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V-MODULES AND GENERALIZED V-MODULES

by

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ABSTRACT

In this thesis we obtain results about V-modules, GV-modules and P-V-modules complementing results of K.R. Fuller and Y. Hirano. We also introduce the notions of weakly GV-modules, DSI-modules and P-V'-modules.

The class of V-modules turns out to be a hereditary pretorsion class and thus gives rise to a left exact preradical ν. In general ν is not a radical. We study the associated hereditary torsion class and the arising Loewy series of modules. We introduce the notions of semi-V-modules and semi-V-rings, and generalize some results of H. Bass on perfect rings.

We also introduce the concept of an SI-module, extending the notion of an SI-ring introduced by K.R. Goodearl. The connections between SI-modules, regular modules and the preceeding modules are studied.

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To my wife Lois

TABLE OF CONTENTS

INTRODUCTION

A ring R is called a (Von Neumann) regular ring if for each a in R there exists an x in R such that a = axa. If R is commutative, I. Kaplansky has proved that R is regular if and only if every simple R-module is injective. Subsequently a ring R is called a left V-ring if every simple left R-module is injective. Such rings were called V-rings after O.Villamayor, who characterized left V-rings as those rings in which every proper left ideal is an intersection of maximal left ideals.

The notion of regularity has been extended to modules in [18], [50] and [60], while the notion of a V-ring has been extended to modules in [21], [35] and [46]. In this thesis, following H. Tominaga [46], we call a module ${}_RM$ a V-module if every proper R-submodule is an intersection of maximal submodules. Such a module M has also been called "co-semisimple" by K.R. Fuller in [21]. A result of Fuller asserts that the class of V-modules is closed under submodules, homomorphic images and arbitrary direct sums. A class with these properties is defined by Stenström [44] to be a hereditary pretorsion class.

This thesis is intended to give further contributions to the study of V-modules and their generalizations. We shall also introduce and study the left exact preradical associated with the pretorsion class of V-modules.

In Chapter 1, several characterizations of V-modules are given and the relationship between V-modules and M-flatness is studied. We prove, among other things, that a module M is a V-module if and only if every cofinitely generated module is M-injective. We also prove that if R is a commutative ring and ${}_RM$ is a projective module then M is a V-module if and only if every simple R-module is M-flat.

Chapter 2 is devoted to the study of Noetherian V-modules. We characterize them in terms of semisimple modules as well as minimal generating sets. We prove that a finitely generated module M is a Noetherian V-module if and only if every semisimple module is M-injective, which extends a similar result for rings in [8] and [40]. It is also proven that a finitely generated module M is a Noetherian V-module if and only if every submodule of M has a minimal generating set and if L is a homomorphic image of M, then every minimal generating set of any submodule of L can be extended to a minimal generating set for L, which extends a similar result for rings by B. Sarath in [39].

In Chapter 3, we study Generalized V-modules (GV-modules) and introduce the notion of weakly GV-modules. Following Y. Hirano [28], a module ${}_RM$ is called a GV-module if every simple singular left R-module is M-injective. Many known results on GV-rings will be extended to GV-modules. We will call a module M a Weakly GV-module (WGV-module) if every proper essential submodule of M is an intersection of maximal submodules. It is shown that a module M is a GV-module if and only if M is a WGV-module and $J(M) \cap Z(M) = 0$. We also prove that a module M is a WGV-module if and only if $M|Soc(M)$ is a V-module. A ring R is called a left WGV-ring if the left R-module ${}_RR$ is a WGV-module. The ring R is shown to be left WGV-ring if and only if all left R-modules are WGV-modules. The class of WGV-modules turns out to be a hereditary pretorsion class. A necessary and sufficient condition for a WGV-module to be a V-module is given.

In Chapter 4, we consider the notion of P-M-injectivity. A module ${}_RU$ is called P-M-injective if every non-zero R-homomorphism of any cyclic submodule of M into U can be extended to an R-homomorphism of M into U. If every simple (resp. simple singular) module is P-M-injective, M is called a P-V-module (resp. a P-V'-module). Known results on P-V-rings and P-V'-rings are extended to modules. We will also introduce the notion of P-M-flatness and as in Chapter 1, we prove that if R is a commutative ring and ${}_RM$ is a projective module then M is a P-V-module if and only if every simple R-module is P-M-flat. Using this result and a result of Y. Hirano [28], we prove that if R is a commutative ring and M is a projective R-module then the following conditions are equivalent:

(i) M is a V-module.

(ii) M is a GV-module.

(iii) M is a P-V-module.

(iv) M is a P-V'-module.

Chapter 5 consists of two sections. In Section 1, we introduce the notions of SI-modules and P-SI-modules. SI-modules are natural extensions of Goodearl's SI-rings [22]. A module M will be called an SI-module (resp. P-SI-module) if every singular module is M-injective (resp. P-M-injective). Many known results on SI-rings are extended to SI-modules. The connections between regular modules, V-modules, GV-modules and SI-modules are studied. A structure theorem for finitely generated projective SI-modules over commutative rings is obtained. In Section 2, we introduce a generalization of SI-rings. A ring R will be called a left P-SI-ring if the left R-module ${}_R R$ is a P-SI-module. We prove, among others, if R is a ring with essential left socle then R is a left P-SI-ring if and only if $\mathrm{Soc}_R R$ is projective and $R|\mathrm{Soc}_R R$ is a regular ring. We also prove that if $R|J(R)$ is semisimple then R is a left P-SI-ring if and only if R is a right P-SI-ring.

In Chapter 6, the focus is once again on V-modules. We show that V-modules can be as useful as semisimple modules in characterizing various types of rings. We characterize rings whose V-modules are injective, rings whose singular V-modules are injective and non-singular rings whose singular modules are V-modules.

Chapter 7 is divided into three sections. In Section 1, we introduce the left exact preradical ν associated with the hereditary pretorsion class $\underline{\underline{C}}_\nu$ of V-modules. For every left R-module M, $\nu(M)$ denotes the sum of all submodules of M belonging to $\underline{\underline{C}}_\nu$. An example is given to show that in general ν is not a radical. We shall give necessary and sufficient conditions for the class $\underline{\underline{C}}_\nu$ to be closed under extensions, injective hulls and respectively direct products. We prove, among other things, a ring R is a left V-ring if and only if the class $\underline{\underline{C}}_{\nu}$ has the lifting property [48]. In Section 2, we consider Amitsur's transfinite process of associating a left exact radical $\bar{\nu}$ with ν, which yields an ascending chain of preradicals $\{\nu_{\alpha}\}$ for each ordinal α, thus gives rise to a ν-Loewy series for each module M. We shall study the ν-Loewy series and obtain results similar to known results on the usual Loewy series associated with the left exact preradical Soc. We will introduce the notions of semi-V-modules and semi-V-rings. A module M will be called semi-V-module if $\nu_{\alpha}(M) = M$, for some ordinal α; and a ring R will be called a left semi-V-ring (or a ν-Loewy ring) if the left R-module $_{R}R$ is a semi-V-module. An example is given to show that there are V-modules with zero socle. Thus every semiartinian ring is a semi-V-ring but not vice-versa. In his work on perfect rings, H. Bass has proved that if R is a right semiartinian ring then J(R) is left T-nilpotent. We shall extend this result to the class of semi-V-rings. We show that a ring R is a left semi-V-ring if and only if J(R) is right T-nilpotent and $R|J(R)$ is a left semi-V-ring. We also prove that if R is a commutative Noetherian ring then R is a semi-V-ring if and only if R is a perfect ring.

In Section 3, we shall investigate finite or infinite sequences of submodules of a given module M, of the form $\{0\} = M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots$ or of the form $M = M^0 \supseteq M^1 \supseteq M^2 \supseteq \ldots$, where all the factor modules $M_{i+1}|M_i$ or $M^i|M^{i+1}$ are V-modules.

CHAPTER 0

NOTATIONS AND DEFINITIONS

Throughout this thesis, unless otherwise indicated, a ring R is an associative ring with identity; all modules are unitary left R-modules. For any ring R, R-mod denotes the category of left R-modules. For any module M we denote by Z(M), J(M), Soc(M) and E(M) the singular submodule, the Jacobson radical, the socle and the injective hull respectively of M. A module ${}_RM$ is semisimple if it is a direct sum of simple modules. ${}_RM$ is called semiartinian if every non-zero homomorphic image of M has a non-zero socle. A submodule N of M is "large" or "essential" in M if for all nonzero x in M, $Rx \cap N \neq 0$. Given a subset A of M, we denote the submodule generated by A by $\langle A \rangle$. Given a submodule L of M, we write L^* for the intersection of all maximal submodules of M containing L. Given a subset N of a module M, the annihilator of N in R, denoted by $\mathrm{Ann}_R(N)$, is the set of those $r \in R$ such that $rx = 0$ for all $x \in N$. A module M is indecomposable if the only direct sum decompositions $M = M_1 \oplus M_2$ are those in which $M_1 = 0$ or $M_2 = 0$. If M and N are modules, then the phrase "map from M to N" or the notation "$f : M \longrightarrow N$", refers to an R-homomorphism. When $N \subseteq M$, we sometimes use the notation $x \mapsto \bar{x}$ for the natural homomorphism $M \longrightarrow M|N$. The ring of all endomorphisms of an R-module M is denoted $\mathrm{End}_R(M)$.

Let M and U be R-modules. Following G. Azumaya [3], we say that U is M-injective if for each submodule K of M every R-homomorphism from K into U can be extended to an R-homomorphism from M into U. According to Sandomierski [38] U is M-injective if and only if every R-homomorphism $\Upsilon : M \longrightarrow E(U)$ has its image in U.

An R-module U is said to be injective if given any exact sequence $0 \longrightarrow A \xrightarrow{i} B \longrightarrow C \longrightarrow 0$ of left R-modules and any map $g : A \longrightarrow U$, there exists a map $f : B \longrightarrow U$ satisfying $g = f \circ i$. It is well-known (Baer's criterion) that U is injective if and only if U is R-injective. We refer to [2] for the definition and properties of M-injective modules.

A module M is called cofinitely generated, if $E(M) = E(S_1) \oplus \cdots \oplus E(S_k)$ for some integer $k > 0$, with each S_i simple. Equivalently if every family of submodules of M with intersection 0 contains a finite subfamily with zero intersection. Such a module M has also been called "finitely embedded (f.e.)" by P. Vámos in [47] and "finitely cogenerated" by K.R. Fuller in [21].

A ring R is called (Von Neumann) regular ring if given any $x \in R$ there exists $a \in R$ with $x = xax$. Equivalently if every finitely generated left ideal of R is generated by an idempotent. The notion of regularity has been extended to modules by D. Fieldhouse [18], J. Zelmanowitz [60] and R. Ware [50]. The first two authors considered arbitrary modules while the third author dealt with projective modules only. However their definitions agree for projective modules. In this thesis, following Zelmanowitz [60], we call a module ${}_RM$ regular if given any $m \in M$ there exists $f \in \mathrm{Hom}_R(M,R)$ with $(m)fm = m$. The following proposition is needed for our later purposes. For the proof see [50, Theorem 2.2], [60, Proposition 2.1] and [18, Theorem 1].

<u>Proposition 0.1</u>: Let R be a ring and ${}_RM$ be a projective module. Then the following statements are equivalent:

(i) M is a regular module.

(ii) Every homomorphic image of M is flat.

(iii) Every cyclic submodule of M is a direct summand.

(iv) Every finitely generated submodule of M is a direct summand.

(v) For every submodule K of M and every right ideal I of R, $IM \cap K = IK$.

(vi) For every submodule K of M, the sequence $0 \longrightarrow E \otimes_R K \longrightarrow E \otimes_R M$ is exact for all right R-modules E (i.e. every submodule K of M is pure in the sense of P.M. Cohn [18]).

Following B. Zimmermann-Huisgen [62] we say that a module ${}_RM$ is locally projective if M satisfies the following condition:

For all diagrams

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r] & 0 \\ F \arrow[r] & M \arrow[u, "g"'] & \end{tikzcd}$$

with exact upper row and a finitely generated submodule F of M there is a map $g' \in \mathrm{Hom}_R(M,A)$ such that $g|F = f \circ g'|F$. It is known that every regular module is locally projective.

A preradical σ of R-mod assigns to each module M a submodule $\sigma(M)$ in such a way that every homomorphism $M \longrightarrow N$ induces $\sigma(M) \longrightarrow \sigma(N)$ by restriction. In other words, a preradical is a subfunctor of the identity functor of R-mod. A preradical σ is idempotent if $\sigma\sigma = \sigma$ and is called a radical if $\sigma(M|\sigma(M)) = 0$ for every module M. A preradical σ is called left exact if $\sigma(N) = N \cap \sigma(M)$ for every submodule N of M.

A class $\underline{\underline{C}}$ of modules is called a pretorsion class if it is closed under homomorphic images and direct sums, and is a pretorsion-free class if it is closed under submodules and direct products. There is a bijective correspondence between idempotent preradicals of R-mod and pretorsion classes of R-modules. A pretorsion class is called hereditary if it is closed under submodules. There is a bijective correspondence between left exact preradicals and hereditary pretorsion classes. A pretorsion class (resp. a pretorsion-free class) is called a torsion (resp. a torsion-free) class if it is closed under extensions. A torsion theory for R-mod is a pair $(\underline{\underline{C}},\underline{\underline{F}})$ of classes of R-modules such that $\underline{\underline{C}}$ is a torsion class and $\underline{\underline{F}} = [N \in \text{R-mod}: \text{Hom}_R(M,N) = 0 \text{ for all } M \in \underline{\underline{C}}\}$. Then $\underline{\underline{F}}$ is automatically a torsion free class. There is a bijective correspondence between torsion theories and idempotent radicals. A torsion theory $(\underline{\underline{C}},\underline{\underline{F}})$ is called hereditary if $\underline{\underline{C}}$ is hereditary, and is called stable if $\underline{\underline{C}}$ is closed under injective hulls.

In this thesis we will follow the terminology of B. Stenström [44] regarding "torsion theories".

CHAPTER 1

V-MODULES

A ring R is called a left (right) V-ring if every simple left (right) R-module is injective. Life was given to this class of rings by Kaplansky [19] when he proved that a commutative ring R is regular in the sense of Von Neumann if and only if every simple R-module is injective. Such rings were called V-rings (by C. Faith in [17]) after Villamayor who characterized left V-rings as those in which every proper left ideal is an intersection of maximal left ideals. V-rings have been extensively studied by many authors. The notion of V-rings has been extended to modules by V.S. Ramamurthi in [35], K.R. Fuller in [21] and H. Tominaga in [46]. In this thesis, following H. Tominaga [46], we call a module ${}_RM$ a V-module if every proper submodule of M is an intersection of maximal submodules. Such a module M has also been called "Co-semisimple" by K.R. Fuller in [21]. The connections between regular modules, V-modules and their endomorphism rings are studied by Y. Hirano in [28] and R. Wisbauer in [51]. In [28], known results on V-rings are extended to modules. In this chapter several new characterizations of V-modules are given. We prove among others that a module M is a V-module if and only if every Artinian module is M-injective (Proposition 1.1) extending a similar result for rings by A.K. Gupta and K. Varadarajan [25]. We also prove that a module M is a

V-module if and only if for any essential submodule L of M and for any maximal submodule K of L, $K^* \neq L^*$ (Proposition 1.5) extending a similar result due to Yue Chi Ming for rings [58]. In Proposition 1.14, we show that if ${}_RM$ is a projective module over a commutative ring R then M is a V-module if and only if every simple R-module is M-flat, which extends a well-known result for V-rings by R. Ware in [50].

Now we begin with the following proposition.

Proposition 1.1.: Let ${}_RM$ be a left R-module. Then the following statements are equivalent:

(i) Every simple R-module is M-injective.

(ii) $J(A) = 0$ for every factor module A of M.

(iii) Every proper submodule of M is an intersection of maximal submodules.

(iv) If $K \subseteq M$, $x \in M$, $x \notin K$ there is an R-homomorphism $\Upsilon : M \longrightarrow S$, with S simple, such that $\Upsilon(K) = 0$ and $\Upsilon(x) \neq 0$.

(v) If $K \subseteq M$, $x \in M$, $x \notin K$ there is a maximal submodule L of M with $K \subseteq L$ and $x \notin L$.

(vi) Every cofinitely generated factor module of M is a finite direct sum of simple modules.

(vii) Every cofinitely generated module is M-injective.

(viii) Every Artinian module is M-injective.

(The equivalence of conditions (i) to (vi) is due to K.R. Fuller [21, Proposition 3.1]).

<u>Proof</u>: <u>(i) $\longrightarrow$ (ii)</u>: Let $\eta : M \longrightarrow A$ be an R-epimorphism of R-modules. If $A = 0$, clearly $J(A) = 0$. If $A \neq 0$, let x be any non-zero element of A. By Zorn's lemma choose a submodule B of A maximal with respect to $x \notin B$. Let $-: A \longrightarrow A|B$ denote the quotient map and write $\bar{x} = x + B$. Clearly $R\bar{x}$ is a simple module which is contained in every non-zero submodule of $A|B$. Then by (i), $R\bar{x}$ is M-injective and so $A|B$-injective by [2, Proposition 16.13, p.188]. Therefore $R\bar{x}$ is a direct summand of $A|B$. But since $R\bar{x}$ is an essential submodule of $A|B$, it follows that $A|B = R\bar{x}$. This means that B is a maximal submodule of A with $x \notin B$. whence $x \notin J(A)$, and so $J(A) = 0$.

<u>(ii) $\longrightarrow$ (iii)</u>: Clear.

<u>(iii) $\longrightarrow$ (iv)</u>: Let K be a submodule of M, $x \in M$ and $x \notin K$. Since K is an intersection of maximal submodules of M and $x \notin K$, there exists a maximal submodule L of M with $K \subseteq L$ and $x \notin L$. Let $S = M|L$ and $\Upsilon : M \longrightarrow S$ denote the quotient map. Clearly $\Upsilon(K) \subseteq \Upsilon(L) = 0$ and $\Upsilon(x) = x + L \neq 0$.

<u>(iv) $\longrightarrow$ (v)</u>: Let K be a submodule of M, $x \in M$ and $x \notin K$. By (iv), there exists a simple module S and an R-homomorphism $\Upsilon : M \longrightarrow S$, such that $\Upsilon(K) = 0$ and $\Upsilon(x) \neq 0$. This implies that $\Upsilon \neq 0$ and $L = \ker(\Upsilon)$ is a maximal submodule of M such that $K \subseteq L$ and $x \notin L$.

<u>(v) $\longrightarrow$ (iii)</u>: Let K be a proper submodule of M. By (v), $\forall\, y \notin K$ there exists a maximal submodule L_y of M such that $y \notin L_y$ and $K \subseteq L_y$. Now, it is an easy task to see that $K = \bigcap_{y \notin K} L_y$. Whence every proper submodule of M is an intersection of maximal submodules.

<u>(iii) $\longrightarrow$ (i)</u>: Let S be a simple R-module and f be a non-zero R-homomorphism from a submodule N of M into S. Let $K = \ker(f)$. By (iii), since $K \neq N$, there exists a maximal submodule L of M with $K \subseteq L$ and $N \not\subset L$, it follows that $L \cap N = K$. Thus $M|K = (L+N)|K = L|K \oplus N|K$. If $\tilde{f} : N|K \longrightarrow S$ is the map induced by f in the obvious way, define $\tilde{g} : M|K \longrightarrow S$ by $\tilde{g}|(N|K) = \tilde{f}$ and $\tilde{g}|(L|K) = 0$. Thus the map $g : M \longrightarrow S$, defined by $g(m) = \tilde{g}(m+K)$, $\forall m \in M$, extends f.

<u>(iii) $\longrightarrow$ (vi)</u>: Let $M \xrightarrow{\epsilon} A \longrightarrow 0$ be an exact sequence of left R-modules, with A cofinitely generated. If $N = \operatorname{Ker} \epsilon$, then N is an intersection of maximal submodules. Let $N = \bigcap_{i \in I} L_i$, for some set I, where each L_i is a maximal submodule of M. Since $M|N$ is cofinitely generated and $\bigcap_{i \in I} (L_i|N) = 0$, there exists a finite subset $J \subseteq I$, such that $N = \bigcap_{i \in J} L_i$. Define $\phi : M \longrightarrow \bigoplus_{i \in J} (M|L_i)$ by $\phi(m) = \sum_{i \in J} (m + L_i)$. Clearly $\operatorname{Ker} \phi = N$. Whence A can be embedded in a finite product of simple modules.

<u>(vi) $\longrightarrow$ (i)</u>: Let S be a simple module and $\Upsilon : M \longrightarrow E(S)$ be a non-zero R-homomorphism. Since S is simple, we get $S \subseteq \Upsilon(M) \subseteq E(S)$. Thus $\Upsilon(M)$ is a cofinitely generated homomorphic image of M and hence semisimple by (vi). Since $\operatorname{Soc}(\Upsilon(M)) = S$, it follows that $\Upsilon(M) = S$ and hence S is M-injective (Proposition 3.21 of [25]).

<u>(vii) $\longrightarrow$ (viii)</u>: Clear, since every Artinian module is cofinitely generated.

<u>(viii) $\longrightarrow$ (i)</u>: Clear, since every simple module is Artinian.

