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REISSNER-SAGOCI PROBLEM IN THE LINEAR THEORY OF ELASTICITY

by

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Reissner-Sagoci Problem in Linear Theory of Elasticity" submitted by Anjali Singh in partial fulfillment of the requirements for the degree of Master of Science.

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ABSTRACT

The present thesis is to survey the Reissner-Sagoci problem in the linear theory of elasticity. In Chapter I we have given a brief summary of the various Reissner-Sagoci type problems solved so far. Chapter II deals with the derivation of the basic equations of linear theory of elasticity. In Chapter III we have given the solution of the classical Reissner-Sagoci problem as discussed by Reissner and Sagoci and later on by Sneddon. In Chapter IV we have solved a Reissner-Sagoci type problem for a homogeneous elastic layer bonded to another homogeneous elastic layer. The numerical values have been tabulated and displayed graphically.

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CHAPTER I

INTRODUCTION

In 1944 Reissner and Sagoci [22] investigated the torsional oscillations produced in a semi-infinite, homogeneous, isotropic medium by a periodic shear stress applied in an axially symmetrical manner to a circular area of the plane surface of the medium. Earlier in 1937 the same problem was studied by Reissner [21] under the assumption that the law of variation of shear stresses over the surface is given. Reissner [21] obtained a solution of the problem for the case of shear stresses increasing linearly from the centre of the stressed surface region to the edge of the stressed surface region, by means of the Fourier-Bessel integral method. He also obtained the solution of the problem when the law of variation of displacement over the loaded portion of the surface is prescribed. Mathematically this is a mixed boundary problem and the Fourier-Bessel method reduces the problem to an integral equation problem which may be further reduced to the problem of solving an infinite number of linear algebraic equations for an infinite number of unknowns. Reissner and Sagoci [22] in their paper obtained solution of the mixed boundary problem by introducing in a suitable manner a system of oblate spheroidal coordinates, for static case of torsional deformation. This problem is now known as the Reissner-Sagoci problem.

Sagoci [23] also studied the forced torsional oscillations of an elastic half-space under the action of a rigid circular disk oscillating periodically about an axis through its centre. He found the expressions for displacement and shear stresses at any point of the surface and derived the relation between the angle of rotation and the resultant moment of the surface shear stress. He also gave a method for the determination of the shear modulus of the half-space through a study of its torsional oscillations.

Later on in 1947 Sneddon [26] solved the same Reissner-Sagoci problem by a different approach. He reduced the problem to a pair of dual integral equations by using the Hankel transform method. In the static case these dual integral equations reduce to the known dual integral equations whose solution was given by Titchmarch [33]. Bycroft [1] gave an approximate treatment of the oscillations in both the half-space and a stratum by assuming the distributions of the shear stress under the disk in both cases to be the same as the distribution in the static case for the half-space and then applying the Hankel transform to obtain the displacement which is only valid if the frequency of the oscillations is very small and the thickness of the stratum is very large. Collins [3] solved the torsional problem of an elastic half-space by supposing the displacement at any point in the half-space to be due to a distribution of wave sources over the part of the free surface in contact with the disk. He obtained the integral equation governing the problem, holding for all frequencies of oscillations, which he solved approximately by iteration for small frequencies. The same method was extended to the oscillations in an elastic stratum. Gladwell [15] solved the Reissner-Sagoci problem for an elastic layer of finite thickness, when the lower face is either stress free or rigidly clamped, by reducing the mixed boundary value problem to a Fredholm integral equation by Noble's [19] method and gave an approximate solution for small values of the reduced frequency and large values of the stratum depth. His analysis was much simpler for the case when the elastic layer was rigidly clamped to a rigid foundation, compared to that of Collins [3].

Sneddon [29] in his book on mixed boundary value problems of potential theory used Hankel transforms to solve mixed boundary value problems of the linear theory of elasticity. It was shown by him that the mixed boundary value problems reduce to the solution of dual integral equations. A simple method for the solution of dual integral equations was developed by Sneddon [27]. Making use of his own method Sneddon [28] in 1966 obtained the solution of the Reissner-Sagoci problem. He [31] also considered the problem of determining the distribution of stress in the interior of a very long circular cylinder of homogeneous isotropic material when a circular area of its flat surface is forced to rotate through an angle about the axis of the cylinder, whose curved surface is fixed.

Freeman and Keer [14] investigated a torsion problem of an elastic cylindrical rod welded to an elastic half-space. The problem was formulated so as to involve coupling between dual integral equations and Dini-Series, and these equations were reduced to a single integral equation which was solved numerically.

Keer, Jabali and Chantaramungkorn [18] considered the problem of a layer bonded to an elastic half-space, where the layer is driven by torsional oscillations of a bonded rigid circular disk. They reduced the problem to a Fredholm integral equation of the second kind which was solved numerically. They also developed dynamic stiffnesses for a range of layer thicknesses, material properties and frequencies. Jabali [16] considered the static solution to the problem of a layer bonded to an elastic half-space, when the layer is driven by torsional rotation of a bonded rigid circular disk. The problem was reduced to the Fredholm integral equation of the second kind and an iterative solution, had been obtained for large values of the ratio of the stratum depth to the radius of the disk.

For a non-homogeneous, isotropic, elastic half-space Reissner-Sagoci problem has been first solved by Kassir [17]. He assumed the modulus of rigidity of the medium in the form $\mu(z) = \mu_o z^\alpha$ $(0 \leq \alpha \leq 1)$ where μ_0 is some real constant, and z is the coordinate perpendicular to the plane boundary of the half-space. He solved the problem by reducing it to a pair of dual integral equations and then solving it by Copson's [5] elementary solutions. He also solved the problem for a long circular cylinder of finite radius whose curved surface may be either clamped or stress free by reducing the problem to a Fredholm integral equation of the second kind and using the method developed by Sneddon and Srivastava [30].

Dhaliwal and Singh [7] considered the problem of torsion, by an annular die, of an elastic layer of assumed thickness bonded to an elastic half-space when both the elastic layer and half-space are assumed to be isotropic, homogeneous and consisting of different elastic constants. The problem was reduced to the solution of a system of four Fredholm integral equations of the second kind in four unknown functions. An iterative solution of these integral equations was obtained for the case of simple rotation of the annular die through a small angle for $a/b \ll 1$ and $b/h \ll 1$ where h is the layer thickness and a and b are the inner and outer radii of the annular die.

Protsenko [20] considered the elastic equilibrium of a half-space being twisted by the rotation of a rigid cylindrical die, with flat base, bonded to the half-space where the modulus of elasticity of the half-space varies with the depth by $\mu(z) = \mu_0 z^{\alpha}$ $(0 \leq \alpha \leq 1)$ and μ_0 is a constant. He solved the problem by reducing it to a pair of dual integral equations and solving them by Sneddon's method. Later on Chuaprasert and Kassir [2] solved the same problem by assuming the modulus of rigidity $\mu(z) = \mu_0(c+z)^{\alpha}$ where μ_0, c and α are real constants. By employing the Hankel transform and Fourier-Bessel series method to the problem, for both half-space and a semi-infinite circular cylinder whose lateral surface is clamped, they reduced the problem to Fredholm integral equation of the second kind.

Singh and Dhaliwal [24] considered the torsion of an elastic layer by two circular disks of rigid material and of different radii bonded to the opposite faces of an infinite elastic layer, rotated through different angles. The solution of the problem was reduced to a pair of simultaneous Fredholm integral equations which were then solved by the method of iteration as well as numerically.

Dhaliwal and Singh [6] considered the Reissner-Sagoci problem for an isotropic, nonhomogeneous, elastic layer of finite thickness and modulus of rigidity $\bar{\mu}_1(z) = \mu_1(a+z)^\alpha$ perfectly bonded to an isotropic, nonhomogeneous elastic half-space of modulus of rigidity $\bar{\mu}_2(z) = \mu_2(b+z)^\beta$ where μ_1, μ_2, a, b, α and β are real constants. They reduced the problem to the solution of a Fredholm integral equation of the second kind which was solved iteratively. By assigning different values to α, β, a, b, μ_1 and μ_2, they derived solutions of many of the earlier solved problems by Reissner and Sagoci [22], Jabali [16], Chuaprasert and Kassir [2] and Gladwell [15] for a half-space and a layer of finite thickness.

Dhaliwal, Singh and Sneddon [10] considered a Reissner-Sagoci type problem for an elastic cylinder embedded in an elastic half-space of different modulus of rigidity, assuming that there is perfect bonding at the common cylindrical surface and torque is applied to the cylinder through a rigid disk bonded to its flat surface. They reduced the problem to a pair of dual integral equations, by means of the integral transforms, which were then reduced to a Fredholm integral equation of the second kind and solved numerically. Also Dhaliwal, Singh and Rokne [8] considered the torsion of a homogeneous isotropic elastic hemisphere embedded in a semi-infinite isotropic elastic medium when a rigid circular disk is clamped to the plane face of the hemisphere and the stresses are caused by the rotation of the disk through a small angle. They reduced the problem to a pair of dual integral equations by assuming appropriate solutions of the two regions. These dual integral equations were further reduced to the solution of a Fredholm integral equation of the second kind which was solved numerically. By taking the same values of modulus of rigidities of the two regions, they derived the solution to the classical Reissner-Sagoci problem which was in agreement with the solution of Reissner and Sagoci [22].

In 1982 Erguven [11] considered the Reissner-Sagoci problem for a transversely isotropic, nonhomogeneous elastic half-space. The modulus of rigidity of the medium was assumed to be variable as a power of the radial coordinate in the form r^{β} ($\beta \geq 0$). He reduced the problem to the solution of dual integral equations for the determination of an unknown function which were then solved by taking an appropriate form of the unknown function. The expressions for stress, torque and the displacement were found.

Dhaliwal, Singh, Rokne and Vrbik [9] found the stress distribution in an homogeneous isotropic elastic hemisphere embedded in another semi-infinite homogeneous isotropic elastic medium when a rigid annular disk is clamped to the plane face of the hemisphere and the stresses were caused by the rotation of the annular disk through an angle β. By assuming appropriate solution for two regions, they reduced the solution of the problem into triple integral equations. These triple integral equations were further reduced into a Fredholm integral equation of the second kind by Cook's method [4].

Erguven [12] considered the torsional stresses and displacement of a transversely isotropic elastic layer of finite thickness for which torsional shearing forces are prescribed on its boundary surface. He solved the problem by reducing it to a pair of dual integral equations which were further reduced to a Fredholm equation of the second kind. He also derived solutions for some particular cases.

The intent of the present thesis is to study the Reissner-Sagoci problem for various configurations. The organization of the chapters of the thesis are as follows.

The second chapter deals with the basic definitions and derivation of fundamental equations of elasticity to be used in later chapters.

In the third chapter we have derived a complete solution of Reissner-Sagoci problem solved by Reissner and Sagoci [22] and Sneddon [26].

The fourth chapter forms the main contribution of this thesis in which we have considered a Reissner-Sagoci type problem, for a homogeneous isotropic, elastic layer bonded to another homogeneous isotropic elastic layer of different modulus of rigidity and thickness, whose lower surface is bonded to a rigid foundation. A rigid circular cylinder is bonded to the top surface of the upper layer and it is rotated through a small angle and the rest of the surface is stress free. By the use of Hankel transforms the problem is reduced to the solution of a pair of dual integral equations. The dual integral equations are further reduced to a Fredholm equation of the second kind which has been solved numerically to find the numerical values of the torque required to rotate the rigid cylinder through a small angle.

CHAPTER II

BASIC EQUATIONS OF ELASTICITY

§2.1 Deformation and Strain Tensor

In the formulation of continuum mechanics the configuration of a solid body is described by a continuous mathematical model whose geometrical points are identified with the place of the material particles of the body. When such a continuous body changes its configuration under some physical actions, we impose the assumption that the change is continuous; that is, neighbourhoods are changed into its neighbourhoods. Thus, when the particles of a continuous body move so that the distance between particles is changed, the body is said to be deformed. When the distance between the particles is unchanged, the body is said to be undeformable or a rigid body.

Let a system of coordinates a_1, a_2, a_3 be chosen so that a point P of a body at a certain instant of time is described by the coordinates a_i $(i = 1,2,3)$. At a later instant of time, the body moves to a new configuration; the point P moves to Q with coordinates x_i $(i = 1,2,3)$ with respect to new system of coordinates x_1, x_2, x_3. The coordinates a_1, a_2, a_3 and x_1, x_2, x_3 may be curvilinear; and describe a Euclidean space.

The change of configuration of the body will be assumed continuous and the mapping from P to Q is assumed to be one-to-one. The equation of transformation can be written as

$$x_i = \bar{x}_i(a_1,a_2,a_3), \quad (i = 1,2,3) \tag{2.1.1}$$

which has a unique inverse

$$a_i = \bar{a}_i(x_1,x_2,x_3), \quad (i = 1,2,3) \tag{2.1.2}$$

for every point of the body. The functions $\bar{x}_i(a_j)$ and $\bar{a}_i(x_j)$ are assumed to be continuous and differentiable.

We shall be concerned with the description of the strain of the body, that is, with the stretching and distortion of the body. If P, P', P'' are three neighbouring points forming a triangle in the original configuration, and if they are transformed to points Q, Q', Q'' in the deformed configuration, the change in area and angles of the triangle is completely determined if we know the change in length of the sides. But the "location" of the triangle is undetermined by the change of the sides. Similarly, if the change of the length between any two arbitrary points of the body is known, the new configuration of the body will be completely defined except for the location of the body in space. Therefore the description of the change in distance between any two points of the body is the key to the analysis of deformation.

Let us consider a cartesian coordinate system in which the point P^0 (a_1^0,a_2^0,a_3^0) is moved to the point Q^0 (x_1^0,x_2^0,x_3^0) after deformation. We denote the small displacement of the point P^0 by

$$u_i(a_1^0,a_2^0,a_3^0) = x_i^0 - a_i^0, \quad (i = 1,2,3). \tag{2.1.3}$$

Let us consider a neighbourhood point $P(a_1,a_2,a_3)$ and let $\vec{A}$ be the vector joining P^0 and P. Let $Q(x_1,x_2,x_3)$ be the deformed position of P. The displacement u_i at the point P is

$$u_i(a_1,a_2,a_3) = u_i(a_1^0+A_1,a_2^0+A_2,a_3^0+A_3) = x_i - a_i, \quad (i = 1,2,3) \tag{2.1.4}$$

where A_1, A_2, A_3 are the components of $\vec{A}$. The deformed vector $\vec{X}$ has components

$$X_i = x_i - x_i^0 \tag{2.1.5}$$

and $\delta\vec{A} = \vec{X} - \vec{A}$ has component

$$\begin{aligned}
\delta A_i &= (x_i-x_i^0) - (a_i-a_i^0) \\
&= (x_i-a_i) - (x_i^0-a_i^0) \\
&= u_i(a_1^0+A_1,a_2^0+A_2,a_3^0+A_3) - u_i(a_1^0,a_2^0,a_3^0) \\
&= \left[\frac{\partial u_i}{\partial a_j}\right]_0 A_j
\end{aligned} \tag{2.1.6}$$

plus the remainder in the Taylor's expansion of the function $u_i(a_1^0+A_1,a_2^0+A_2,a_3^0+A_3)$ and the subscript 0 indicates that the derivatives are to be evaluated at the point P_0. Writing

$$\frac{\partial u_i}{\partial a_j} = u_{i,j} \tag{2.1.7}$$

and dropping the subscript, we have

$$\delta A_i = u_{i,j} A_j \; . \tag{2.1.8}$$

Now if we assume that the displacement u_i, and its partial derivatives are so small that their product can be neglected then (2.1.8) defines an infinitesimal affine transformation of the neighbourhood of the point. Now

$$\begin{aligned}\delta A_i &= u_{i,j} A_j \\ &= \left[\frac{u_{i,j}+u_{j,i}}{2} + \frac{u_{i,j}-u_{j,i}}{2}\right] A_j \\ &= (e_{ij}-\omega_{ij}) A_j \end{aligned} \tag{2.1.9}$$

where

$$e_{ij} = \frac{1}{2}\,(u_{i,j}+u_{j,i}) \tag{2.1.10}$$

$$\omega_{ij} = \frac{1}{2}\,(u_{i,j}-u_{j,i}). \tag{2.1.11}$$

The symmetric coefficient e_{ij} is called strain tensor and it characterizes pure deformation. The skew-symmetric coefficient ω_{ij} corresponds to rigid body motion.

In general consider an infinitesimal line element connecting the point $P(a_i)$ to a neighbouring point $P'(a_i+da^i)$. The square of the length ds_0 of PP' in the original configuration is given by

$$ds_0^2 = a_{ij}da^i da^j \tag{2.1.12}$$

where a_{ij} evaluated at the point P is the Euclidean metric tensor for the coordinate system a_i. When the points P, P' are deformed to the points $Q(x_i)$ and $Q'(x_i+dx^i)$, respectively, the square of the length ds of the new element QQ' is

$$ds^2 = g_{ij}dx^i dx^j \tag{2.1.13}$$

where g_{ij} is the Euclidean metric tensor for the coordinate system x_i.

