsilon^{2m+2}\frac{1}{2}y^2(\varphi+\varepsilon^{2m}\psi)^{-1}\left[\varphi'\psi\frac{d\psi'}{dy}-\frac{1}{2}\varphi\frac{d}{dy}(\psi')^2\right]$$

$$=\overline{f}'(y)+\varepsilon^2 G_1(y,\psi,\psi')+\varepsilon^{2m}(\varphi+\varepsilon^{2m}\psi)(\psi)\left(G_2(y,\psi,\psi')-\varepsilon^2 g_2\right)$$

$$+\,\varepsilon^{2m+2}(\varphi+\varepsilon^{2m}\psi)^{-1}\frac{1}{4}y^2\left[2\varphi'\psi\frac{d\psi'}{dy}-\varphi\frac{d}{dy}(\psi')^2\right] \tag{4.14}$$

Notice that our definition for G does not allow us to write $G_2(y,\psi,\psi')-\varepsilon^2 g_2=G_2$. We will solve this problem in the following section, for now we may just write out anything that does not fit explicitly. Inverting equation 4.14 we get

$$\psi'=-\frac{1}{\varepsilon}\frac{1}{y^2}\int_0^y \sin[\frac{1}{\varepsilon}\Omega]\frac{t}{\sqrt{\varphi}}\left[\overline{f}'(y)+\varepsilon^2 G_1(y,\psi,\psi')\right.$$

$$+\varepsilon^{2m}(\varphi+\varepsilon^{2m}\psi)(\psi)\left(G_2(y,\psi,\psi')-\varepsilon^2 g_2\right)$$

$$\left.+\varepsilon^{2m+2}(\varphi+\varepsilon^{2m}\psi)^{-1}\frac{1}{4}y^2\left[2\varphi'\psi\frac{d\psi'}{dy}-\varphi\frac{d}{dy}(\psi')^2\right]\right]$$

Consider the term

$$-\frac{1}{\varepsilon}\frac{1}{y^2}\int_0^y \sin[\frac{1}{\varepsilon}\Omega]\frac{t}{\sqrt{\varphi}}\varepsilon^{2m+2}(\varphi+\varepsilon^{2m}\psi)^{-1}\frac{1}{4}t^2\left[2\varphi'\psi\frac{d\psi'}{dt}-\varphi\frac{d}{dt}(\psi')^2\right]dt$$

$$= -\varepsilon^{2m+1}\frac{1}{y^2}\frac{1}{4}\int_0^y \sin[\frac{1}{\varepsilon}\Omega](\varphi+\varepsilon^{2m}\psi)^{-1}\frac{t^3}{\sqrt{\varphi}}2\varphi'\psi\frac{d\psi'}{dt}dt$$

$$+\,\varepsilon^{2m+1}\frac{1}{y^2}\frac{1}{4}\int_0^y \sin[\frac{1}{\varepsilon}\Omega](\varphi+\varepsilon^{2m}\psi)^{-1}\frac{t^3}{\sqrt{\varphi}}\varphi\frac{d}{dt}(\psi')^2dt \tag{4.15}$$

Tackle the first integral by parts

$$-\varepsilon^{2m+1}\frac{1}{y^2}\frac{1}{4}\int_0^y \sin[\frac{1}{\varepsilon}\Omega](\varphi+\varepsilon^{2m}\psi)^{-1}\frac{t^3}{\sqrt{\varphi}}2\varphi'\psi\frac{d\psi'}{dt}dt$$

$$= \varepsilon^{2m+1}\frac{1}{y^2}\frac{1}{4}\int_0^y \psi'\sin[\frac{1}{\varepsilon}\Omega]\frac{1}{\varphi+\varepsilon^{2m}\psi}2\frac{t^3}{\sqrt{\varphi}}(\varphi'\psi'+\varphi''\psi)dt$$

$$+\varepsilon^{2m+1}\frac{1}{y^2}\frac{1}{4}\int_0^y \psi'\sin[\frac{1}{\varepsilon}\Omega]\frac{1}{\varphi+\varepsilon^{2m}\psi}\frac{6t^2\varphi-t^3\varphi'}{\varphi\sqrt{\varphi}}\varphi'\psi dt$$

$$-\varepsilon^{2m+1}\frac{1}{y^2}\frac{1}{4}\int_0^y \psi'\sin[\frac{1}{\varepsilon}\Omega]\frac{\varphi'+\varepsilon^{2m}\psi'}{(\varphi+\varepsilon^{2m}\psi)^2}\frac{t^3}{\sqrt{\varphi}}2\varphi'\psi dt$$

$$+\varepsilon^{2m}\frac{1}{y^2}\frac{1}{4}\int_0^y \psi'\cos[\frac{1}{\varepsilon}\Omega]\frac{1}{\varphi+\varepsilon^{2m}\psi}\frac{t^2}{\varphi}2\varphi'\psi dt$$

Now we do the second integral in 4.15 by parts to get

$$+\varepsilon^{2m+1}\frac{1}{y^2}\frac{1}{4}\int_0^y \sin[\frac{1}{\varepsilon}\Omega](\varphi+\varepsilon^{2m}\psi)^{-1}\frac{t^3}{\sqrt{\varphi}}\varphi\frac{d}{dt}(\psi')^2dt$$

$$= -\varepsilon^{2m+1}\frac{1}{y^2}\frac{1}{4}\int_0^y (\psi')^2\sin[\frac{1}{\varepsilon}\Omega]\frac{1}{(\varphi+\varepsilon^{2m}\psi)}\frac{t^3}{\sqrt{\varphi}}\varphi' dt$$

$$- -\,\varepsilon^{2m+1}\frac{1}{y^2}\frac{1}{4}\int_0^y (\psi')^2\sin[\frac{1}{\varepsilon}\Omega]\frac{1}{(\varphi+\varepsilon^{2m}\psi)}\left(3t^2\sqrt{\varphi}-\frac{t^3\varphi'}{2\sqrt{\varphi}}\right)dt$$

$$+\varepsilon^{2m+1}\frac{1}{y^2}\frac{1}{4}\int_0^y (\psi')^2\sin[\frac{1}{\varepsilon}\Omega]\frac{\varphi'+\varepsilon^{2m}\psi'}{(\varphi+\varepsilon^{2m}\psi)^2}t^3\sqrt{\varphi}dt$$

$$-\varepsilon^{2m}\frac{1}{y^2}\frac{1}{4}\int_0^y (\psi')^2\cos[\frac{1}{\varepsilon}\Omega]\frac{1}{(\varphi+\varepsilon^{2m}\psi)}t^2dt$$