(iv) $\longrightarrow$ (vii): Let N be a cofinitely generated module and write $E(N) = E(S_1) \oplus \cdots \oplus E(S_k)$ for a finite set of simple modules S_i, $1 \le i \le k$. Let L be a non-zero submodule of M and $f : L \longrightarrow N$ a non-zero R-homomorphism. We want to show that f can be extended to an R-homomorphism $g : M \longrightarrow N$. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "\text{inc.}"] \arrow[d, "f"'] & M \arrow[ddl, dashed, "\exists g"] \\
& N \arrow[d, "\text{inc.}"'] & \\
& E(N) &
\end{tikzcd}$$

Since E(N) is injective, there exists an R-homomorphism $g : M \longrightarrow E(N)$ such that $g(x) = f(x)$, $\forall x \in L$. For each i, $1 \le i \le n$, denote by $\pi_i : E(N) \longrightarrow E(S_i)$ the projection map, and consider the R-homomorphisms $\pi_i \circ g : M \longrightarrow E(S_i)$. By (i), since every S_i is M-injective, we get

$$\pi_i \circ g(M) \subseteq S_i, \qquad 1 \le i \le n.$$

Whence $g(M) \subseteq S_1 \oplus \cdots \oplus S_n$. But since $\mathrm{Soc}(N) = \mathrm{Soc}(E(N)) = S_1 \oplus \cdots \oplus S_n$ we get $g(M) \subseteq \mathrm{Soc}(N) \subseteq N$. Thus the map $g : M \longrightarrow N$ is the required map. □

A result of K.R. Fuller asserts that the class of V-modules is closed under submodules, homomorphic images and arbitrary direct sums. We include a proof here.

Proposition 1.2 (K.R. Fuller [21]): (i) Submodules and homomorphic images of V-modules are also V-modules.

(ii) $\bigoplus_{i \in I} M_i$ is a V-module if and only if each M_i is a V-module.

Proof: (i) Let M be a V-module and N a non-zero submodule of M. Let S be a simple R-module and f a non-zero R-homomorphism from a submodule K of N into S. We want to show f can be extended to an R-homomorphism $g : N \longrightarrow S$. Since M is a V-module, the map f can be extended to an R-homomorphism $\bar{f} : M \longrightarrow S$. Thus the map $g = (\bar{f}|N) : N \longrightarrow S$ is the required map.

Now, let $M \xrightarrow{\epsilon} A \longrightarrow 0$ be an exact sequence of non-zero R-modules. We want to show that A is a V-module. Let S be a simple R-module and $f : A \longrightarrow E(S)$ be a non-zero R-homomorphism. We must show $f(A) \subseteq S$. But since M is a V-module, the map $f \circ \epsilon : M \longrightarrow E(S)$ has its image in S. Thus $f(\epsilon(M)) = f(A) \subseteq S$.

(ii) Suppose $M = \bigoplus_{i \in I} M_i$ is a V-module. By (i), since every submodule of a V-module is also a V-module, it follows that each M_i is a V-module. Conversely, suppose that each M_i is a V-module. Let S be a simple module and $\Upsilon : M \longrightarrow E(S)$ be a non-zero R-homomorphism. For each $i \in I$, denote by Υ_i to the restriction of the map Υ to M_i. Then $\Upsilon_i(M_i) \subseteq S$, $\forall i \in I$, since S is M_i-injective. Therefore $\Upsilon(M) \subseteq S$, which implies that S is M-injective and hence M is a V-module. □

Proposition 1.3: For any ring R the following statements are equivalent:

(i) R is a left V-ring.

(ii) Every left R-module is a V-module.

(iii) Every cyclic left R-module is a V-module.

Proof: (i) $\longrightarrow$ (ii). Let $M \in R$-mod and S any simple R-module. Then S is injective and hence M-injective.

(ii) $\longrightarrow$ (iii) and (iii) $\longrightarrow$ (i) are trivial. □

Definition 1.4: Let N be a submodule of a module M. A relative complement for N in M is any submodule L of M which is maximal with respect to the property $N \cap L = 0$. Such submodules L always exist, by virtue of Zorn's lemma. And it is easy to see that $N \oplus L$ is essential in M.

Proposition 1.5: (cf. [58, Theorem 3]): The following conditions are equivalent:

(i) M is a V-module.

(ii) If L is either a proper essential submodule or a relative complement of a simple submodule of M, then $L = L^*$. (Here L^* = intersection of maximal submodules of M containing L).

(iii) If K is a maximal submodule of a proper essential submodule L of M, then $K^* \neq L^*$.

Proof: (i) $\longrightarrow$ (ii): Clear, since in a V-module every proper submodule is an intersection of maximal submodules.

(ii) $\longrightarrow$ (iii): Let L be an essential submodule of M and K a maximal submodule of L. If K were essential in L, then K is essential in M and hence $K = K^*$ and $L = L^*$ which implies that $K^* \neq L^*$. Otherwise, suppose that $K \cap N = 0$ for some non-zero submodule N of L. Since K is a maximal submodule of L, $L = K \oplus N$ and N is a simple submodule of M. Let T be a submodule of M, maximal with respect to $K \subseteq T$ and $T \cap N = 0$. Since T is a relative complement of the simple module N, it follows that $T = T^*$. Thus there exists a maximal submodule Q of M such that $T \subseteq Q$ but $L \not\subseteq Q$ (otherwise if $L \subseteq T^*$ then $N \subseteq L \subseteq T^* = T$, a contradiction with $T \cap N = 0$). Therefore $K \subseteq Q$ and $L \not\subseteq Q$, and hence $K^* \neq L^*$.

<u>(iii) $\longrightarrow$ (i)</u>: Let S be a simple module, N a proper essential submodule of M and $f : N \longrightarrow S$ a non-zero homomorphism. If $K = \mathrm{Ker}(f)$ then K is a maximal submodule of N and so $K^* \neq N^*$. Choose a maximal submodule T of M with $K \subseteq T$ and $N \not\subseteq T$. The maximality of T in M and of K in N implies that $M = T + N$ and $T \cap N = K$; hence $\frac{M}{K} = \frac{T}{K} \oplus \frac{N}{K}$. Thus the map f can be extended to a map $g : M \longrightarrow S$ in the obvious way. This proves that S is M-injective. Thus M is a V-module. □

<u>Corollary 1.6</u> (cf. [58, Corollary 3.1]) If M is a regular module, then M is a V-module if and only if given any essential submodule L of M either L is finitely generated or $K = K^*$ for every maximal submodule K of L.

<u>Proof</u>: <u>"only if" part</u>: Obvious.

<u>"If" part</u>: Let S be a simple module, N an essential submodule of M and $f : N \longrightarrow S$ a non-zero homomorphism. If N were finitely generated then from the regularity of M and by [60, Theorem 1.6] it follows that $M = N \oplus T$ for some submodule T of M. Thus f can be extended to a homomorphism $\tilde{f} : M \longrightarrow S$. Otherwise, suppose that $K = K^*$, where $K = \mathrm{Ker}(f)$. Then there is a maximal submodule L of M such that $K \subseteq L$ and $N \not\subseteq L$, from which we infer that the map f may be extended to a homomorphism $\tilde{f} : M \longrightarrow S$. □

Let us recall the definitions of Co-Noetherian and Co-Artinian modules as they were introduced by A.K. Gupta and K. Varadarajan [25].

Definition 1.7: (i) Let $\underline{\underline{C}}_a(R)$ (resp. $\underline{\underline{C}}_n(R)$) denote the class of all Artinian (resp. Noetherian) R-modules. For any R-module M, we set

$$\sigma_a(M) = \cap\{N : N \subseteq M,\ M|N \in \underline{\underline{C}}_a(R)\}$$

$$\sigma_n(M) = \cap\{N : N \subseteq M,\ M|N \in \underline{\underline{C}}_n(R)\}.$$

It is clear that both σ_a and σ_n are radicals and that $\sigma_a(M) \subseteq J(M)$ and $\sigma_n(M) \subseteq J(M)$.

(ii) A left R-module M is said to be Co-Noetherian (Co-Artinian) if $\sigma_a(N) = 0$ (resp. $\sigma_n(N) = 0$) for any factor module N of $M^{(I)}$ (direct sum of I-copies of M), where I is any set.

Proposition 1.8: Every V-module is Co-Noetherain and Co-Artinian.

Proof: Immediate consequence of Proposition 1.1, Proposition 1.2 and the observations $\sigma_a \leq J$ and $\sigma_n \leq J$. □

A result, originally due to Roger Ware [50, Proposition 2.5] asserts that if R is a commutative ring and S is a simple R-module then S is flat if and only if S is injective. In particular a commutative ring R is a V-ring if and only if every simple R-module is flat. Our aim is to extend this result to modules.

Following P.M. Cohn, a submodule K of a left R-module M is called pure if the sequence $0 \longrightarrow E \otimes_R K \longrightarrow E \otimes_R M$ is exact for every right R-module E. Dually, we have the following:

Definition 1.9 [2]: Let U be a right R-module and M be a left R-module. U is said to be flat relative to M (or M-flat) if for every submodule K of M, the sequence $0 \longrightarrow U \otimes_R K \longrightarrow U \otimes_R M$ is exact.

The following is an immediate consequence of (i) $\leftrightarrow$ (vi) of Proposition 0.1.

<u>Proposition 1.10</u>: If M is a projective left R-module, then M is a regular module if and only if every right R-module is M-flat.

<u>Lemma 1.11</u>: Let ${}_RM$ be a projective module and ${}_RU$ any left R-module. Then the following are equivalent:

(i) U is M-injective.

(ii) $\mathrm{Ext}^1_R(M|N,U) = 0$ for every submodule N of M.

<u>Proof</u>: If $0 \longrightarrow N \longrightarrow M \longrightarrow M|N \longrightarrow 0$ is an exact sequence of R-modules, then there is a long exact sequence with natural connecting homomorphisms:

$$0 \longrightarrow \mathrm{Hom}_R(M|N,U) \longrightarrow \mathrm{Hom}_R(M,U) \longrightarrow \mathrm{Hom}_R(N,U) \longrightarrow \mathrm{Ext}^1_R(M|N,U)$$
$$\longrightarrow \mathrm{Ext}^1_R(M,U) \longrightarrow \mathrm{Ext}^1_R(N,U) \longrightarrow \cdots \quad .$$

Since M is projective, $\mathrm{Ext}^1_R(M,U) = 0$, and so U is M-injective if and only if $\mathrm{Ext}^1_R(M|N,U) = 0$, for every submodule N of M. □

<u>Lemma 1.12</u>: Let M_R be a flat right R-module and ${}_RU$ a left R-module. Then the following are equivalent:

(i) U is M-flat.

(ii) $\mathrm{Tor}^R_1(M|N,U) = 0$, for every submodule N of M.

<u>Proof</u>: Given an exact sequence $0 \longrightarrow N \longrightarrow M \longrightarrow M|N \longrightarrow 0$ there exists a long exact sequence:

$$\cdots \longrightarrow \mathrm{Tor}^R_1(M,U) \longrightarrow \mathrm{Tor}^R_1(M|N,U) \longrightarrow N \otimes U \longrightarrow M \otimes U \longrightarrow M|N \otimes U \longrightarrow 0 \ .$$

Since M is flat, $\mathrm{Tor}^R_1(M,U) = 0$, and so U is M-flat if and only if $\mathrm{Tor}^R_1(M|N,U) = 0$ for every submodule N of M. □

The next proposition is an extension of [50, Lemma 2.6] to modules and the proof is patterned after that of Lemma 2.6 of [50].

Proposition 1.13: Let R be a commutative ring, M a projective R-module and S a simple R-module. Then S is M-flat if and only if S is M-injective.

Proof: Let S_i, $i \in I$, be a set of representatives of the distinct isomorphism classes of simple R-modules and set $E = E(\bigoplus_{i\in I} S_i)$. It is easy to see that for any R-module L, $\mathrm{Hom}_R(L,E) = 0$ if and only if $L = 0$. Now if S is any simple submodule of E then $S \cap (\bigoplus_{i\in I} S_i) \neq 0$ since $\bigoplus_{i\in I} S_i$ is an essential submodule of E. Since S is simple and hence cyclic, there exist finitely many indices $i_1,\ldots,i_n$ in I with $S \subset S_{i_1} \oplus \cdots \oplus S_{i_n}$. Let $0 \neq x \in S$. Then $S = Rx$ and $x = x_{i_1} + \cdots + x_{i_n}$ with $x_{i_\mu} \in S_{i_\mu}$ and not all x_{i_μ} zero. Let $\lambda \in \mathrm{Ann}_R(x)$. Then $\lambda x_{i_1} + \cdots + \lambda x_{i_n} = 0$. Hence $\lambda x_{i_\mu} = -\sum_{j\neq\mu} \lambda x_{i_j} \in S_{i_\mu} \cap (\sum_{j\neq\mu} S_{i_j}) = 0$. This means $\lambda x_{i_\mu} = 0$ for $1 \leq \mu \leq n$. Since S is simple, $\mathrm{Ann}_R(x)$ is a maximal ideal in R. Hence either $\mathrm{Ann}_R x_{i_\mu} = R$ or $\mathrm{Ann}_R(x)$. Since $S_{i_1},\ldots,S_{i_n}$ are mutually non-isomorphic, we get $\mathrm{Ann}_R(x) = \mathrm{Ann}_R(x_{i_k})$ for some k and $x_{i_\mu} = 0$ for $\mu \neq k$. Thus $x = x_{i_k}$ and hence $S = S_{i_k}$.

Now let S be any simple R-module and let S_{i_k} be the copy of S in E. Then $\mathrm{Hom}_R(S,S_{i_k}) \subseteq \mathrm{Hom}_R(S,E)$ and if $0 \neq f \in \mathrm{Hom}_R(S,E)$ then

$S \cong \mathrm{Im}(f) \subseteq E$, and so by the above we must have $\mathrm{Im}(f) = S_{i_k}$. Therefore $\mathrm{Hom}_R(S,S_{i_k}) = \mathrm{Hom}_R(S,E)$. Thus we have

$\mathrm{Hom}_R(S,E) = \mathrm{Hom}_R(S,S_{i_k}) \cong \mathrm{Hom}_R(S,S)$ and since R is commutative $\mathrm{Hom}_R(S,S) \cong S$ as R-modules. And since E is injective we have an isomorphism:

$$\mathrm{Ext}^1_R(X, \mathrm{Hom}_R(S,E)) \cong \mathrm{Hom}_R(\mathrm{Tor}^R_1(X,S),E),$$

for any R-module X. Whence for any submodule N of M we have:

$$\begin{aligned}\mathrm{Ext}^1_R(M|N,S) &\cong \mathrm{Ext}^1_R(M|N, \mathrm{Hom}_R(S,E)) \\ &\cong \mathrm{Hom}_R(\mathrm{Tor}^R_1(M|N, S),E).\end{aligned}$$

Now the result follows from Lemma 1.11, Lemma 1.12 and the fact that E is a cogenerator for R-mod. □

Corollary 1.14 (cf. [50, Lemma 2.6]). Suppose R is a commutative ring and S is a simple R-module. Then S is flat if and only if S is injective. □

Proposition 1.15: Let R be a commutative ring and M a projective R-module. Then the following are equivalent:

(i) M is a V-module.

(ii) M is a regular module.

(iii) Every simple R-module is M-flat.

(iv) Every simple homomorphic image of M is injective.

(v) Every simple homomorphic image of M is flat.

(vi) Every simple homomorphic image of M is M-injective.

(vii) Every simple homomorphic image of M is M-flat.

(The equivalence between (i), (ii) and (vi) has been given by Y. Hirano in [28, Theorem 4.8] using different techniques.)

Proof: (i) $\leftrightarrow$ (iii): By Proposition 1.13.

(iv) $\leftrightarrow$ (v): By Corollary 1.14.

(vi) $\leftrightarrow$ (vii): By Proposition 1.13.

(ii) $\rightarrow$ (i): If M is a regular module, then by Proposition 1.10, every R-module is M-flat, and by Proposition 1.13, it follows that every simple R-module is M-injective. Therefore M is a V-module.

(iii) $\rightarrow$ (vii): Obvious.

(iv) $\leftrightarrow$ (ii): By [35, Theorem 4].

(vi) $\rightarrow$ (iv): By [28, Theorem 4.8]. $\square$

CHAPTER 2.

NOETHERIAN V-MODULES

In this chapter we study modules with the property that arbitrary direct sums of simple left R-modules are M-injective. We call them DSI-modules. We prove that a finitely generated left R-module M is a DSI-module if and only if M is a Noetherian V-module, which extends a similar result for rings by B. Sarath and K. Varadarajan in [40] and K.A. Byrd in [8]. We also prove that a finitely generated left R-module M is a DSI-module if and only if every submodule N of M has a minimal generating set and if L is any homomorphic image of M then every minimal generating set of any submodule of L can be extended to a minimal generating set for L, which extends a similar result for rings by B. Sarath in [39].

Definition 2.1 (B. Sarath and K. Varadarajan [40]). A ring R is called a left DSI-ring if every direct sum of simple left R-modules is injective. Such rings were also called "SSI-rings" by K.A. Byrd in [8]. It was proved in [8] and [40] that for a ring R the Following statements are equivalent:

(i) R is a left Noetherian left V-ring.

(ii) Every semisimple left R-module is injective.

In the next proposition we extend this result to modules.

Proposition 2.2: The following conditions are equivalent for a finitely generated R-module M.

(i) M is a Noetherian V-module.

(ii) Every semisimple left R-module is M-injective.

(iii) Every countably generated semisimple left R-module is M-injective.

Proof: (i) $\longrightarrow$ (ii): Since M is Noetherian, it follows from [4, Theorem 2.5] that, any direct sum of M-injective modules is M-injective. And since M is a V-module it follows from Proposition 1.1, that every semisimple left R-module is M-injective.

(ii) $\longrightarrow$ (iii): Obvious.

(iii) $\longrightarrow$ (i): Inasmuch as any simple module is cyclic and semisimple it follows from Proposition 1.1, that M is a V-module. Now, to see that M is Noetherian, let $N_1 \subsetneq N_2 \subsetneq \cdots \subsetneq N_k \subsetneq \cdots$ be an ascending chain of distinct submodules of M. Since M is a V-module, by Proposition 1.1 there are maximal submodules $L_k (k = 1,2,\ldots)$ such that $N_k \subsetneq L_k$ and $N_{k+1} \not\subset L_k$. Let $\pi_k : M \longrightarrow M|L_k$ denotes the quotient map, $k = 1,2,\ldots$. Set $N = \bigcup_{k=1}^{\infty} N_k$ and define the homomorphism $f : N \longrightarrow \bigoplus_{k=1}^{\infty} M|L_k$, by $f(x) = \sum_{k=1}^{\infty} \pi_k(x)$, (note that $\pi_k(x) = 0$ for all but a finite number of the k's). Since $\bigoplus_{k=1}^{\infty} (M|L_k)$ is M-injective, there exists an R-homomorphism $g : M \longrightarrow (\bigoplus_{k=1}^{\infty} M|L_k)$ extending f. But since M is finitely generated, $g(M) \subseteq \bigoplus_{k=1}^{S} M|L_k$, for some positive integer S. Whence the above chain of submodules is finite. $\square$

Let us call a module $_RM$ a DSI-module if every semisimple R-module is M-injective.

With the same argument used in the proof of Proposition 1.2 one obtains the following:

Proposition 2.3: (i) Submodules and homomorphic images of DSI-modules are also DSI-modules.

(ii) $\bigoplus_{i \in I} M_i$ is a DSI-module if and only if every M_i is a DSI-module.

The next proposition can easily be verified.

Proposition 2.4: For any ring R the following are equivalent:

(i) R is a left DSI-ring.

(ii) Every left R-module is a DSI-module.

(iii) Every cyclic left R-module is a DSI-module.

Lemma 2.5: A module M is finitely generated semisimple if and only if M is finite dimensional and every cyclic submodule of M is a direct summand of M.

Proof: (cf. [22, Proposition 1.22]): Clearly finitely generated semisimple modules are finite dimensional. Conversely, when M is finite dimensional it is a finite direct sum of indecomposable modules, hence it suffices to assume that M is indecomposable with all cyclic submodules being direct summands, and then show that M is simple. But this is clear, since under these hypotheses any cyclic submodule of M must be 0 or M. □

Proposition 2.6: Suppose that R is a commutative ring and M is a finitely generated projective R-module. Then M is a DSI-module if and only if M is a finite direct sum of simple R-modules.

Proof: If M is a DSI-module, then by Proposition 2.2, M is a Noetherian V-module and by Proposition 1.15, M is a Noetherian regular module. Now by Proposition 0.1 and Lemma 2.5, M is a finite direct sum of simple modules. □

B. Sarath [39, Theorem 1.6] proved that a ring R is a left Noetherian left V-ring if and only if given any minimal generating set of a submodule N of any module M, it can be extended to a minimal generating set for M. In the next proposition we shall extend this result for modules.

Definition 2.7: Let M be a left R-module and B a subset of M. We say B is "irredundant" if and only if $A \subseteq B$, $\langle A\rangle = \langle B\rangle \Rightarrow A = B$. If B is not irredundant we call it redundant. A subset B of M will be called a "minimal generating set" for M if B is irredundant and $M = \langle B\rangle$.

Lemma 2.8: (i) If $B \subseteq M$ is irredundant and $A \subseteq B$, then A is irredundant.

(ii) If $\{B_\alpha\}_{\alpha\in J}$ is a family of irredundant subsets of M totally ordered by inclusion then $\bigcup_{\alpha\in J} B_\alpha$ is irredundant.

(iii) B is redundant if and only if for some subset $A \subseteq B$, $\langle A\rangle = \langle A\backslash\{a\}\rangle$, for some $a \in A$.

Lemma 2.9: Let B be an irredundant subset of L, $\{L_b\}_{b\in B}$ a collection of maximal submodules of L satisfying $b \notin L_b$ and $\langle B\backslash\{b\}\rangle \subset L_b$. Let $I = \langle B\rangle$, $N = \bigcap_{b\in B} L_b$ and $j : L|N \longrightarrow \prod_{b\in B} L|L_b$ the natural embedding. Then j maps $(I+N)|N$ isomorphically onto $\bigoplus_{b\in B} L|L_b$.

The above two lemmas are due to B. Sarath and the proofs are straightforward, see [39, Remark 1.2 and Lemma 1.3].

<u>Proposition 2.10</u>: The following conditions are equivalent for a finitely generated left R-module M.

(i) M is a DSI-module.

(ii) If K is a submodule of M and L is a homomorphic image of K then given any irredundant generating set of any submodule N of L, it can be extended to an irredundant generating set for L.