The equations (2.1.12) and (2.1.13) may also be written as

$$ds_0^2 = a_{ij} \frac{\partial a_i}{\partial x_\ell} \cdot \frac{\partial a_j}{\partial x_m} dx^\ell dx^m \tag{2.1.14}$$

$$ds^2 = g_{ij} \frac{\partial x_i}{\partial a_\ell} \cdot \frac{\partial x_j}{\partial a_m} da^\ell da^m. \tag{2.1.15}$$

The difference between the squares of the length elements may be written either as

$$ds^2-ds_0^2 = \left[g_{\alpha\beta} \frac{\partial x_\alpha}{\partial a_i} \cdot \frac{\partial x_\beta}{\partial a_j} - a_{ij}\right] da^i da^j \tag{2.1.16}$$

or as

$$ds^2 - ds_0^2 = \left[g_{ij} - a_{\alpha\beta}\frac{\partial a_\alpha}{\partial x_i}\cdot\frac{\partial a_\beta}{\partial x_j}\right]dx^i dx^j \ . \tag{2.1.17}$$

Now we define the strain tensors as

$$E_{ij} = \frac{1}{2}\left[g_{\alpha\beta}\frac{\partial x_\alpha}{\partial a_i}\cdot\frac{\partial x_\beta}{\partial a_j} - a_{ij}\right] \tag{2.1.18}$$

$$\epsilon_{ij} = \frac{1}{2}\left[g_{ij} - a_{\alpha\beta}\frac{\partial a_\alpha}{\partial x_i}\cdot\frac{\partial a_\beta}{\partial x_j}\right] \tag{2.1.19}$$

so that

$$ds^2 - ds_0^2 = 2E_{ij}da^i da^j \tag{2.1.20}$$

$$ds^2 - ds_0^2 = 2\epsilon_{ij}dx^i dx^j \ . \tag{2.1.21}$$

The strain tensor E_{ij} was introduced by Green and St. Venant and is called Green's strain tensor [13]. The strain tensor ϵ_{ij} was introduced by Cauchy for infinitesimal strains, and by Almansi and Hamel for finite strain, and is known as Almansi's strain tensor [13].

The tensors E_{ij} and ϵ_{ij} are symmetric, that is

$$E_{ij} = E_{ji}, \quad \epsilon_{ij} = \epsilon_{ji} \ . \tag{2.1.22}$$

If we use the rectangular cartesian (rectilinear and orthogonal) coordinate system to describe both the original and the deformed configurations of the body, then

$$g_{ij} = a_{ij} = \delta_{ij}\begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases} . \tag{2.1.23}$$

Furthermore, if we introduce the displacement vector $\vec{u}$ with the components

$$u_i = x_i - a_i \quad (i = 1,2,3) \tag{2.1.24}$$

then

$$\left.\begin{aligned} \frac{\partial x_\alpha}{\partial a_i} &= \frac{\partial u_\alpha}{\partial a_i} + \delta_{\alpha i} \\ \frac{\partial a_\alpha}{\partial x_i} &= \delta_{\alpha i} - \frac{\partial u_\alpha}{\partial x_i} \end{aligned}\right\} \tag{2.1.25}$$

and the strain tensors reduce to a simpler form

$$\begin{aligned} E_{ij} &= \frac{1}{2}\left[\delta_{\alpha\beta}\frac{\partial x_\alpha}{\partial a_i}\frac{\partial x_\beta}{\partial a_j} - \delta_{ij}\right] \\ &= \frac{1}{2}\left[\delta_{\alpha\beta}\left[\frac{\partial u_\alpha}{\partial a_i} + \delta_{\alpha i}\right]\left[\frac{\partial u_\beta}{\partial a_j} + \delta_{\beta j}\right] - \delta_{ij}\right] \\ &= \frac{1}{2}\left[\frac{\partial u_j}{\partial a_i} + \frac{\partial u_i}{\partial a_j} + \frac{\partial u_\alpha}{\partial a_i}\frac{\partial u_\alpha}{\partial a_j}\right] \end{aligned} \tag{2.1.26}$$

and

$$\begin{aligned} \epsilon_{ij} &= \frac{1}{2}\left[\delta_{ij} - \delta_{\alpha\beta}\frac{\partial a_\alpha}{\partial x_i}\frac{\partial a_\beta}{\partial x_j}\right] \\ &= \frac{1}{2}\left[\delta_{ij} - \delta_{\alpha\beta}\left[\delta_{\alpha i} - \frac{\partial u_\alpha}{\partial x_i}\right]\left[\delta_{\beta j} - \frac{\partial u_\beta}{\partial x_j}\right]\right] \\ &= \frac{1}{2}\left[\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} - \frac{\partial u_\alpha}{\partial x_i}\frac{\partial u_\beta}{\partial x_j}\right] . \end{aligned} \tag{2.1.27}$$

Now if the components of the displacement u_i are such that their first derivatives are so small that the squares and products of the partial derivatives of u_i are negligible then ϵ_{ij} reduces to Cauchy's infinitesimal strain tensor, given by (2.1.10)

$$\epsilon_{ij} = e_{ij} = \frac{1}{2}\,[u_{i,j}+u_{j,i}] \qquad (2.1.28)$$

since if the displacement and its derivatives are small it is "immaterial" whether the derivatives of the displacements are calculated at the position of the point before or after the deformation.

The component of the strain tensor cannot be arbitrary. In order to find this condition, let $P^0(x_1^0,x_2^0,x_3^0)$ be some point of a simply connected region, at which the displacements $u_j^0(x_1^0,x_2^0,x_3^0)$ and the components of rotation $\omega_{ij}^0(x_1^0,x_2^0,x_3^0)$ are known. The displacements u_j at any other point $P'(x_1',x_2',x_3')$ are given by

$$\begin{aligned} u_j(x_1',x_2',x_3') &= u_j^0 + \int_{P^0}^{P'} du_j \\ &= u_j^0 + \int_{P^0}^{P'} u_{j,k}dx_k \\ &= u_j^0 + \int_{P^0}^{P'} e_{jk}dx_k + \int_{P^0}^{P'} \omega_{jk}dx_k\,. \end{aligned} \qquad (2.1.29)$$

Now

$$\int_{P^0}^{P'} \omega_{jk}dx_k = \int_{P^0}^{P'} \omega_{jk}d(x_k-x_k')$$

$$= (x_k'-x_k^0)\omega_{jk}^0 + \int_{P^0}^{P'} (x_k'-x_k)\omega_{jk,\ell}\,dx_\ell \qquad (2.1.30)$$

and hence

$$u_j(x_1',x_2',x_3') = u_j^0+(x_k'-x_k^0)\omega_{jk}^0 + \int_{P^0}^{P'} \left[e_{j\ell}+(x_k'-x_k)\omega_{jk,\ell}\right]dx_\ell \;. \qquad (2.1.31)$$

Also from (2.1.11) we have

$$\omega_{jk,\ell} = \frac{\partial}{\partial x_\ell}\frac{1}{2}(u_{j,k}-u_{k,j})$$

$$= \frac{1}{2}(u_{j,k\ell}-u_{k,j\ell}) + \frac{1}{2}(u_{\ell,jk}-u_{\ell,jk})$$

$$= \frac{\partial}{\partial x_k}\frac{1}{2}(u_{\ell,j}+u_{j,\ell}) - \frac{\partial}{\partial x_j}\frac{1}{2}(u_{k,\ell}+u_{\ell,k}) \qquad (2.1.32)$$

or

$$\omega_{jk,\ell} = e_{\ell j,k}-e_{k\ell,j} \qquad (2.1.33)$$

so now (2.1.31) becomes

$$u_j(x_1',x_2',x_3') = u_j^0 + (x_k'-x_k^0)\omega_{jk}^0 + \int_{P^0}^{P'} U_{j\ell}\,dx_\ell \qquad (2.1.34)$$

where

$$U_{j\ell} = e_{j\ell} + (x_k'-x_k)(e_{\ell j,k}-e_{k\ell,j}). \qquad (2.1.35)$$

Now as the displacements u_i must be independent of the path of integration, $U_{j\ell}dx_\ell$ must be exact differential, so we must have

$$u_{ji,\ell} - u_{j\ell,i} = 0 \qquad (2.1.36)$$

or

$$e_{ji,\ell} - \delta_{\ell k}(e_{ij,k}-e_{ki,j}) - e_{j\ell,i} + \delta_{ki}(e_{\ell j,k}-e_{k\ell,j})$$

$$+ (x_k'-x_k)(e_{ij,k\ell}-e_{ki,j\ell}-e_{\ell j,k\ell}+e_{k\ell,ji}) = 0. \qquad (2.1.37)$$

In equation (2.1.37) the first line is identically zero, and since this is true for arbitrary choice of $(x_k'-x_k)$, we must have

$$e_{ij,k\ell} - e_{ki,j\ell} - e_{\ell j,ki} + e_{k\ell,ji} = 0 \qquad (2.1.38)$$

which is called the compatability condition and the components of strain tensor must satisfy this condition.

§2.2 Stress Tensor

(A) Stresses:

Consider a configuration occupied by a body B at some time. Imagine a closed surface S within B. We would like to know the iteraction between the material exterior to this surface and that in the interior. Consider now a small surface element ΔS on the outside of imagined surface S. Let us draw a unit normal vector $\vec{\nu}$ on ΔS with its direction outward from the interior of S. Then we can distinguish the two sides of ΔS according to the sense of $\vec{\nu}$. Now consider the portion of the material lying on the positive side of the normal. This portion exerts a force say $\Delta\vec{F}$ on the other portion of the material, which is the negative side of the normal $\vec{\nu}$. The force $\Delta\vec{F}$ is a function of the area and the orientation of the surface considered. We make the following assumption:

As ΔS tends to zero, the ratio $\frac{\Delta\vec{F}}{\Delta s}$ tends to a definite limit $d\vec{F}/ds$ and that the moment of the forces acting on the elementary surface Δs about any point within the area vanishes in the limit.

Thus the limiting vector will be written as

$$\overset{\nu}{\vec{T}} = \frac{d\vec{F}}{ds}$$

where the subscript ν is introduced to denote the direction of the unit normal $\vec{\nu}$ of the surface ΔS. This limiting vector $\overset{\nu}{\vec{T}}$ is called the "stress vector" or "traction", and represents a force per unit area acting on the surface.

(B) <u>Components of stresses</u>:

Consider a special case in which the surface ΔS_k is parallel to one of the coordinate planes. Let the normal ΔS_k be in the positive direction of the x_k axis.

Let the stress vector acting on Δs_k be denoted by $\overset{k}{\vec{T}}$, with component T_1^k, T_2^k, T_3^k along the directions of the coordinate axis x_1, x_2, x_3 respectively, the index i of T_i^k denoting the components of the force, and the symbol k indicating the surface on which the force acts. In this case we introduce a new notation for the stress components

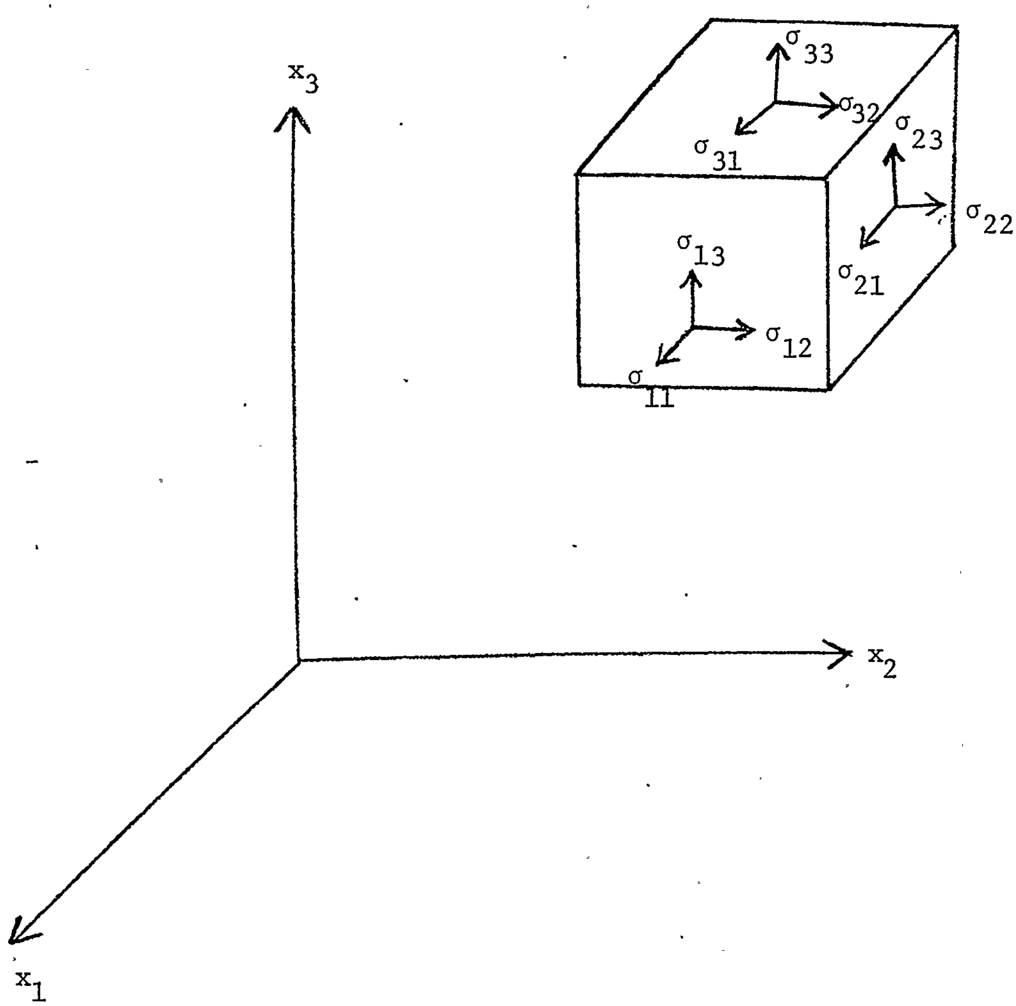

Figure 1: Notation of Stress Components.

$$T_1^k = \sigma_{k1},\ T_2^k = \sigma_{k2},\ T_3^k = \sigma_{k3}\ .$$

If we arrange the components of tractions acting on the surfaces $k = 1$, $k = 2$, $k = 3$ in a square matrix as shown below:

	Components of Stress		
	1	2	3
surface normal to x_1	σ_{11}	σ_{12}	σ_{13}
surface normal to x_2	σ_{21}	σ_{22}	σ_{23}
surface normal to x_3	σ_{31}	σ_{32}	σ_{33} .

The components σ_{11}, σ_{22}, σ_{33} are called "normal stresses" and the remaining components σ_{12}, σ_{13}, etc. are called "shearing stresses".

(C) Stress at a point:

In this we shall show that knowing the components σ_{ij}, we can write the stress vector acting on any surface with unit outer normal vector $\vec{\nu}$ whose components are ν_1, ν_2, ν_3. This stress vector is denoted by $\overset{\rightarrow\nu}{T}$ with components T_1^ν, T_2^ν, T_3^ν given by Cauchy's formula:

$$T_i^\nu = \nu_j \sigma_{ij}\ . \qquad (2.2.1)$$

Let us consider an infinitesimal tetrahedron formed by three surfaces parallel to the coordinate planes and one normal to the unit vector $\vec{\nu}$. Let the area of the surface normal to $\vec{\nu}$ be ds. Then the area

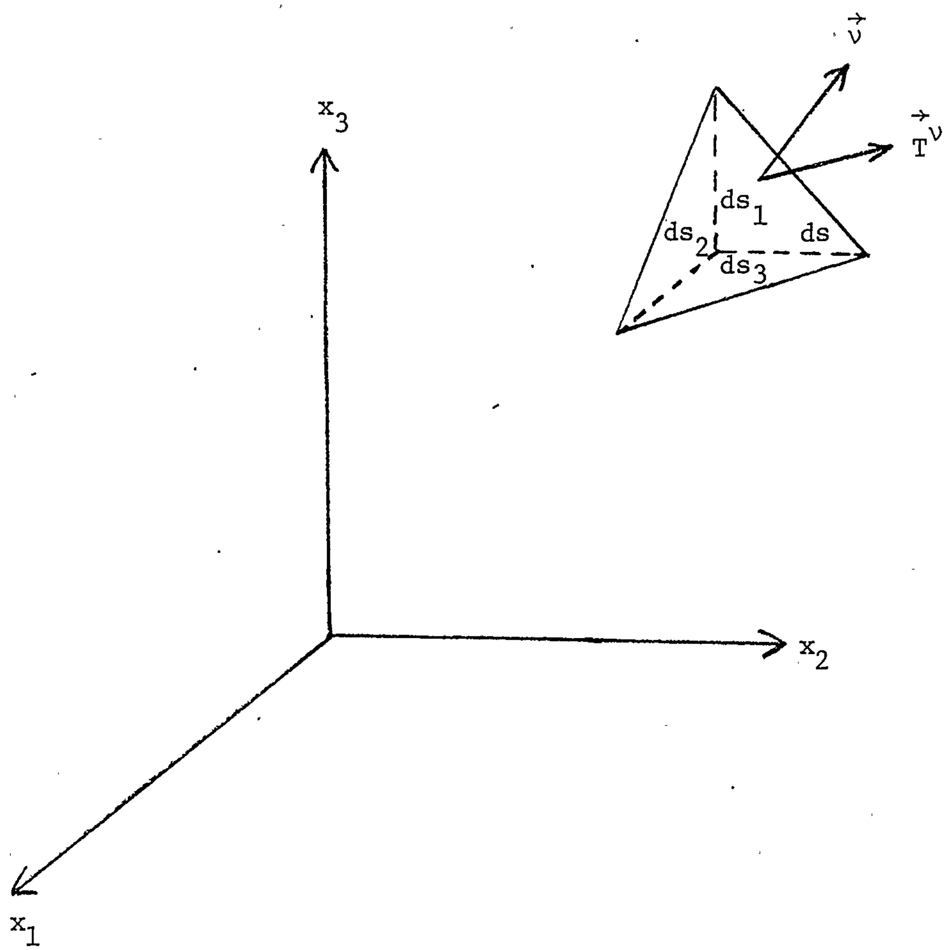

Figure 2: Stress at a Point.

of the other remaining surfaces are:

$$ds_1 = ds\cos(\vec{\nu},\vec{x}_1) = \nu_1 ds = \text{area of the surface} \parallel \text{to } x_2x_3 \text{ plane}$$
$$ds_2 = \nu_2 ds = \text{area of the surface} \parallel \text{to } x_1x_3 \text{ plane}$$
$$ds_3 = \nu_3 ds = \text{area of the surface} \parallel \text{to } x_1x_2 \text{ plane.}$$

The volume of this tetrahedron is

$$d\nu = \frac{1}{3}\,hds$$

where h is the height of the vertex from the base ds.