Putting the two together we get

$$-\varepsilon^{2m}\frac{1}{y^2}\frac{1}{4}\int_0^y \cos[\frac{1}{\varepsilon}\Omega]\frac{1}{(\varphi+\varepsilon^{2m}\psi)}\frac{t^2}{\varphi}\left[\varphi\psi'-2\varphi'\psi\right]\psi'dt$$

$$+\varepsilon^{2m+1}\frac{1}{y^2}\frac{1}{4}\int_0^y \sin[\frac{1}{\varepsilon}\Omega]\frac{\varphi'+\varepsilon^{2m}\psi'}{(\varphi+\varepsilon^{2m}\psi)}\frac{t^3}{\sqrt{\varphi}}\left[\varphi\psi'-2\varphi'\psi\right]\psi'dt$$

$$-\varepsilon^{2m+1}\frac{1}{y^2}\frac{1}{4}\int_0^y \sin[\frac{1}{\varepsilon}\Omega]\frac{1}{(\varphi+\varepsilon^{2m}\psi)}\left[3\psi't^2\sqrt{\varphi}-\frac{t^3\varphi'\psi'}{2\sqrt{\varphi}}-\frac{6t^2\varphi'\psi}{\sqrt{\varphi}}+\frac{t^3(\varphi')^2}{\varphi\sqrt{\varphi}}\psi\right]\psi'dt$$

$$-\varepsilon^{2m+1}\frac{1}{y^2}\frac{1}{4}\int_0^y \sin[\frac{1}{\varepsilon}\Omega]\frac{1}{(\varphi+\varepsilon^{2m}\psi)}\frac{t^3}{\sqrt{\varphi}}\left[\psi'\varphi'-2\varphi'\psi'-2\varphi''\psi\right]\psi'dt$$

which can be written as

$$-\varepsilon^{2m}S_2\left[\frac{1}{4}\frac{t}{\sqrt{\varphi}}(\varphi+\varepsilon^{2m}\psi)^{-1}\left[2\varphi'\psi-\varphi\psi'\right]\psi'\right]$$

$$-\varepsilon^{2m+1}S_1\left[\frac{3}{4}t(\varphi+\varepsilon^{2m}\psi)^{-1}\left[2\varphi'\psi-\varphi\psi'\right]\psi'\right]$$

$$+\varepsilon^{2m+1}S_1\left[\frac{1}{8}\frac{t^2\varphi'}{\varphi}(\varphi+\varepsilon^{2m}\psi)^{-1}\left[2\varphi'\psi-\varphi\psi'\right]\psi'\right]$$

$$+\varepsilon^{4m+1}S_1\left[\frac{1}{4}t^2(\varphi+\varepsilon^{2m}\psi)^{-2}\left[2\varphi'\psi-\varphi\psi'\right](\psi')^2\right]$$

$$-\varepsilon^{2m+1}\frac{1}{y^2}\frac{1}{4}\int_0^y \sin[\frac{1}{\varepsilon}\Omega](\varphi+\varepsilon^{2m}\psi)^{-1}\frac{t^3}{\sqrt{\varphi}}\left[2\varphi'\varphi'\psi-\varphi\varphi'\psi'-\varphi'\psi'-2\varphi''\psi\right]\psi'dt$$

and we can write the last term as

$$+\varepsilon^{2m+1}S_1\left[\frac{t^2}{4}(\varphi+\varepsilon^{2m}\psi)^{-1}\left[2\psi\left((\varphi')^2-\varphi''\right)-\varphi'\psi'(\varphi+1)\right]\psi'\right]$$

All that we are interested in is the structure of the equations, and so we extend the definition for the generic G. By doing so, we will also solve the earlier problem of certain terms not fitting with the old definition for G.

$$\text{Define } \widetilde{G}(y,\psi,\psi') = \sum_{l+q=1}^{l+q=3} P_{l,q}(y)(\psi)^l(\psi')^q$$

where $P_{l,q}(y)$ are polynomials in y which depend on ε in a regular way.We can invert fully the equation 4.14 to get

$$\begin{aligned} \psi' &= \psi_0' + \varepsilon S_1\left[\widetilde{G}_1(t,\psi,\psi')\right] \\ &+\varepsilon^{2m-1}S_1\left[(\varphi+\varepsilon^{2m}\psi)^{-1}\widetilde{G}_2(t,\psi,\psi')\right] \\ &+\varepsilon^{2m}S_2\left[\varphi^{-\frac{1}{2}}(\varphi+\varepsilon^{2m}\psi)^{-1}\widetilde{G}_3(t,\psi,\psi')\right] \\ &+\varepsilon^{2m+1}S_1\left[\varphi^{-1}(\varphi+\varepsilon^{2m}\psi)^{-1}\widetilde{G}_4(t,\psi,\psi')\right] \\ &+\varepsilon^{4m+1}S_1\left[(\varphi+\varepsilon^{2m}\psi)^{-2}\widetilde{G}_5(t,\psi,\psi')\right] \end{aligned} \tag{4.16}$$

where $\psi_0' = \frac{1}{\varepsilon}S_1\left[\overline{f}'\right] = \overline{f}' + O(\varepsilon)$.