(iii) Every submodule N of M has a minimal generating set, and if L is any homomorphic image of M then every minimal generating set of any submodule of L can be extended to a minimal generating set for L.

<u>Proof</u>: <u>(i) $\rightarrow$ (ii)</u>: Adapted from [39, Theorem 2.8]. Let C be an irredundant generating set for a submodule N of L, where L is a homomorphic image of a submodule K of M. Let $E = \{B : C \subseteq B \subseteq L,$ with B irredundant$\}$. E is non-empty, and when partially ordered by inclusion, by Lemma 2.8 (ii) and Zorn's lemma, has a maximal element say B. Suppose $\langle B\rangle \neq L$. Since L is a V-module and B is irredundant, there exist maximal submodules $\{L_b\}_{b\in B}$ of L with $b \notin L_b$ and $\langle B\backslash\{b\}\rangle \subseteq L_b$. Let I,N and j be as in Lemma 2.9. We now consider two cases:

<u>Case 1</u>: $I \supset N$.

By Lemma 2.9, $(I+N)|N$ is isomorphic to $\bigoplus_{b\in B} (L|L_b)$ whence L-injective, since L is a Noetherian V-module and each $L|L_b$ is simple. Therefore $(I+N|N) = I|N$ is a direct summand of $L|N$. Write

$L|N = (I|N) \oplus (I'|N)$ for some non-zero submodule I' of L with $N \subsetneq I'$ (note, $I = \langle B \rangle \neq L$). Then $I \cap I' = N$. Let $u \in I' \backslash N$. Now $B' = B \cup \{u\}$ is irredundant, since $u \notin \langle B \rangle = I$ and $\langle B' \backslash \{b\} \rangle \cap I \subseteq (\langle B \backslash \{b\} \rangle + I') \cap I \subseteq \langle B \backslash \{b\} \rangle + N \subseteq L_b$ and hence $b \notin \langle B' \backslash \{b\} \rangle$ for all $b \in B$. This contradicts the maximality of B, and hence it follows that $\langle B \rangle = L$.

Case 2: $I \not\supseteq N$.

Pick $u \in N$, $u \notin I$. Then $B' = B \cup \{u\}$ is irredundant, since $u \notin \langle B \rangle = I$ and $b \notin \langle B' \backslash \{b\} \rangle \subseteq \langle B \backslash \{b\} \rangle + N \subseteq L_b$, $\forall b \in B$. This contradicts the maximality of B.

(ii) $\longrightarrow$ (iii): Inasmuch as the zero submodule has a minimal generating set, namely $\{0\}$, we infer from the hypothesis that every submodule N of M has a minimal generating set. The rest of the assertion is clear.

(iii) $\longrightarrow$ (i): We first show that M is a V-module. We do this by proving that every cofinitely generated homomorphic image of M is a finite direct sum of simple modules and hence by Proposition 1.1, M is a V-module. Let L be a cofinitely generated homomorphic image of M. Then $S = \mathrm{Soc}(L)$ is finitely generated and essential in L. Write $S = S_1 \oplus \cdots \oplus S_n$, with each S_i simple. We must show $L = S$. Suppose $L \neq S$. Let $0 \neq x_k \in S_k$, $1 \leq k \leq n$. Then $C = \{x_i\}_{i=1}^n$ is an irredundant generating set of S. S is a submodule of L and L is a quotient of M, hence there exists an irredundant generating set D of L with $D \supsetneq C$ (if $D = C$, then $L = S$). Let $x \in D \backslash C$. Since $x \neq 0$ and S is essential in L, there exist $\lambda, \lambda_i \in R$, $1 \leq i \leq n$, with $0 \neq \lambda x = \sum_{i=1}^n \lambda_i x_i$. Hence $\lambda_i x_i \neq 0$ for some i, $1 \leq i \leq n$. Without loss of generality we may assume that $\lambda_n x_n \neq 0$. Since S_n is simple, there exists $\mu \in R$ with $\mu\lambda_n x_n = x_n$, so $\mu(\lambda x - \sum_{i=1}^{n-1} \lambda_i x_i) = \mu\lambda_n x_n = x_n$. Then $x_n \in \langle x_1,\dots,x_{n-1},x\rangle$, but this means that $\{x_1,\dots,x_n,x\}$ is redundant, a contradiction with Lemma 2.8 (i) and the irredundancy of D.

Now to see that M is Noetherian, we will prove that every submodule N of M is finitely generated. But if N is any submodule of M, then by hypotheses N has a minimal generating set say C. Extend C to a minimal generating set D for M. Inasmuch as M is finitely generated and D is irredundant, we infer that D must be finite. Thus C is finite and N is finitely generated. □

Corollary 2.11: For any ring R the following conditions are equivalent:

(i) R is a left Noetherian left V-ring.

(ii) If $I \supseteq J$ are left ideals of R, then every minimal generating set of any R-submodule of the left R-module $I|J$ can be extended to a minimal generating set for $I|J$.

(iii) Every left ideal I of R has a minimal generating set and given any minimal generating set of a submodule N of any cyclic R-module M, it can be extended to a minimal generating set for M.

(iv) Given any minimal generating set of a submodule N of any R-module M, it can be extended to a minimal generating set for M. □

CHAPTER 3.

GENERALIZED V-MODULES

According to V.S. Ramamurthi and K.M. Rangaswamy [36], a ring R is called a Generalized left V-ring (left GV-ring) if every simple singular left R-module is injective. GV-rings were also studied by J.S. Alin and E.P. Armendariz in [1], H. Tominaga in [46], Yue Chi Ming in [57] and many other authors. The following theorem characterizes GV-rings and is due to Ramamurthi and Rangaswamy [36].

Theorem 3.1: For any ring R the following conditions are equivalent:

(i) $Z(R) \cap J(R) = 0$, and every proper essential left ideal of R is an intersection of maximal left ideals.

(ii) R is a left GV-ring.

(iii) The module $J(M)$ vanishes for any left R-module M with $Z(M)$ essential in M.

(iv) If M is any left R-module, then every proper essential submodule of M is an intersection of maximal submodules of M and $Z(M) \cap J(M) = 0$.

In [49], K. Varadarajan has proved that the condition $Z(M) \cap J(M) = 0$ for all $M \in$ R-mod, automatically implies that any proper essential submodule of a module M is an intersection of maximal submodules of M.

In [5], G. Baccella has given an alternative description of GV-rings which involves the socle. It was proved in [5] that for a ring R the following statements are equivalent:

(i) R is a left GV-ring.

(ii) $Soc(_RR) \cap Z(_RR) = 0$, and every proper essential left ideal of R is an intersection of maximal left ideals.

(iii) $Soc(_RR)$ is projective and $R|Soc(_RR)$ is a left V-ring.

In [28], GV-modules were introduced and studied. Following Y. Hirano [28], a module M is called a GV-module if every simple singular left R-module is M-injective. The present chapter is intended to give further contributions to the study of GV-modules. We also study modules with the property that proper essential submodules are intersections of maximal submodules, we call them weakly GV-modules (WGV-modules). We prove that a module M is a WGV-module if and only if $M|Soc(M)$ is a V-module, then using this result we show that the class of weakly GV-modules is closed under taking submodules, factor modules and arbitrary direct sums.

We now begin with the following proposition.

Proposition 3.2: The following conditions are equivalent.

(i) Every simple singular left R-module is M-injective.

(ii) $Z(M) \cap J(M) = 0$ and $J(M|N) = 0$ for any essential submodule N of M.

(iii) Every simple singular submodule of M is a direct summand of M and $J(M|N) = 0$ for any essential submodule N of M.

(iv) Every singular cofinitely generated R-module is M-injective.

(v) Every singular Artinian module is M-injective.

<u>Proof</u>: (The equivalence between (i), (ii) and (iii) is due to Y. Hirano [28, Theorem 3.15].)

<u>(i) $\longrightarrow$ (ii)</u>: Let N be an essential submodule of M. Want to show that $J(M|N) = 0$. Suppose not, and let $\tilde{x} = x + N$ be a non-zero element of $J(M|N)$. Let $\mathcal{F} = \{K \subseteq M$: K is a submodule of M with $x \notin K$ and $N \subseteq K\}$. $\mathcal{F}$ is non-empty and when partially ordered by inclusion, it is easy to see that every totally ordered subset of $\mathcal{F}$ has an upper bound, and so by Zorn's lemma $\mathcal{F}$ has a maximal element T. Clearly T is essential in M with $x \notin T$. Let $\bar{x} = x + T \in M|T$. Then $R\bar{x}$ is a simple singular essential submodule of $M|T$. By assumption $R\bar{x}$ is M-injective and by [2, Proposition 16.13, p.188], it follows that $R\bar{x}$ is $M|T$-injective. Hence $R\bar{x}$ is a direct summand of $M|T$. But since $R\bar{x}$ is essential in $M|T$, it follows that $R\bar{x} = M|T$ which implies that T is a maximal submodule of M with $x \notin T$ and $N \subseteq T$. This is a contradiction to the fact that $\tilde{x} \in J(M|N)$ = the intersection of all maximal submodules of M containing N.

Now suppose on the contrary, $Z(M) \cap J(M)$ contains a non-zero element x. Then by Zorn's lemma, there is a submodule Y of M which is maximal among the submodules X of M with $x \notin X$. Write $\bar{x} = x + Y \in M|Y$. Then $R\bar{x}$ is a simple singular submodule of $M|Y$, and so the map $\bar{\ }: Rx \longrightarrow R\bar{x}$ can be extended to an R-homomorphism $g : M \longrightarrow R\bar{x}$. Therefore $K = \mathrm{Ker}(g)$ is a maximal submodule of M with $x \notin K$, a contradiction with the fact that $x \in J(M)$.

<u>(ii) $\longrightarrow$ (iii)</u>: Let S be a simple singular submodule of M. Since $Z(M) \cap J(M) = 0$, there is a maximal submodule L of M such that $S \cap L = 0$. Then clearly $M = S \oplus L$.

<u>(iii) $\longrightarrow$ (i)</u>: Let S be a simple singular module, N an essential submodule of M and $f : N \longrightarrow S$ any non-zero R-homomorphism. We want to show that f can be extended to a map $g : M \longrightarrow S$. Let $K = \mathrm{Ker}(f)$. If K is essential in N then K is essential in M and so there exists a maximal submodule L of M with $K \subseteq L$ and $N \not\subset L$. Since K is a maximal submodule of N it follows that $K = N \cap L$. And since $N \not\subset L$ and L is a maximal submodule of M it follows that $M = N + L$ and so $\frac{M}{K} = \frac{N + L}{K} = \frac{N}{K} \oplus \frac{L}{K}$. Now if $\tilde{f} : N|K \longrightarrow S$, is the map induced by f in the obvious way, then clearly $\tilde{f}$ can be extended to an R-homomorphism $\tilde{g} : M|K \longrightarrow S$. And if we define $g : M \longrightarrow S$, by $g(m) = \tilde{g}(m+K)$ for every $m \in M$, then clearly g is an extension of f.

Now suppose $K \cap I = 0$ for some non-zero submodule I of N. Thus $I(\cong S)$ is a simple singular submodule of M, and by hypothesis we see that $M = I \oplus L$ for some submodule L of M. Then f can be extended to an R-homomorphism of M to S.

<u>(i) $\longrightarrow$ (iv)</u>: Let N be a singular cofinitely generated left R-module. Write $N \subseteq E(N) = E(S_1) \oplus \cdots \oplus E(S_n)$, for a finite family of simple R-modules S_j, $1 \leq j \leq n$. Since $\mathrm{Soc}(N) = \mathrm{Soc}(E(N)) = S_1 \oplus \cdots \oplus S_n \subseteq N$, it follows from the singularity of N that each S_i is a simple singular module and hence M-injective. Let $f : K \longrightarrow N$ be a non-zero R-homomorphism, where $K \neq 0$ is a submodule of M, and consider the

following diagram:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "\text{inc.}"] \arrow[d, "f"'] & M \arrow[ddl, dashed, "\exists g."] \\
& N \arrow[d] & \\
& E(N) &
\end{tikzcd}$$

Since $E(N)$ is injective, there exists a map $g : M \longrightarrow E(N)$ such that $g(x) = f(x)$, $\forall x \in K$. If $\pi_i : E(N) \longrightarrow E(S_i)$ denotes the projection map, $1 \leq i \leq n$, then $\pi_i \circ g: M \longrightarrow E(S_i)$ is an R-homomorphism, and so $\pi_i \circ g(M) \subseteq S_i$ by the M-injectivity of S_i. Thus $g(M) \subseteq S_1 \oplus \cdots \oplus S_n \subseteq N$ and the map g is the required map.

<u>(iv) $\longrightarrow$ (v)</u>: Obvious, since every Artinian module is cofinitely generated.

<u>(v) $\longrightarrow$ (i)</u>: Clear, since every simple module is Artinian. □

<u>Remarks 3.3</u>: From Proposition 3.2 (ii), it follows that if M is a GV-module then $J(M) \subseteq Soc(M)$. This is because $Soc(M)$ is the intersection of all essential submodules of M and every proper essential submodule of M is an intersection of maximal submodules. And since $J(M) \cap Z(M) = 0$, it follows that $J(M)$ is a direct sum of simple projective modules.

With the same argument used in the proof of Proposition 1.2 one can easily prove the following:

<u>Proposition 3.4</u>: (i) Submodules and homomorphic images of GV-modules are also GV-modules.

(ii) $\bigoplus_{i\in I} M_i$ is a GV-module if and only if every M_i is a GV-module. □

Proposition 3.5: For any ring R the following statements are equivalent:

(i) R is a left GV-ring.

(ii) Every left R-module is a GV-module.

(iii) Every cyclic left R-module is a GV-module. □

The next proposition is an extension of [49, Theorem 4.2(2)] to modules.

Proposition 3.6: For any module ${}_RM$ the following are equivalent:

(i) M is a GV-module.

(ii) $Z(L) \cap J(L) = 0$, for every homomorphic image L of M.

Proof: (i) $\rightarrow$ (ii): If M is a GV-module then by Proposition 3.3(ii), every homomorphic image L of M is a GV-module. Hence $Z(L) \cap J(L) = 0$.

(ii) $\rightarrow$ (i): Let N be a proper essential submodule of M. Since $M|N$ is singular, it follows that $Z(M|N) = M|N$ and so by assumption $J(M|N) = 0$. Whence by Proposition 3.2, M is a GV-module. □

In [21, Theorem 3.1], K.R. Fuller proved that a module M is a V-module if and only if every cofinitely generated factor module of M is a finite direct sum of simple modules (see Proposition 1.1(vi)). For GV-modules we have the following:

Proposition 3.7: Suppose M is a GV-module. Then every singular cofinitely generated factor module of M is a finite direct sum of simple modules.

Proof: Let $M \xrightarrow{\delta} A \longrightarrow 0$ be an exact sequence of R-modules with A cofinitely generated and $Z(A) = A$. By Proposition 3.3, A is a GV-module and hence $Z(A) \cap J(A) = 0$. But since $Z(A) = A$, $J(A) = 0$. If $N = \mathrm{Ker}(\delta)$, then N is an intersection of maximal submodules of M. Write $N = \bigcap_{i \in I} L_i$, for some set I, where each L_i is a maximal submodule of M. Now, since $M|N$ is cofinitely generated and $\bigcap_{i \in I} (L_i|N) = 0$, there exists a finite subset $J \subseteq I$, such that $N = \bigcap_{i \in J} L_i$ and hence A can be embedded in a finite direct sum of simple modules. □

We do not know whether the converse to Proposition 3.7 holds. However, for non-singular rings we have the following:

Proposition 3.8: Suppose R is a left non-singular ring. Then the following conditions are equivalent:

(i) ${}_RM$ is a GV-module.

(ii) Every singular cofinitely generated homomorphic image of M is a finite direct sum of simple modules.

Proof: (i) $\longrightarrow$ (ii): Follows from Proposition 3.7.

(ii) $\longrightarrow$ (i): Let S be a simple singular module and $f : M \longrightarrow E(S)$ be a non-zero R-homomorphism. Since S is simple, we get $S \subseteq f(M) \subseteq E(S)$. Since R is non-singular, f(M) is singular. Whence f(M) is a cofinitely generated singular homomorphic image of M and so by (ii), f(M) is semisimple. But since $\mathrm{Soc}(E(S)) = S$, $f(M) = S$; and S is M-injective. □

As immediate corollaries to Proposition 3.2 and Proposition 3.8, we have the following.

Corollary 3.9: For any ring R the following statements are equivalent:

(i) R is a left GV-ring.

(ii) Every singular cofinitely generated R-module is injective.

(iii) Every singular Artinian module is injective.

Corollary 3.10: If R is a left non-singular ring then R is a left GV-ring if and only if every singular cofinitely generated R-module is a finite direct sum of simple modules.

Proof: Let L be a singular cofinitely generated module.
Write $L \subseteq E(L) = E(S_1) \oplus \cdots \oplus E(S_n)$, with each S_i being simple. If every simple singular module is injective then $S_i = E(S_i)$, for each i, and hence $L = E(L) = S_1 \oplus \cdots \oplus S_n$.
Conversely, let S be a simple singular module. Since R is non-singular, E(S) is a singular cofinitely generated module and hence semisimple by assumption. Thus S = E(S) and S is injective. Therefore R is a left GV-ring. □

As we have mentioned at the beginning of this chapter, G. Baccella [5], has given an alternative description of GV-rings in terms of the socle. For locally projective modules we have the following proposition which corresponds to [5, Theorem 2.2].

Proposition 3.11: If M is a locally projective module. Then the following are equivalent:

(i) M is a GV-module.

(ii) $Soc(M) \cap Z(M) = 0$ and every proper essential submodule of M is an intersection of maximal submodules.

Proof: (i) $\longrightarrow$ (ii): We claim that every simple submodule of M is projective. For if S were a simple singular submodule of M then by Proposition 3.2, S is direct summand of M. But since M is locally projective then clearly S is projective. Contradicting the singular nature of S. Whence Soc(M) is projective and so Soc(M) $\cap$ Z(M) = 0. The rest of the assertion is clear.

(ii) $\longrightarrow$ (i): Note that Soc(M) is the intersection of all the essential submodules of M. And since every proper essential submodule of M is an intersection of maximal submodules, it follows that J(M) $\subseteq$ Soc(M), and hence J(M) $\cap$ Z(M) = 0, and by Proposition 3.2, M is a GV-module. $\square$

Remark 3.12: If M is a locally projective GV-module then Soc(M) is projective.

It was proved in [36, Proposition 3.7] that a ring R is left V-ring if and only if R is a left GV-ring and every minimal left ideal of R is an absolute summand of R. In the next proposition we extend this result to modules.

Definition 3.13: Let M be a left R-module. A submodule L of M will be called an absolute summand if for any submodule T of M, such that T is maximal with respect to L $\cap$ T = 0, we have L $\oplus$ T = M.

Proposition 3.14: The following conditions are equivalent:

(i) M is a V-module.

(ii) M is a GV-module, and every simple submodule of M is an absolute summand.

Proof: (i) $\longrightarrow$ (ii): Let M be a V-module and let S be a simple submodule of M. Let T be a submodule of M maximal with respect to $S \cap T = 0$. If $A = S \oplus T$ and $\pi : A \longrightarrow S$ denotes the projection map, then π can be extended to a map $\tilde{\pi} : M \longrightarrow S$. Since $\tilde{\pi}|S = \pi$, $\mathrm{Ker}(\tilde{\pi}) \cap S = 0$, and since $\pi(T) = 0$, $T \subseteq \mathrm{Ker}(\tilde{\pi})$. Thus by the choice of T it follows that $T = \mathrm{Ker}(\tilde{\pi})$. Whence T is a maximal submodule of M and therefore $M = T \oplus S$.

(ii) $\longrightarrow$ (i): Let M be a GV-module and assume that every simple submodule of M is an absolute summand. Let S be a simple module. If S is singular then it is M-injective. Suppose S is a simple projective module and $f : N \longrightarrow S$ a non-zero R-homomorphism, where N is a submodule of M. Let $K = \mathrm{Ker}(f)$. By the projectivity of S, the following exact sequence $0 \longrightarrow K \longrightarrow N \longrightarrow S \longrightarrow 0$ splits. Write $N = K \oplus L$, for some submodule $L(\cong N|K \cong S)$ of N. Inasmuch as L is a simple submodule of M we infer that if T is a submodule of M containing K and maximal with respect to $T \cap L = 0$, then $L \oplus T = M$.

Now if $g : M \longrightarrow L$ denotes the projection map then the map $f \circ g : M \longrightarrow S$ extends f, and hence every simple module is M-injective. By Proposition 1.1, M is a V-module. □

The following proposition is an extension of [58, Theorem 3'] to modules.

Proposition 3.15: The following conditions on a left R-module M are equivalent:

(i) M is a GV-module.

(ii) If K is a submodule of any essential submodule L of M such that $L|K$ is simple singular then $K^* \neq L^*$.

Proof: (i) $\longrightarrow$ (ii): Let L be an essential submodule of M and K be a submodule of L such that $L|K$ is simple singular. If K is essential in L, then K is essential in M and hence $K = K^*$ and $L = L^*$ which implies that $K^* \neq L^*$ (since K is a maximal submodule of L). Otherwise, let $K \cap N = 0$ for some non-zero submodule N of L. Since K is a maximal submodule of L, $L = K \oplus N$, and N is a simple singular submodule of M. Let $\eta : L \longrightarrow N$ denotes the projection map. Since M is a GV-module, η can be extended to a map $g : M \longrightarrow N$. Inasmuch as $g(x) = x$, $\forall x \in N$, we infer that the submodule N is a direct summand of M, in fact $M = N \oplus \mathrm{Ker}(g)$. Now, since $g(k) = \eta(k) = 0$ $\forall k \in K$, it follows that $K \subseteq \mathrm{Ker}(g)$. Let $T = \mathrm{Ker}(g)$, then T is a maximal submodule of M with $K \subseteq T$ and $L \nsubseteq T$. Thus $K^* \neq L^*$.