On account of the assumed continuity of the stress vector T_i^{ν}, the x_i-component of the force acting on the face ABC of the tetrahedron is $(T_i^{\nu}+\epsilon_i)ds$ where $\lim\limits_{h\to 0}\epsilon_i = 0$. The corresponding component of force due to stresses acting on the faces of area ds_i is $(-\sigma_{ji}+\epsilon_{ji})ds_j$ where $\lim\limits_{h\to 0}\epsilon_{ij} = 0$ and σ_{ji} are taken with negative sign due to the outer normals to the three surfaces which are opposite in sense with respect to the coordinate axis. Finally the contribution of the body force X_i to the x_i-component of the resultant force is $(X_i+\epsilon_i')\,\frac{1}{3}\,hds$, where $\lim\limits_{h\to 0}\epsilon_i' = 0$. Thus for the equilibrium of the tetrahedron we must have

$$(T_i^{\nu}+\epsilon_i)ds + (-\sigma_{ji}+\epsilon_{ji})ds_j + (X_i+\epsilon_i')\,\frac{1}{3}\,hds = 0\ .$$

Now $ds_j = \nu_j ds$ so dividing by ds we have

$$(T_i^{\nu}+\epsilon_i) + (-\sigma_{ji}+\epsilon_{ji})\nu_j + (X_i+\epsilon_i') \frac{1}{3} h = 0.$$

Taking limit as $h \to 0$, we obtain

$$T_i^{\nu} = \sigma_{ji}\nu_j \quad . \tag{2.2.2}$$

§2.3 Equations of Motion

Consider a continuous medium, every portion of which is contained within the volume V bounded by the closed surface S. Each point P of S is subjected to traction T_i^{ν} and each mass element of the medium is subjected to a body force per unit mass f_i, which includes any inertia forces present. Then for equilibrium, both the resultant force acting on the body within V and the resultant moment of all the forces acting (produced by body and surface forces) on the body must vanish, that is

$$\int_V \rho f_i dV + \int_S T_i^{\nu} dS = 0 \tag{2.3.1}$$

$$\int_V \rho\gamma_{ijk} f_j x_k dV + \int_S \gamma_{ijk} T_j^{\nu} x_k dS = 0 \tag{2.3.2}$$

where $\rho = \rho(x_1,x_2,x_3)$ is the mass density of the body under consideration at the point in space with coordinate x_k; and γ_{ijk} are defined as:

$$\gamma_{ijk} = \begin{cases} +1 & \text{if ijk represent an even permutation of 123} \\ 0 & \text{if any two of ijk indices are zero} \\ -1 & \text{if ijk represent an odd permutation of 123} \end{cases} .$$

Substituting for T_i^{ν} from equation (2.2.2) into equation (2.3.1) we get

$$\int_V \rho f_i dV + \int_S \sigma_{ji}\nu_j dS = 0. \tag{2.3.3}$$

Now, since the function σ_{ji} and their first partial derivatives are continuous and single-valued in V, the divergence theorem can be applied to the surface integral in (2.3.3) and we have

$$\int_V (\rho f_i + \sigma_{ji,j}) dV = 0 \tag{2.3.4}$$

since the region of integration V is arbitrary and integrand of (2.3.4) is continuous. Thus at every interior point in V, we have

$$\sigma_{ji,j} + \rho f_i = 0. \tag{2.3.5}$$

Next consider the consequence of vanishing of the resultant moment, that is, the equation (2.3.2). Using equation (2.2.2) and the divergence theorem we have

$$\int_S \gamma_{ijk} T_j^{\nu} x_k dS = \int_S \gamma_{ijk}\sigma_{mj}\nu_m x_k dS$$

$$= \int_V (\gamma_{ijk} x_k \sigma_{mj})_m dV$$

$$= \int_V (\gamma_{ijk} x_k \sigma_{mj,m} + \gamma_{ijk} \sigma_{mj} \delta_{km}) dV. \qquad (2.3.6)$$

Here the relation $x_{k,m} = \delta_{km}$ has been used where δ_{km} is the usual Kronecker delta defined by

$$\delta_{km} = \begin{cases} 1 & \text{if} \quad k = m \\ 0 & \text{if} \quad k \neq m \end{cases} .$$

Since $\delta_{km}\sigma_{mj} = \sigma_{kj}$ and from equation (2.3.5) we have

$$\sigma_{mj,m} = -\rho f_j$$

the equation (2.3.6) becomes

$$\int_S \gamma_{ijk} T_j^{\nu} x_k dS = \int_V (-\rho\gamma_{ijk} x_k f_j + \gamma_{ijk}\sigma_{kj}) dV .$$

Using this relation in equation (2.3.2) we get

$$\int_V \gamma_{ijk}\sigma_{kj} dV = 0 .$$

Since this integrand is continuous and the volume V is arbitrary, we must have

$$\gamma_{ijk}\sigma_{kj} = 0 \qquad (2.3.7)$$

for every point in V. Expanding this equation, we get

$$\sigma_{ij} = \sigma_{ji} \quad (i,j = 1,2,3) \tag{2.3.8}$$

that is to say, that the stress tensor is symmetric.

Equations (2.3.5) and (2.3.8) are equations of motion and are true for any continuous medium. If, in addition, to the body force F_i per unit mass exerted by some external agency, interia forces are present, then we can write

$$f_i = F_i - \ddot{u}_i$$

and equation (2.3.5) takes the form

$$\sigma_{ij,j} + \rho F_i = \rho \ddot{u}_i \tag{2.3.9}$$

where $u_i = u_i(x_1,x_2,x_3,t)$ is the displacement vector of a particle at the point P with coordinate x_i, at any time t and a dot denotes a derivative with respect to time.

§2.4 <u>Generalized Hook's law</u>

Generalized Hook's law states that the components of stresses are linearly related to the components of strains. That is to say

$$\sigma_{ij} = c_{ijk\ell} e_{k\ell} \tag{2.4.1}$$

where $c_{ijk\ell}$ is called the tensor of the "elastic constants" or "moduli" of the material. Also since $\sigma_{ij} = \sigma_{ji}$, we have

$$c_{ijk\ell} = c_{jik\ell} . \tag{2.4.2}$$

Further, since $e_{k\ell} = e_{\ell k}$ and in equation (2.4.2) the indices k and ℓ are dummies, we can symmetrize $c_{ijk\ell}$ with respect to k and ℓ without altering the sum. Thus

$$c_{ijk\ell} = c_{ij\ell k} .$$

According to these symmetric properties the maximum number of independent elastic constants is 36.

Again, if we define a strain energy function ω by

$$\omega = \frac{1}{2} c_{ijk\ell} e_{ij} e_{k\ell} \tag{2.4.3}$$

with the property

$$\frac{\partial \omega}{\partial e_{ij}} = \sigma_{ij} \tag{2.4.4}$$

then the quadratic form (2.4.3) is symmetric and it follows that

$$c_{ijk\ell} = c_{k\ell ij} \tag{2.4.5}$$

and the number of independent elastic constants is further reduced to 21. Now introducing the notation

$$\sigma_{11} = \sigma_1,\ \sigma_{22} = \sigma_2,\ \sigma_{33} = \sigma_3,\ \sigma_{23} = \sigma_4,\ \sigma_{31} = \sigma_5,\ \sigma_{12} = \sigma_6$$
$$e_{11} = e_1,\ e_{22} = e_2,\ e_{33} = e_3,\ e_{23} = e_4,\ e_{31} = e_5,\ e_{12} = e_6$$

the equation (2.4.1) can be written as

$$\sigma_i = c_{ij} e_j \quad (i,j = 1,2,\cdots,6). \tag{2.4.6}$$

If the medium is elastically symmetric in certain direction, then the number of independent constants c_{ij} in (2.4.6) is further reduced. Consider a substance elastically symmetric with respect to the x_1x_2 plane. This symmetry is expressed by the statement that the c_{ij} are invariant under the transformation

$$x_1 = x_1',\ x_2 = x_2',\ x_3 = -\ x_3'$$

and by using the transformation of coordinates we have

$$\sigma_i' = \sigma_i,\ e_i' = e_i \quad (i = 1,2,3,6)$$

$$\sigma_4' = -\ \sigma_4,\ e_4' = -\ e_4,\ \sigma_5' = -\ \sigma_5,\ e_5' = -\ e_5$$

and equation (2.4.6) for i = 1 becomes

$$\sigma_1' = c_{11}e_1' + c_{12}e_2' + c_{13}e_3' + c_{14}e_4' + c_{15}e_5' + c_{16}e_6'$$

or

$$\sigma_1 = c_{11}e_1 + c_{12}e_2 + c_{13}e_3 - c_{14}e_4 - c_{15}e_5 + c_{16}e_6 \ .$$

Also from equation (2.4.6) for i = 1, we have

$$\sigma_1 = c_{11}e_1 + c_{12}e_2 + c_{13}e_3 + c_{14}e_4 + c_{15}e_5 + c_{16}e_6 \ .$$

Thus on comparison of these two we get

$$c_{14} = c_{15} = 0.$$

Similarly taking equations for $\sigma_2',\cdots,\sigma_6'$ we get

$$c_{24} = c_{25} = c_{34} = c_{35} = c_{64} = c_{65} = 0$$
$$c_{41} = c_{42} = c_{43} = c_{46} = c_{51} = c_{52} = c_{53} = c_{56} = 0 \ .$$

Hence for a material with one plane of elastic symmetry (taken to be x_1x_2 plane) the matrix of the coefficients of the linear forms in (2.4.6) can be written as

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{21} & c_{22} & c_{23} & 0 & 0 & c_{26} \\ c_{31} & c_{32} & c_{33} & 0 & 0 & c_{36} \\ 0 & 0 & 0 & c_{44} & c_{45} & 0 \\ 0 & 0 & 0 & c_{54} & c_{55} & 0 \\ c_{61} & c_{62} & c_{63} & 0 & 0 & c_{66} \end{bmatrix} . \qquad (2.4.7)$$

In case of an orthotropic materials, that is, the material which have three mutually orthogonal planes of elastic symmetry, if we choose the axis of coordinates so that the coordinate planes coincide with the plane of elastic symmetry, then some of the coefficients c_{ij} in (2.4.7) vanish. In this case the matrix is given by

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{21} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{31} & c_{32} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} . \qquad (2.4.8)$$

In the case of an isotropic media, that is, the material in which the elastic properties of a body are identical in all directions, these constants reduce to 2. Now from the definition of the isotropic media, its elastic properties are independent of the orientation of the coordinate axes. In particular, the coefficients c_{ij} must remain invariant when we introduce new coordinates axes x_1', x_2', x_3', obtained by rotating the x_1, x_2, x_3-system through a right angle about the x_1-axis. By considering σ_1', we have

$$c_{12} = c_{13},\ c_{31} = c_{21},\ c_{32} = c_{23},\ c_{33} = c_{22},\ c_{66} = c_{55}.$$

Similarly, a rotation of axes through a right angle about the x_3-axis leads to

$$c_{21} = c_{12},\ c_{22} = c_{11},\ c_{23} = c_{13},\ c_{31} = c_{32},\ c_{55} = c_{44}.$$

Finally consider the coordinate system x_1', x_2', x_3' obtained from x_1, x_2, x_3-system by rotating the latter through an angle of 45° about the x_3-axis. In this case,

$$\sigma_6' = -\frac{1}{2}\sigma_1 + \frac{1}{2}\sigma_2,\ e_6' = -e_1 + e_2 .$$

From equation (2.4.6) we have

$$\sigma_6 = c_{44}e_6$$

and referred to x_1', x_2', x_3'-axis we have

$$\sigma_6' = c_{44}e_6'$$

or

$$\frac{1}{2}(-\sigma_1+\sigma_2) = c_{44}(-e_1+e_2) . \qquad (2.4.9)$$

Now from (2.4.7) we have

$$\sigma_1 = c_{11}e_1 + c_{12}e_2 + c_{13}e_3$$
$$\sigma_2 = c_{21}e_1 + c_{22}e_2 + c_{23}e_3$$

and from above relations, namely,

$$c_{22} = c_{11},\ c_{23} = c_{13} = c_{12} = c_{21}$$

we get

$$\begin{aligned}\frac{1}{2}(-\sigma_1+\sigma_2) &= \frac{1}{2}(c_{21}e_1+c_{22}e_2) - \frac{1}{2}(c_{11}e_1+c_{12}e_2)\\ &= \frac{1}{2}\left[c_{11}(e_2-e_1)-c_{12}(e_2-e_1)\right]\\ &= \frac{1}{2}(c_{11}-c_{12})(e_2-e_1)\ .\end{aligned} \tag{2.4.10}$$

Comparing (2.4.9) and (2.4.10) we get

$$c_{44} = \frac{1}{2}(c_{11}-c_{12}) \equiv \mu.$$

Also writing $c_{12} = \lambda$, we have from equation (2.4.6)

$$\begin{aligned}\sigma_{11} &= c_{11}e_{11} + c_{12}e_{22} + c_{12}e_{33}\\ &= c_{12}(e_{11}+e_{22}+e_{33}) + (c_{11}-c_{12})e_{11}\\ &= \lambda\Delta + 2\mu e_{11},\end{aligned}$$

where $\Delta = e_{11}+e_{22}+e_{33} = e_{ii}$. So the generalized Hook's law for a homogeneous isotropic body can be written as

$$\sigma_{ij} = \lambda\delta_{ij}\Delta + 2\mu e_{ij} \quad (i,j = 1,2,3). \tag{2.4.11}$$

The constants λ and μ were introduced by G. Lamé and are called the Lamé constants.