4.4 The Contraction Mapping

We can write our results as follows

$$\psi = \psi_0 + \varepsilon S_1\left[P_1(t,\psi,\psi')\right] + \varepsilon S_2\left[P_2(t,\psi,\psi')\right] \tag{4.17}$$

$$\psi' = \psi_0' + \varepsilon S_1\left[\widetilde{P}_1(t,\psi,\psi')\right] + \varepsilon S_2\left[\widetilde{P}_2(t,\psi,\psi')\right] \tag{4.18}$$

where $P_{1,2}$ and $\widetilde{P}_{1,2}$ are explicitly given in 4.13 and 4.16. Let Λ be the closed interval $[0, y_0]$ and consider pairs of continuous functions $f, g \in C(\Lambda)$. We define a mapping T with components $T^{(1)}$ and $T^{(2)}$ by the formulas

$$T^{(1)}(f,g) = \psi_0 + \varepsilon S_1 \left[P_1(t,f,g)\right] + \varepsilon S_2 \left[P_2(t,f,g)\right]$$

$$T^{(2)}(f,g) = \psi_0' + \varepsilon S_1 \left[\widetilde{P}_1(t,f,g)\right] + \varepsilon S_2 \left[\widetilde{P}_2(t,f,g)\right]$$

The operators S_1 and S_2 map continuous functions into continuous functions, so we must check that $P_{1,2}(t,f,g)$ and $\widetilde{P}_{1,2}(t,f,g)$ are continuous functions. By inspection, we see that $(\varphi + \varepsilon^{2m}\psi) \neq 0$ thus $|\varphi + \varepsilon^{2m}\psi|$ must be positive, and for this we must restrict the interval Λ such that $\varphi(y) \geq c\varepsilon^{2m-1}$, $c > 0$ for $y \in \Lambda$.

Consider T as an operator in the Banach space of continuous functions (f, g) with the usual norm

$$\sup_{\Lambda} |f| + \sup_{\Lambda} |g|$$

In this space, we consider the ball B_ε which are centered at the pair ψ_0, ψ_0' and has a radius of $\varepsilon^{1-\mu}$ with μ positive and arbitrarily small. We want T to map the ball into itself. We have the estimate

$$|S_{1,2}[R]| \leq c \sup |R|$$

and so we must have that $P_{1,2}(t,f,g) = O(1)$ and $\widetilde{P}_{1,2}(t,f,g) = O(1)$. Analysis of the terms in 4.16 will give us a more severe restriction on the interval Λ.

$$\varphi(y) \geq c\varepsilon^m, \;\; c > 0, \text{ for } y \in \Lambda \tag{4.19}$$

with the side condition that $m \geq 2$. We can now show that T is indeed a contraction mapping in B_ε. This means that for (f_1, g_1) and $(f_2, g_2) \in B_\varepsilon$ we have

$$\left|T^{(1,2)}(f_1, g_1) - T^{(1,2)}(f_2, g_2)\right| \leq \varepsilon C \left\{\sup|f_1 - f_2| + \sup|g_1 - g_2|\right\}$$

with C independent of ε. So we consider a representative term of 4.16

$$\varepsilon\left|\varepsilon^{2m} S_1\left[\varphi^{-1}(\varphi + \varepsilon^{2m} f_1)^{-1}\tilde{G}(t, f_1, g_1) - \varphi^{-1}(\varphi + \varepsilon^{2m} f_2)^{-1}\tilde{G}(t, f_2, g_2)\right]\right|$$

$$= \varepsilon\left|-\frac{\varepsilon^{2m}}{y^2}\int_0^y \sin[\frac{1}{\varepsilon}\Omega(t,y)]\frac{t}{\sqrt{\varphi}}\left[\frac{(\varphi + \varepsilon^{2m} f_1)^{-1}}{\varphi}\tilde{G}(t, f_1, g_1) - \frac{(\varphi + \varepsilon^{2m} f_2)^{-1}}{\varphi}\tilde{G}(t, f_2, g_2)\right]\right| dt$$

$$\leq \varepsilon\frac{\varepsilon^{2m}}{y^2}\int_0^y \frac{t}{\sqrt{\varphi}}\frac{(\varphi + \varepsilon^{2m} f_1)^{-1}}{\varphi}\left|\tilde{G}(t, f_1, g_1) - \tilde{G}(t, f_2, g_2)\right| dt$$

$$+\varepsilon\frac{\varepsilon^{2m}}{y^2}\int_0^y \frac{t}{\sqrt{\varphi}}\frac{1}{\varphi}\left|\tilde{G}(t, f_2, g_2)\right|\left|\frac{1}{(\varphi + \varepsilon^{2m} f_2)} - \frac{1}{(\varphi + \varepsilon^{2m} f_1)}\right| dt$$

In the first term, $\tilde{G}(t, f, g)$ is Lipschitz continuous with respect to f and g. So this term is majorized by

$$\varepsilon\frac{1}{y^2}\int_0^y \frac{t}{\sqrt{\varphi}}dt \cdot \sup\frac{\varepsilon^{2m}}{(\varphi + \varepsilon^{2m} f_1)\varphi} \cdot C\left\{\sup|f_1 - f_2| + \sup|g_1 - g_2|\right\}$$

Now we also have that

$$\sup\frac{\varepsilon^{2m}}{(\varphi + \varepsilon^{2m} f_1)} \cdot \frac{1}{\varphi} = O(1)$$

so that the term is then majorized by

$$\varepsilon C \left\{ \sup |f_1 - f_2| + \sup |g_1 - g_2| \right\}$$

with C a constant independent of ε: We have that the second term is majorized by

$$\varepsilon \frac{1}{y^2} \int_0^y \frac{t}{\sqrt{\varphi}} dt \cdot \sup \frac{\varepsilon^{4m}}{\varphi} (\varphi + \varepsilon^{2m} f_1)^{-1} (\varphi + \varepsilon^{2m} f_2)^{-1} \cdot \sup |f_1 - f_2|$$

With the condition 4.19 we have a much stronger contraction, the term is majorized by

$$\varepsilon^{1+m} \cdot C \cdot \sup |f_1 - f_2|$$

with C independent of ε.