(ii) $\longrightarrow$ (i): Let S be a simple singular R-module, N an essential submodule of M and $f : N \longrightarrow S$ a non-zero R-homomorphism. If $K = \mathrm{Ker}(f)$, then $N|K$ is a simple singular module, and thus by hypothesis $N^* \neq K^*$. Choose a maximal submodule L of M with $K \subseteq L$ and $N \not\subset L$. By the maximality of L, we have $M = L + N$, and since K is maximal in N, it follows that $K = L \cap N$ and so $\frac{M}{K} = \frac{L}{K} \oplus \frac{N}{K}$. Thus the map f can be extended to a map $g : M \longrightarrow S$ in the obvious way. □

Proposition 3.16: Let R be a commutative ring and ${}_RM$ a projective module. Then the following are equivalent:

(i) M is a regular module.

(ii) M is a V-module.

(iii) M is a GV-module.

Proof: (i) $\leftrightarrow$ (ii): By Proposition 1.15.

(ii) $\rightarrow$ (iii): Clear.

(iii) $\rightarrow$ (ii): Let S be a simple R-module. If S is singular then it is M-injective. If S is projective then it is M-flat and by Proposition 1.13, S is M-injective. Thus every simple module is M-injective and hence M is a V-module. □

The following proposition has been proved in [5, Proposition 2.1]. However we shall reprove it here because of the important role it plays in what follows.

Proposition 3.17 (G. Baccella): For any ring R the following conditions are equivalent:

(i) $R|\mathrm{Soc}_R R$ is a left V-ring.

(ii) If M is a left R-module, then every essential submodule of M is an intersection of maximal submodules.

Proof: (i) $\rightarrow$ (ii): Let M be a left R-module and L an essential submodule of M. Since $(\mathrm{Soc}_R R)M \subseteq \mathrm{Soc}(M)$, $(\mathrm{Soc}_R R)M \subseteq L$. If $R|\mathrm{Soc}_R R$ is a left V-ring, then $L|(\mathrm{Soc}_R R)M$, as left $(R|\mathrm{Soc}_R R)$-submodule of $M|(\mathrm{Soc}_R R)M$, is an intersection of maximal $R|(\mathrm{Soc}_R R)$-submodules of $M|(\mathrm{Soc}_R R)M$. This is enough to conclude that L is an intersection of maximal submodules of M.

(ii) $\rightarrow$ (i): Let S be a simple left $R|(\mathrm{Soc}_R R)$-module, let $\underline{a}$ be a left ideal of R, with $(\mathrm{Soc}_R R) \subseteq \underline{a}$ and $\underline{a}|(\mathrm{Soc}_R R)$ essential in $R|(\mathrm{Soc}_R R)$, and let $f : \underline{a}|(\mathrm{Soc}_R R) \longrightarrow S$ be a non-zero $(R|\mathrm{Soc}_R R)$-homomorphism. We claim that if $\pi : \underline{a} \longrightarrow \underline{a}|(\mathrm{Soc}_R R)$ is the canonical epimorphism and $\underline{b} = \mathrm{Ker}(f \circ \pi)$, then $\underline{b}$ is essential in $\underline{a}$. If not, then by the definition of $\underline{b}$, there is a minimal left ideal $\underline{n}$ of R such that $\underline{a} = \underline{b} \oplus \underline{n}$, in contradiction with the fact that $\mathrm{Soc}_R R \subseteq \underline{b}$. Inasmuch as $\underline{a}|(\mathrm{Soc}_R R)$ is essential in $R|(\mathrm{Soc}_R R)$, $\underline{a}$ is essential in R and hence $\underline{b}$ is essential in R. Since $\underline{b} \neq \underline{a}$, from (ii) it follows that there is a maximal left ideal $\underline{m}$ of R such that $\underline{b} \subseteq \underline{m}$ and $\underline{a} \not\subseteq \underline{m}$. $\underline{b}$ being maximal in $\underline{a}$, we have that $\underline{b} = \underline{a} \cap \underline{m}$. It follows that $\underline{a} + \underline{m} = R$ and hence $\frac{R}{\underline{b}} = \frac{\underline{a}}{\underline{b}} \oplus \frac{\underline{m}}{\underline{b}}$.

$f \circ \pi : \underline{a} \longrightarrow S$ is zero on $\underline{b}$, hence induces $\bar{f} : \frac{\underline{a}}{\underline{b}} \longrightarrow S$. Then $\bar{g} : \frac{R}{\underline{b}} \longrightarrow S$ given by $\bar{g}|\left[\frac{\underline{a}}{\underline{b}}\right] = \bar{f}$, $\bar{g}|\left[\frac{\underline{m}}{\underline{b}}\right] = 0$ extends $\bar{f}$. If $\eta : \frac{R}{(\mathrm{Soc}_R R)} \longrightarrow \frac{R}{\underline{b}}$ is the quotient map then $g = \bar{g} \circ \eta$ extends f, from $R|(\mathrm{Soc}_R R)$ to S. This shows that S is injective as a left $R|(\mathrm{Soc}_R R)$-module. □

As a result of the preceding proposition, we are now in a position to introduce the notion of weakly GV-modules.

Definition 3.18: A module $_RM$ is called a weakly GV-module (WGV-module) if every proper essential submodule of M is an intersection of maximal submodules.

R is said to be a left WGV-ring if the left R-module $_RR$ is a WGV-module.

Clearly every GV-module is a WGV-module. The next result is an extension of Proposition 3.17 to modules.

Proposition 3.19: For a module ${}_RM$ the following are equivalent.

(i) M is a WGV-module.

(ii) $M|Soc(M)$ is a V-module.

Proof: (ii) $\rightarrow$ (i): If L is a proper essential submodule of M then $Soc(M) \subseteq L$; whence $L|Soc(M)$, as a submodule of $M|Soc(M)$, is an intersection of maximal submodules of $M|Soc(M)$, and so L is an intersection of maximal submodules of M.

(i) $\rightarrow$ (ii): Let S be a simple R-module. We want to show that S is $M|Soc(M)$-injective. Let $N|Soc(M)$ be an essential submodule of $M|Soc(M)$ and $f : N|Soc(M) \longrightarrow S$ be any non-zero R-homomorphism. If $Ker(f) = K|Soc(M)$, then K is a maximal submodule of N. We claim that K is an essential submodule of N. For if not, then $K \cap I = 0$ for some non-zero submodule I of N. Whence $N = K \oplus I$ and I is a simple submodule of M, i.e. $I \subseteq Soc(M) \subseteq K$ – a clear contradiction.

Now, since K is a proper essential submodule of M and a maximal submodule of N, by (i) there exists a maximal submodule L of M, such that $K \subseteq L$ and $N \not\subseteq L$. If $\bar{\ }: M \longrightarrow M|Soc(M)$ denotes the quotient map, then $\dfrac{\bar{M}}{\bar{K}} = \dfrac{\bar{N} + \bar{L}}{\bar{K}} = \dfrac{\bar{N}}{\bar{K}} \oplus \dfrac{\bar{L}}{\bar{K}}$. And if $\tilde{f} : \bar{N}|\bar{K} \longrightarrow S$ is the map induced by f in the obvious way, then clearly $\tilde{f}$ can be extended to an R-homomorphism $\tilde{g} : \bar{M}|\bar{K} \longrightarrow S$. And if we define $g : \bar{M} \longrightarrow S$, by $g(\bar{m}) = \tilde{g}(\bar{m} + \bar{K})$ for every $m \in M$, then clearly g is an R-homomorphism which extends f. □

Corollary 3.20: For any ring R the following are equivalent:

(i) $R|(Soc_R R)$ is a left V-ring.

(ii) R is a left WGV-ring.

(iii) Every left R-module is a WGV-module.

(iv) Every cyclic left R-module is a WGV-module. □

In the next proposition we show that the class of WGV-modules is closed under taking submodules, factor modules and arbitrary direct sums – a fact that is hardly obvious from the definition of WGV-modules.

<u>Proposition 3.21</u>: (i) Submodules and homomorphic images of WGV-modules are also WGV-modules.

(ii) $\bigoplus_{i\in I} M_i$ is a WGV-module if and only if each M_i is a WGV-module.

<u>Proof</u>: (i) Let M be a WGV-module and N be a submodule of M. Since $\mathrm{Soc}(N) = N \cap \mathrm{Soc}(M)$ it follows that $N|\mathrm{Soc}(N) = N|(N \cap \mathrm{Soc}(M))$ $\cong (N + \mathrm{Soc}(M))|\mathrm{Soc}(M)$ and since the latter is a submodule of the V-module $M|\mathrm{Soc}(M)$, it follows from Proposition 1.2(i) that $N|\mathrm{Soc}(N)$ is a V-module and by Proposition 3.19 that N is a WGV-module.

Now, let $M \xrightarrow{\epsilon} A \longrightarrow 0$ be an exact sequence of left R-modules, with M a WGV-module. Then $A \cong M|N$ for some submodule N of M. If $L|N$ is a proper essential submodule of $M|N$, then L is a proper essential submodule of M, and so L is an intersection of maximal submodules of M. Whence $L|N$ is an intersection of maximal submodules of $M|N$.

(ii) Let $M = \bigoplus_{i\in I} M_i$. If M is a weakly GV-module then by (i), each M_i is a weakly GV-module. Conversely, suppose each M_i is a WGV-module. Then $M|\mathrm{Soc}(M) = (\bigoplus_{i\in I} M_i)|\mathrm{Soc}(\bigoplus_{i\in I} M_i) = (\bigoplus_{i\in I} M_i)|(\bigoplus_{i\in I} \mathrm{Soc}\, M_i)$ and since the latter is isomorphic to $\bigoplus_{i\in I} (M_i|\mathrm{Soc}\, M_i)$, it follows from Proposition 1.2(ii) that $M|\mathrm{Soc}(M)$ is a V-module and hence M is a WGV-module. □

In the next proposition we give a necessary and sufficient condition for a WGV-module to be a V-module.

<u>Proposition 3.22</u>: For a module ${}_R M$ the following are equivalent:

(i) M is a V-module.

(ii) M is a WGV-module and every simple submodule of M is M-injective.

<u>Proof</u>: <u>(i) $\longrightarrow$ (ii)</u>: Clear.

<u>(ii) $\longrightarrow$ (i)</u>: Let S be a simple R-module and let $f : N \longrightarrow S$ be any non-zero R-homomorphism where N is any proper essential submodule of M. Let $K = \mathrm{Ker}(f)$. If K were not essential in N then $K \cap I = 0$ for some non-zero submodule I of N. Then $f|I : I \cong S$, and I is a simple submodule of M. By (ii), it follows that S is M-injective. If K is essential in N then K is a proper essential submodule of M, and since M is a WGV-module, there is a maximal submodule L of M such that $K \subseteq L$ and $N \not\subset L$. Hence $M|K = (L|K) \oplus (N|K)$ and the map f can be extended to an R-homomorphism $g : M \longrightarrow S$. Whence every simple R-module is M-injective. □

As we have pointed out before, G. Baccella has charactrized GV-rings in terms of the socle. It was proven, among other things, that a ring R is a left GV-ring if and only if $\mathrm{Soc}_R R$ is projective and $R|(\mathrm{Soc}_R R)$ is a left V-ring - see [5, Theorem 2.2]. In the next proposition we extend this result to modules and the proof follows from Proposition 3.11 and Proposition 3.19.

Proposition 3.23: For a locally projective R-module M the following are equivalent:

(i) M is a GV-module.

(ii) Soc(M) is projective and M|Soc(M) is a V-module. □

Example 3.24: The following is an example of a WGV-module which is not a GV-module.

Let $M = Z_{p^2}$. Then $J(Z_{p^2}) = Z_p$ and $Z(Z_{p^2}) = Z_{p^2}$. Thus $J(Z_{p^2}) \cap Z(Z_{p^2}) = Z_p \neq 0$ and hence Z_{p^2} is not a GV-module. But $\mathrm{Soc}(Z_{p^2}) = Z_p$, $Z_{p^2}|\mathrm{Soc}(Z_{p^2}) \cong Z_p$ whence M is a WGV-module. In fact the same example shows that the class of GV-modules is not closed under extensions.

CHAPTER 4

P-V-MODULES AND P-V'-MODULES

A module M is said to be P-injective if for any principal left ideal I of R and $f \in \mathrm{Hom}_R(I,M)$ there exists an element $m \in M$ such that $f(x) = xm$, for all $x \in I$. Equivalently M is P-injective if $\mathrm{Ext}_R^1(R|xR,M) = 0$ for each $x \in R$. A ring R is defined to be a left P-V-ring (resp. P-V'-ring) if every simple (resp. simple singular) left R-module is P-injective. Such rings were introduced and studied by H. Tominaga in [46]; and by Yue Chi Ming in [55], [56], [57] and [58].

In [28], Y. Hirano has introduced the notion of P-V-modules. In this chapter we introduce the notions of P-V'-modules, f-V-modules and f-V'-modules. Known results for P-V-rings (resp. P-V'-ring) are extended to modules. The connections between regular modules, V-modules, P-V-modules and P-V'-modules are given. We also introduce the notion of P-M-flatness and prove that if ${}_RM$ is a projective module over a commutative ring R, then M is a P-V-module if and only if M is a P-V'-module if and only if every simple R-module is P-M-flat; from which we infer that M is a P-V'-module if and only if M is a V-module.

Definition 4.1 [28]: Let M and U be R-modules. U is said to be P-M-injective if every R-homomorphism of any cyclic submodule of M into U can be extended to an R-homomorphism of M into U. U is said to be P-injective if it is P-R-injective.

<u>Definition 4.2</u>: Let M and U be R-modules. U is said to be f-M-injective if every R-homomorphism of any finitely generated submodule of M into U can be extended to an R-homomorphism of M into U. U is said to be f-injective if it is f-R-injective.

<u>Definition 4.3</u> [28]: Let M be a left R-module. If every simple R-module is P-M-injective, M is called a P-V-module.

<u>Definition 4.4</u>: A module ${}_RM$ is called a P-V'-module if every simple singular R-module is P-M-injective.

<u>Definition 4.5</u>: A module M is called an f-V-module (resp. f-V'-module) if every simple (resp. simple singular) module is f-M-injective.

<u>Proposition 4.6</u>: The following conditions are equivalent for a locally projective R-module M.

(i) Every cyclic submodule of M is projective.

(ii) Every quotient of a P-M-injective module is P-M-injective.

(iii) Every quotient of an injective module is P-M-injective.

<u>Proof</u>: <u>(i) $\rightarrow$ (ii)</u>: Let $K \xrightarrow{\epsilon} L \longrightarrow 0$ be an exact sequence of left R-modules with K being P-M-injective. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "\text{inc.}"] \arrow[d, "f"] \arrow[dl, dashed, "\exists\, g"'] & M \\
K \arrow[r, "\epsilon"] & L \arrow[r] & 0
\end{tikzcd}$$

with exact rows and a cyclic submodule N of M. Since N is projective, there exists a map $g : N \longrightarrow K$ such that $\epsilon \circ g = f$. Now since K is P-M-injective, the map g can be extended to a map $\tilde{g} : M \longrightarrow K$. Now the map $\epsilon \circ \tilde{g} : M \longrightarrow L$ is an extension of f.

<u>(ii) $\longrightarrow$ (iii)</u>: Clear.

<u>(iii) $\longrightarrow$ (i)</u>: Let N be a cyclic submodule of M and consider the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, "f"] & M \arrow[dl, dashed, "\exists\, g"] \\ B \arrow[r, "\epsilon"'] & A \arrow[r] & 0 \end{tikzcd}$$

with exact rows and with B being injective. Since A is P-M-injective, the map f can be extended to a map $g : M \longrightarrow A$. And since M is locally projective there exists a map $h : M \longrightarrow B$ such that $(\epsilon \circ h)|N = g|N$. If we set $\tilde{h} = h|N$, then $\epsilon \circ \tilde{h} = f$. By [10, Proposition 5.1, Chap I], it follows that N is projective. □

It was proved in [55] that a ring R is regular if and only if every R-module is P-injective. The following proposition is an extension of this result to modules.

<u>Proposition 4.7</u> (cf [56,Lemma 2]): The following statements are equivalent for any projective R-module ${}_RM$.

(i) M is a regular module.

(ii) Every R-module is P-M-injective.

(iii) Every cyclic R-module is P-M-injective.

(iv) Every cyclic module L with $J(L) = 0$ is P-M-injective.

<u>Proof</u>: <u>(i) $\longrightarrow$ (ii)</u>: If M is a regular module then by Proposition 0.1, every cyclic submodule of M is a direct summand of M, therefore any R-homomorphism of any cyclic submodule of M into a module U can be extended to an R-homomorphism of M into U. Thus every R-module U is

P-M-injective.

<u>(ii) $\longrightarrow$ (iii)</u>: Obvious.

<u>(iii) $\longrightarrow$ (iv)</u>: Obvious.

<u>(iv) $\longrightarrow$ (i)</u> : Note that by hypothesis, every simple module is P-M-injective. We show that if L is any cyclic submodule of M then $J(L) = 0$. Then it will follow that every cyclic submodule of M is a direct summand of M and by Proposition 0.1, M would be a regular module.

Now, let $0 \neq b \in L$, and let $\mathcal{F}$ be the set of all submodules K of Rb such that $b \notin K$. Clearly $\mathcal{F}$ is non-empty and when partially ordered by inclusion it is easy to see that every chain of elements of $\mathcal{F}$ has an upper bound. By Zorn's lemma, $\mathcal{F}$ has a maximal member T. Then $Rb|T$ is a simple module, hence $J(Rb|T) = 0$ and therefore $Rb|T$ is P-M-injective. Hence the quotient map $\eta : Rb \longrightarrow Rb|T$ can be extended to an R-homomorphism $\tilde{\eta} : M \longrightarrow Rb|T$. Let $\phi = \tilde{\eta}|L$. Then $\phi : L \longrightarrow Rb|T$ is an onto map and hence $L|\mathrm{Ker}\ \phi$ is a simple module. Thus $\mathrm{Ker}\ \phi$ is a maximal submodule of L with $b \notin \mathrm{Ker}\ \phi$, and so $J(L) = 0$. □

<u>Proposition 4.8</u>: If M is a P-V-module (resp. a P-V'-module) then every submodule of M is a P-V-module (resp. a P-V'-module).

<u>Proof</u>: Let N be a submodule of M. We want to show that every simple (resp. simple singular) module is P-N-injective. Let S be a simple (resp. simple singular) module, Rm a cyclic submodule of N and $f : Rm \longrightarrow S$ a non-zero homomorphism. Since M is a P-V-module (resp. a P-V'-module), f can be extended to a map $g : M \longrightarrow S$. Then the map $\tilde{f} = (g|N)$ extends f from N into S. □

Proposition 4.9: If M is a P-V-module, then $J(M) = 0$.

Proof: Suppose on the contrary, there is a non-zero element $x \in J(M)$. Since Rx is finitely generated, it has a maximal submodule N. Let $\eta : Rx \longrightarrow Rx|N$ denotes the canonical quotient map. Extend η to a map $\tilde{\eta} : M \longrightarrow Rx|N$. Then $\operatorname{Ker} \tilde{\eta}$ is a maximal submodule of M with $x \notin \operatorname{Ker} \tilde{\eta}$, this is a clear contradiction. □

Remark 4.10: From Proposition 4.9, we can see that if N is a submodule of a P-V-module M then $J(N) = 0$, in particular every non-zero submodule of a P-V-module contains a maximal submodule.

Proposition 4.11: Let M be a P-V'-module. Then $J(M) \cap Z(M) = 0$ and $J(M) \subseteq \operatorname{Soc}(M)$. In particular $J(M)$ is a direct sum of simple projective modules.

Proof: Suppose on the contrary, there exists a non-zero element $x \in J(M) \cap Z(M)$. By Zorn's lemma choose a submodule L of M maximal with respect to $x \notin L$. Let $\eta : M \longrightarrow M|L$ denotes the quotient map and write $\bar{x} = x + L$. Then $R\bar{x}$ is a simple singular submodule of the factor module $M|L$. Let $\phi = \eta|Rx$. Since M is a P-V'-module, ϕ can be extended to an epimorphism $\Psi : M \longrightarrow R\bar{x}$. Thus $M|\operatorname{Ker} \Psi \cong R\bar{x}$ and $\operatorname{Ker} \Psi$ is a maximal submodule of M with $x \notin \operatorname{Ker} \Psi$, a clear contradiction with the choice of x.

To see that $J(M) \subseteq \operatorname{Soc}(M)$, suppose on the contrary there exists an element $x \in M$ with $x \in J(M)$ and $x \notin \operatorname{Soc}(M)$. Since $\operatorname{Soc}(M)$ is the intersection of all the essential submodules of M, it follows that $x \notin T$ for some proper essential submodule T of M. By Zorn's lemma, the set of all essential submodules I of M such that $x \notin I$ has a maximal member L. Let $\Pi : M \longrightarrow M|L$ denote the canonical quotient map and write $\bar{x} = \Pi(x) = x + L$. Writing $\bar{M}$ for the factor module $M|L$, we see that $0 \neq \bar{x} \in \bar{M}$ and any non-zero submodule of $\bar{M}$ must contain $\bar{x}$. Therefore $R\bar{x}$ is a simple singular submodule of $\bar{M}$. Let η denote the restriction of the map Π to the submodule Rx. Clearly $\eta : Rx \longrightarrow R\bar{x}$ is onto. Since M is a P-V'-module, η can be extended to a map $\bar{\eta} : M \longrightarrow R\bar{x}$. Clearly $\bar{\eta}$ is onto. If $N = \mathrm{Ker}(\bar{\eta})$ then $M|N \cong R\bar{x}$ and N is a maximal submodule of M with $x \notin N$, a contradiction with the fact that $x \in J(M)$. □

<u>Remark 4.12</u>: If M is a locally projective P-V'-module then $\mathrm{Soc}(M) \cap Z(M) = 0$. For, if S were a singular simple submodule of M then S is a direct summand of M. And since M is locally projective, it follows that S is projective, a contradiction. Thus every simple submodule of M is projective.