§2.5 <u>Equations of motion in terms of displacements</u>

The equation of motion is given by the equation (2.3.9) as

$$\sigma_{ji,j} + \rho F_i = \rho \frac{\partial^2 u_i}{\partial t^2} .$$

Now substituting for σ_{ji} from equation (2.4.11) we have

$$\lambda\delta_{ij}\Delta_j + 2\mu e_{ij,j} + \rho F_i = \rho \frac{\partial^2 u_i}{\partial t^2} .$$

These equations can be written as

$$(\lambda+\mu) \frac{\partial\Delta}{\partial x_1} + \mu\left[\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2}\right] + \rho F_1 = \rho \frac{\partial^2 u_1}{\partial t^2}$$

$$\left.\begin{aligned}(\lambda+\mu)\frac{\partial\Delta}{\partial x_2}+\mu\left[\frac{\partial^2u_2}{\partial x_1^2}+\frac{\partial^2u_2}{\partial x_2^2}+\frac{\partial^2u_2}{\partial x_3^2}\right]+\rho F_2&=\rho\frac{\partial^2u_2}{\partial t^2}\\(\lambda+\mu)\frac{\partial\Delta}{\partial x_3}+\mu\left[\frac{\partial^2u_3}{\partial x_1^2}+\frac{\partial^2u_3}{\partial x_2^2}+\frac{\partial^2u_3}{\partial x_3^2}\right]+\rho F_3&=\rho\frac{\partial^2u_3}{\partial t^2}\end{aligned}\right\}. \qquad (2.5.1)$$

These are the equations of motion in terms of displacements. Now from equation (2.1.11) we have

$$\omega_{ij}=\frac{1}{2}(u_{i,j}-u_{j,i})$$

so that

$$\nabla^2u_1=\frac{\partial^2u_1}{\partial x_1^2}+\frac{\partial^2u_1}{\partial x_2^2}+\frac{\partial^2u_1}{\partial x_3^2}=\frac{\partial\Delta}{\partial x_1}-2\left[\frac{\partial\omega_{21}}{\partial x_2}-\frac{\partial\omega_{13}}{\partial x_3}\right]$$

and hence equations (2.5.1) may be rewritten as

$$\left.\begin{aligned}(\lambda+2\mu)\frac{\partial\Delta}{\partial x_1}-2\mu\left[\frac{\partial\omega_{21}}{\partial x_2}-\frac{\partial\omega_{13}}{\partial x_3}\right]+\rho F_1&=\rho\frac{\partial^2u_1}{\partial t^2}\\(\lambda+2\mu)\frac{\partial\Delta}{\partial x_2}-2\mu\left[\frac{\partial\omega_{32}}{\partial x_3}-\frac{\partial\omega_{21}}{\partial x_1}\right]+\rho F_2&=\rho\frac{\partial^2u_2}{\partial t^2}\\(\lambda+2\mu)\frac{\partial\Delta}{\partial x_3}-2\mu\left[\frac{\partial\omega_{13}}{\partial x_1}-\frac{\partial\omega_{32}}{\partial x_2}\right]+\rho F_3&=\rho\frac{\partial^2u_3}{\partial t^2}\end{aligned}\right\}. \qquad (2.5.2)$$

If we write $\omega_{21}=\omega_3$, $\omega_{13}=\omega_2$, and $\omega_{32}=\omega_1$ these equations become

$$\left.\begin{aligned}
(\lambda+2\mu)\,\frac{\partial\Delta}{\partial x_1} - 2\mu\left[\frac{\partial\omega_3}{\partial x_2} - \frac{\partial\omega_2}{\partial x_3}\right] + \rho F_1 &= \rho\,\frac{\partial^2 u_1}{\partial t^2} \\
(\lambda+2\mu)\,\frac{\partial\Delta}{\partial x_2} - 2\mu\left[\frac{\partial\omega_1}{\partial x_3} - \frac{\partial\omega_3}{\partial x_1}\right] + \rho F_2 &= \rho\,\frac{\partial^2 u_2}{\partial t^2} \\
(\lambda+2\mu)\,\frac{\partial\Delta}{\partial x_3} - 2\mu\left[\frac{\partial\omega_2}{\partial x_1} - \frac{\partial\omega_1}{\partial x_2}\right] + \rho F_3 &= \rho\,\frac{\partial^2 u_3}{\partial t^2}
\end{aligned}\right\} \qquad (2.5.3)$$

§2.6 Curvilinear orthogonal coordinates

Let $f(x,y,z) = \alpha$; where α is some constant, be the equation of a surface. If α is allowed to vary we obtain a family of surfaces. If $\alpha+d\alpha$ is the parameter of that surface of the family which passes through $(x+dx, y+dy, z+dz)$ we have

$$d\alpha = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz = \frac{\partial\alpha}{\partial x}\,dx + \frac{\partial\alpha}{\partial y}\,dy + \frac{\partial\alpha}{\partial z}\,dz. \qquad (2.6.1)$$

If we have three independent families of surfaces given by equations

$$f_1(x,y,z) = \alpha,\ f_2(x,y,z) = \beta,\ f_3(x,y,z) = \gamma$$

so that in general one surface of each family passes through a chosen point, then a point may be determined by the values of α, β, γ which belongs to the surface that passes through it and the neighbouring point will be determined by the neighbouring values $\alpha+d\alpha$, $\beta+d\beta$, $\gamma+d\gamma$. Such quantities as α, β, γ are called "curvilinear coordinates" of the point.

When the families of surfaces cut each other everywhere at right angles it is called an "orthogonal curvilinear coordinates". So let us take α, β, γ to be parameters of such a set of surfaces, so that the following relations hold:

$$\frac{\partial\beta}{\partial x}\frac{\partial\gamma}{\partial x} + \frac{\partial\beta}{\partial y}\frac{\partial\gamma}{\partial y} + \frac{\partial\beta}{\partial z}\frac{\partial\gamma}{\partial z} = 0$$

$$\frac{\partial\gamma}{\partial x}\frac{\partial\alpha}{\partial x} + \frac{\partial\gamma}{\partial y}\frac{\partial\alpha}{\partial y} + \frac{\partial\gamma}{\partial z}\frac{\partial\alpha}{\partial z} = 0$$

$$\frac{\partial\alpha}{\partial x}\frac{\partial\beta}{\partial x} + \frac{\partial\alpha}{\partial y}\frac{\partial\beta}{\partial y} + \frac{\partial\alpha}{\partial z}\frac{\partial\beta}{\partial z} = 0 .$$

Now the direction cosines of the normal to α at the point (x,y,z) are

$$\frac{1}{h_1}\frac{d\alpha}{dx} , \quad \frac{1}{h_1}\frac{d\alpha}{dy} , \quad \frac{1}{h_1}\frac{d\alpha}{dz}$$

where h_1 is given by equation (2.6.2) below. By projecting the line joining two neighbouring points on the normal n_1 to α, we have

$$dn_1 = \frac{1}{h_1}\left[\frac{\partial\alpha}{\partial x}dx + \frac{\partial\alpha}{\partial y}dy + \frac{\partial\alpha}{\partial z}dz\right] = \frac{d\alpha}{h_1}$$

by equation (2.6.1). In the same way the elements dn_2, dn_3 of the normals to β and γ are $d\beta/h_2$ and $d\beta/h_3$ where

$$h_1^2 = \left[\frac{\partial\alpha}{\partial x}\right]^2 + \left[\frac{\partial\alpha}{\partial y}\right]^2 + \left[\frac{\partial\alpha}{\partial z}\right]^2$$
$$h_2^2 = \left[\frac{\partial\beta}{\partial x}\right]^2 + \left[\frac{\partial\beta}{\partial y}\right]^2 + \left[\frac{\partial\beta}{\partial z}\right]^2$$
$$h_3^2 = \left[\frac{\partial\gamma}{\partial x}\right]^2 + \left[\frac{\partial\gamma}{\partial y}\right]^2 + \left[\frac{\partial\gamma}{\partial z}\right]^2 . \tag{2.6.2}$$

Now the distance between two neighbouring points is $(dn_1^2+dn_2^2+dn_3^2)^{1/2}$ so that line element ds is given by

$$ds^2 = dn_1^2 + dn_2^2 + dn_3^2 . \tag{2.6.3}$$

§2.7 <u>Components of strain referred to curvilinear orthogonal coordinates</u>

Let $P(\alpha,\beta,\gamma)$ and $Q(\alpha+a,\beta+b,\gamma+c)$ be two points at a small distance r apart, and let the direction cosines of PQ, referred to the normals at P to those surfaces of the α, β and γ families which passes through P, be ℓ, m, n. Then, to the first order in r

$$a = \ell r h_1, \quad b = m r h_2, \quad c = n r h_3.$$

Let the particle which is at P, Q in the unstrained state be displaced to P_1, Q_1 and u_α, u_β, u_γ be the projections of the displacement PP_1 on the same three normals, and let $\alpha+\xi$, $\beta+\eta$, $\gamma+\zeta$ be the curvilinear coordinates of P. If the displacement is small, so that u_α, u_β, u_γ and ξ, η, ζ are small quantities of same order, we have

$$\xi = h_1 u_\alpha, \quad \eta = h_2 u_\beta, \quad \zeta = h_3 u_\gamma .$$

The curvilinear coordinates of Q are expressed with sufficient approximation as

$$\alpha + a + \xi + a \frac{\partial \xi}{\partial \alpha} + b \frac{\partial \xi}{\partial \beta} + c \frac{\partial \xi}{\partial \gamma} ,$$

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and the values of $1/h_1, \cdots$ at P_1 are expressed with sufficient approximations as

$$\frac{1}{h_1} + \xi \frac{\partial}{\partial \alpha} \left[\frac{1}{h_1}\right] + \eta \frac{\partial}{\partial \beta} \left[\frac{1}{h_1}\right] + \zeta \frac{\partial}{\partial \gamma} \left[\frac{1}{h_1}\right] ,$$

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so the projection of P_1Q_1 on the normal at P_1, to those surfaces of the α, β and γ families which pass through P_1 are expressed with sufficient approximation by three formulas of type

$$\left\{a\left[1 + \frac{\partial \xi}{\partial \alpha}\right] + b \frac{\partial \xi}{\partial \beta} + c \frac{\partial \xi}{\partial \gamma}\right\}\left\{\frac{1}{h_1} + \xi \frac{\partial}{\partial \alpha} \left[\frac{1}{h_1}\right] + \eta \frac{\partial}{\partial \beta} \left[\frac{1}{h_1}\right] + \zeta \frac{\partial}{\partial \gamma} \left[\frac{1}{h_1}\right]\right\} ,$$

which can be simplified by neglecting the terms of order higher than one in ξ, η, ζ. On substituting for a, b, c and ξ, η, ζ and squaring and adding three formulas of this type, we obtain an expression for the square of the length P_1Q_1. This length is r(1+e), where e is the extension of a linear element along PQ. Hence e is given by

$$(1+e)^2 = \left[\ell\left\{1 + h_1 \frac{\partial u_2}{\partial\alpha} + h_1h_2u_\beta \frac{\partial}{\partial\beta}\left[\frac{1}{h_1}\right] + h_1h_3u_\gamma \frac{\partial}{\partial\gamma}\left[\frac{1}{h_1}\right]\right\}\right.$$
$$\left. + m\frac{h_2}{h_1}\frac{\partial}{\partial\beta}(h_1u_2) + n\frac{h_3}{h_1}(h_1u_2)\right]^2 + \cdots$$

neglecting squares and products of u_α, u_β, u_γ, we may write the results as

$$e^2 = e_{\alpha\alpha}\ell^2 + e_{\beta\beta}m^2 + e_{\gamma\gamma}n^2 + e_{\beta\gamma}mn + e_{\gamma\alpha}n\ell + e_{\alpha\beta}\ell m$$

where

$$\left.\begin{aligned}
e_{\alpha\alpha} &= h_1\frac{\partial u_\alpha}{\partial\alpha} + h_1h_2u_\beta\frac{\partial}{\partial\beta}\left[\frac{1}{h_1}\right] + h_3h_1u_\gamma\frac{\partial}{\partial\gamma}\left[\frac{1}{h_1}\right]\\
e_{\beta\beta} &= h_2\frac{\partial u_\beta}{\partial\beta} + h_2h_3u_\gamma\frac{\partial}{\partial\gamma}\left[\frac{1}{h_2}\right] + h_1h_2u_\alpha\frac{\partial}{\partial\alpha}\left[\frac{1}{h_2}\right]\\
e_{\gamma\gamma} &= h_3\frac{\partial u_\gamma}{\partial\gamma} + h_3h_1u_\alpha\frac{\partial}{\partial\alpha}\left[\frac{1}{h_3}\right] + h_2h_3u_\beta\frac{\partial}{\partial\beta}\left[\frac{1}{h_3}\right]\\
e_{\beta\gamma} &= \frac{h_2}{h_3}\frac{\partial}{\partial\beta}(h_3u_\gamma) + \frac{h_3}{h_2}\frac{\partial}{\partial\gamma}(h_2u_\beta)\\
e_{\gamma\alpha} &= \frac{h_3}{h_1}\frac{\partial}{\partial\gamma}(h_1u_\alpha) + \frac{h_1}{h_3}\frac{\partial}{\partial\alpha}(h_3u_\gamma)\\
e_{\alpha\beta} &= \frac{h_1}{h_2}\frac{\partial}{\partial\alpha}(h_2u_\beta) + \frac{h_2}{h_1}\frac{\partial}{\partial\beta}(h_1u_\alpha)
\end{aligned}\right\} . \qquad (2.7.1)$$

The quantities $e_{\alpha\alpha} \cdots e_{\beta\gamma} \cdots$ are the six components of strain referred to the orthogonal coordinates.

§2.8 Equations of motion in cylindrical coordinates

In the case of cylindrical coordinates r, θ, z, which is a special case of an orthogonal curvilinear coordinate, we have the line element

$$\{(dr)^2 + r^2(d\theta)^2 + (dz)^2\}^{1/2}$$

and the displacements are u_r, u_θ, u_z. Then from equation (2.7.1) we have

$$\left.\begin{aligned} e_{rr} &= \frac{\partial u_r}{\partial r}, \qquad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \qquad e_{zz} = \frac{\partial u_z}{\partial z} \\ e_{\theta z} &= \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}, \quad e_{zr} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}, \quad e_{r\theta} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} \end{aligned}\right\} \quad (2.8.1)$$

and the dilatation is given by

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}. \qquad (2.8.2)$$

By substituting from equations (2.8.1) and (2.8.2) into (2.5.3) we get

$$\left.\begin{aligned} (\lambda+2\mu)\frac{\partial \Delta}{\partial r} - \frac{2\mu}{r}\frac{\partial \omega_z}{\partial \theta} + 2\mu\frac{\partial \omega_\theta}{\partial z} &= \rho\frac{\partial^2 u_r}{\partial t^2} \\ (\lambda+2\mu)\frac{\partial \Delta}{\partial \theta} - 2\mu\frac{\partial \omega_r}{\partial z} + 2\mu\frac{\partial \omega_z}{\partial r} &= \rho\frac{\partial^2 u_\theta}{\partial t^2} \\ (\lambda+2\mu)\frac{\partial \Delta}{\partial z} - \frac{2\mu}{r}\frac{\partial}{\partial r}(r\omega_\theta) + 2\mu\frac{\partial \omega_r}{\partial \theta} &= \rho\frac{\partial^2 u_z}{\partial t^2} \end{aligned}\right|$$

where

$$\left.\begin{aligned} 2\omega_r &= \frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \\ 2\omega_\theta &= \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \\ 2\omega_z &= \frac{1}{r}\frac{\partial}{\partial r}(ru_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta} \end{aligned}\right\} . \qquad (2.8.3)$$

These are the equations of motion in the cylindrical coordinates.

CHAPTER III

REISSNER-SAGOCI PROBLEM

§3.1 Introduction

Reissner and Sagoci [22] in 1944 investigated the torsional oscillations of a homogeneous, isotropic elastic half-space under the influence of periodic shear stresses applied in a rotationally symmetric manner to a circular portion of the surface of the half-space. They obtained solution for the static case of the above mixed boundary value problem by introducing in a suitable manner a system of oblate spheroidical coordinates. Later on in 1947 Sneddon [26] solved the same problem by a different approach. He reduced the problem to a solution of a pair of dual integral equations by using the Hankel transform method. In the static case these dual integral equations reduce to simple dual integral equations whose solution is known. This problem is now known as the Reissner-Sagoci problem. In this chapter I have discussed the solution of the Reissner-Sagoci problem given by Reissner and Sagoci [22] and Sneddon [26].