Now that we have proved the contraction property of the mapping T we have proved that there exists a unique solution $\psi(y), \psi'(y)$ of 4.17 and 4.18 in an interval Λ limited by the condition 4.19 and that

$$\psi = \psi_0 + O(\varepsilon)$$

$$\psi' = \psi_0' + O(\varepsilon)$$

which concludes the proof.

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Appendix A

Computation of Some Integrals

In this appendix, we will evaluate two integrals which occur in the analysis. The outline for this work was given by Dr. Nico Temme, C.W.I. Amsterdam. We are looking at

$$\left(\frac{dy}{dx}\right)_{\pm} = \pm\sqrt{z(y)}$$

We replace $z(y)$ by its asymptotic approximation and so we have for $y^m(x), x < 0$.

$$\frac{dy^{(m)}}{dx} = \sqrt{\Phi_m(y;\varepsilon)}$$

$$\Phi_m = \sum_{n=0}^{m-1} \varepsilon^{2n} z_n$$

For $m = 1$

$$\Phi_1 = z_0 = \left(y^{(1)}\right)^2 (1 - \frac{2}{3}y^{(1)})$$

and so

$$\frac{dy^{(1)}}{dx} = \sqrt{\left(y^{(1)}\right)^2 - \frac{2}{3}\left(y^{(1)}\right)^3}$$

For $m = 2$ we have

$$\Phi_2 = z_0 + \varepsilon^2 z_1$$

$$= \left(y^{(2)}\right)^2 (1 - \frac{2}{3}y^{(2)}) + \varepsilon^2 \left(-z_0 \frac{d^2 z_0}{dy^2} + \frac{1}{4}\left(\frac{dz_0}{dy}\right)^2\right)$$

$$= \left(y^{(2)}\right)^2 - \frac{2}{3}\left(y^{(2)}\right)^3 - \varepsilon^2 \left[\left(y^{(2)}\right)^2 - \frac{2}{3}\left(y^{(2)}\right)^3\right]\left[2 - 4y^{(2)}\right] + \frac{\varepsilon^2}{4}\left[2y^{(2)} - 2\left(y^{(2)}\right)^2\right]^2$$

$$= \left(y^{(2)}\right)^2 (1 - \varepsilon^2) + \left(y^{(2)}\right)^3 (-\frac{2}{3} + 4\varepsilon^2 + \frac{4}{3}\varepsilon^2 - 2\varepsilon^2) + \left(y^{(2)}\right)^4 (-\frac{5}{3}\varepsilon^2)$$

$$= \left(y^{(2)}\right)^2 (1 - \varepsilon^2) + \left(y^{(2)}\right)^3 (-\frac{2}{3} + \frac{10}{3}\varepsilon^2) + \left(y^{(2)}\right)^4 (-\frac{5}{3}\varepsilon^2)$$

We wish to calculate

$$I^{(m)}(\varepsilon) = \frac{1}{\varepsilon^2}\int_{-\infty}^{0} \cos[\frac{1}{\varepsilon}\xi]\left[y^{(m)}(\xi) - \left(y^{(m)}(\xi)\right)^2\right] d\xi$$

for $m = 1$ and $m = 2$. As

$$\left(\frac{dy^{(1)}}{dx}\right)^2 = \left(y^{(1)}\right)^2 - \frac{2}{3}\left(y^{(1)}\right)^3$$

so we need to integrate

$$\int \frac{1}{y^{(1)}(1 - \frac{2}{3}y^{(1)})^{\frac{1}{2}}} dy$$

$$= -\ln\left|\frac{1 + \sqrt{1 - \frac{2}{3}y^{(1)}}}{1 - \sqrt{1 - \frac{2}{3}y^{(1)}}}\right|$$

and so we have

$$x = -\ln\left|\frac{1 + \sqrt{1 - \frac{2}{3}y^{(1)}}}{1 - \sqrt{1 - \frac{2}{3}y^{(1)}}}\right|$$

Solving for $y^{(1)}$ yields

$$\frac{2}{3}y^{(1)} = \frac{4\varepsilon^x}{\varepsilon^{2x} + 2\varepsilon^x + 1}$$

Thus

$$y^{(1)} = \frac{3}{2}\,\mathrm{sech}^2(\frac{1}{2}x) = \frac{3}{\cosh[x]+1}$$

and so

$$I^{(1)}(\varepsilon) = \frac{1}{\varepsilon^2}\int_{-\infty}^{0} \cos[\frac{1}{\varepsilon}\xi]\frac{d^2y^{(1)}}{d\xi^2}d\xi$$

which we integrate by parts to get

$$I^{(1)}(\varepsilon) = \frac{1}{\varepsilon^2}\left(\left[\cos[\frac{1}{\varepsilon}\xi]\frac{dy^{(1)}}{d\xi}\right]_{-\infty}^{0} + \frac{1}{\varepsilon}\int_{-\infty}^{0}\frac{dy^{(1)}}{d\xi}\sin[\frac{1}{\varepsilon}\xi]d\xi\right)$$

$$= \frac{1}{\varepsilon^3}\int_{-\infty}^{0}\sin[\frac{1}{\varepsilon}\xi]\frac{dy^{(1)}}{d\xi}d\xi$$