In the next proposition we show that every Artinian P-V'-module is Noetherian. In particular every Artinian GV-module is Noetherian.

<u>Proposition 4.13</u>:: Every Artinian P-V'-module is Noetherian.

<u>Proof</u>: Let M be a P-V'-module. If M is semisimple then we are done. Otherwise, suppose M has a proper essential submodule L and let x be a non-zero element of M which is not contained in L.

Let $\mathcal{F} = \{K \subseteq M : K$ is a submodule of M with $L \subseteq K$ and $x \notin K\}$. Since $L \in \mathcal{F}$, $\mathcal{F}$ is not empty, and it is easy to see that every totally ordered subset of $\mathcal{F}$ has an upper bound. By Zorn's lemma let K be a maximal element of $\mathcal{F}$. Let $\eta : M \longrightarrow M|K$ denote the canonical quotient map and write $\bar{x} = \eta(x) = x + K$. It is not difficult to see that $R\bar{x}$ is a simple singular submodule of the factor module $M|K$. If we define f to be the restriction of the map η to Rx, then f is an R-epimorphism. And since M is a P-V'-module, f can be extended to an R-epimorphism $g : M \longrightarrow R\bar{x}$. Whence $M|\text{Ker } g \cong R\bar{x}$. Thus M has a maximal submodule, namely Ker g. Whence $J(M) \neq M$.

Now, since every submodule of a P-V'-module is also a P-V'-module, $J(N) \neq N$ for every submodule N of M. Let L_1 be a maximal submodule of M. If L_1 is not simple, let L_2 be a maximal submodule of L_1, and so on. Since M is Artinian we must stop after a finite number of steps and $M = L_b \supset L_1 \supset L_2 \supset \cdots \supset L_n = 0$ is a composition series for M. Whence M is Noetherian. □

Remark 4.14: Note that along the lines of the above proof we have shown that every submodule of a P-V'-module contains a maximal submodule. In particular if R is a left GV-ring then every R-module is a GV-module and hence contains a maximal submodule. Thus every left GV-ring is a B-ring (max-ring) in the sense of [17].

Proposition 4.15: The following conditions are equivalent:

(i) M is a P-V-module.

(ii) If K is a maximal submodule of a cyclic submodule N of M, then $K^* \neq N^*$. (Here K^* = intersection of maximal submodules of M containing K, similar definition for N^*).

Proof: (i) $\longrightarrow$ (ii): Suppose on the contrary there exists a cyclic submodule N of M and a maximal submodule K of N such that $K^* = N^*$. Let $f : N \longrightarrow N|K$ denote the quotient map. Since M is a P-V-module, f can be extended to a map $g : M \longrightarrow N|K$. Let $h = g|N^*$. Then $H = \mathrm{Ker}(h)$ contains K and $H \subseteq N^* = K^*$. Whence $K \subseteq H \subseteq K^*$, which implies that $H^* = K^*$. Now, if $G = \mathrm{Ker}(g)$, then G is a maximal submodule of M with $G \cap N^* = H$, and $N^* = H^* \subseteq G$ ($H^* \subseteq G$, since G is a maximal submodule of M containing H). Thus $h(N^*) = 0$, consequently $h(N) = 0$. But $h|N = f$ the quotient map $N \longrightarrow N|K$. Therefore $N = K$ - a clear contradiction.

(ii) $\longrightarrow$ (i): Let S be a simple R-module, L a cyclic submodule of M, and $f : L \longrightarrow S$ a non-zero R-homomorphism. Let $K = \mathrm{Ker}(f)$. Since K is a maximal submodule of L, $K^* \neq L^*$. Hence there is a maximal submodule T of M with $K \subseteq T$ and $L \not\subset T$. Then $M = T + L$ and $M|K = T|K \oplus L|K$, which shows that f can be extended to a map $g : M \longrightarrow S$. This proves that every simple module is P-M-injective. □

Corollary 4.16: Let M be a P-V-module. Then for any submodule L of M either $L = L^*$ or L^* is not cyclic.

Proof: Suppose there exists a submodule L of M such that $L \neq L^*$ and $L^* = N$ is cyclic. Since $N|L$ is a cyclic module, it has a maximal submodule $T|L$. By Proposition 4.14, $T^* \neq N^*$. But since $L \subseteq T \subseteq N$, it follows that $L^* \subseteq T^* \subseteq N^*$ and since $L^* = N^*$, we get $T^* = L^*$ and hence $T^* = N^*$, a clear contradiction. □

The next three results will be stated without proofs. The proofs are similar to the proof of Proposition 4.15.

Proposition 4.17: The following are equivalent for a left R-module M.

(i) M is a P-V'-module.

(ii) If K is a submodule of any cyclic submodule L of M, such that $L|K$ is simple singular then $K^* \neq L^*$.

Proposition 4.18: The following are equivalent for a left R-module M.

(i) M is an f-V-module.

(ii) If K is a maximal submodule of a finitely generated submodule L of M then $K^* \neq L^*$.

Proposition 4.19: The following are equivalent for a left R-module M.

(i) M is an f-V'-module.

(ii) If K is a submodule of any finitely generated submodule L of M such that $L|K$ is simple singular then $K^* \neq L^*$.

Proposition 4.20: Let M be left R-module. Then the following conditions are equivalent:

(i) M is a V-module.

(ii) Every simple submodule of M is M-injective and every singular homomorphic image of M has zero radical.

Proof: (i) $\longrightarrow$ (ii): Immediate consequence of Proposition 1.1 and 1.2.

(ii) $\longrightarrow$ (i): Let S be a simple module, N an essential submodule of M and $f : N \longrightarrow S$ a non-zero homomorphism. Let $K = \text{Ker}(f)$. If $K \cap T = 0$ for some non-zero submodule T of N then by the maximality of K in N we infer that T is a simple submodule of M with $T \overset{f}{\cong} S$. Thus S is M-injective. Otherwise suppose K is essential in N. In this case both K and N are essential submodules of M and hence $J(M|K) = J(M|N) = 0$ yielding $K = K^*$, $N = N^*$. Since $K \neq N$ there is a maximal submodule L of M such that $K \subseteq L$ and $N \not\subset L$. Thus $\frac{M}{K} = \frac{L}{K} \oplus \frac{N}{K}$ and the map f can be extended to an R-homomorphism from M into S. Whence M is a V-module. □

<u>Definition 4.21</u>: Let M be a right R-module and U be a left R-module. U is said to be P-M-flat if for every cyclic submodule K of M the sequence $0 \longrightarrow K \otimes_R U \longrightarrow M \otimes_R U$ is exact. U is said to be P-flat if it is P-R-flat.

<u>Lemma 4.22</u>: ([18, Theorem 9]) Suppose $0 \longrightarrow K \longrightarrow M \longrightarrow L \longrightarrow 0$ is exact with K finitely generated and M projective. Then K is a pure submodule of M if and only if K is a direct summand.

<u>Proof</u>: Since any direct summand is pure, it suffices to show the converse. Suppose then that K is a pure submodule of M and let $\{x_i : x_i \in K, 1 \leq i \leq n\}$ be a generating set for K. Since M is projective, it is isomorphic to a direct summand of a free module F. Without loss of generality we may assume that $F = M \oplus M'$. Since K is pure in M and M is pure in F, it follows that K is pure in F. Then, by [18, Theorem 8], there exists $\Upsilon : F \longrightarrow K$ such that $\Upsilon(x_i) = x_i$, $1 \leq i \leq n$. Let $\delta = \Upsilon|M$. Then $\delta : M \longrightarrow K$, with $\delta(x_i) = \Upsilon(x_i) = x_i$ for all i. If $\Pi : K \longrightarrow M$ is the natural injection, then we have $\delta \circ \Pi = Id_K$, whence K is a direct summand of M. □

From the above lemma we can easily see the following.

<u>Proposition 4.23</u>: A projective module ${}_RM$ is regular if and only if every right R-module is P-M-flat.

<u>Corollary 4.24</u>: A ring R is regular if and only if every R-module is P-flat.

The next three results will be stated without proof. The proofs are similar to those of corresponding results in Chapter 1.

<u>Lemma 4.25</u>: If M is a projective module and U is any R-module, then the following are equivalent:

(i) U is P-M-injective.

(ii) $\mathrm{Ext}_R^1(M|N,U) = 0$, for every cyclic submodule N of M.

<u>Lemma 4.26</u>: If M is a flat right R-module and U is any left R-module. Then the following are equivalent:

(i) U is P-M-flat.

(ii) $\mathrm{Tor}_1^R(M|N,U) = 0$, for every cyclic submodule N of M.

<u>Proposition 4.27</u>: Let R be a commutative ring and M a projective R-module. If S is any simple R-module then the following are equivalent:

(i) S is P-M-injective.

(ii) S is P-M-flat.

<u>Proposition 4.28</u>: Let R be a commutative ring and M a projective R-module. Then the following are equivalent:

(i) M is a regular module.

(ii) M is a V-module.

(iii) M is a GV-module.

(iv) M is a P-V-module.

(v) M is a P-V'-module.

(vi) M is an f-V-module.

(vii) M is an f-V'-module.

(viii) Every simple R-module is M-flat.

(ix) Every simple singular R-module is M-flat.

(x) Every simple R-module is P-M-flat.

(xi) Every simple singular R-module is P-M-flat.

Proof: (i) $\leftrightarrow$ (ii) $\leftrightarrow$ (iii) $\leftrightarrow$ (viii) By Proposition 1.15 and Proposition 3.16.

(iii) $\leftrightarrow$ (ix): By Proposition 1.13.

(iv) $\leftrightarrow$ (ix): By Proposition 4.27.

(v) $\leftrightarrow$ (xi): By Proposition 4.27.

(iv) $\rightarrow$ (v): Obvious.

(v) $\rightarrow$ (iv): Let S be a simple R-module. If S is projective, then S is flat and hence P-M-flat; and by Proposition 4.26, S is P-M-injective. If S is singular then automatically S is P-M-injective. Thus every simple module is P-M-injective.

(ii) $\rightarrow$ (iv): Clear.

(iv) $\rightarrow$ (ii): By [28, Theorem 4.8 and Proposition 3.7].

(ii) $\rightarrow$ (vi) $\rightarrow$ (vii) $\rightarrow$ (v): Clear. □

CHAPTER 5.

SI-MODULES

A ring R is called a left SI-ring if every singular left R-module is injective. SI-rings were introduced and studied by K.R. Goodearl. In this chapter we say that a left R-module M is an SI-module provided that every singular left R-module is M-injective. It was shown by K.R. Goodearl [22] that a ring R is a left SI-ring if and only if $Z({}_RR) = 0$ and for every essential left ideal I of R, $R|I$ is semisimple. Commutative SI-rings were also investigated by V.C. Cateforis and F.L. Sandomierski in [11] and [12]. It was proved in [12] that for a commutative ring R the following are equivalent:

(i) R is an SI-ring.

(ii) R is (von Neumann) regular and $R|\mathrm{Soc}(R)$ is semisimple.

In Section 1, we show that results of this type can be obtained for SI-modules. The connections between regular modules, V-modules, GV-modules and SI-modules are studied. We show, among other things, that a finitely generated projective module over a commutative ring is an SI-module if and only if it is a finite direct sum of regular modules each of which has at most two essential submodules.

In Section 2, we introduce and study P-SI-rings. A ring R will be called a left P-SI-ring if every singular left R-module is P-injective. We prove, among many other things, that if R is a ring with essential left socle then R is a left P-SI-ring if and only if $\mathrm{Soc}({}_RR)$ is projective and $R|(\mathrm{Soc}_R R)$ is a regular ring. Known results for SI-rings are extended to P-SI-rings.

Section 1. SI-modules.

Definition 5.1.1: A left R-module M is called an SI-module (resp. P-SI-module) if every singular left R-module is M-injective (resp. P-M-injective). Clearly every SI-module (resp. P-SI-module) is a GV-module (resp. P-V'-module). A ring R is called a left SI-ring (resp. P-SI-ring) if the left R-module $_R R$ is an SI-module (resp. P-SI-module).

With the same argument used in the proof of Proposition 1.2 one can easily verify the following:

Proposition 5.1.2: (i) Submodules and homomorphic images of SI-modules are also SI-modules.

(ii) $\bigoplus_{i \in I} M_i$ is an SI-module if and only if each M_i is an SI-module.

Proposition 5.1.3: Suppose that $_R M$ is an SI-module. Then the following statements are true.

(i) Every singular homomorphic image of M is semisimple.

(ii) $M|N$ is semisimple for every essential submodule N of M.

(iii) $J(M) \subseteq \mathrm{Soc}(M)$, $Z(M) \subseteq \mathrm{Soc}(M)$ and $J(M) \cap Z(M) = 0$.

Proof: (i) If L is a singular homomorphic image of M then by Proposition 5.1.2 (i), L is a singular SI-module. Whence every submodule of L, which necessarily has to be singular, is L-injective. Hence every submodule of L is a direct summand of L, and so L is semisimple.

(ii) If N is an essential submodule of M then $M|N$ is a singular homomorphic image of M, whence semisimple from above.

(iii) Since Soc(M) is an intersection of essential submodules of M and every proper essential submodule of M is an intersection of maximal submodules, it follows that $J(M) \subseteq \mathrm{Soc}(M)$. Since Z(M) is a singular SI-module (since submodules of SI-modules are again SI-modules), by (i) we infer that Z(M) is semisimple, and hence $Z(M) \subseteq \mathrm{Soc}(M)$. Since every SI-module is a GV-module, it follows from Proposition 3.2(ii) that $J(M) \cap Z(M) = 0$. □

Proposition 5.1.4: For a locally projective module ${}_RM$ the following conditions are equivalent:

(i) M is an SI-module.

(ii) $Z(M) = 0$ and every singular homomorphic image of M is semisimple.

(iii) $Z(M) = 0$ and $M|N$ is semisimple for every essential submodule N of M.

Proof: (i) → (ii): Suppose $Z(M) \neq 0$ and let x be a non-zero element of Z(M). Then Rx is a singular submodule of M and hence a direct summand of M. Since M is locally projective it follows that Rx is projective. Now consider the following exact sequence of left R-modules $0 \longrightarrow \mathrm{Ann}_R(x) \longrightarrow R \xrightarrow{\eta} Rx \longrightarrow 0$, where η is given by $\eta(r) = rx$, $\forall\, r \in R$. Since Rx is projective the sequence splits, and hence $\mathrm{Ann}_R(x)$ is not essential in ${}_RR$, contradicting the choice of x. Now the rest of the assertion follows from Proposition 5.1.3(i).

<u>(ii) $\longrightarrow$ (iii)</u>: Clear.

<u>(iii) $\longrightarrow$ (i)</u>: Let L be a singular R-module. We want to show that L is M-injective. So, let N be a proper essential submodule of M and $f : N \longrightarrow L$ be any non-zero homomorphism. Let $K = \mathrm{Ker}(f)$. We claim that K is essential in N. For, if $K \cap I = 0$ for some non-zero submodule I of N, then $f|I : I \longrightarrow L$ is a monomorphism. So I is a non-zero singular submodule of M, a clear contradiction since $Z(M) = 0$. Now, since K is essential in M, it follows that $M|K$ is semisimple and $N|K$ is a direct summand of $M|K$. Whence f can be extended to a map $g : M \longrightarrow L$ in the obvious way. □

Note that along the lines of the above proof we have shown that every locally projective SI-module is non-singular. In fact with the same argument one can prove the following:

<u>Proposition 5.1.5</u>: Every locally projective P-SI-module is non-singular.

<u>Proposition 5.1.6</u>: Let M be a non-singular module. Then the following conditions are equivalent:

(i) M is an SI-module.

(ii) $Z(L) \subseteq \mathrm{Soc}(L)$, for every homomorphic image L of M.

(iii) Every singular homomorphic image of M is semisimple.

(iv) $M|N$ is semisimple, for every essential submodule N of M.

<u>Proof</u>: <u>(i) $\longrightarrow$ (ii)</u>: If L is a homomorphic image of M then L is an SI-module and hence $Z(L) \subseteq \mathrm{Soc}(L)$, by Proposition 5.1.3 (iii). The proof of the other implications is similar to that of Proposition 5.1.4. □

Observe that if R is a left SI-ring then for any left R-module M, every singular module is M-injective. As a consequence of this observation we have the following:

<u>Proposition 5.1.7</u>: For any ring R the following are equivalent:

(i) R is a left SI-ring.

(ii) Every left R-module is an SI-module.

(iii) Every cyclic left R-module is an SI-module. □

<u>Proposition 5.1.8</u>: For a locally projective module M the following conditions are equivalent:

(i) M is an SI-module with essential socle.

(ii) Soc(M) is projective and $M|Soc(M)$ is semisimple.

<u>Proof</u>: <u>(i) $\longrightarrow$ (ii):</u> Since M is a locally projective SI-module, $Z(M) = 0$ by Proposition 5.1.4, and hence Soc(M) is projective. Since Soc(M) is essential in M, it follows from Proposition 5.1.3(ii) that $M|Soc(M)$ is semisimple.

<u>(ii) $\longrightarrow$ (i)</u>: If $Soc(M) \cap I = 0$ for some non-zero submodule I of M, then $I \cong (I + Soc(M))|Soc(M) \subseteq M|Soc(M)$ which implies that I is semisimple and hence $I \subseteq Soc(M)$, a contradiction. Thus Soc(M) is essential in M. Now, if Z(M) is non-zero, then $Z(M) \cap Soc(M) \neq 0$, a contradiction with the projectivity of Soc(M). Thus $Z(M) = 0$. Now if N is any essential submodule of M then $Soc(M) \subseteq N$ and hence $M|N$, being a factor module of $M|Soc(M)$, is semisimple, and we can apply Proposition 5.1.6. □

Proposition 5.1.9: Let M be a locally projective module with $M|J(M)$ semisimple. Then the following are equivalent:

(i) M is a GV-module.

(ii) M is an SI-module.

Proof: (i) $\longrightarrow$ (ii): Since M is a GV-module, by Proposition 3.2(ii), it follows that $Z(M) \cap J(M) = 0$, and hence $Z(M) \cong (Z(M) \oplus J(M))|J(M)$ is a semisimple module being isomorphic to a submodule of the semisimple module $M|J(M)$. This means that $Z(M) \subseteq \text{Soc}(M)$. But since M is a locally projective GV-module, by Proposition 3.11, it follows that $Z(M) \cap \text{Soc}(M) = 0$, and hence $Z(M)$ must be zero.

Now Let L be any singular R-module, N any essential submodule of M and $f : N \longrightarrow L$ any non-zero R-homomorphism. Let $K = \text{Ker}(f)$. Then one can easily see that K is essential in M and hence $J(M) \subseteq \text{Soc}(M) \subseteq K$. Whence $N|K$ is a direct summand of $M|K$ and the map f can be extended to a map $g : M \longrightarrow L$. Therefore M is an SI-module.

(ii) $\longrightarrow$ (i): Obvious. $\square$

It was proved in [12, Theorem 1 and Theorem 5] that for a commutative ring R the following conditions are equivalent:

(i) R is an SI-ring.

(ii) R is a regular ring and $R|\text{Soc}(R)$ is semisimple.

In [22, Theorem 3.9] K.R. Goodearl has proved that the above conditions are equivalent to saying that:

(iii) R is a finite direct sum of non-singular rings which have at most two essential ideals.

In our next proposition we shall extend these results to modules. But first we need the following lemma which extends [22, Proposition 3.6] to modules.

Lemma 5.1.10: If M is a finitely generated SI-module then $M|Soc(M)$ is Noetherian.

Proof: (Adapted from [22, Proposition 3.6])

We will show that every submodule of $M|Soc(M)$ is finitely generated. Let $J = Soc(M)$ and I be a submodule of M with $I \supseteq J$. Let K be a submodule of I maximal with respect to $K \cap J = 0$. Then $J \oplus K$ is essential in I and $I|(J \oplus K)$ is a singular module. Since $M|(J \oplus K)$ is an SI-module we see that $I|(J \oplus K)$ is a direct summand of $M|(J \oplus K)$. Thus $I|(J \oplus K)$ is finitely generated. Our aim is to show that $I|J$ is finitely generated. From the exactness of the sequence $0 \longrightarrow K \longrightarrow I|J \longrightarrow I|(J \oplus K) \longrightarrow 0$, it suffices to prove that K is finitely generated. We first show that K is finite dimensional. If not, then there exists an infinite direct sum $K_1 \oplus K_2 \oplus \cdots$ of non-zero submodules of K. Since $K \cap J = 0$, none of the K_i are semisimple; whence each K_i has a proper essential submodule H_i. Inasmuch as $(\bigoplus_{i=1}^{\infty} K_i)|(\bigoplus_{i=1}^{\infty} H_i) \cong \bigoplus_{i=1}^{\infty} (K_i|H_i)$ is a singular module and hence is $M|(\bigoplus_{i=1}^{\infty} H_i)$-injective, it follows that $(\bigoplus_{i=1}^{\infty} K_i)|(\bigoplus_{i=1}^{\infty} H_i)$ is a direct summand of $M|(\bigoplus_{i=1}^{\infty} H_i)$ and so is finitely generated, which contradicts the fact that it is an infinite direct sum of non-zero modules. By the finite dimensionality of K, Let $\{E_i\}_{i=1}^{n}$ be a maximal family of non-zero cyclic submodules of K such that the sum $\sum_{i=1}^{N} E_i$ is direct. Clearly $E = \bigoplus_{i=1}^{n} E_i$ is essential in K, and hence $K|E$ is singular. Inasmuch as $M|E$ is an SI-module, it follows that $K|E$ is a direct summand of $M|E$ and thus is finitely generated. Whence K is finitely generated. □

<u>Corollary 5.1.11</u>: If M is a finitely generated regular module then the following statements are equivalent:

(i) M is an SI-module.