§3.2 Basic equations

The equations of motion in cylindrical polar coordinates (r,θ,z) in the absence of body forces are given by (2.8.3)

$$\left.\begin{aligned}
(\lambda+2\mu)\frac{\partial \Delta}{\partial r} - \frac{2\mu}{r}\frac{\partial \omega_z}{\partial \theta} + 2\mu\frac{\partial \omega_\theta}{\partial z} &= \rho\frac{\partial^2 u_r}{\partial t^2} \\
(\lambda+2\mu)\frac{\partial \Delta}{\partial \theta} - 2\mu\frac{\partial \omega_r}{\partial z} + 2\mu\frac{\partial \omega_z}{\partial r} &= \rho\frac{\partial^2 u_\theta}{\partial t^2} \\
(\lambda+2\mu)\frac{\partial \Delta}{\partial z} - \frac{2\mu}{r}\frac{\partial (r\omega_\theta)}{\partial r} + 2\mu\frac{\partial \omega_r}{\partial \theta} &= \rho\frac{\partial^2 u_z}{\partial t^2}
\end{aligned}\right\} \tag{3.2.1}$$

where λ and μ are elastic constants, ρ the mass density, u_r, u_θ, u_z the displacements in the corresponding coordinates, Δ is the dilatation and ω_r, ω_θ, ω_z the components of the rotational vector which are given by

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \tag{3.2.2}$$

$$\left.\begin{aligned}
2\omega_r &= \frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \\
2\omega_\theta &= \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \\
2\omega_z &= \frac{1}{r}\frac{\partial}{\partial r}(ru_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta}
\end{aligned}\right\} . \tag{3.2.3}$$

If the problem is symmetric about the z-axis then the rotation, dilatation and displacement components must be independent of θ and hence equations (3.2.1), (3.2.2) and (3.2.3) above may be written as

$$\left.\begin{aligned}
(\lambda+2\mu)\frac{\partial\Delta}{\partial r} + 2\mu\frac{\partial\omega_\theta}{\partial z} &= \rho\frac{\partial^2 u_r}{\partial t^2} \\
-2\mu\frac{\partial\omega_r}{\partial z} + 2\mu\frac{\partial\omega_z}{\partial r} &= \rho\frac{\partial^2 u_\theta}{\partial t^2} \\
(\lambda+2\mu)\frac{\partial\Delta}{\partial z} - \frac{2\mu}{r}\frac{\partial}{\partial r}(r\omega_\theta) &= \rho\frac{\partial^2 u_z}{\partial t^2}
\end{aligned}\right\} \tag{3.2.4}$$

with

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z} \tag{3.2.5}$$

and

$$\begin{aligned}
2\omega_r &= -\frac{\partial u_\theta}{\partial z} \\
2\omega_\theta &= \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \\
2\omega_z &= \frac{1}{r}\frac{\partial(ru_\theta)}{\partial r}
\end{aligned} \tag{3.2.6}$$

Also the expressions for stresses are given by

$$\left.\begin{aligned}
\sigma_{zz} &= \lambda\Delta + 2\mu\frac{\partial u_z}{\partial z} \\
\sigma_{rr} &= \lambda\Delta + 2\mu\frac{\partial u_r}{\partial r} \\
\sigma_{\theta\theta} &= \lambda\Delta
\end{aligned}\right\} \tag{3.2.7}$$

$$\sigma_{rz} = \mu\left[\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right]$$

$$\sigma_{r\theta} = \mu r \frac{\partial}{\partial r}\left[\frac{u_\theta}{r}\right] \tag{3.2.8}$$

$$\sigma_{z\theta} = \mu \frac{\partial u_\theta}{\partial z}$$

Substituting from (3.2.5) and (3.2.6) into (3.2.4), we get

$$\left.\begin{aligned}
(\lambda+2\mu)\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z}\right] + \mu\frac{\partial}{\partial z}\left[\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right] &= \rho\frac{\partial^2 u_r}{\partial t^2}\\
\mu\frac{\partial}{\partial z}\left[\frac{\partial u_\theta}{\partial z}\right] + \mu\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial(ru_\theta)}{\partial r}\right] &= \rho\frac{\partial^2 u_\theta}{\partial t^2}\\
(\lambda+2\mu)\frac{\partial}{\partial z}\left[\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{\partial u_z}{\partial z}\right] - \mu\frac{\partial}{\partial r}\left[r\left[\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right]\right] &= \rho\frac{\partial^2 u_z}{\partial t^2}
\end{aligned}\right\} \tag{3.2.9}$$

In the equations (3.2.9) the first and last equations contain only u_r and u_z and the second equation contains only u_θ. Also from (3.2.7) and (3.2.8) the stresses σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz}, σ_{rz} are functions of u_r and u_z only and the stresses $\sigma_{r\theta}$, $\sigma_{\theta z}$ are functions of u_θ only. Hence in axisymmetric problems there occur two systems of displacements which do not influence each other in any way. So in order to obtain the most general axisymmetric solution they can be superimposed. Let us consider the case in which only u_θ is present, and we will denote this by $v(r,z)$. In this case the first and third equations of (3.2.9) become identically zero and the second equation becomes

$$\mu\left[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2}\right] = \rho\,\frac{\partial^2 v}{\partial t^2} \tag{3.2.10}$$

and the non-zero stresses are given by

$$\begin{aligned} \sigma_{\theta z} &= \mu\,\frac{\partial v}{\partial z} \\ \sigma_{r\theta} &= \mu\left[\frac{\partial v}{\partial r} - \frac{v}{r}\right] \end{aligned} \tag{3.2.11}$$

§3.3 <u>Formulation of the problem</u>

Let us take a homogeneous isotropic elastic half-space ($z \geq 0$), a circular portion of whose surface ($r \geq r_0$) is forced to rotate through an angle ω, the axis of rotation being perpendicular to the surface of the half-space. The remaining portion of the surface ($r > r_0$) is assumed to be free of stress. The problem is to find the stresses and displacement in the half-space.

In this case only the circumferential displacement component v occurs and that all components of stresses are zero except the following two components of shear stress

$$\begin{aligned} \sigma_{\theta z} &= \mu\,\frac{\partial v}{\partial z} \\ \sigma_{r\theta} &= \mu\left[\frac{\partial v}{\partial r} - \frac{v}{r}\right] \end{aligned} \tag{3.3.1}$$

where v satisfies the following equation of equilibrium

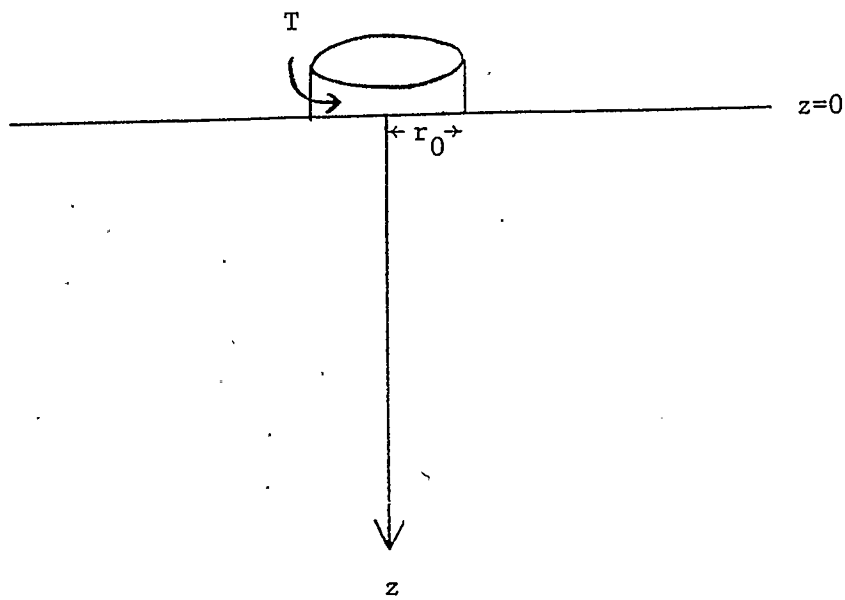

Figure 3: Semi-Infinite Space $z \geq 0$.

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2} = 0 \ . \tag{3.3.2}$$

The boundary conditions are given by

$$v(r,0) = \omega r \ , \qquad r \le r_0 \tag{3.3.3}$$

$$\sigma_{\theta z}(r,0) = \mu \left.\frac{\partial v}{\partial z}\right|_{z=0} = 0, \quad r > r_0 \tag{3.3.4}$$

and the applied torque T required to rotate the circular portion $r \le r_0$ of the boundary through an angle ω, is given by

$$T = -\ 2\pi \int_0^{r_0} r^2 \sigma_{\theta z}(r,0)dr \ . \tag{3.3.5}$$

§3.4 Reissner-Sagoci solution

Reissner and Sagoci solved the equation (3.3.2) by introducing a system of curvilinear coordinates in which the circular disk becomes one of the coordinate surfaces. Such coordinates ξ, η are [32] the following

$$\frac{r}{r_0} = [(1+\xi^2)(1-\eta^2)]^{1/2}, \quad \frac{z}{r_0} = \xi\eta \ . \tag{3.4.1}$$

The surface ξ = constant represents ellipsoids of revolution, the surface η = constant represents hyperboloids of one sheet. The half-space $z \ge 0$ is defined in the new coordinates by $\eta \ge 0$ while the portion $r \le r_0$ of the surface of the half-space is characterized by

$\xi = 0$.

In terms of the coordinates ξ, η the boundary conditions (3.3.3) and (3.3.4) become

$$v = \omega r_0 (1-\eta^2)^{1/2} \quad \text{at } \xi = 0 \tag{3.4.2}$$

$$\frac{\partial v}{\partial \eta} = 0 \quad \text{at } \eta = 0 \tag{3.4.3}$$

Let us consider solution of equation (3.3.2) in the form

$$v(r,z) = f(\xi)g(\eta) \tag{3.4.4}$$

and solve it by separation of variables. Now treating $\xi = \xi(r,z)$ and $\eta = \eta(r,z)$, we obtain

$$\left.\begin{array}{ll} \dfrac{\partial \xi}{\partial r} = \dfrac{r\xi}{\xi^2+\eta^2}, & \dfrac{\partial \eta}{\partial r} = -\dfrac{r\eta}{\xi^2+\eta^2} \\[2ex] \dfrac{\partial \xi}{\partial z} = \dfrac{\eta(1+\xi^2)}{r_0(\xi^2+\eta^2)}, & \dfrac{\partial \eta}{\partial z} = \dfrac{\xi(1-\eta^2)}{r_0(\xi^2+\eta^2)} \end{array}\right\} . \tag{3.4.5}$$

Differentiating (3.4.4) with respect to r and z partially, we find that

$$\frac{\partial v}{\partial r} = \frac{r\xi f'(\xi)g(\eta) - r\eta g'(\eta)f(\xi)}{r_0^2(\xi^2+\eta^2)} \tag{3.4.6}$$

$$\frac{\partial^2 v}{\partial r^2} = \frac{r^2\xi^2}{r_0^4(\xi^2+\eta^2)^2} f''(\xi)g(\eta) + \frac{r^2\eta^2}{r_0^4(\xi^2+\eta^2)^2} g''(\eta)f(\xi)$$

$$- \frac{2r^2\xi\eta}{r_0^4(\xi^2+\eta^2)^2} f'(\xi)g'(\eta) + \left[\frac{\xi}{r_0^2(\xi^2+\eta^2)} + \frac{r^2\xi(3\eta^2-\xi^2)}{r_0^4(\xi^2+\eta^2)^3}\right] f'(\xi)g(\eta)$$

$$+ \left[\frac{r^2\eta(3\xi^2-\eta^2)}{r_0^4(\xi^2+\eta^2)^3} - \frac{\eta}{r_0^2(\xi^2+\eta^2)}\right] g'(\eta)f(\xi) \qquad (3.4.7)$$

$$\frac{\partial^2 v}{\partial z^2} = \frac{\eta^2(1+\xi^2)^2}{r_0^2(\xi^2+\eta^2)^2} f''(\xi)g(\eta) + \frac{\xi^2(1-\eta^2)^2}{r_0^2(\xi^2+\eta^2)^2} g''(\eta)f(\xi)$$

$$+ \frac{2r^2\xi\eta}{r_0^4(\xi^2+\eta^2)^2} f'(\xi)g'(\eta) + \frac{r^2\xi(\xi^2-3\eta^2)}{r_0^4(\xi^2+\eta^2)^3} f'(\xi)g(\eta)$$

$$+ \frac{r^2\eta(\eta^2-3\xi^2)}{r_0^4(\xi^2+\eta^2)^3} g'(\eta)f(\xi) \qquad (3.4.8)$$

where prime denotes the derivative with respect to the argument and we have used (3.4.5) to obtain the above expressions.

Substituting from (3.4.4) and (3.4.6)-(3.4.8) into (3.3.2) and simplifying we obtain

$$(1+\xi^2)f''(\xi)g(\eta) + (1-\eta^2)g''(\eta)f(\xi) + 2\xi f'(\xi)g(\eta)$$

$$- 2\eta g'(\eta)f(\xi) - \frac{\xi^2+\eta^2}{(1+\xi^2)(1-\eta^2)} f(\xi)g(\eta) = 0. \qquad (3.4.9)$$

Dividing (3.4.9) by $f(\xi)g(\eta)$, we get

$$\frac{(1+\xi^2)f''(\xi)}{f(\xi)} + \frac{(1-\eta^2)}{g(\eta)}g''(\eta) + \frac{2\xi f'(\xi)}{f(\xi)} - \frac{2\eta g'(\eta)}{g(\eta)}$$

$$- \left[\frac{1}{1-\eta^2} - \frac{1}{1+\xi^2}\right] = 0 \tag{3.4.10}$$

which may be rewritten as

$$-\frac{[(1+\xi^2)f'(\xi)]'}{f(\xi)} - \frac{1}{1+\xi^2} = \frac{[(1-\eta^2)g'(\eta)]'}{g(\eta)} - \frac{1}{1-\eta^2} . \tag{3.4.11}$$

Now the variables are separated and, by equating the left hand side and right hand side of (3.4.11) to $-d_\ell$, the separation constant, we obtain the following two differential equations for $f(\xi)$ and $g(\eta)$:

$$[(1+\xi^2)f'(\xi)]' - \left[d_\ell - \frac{1}{1+\xi^2}\right]f(\xi) = 0 \tag{3.4.12}$$

$$[(1-\eta^2)g'(\eta)]' + \left[d_\ell - \frac{1}{1-\eta^2}\right]g(\eta) = 0 . \tag{3.4.13}$$

The values of the separation constant d_ℓ, insuring periodicity in η, are

$$d_\ell = (\ell+1)(\ell+2), \qquad \ell = 0,1,2,\cdots . \tag{3.4.14}$$

Now equations (3.4.12) and (3.4.13) are associated Legendre's equations of degree $(\ell+1)$ and order one whose solution is given by

$$g_\ell(\eta) = A_\ell P^1_{\ell+1}(\eta) + B_\ell Q^1{}_{\ell+1}(\eta) \tag{3.4.15}$$

$$f_\ell(\xi) = C_\ell P^1_{\ell+1}(i\xi) + D_\ell Q^1_{\ell+1}(i\xi) \tag{3.4.16}$$

where $P^m_n(z)$ and $Q^m_n(z)$ are defined by

$$\left.\begin{aligned} P^m_n(z) &= (1-z^2)^{m/2} \frac{d^m P_n(z)}{dz^m} \\ Q^m_n(z) &= (1-z^2)^{m/2} \frac{d^m Q_n(z)}{dz^m} \end{aligned}\right\} \tag{3.4.17}$$

m is a positive integer, $-1 < z < 1$, n being unrestricted. When z is not a real number, these functions are defined by

$$P^m_n(z) = (z^2-1)^{m/2} \frac{d^m P_n(z)}{dz^m}$$

$$Q^m_n(z) = (z^2-1)^{m/2} \frac{d^m Q_n(z)}{dz^m}$$

m is a positive integer, n is unrestricted and arg z, arg(z+1), arg(z-1) have their principal values. The Legendre functions $P_n(z)$ and $Q_n(z)$ are defined by

$$P_n(z) = \sum_{r=0}^{k} (-1)^r \frac{(2n-2r)!}{2^n.r!(n-r)!(n-2r)!} z^{n-2r}$$

where $k = \frac{1}{2}$ n or $\frac{1}{2}$ (n-1), whichever is an integer and

$$Q_n(z) = \frac{1}{2}\int_{-1}^{1} P_n(y)\,\frac{dy}{z-y}$$

where n is a positive integer, and z is not a real number.

Since $P^1_{\ell+1}(i\xi)$ becomes infinity as ξ approaches infinity and $Q^1_{\ell+1}(\eta)$ becomes singular for $\eta = 1$, we must have

$$B_\ell = C_\ell = 0. \tag{3.4.18}$$

Also, the condition that v is an even function of η requires that only even values of the subscript ℓ occur. The series solution for v possessing appropriate behavior is then given by

$$v(r,z) = \bar{v}(\xi,\eta) = \sum_{\ell=0,2,\cdots} A_\ell P^1_{\ell+1}(\eta)Q^1_{\ell+1}(i\xi)\ , \tag{3.4.19}$$

where we have taken $D_\ell = 1$.