Integrating by parts once again will yield

$$I^{(1)}(\varepsilon) = \frac{1}{\varepsilon^3}\left(\left[y^{(1)}\sin[\frac{1}{\varepsilon}\xi]\right]_{-\infty}^{0} - \frac{1}{\varepsilon}\int_{-\infty}^{0}\cos[\frac{1}{\varepsilon}\xi]y^{(1)}d\xi\right)$$

$$= -\frac{1}{\varepsilon^4}\int_{-\infty}^{0}\cos[\frac{1}{\varepsilon}\xi]y^{(1)}d\xi$$

now recall that

$$\cos(y) = \frac{e^{iy}+e^{-iy}}{2}$$

so

$$\cos(\frac{1}{\varepsilon}\xi) = \frac{e^{\frac{i\xi}{\varepsilon}}+e^{-\frac{i\xi}{\varepsilon}}}{2}$$

and thus

$$-\frac{1}{\varepsilon^4}\int_{-\infty}^{0}\cos[\frac{1}{\varepsilon}\xi]y^{(1)}d\xi = -\frac{1}{2\varepsilon^4}\int_{-\infty}^{0}\left(e^{\frac{i\xi}{\varepsilon}}+e^{-\frac{i\xi}{\varepsilon}}\right)y^{(1)}d\xi$$

$$= -\frac{1}{2\varepsilon^4}\left[\int_{-\infty}^{0} e^{\frac{i\xi}{\varepsilon}} y^{(1)} d\xi + \int_{-\infty}^{0} e^{-\frac{i\xi}{\varepsilon}} y^{(1)} d\xi\right]$$

$$= -\frac{1}{2\varepsilon^4}\left[\int_{-\infty}^{0} e^{\frac{i\xi}{\varepsilon}} y^{(1)}(\xi) d\xi - \int_{\infty}^{0} e^{-\frac{i\xi}{\varepsilon}} y^{(1)}(-\xi) d\xi\right]$$

$$= -\frac{1}{2\varepsilon^4}\left[\int_{-\infty}^{0} e^{\frac{i\xi}{\varepsilon}} y^{(1)}(\xi) d\xi + \int_{0}^{\infty} e^{-\frac{i\xi}{\varepsilon}} y^{(1)}(\xi) d\xi\right]$$

$$= -\frac{1}{2\varepsilon^4}\int_{-\infty}^{\infty} e^{\frac{i\xi}{\varepsilon}} y^{(1)}(\xi) d\xi$$

Now, set

$$\xi = \ln(u) \text{ so } d\xi = \frac{1}{u} du$$

$$I^{(1)}(\varepsilon) = -\frac{1}{2\varepsilon^4}\int_{0}^{\infty} e^{\frac{\ln(u) i}{\varepsilon}} y^{(1)} \ln(u) \frac{1}{u} du$$

$$= -\frac{1}{2\varepsilon^4}\int_{0}^{\infty} u^{\frac{i}{\varepsilon}} \frac{6u}{(u+1)^2} \frac{1}{u} du$$

$$= -\frac{3}{\varepsilon^4}\int_{0}^{\infty} \frac{u^{\frac{i}{\varepsilon}}}{(u+1)^2} du$$

now we substitute

$$u = \frac{t}{(1-t)} \text{ so } du = \frac{1}{(1-t)^2} dt$$

to get

$$-\frac{3}{\varepsilon^4}\int_{0}^{1} \frac{\left(\frac{t}{1-t}\right)^{\frac{i}{\varepsilon}}}{\frac{1}{(1-t)^2}} \frac{1}{(1-t)^2} dt$$

$$= -\frac{3}{\varepsilon^4}\int_{0}^{1} \left(\frac{t}{1-t}\right)^{\frac{i}{\varepsilon}} dt$$

$$= -\frac{3}{\varepsilon^4}\int_{0}^{1} t^{\frac{i}{\varepsilon}} (1-t)^{\frac{-i}{\varepsilon}} dt$$

$$= -\frac{3}{\varepsilon^4}\int_{0}^{1} t^{(\frac{i}{\varepsilon}+1)-1} (1-t)^{(1-\frac{i}{\varepsilon})-1} dt \tag{A.1}$$

notice that the above equation is in the form of a Beta function and recall the formula

$$\Gamma(x)\Gamma(y) = B(x,y)\Gamma(x+y)$$

where

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$$

so that we get for A.1

$$\frac{\left(-\frac{3}{\varepsilon^4}\right)\Gamma(\frac{i}{\varepsilon}+1)\Gamma(1-\frac{i}{\varepsilon})}{\Gamma(\frac{i}{\varepsilon}+1+1-\frac{i}{\varepsilon})}$$

$$= \frac{(-3)\Gamma(\frac{i}{\varepsilon}+1)\Gamma(1-\frac{i}{\varepsilon})}{\varepsilon^4\Gamma(2)} \tag{A.2}$$

Now we also recall that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

thus

$$z\left(\Gamma(z)\Gamma(1-z)\right) = \frac{\pi z}{\sin(\pi z)}$$

and so

$$\Gamma(z+1)\Gamma(1-z) = \frac{\pi z}{\sin(\pi z)}$$

so A.2 becomes

$$I^{(1)}(\varepsilon) = \left(-\frac{3}{\varepsilon^4}\right)\frac{\pi(\frac{i}{\varepsilon})}{\sin(\frac{i\pi}{\varepsilon})} = -\frac{3\pi i}{\varepsilon^5 i\sinh(\frac{\pi}{\varepsilon})} = -\frac{3\pi}{\varepsilon^5\sinh(\frac{\pi}{\varepsilon})} \tag{A.3}$$