(ii) $M|\mathrm{Soc}(M)$ is semisimple.

<u>Proof</u>: <u>(i) → (ii)</u>: Note first $M|\mathrm{Soc}(M)$ is Noetherian, by Lemma 5.1.10. We claim that Soc(M) is essential in M, for if $I \cap \mathrm{Soc}(M) = 0$ for some non-zero submodule I of M it follows that $I \cong \frac{I \oplus \mathrm{Soc}(M)}{\mathrm{Soc}(M)} \subseteq M|\mathrm{Soc}(M)$, which implies that I is a Noetherian module. And since submodules of regular modules are again regular, we conclude from Lemma 2.5 that I is semisimple. Whence $I \subseteq \mathrm{Soc}(M)$, a clear contradiction. Now by Proposition 5.1.3(ii) it follows that $M|\mathrm{Soc}(M)$ is semisimple.

<u>(ii) → (i)</u>: Since M is a regular module, it follows that every simple submodule is a direct summand and hence projective. Hence Soc(M) is projective. Since $M|\mathrm{Soc}(M)$ is semisimple, Soc(M) is essential in M. Inasmuch as M is regular, and hence locally projective, it follows from Proposition 5.1.8 that M is an SI-module. □

Following M.S. Shirkhande [41], a module M is called hereditary (resp. semihereditary) if every submodule (resp. finitely generated submodule) of M is projective.

<u>Proposition 5.1.12</u>: If R is a commutative ring and M is a finitely generated projective R-module. Then the following conditions are equivalent:

(i) M is an SI-module.

(ii) M is a regular module and $M|\mathrm{Soc}(M)$ is semisimple.

(iii) M is a semihereditary module and $M|\mathrm{Soc}(M)$ is semisimple.

(iv) M is non-singular and $M|\mathrm{Soc}(M)$ is semisimple.

(v) M is a finite direct sum of regular modules each of which has at most two essential submodules.

(vi) M is a finite direct sum of non-singular modules each of which has at most two essential submodules.

<u>Proof</u>: <u>(i) $\rightarrow$ (ii)</u>: Since every SI-module is a GV-module it follows from Proposition 1.1 that M is a regular module, and hence $M|\mathrm{Soc}(M)$ is semisimple by Corollary 5.1.11.

<u>(ii) $\rightarrow$ (iii)</u>: Clear, since every regular module is semihereditary

<u>(iii) $\rightarrow$ (iv)</u>: Clear, since every semihereditary module is non-singular.

<u>(v) $\rightarrow$ (vi)</u>: Obvious since every regular module is non-singular.

<u>(vi) $\rightarrow$ (i)</u>: Let $M = M_1 \oplus \cdots \oplus M_n$, where each M_i is non-singular and has at most two essential submodules. By Proposition 5.1.2 (ii), it is enough to show that each M_i is an SI-module. But if I is any essential submodule of M_i then $M_i|I$ is either zero or simple, and by Proposition 5.1.4 (iii) it follows that each M_i is an SI-module.

<u>(iv) $\longrightarrow$ (i)</u>: Let L be any non-zero singular R-module, N any essential submodule of M and $f : N \longrightarrow L$ any non-zero R-homomorphism. Let $K = \text{Ker}(f)$. Since M is non-singular, it is not difficult to see that K is essential in M, and so $\text{Soc}(M) \subseteq K$. Now since $M|K \cong (M|\text{Soc}(M))|(K|\text{Soc}(M))$ is a semisimple module, we see that $N|K$ is a direct summand of $M|K$ and the map f can be extended to a map $g : M \longrightarrow L$. Whence every singular module is M-injective, and so M is an SI-module.

<u>(ii) $\longrightarrow$ (v)</u>: Since $M|\text{Soc}(M)$ is a finite direct sum of simple modules, it has a composition series. We shall prove our assertion by induction on the composition length of $M|\text{Soc}(M)$. If $\ell(M|\text{Soc}(M)) = 0$, then $M = \text{Soc}(M)$ and M is a finite direct sum of simple projective modules. Assume that $\ell(M|\text{Soc}(M)) > 0$, then $M|\text{Soc}(M)$ has a non-zero simple submodule $I|\text{Soc}(M)$. Let $K = \text{Soc}(M)$ and choose some $x \in I$ with $x \notin K$. Thus $Rx|(K \cap Rx) \neq 0$. Hence $Rx|(K \cap Rx) \cong I|K$. Because $I|K$ is simple, it follows that $\text{Soc}(Rx) = K \cap Rx$ is a maximal submodule of Rx. Inasmuch as M is a regular module we see that Rx is a projective summand of M. Write $M = Rx \oplus N$, for some submodule N of M. Since $\text{Soc}(Rx)$ is an intersection of essential submodules of Rx and $\text{Soc}(Rx)$ is a maximal submodule of Rx, it follows that Rx has only two essential submodules, namely Rx and $\text{Soc}(Rx)$. Since $M|K = \dfrac{Rx \oplus N}{\text{Soc}(Rx \oplus N)} \cong \dfrac{Rx}{\text{Soc}(Rx)} \oplus \dfrac{N}{\text{Soc}(N)}$, We have $\ell(N|\text{Soc}(N)) = \ell(M|K) - 1$, and hence may use an inductive hypothesis on the module N. $\square$

Remark 5.1.13: The above proposition remains valid if we replace "regular module" by "λ-module", where λ stands for one of the symbols V, GV, P-V, P-V' or P-SI, see Proposition 4.28 and the next proposition.

Proposition 5.1.14: If R is a commutative ring and M is a projective R-module then the following are equivalent:

(i) M is a regular module.

(ii) M is a P-SI-module.

In particular a commutative ring R is regular if and only if R is a P-SI-ring.

Proof: (i) $\rightarrow$ (ii): By Proposition 4.7, if M is a projective regular module then every R-module is P-M-injective. Thus M is a P-SI-module.
(ii) $\rightarrow$ (i): If M is a P-SI-module then M is a P-V'-module and hence by Proposition 4.28 M is a regular module. $\square$

Section 2. P-SI-rings.

Recall that a module ${}_RM$ is said to be P-injective if for any principal left ideal I of R and $f \in \mathrm{Hom}_R(I,M)$ there exists an element $m \in M$ such that $f(x) = xm$, for all $x \in I$. It was proved in [56] that a ring R is regular if and only if every R-module is P-injective. A ring R is defined to be a left P-V-ring if every simple left R-module is P-injective. P-V-rings were introduced and studied by Yue Chi Ming in [55] and [56], and by H. Tominaga in [46]. We defined a ring R to be a left P-SI-ring if every singular left R-module is P-injective (Definition 5.1.1). In this section we establish the following characterization:

Proposition 5.2.1 For a ring R with essential left socle, the following statements are equivalent:

(i) R is a left P-SI-ring.

(ii) $\mathrm{Soc}({}_R R)$ is projective and $R|(\mathrm{Soc}_R R)$ is a regular ring.

(iii) $R|(\mathrm{Soc}_R R)^2$ is a regular ring.

We postpone the proof until some of the ideas involved have been sufficiently developed below.

Let $\underline{K}$ be a two-sided ideal of R. G. Azumaya has proved in [3(II), Proposition 10(ii)] that, every injective right $R|\underline{K}$-module is injective as a right R-module if and only if $R|\underline{K}$ is flat as a left R-module. For P-injective modules we have the following:

Proposition 5.2.2 Let $\underline{K}$ be a two sided ideal of R. Then every P-injective right $R|\underline{K}$-module is P-injective as a right R-module if and only if $R|\underline{K}$ is flat as a left R-module.

Proof: "Only if" part: adapted from [3(II), Proposition 10]. Let $a \in \underline{K}$ and consider the right R-modules $aR, a\underline{K}$ and $aR|a\underline{K}$. Let $\phi : aR \longrightarrow aR|a\underline{K}$ denote the canonical quotient mapping. $aR|a\underline{K}$ is annihilated by $\underline{K}$, and so can be regarded as a right $R|\underline{K}$-module. Let $Q = E(aR|a\underline{K})$ be the injective hull of the right $R|\underline{K}$-module $aR|a\underline{K}$. Then Q is P-injective as a right $R|\underline{K}$-module, whence P-injective as a right R-module, by assumption. Now the map $\phi : aR \longrightarrow Q$ can be regarded as a map of R-modules. Therefore ϕ can be extended to an R-homomorphism $\bar{\phi} : R \longrightarrow Q$. Let $\bar{\phi}(1) = y$, $y \in Q$. Then $\phi(x) = yx$, $\forall x \in aR$. But $aR \subseteq \underline{K}$, and Q is annihilated by $\underline{K}$, so $yx = 0$ $\forall x \in aR$. Thus $\phi = 0$, and

$aR = a\underline{K}$. Since a was arbitrarly chosen from $\underline{K}$, $a \in a\underline{K}$ $\forall$ $a \in K$ and it follows from a well-known result of G. Azumaya [3(II), Proposition 5] that ${}_R(R|\underline{K})$ is flat.

<u>"if" part</u>: Suppose ${}_R(R|\underline{K})$ is flat as a left R-module. And let Q be a P-injective right $R|\underline{K}$-module. Want to show $\mathrm{Ext}^1_R(R|xR,Q) = 0$ for every $x \in R$. So, let x be any element of R and consider the following exact sequence of right R-modules $0 \longrightarrow xR \longrightarrow R \longrightarrow R|xR \longrightarrow 0$. Since ${}_R(R|\underline{K})$ is flat, it follows that: $(R|\underline{K})|(\underline{K}+xR|\underline{K}) \cong (R|xR) \otimes_R (R|\underline{K})$ and that $\mathrm{Ext}^1_R(R|xR,Q) \cong \mathrm{Ext}^1_{R|\underline{K}}(R|xR \otimes_R R|\underline{K},Q)$, whence

$\mathrm{Ext}^1_R(R|xR,Q) \cong \mathrm{Ext}^1_{R|\underline{K}}(R|(\underline{K}+xR),Q)$. Now since Q is P-injective as a right $R|\underline{K}$-module and $(\underline{K}+xR)|\underline{K}$ is a principal right ideal of $R|\underline{K}$ we get $\mathrm{Ext}^1_{R|\underline{K}}(R|(\underline{K}+xR),Q) = 0$, and so $\mathrm{Ext}^1_R(R|xR,Q) = 0$ for every $x \in R$ and Q is P-injective as a right R-module. □

With the same argument used in the "if" part of the above proof one can also verify the following:

<u>Proposition 5.2.3</u>: Let $\underline{K}$ be a two-sided ideal of R, $R|\underline{K}$ flat as a left R-module and Q a right $R|\underline{K}$-module. If Q is P-injective as a right R-module then it is also P-injective as a right $R|\underline{K}$-module. □

We shall also make use of the following result, which was proved in [6, Proposition 1.4 and Proposition 1.10].

<u>Proposition 5.2.4</u>: For every ring R one has $\mathrm{Soc}_{\underline{\underline{P}}}({}_RR) = (\mathrm{Soc}({}_RR))^2$, where $\mathrm{Soc}_{\underline{\underline{P}}}({}_RR)$ denotes the projective homogeneous component of the left

socle of R. Moreover, if $\underline{K}$ is a two-sided ideal contained in $\mathrm{Soc}({}_R R)$, then the following conditions are equivalent:

(i) $\underline{K}^2 = \underline{K}$.

(ii) $(R|\underline{K})_R$ is flat as a right R-module. □

We can now prove Proposition 5.2.1:

<u>(i) ⟶ (ii)</u>: By Proposition 5.1.5, since R is a left P-SI-ring, R is left non-singular and so $\mathrm{Soc}_R R$ is projective. Now, in order to show that $R|(\mathrm{Soc}_R R)$ is a regular ring we must prove that every left $R|(\mathrm{Soc}_R R)$-module is P-injective. So, let M be a left $R|(\mathrm{Soc}_R R)$-module. Since $\mathrm{Soc}({}_R R)$ is essential in ${}_R R$ it follows that M is a singular left R-module, whence M is P-injective as a left R-module. Now since $\mathrm{Soc}({}_R R)$ is projective, it follows from Proposition 5.2.4 that $R|(\mathrm{Soc}_R R)$ is flat as a right R-module and so by Proposition 5.2.3, it follows that M is P-injective as a left $R|(\mathrm{Soc}_R R)$-module.

<u>(ii) ⟶ (iii)</u>: Inasmuch as $\mathrm{Soc}_R R$ is projective, it follows from Proposition 5.2.4 that $\mathrm{Soc}_R R = (\mathrm{Soc}_R R)^2$ and hence $R|(\mathrm{Soc}_R R)^2$ is a regular ring.

<u>(iii) ⟶ (i)</u>: Since $R|(\mathrm{Soc}_R R)^2$ is a regular ring, and hence fully right idempotent, it follows from [5, Proposition 1.4] that $\mathrm{Soc}_R R$ is projective and hence by Proposition 5.2.4, we have $(\mathrm{Soc}_R R)^2 = \mathrm{Soc}_R R$, whence $R|(\mathrm{Soc}_R R)$ is a regular ring. Now let M be any singular left R-module. By the singularity of M we have $(\mathrm{Soc}_R R)\cdot M = 0$, and so M can be regarded as a left $R|(\mathrm{Soc}_R R)$-module. Since $R|(\mathrm{Soc}_R R)$ is a regular ring, M is P-injective as a left $R|(\mathrm{Soc}_R R)$-module. By Proposition

5.2.4, since $Soc_R R$ is projective, $(R|Soc_R R)_R$ is flat as a right R-module. Now by Proposition 5.2.2, it follows that M is P-injective as a left R-module. Hence R is a left P-SI-ring. □

We do not know whether Proposition 5.2.1 holds for modules. However we have the following:

<u>Proposition 5.2.5</u>: Let M be a left R-module. If Soc(M) is projective and M|Soc(M) is a regular module then M is a P-SI-module.

<u>Proof</u>: Let N be a cyclic submodule of M, L a singular R-module and $f : N \longrightarrow L$ a non-zero homomorphism. We want to show that f can be extended to a map $g : M \longrightarrow L$. Let K = Ker(f). If $K \cap I = 0$ for some non-zero submodule I of N, then $f : I \longrightarrow L$ is a monomorphism and I is a non-zero singular submodule of M. Thus $I \cap Soc(M) = 0$, and hence $I \cong (I + Soc(M))|Soc(M) \subseteq M|Soc(M)$, which implies that I is a regular submodule of M. But since every regular module is non-singular, it follows that $Z(I) = 0$, a clear contradiction with the singularity of I. Thus K is essential in N, and hence $Soc(N) \subseteq K$.

Now define $\phi : N|Soc(N) \longrightarrow (N + Soc(M))|Soc(M)$, by $\phi(n + Soc(N)) = n + Soc(M)$. Then ϕ is an isomorphism. Let $\bar{\ } : M \longrightarrow M|Soc(M)$ denote the canonical quotient map, and write $\bar{M} = M|Soc(M)$. Since $\bar{M}$ is a regular module and $\bar{N}$ is a cyclic submodule of $\bar{M}$, we can write $\bar{M} = \bar{N} \oplus \bar{T}$, for some submodule $\bar{T}$ of $\bar{M}$. Since $soc(N) \subseteq Ker(f)$, there is a map $\tilde{f} : N|Soc(N) \longrightarrow L$, such that $\tilde{f}(n + Soc(N)) = f(n)$. Thus $\tilde{f} \circ \phi^{-1} : \bar{N} \longrightarrow L$. Extend $(\tilde{f} \circ \phi^{-1})$ to a map $\tilde{g} : \bar{M} = \bar{N} \oplus \bar{T} \longrightarrow L$ in the obvious way. Define $g : M \longrightarrow L$, by

$g(m) = \tilde{g}(\bar{m})$, $\forall\, m \in M$. Now if $x \in N$ then:

$$\begin{aligned} g(x) = \tilde{g}(\bar{x}) &= g(x + \mathrm{Soc}(M)) \\ &= (\tilde{f} \circ \phi^{-1})(x + \mathrm{Soc}(M)) \\ &= \tilde{f}(\phi^{-1}(x + \mathrm{Soc}(M))) \\ &= \tilde{f}(x + \mathrm{Soc}(N)) \\ &= f(x). \end{aligned}$$

Thus the map g is the required map. □

It was proved in [22, Proposition 3.5] that for a ring R with $R|J(R)$ semisimple, the following statements are equivalent:

(i) $Z_r(R) = 0$ and R is a right SI-ring.

(ii) $Z_r(R) = 0$ and $[J(R)]^2 = 0$.

(iii) $Z_\ell(R) = 0$ and $[J(R)]^2 = 0$.

(iv) $Z_\ell(R) = 0$ and R is a left SI-ring.

However in view of our Proposition 5.1.5, R is a right SI-ring $\Rightarrow Z_r(R) = 0$ (similarly R is a left SI-ring $\Rightarrow Z_\ell(R) = 0$). Thus in (i) we can remove the condition $Z_r(R) = 0$ (similarly in (iv) we can remove the condition $Z_\ell(R) = 0$).

In the next Proposition we shall prove also that, under the same hypothesis, a ring R is a right P-SI-ring if and only if R is a left P-SI-ring.

<u>Proposition 5.2.6</u>: If R is a ring with $R|J(R)$ semisimple, then the following conditions are equivalent:

(i) R is a right SI-ring.

(ii) R is a left SI-ring.

(iii) $Z_r(R) = 0$ and $[J(R)]^2 = 0$.

(iv) $Z_\ell(R) = 0$ and $[J(R)]^2 = 0$.

(v) R is a right P-SI-ring.

(vi) R is a left P-SI-ring.

(vii) R is a right GV-ring.

(viii) R is a left GV-ring.

(ix) R is a right P-V'-ring.

(x) R is a left P-V'-ring.

(xi) R is right semihereditary and $[J(R)]^2 = 0$.

(xii) R is left semihereditary and $[J(R)]^2 = 0$.

(xiii) R is right hereditary and $[J(R)]^2 = 0$.

(xiv) R is left hereditary and $[J(R)]^2 = 0$.

Proof: (v) $\longrightarrow$ (ix): Clear.

(ix) $\longrightarrow$ (iii): Inasmuch as R is a right P-V'-ring, $[J(R)]^2 = 0$ and $J(R) \cap Z_r(R) = 0$, by Proposition 4.11. Hence $Z_r(R) \cong \frac{J(R) \oplus Z_r(R)}{J(R)} \subseteq R|J(R)$. Whence $Z_r(R)$ is a semisimple right R-module and so $Z_r(R) \subseteq \mathrm{Soc}(R_R)$. But since R is a right P-V'-ring it follows that every minimal right ideal of R must be projective. Therefore $Z_r(R) = 0$.

(iii) $\longrightarrow$ (i): By [22, Proposition 3.5].

(i) $\longrightarrow$ (v): Obvious.

(x) $\longleftrightarrow$ (vi) $\longleftrightarrow$ (iv) $\longleftrightarrow$ (ii): By symmetry.

(i) $\longleftrightarrow$ (ii): By [22, Proposition 3.5].

(i) $\longleftrightarrow$ (vii): By Proposition 5.1.8.

<u>(xiii) $\longrightarrow$ (xi)</u>: Clear.

<u>(xi) $\longrightarrow$ (iii)</u>: If x is any non-zero element of R then the sequence $0 \longrightarrow \mathrm{Ann}_R(x) \longrightarrow R \longrightarrow xR \longrightarrow 0$ splits, where $\mathrm{Ann}_R(x)$ denotes the right annihilator of x in R. Whence $Z_r(R) = 0$.

<u>(i) $\longrightarrow$ (xiii)</u>: By [22, Proposition 3.3].

<u>(xiv) $\longleftrightarrow$ (xii) $\longleftrightarrow$ (iv) $\longleftrightarrow$ (i)</u>: By symmetry.

Finally we conclude this section with the following.

<u>Proposition 5.2.7</u>: For a left self-injective ring R, the following conditions are equivalent:

(i) R is a left P-SI-ring.

(ii) R is a regular ring.

<u>Proof</u>: <u>(i) $\longrightarrow$ (ii)</u>: By Proposition 5.1.5, since R is a left P-SI-ring it follows that R is left non-singular. And since R is left self-injective, J(R) = 0 and R is a regular ring.

<u>(ii) $\longrightarrow$ (i)</u>: Since R is a regular ring, every R-module is P-injective, in particular every singular left R-module is P-injective, and hence R is a left P-SI-ring. □

CHAPTER 6.

MORE ON V-MODULES

In this chapter we show that V-modules can be as useful as semisimple modules in characterizing different types of rings. We characterize rings whose V-modules are injective, rings whose singular V-modules are injective and non-singular rings whose singular modules are V-modules.

Proposition 6.1: A ring R is semisimple Artinian if and only if every V-module is injective.

Proof: If R is semisimple Artinian then every R-module is injective. Conversely, if every V-module is injective then in particular every simple R-module is injective and hence R is a left V-ring. Therefore, every R-module is a V-module and hence injective. Thus R is semisimple Artinian. □

Recall that a ring R is a left SI-ring if every singular left R-module is injective. In the next proposition we characterize SI-rings in terms of V-modules.

Propsition 6.2: The following are equivalent for a ring R.

(i) R is a left SI-ring.

(ii) Every singular V-module is injective.

Proof: (i) $\longrightarrow$ (ii): Clear.

(ii) $\longrightarrow$ (i): Let M be a singular R-module. We want to show that $J(M) = 0$. Let $0 \neq x \in M$. By Zorn's lemma, let L be a submodule of M maximal with respect to $x \notin L$. Let $\bar{\ }$: $M \longrightarrow M|L$ denote the canonical quotient map and write $\bar{x} = x + L \in M|L$. Clearly the left R-module $R\bar{x}$ is a simple singular essential submodule of $M|L$. By hypothesis, since $R\bar{x}$ is injective, it is a direct summand of $M|L$. But since $R\bar{x}$ is essential in $M|L$, $R\bar{x} = M|L$ and hence L is a maximal submodule of M with $x \notin L$. Therefore $J(M) = 0$.