The first term of the above series is given by $A_0P^1_1(\eta)Q^1_1(i\xi)$ and we find that

$$P^1_1(\eta) = (1-\eta^2)^{1/2}\,\frac{dP_1(\eta)}{d\eta} = (1-\eta^2)^{1/2}\,\frac{d}{d\eta}\,\eta = (1-\eta^2)^{1/2} \tag{3.4.20}$$

$$Q^1_1(i\xi) = (1+\xi^2)^{1/2}\,\frac{d}{d\xi}\,Q_1(i\xi) = (1+\xi^2)^{1/2}\,\frac{d}{d\xi}\left[\frac{i\xi}{2}\log\frac{i\xi+1}{i\xi-1} - 1\right]$$

$$= (1+\xi^2)^{1/2}\left[-\frac{\xi}{\xi^2+1} + \frac{i}{2}\log\frac{i\xi+1}{i\xi-1}\right] \tag{3.4.21}$$

since

$$\frac{i}{2} \log \frac{i\xi+1}{i\xi-1} = \frac{\pi}{2} - \tan^{-1}\xi \tag{3.4.22}$$

we may write

$$Q_1^1(i\xi) = (1+\xi^2)^{1/2}\left[\frac{\pi}{2} - \frac{\xi}{\xi^2+1} - \tan^{-1}\xi\right] . \tag{3.4.23}$$

Now from (3.4.19), (3.4.20) and (3.4.23) we find that

$$\bar{v}(0,\eta) = \frac{\pi}{2} A_0(1-\eta^2)^{1/2} + \sum_{\ell=2,4,\cdots} A_\ell P_{\ell+1}^1(\eta)Q_{\ell+1}^1(0) . \tag{3.4.24}$$

It is easy to see that the boundary condition (3.4.2) will be satisfied if

$$\left.\begin{aligned} A_0 &= \frac{2\omega r_0}{\pi} \\ A_\ell &= 0, \qquad \ell = 2,4,\cdots \end{aligned}\right\} . \tag{3.4.25}$$

Hence the expression for v is given by

$$\bar{v}(\xi,\eta) = \frac{2}{\pi}\,\omega r_0\,(1-\eta^2)^{1/2}(1+\xi^2)^{1/2}\left[\frac{\pi}{2} - \frac{\xi}{\xi^2+1} - \tan^{-1}\xi\right] . \tag{3.4.26}$$

The above equation for $\bar{v}(\xi,\eta)$ satisfies the boundary condition (3.4.3) identically. From equation (3.4.26) we find that the surface displacement in cylindrical coordinates is given by

$$v(r,0) = \begin{cases} \omega r & r \leq r_0 \\ \omega r_0 \left\{ \frac{r}{r_0} \left[1 - \frac{2}{\pi} \tan^{-1} \left[\left[\frac{r}{r_0}\right]^2 - 1 \right]^{1/2} \right] - \frac{2}{\pi} \left[1 - \left[\frac{r_0}{r}\right]^2 \right]^{1/2} \right\} & \\ & r_0 \leq r \, . \end{cases} \tag{3.4.27}$$

The shear stress distribution under the plate is given by

$$\begin{aligned} \sigma_{\theta z}(r,0) &= \mu \left[\frac{\partial v}{\partial z}\right]_{z=0} \\ &= \mu \left[\frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial v}{\partial \eta} \cdot \frac{\partial \eta}{\partial z}\right]_{\xi=0} \\ &= \frac{\mu}{r_0 \eta}\left[\frac{\partial v}{\partial \xi}\right]_{\xi=0} \\ &= -\frac{4\omega\mu}{\pi}\frac{(1-\eta^2)^{1/2}}{\eta} \, , \quad r \leq r_0 \end{aligned} \tag{3.4.28}$$

and changing to cylindrical coordinates we find that

$$\sigma_{\theta z}(r,0) = -\frac{4\omega\mu}{\pi}\frac{1}{\left[\left[\frac{r_0}{r}\right]^2 - 1\right]^{1/2}} \, , \quad r \leq r_0 \, . \tag{3.4.29}$$

The torque T required to rotate the circular area through an angle ω is given by

$$\begin{aligned} T &= -2\pi \int_0^{r_0} r^2 \sigma_{\theta z}(r,0)\,dr \\ &= 8\omega\mu \int_0^{r_0} \frac{r^2}{\left[\left[\frac{r_0}{r}\right]^2 - 1\right]^{1/2}}\,dr \end{aligned}$$

$$= \frac{16}{3} \omega\mu r_0^3 \ . \tag{3.4.30}$$

§3.5 Sneddon's Solution

Sneddon [26] solved the problem by reducing it to a pair of dual integral equations. In order to find solution of the equation (3.3.2), he introduced the Hankel-transform

$$\bar{v}(s,z) = \int_0^\infty rv(r,z)J_1(sr)dr \tag{3.5.1}$$

of the circumferential component v of the displacement vector. Multiplying both sides of equation (3.3.2) by $rJ_1(sr)$ we get

$$rJ_1(sr)\frac{\partial^2 v}{\partial r^2} + J_1(sr)\frac{\partial v}{\partial r} - \frac{v}{r}J_1(sr) + rJ_1(sr)\frac{\partial^2 v}{\partial z^2} = 0 \ . \tag{3.5.2}$$

Integrating the above equation with respect to r from zero to infinity we get

$$\int_0^\infty r\left[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right]J_1(sr)dr + \int_0^\infty rJ_1(sr)\frac{\partial^2 v}{\partial z^2}dr = 0 \ . \tag{3.5.3}$$

Now [25]

$$\int_0^\infty r\left[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right]J_1(sr)dr = -s^2\bar{v} \tag{3.5.4}$$

so that we have

$$\left[s^2 - \frac{\partial^2}{\partial z^2}\right]\bar{v} = 0 \tag{3.5.5}$$

where $\bar{v}$ is the Hankel transform of v and is given by (3.5.1). By means of the Hankel inversion theorem [33] we have

$$v(r,z) = \int_0^\infty s\bar{v}(s,z)J_1(sr)ds \tag{3.5.6}$$

and

$$\sigma_{\theta z}(r,z) = \mu \int_0^\infty s \frac{\partial \bar{v}(s,z)}{\partial z} J_1(sr)ds \; . \tag{3.5.7}$$

The solution of equation (3.5.5) is given by

$$\bar{v}(s,z) = A(s)e^{-sz} + B(s)e^{sz} \tag{3.5.8}$$

where A(s) and B(s) are unknown functions to be determined by the boundary conditions (3.3.3) and (3.3.4) and the condition that v and hence $\bar{v}$ tends to zero as z tends to infinity. This last condition requires that we must take B(s) to be identically zero and hence

$$\bar{v}(s,z) = A(s)e^{-sz} \; . \tag{3.5.9}$$

Substituting the value of v and $\sigma_{\theta z}$ from equation (3.5.6) and (3.5.7) into (3.3.3) and (3.3.4), we get

$$\left.\begin{aligned} &\int_0^\infty s\bar{v}(s,0)J_1(sr)ds = \omega r, && r \leq r_0 \\ &\int_0^\infty \left[\frac{\partial \bar{v}(s,z)}{\partial z}\right]_{z=0} J_1(sr)ds = 0, && r > r_0 \end{aligned}\right\} . \tag{3.5.10}$$

Substituting the value of $\bar{v}$ from equation (3.5.9) into equations (3.5.10) we get

$$\left.\begin{aligned} &\int_0^\infty sA(s)J_1(sr)ds = \omega r && r \leq r_0 \\ &\int_0^\infty s^2A(s)J_1(sr)ds = 0 && r > r_0 \end{aligned}\right\} . \tag{3.5.11}$$

Making the substitution

$$r = \rho r_0, \quad s = \varsigma/r_0, \quad \varsigma^2 A(\varsigma/r_0) = r_0{}^3 F(\varsigma) \tag{3.5.11a}$$

the equations (3.5.11) become

$$\left.\begin{aligned} &\int_0^\infty \varsigma^{-1}F(\varsigma)J_1(\varsigma\rho)d\varsigma = \omega\rho && \rho \leq 1 \\ &\int_0^\infty F(\varsigma)J_1(\varsigma\rho)d\varsigma = 0 && \rho > 1 \end{aligned}\right\} . \tag{3.5.12}$$

In order to solve the above dual integral equations let us take

$$F(\varsigma) = \varsigma \int_0^1 \phi(t)\sin \varsigma t \, dt . \tag{3.5.13}$$

Substituting the value of $F(\varsigma)$ from (3.5.13) into the second equation of (3.5.12) we get

$$\int_0^\infty \varsigma \int_0^1 \Phi(t)\sin \varsigma t \, dt \, J_1(\varsigma\rho)d\varsigma = 0 \qquad \rho > 1. \qquad (3.5.14)$$

Changing the order of integration in (3.5.14) we get

$$\int_0^1 \Phi(t)dt \int_0^\infty \varsigma \sin \varsigma t \, J_1(\varsigma\rho)d\varsigma = 0 \qquad \rho > 1. \qquad (3.5.15)$$

Since [29]

$$\int_0^\infty \varsigma \sin \varsigma t \, J_1(\varsigma\rho)d\varsigma = \begin{cases} \dfrac{-\rho}{(t^2-\rho^2)^{3/2}} & 0 \le \rho < t \\ 0 & \rho > t \end{cases} \qquad (3.5.16)$$

the equation (3.5.15) is identically satisfied. By substituting the value of $F(\varsigma)$ from (3.5.13) into the first equation of (3.5.12) we have

$$\int_0^\infty \int_0^1 \Phi(t)\sin \varsigma t \, dt \, J_1(\varsigma\rho)d\varsigma = \omega\rho \qquad \rho < 1. \qquad (3.5.17)$$

Changing the order of integration in the above equation, and making use of the following relation [29]:

$$\int_0^\infty \sin \varsigma t \, J_1(\varsigma\rho)d\varsigma = \begin{cases} 0 & \text{if} \quad 0 \le \rho < t \\ \dfrac{t}{\rho(\rho^2-t^2)^{1/2}} & \rho > t \end{cases} \qquad (3.5.18)$$

the equation (3.5.17) gives

$$\int_0^\rho \frac{t\Phi(t)}{\rho(\rho^2-t^2)^{1/2}} dt = \omega\rho \ , \quad p < 1. \qquad (3.5.19)$$

We may write

$$\int_0^{\rho} \frac{t\phi(t)}{\rho(\rho^2-t^2)^{1/2}}\, dt = \frac{\sqrt{\pi}}{2}\, I_{0,\frac{1}{2}}\, (\phi(t),\rho) \tag{3.5.20}$$

where $I_{\eta,\alpha}$ is called Erdelyi-Kober operator defined by

$$I_{\eta,\alpha}(f(u),x) = \frac{2x^{-2\alpha-2\eta}}{\Gamma(\alpha)} \int_0^x (x^2-u^2)^{\alpha-1}\, u^{2\eta+1}\, f(u)du, \quad \alpha > 0 \tag{3.5.21}$$

and

$$I_{\eta,\alpha}(f(u),x) = \frac{x^{-2\alpha-2\eta-1}}{\Gamma(1+\alpha)} \frac{d}{dx} \int_0^x u^{2\eta+1}(x^2-u^2)^{\alpha} f(u)du, \; -1 < \alpha < 0. \tag{3.5.22}$$

Multiplying equation (3.5.19) by $\frac{2}{\sqrt{\pi}}\, I^{-1}_{0,\frac{1}{2}}$, that is, by $\frac{2}{\sqrt{\pi}}\, I_{\frac{1}{2},-\frac{1}{2}}$ where $I^{-1}_{\eta,\alpha}$ is the inverse of $I_{\eta,\alpha}$ and is defined by $I^{-1}_{\eta,\alpha} = I_{\eta+\alpha,-\alpha}$, we get

$$\phi(t) = \frac{2}{\pi}\, \frac{1}{t}\, \frac{d}{dt} \int_0^t \frac{\rho^3 \omega}{(t^2-\rho^2)^{1/2}}\, d\rho \tag{3.5.23}$$

which gives

$$\phi(t) = \frac{4\omega t}{\pi}\, . \tag{3.5.24}$$

Substituting the value of $\phi(t)$ into equation (3.5.13) we get

$$F(\varsigma) = \frac{4\omega\varsigma}{\pi} \int_0^1 t \sin \varsigma t \, dt\, . \tag{3.5.25}$$

Integrating (3.5.25) by parts we get

$$F(\varsigma) = \frac{4\omega\varsigma}{\pi}\left[\frac{\sin\varsigma}{\varsigma^2} - \frac{\cos\varsigma}{\varsigma}\right]$$

$$= \frac{4\omega}{\pi}\left[\frac{\sin\varsigma}{\varsigma} - \cos\varsigma\right] . \tag{3.5.26}$$

Substituting from equations (3.5.9), (3.5.11a) and (3.5.26) into equations (3.5.6) and (3.5.7) we get

$$v(r,z) = \frac{4\omega r_0}{\pi}\int_0^\infty \left[\frac{\sin\varsigma}{\varsigma^2} - \frac{\cos\varsigma}{\varsigma^2}\right] e^{-u\varsigma} J_1(\varsigma\rho)d\varsigma \tag{3.5.27}$$

and

$$\sigma_{\theta z}(r,z) = -\frac{4\omega\mu}{\pi}\int_0^\infty \left[\frac{\sin\varsigma}{\varsigma} - \cos\varsigma\right] e^{-u\varsigma} J_1(\varsigma\rho)d\varsigma \tag{3.5.28}$$

where

$$u = z/r_0 . \tag{3.5.29}$$

In order to evaluate the integrals in equations (3.5.27) and (3.5.28) let us take

$$I_n(p) = \int_0^\infty e^{-pt} t^n J_1(\rho t)dt . \tag{3.5.30}$$

Substituting the value of $J_1(\rho t)$ given by

$$J_1(\rho t) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu (\rho t/2)^{1+2\nu}}{\nu!\ \Gamma(\nu+2)} \tag{3.5.31}$$

in (3.5.30) we get

$$I_n(p) = \int_0^\infty \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \rho^{1+2\nu}}{2^{1+2\nu}\nu!\Gamma(\nu+2)} \cdot e^{-pt}\, t^{n+1+2\nu}dt$$

$$= \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \rho^{1+2\nu}}{2^{1+2\nu}\nu!\ \Gamma(\nu+2)} \int_0^\infty e^{-pt}t^{n+1+2\nu}dt \ . \qquad (3.5.32)$$

By taking $t = u/p$ and making use of the following integral

$$\int_0^\infty e^{-t}t^{m-1}dt = \Gamma(m) \qquad (3.5.33)$$

we obtain

$$I_n(p) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \rho^{1+2\nu}}{2^{1+2\nu}\nu!\ \Gamma(\nu+2)} \cdot \frac{1}{p^{n+2+2\nu}} \int_0^\infty e^{-u}u^{n+1+2\nu}du$$

$$= \sum_{\nu=0}^{\infty} \frac{\Gamma(n+2+2\nu)(-1)^\nu}{2^{1+2\nu}\nu!\Gamma(\nu+2)} \cdot \frac{\rho^{1+2\nu}}{p^{n+2+2\nu}} \ . \qquad (3.5.34)$$

Taking $n = 0$ in (3.5.34) we have

$$I_0(p) = \int e^{-pt}J_1(\rho t)dt = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \Gamma(2+2\nu)}{\nu!\ 2^{1+2\nu}\Gamma(\nu+2)} \cdot \frac{\rho^{1+2\nu}}{p^{2+2\nu}}$$

$$= \frac{\rho}{p^2} \sum_{\nu=0}^{\infty} \frac{(2\nu+1)!}{2^{1+2\nu}\nu!\ (\nu+1)!} \left[-\frac{\rho^2}{p^2}\right]^\nu$$

$$= \frac{\rho}{p^2}\left[\frac{1}{2} - \frac{3}{2^3}\frac{\rho^2}{p^2} + \frac{5}{2^4}\frac{\rho^4}{p^4} \cdots \right] . \qquad (3.5.35)$$

Noting that the expansion of

$$(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{3}{2^3}x^2 - \frac{5}{2^4}x^3 + \cdots , \qquad |x| < 1 \tag{3.5.36}$$

we may write

$$\int_0^\infty e^{-pt}J_1(\rho t)dt = \frac{1}{\rho} - \frac{p}{\rho(\rho^2+p^2)^{1/2}} , \qquad \rho < p . \tag{3.5.37}$$

Similarly, taking n = -1, we get

$$\begin{aligned} I_{-1}(p) &= \int_0^\infty e^{-pt}t^{-1}J_1(\rho t)dt \\ &= \sum_{\nu=0}^{\infty} \frac{\Gamma(2\nu+1)\ (-1)^\nu}{2^{1+2\nu}\nu!\Gamma(\nu+2)} \cdot \frac{\rho^{1+2\nu}}{p^{1+2\nu}} \\ &= \frac{\rho}{p} \sum_{\nu=0}^{\infty} \frac{2\nu!}{\nu!\ (\nu+1)!\ 2^{1+2\nu}} \left[-\frac{\rho^2}{p^2}\right]^\nu \\ &= \frac{\rho}{p}\left[\frac{1}{2} - \frac{1}{2^3}\frac{\rho^2}{p^2} + \frac{1}{2^4}\frac{\rho^4}{p^4} + \cdots \right] . \end{aligned} \tag{3.5.38}$$

Again comparing the expansion of $(1+x)^{1/2}$ with (3.5.38) we can write

$$\int_0^\infty e^{-pt}t^{-1}J_1(\rho t)dt = \frac{(\rho^2+p^2)^{1/2} - p}{\rho} , \qquad \rho < p . \tag{3.5.39}$$

Taking p = (u-i) in equation (3.5.37), we obtain

$$\int_0^\infty e^{-ut}e^{it}J_1(\rho t)dt = \frac{1}{\rho}\left[1 - \frac{u-i}{[\rho^2+(u-i)^2]^{1/2}}\right] . \qquad (3.5.40)$$

If we take

$$[(\rho^2+u^2-1) - 2ui]^{1/2} = c+id \qquad (3.5.41)$$

then we have

$$\left.\begin{aligned} c^2 &= \frac{(\rho^2+u^2-1) + [(\rho^2+u^2-1)^2+4u^2]^{1/2}}{2} \\ d &= - u/c \end{aligned}\right\} . \qquad (3.5.42)$$

Now we may write (3.5.40) in the form

$$\int_0^\infty (\cos t + i \sin t)e^{-ut}J_1(\rho t)dt = \frac{1}{\rho}\left[1 - \frac{(u-i)(c-id)}{c^2+d^2}\right] \qquad (3.5.43)$$

which, on equating the real part from both sides, gives

$$\int_0^\infty \cos t \; e^{-ut}J_1(\rho t)dt = \frac{1}{\rho}\left[1 - \frac{uc-d}{c^2+d^2}\right] . \qquad (3.5.44)$$

Again taking p = u-i in (3.5.39) we get

$$\begin{aligned} \int_0^\infty (\cos t + i \sin t) \frac{e^{-ut}}{t} J_1(\rho t)dt &= \frac{(\rho^2+(u-i)^2)^{1/2} - (u-i)}{\rho} \\ &= \frac{(c+id) - (u-i)}{\rho} \end{aligned} \qquad (3.5.45)$$

where c and d are given by (3.5.42).