Now we wish to deal with the second integral so we start by considering

$$\frac{dy^{(2)}}{dx} = \sqrt{\left(y^{(2)}\right)^2(1-\varepsilon^2) + \left(y^{(2)}\right)^3(-\frac{2}{3}+\frac{10}{3}\varepsilon^2) + \left(y^{(2)}\right)^4(-\frac{5}{3}\varepsilon^2)}$$

$$= y^{(2)}\sqrt{(1-\varepsilon^2) + y^{(2)}(-\frac{2}{3}+\frac{10}{3}\varepsilon^2) + \left(y^{(2)}\right)^2(-\frac{5}{3}\varepsilon^2)}$$

To help simplify the notation, we will drop the superscript (2) on the y and just use y where it will not be confusing. We must integrate

$$\int \frac{dy}{y\sqrt{(1-\varepsilon^2) + y(-\frac{2}{3}+\frac{10}{3}\varepsilon^2) + y^2(-\frac{5}{3}\varepsilon^2)}}$$

by making the substitution $y = \frac{1}{u}$ we get

$$\int \frac{-du}{\sqrt{(1-\varepsilon^2)u^2 + (-\frac{2}{3}+\frac{10}{3}\varepsilon^2)u + (-\frac{5}{3}\varepsilon^2)}}$$

So evaluation of this integral yields

$$y^{(2)} = \frac{2(1-\varepsilon^2)}{\cosh\left(x\sqrt{1-\varepsilon^2}\right)\left[\frac{4}{9}\left(10\varepsilon^4+5\varepsilon^2+1\right)\right]^{\frac{1}{2}} - \left(\frac{10\varepsilon^2-2}{3}\right)}$$

$$= \frac{6(1-\varepsilon^2)}{\cosh\left(x\sqrt{1-\varepsilon^2}\right)2\left(10\varepsilon^4+5\varepsilon^2+1\right)^{\frac{1}{2}} - (10\varepsilon^2-2)}$$

$$= \frac{3(1-\varepsilon^2)}{\cosh\left(x\sqrt{1-\varepsilon^2}\right)\left(10\varepsilon^4+5\varepsilon^2+1\right)^{\frac{1}{2}} - (5\varepsilon^2-1)}$$

$$= \frac{3\lambda(1-\varepsilon^2)}{\cosh\left(x\sqrt{1-\varepsilon^2}\right) + \lambda(1-5\varepsilon^2)}$$

where

$$\lambda = \frac{1}{\sqrt{10\varepsilon^4 + 5\varepsilon^2 + 1}}$$

To solve

$$I^{(2)}(\varepsilon) = \frac{1}{\varepsilon^2} \int_{-\infty}^{0} \cos[\frac{1}{\varepsilon}\xi] y^{(2)}(\xi) \left[1 - y^{(2)}(\xi)\right] d\xi$$

we have to look at integrals of the type

$$J_k(a, \gamma) = \int_0^\infty \frac{\cos(ax)}{(\cosh(x) + \cos(\gamma))^k} dx, \text{ for } a \in \mathbf{C}, |Im\ a| < k, \gamma \in (-\pi, \pi), k = 1, 2 \tag{A.4}$$

We also have that

$$\cos(ax) = \frac{e^{iax} + e^{-iax}}{2}$$

for any complex number a. Thus

$$J_k(a, \gamma) = \frac{1}{2} \int_0^\infty \frac{e^{iax} + e^{-iax}}{(\cosh(x) + \cos(\gamma))^k} dx$$

$$= \frac{1}{2} \int_{-\infty}^\infty \frac{e^{iax}}{(\cosh(x) + \cos(\gamma))^k} dx$$

Let $k = 1$ and consider

$$\frac{1}{2} \int_{-\infty}^\infty \frac{e^{iax}}{\cosh(x) + \cos(\gamma)} dx$$

substitute $x = \ln(t)$ to get

$$\frac{1}{2} \int_0^\infty \frac{1}{t} \frac{t^{ia}}{\left(\frac{t+t^{-1}}{2}\right) + \left(\frac{e^{i\gamma}+e^{-i\gamma}}{2}\right)} dt$$

$$= \int_0^\infty \frac{1}{t} \frac{t^{ia}}{t + t^{-1} + e^{i\gamma} + e^{-i\gamma}} dt$$

$$= \int_0^\infty \frac{t^{ia}}{t^2 + te^{i\gamma} + te^{-i\gamma} + 1} dt$$

$$= \int_0^\infty \frac{t^{ia}}{(t + e^{i\gamma})(t + e^{-i\gamma})} dt$$

$$= \int_0^\infty \frac{t^{ia}(e^{i\gamma} - e^{-i\gamma})}{(e^{i\gamma} - e^{-i\gamma})(t + e^{i\gamma})(t + e^{-i\gamma})} dt$$

$$= \int_0^\infty \frac{e^{i\gamma}t^{ia} + t^{ia-1} - t^{ia}e^{-i\gamma} - t^{ia-1}}{(e^{i\gamma} - e^{-i\gamma})(t + e^{i\gamma})(t - e^{-i\gamma})} dt$$

$$= \int_0^\infty \frac{t^{ia-1}e^{i\gamma}(t + e^{-i\gamma}) - e^{-i\gamma}t^{ia-1}(t + e^{i\gamma})}{(e^{i\gamma} - e^{-i\gamma})(t + e^{i\gamma})(t - e^{-i\gamma})} dt$$

$$= \frac{e^{i\gamma}}{e^{i\gamma} - e^{-i\gamma}} \int_0^\infty \frac{t^{ia-1}}{t + e^{i\gamma}} dt - \frac{e^{-i\gamma}}{e^{i\gamma} - e^{-i\gamma}} \int_0^\infty \frac{t^{ia-1}}{t + e^{-i\gamma}} dt$$

$$= \frac{e^{i\gamma}}{2i\sin(\gamma)} \int_0^\infty \frac{t^{ia-1}}{t + e^{i\gamma}} dt - \frac{e^{-i\gamma}}{2i\sin(\gamma)} \int_0^\infty \frac{t^{ia-1}}{t + e^{-i\gamma}} dt$$