Now if N is any submodule of M then $M|N$ is singular, and hence $J(M|N) = 0$ by the earlier paragraph. Whence every proper submodule of M is an intersection of maximal submodules; therefore M is a V-module, and so injective by hypothesis. Hence R is a left SI-ring. □

<u>Proposition 6.3</u>: If R is a left GV-ring, then every singular R-module is a V-module.

<u>Proof</u>: Let M be a singular R-module. Since R is a left GV-ring, every R-module is a GV-module. Therefore $J(M|N) \cap Z(M|N) = 0$ for every submodule N of M, see Proposition 3.2(ii). Since M is singular, $J(M|N) = 0$ for every submodule N of M. Thus M is a V-module. □

We do not know whether the converse to Proposition 6.3 holds. However, for non-singular rings we have the following.

<u>Proposition 6.4</u>: If R is a left non-singular ring then the following conditions are equivalent:

(i) R is a left GV-ring.

(ii) Every singular left R-module is a V-module.

<u>Proof</u>: <u>(i) $\longrightarrow$ (ii)</u>: By Proposition 6.3.

<u>(ii) $\longrightarrow$ (i)</u>: By Proposition 3.10 it is enough to show that every singular cofinitely generated left R-module is semisimple.

Let L be a singular cofinitely generated left R-module. By hypothesis L is a cofinitely generated V-module and hence a finite direct sum of simple modules by Proposition 1.1(vi). □

CHAPTER 7.

V-TORSION THEORY

In this chapter we will follow the terminology of Stenström [44] and Varadarajan [48]. As we have seen in Proposition 1.2, the class of left V-modules is closed under submodules, homomorphic images and arbitrary direct sums, and so is a herditary pretorsion class which will be denoted by $\underline{\underline{C}}_{\nu}$. If M is an arbitrary left R-module and $\nu(M)$ denotes the sum of all submodules of M belonging to $\underline{\underline{C}}_{\nu}$, then clearly $\nu(M) \in \underline{\underline{C}}_{\nu}$ as well. In this way $\underline{\underline{C}}_{\nu}$ gives rise to a preradical ν of R-mod, and ν is clearly left exact. By [44, Proposition 4.2] we get a pretorsion theory $(\underline{\underline{C}}_{\nu}, \underline{\underline{F}}_{\nu})$ for R-mod with

$$\underline{\underline{C}}_{\nu} = \{M \in R\text{-mod}: \ \nu(M) = M\}$$

$$\underline{\underline{F}}_{\nu} = \{M \in R\text{-mod}: \ \nu(M) = 0\}$$

and $\mathcal{F} = \{I : I \text{ is a left ideal of } R \text{ with } R|I \in \underline{\underline{C}}_{\nu}\}$ the corresponding linear topology.

In 7.1.1, an example is given to show that $\underline{\underline{C}}_{\nu}$ is not necessarily closed under extensions, and so in general ν is not a radical. Thus, Amitsur's transfinite process of associating a left exact radical $\bar{\nu}$ with ν yields an ascending series of preradicals $\{\nu_{\alpha}\}$ for each ordinal α, and gives rise to a ν-Lowey series for each module M.

In the first part of this chapter we study the class $\underline{\underline{C}}_{\nu}$ and its associated left exact preradical ν. We prove, among other things, that $\underline{\underline{C}}_{\nu}$ is closed under direct products if and only if $R|J(R)$ is a left

V-ring, and in this case $\nu(M) = r_M(J(R))$, a result which was noted by K.R. Fuller in [21]. We also show that $\underline{\underline{C}}_\nu$ is closed under injective envelopes (i.e. stable) if and only if R is a left V-ring. In Proposition 7.1.10, it is proved that a ring R is a left V-ring if and only if the class $\underline{\underline{C}}_\nu$ has the lifting property (L.P), (see [48]).

In the second part, we study the ν-Loewy series and obtain results similar to known results on the usual Loewy series associated to the left exact preradical Soc. An example is given to show that there are V-modules with zero socle. A ring R will be called a left semi-V-ring if every left R-module has a V-submodule. Clearly every semiartinian ring is a semi-V-ring but not vice-versa. In his work on perfect rings, H. Bass has proved that if R is a semiartinian ring then J(R) is left T-nilpotent. We shall extend this result to a larger class of rings, namely the class of semi-V-rings. We show that a ring R is a left semi-V-ring if and only if J(R) is left T-nilpotent and $R|J(R)$ is a left semi-V-ring.

We shall also investigate finite or infinite sequences of submodules, of a given module M, of the form $\{0\} = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$ or of the form $M = M^0 \supseteq M^1 \supseteq \cdots$, where all the factor modules $M_{i+1}|M_i$ or $M^i|M^{i+1}$ are V-modules. Many known properties of such series (with factor modules being semisimple) for a module over a ring R with $R|J(R)$ semisimple will be generalized.

Section 1. On the preradical ν.

We start this section with an example to show that in general ν is not a radical.

<u>Example 7.1.1</u>: Consider the following exact sequence of abelian groups $0 \longrightarrow Z_p \longrightarrow Z_{p^2} \longrightarrow Z_p \longrightarrow 0$. Since every simple module is a V-module, $\nu(Z_p) = Z_p$. And since $J(Z_{p^2}) = Z_p \neq 0$, Z_{p^2} is not a V-module (in fact $\nu(Z_{p^2}) = Z_p$). Thus the class of V-modules is not necessarily closed under extensions, whence in general ν is not a radical. □

Note that since there are left V-rings which are not right V-rings (and vise-versa), it follows that $\nu({}_RR) \neq \nu(R_R)$, i.e. ν is not left-right symmetric.

<u>Proposition 7.1.2</u>: (i) $J(R) = \bigcap\limits_{I \in \mathcal{F}} I$.

(ii) $J(R) = \bigcap\limits_{M \in \underline{\underline{C}}_{\nu}} \mathrm{Ann}_R(M)$.

(iii) $\nu({}_RR)M \subseteq \nu(M)$, for every left R-module M, (a fact which is valid for any preradical σ).

(iv) If M is a cofinitely generated module then $\nu(M) = \mathrm{Soc}(M)$.

(v) $\nu({}_RR) \cap J(R)$ and $\nu(R_R) \cap J(R)$ are nilpotent ideals. In particular if R is a semiprime ring then $\nu({}_RR) \cap J(R) = \nu(R_R) \cap J(R) = 0$.

(vi) If R is a left Noetherian ring with $\mathrm{Soc}_R R \neq 0$, then $\mathrm{Soc}_R R$ is a direct summand of $\nu({}_RR)$.

<u>Proof</u>: The proof of (i), (ii), (iii) and (iv) are straightforward.

<u>(v)</u>: Set $A = \nu({}_RR) \cap J(R)$. Since ${}_RA$ is a left V-module it follows that $J(A) = 0$ and hence $J(R)A = 0$. But since $A \subseteq J(R)$, we get $AA = 0$, i.e. $A^2 = 0$. Similarly $[\nu(R_R) \cap J(R)]^2 = 0$.

<u>(iv)</u> Let $J = \mathrm{Soc}({}_RR)$ and $K = \nu({}_RR)$. Since ${}_RK$ is a noetherian left V-module and ${}_RJ$ is a semisimple submodule of ${}_RK$, it follows from Proposition 2.2 that ${}_RJ$ is a direct summand of ${}_RK$. □

Note that if M is a V-module then $J(M) = 0$ and hence $J(R)M = 0$, i.e. every V-module is an $R|J(R)$-module. Also if $R|J(R)$ is a left V-ring and M is a left R-module with $J(R)M = 0$, then M is an $R|J(R)$-module and hence M is a V-module as an $R|J(R)$-module, whence a V-module as a left R-module, observing that R-submodules of M are the same as $R|J(R)$-submodules of M.

Now if $R|J(R)$ is a semisimple ring and M is a left V-module then by the above remarks M is semisimple. In particular if ${}_RM$ is a V-module over a semiperfect ring then ${}_RM$ is semisimple. In the next proposition we show that if $R|J(R)$ is a left V-ring then $\nu(M) = r_M(J(R))$ (a fact which was noted by K.R. Fuller in [21]). In particular if $R|J(R)$ is semisimple then $\nu(M) = \mathrm{Soc}(M) = r_M(J(R))$, for every R-module M.

Proposition 7.1.3: The following conditions on a ring R are equivalent:

(i) $R|J(R)$ is a left V-ring.

(ii) $J(M) = J(R)M$ for every left R-module M.

(iii) $\underline{\underline{C}}_\nu$ is closed under direct products.

(iv) $J(M|N) = \dfrac{J(M) + N}{N}$, for every ${}_RM$ and every submodule N of M.

(v) The Jacobson radical J preserves epimorphisms (i.e., if $M \xrightarrow{g} L \longrightarrow 0$ is exact then $J(M) \xrightarrow{g} J(L) \longrightarrow 0$ is exact).

(vi) The class $\tau = \{M \in R\text{-mod}: J(M) = 0\}$ is closed under quotients.

And in this case $\nu(M) = r_M(J(R))$, for every left R-module M.

Proof <u>(i) $\longrightarrow$ (ii)</u>: Let ${}_RM$ be a left R-module. The factor module $M|J(R)M$ is an $R|J(R)$-module, and since $R|J(R)$ is a left V-ring, $M|J(R)M$ is a V-module. Thus $J(M|J(R)M) = 0$. But since $J(R)M \subseteq J(M)$, for every module M, it follows that $0 = J(M|J(R)M) = \frac{J(M)}{J(R)M}$, and hence $J(M) = J(R)M$.

<u>(ii) $\longrightarrow$ (i)</u>: Suppose $J(M) = J(R)M$ for every left R-module M. Now, if M is an $R|J(R)$-module then $J(R)M = 0$ and hence $J(M) = 0$. Thus $R|J(R)$ is a left V-ring.

<u>(i) $\longrightarrow$ (iii)</u>: Let $M = \prod_{i\in I} M_i$, with each M_i a V-module. Thus $J(R)M_i \subseteq J(M_i) = 0$, for each $i \in I$. From which we infer that M can be regarded as an $R|J(R)$-module, and hence M is a V-module.

<u>(iii) $\longrightarrow$ (i)</u>: Define $\phi : R \longrightarrow \prod R|L$, by, $\phi(r) = \langle r + L\rangle$, $\forall\ r \in R$, where the product ranges over maximal left ideals L of R.

Clearly ϕ is an R-homomorphism with $J(R) = \text{Ker}(\phi)$. Thus $R|J(R)$ is isomorphic to a submodule of the V-module $\prod R|L$. Thus $R|J(R)$ is a left V-ring.

<u>(i) $\longrightarrow$ (iv)</u>: Let M be a left R-module and N a submodule of M. Let $\phi : M|N \longrightarrow M|(J(M) + N)$ denote the canonical quotient map. Then $\text{Ker}(\phi) = \frac{J(M) + N}{N}$. Inasmuch as $R|J(R)$ is a left V-ring, we infer that $M|(J(M) + N)$ is a V-module (being isomorphic to a factor module of the V-module $M|J(M)$). Thus $J\left[\frac{M}{J(M) + N}\right] = 0$, and hence $\phi(J(M|N)) = 0$, which implies that $J(M|N) \subseteq \text{Ker}(\phi) = \frac{J(M) + N}{N}$. Since $\frac{J(M) + N}{N} \subseteq J(M|N)$ is always true, we conclude that $J(M|N) = \frac{J(M) + N}{N}$.

(iv) $\longrightarrow$ (i): Let $A|J(R)$ be a left ideal of $R|J(R)$. Since $J(R|A) = \frac{J(R) + A}{A} = 0$, it follows that A is an intersection of maximal left ideals of R and hence $A|J(R)$ is an intersection of maximal left ideals of $R|J(R)$. Thus $R|J(R)$ is a left V-ring.

(ii) $\longrightarrow$ (v): Let $M \xrightarrow{f} N \longrightarrow 0$ be exact. Then assuming (ii), $f(J(M)) = f(J(R)M) = J(R)f(M) = J(R)N = J(N)$, whence $J(M) \xrightarrow{f} J(N) \longrightarrow 0$ is exact.

(v) $\longrightarrow$ (ii): For any $m \in M$, define $\mu_m : R \longrightarrow M$ by $\mu_m(r) = rm$. Then $\mu_m(J(R)) = J(R)m$ and the maps $\{\mu_m\}_{m \in M}$ determine an epimorphism $\mu : R^{(M)} \longrightarrow M$, where $R^{(M)}$ denote the direct sum of M copies of R. By (v), we have $J(M) = \mu(J(R^{(M)})) = \mu((J(R))^{(M)}) = J(R)M$.

(v) $\longrightarrow$ (vi): Let $M \xrightarrow{f} N \longrightarrow 0$ be an exact sequence in R-mod with $J(M) = 0$. By (v), $0 = f(J(M)) = J(N)$. Whence $N \in \tau$.

(vi) $\longrightarrow$ (v): Let $M \xrightarrow{f} N \longrightarrow 0$ be an exact sequence in R-mod. We must show that $f(J(M)) = J(N)$. Inasmuch as J is a preradical, we have $f(J(M)) \subseteq J(N)$. And since J is a radical, we have $J(N|f(J(M))) = J(N)|f(J(M))$. Let $M|J(M) \xrightarrow{\bar{f}} N|f(J(M)) \longrightarrow 0$ be the map induced by f in the obvious way. Since $J(M|J(M)) = 0$, it follows from (vi) that $J(N|f(J(M))) = 0$. Whence $J(N)|f(J(M)) = 0$, and so $J(N) = f(J(M))$.

Now suppose that one of the above conditions is satisfied. We want to show that $\nu(M) = r_M(J(R))$. Clearly $\nu(M)$ is contained in $r_M(J(R))$. And if $m \in r_M(J(R))$ then Rm is an $R|J(R)$-module and hence a V-module, therefore $Rm \subseteq \nu(M)$, i.e. $m \in \nu(M)$. Thus $\nu(M) = r_M(J(R))$. $\square$

<u>Corollary 7.1.4</u>: Let R be a ring with $R|J(R)$ semisimple. Then $\nu(M) = \mathrm{Soc}(M) = r_M(J(R))$.

<u>Proof</u>: By Proposition 7.1.3 and [2, Proposition 15.17]. □

<u>Proposition 7.1.5</u>: The following conditions on a ring R are equivalent:

(i) R is a left V-ring.

(ii) $\underline{\underline{C}}_\nu$ is closed under injective envelopes.

<u>Proof</u>: <u>(i) $\longrightarrow$ (ii)</u>: Clear, since $\underline{\underline{C}}_\nu$ = R-mod, when R is a left V-ring.
<u>(ii) $\longrightarrow$ (i)</u>: Let S be a simple R-module. Since E(S) is a cofinitely generated V-module, it is semisimple by Proposition 1.1. Therefore S = E(S) and hence S is injective. Whence R is a left V-ring. □

<u>Proposition 7.1.6</u>: The following conditions on a ring R are equivalent:

(i) R is a left V-ring.

(ii) R is a left GV-ring and $\underline{\underline{C}}_\nu$ is closed under extensions.

<u>Proof</u>: <u>(i) $\longrightarrow$ (ii)</u>: Clear, since $\underline{\underline{C}}_\nu$ = R-mod, when R is a left V-ring.
<u>(ii) $\longrightarrow$ (i)</u>: Let S be a simple left R-module and consider the exact sequence $0 \longrightarrow S \longrightarrow E(S) \longrightarrow E(S)|S \longrightarrow 0$. Inasmuch as R is a left GV-ring and $E(S)|S$ is a singular module, it follows from Proposition 6.3 that $E(S)|S$ is a V-module. Whence E(S) is a cofinitely generated V-module and hence semisimple by Proposition 1.1. Therefore S = E(S) and S is injective. Whence R is a left V-ring. □

<u>Proposition 7.1.7</u>: For a left non-singular ring R the following statements are true:

(i) R is a left SI-ring if and only if $Z(L) \subseteq Soc(L)$ for every left R-module L.

(ii) R is a left GV-ring if and only if $Z(L) \subseteq \nu(L)$ for every left R-module L.

Proof: See Proposition 5.1.6 (ii) and Proposition 6.4. □

Now, as in [48], let $\underline{\underline{C}}_c$, $\underline{\underline{C}}_f$, $\underline{\underline{C}}_n$, $\underline{\underline{C}}_{ss}$ and respectively $\underline{\underline{C}}_a$ denote the class of cyclic, finitely generated, noetherian, semisimple, respectively artinian R-modules, and let $\underline{\underline{C}}_s$ denote the class constituted by all simple R-modules and the zero module. Define the classes $\underline{\underline{T}}^\lambda$ and the functions G_λ in R-mod as follows:

$\underline{\underline{T}}^\lambda = \{M : \forall\ N \subsetneqq M,\ M|N \notin C_\lambda\}$ and $G_\lambda(M) = \cap\{N : N \subseteq M \text{ and } M|N \in \underline{\underline{C}}_\lambda\}$, where λ stands for any one of the symbols c, f, s, ss, a or ν. Also let $\underline{\underline{T}}_J = \{M : J(M) = M\}$.

It was proved in [48, Proposition 1.3] that for any ring R, $\underline{\underline{T}}^c = \underline{\underline{T}}^f = \underline{\underline{T}}^s = \underline{\underline{T}}^{ss}$. In the next proposition we show that $\underline{\underline{T}}^\nu = \underline{\underline{T}}^\lambda$, where λ stands for one of the symbols c, f, n, s or ss.

Proposition 7.1.8: $\underline{\underline{T}}^\nu = \underline{\underline{T}}^s$.

Proof: Since every simple R-module is a V-module then clearly $\underline{\underline{C}}_s \subseteq \underline{\underline{C}}_\nu$ and $\underline{\underline{T}}^\nu \subseteq \underline{\underline{T}}^s$. Conversely, if there is an R-module M with $M \in \underline{\underline{T}}^s$ and $M \notin \underline{\underline{T}}^\nu$, then there exists a proper submodule N of M with $M|N \in \underline{\underline{C}}_\nu$. Since V-modules have maximal submodules, let L|M be a maximal submodule of M|N. Then L is a proper submodule of M with $M|L \in \underline{\underline{C}}_s$, which is a clear contradiction. □

Remarks: (i) Since $\underline{\underline{T}}_J = \{M : J(M) = M\} = \{M : \forall\, N \subsetneqq M,\ M|N \notin \underline{\underline{C}}_s\}$. Then $\underline{\underline{T}}^{\nu} = \underline{\underline{T}}^{s} = \underline{\underline{T}}_J$. Whence by [48, Corollary 1.2(i)] $\underline{\underline{T}}^{\nu}$ is a torsion class.

(ii) Since the class of V-modules is closed under submodules it follows from [48, Proposition 1.5] that G_ν is a radical. And we have the following:

Proposition 7.1.9: For any R-module M we have

$$J(M) = G_s(M) = G_{ss}(M) = G_\nu(M).$$

Proof: $G_\nu(M) = \cap\{N : N \subseteq M \text{ and } M|N \in \underline{\underline{C}}_\nu\}$. Clearly if L is a maximal submodule of M then $G_\nu(M) \subseteq L$, and hence $G_\nu(M) \subseteq J(M)$. Conversely, if N is a submodule of M with $M|N \in \underline{\underline{C}}_\nu$, then N is an intersection of maximal submodules of M, thus $J(M) \subseteq N$. Whence $J(M) \subseteq G_\nu(M)$. □

Following K. Varadarajan [48, Definition 2.3], a class $\underline{\underline{C}}$ of modules is said to have the lifting property (L.P) if $M \xrightarrow{\phi} N \longrightarrow 0$ is exact in R-mod, and $B \subseteq N$, $B \in \underline{\underline{C}}$ implies the existence of an $A \subseteq M$ with $A \in \underline{\underline{C}}$ and $\phi(A) = B$.

It was proved in [48, Theorem 2.6] that for a ring R the following are equivalent:

(i) R is semisimple artinian.

(ii) The class $\underline{\underline{C}}_{ss}$ has the L.P.

(iii) The class $\underline{\underline{C}}_s$ has the L.P.

For left V-rings we obtain the following.

Proposition 7.1.10: For any ring R the following are equivalent:

(i) R is a left V-ring.

(ii) The class $\underline{\underline{C}}_\nu$ has the L.P.

<u>Proof</u>: <u>(i) $\longrightarrow$ (ii)</u>: Since $\underline{\underline{C}}_{\nu}$ = R-mod, when R is a left V-ring.

<u>(ii) $\longrightarrow$ (i)</u>: First we show that $\nu(M) \neq 0$ for any non-zero left R-module M. Let M be a non-zero left R-module and let $0 \neq x \in M$. Since Rx is finitely generated, it has a maximal submodule L. If $S = Rx|L$, then S is a simple R-module and hence a V-module. By (ii) and the exactness of the sequence $Rx \xrightarrow{\phi} Rx|L = S \longrightarrow 0$, there exists a submodule N of Rx with $N \in \underline{\underline{C}}_{\nu}$ and $\phi(N) = S$. Now, since N is a V-module, $\nu(Rx) \neq 0$, and since ν is a preradical, $\nu(M) \neq 0$.

Now, we want to show that every left R-module is a V-module. Suppose M is a non-zero left R-module with $\nu(M) \neq M$. Then $N = M|\nu(M) \neq 0$ and hence $\nu(N) \neq 0$ by the earlier paragraph. Let $\bar{x} = x + \nu(M)$ be a non-zero element of $\nu(N)$. Then $R\bar{x}$ is a left V-module. If the map $\eta : M \longrightarrow M|\nu(M)$ denotes the canonical mapping, then the sequence $\eta^{-1}(R\bar{x}) \xrightarrow{\eta'} R\bar{x} \longrightarrow 0$ is exact, where $\eta' = \eta|\eta^{-1}(R\bar{x})$. Now, since $R\bar{x} \in \underline{\underline{C}}_{\nu}$, there exists a submodule $A \subseteq \eta^{-1}(R\bar{x}) \subseteq M$ with $A \in \underline{\underline{C}}_{\nu}$ and $\eta(A) = R\bar{x}$. But since A is a V-submodule of M, $A \subseteq \nu(M)$. And since $\nu(M) \subseteq \mathrm{Ker}(\eta')$, $\eta'(A) = 0$; whence $R\bar{x} = 0$, a clear contradiction. Thus $\nu(M) = M$ for every left R-module M. Whence R is a left V-ring. □

<u>Section 2</u>. <u>ν-Loewy series</u>.