Equating the real and imaginary parts from both sides of equation (3.5.45) gives

$$\int_0^\infty \frac{\cos t}{t} e^{-ut} J_1(\rho t)dt = \frac{c-u}{\rho} \tag{3.5.46}$$

and

$$\int_0^\infty \frac{\sin t}{t} e^{-ut} J_1(\rho t)dt = \frac{d+1}{\rho} . \tag{3.5.47}$$

Integrating equation (3.5.39) with respect to p between 0 to p we get

$$\int_0^\infty \frac{1-e^{-pt}}{t^2} J_1(\rho t)dt$$
$$= \frac{1}{2\rho} \left[p(p^2+\rho^2)^{1/2} + \rho^2 \ln \left| \frac{p+(p^2+\rho^2)^{1/2}}{\rho} \right| - p^2 \right] . \tag{3.5.48}$$

Again taking p = (u-i) in the above equation, we obtain

$$\int_0^\infty \frac{(1-e^{-ut}e^{it})}{t^2} J_1(\rho t)dt$$
$$= \frac{1}{2\rho} \left[(u-i)(c+di) + \rho^2 \ln \left| \frac{(u-i)+(c+di)}{\rho} \right| \right] - \frac{u^2-1-2ui}{2\rho} \tag{3.5.49}$$

where c and d are given by (3.5.42).

Equating the imaginary parts from both sides of equation (3.5.49) yields

$$- \int_0^\infty \frac{\sin t}{t^2} e^{-ut} J_1(\rho t)dt = \frac{1}{2\rho} \left[(ud-c) + \rho^2 \tan^{-1} \frac{d-1}{u+c} \right] + \frac{u}{\rho} . \tag{3.5.50}$$

Substituting from (3.5.44), (3.5.46), (3.5.47) and (3.5.50) into (3.5.27) and (3.5.28), we obtain the following expressions for the displacement and the shearing stress:

$$v(r,z) = -\frac{4\omega r_0}{\pi}\left[\frac{1}{2\rho}\left[(ud+c) + \rho^2 \tan^{-1}\frac{d-1}{u+c}\right]\right] \tag{3.5.51}$$

$$\sigma_{\theta z}(r,z) = -\frac{4\omega\mu}{\pi\rho}\left[d + \frac{uc-d}{c^2+d^2}\right] \tag{3.5.52}$$

where c and d are given by (3.5.42).

When z = 0 that is u = 0, the expressions for v and $\sigma_{\theta z}$ on the surface are given by

$$v(r,0) = \omega r_0\rho\left[\left\{1 - \frac{2}{\pi}\tan^{-1}(\rho^2-1)^{1/2}\right\} - \frac{2}{\pi}\left[1 - \frac{1}{\rho^2}\right]^{1/2}, \ \rho > 1 \tag{3.5.53}$$

and

$$\sigma_{r\theta}(r,0) = -\frac{4\omega\mu}{\pi}\left[\frac{1}{\rho^2} - 1\right]^{-1/2}, \quad \rho < 1 \tag{3.5.54}$$

which agree with those obtained by Reissner and Sagoci.

The expression for torque T required to rotate the circular area through an angle ω is given by

$$\begin{aligned} T &= -2\pi\int_0^{r_0} r^2\sigma_{\theta z}(r,0)dr \\ &= 8\omega\mu\int_0^{r_0}\frac{r^2}{\left[\left[\frac{r_0}{r}\right]^2 - 1\right]^{1/2}}dr \\ &= \frac{16}{3}\,\omega\,\mu\,r_0^3\ . \end{aligned}$$

CHAPTER IV

A REISSNER-SAGOCI PROBLEM FOR AN ELASTIC LAYER BONDED TO ANOTHER ELASTIC LAYER

§4.1 Introduction

In this chapter we have considered the Reissner-Sagoci type problem for two, isotropic, homogeneous elastic layers of different thicknesses and modulus of rigidities bonded to a rigid foundation. The problem has been reduced to the solution of a Fredholm integral equation of the second kind, which has been solved numerically. The expression for the torque required to rotate the circular portion of the boundary through an angle ω has been obtained in the closed form. Numerical values of the torque T for various ratios of the thicknesses and elastic moduli of the two layers has been given in tabulated form as well as displayed graphically.

The results for the following four problems have been derived as particular cases:

(i) Reissner-Sagoci problem for the half-space.

(ii) Reissner-Sagoci problem for a layer bonded to a rigid foundation.

(iii) Reissner-Sagoci problem for a layer with lower face stress free.

(iv) Reissner-Sagoci problem for a layer bonded to an elastic half-space.

The results for these cases have been shown to agree with the known results.

§4.2 Statement of the problem and solution

Let us consider two isotropic homogeneous elastic layers of thickness h_1, h_2 and modulus of rigidity μ_1, μ_2 respectively, bonded together and to a rigid foundation. A rigid circular cylinder of unit radius bonded to the upper layer of thickness h_1 is forced to rotate through an angle ω and the remaining portion of the surface of the layer is assumed to be free of stress. The axis of rotation is perpendicular to the surface of the layer. It is assumed that the lower face of the layer of thickness h_2 is rigidly fixed. Let us take the common boundary of the two layers as the plane $z = h_1$ and, taking the z-axis downwards as shown in the Figure 4, we denote the upper face by $z = 0$ and the lower face by $z = h_1+h_2$. Now we denote the region $0 \leq z \leq h_1$ by R_1 and the region $h_1 \leq z \leq h_1+h_2$ by R_2. The physical quantities for the region R_i $(i = 1,2)$ are denoted by the subscript/superscript (i), namely μ_i, v_i, $\sigma_{\theta z}^{(i)}$, $\sigma_{r\theta}^{(i)}$ $(i = 1,2)$.

As shown by Reissner [21] in this case only the circumferential displacement v_i occurs and all the components of stresses are zero except the following two components of shearing stress:

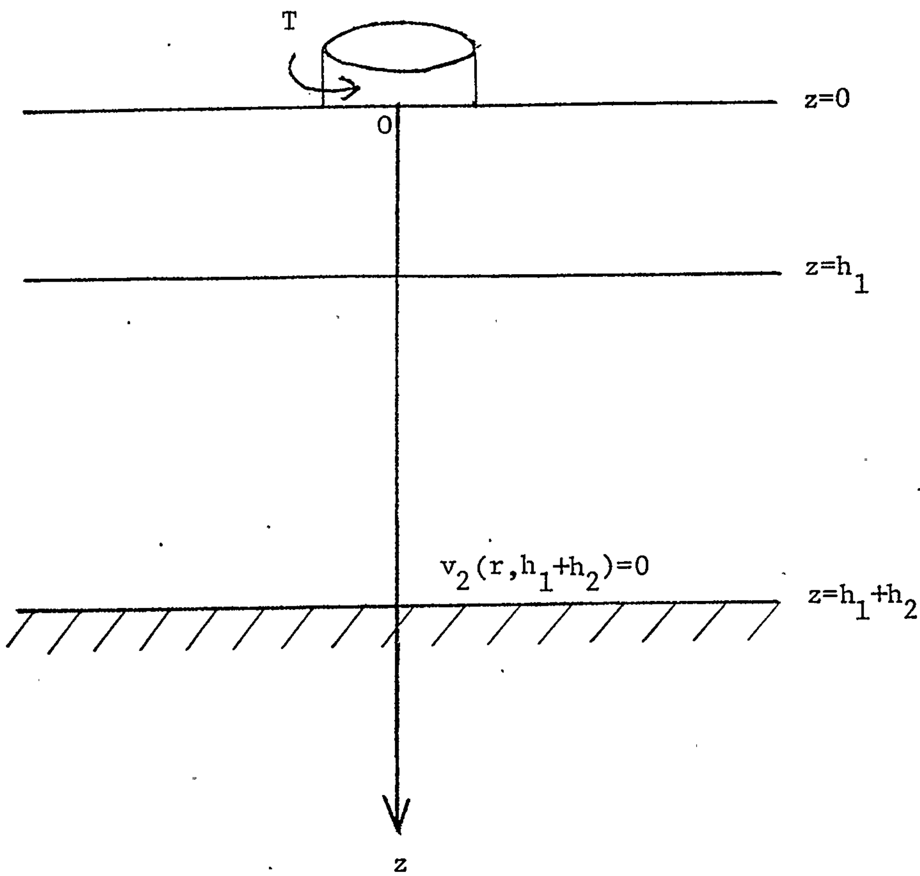

Figure 4: Two Layers $0 \le z \le h_1$ and $h_1 \le z \le h_1+h_2$ Bonded to a Rigid Foundation.

$$\left.\begin{aligned} \sigma_{\theta z}^{(i)} &= \mu_i \frac{\partial v_i}{\partial z} \\ \sigma_{r\theta}^{(i)} &= \mu_i \left[\frac{\partial v_i}{\partial r} - \frac{v_i}{r}\right] \end{aligned}\right\} \tag{4.2.1}$$

and v_i satisfies the partial differential equation

$$\frac{\partial^2 v_i}{\partial r^2} + \frac{1}{r}\frac{\partial v_i}{\partial r} - \frac{v_i}{r^2} + \frac{\partial^2 v_i}{\partial z^2} = 0 \quad \text{in } R_i. \tag{4.2.2}$$

The boundary conditions of the problem are

$$\left.\begin{aligned} v_1(r,0) &= \omega\, f(r) & 0 \leq r \leq 1 \\ \sigma_{\theta z}^{(1)}(r,0) &= 0 & r > 1 \end{aligned}\right\} \tag{4.2.3}$$

$$v_2(r,h_1+h_2) = 0 \qquad 0 \leq r \leq \infty . \tag{4.2.4}$$

The continuity conditions at the interface $z = h_1$ are

$$\left.\begin{aligned} [v_i(r,h_1)]_1^2 &= 0 & 0 \leq r < \infty \\ [\sigma_{\theta z}^{(i)}(r,h_1)]_1^2 &= 0 & 0 \leq r < \infty \end{aligned}\right\} . \tag{4.2.5}$$

In order to solve the differential equation (4.2.2) let us multiply this equation by $rJ_1(\xi r)$ and integrate with respect to r from 0 to ∞, to obtain

$$\left[\frac{\partial^2}{\partial z^2} - \xi^2\right]\bar{v}_i = 0 \tag{4.2.6}$$

where

$$\bar{v}_i(\xi,z) = \int_0^\infty r v_i(r,z) J_1(\xi r) dr \tag{4.2.7}$$

denotes the Hankel transform of order one of the function $v_i(r,z)$. By the Hankel inversion theorem, we have

$$v_i(r,z) = \int_0^\infty \xi \bar{v}_i(\xi,z) J_1(\xi r) d\xi \tag{4.2.8}$$

and we have

$$\sigma_{\theta z}^{(i)}(r,z) = \mu_i \frac{\partial v_i}{\partial z} = \mu_i \int_0^\infty \xi \frac{\partial \bar{v}_i}{\partial z} J_1(\xi r) d\xi \ . \tag{4.2.9}$$

The solution of equation (4.2.6) is given by

$$\bar{v}_i(\xi,z) = A_i(\xi)e^{\xi z} + B_i(\xi)e^{-\xi z}, \quad i = 1,2 \tag{4.2.10}$$

where A_i and B_i $(i = 1,2)$ are the unknown functions which are to be determined by using the boundary and continuity conditions. Substituting (4.2.10) in (4.2.8) and (4.2.9) we get

$$v_i(r,z) = \int_0^\infty \xi \ [A_i(\xi)e^{\xi z} + B_i(\xi)e^{-\xi z}] J_1(\xi r) d\xi \tag{4.2.11}$$

$$\sigma_{\theta z}^{(i)}(r,z) = \mu_i \int_0^\infty \xi^2 [A_i(\xi)e^{\xi z} - B_i(\xi)e^{-\xi z}] J_1(\xi r) d\xi . \tag{4.2.12}$$

Now substituting from (4.2.11) and (4.2.12) into the boundary condition (4.2.4) and continuity conditions (4.2.5) we get

$$\left.\begin{aligned}
&\int_0^\infty \xi\left[A_2 e^{\xi(h_1+h_2)}+B_2 e^{-\xi(h_1+h_2)}\right]J_1(\xi r)d\xi = 0\\
&\int_0^\infty \xi\left[\left[A_1 e^{\xi h_1}+B_1 e^{-\xi h_1}\right]-\left[A_2 e^{\xi h_1}+B_2 e^{-\xi h_1}\right]\right]J_1(\xi r)d\xi = 0\\
&\int_0^\infty \xi^2\left[\mu_1\left[A_1 e^{\xi h_1}-B_1 e^{-\xi h_1}\right]-\mu_2\left[A_2 e^{\xi h_1}-B_2 e^{-\xi h_1}\right]\right]J_1(\xi r)d\xi=0
\end{aligned}\right\},\quad 0\le r<\infty \qquad (4.2.13)$$

Since the equations (4.2.13) hold true for all values of r, we must have

$$\left.\begin{aligned}
&A_2 e^{\xi(h_1+h_2)} + B_2 e^{-\xi(h_1+h_2)} = 0\\
&A_1 e^{\xi h_1} + B_1 e^{-\xi h_1} = A_2 e^{\xi h_1} + B_2 e^{-\xi h_1}\\
&\mu_1\left[A_1 e^{\xi h_1} - B_1 e^{-\xi h_1}\right] = \mu_2\left[A_2 e^{\xi h_1} - B_2 e^{-\xi h_1}\right]
\end{aligned}\right\} \qquad (4.2.14)$$

Solving equations (4.2.14) for A_1, B_1 and A_2 in terms of B_2, we find

$$\left.\begin{aligned}
&A_2 = - B_2 e^{-2\xi(h_1+h_2)}\\
&A_1 = \frac{B_2}{2}\left[(1-\delta)e^{-2\xi h_1} - (1+\delta)e^{-2\xi(h_1+h_2)}\right]\\
&B_1 = \frac{B_2}{2}\left[(1+\delta) - (1-\delta)e^{-2\xi h_2}\right]
\end{aligned}\right\} \qquad (4.2.15)$$

where

$$\delta = \mu_2/\mu_1 \ . \tag{4.2.16}$$

Substituting the values of A_1 and B_1 from (4.2.15) into (4.2.11) and (4.2.12), we find the following expressions for $v_1(r,0)$ and $\sigma_{\theta z}^{(1)}(r,0)$.

$$v_1(r,0) = \frac{1}{2}\int_0^\infty \xi B_2(\xi)\Big[(1-\delta)\left[e^{-2\xi h_1}-e^{-2\xi h_2}\right] + (1+\delta)\left[1-e^{-2\xi(h_1+h_2)}\right]\Big]J_1(\xi r)d\xi \tag{4.2.17}$$

$$\sigma_{\theta z}^{(1)}(r,0) = \frac{1}{2}\int_0^\infty \xi^2 B_2(\xi)\Big\{(1-\delta)\left[e^{-2\xi h_1}+e^{-2\xi h_2}\right] - (1+\delta)\left[e^{-2\xi(h_1+h_2)}+1\right]\Big\}J_1(\xi r)d\xi \tag{4.2.18}$$

which can be written as

$$v_1(r,0) = \int_0^\infty [H(\xi)-1]P(\xi)J_1(\xi r)d\xi \tag{4.2.19}$$

$$\sigma_{\theta z}^{(1)}(r,0) = \mu_1\int_0^\infty \xi P(\xi)J_1(\xi r)d\xi \tag{4.2.20}$$

where

$$P(\xi) = \frac{\xi}{2}B_2(\xi)\Big\{(1-\delta)\left[e^{-2\xi h_1}+e^{-2\xi h_2}\right]-(1+\delta)\left[1+e^{-2\xi(h_1+h_2)}\right]\Big\} \tag{4.2.21}$$

$$H(\xi) = \frac{2\left[1 + \bar{\mu}\, e^{-2\xi h_2}\right] e^{-2\xi h_1}}{\left[e^{-2\xi h_1} + e^{-2\xi h_2}\right] + \bar{\mu}\left[1 + e^{-2\xi(h_1+h_2)}\right]} \tag{4.2.22}$$

with

$$\bar{\mu} = \frac{\delta+1}{\delta-1} \, . \tag{4.2.23}$$

Substituting from (4.2.19) and (4.2.20) into the boundary conditions (4.2.3), we find that these will be satisfied if $P(\xi)$ is a solution of the following dual integral equations:

$$\int_0^\infty [H(\xi)-1]P(\xi)J_1(\xi r) = \omega f(r), \quad 0 \le r \le 1 \tag{4.2.24}$$

$$\int_0^\infty \xi \; P(\xi)J_1(\xi r)d\xi = 0, \qquad r > 1 \, . \tag{4.2.25}$$

In order to solve these dual integral equations, let us take

$$P(\xi) = \int_0^1 \Phi(t) \sin \xi t \, dt \, . \tag{4.2.26}$$

Now substituting for $P(\xi)$ from (4.2.26) into (4.2.25) we have

$$\int_0^\infty \xi \int_0^1 \Phi(t) \sin \xi t \, dt \; J_1(\xi r) d\xi = 0 \, , \qquad r > 1. \tag{4.2.27}$$

Changing the order of integration, we have

$$\int_0^1 \Phi(t)dt \int_0^\infty \xi \sin(\xi t)J_1(\xi r)d\xi = 0 , \quad r > 1. \tag{4.2.28}$$

Now [29]

$$\int_0^\infty \xi \sin(\xi t)J_1(\xi r)d\xi = \begin{cases} -\dfrac{r}{(t^2-r^2)^{3/2}} , & r < t \\ 0 , & r > t. \end{cases} \tag{4.2.29}$$

Hence we find that the equation (4.2.25) is identically satisfied.