When a is real, two divergent integrals appear, so to ensure convergence we will temporarily assume that $-1 < Im\ a < 0$. Now set $t = se^{i\gamma}$, $t = se^{-i\gamma}$ respectively, so that we can get

$$\frac{1}{2i\sin(\gamma)} \left[\int_0^\infty \frac{(se^{-i\gamma})^{ia}}{se^{-i\gamma} + e^{-i\gamma}} e^{-i\gamma} ds - \int_0^\infty \frac{(se^{-i\gamma})^{ia}}{se^{i\gamma} + e^{i\gamma}} e^{i\gamma} ds \right]$$

$$= \frac{e^{a\gamma} - e^{-a\gamma}}{2i\sin(\gamma)} \int_0^\infty \frac{s^{ia}}{s+1} ds \tag{A.5}$$

Consider

$$\int_0^\infty \frac{s^{ia}}{s+1} ds$$

with the substitution

$$s = \frac{w}{1-w}$$

to get

$$\int_0^1 \frac{w^{ia}}{(1-w)^{ia}(1-w)^{-1}(1-w)^2} dw$$

$$= \int_0^1 \frac{w^{ia}}{(1-w)^{ia+1}} dw$$

$$= \int_0^1 w^{(ia+1)-1}(1-w)^{-ia-1} dw$$

and since this is a Beta function, we get

$$\frac{\Gamma(ia+1)\Gamma(-ia)}{\Gamma(1)} = \Gamma(-ia)\Gamma(1-(-ia)) = \frac{\pi}{\sin(-\pi ia)}$$

so that A.5 becomes

$$\frac{e^{a\gamma} - e^{-a\gamma}}{2i\sin(\gamma)} \frac{\pi}{-\sin(\pi ia)}$$

$$= -\frac{2\sinh(a\gamma)\pi}{\sin(\gamma)2i^2\sinh(\pi a)} = \frac{\pi\sinh(a\gamma)}{\sin(\gamma)\sinh(\pi a)}$$

so that

$$J_1(a,\gamma) = \frac{\pi\sinh(a\gamma)}{\sin(\gamma)\sinh(\pi a)} \tag{A.6}$$

The above equation was derived under the condition $-1 < Im\ a < 0$, however, both this final expression and the integral representation given in A.4 are analytic functions of a in the strip $|Im\ a| < 1$,and so we can conclude that A.6 also holds in this strip. Thus it is true for real values of a.

Now we wish to evaluate $J_2(a,\gamma)$, this can be accomplished by differentiating

$J_1(a,\gamma)$ with respect to γ. We notice that

$$\frac{\partial}{\partial\gamma}J_1(a,\gamma) = \sin(\gamma)J_2(a,\gamma)$$

so that

$$J_2(a,\gamma) = \frac{1}{\sin(\gamma)}\frac{\partial}{\partial\gamma}J_1(a,\gamma)$$

and

$$\frac{\partial}{\partial\gamma}J_1(a,\gamma) = \frac{\partial}{\partial\gamma}\frac{\pi\sinh(a\gamma)}{\sin(\gamma)\sinh(\pi a)}$$

$$= \frac{\pi a\cosh(a\gamma)}{\sin(\gamma)\sinh(\pi a)} - \frac{\pi\sinh(a\gamma)\cos(\gamma)}{\sin^2(\gamma)\sinh(\pi a)}$$

$$= \frac{\pi}{\sin^2(\gamma)\sinh(\pi a)}\left(a\cosh(a\gamma)\sin(\gamma) - \sinh(a\gamma)\cos(\gamma)\right)$$

and so

$$J_2(a,\gamma) = \frac{\pi}{\sin^3(\gamma)\sinh(\pi a)}\left(a\cosh(a\gamma)\sin(\gamma) - \sinh(a\gamma)\cos(\gamma)\right) .$$

We can return to the evaluation of $I^{(2)}(\varepsilon)$

$$I^{(2)}(\varepsilon) = \frac{1}{\varepsilon^2}\int_0^{-\infty}\cos[\frac{1}{\varepsilon}\xi]y^{(2)}(\xi)\left[1 - y^{(2)}(\xi)\right]d\xi$$

where

$$y^{(2)}(x) = \frac{3\lambda(1-\varepsilon^2)}{\cosh\left(x\sqrt{1-\varepsilon^2}\right) + \lambda(1-5\varepsilon^2)}$$

with

$$\lambda = \frac{1}{\sqrt{1+5\varepsilon^2+10\varepsilon^4}}$$

We set

$$a = \frac{1}{\varepsilon\sqrt{1-\varepsilon^2}}, \cos(\gamma) = \lambda(1-5\varepsilon^2)$$

and consider

$$\frac{1}{\varepsilon^2}3\lambda\sqrt{1-\varepsilon^2}\left[J_1(a,\gamma) - 3\lambda(1 - \frac{1}{\varepsilon^2}3\lambda(1-\varepsilon^2)J_2(a,\gamma)\right]$$

to get

$$\frac{1}{\varepsilon^2}3\lambda\sqrt{1-\varepsilon^2}\left[\int_0^\infty \frac{\cos\frac{x}{\varepsilon\sqrt{1-\varepsilon^2}}}{\cosh(x)+\lambda(1-5\varepsilon^2)}dx\right.$$

$$\left.-3\lambda(1-\varepsilon^2)\int_0^\infty \frac{\cos\frac{x}{\varepsilon\sqrt{1-\varepsilon^2}}}{(\cosh(x)+\lambda(1-5\varepsilon^2))^2}dx\right]$$