The socle series for a module M is defined transfinitely by $Soc_0(M) = 0$, $Soc_{\alpha+1}(M)|Soc_{\alpha}(M) = Soc(M|Soc_{\alpha}(M))$ and, if α is a limit ordinal, $Soc_{\alpha}(M) = \bigcup_{\beta<\alpha} Soc_{\beta}(M)$, see [17, P.470]. If $M = Soc_{\alpha}(M)$ for some ordinal α, M is called a Loewy module [9], [20] and its Loewy length is the smallest such ordinal α. A ring R is called a left Loewy ring (or said to be left semi-artinian) in case ${}_RR$ is a Loewy module or, equivalently, every non-zero left R-module contains a simple submodule, such rings were also called left socular rings by C. Faith in [17].

Loewy rings and Loewy series have been studied by many authors (e.g. H. Bass [7], S.E. Dickson [16], M Teply [45], C. Nastasescu and N. Popescu [33], T. Shores [42], [43], L. Fuchs [20], V.P. Camillo and K.R. Fuller [9] and John Dauns [15]).

The aim of this section is to introduce the notion of ν-Loewy series, ν-Loewy rings and obtain results similar to known results on the usual Loewy series and Loewy rings.

<u>Definition 7.2.1</u>: Let M be a left R-module. The ν-Loewy series for M is defined transfinitely by

$$\nu_0(M) = 0$$

$$\nu_{\alpha+1}(M)|\nu_\alpha(M) = \nu(M|\nu_\alpha(M)), \text{ and}$$

$$\nu_\alpha(M) = \bigcup_{\beta<\alpha} \nu_\beta(M), \text{ if } \alpha \text{ is a limit ordinal.}$$

The set $\{\nu_i(M)\}_i$ is sometimes called the ascending ν-Loewy chain of M.

For each module ${}_RM$ there is a smallest ordinal λ, not exceeding the cardinality of M, such that $\nu_\lambda(M) = \nu_{\lambda+1}(M)$. In this case $\lambda = \lambda(M)$ will be called the ν-length of M (is also called the ν-Loewy length of M). If $\nu_\lambda(M) = M$, we shall say M is a ν-Loewy module (or a semi-V-module). A ring R is called a ν-Lowey ring (or a semi-V-ring) if ${}_RR$ is a ν-Loewy module.

The functor $\bar{\nu}$ on R-mod defined by

$$\bar{\nu}(M) = \nu_{\lambda(M)}(M)$$

is the smallest radical such that $\nu \leq \bar{\nu}$. A module M will be called a $\bar{\nu}$-module if $\bar{\nu}(M) = M$. We state some useful remarks:

Remarks 7.2.2: (i) $Soc_{\alpha}(M) \subseteq \nu_{\alpha}(M)$, $\forall\alpha$.

(ii) Each ν_{α} is a left exact preradical.

(iii) A left R-module M is a $\bar{\nu}$-module if and only if M is a semi-V-module if and only if every non-zero homomorphic image of M has a non-zero V-submodule.

(iv) A ring R is a left semi-V-ring if and only if every left R-module has a V-submodule, if and only if $\nu(M)$ is essential in M, for every left R-module M, if and only if every left R-module is a semi-V-module.

(v) $\bar{\nu}$ is a left exact radical.

(vi) $\nu(M)$ is an essential submodule of $\bar{\nu}(M)$.

(vii) For every left R-module M, $\bar{\nu}(M)$ is the smallest submodule L of M such that $M|L \in \underline{\underline{F}}_{\nu}$ (i.e. $\nu(M|L) = 0$).

Next we give an example of a left semi-V-ring which is not left semi-artinian. Thus there are V-modules with zero socle.

Example 7.2.3: consider the ring R = k[y,D] of differential polynomials over a universal field k. In [14], Cozzens has proved that R has the following properties:

(i) R is a left Noetherian ring.

(ii) R is a left V-ring.

(iii) R is not regular.

It follows from (ii) that, every left R-module is a V-module. Thus R is a semi-V-ring. If R is left semi-artinian then Soc(M) is essential in M, for every left R-module M. Inasmuch as R is left Noetherian left V-ring, and hence every semisimple module is injective, it follows that Soc(M) is a direct summand of M, for every left R-module M. Thus M = Soc(M) for every left R-module M, and therefore R is a semisimple ring - a clear contradiction with (iii). Hence R is not semiartinian. Thus there exists a left R-module M with soc(M) = 0, in particular R is not a right perfect ring. □

Recall that a module M is called a weakly GV-module (WGV-module) if every proper essential submodule of M is an intersection of maximal submodules. A ring R is said to be a left WGV-ring if the left R-module ${}_RR$ is a WGV-module.

Proposition 7.2.4: If M is a left WGV-module then $\nu_2(L) = L$, for every homomorphic image L of M. In particular every WGV-module is a semi-V-module.

Proof: Let M be a WGV- module and L be a homomorphic image of M. By Proposition 3.21 (i), L is a WGV-module and by Proposition 3.19, $L|\mathrm{Soc}(L)$ is a V-module. Since Soc $\leq \nu$, it follows that $L|\nu(L)$ is a V-module, and hence $\nu_2(L) = L$. Whence M is a semi-V-module. □

Corollary 7.2.5: If R is a left WGV-ring then $\nu_2(M) = M$ for every left R-module M. In particular $\nu(M)$ is essential in M for every ${}_RM$.

In [7], Bass proved that a ring R is left perfect (i.e. J(R) is left T-nilpotent and $R|J(R)$ is semisimple) if and only if R is right

Loewy and contains no infinite set of orthogonal idempotents. In [33], Nastasescu and Popescu proved that a ring R is right Loewy ring if and only if its radical J is left T-nilpotent and $R|J$ is right Loewy. In the next proposition we extend this result to semi-V-rings.

Proposition 7.2.6: The following conditions on a ring R are equivalent:

(i) R is a left semi-V-ring.

(ii) $J(R)$ is right T-nilpotent and $R|J(R)$ is a left semi-V-ring.

Proof: (i) $\longrightarrow$ (ii) (Adopted from [7, Theorem P]).

Let $\{\nu_\alpha\}_\alpha$ be the ascending ν-Loewy series of the left R-module ${}_RR$. Since R is a left semi-V-ring, $R = \nu_\alpha$ for some ordinal α. For each $a \in R$, define $h(a)$ to be the smallest ordinal α such that $a \in \nu_{h(a)}$. Then it is easy to see that $h(a)$ is not a limit ordinal, for any $a \in R$. Write $h(a) = \beta + 1$, for some ordinal β, and let $J = J(R)$. Inasmuch as $\nu_{\beta+1}|\nu_\beta = \nu(R|\nu_\beta)$ is a V-module, it follows that $J\cdot(\nu_{\beta+1}|\nu_\beta) = 0$, and hence $J\cdot\nu_{\beta+1} \subseteq \nu_\beta$. Thus $h(ba) < h(a)$ for every $b \in J$, unless $a = 0$. Now, suppose that there is an infinite sequence $\{a_n\}$ of elements of J such that $a_n \cdots a_1 \neq 0$ for every $n \in N$. Then there is a strictly decreasing chain of ordinals $h(a_1) > h(a_2a_1) > \cdots > h(a_n \cdots a_1) > \cdots$, which is impossible. Hence $J(R)$ is right T-nilpotent. Clearly R a left semi-V-ring implies that $R|J(R)$ is a left semi-V-ring.

(ii) $\longrightarrow$ (i): We want to show that $\nu(M) \neq 0$ for every non-zero left R-module M. Let ${}_RM$ be a non-zero module and suppose $J(R)N \neq 0$ for every submodule N of M. Then there exists $a_1 \in J(R)$, such that

$a_1M \neq 0$. Thus $Ra_1M \neq 0$, and there is $a_2 \in J(R)$ such that $a_2a_1M \neq 0$. Proceeding this way, we can find $a_1, a_2, \ldots$ a sequence of non-zero elements of $J(R)$ such that $a_n \ldots a_1 \neq 0$ for each $n \in \mathbb{N}$, a contradiction with the T-nilpotence of $J(R)$. Thus there is a non-zero submodule N of M with $J(R)N = 0$, i.e. N can be regarded as an $R|J(R)$-module, and hence N has a V-submodule, i.e. $0 \neq \nu(N) \subseteq \nu(M)$. □

Corollary 7.2.7: If $R|J(R)$ is a left V-ring. Then the following conditions are equivalent:

(i) R is a semi-V-ring.

(ii) $J(R)$ is right T-nilpotent.

(iii) Every left R-module has a maximal submodule.

Proof: Since $R|J(R)$ is a left semi-V-ring, the equivalence between (i) and (ii) is an immediate consequence of Proposition 7.2.6.

(ii) $\rightarrow$ (iii): Let M be a non-zero left R-module. From the right T-nilpotency of $J(R)$, it follows that $J(R)M \neq M$, and hence $M|J(R)M$ is a non-zero $R|J(R)$-module. Since $R|J(R)$ is a left V-ring, $M|J(R)M$ has a maximal submodule, $N|J(R)M$ say. Hence N is a maximal submodule of M.

(iii) $\rightarrow$ (ii): a well-known result, due to H.Bass. However the proof included here is due to Rosenberg and Zelinsky [37]). Let $x_1, \ldots, x_n, \ldots$ be a countable basis of a free module P, let $a_1, \ldots, a_n, \ldots$ be an infinite sequence of elements of $J(R)$, and let f be the element of $S = \mathrm{End}_R P$ mapping $x_i \mapsto a_i x_{i+1}$, $i = 1,2,\ldots$. Since $J(\mathrm{Hom}_R(P,P)) = \mathrm{Hom}_R(P, J(R)\cdot P)$ (see[17, Corollary 22.3]), it follows that $f \in J(S)$, hence $(1-f)$ is a unit in S. Let $y = (1-f)^{-1}x_1$, and write $y = \sum_{i=1}^{\infty} b_i x_i$ with $b_i \in R$, $b_n = 0$, $n \geq k$.

Then $$x_1 = (1-f)y = (\Sigma\, b_i x_i) - (\Sigma\, b_i a_i x_{i+1})$$
$$= b_1 x_1 + (b_2 - b_1 a_1)x_2 + \sum_{n>2} (b_n - b_{n-1} a_{n-1})x_n$$

since $\{x_n : n \geq 1\}$ is a free basis, then $b_1 = 1$ and $b_n = a_1 a_2 \cdots a_{n-1}$, $n \geq 2$. Thus $b_k = a_1 a_2 \cdots a_{k-1} = 0$. □

Proposition 7.2.8: If R is a left Noetherian, left semi-V-ring then every R-module has a maximal submodule.

Proof: Let $\{\nu_\alpha({}_R R)\}_\alpha$ be the ν-Loewy series associated with the left R-module ${}_R R$. Since R is a left semi-V-ring, $R = \nu_\lambda(R)$, for some ordinal λ, and since R is left Noetherian λ must be finite. We claim that $\nu_\lambda(M) = M$ for every left R-module M. Suppose on the contrary $\nu_\lambda(M) \neq M$ for some non-zero left R-module M. Let $y \in M \backslash \nu_\lambda(M)$. Then $y \notin \nu_\lambda(Ry)$, since ν_λ is a preradical. Let $g : R \longrightarrow Ry$ be the obvious epimorphism. Then $g(\nu_\lambda(R)) \subseteq \nu_\lambda(Ry)$, and hence $Ry = g(R) = g(\nu_\lambda(R)) \subseteq \nu_\lambda(Ry)$, which implies that $y \in \nu_\lambda(Ry)$, a contradiction. Now, if M is a non-zero V-module then clearly M has a maximal submodule. Otherwise M has a ν-Loewy series of length $n \leq \lambda$, for some positive integer $n > 1$, and in this case $M|\nu_{n-1}(M) = \nu_n(M)|\nu_{n-1}(M)$ is a V-module and so has a maximal submodule, $N|\nu_{n-1}(M)$ say. Thus N is a maximal submodule of M. □

Proposition 7.2.9: For a commutative Noetherian ring R the following conditions are equivalent:

(i) R is a semi-artinian ring.

(ii) R is a semi-V-ring.

(iii) Every R-module has a maximal submodule.

(iv) $J(R)$ is T-nilpotent and $R|J(R)$ is regular.

(v) R is a perfect ring.

(vi) R is an Artinian ring.

Proof: The equivalence between (iii) and (iv) is satisfied for any commutative ring, see Koifman's theorem [31, Theorem 1.8]. For the equivalence between (iii), (v) and (vi), see Hamsher's result [26, Theorem 1]. For the equivalence between (i) and (iii), see [33, Corollary 3.1].

(i) $\longrightarrow$ (ii): Since every simple module is a V-module.

(ii) $\longrightarrow$ (iii): By Proposition 7.2.8. □

Section 3: Chains of modules with V-quotients.

In this section we will study finite or infinite sequences of submodules, of a given module M, of the form $\{0\} = M_0 \subseteq M_1 \subseteq \cdots$ or of the form $M = M^0 \supseteq M^1 \supseteq \cdots$, where all the factor modules $M_{i+1}|M_i$ or $M^i|M^{i+1}$ are V-modules. And we will generalize those results which have been obtained in [15].

From now on it will be assumed that $R|J(R)$ is a left V-ring, $J = J(R)$ and J^k the k-th power of J, where $k > 0$ (if $k = 0$ we define $J^0 = R$).

Theorem 7.3.1 Let R be a ring with $R|J(R)$ a left V-ring and M be a left R-module. Then the following hold for all integers $k = 0,1,2,\ldots$

(i) $\nu_k(M) = \mathrm{Ann}_M(J^k) = \{m \in M : J^k m = 0\}$.

(ii) If $\{0\} = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$ is any series of submodules of M with V-quotient modules $M_{k+1}|M_k$ for $k = 0,1,2,\ldots$, then $M_k \subseteq \nu_k(M)$.

Proof: (i) If $k = 0$ then by definition $J^0 = R$ and $\nu_0(M) = 0$, and hence $\nu_0(M) = \mathrm{Ann}_M(J^0) = 0$.

Assume (i) is true for $k - 1$, i.e. $\nu_{k-1}(M) = \mathrm{Ann}_M(J^{k-1})$. Let $L = \nu_k(M)|\nu_{k-1}(M)$. Since L is a V-module, $J(L) = 0$ and hence $J(R)L = 0$. Whence $J\cdot\nu_k(M) \subseteq \nu_{k-1}(M)$. But $\nu_{k-1}(M) = \mathrm{Ann}_M(J^{k-1})$ and hence $J\cdot\nu_k(M) \subseteq \mathrm{Ann}_M(J^{k-1})$, i.e. $J^{k-1}\cdot J\cdot\nu_k(M) = 0$. Thus $J^k\nu_k(M) = 0$, i.e. $\nu_k(M)\subseteq \mathrm{Ann}_M(J^k)$. On the other hand, since $J^k\cdot\mathrm{Ann}_M(J^k) = 0$, it follows that $J^{k-1}\cdot J\cdot\mathrm{Ann}_M(J^k) = 0$, whence $J\cdot\mathrm{Ann}_M(J^k) \subseteq \mathrm{Ann}_M(J^{k-1})$ and the module $\mathrm{Ann}_M(J^k)|\mathrm{Ann}_M(J^{k-1})$ can be regarded as an $R|J$-module. Since $R|J$ is a left V-ring, $\mathrm{Ann}_M(J^k)|\mathrm{Ann}_M(J^{k-1})$ is a V-module. From the induction step, it follows that $\mathrm{Ann}_M(J^k)|\nu_{k-1}(M)$ is a V-module and hence $\mathrm{Ann}_M(J^k)|\nu_{k-1}(M) \subseteq \nu(M|\nu_{k-1}(M)) = \dfrac{\nu_k(M)}{\nu_{k-1}(M)}$. Therefore, $\mathrm{Ann}_M(J^k) \subseteq \nu_k(M)$. Whence $\nu_k(M) = \mathrm{Ann}_M(J^k)$.

(ii) Clearly $M_0 = \nu_0(M) = 0$ and $\nu_1(M) = \nu(M) \supseteq M_1$. Assume $M_{k-1} \subseteq \nu_{k-1}(M)$. Since $M_k|M_{k-1}$ is a V-module, it follows that $J\cdot M_k \subseteq M_{k-1}$ and hence that $J\cdot M_k \subseteq \nu_{k-1}(M)$. Thus, $J^kM_k \subseteq J^{k-1}\nu_{k-1}(M) = 0$, i.e. $M_k \subseteq \mathrm{Ann}_M(J^k)$, therefore $M_k \subseteq \nu_k(M)$ by (i). □

Corollary 7.3.2: If $R|J(R)$ is semisimple, then the following holds for all integers $k = 0,1,2,\ldots$

(i) $\mathrm{Soc}_k(M) = \mathrm{Ann}_M(J^k)$.

(ii) If $\{0\} = M_0 \subseteq M_1 \subseteq \cdots$ is any series of submodules of M with semisimple quotient modules $M_{k+1}|M_k$ for $k = 0,1,2,\ldots,$ then $M_k \subseteq \mathrm{Soc}_k(M)$.

Proof: Since $R|J(R)$ is semisimple, $\nu(M) = \mathrm{Soc}(M)$ for every R-module M and hence $\nu_k(M) = \mathrm{Soc}_k(M)$ for every $k = 0,1,2,\ldots$. □

This is Theorem 2 in [15].

Corollary 7.3.3: If J is nilpotent with index of nilpotency equal to n (i.e. $J^{n-1} \neq 0$ and $J^n = 0$). Then the ν-length of R is exactly n. In particular the ν-length of any left R-module is at most n.

Definition 7.3.4: For a left R-module M over an arbitrary ring R, set $J_0(M) = M$, $J_1(M) = J(M)$ the intersection of all the maximal submodules of M (the empty intersection is by convention all of M). For any positive integer $k = 1,2,\ldots$ the submodule $J_{k+1}(M)$ is defined inductively by $J_{k+1}(M) = J(J_k(M))$. If $J_\alpha(M)$ has been defined for all ordinals $\alpha < \beta$ where β is a limit ordinal, set $J_\beta(M) = \cap\{J_\alpha(M) : \alpha < \beta\}$ and define $J_{\beta+1}(M)$ to be $J_{\beta+1}(M) = J(J_\beta(M))$. The series $M = J_0(M) \supseteq J_1(M) \supseteq \cdots$ is called the upper Loewy series of M over R (see [15]).

Remark 7.3.5: If $J = J(R)$ then $J^k = J_k$ for every integer $k = 0,1,2,\ldots$ (since $R|J(R)$ is a left V-ring, $J(M) = J(R)M$ for every left R-module M. Thus $J_{k+1}(R) = J(R)\cdot J_k(R)$, and by inductive hypothesis, $J_{k+1}(R) = J\cdot J^k = J^{k+1}$).

Theorem 7.3.6: Let R be a ring with $R|J(R)$ a left V-ring. Write $J = J(R)$. Then the following hold for all $k = 0,1,2,\ldots$

(i) $J_k(M)|J_{k+1}(M)$ is a V-module.

(ii) $J_k(M) = J_k(R)M$.

(iii) If $M = M_0 \supseteq M_1 \supseteq \cdots$ is any series of submodules of M with each quotient $M_k|M_{k+1}$ is a V-module for $k = 0,1,2,\ldots$ then $M_k \supseteq J_k(M)$.

Proof: (i) $J(J_k(M)|J_{k+1}(M)) = J(J_k(M)|J(J_k(M))) = 0$ and hence $J(R)\cdot(J_k(M)|J_{k+1}(M)) = 0$ and consequently the module $J_k(M)|J_{k+1}(M)$ can be regarded as an $R|J(R)$-module and hence a V-module, since $R|J(R)$ is a left V-ring.

(ii) If $k = 1$, then $J_1(M) = J(M)$ and $J_1(R)M = J(R)M = J(M)$, since R is a left V-ring. Assume by induction that $J_{k-1}(M) = J_{k-1}(R)M$. Then

$$\begin{aligned} J_k(R)M &= J(J_{k-1}(R))M \\ &= J(R)\cdot J_{k-1}(R)M, \text{ by Proposition 7.1.3 (i} \longrightarrow \text{ii).} \\ &= J(R)\cdot J_{k-1}(M), \text{ induction step.} \\ &= J(J_{k-1}(M)), \text{ since } R|J(R) \text{ is a left V-ring.} \\ &= J_k(M). \end{aligned}$$

(iii) If $k = 0$, $J_0(M) = M = M_0$.

Assume it is valid for $k - 1$, i.e. $J_{k-1}(M) \subseteq M_{k-1}$. Since $M_{k-1}|M_k$ is a V-module, it follows that $J(M_{k-1}|M_k) = 0$ and hence $J(R)\cdot M_{k-1} \subseteq M_k$. Since $J_k(M) = J(J_{k-1}(M)) = J(R)\cdot J_{k-1}(M)$ we get $J_k(M) \subseteq J(R)M_{k-1} \subseteq M_k$. □

Corollary 7.3.7: For an arbitrary ring R the following conditions are equivalent:

(i) $R|J(R)$ is a left V-ring.

(ii) For any left R-module M and any submodule N of M, $J_k(M|N) = (J_k(M) + N)|N$, for every non-negative integer k.

(iii) For every left R-module M and every $k = 0,1,2,\ldots$ $J_k(M) = J_k(R)M$.

Proof: An immediate consequence of Proposition 7.1.3 and Proposition 7.3.6. □

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