Let

$$\frac{x}{\varepsilon\sqrt{1-\varepsilon^2}} = \frac{\xi}{\varepsilon}$$

to get

$$\frac{1}{\varepsilon^2}3\lambda\sqrt{1-\varepsilon^2}\left[\int_0^\infty \frac{\cos(\frac{\xi}{\varepsilon})\sqrt{1-\varepsilon^2}}{\cosh(\xi\sqrt{1-\varepsilon^2})+\lambda(1-5\varepsilon^2)}d\xi\right.$$

$$\left.-3\lambda(1-\varepsilon^2)\int_0^\infty \frac{\cos(\frac{\xi}{\varepsilon})\sqrt{1-\varepsilon^2}}{\left(\cosh(\xi\sqrt{1-\varepsilon^2})+\lambda(1-5\varepsilon^2)\right)^2}d\xi\right]$$

$$= \frac{1}{\varepsilon^2}3\lambda\sqrt{1-\varepsilon^2}\int_0^\infty \frac{\cos(\frac{\xi}{\varepsilon})\sqrt{1-\varepsilon^2}\left(\cosh(\xi\sqrt{1-\varepsilon^2}+\lambda(1-5\varepsilon^2-3\lambda(1-\varepsilon^2)\right)}{\left(\cosh(\xi\sqrt{1-\varepsilon^2}+\lambda(1-5\varepsilon^2)\right)^2}d\xi$$

and so after substituting $-\xi = \xi$ we get the above equation equal to $I^{(2)}(\varepsilon)$. Thus

$$I^{(2)}(\varepsilon) = \frac{1}{\varepsilon^2}3\lambda\sqrt{1-\varepsilon^2}\left[J_1(a,\gamma) - 3\lambda(1-\varepsilon^2)J_2(a,\gamma)\right]$$

γ is positive since $\cos(\gamma)$ is positive and γ tends to zero as $\varepsilon \to 0$. Moreover, we have

$$\sin(\gamma) = \frac{\varepsilon\sqrt{15(1-\varepsilon^2)}}{\sqrt{1-5\varepsilon^2+10\varepsilon^4}} = \lambda\varepsilon\sqrt{15(1-\varepsilon^2)}$$

It then follows that

$$I^{(2)}(\varepsilon) = \frac{1}{\varepsilon^2}3\lambda\sqrt{1-\varepsilon^2}\left[\frac{\pi\sinh(a\gamma)}{\sin(\gamma)\sinh(\pi a)}\right.$$

$$\left.-\frac{3\lambda(1-\varepsilon^2)\pi}{\sin^3(\gamma)\sinh(\pi a)}\left(a\cosh(a\gamma)\sin(\gamma)-\sinh(a\gamma)\cos(\gamma)\right)\right]$$

$$= \frac{1}{\varepsilon^2}\frac{9\lambda^2\pi(1-\varepsilon^2)^{\frac{3}{2}}}{\sin^3(\gamma)\sinh(\pi a)}\Big(\sinh(a\gamma)5\varepsilon^2\lambda+\sinh(a\gamma)\lambda(1-5\varepsilon^2)-a\cosh(a\gamma)\sin(\gamma)\Big)$$

$$= \frac{1}{\varepsilon^2}\frac{9\lambda^2\pi(1-\varepsilon^2)^{\frac{3}{2}}}{\sin^3(\gamma)\sinh(\pi a)}\left(\lambda\sinh(a\gamma)-a\cosh(a\gamma)\sin(\gamma)\right)$$

Now we compare this result with A.3. Intuitively we would expect $I^{(1)} \sim I^{(2)}$ as $\varepsilon \to 0$ but this actually turns out to be false. Letting $\varepsilon \to 0$ and hence $\gamma \to 0$ we have

$$\frac{\lambda\sinh(a\gamma)-a\cosh(a\gamma)\sin(\gamma)}{\sin^3(\gamma)} \sim -\frac{1}{3}a^3+\frac{1}{6}a$$

and substituting this value back into $I^{(2)}(\varepsilon)$ we get

$$I^{(2)}(\varepsilon) \sim -\frac{1}{\varepsilon^2}\frac{3a^3\pi(1-\frac{1}{2}a^{-2})}{\sinh(\pi a)}$$

which does resemble A.3 if $a = \frac{1}{\varepsilon}$, but this would not be allowed in the limit, instead we consider the true value of a , which is $a = \frac{\lambda\sqrt{15}}{\sin(\gamma)}$ to get

$$I^{(2)}(\varepsilon) = \frac{1}{\varepsilon^2}\frac{9\lambda^2\pi(1-\varepsilon^2)^{\frac{3}{2}}}{\sin^3(\gamma)\sinh(\pi a)}\left[\lambda\sinh(a\gamma)-\lambda\sqrt{15}\cosh(a\gamma)\right]$$

$$= \frac{1}{\varepsilon^2} \frac{9\lambda^2\pi(1-\varepsilon^2)^{\frac{3}{2}}}{\sin^3(\gamma)\sinh(\pi a)} \left[\sinh(a\gamma) - \sqrt{15}\cosh(a\gamma)\right]$$

So now as $\varepsilon \to 0$ we have $a\sin(\gamma) \to a\gamma$ and $a\sin(\gamma) = \lambda\sqrt{15} \to \sqrt{15}$ and so $a\gamma \to \sqrt{15}$. Now when we consider the limit of the ratio $\frac{I^{(2)}}{I^{(1)}}$ we get

$$\lim_{\varepsilon \to 0} \frac{I^{(2)}(\varepsilon)}{I^{(1)}(\varepsilon)} = \frac{3\left[\sinh(\sqrt{15}) - \sqrt{15}\cosh(\sqrt{15})\right]}{15\sqrt{15}}$$

$$\sim \frac{24 - \sqrt{15} \cdot 24}{5\sqrt{15}} \sim -3.6$$