Substituting for $P(\xi)$ from (4.2.26) into (4.2.24), we have

$$\int_0^\infty \int_0^1 \Phi(t) \sin \xi t[H(\xi)-1]J_1(\xi r)dtd\xi = \omega f(r), \ 0 \le r \le 1 \tag{4.2.30}$$

or

$$\int_0^\infty \int_0^1 \Phi(t)\sin \xi t \ J_1(\xi r)dtd\xi$$
$$= - \omega f(r) + \int_0^\infty \int_0^1 \Phi(t)\sin\xi t \ H(\xi)J_1(\xi r)dtd\xi, \ 0 \le r \le 1. \tag{4.2.31}$$

Now

$$\int_0^\infty \int_0^1 \Phi(t)\sin\xi t \ J_1(\xi r)dtd\xi = \int_0^1 \Phi(t)dt \int_0^\infty \sin\xi t J_1(\xi r)d\xi$$
$$= \int_0^r \frac{t \ \Phi(t)}{r(r^2-t^2)^{1/2}} dt , \tag{4.2.32}$$

since [29]

$$\int_0^\infty \sin(\xi t) J_1(\xi r) d\xi = \begin{cases} \dfrac{t}{r(r^2-t^2)^{1/2}} & \text{for} \quad 0 \le t < r \\ \\ 0 & \text{for} \quad t > r \end{cases} . \tag{4.2.33}$$

Therefore equation (4.2.31) may be written as

$$\int_0^r \frac{t\,\phi(t)}{r(r^2-t^2)^{1/2}}\,dt$$
$$= -\,\omega f(r) + \int_0^\infty \int_0^1 \phi(t)\sin(\xi t)H(\xi)J_1(\xi r)\,dt\,d\xi, \; 0 \le r \le 1. \tag{4.2.34}$$

Now in the notation of Erdelyi-Kober operators [32] $I_{\eta,\alpha}$ we may write

$$\int_0^r \frac{t\,\phi(t)}{r(r^2-t^2)^{1/2}}\,dt = \frac{\sqrt{\pi}}{2}\, I_{0,1/2}\{\phi(t);r\} \tag{4.2.35}$$

where

$$I_{\eta,\alpha}\{\phi(u),t\} = \frac{2t^{-2\alpha-2\eta}}{\Gamma(\alpha)} \int_0^t (t^2-u^2)^{\alpha-1} u^{2\eta-1}\phi(u)\,du, \quad \alpha > 0, \tag{4.2.36}$$

$$I_{\eta,\alpha}\{\phi(u),t\} = \frac{t^{-2\alpha-2\eta-1}}{\Gamma(1+\alpha)} \frac{d}{dt} \int_0^t u^{2\eta+1}(t^2-u^2)^{\alpha}\phi(u)\,du, \; -1 < \alpha < 0 \tag{4.2.37}$$

and the inverse operator is given by

$$I^{-1}_{\eta,\alpha} = I_{\eta+\alpha,-\alpha} \, . \tag{4.2.38}$$

Now operating equation (4.2.34) by $\frac{2}{\sqrt{\pi}}\, I^{-1}_{0,1/2}$, that is by

$$\frac{2}{\sqrt{\pi}} I^{-1}_{0,1/2} \{f(r);t\} = \frac{2}{\sqrt{\pi}} I_{1/2,-1/2} \{f(r);t\}$$
$$= \frac{2}{\pi} \frac{1}{t} \frac{d}{dt} \int_0^t \frac{r^2 f(r)}{(t^2-r^2)^{1/2}} dr \qquad (4.2.39)$$

we get

$$\phi(t) = -\frac{2}{\pi} \frac{1}{t} \frac{d}{dt} \int_0^t \frac{r^2 \omega f(r)}{(t^2-r^2)^{1/2}} dr$$
$$+ \frac{2}{\pi} \frac{1}{t} \frac{d}{dt} \int_0^t \frac{r^2}{(t^2-r^2)^{1/2}} dr \int_0^\infty \int_0^1 \phi(u) \sin\xi u H(\xi) J_1(\xi r) du d\xi$$
$$0 \le r \le 1 . \qquad (4.2.40)$$

Interchanging the order of integration in the last term of (2.3.40) we get

$$\phi(t) = -\frac{2}{\pi} \frac{1}{t} \frac{d}{dt} \int_0^t \frac{r^2 \omega f(r)}{(t^2-r^2)^{1/2}} dr$$
$$+ \frac{2}{\pi} \int_0^\infty \int_0^1 \phi(u) \sin(\xi u) H(\xi) \frac{1}{t} \frac{d}{dt} \int_0^t \frac{r^2 J_1(\xi r)}{(t^2-r^2)^{1/2}} dr\, du d\xi \qquad (4.2.41)$$

since [29]

$$\frac{1}{t} \frac{d}{dt} \int_0^t \frac{r^2 J_1(\xi r)}{(t^2-r^2)^{1/2}} dr = \sin \xi t \qquad (4.2.42)$$

we may write (4.2.41) as

$$\Phi(t) = -\frac{2}{\pi}\frac{1}{t}\frac{d}{dt}\int_0^t \frac{r^2\omega f(r)}{(t^2-r^2)^{1/2}}\,dr$$

$$+\frac{2}{\pi}\int_0^\infty\int_0^1 \Phi(u)\sin(\xi u)\sin(\xi t)H(\xi)\,du\,d\xi. \tag{4.2.43}$$

Now if we denote

$$K(u,t) = -\frac{2}{\pi}\int_0^\infty H(\xi)\sin(\xi u)\sin(\xi t)\,d\xi \tag{4.2.44}$$

$$h(t) = -\frac{2}{\pi}\frac{1}{t}\frac{d}{dt}\int_0^t \frac{\omega r^2 f(r)}{(t^2-r^2)^{1/2}}\,dr \tag{4.2.45}$$

then equation (4.2.43) may be rewritten in the form

$$\Phi(t) + \int_0^1 \Phi(u)K(u,t)\,du = h(t), \quad 0 \le t \le 1 \tag{4.2.46}$$

This is called the Fredholm integral equation of the second kind for Φ.

If we take

$$f(r) = r \tag{4.2.47}$$

we find that

$$h(t) = -\frac{4\omega}{\pi}\,t \tag{4.2.48}$$

and hence the integral equation (4.2.46) becomes

$$\Phi(t) + \int_0^1 \Phi(u)K(u,t)du = -\frac{4\omega}{\pi}t, \quad 0 \leq t \leq 1. \tag{4.2.49}$$

Again if we take

$$\Phi(t) = \frac{4\omega}{\pi}\Psi(t) \tag{4.2.50}$$

we may write (4.2.49) in a simplified form

$$\Psi(t) + \int_0^1 \Psi(u)K(u,t)du = -t, \quad 0 \leq t \leq 1. \tag{4.2.51}$$

The torque T required to produce the prescribed rotation of the solid cylinder is given by

$$T = -2\pi\int_0^1 r^2\sigma_{\theta z}^{(1)}(r,0)dr \ . \tag{4.2.52}$$

From (4.2.20), we find that

$$\begin{aligned}\sigma_{\theta z}^{(1)}(r,0) &= \mu_1\int_0^\infty \xi\, P(\xi)J_1(\xi r)d\xi \\ &= -\mu_1\frac{\partial}{\partial r}\left\{\int_0^\infty P(\xi)J_0(\xi r)d\xi\right\} .\end{aligned} \tag{4.2.53}$$

Substituting the value of $P(\xi)$ from equation (4.2.26), in equation (4.2.53), we find that the value of the shear stress under the rigid cylinder is given by

$$\sigma_{\theta z}^{(1)}(r,0) = -\mu_1 \frac{\partial}{\partial r} \int_0^\infty \int_0^1 \phi(t) \sin \xi t \, dt J_0(\xi r) d\xi, \quad 0 \le r < 1, \tag{4.2.54}$$

since

$$\int_0^\infty \sin \xi t \; J_0(\xi r) d\xi = \begin{cases} (t^2-r^2)^{-1/2} & 0 \le r < t \\ \\ 0 & r > t \end{cases} \tag{4.2.55}$$

equation (4.2.54) becomes

$$\sigma_{\theta z}^{(1)}(r,0) = -\mu_1 \frac{\partial}{\partial r} \int_r^1 \frac{\phi(t)dt}{(t^2-r^2)^{1/2}} \, . \tag{4.2.56}$$

Substituting this value of $\sigma_{\theta z}^{(1)}(r,0)$ in equation (4.2.52) we get

$$T = 2\pi\mu_1 \int_0^1 \frac{\partial}{\partial r} \left\{ \int_r^1 \frac{\phi(t)dt}{(t^2-r^2)^{1/2}} \right\} r^2 dr \, . \tag{4.2.57}$$

Integrating (4.2.57) by parts we get

$$T = -4\pi\mu_1 \int_0^1 r \int_r^1 \frac{\phi(t)dt}{(t^2-r^2)^{1/2}} \, dr \, . \tag{4.2.58}$$

Changing the order of integration (4.2.58) becomes

$$T = -\ 4\pi\mu_1 \int_0^1 \phi(t)dt \int_0^t \frac{r\ dr}{(t^2-r^2)^{1/2}}$$

$$= -\ 4\pi\mu_1 \int_0^1 t\phi(t)dt. \tag{4.2.59}$$

Writing $\phi(t)$ in terms of $\psi(t)$ from equation (4.2.50), we get

$$T = -\ 16\omega\mu_1 \int_0^1 t\psi(t)dt\ . \tag{4.2.60}$$

§4.3 Numerical results and discussion

In order to find numerical values of the torque T, we need to find the numerical values of $\psi(t)$. We first of all write the integral equation (4.2.51) as a system of algebraic equations by dividing the interval [0,1] into n equal parts:

$$\psi(t_i) + \frac{1}{n}\sum_{j=0}^{n} \psi(u_j)K(u_j,t_i) = -t_i,\ i = 0,1,\cdots,n \tag{4.3.1}$$

where

$$t_i = \frac{i}{n}\ ,\quad u_j = \frac{j}{n}\ ,\quad (i,j = 0,1,2,\cdots,n). \tag{4.3.2}$$

To obtain the numerical values of $K(u,t)$ we may rewrite its expressions given by equations (4.2.44) and (4.2.22) by making the substitution

$$e^{-\xi} = p\ ,\quad h_2 = h_1\theta \tag{4.3.3}$$

in the following form.

$$K(u,t) = -\frac{4}{\pi}\int_0^1 \frac{(1+\bar{\mu}p^{2h_1\theta})\,p^{2h_1-1}}{[p^{2h_1\theta}+p^{2h_1}]+\bar{\mu}[1+p^{2h_1(1+\theta)}]}\sin(u\log p)\sin(t\log p)\,dp \tag{4.3.4}$$

The numerical values of $K(u_j,t_i)$ have been obtained from (4.3.4) by using Simpson's rule by dividing the interval [0,1] in 24 equal parts. Then the system of equations (4.3.1) has been solved for n = 24 which involves solving 25 equations for 25 unknown φ_i, $i = 0,1,2,\cdots,24$, where

$$\varphi_i = \varphi(t_i) = \varphi(u_i),\ i = 0,1,2,\cdots,24 \tag{4.3.5}$$

The numerical values for φ_i have been obtained for the following combination of the numerical values for the geometrical and physical parameters:

$$\left.\begin{aligned} &\frac{\mu_2}{\mu_1} = \delta = 0.0,0.2,0.25,0.50,0.75,1.0,2.5,5.0,7.5,10.0 \\ &\frac{h_2}{h_1} = \theta = 0.0,0.2,0.4,0.6,0.8,1.0,2.0,4.0,6.0,8.0,10.0 \\ &h_1 = 0.25,0.5,1.0,2.0,4.0. \end{aligned}\right\} \tag{4.3.6}$$

It may be noted that the radius of the circular portion of the upper boundary bonded to a rigid circular cylinder is taken as unity.

The numerical values of the torque $T^* = T/\mu_1\omega$ have now been

TABLE 1

Numerical Values of $T^* = T/\mu_1\omega$ for Various Values of $\theta = h_2/h_1$, h_1 and $\delta = \mu_2/\mu_1$

θ \ δ	0.00	0.10	0.25	0.50	0.75
			$h_1 = 0.25$		
0.2		5.53981	7.00420	8.01186	8.43940
0.4		4.58750	5.90673	7.02497	7.63866
0.6		4.14041	5.30849	6.40352	7.05801
0.8		3.87553	4.92913	5.97788	6.63946
1.0		3.69907	4.66686	5.66984	6.32647
2.0	2.42551	3.29858	4.05021	4.90892	5.52263
4.0		3.15752	3.72079	4.49516	5.07617
6.0		3.02128	3.62967	4.38536	4.95979
8.0		2.99498	3.59411	4.34456	4.91760
10.0		2.98180	3.57734	4.32614	4.89895
			$h_1 = 0.50$		
0.2		5.11165	5.68446	6.07585	6.26167
0.4		4.73595	5.26742	5.70505	5.94295
0.6		4.55878	5.04330	5.48071	5.73643
0.8		4.45476	4.90386	5.33274	5.59512
1.0		4.38657	4.80973	5.22969	5.49463
2.0	3.82346	4.23985	4.60335	4.99838	5.26514
4.0		4.17118	4.51015	4.89635	5.16489
6.0		4.15334	4.48815	4.87403	5.14384
8.0		4.14631	4.48018	4.86647	5.13695
10.0		4.14296	4.47657	4.86321	5.13407
			$h_1 = 1.0$		
0.2		5.21919	5.37024	5.47570	5.52683
0.4		5.12375	5.26413	5.38052	5.44436
0.6		5.08002	5.20973	5.32672	5.39509
0.8		5.05509	5.17727	5.29325	5.36364

1.0		5.03920	5.15618	5.27110	5.34258
2.0	4.88795	5.00695	5.11329	5.22592	5.29955
4.0		4.99320	5.09608	5.20870	5.28362
6.0		4.98983	5.09230	5.20525	5.28059
8.0		4.98853	5.09097	5.20412	5.28225
10.0		4.98790	5.09037	5.20365	5.27924
$h_1 = 2.0$					
0.2		5.30897	5.33508	5.35384	5.36314
0.4		5.29318	5.31720	5.33753	5.34887
0.6		5.28616	5.30840	5.32879	5.34084
0.8		5.28225	5.30331	5.32356	5.33593
1.0		5.27980	5.30008	5.32020	5.33276
2.0	5.25445	5.27499	5.29376	5.31365	5.32660
4.0		5.27301	5.29134	5.31130	5.32446
6.0		5.27253	5.29083	5.31084	5.32407
8.0		5.27234	5.29063	5.31070	5.32395
10.0		5.27225	5.29056	5.31063	5.32390
$h_1 = 4.0$					
0.2		5.32976	5.33360	5.33599	5.33730
0.4		5.32762	5.33092	5.33376	5.33533
0.6		5.32669	5.32974	5.33256	5.33424
0.8		5.32617	5.32906	5.33186	5.33358
1.0		5.32585	5.32864	5.33102	5.33289
2.0	5.32242	5.32522	5.32782	5.33057	5.33237
4.0		5.32496	5.32750	5.33027	5.33210
6.0		5.32489	5.32743	5.33021	5.33205
8.0		5.32486	5.32741	5.33019	5.33203
10.0		5.32484	5.32739	5.33018	5.33202
δ / θ	1.0	2.50	5.00	7.50	10.00
$h_1 = 0.25$					
0.2	8.76190	9.34738	9.57324	9.65256	9.69303
0.4	8.02991	8.95450	9.35383	9.50054	9.57675
0.6	7.49866	8.62919	9.16101	9.36395	9.47107
0.8	7.10178	8.36103	8.99423	9.24358	9.37700
1.0	6.79768	8.14030	8.85158	9.13903	9.29462

2.0	5.99209	7.49305	8.40685	8.80443	9.02710
4.0	5.53577	7.09678	8.11938	8.58270	8.84713
6.0	5.41790	6.99530	8.04538	8.52539	8.80070
8.0	5.37579	6.96016	8.02004	8.50583	8.78482
10.0	5.35742	6.94526	8.00941	8.49766	8.77820
$h_1 = 0.5$					
0.2	6.37097	6.60532	6.69717	6.72964	6.74626
0.4	6.09446	6.45508	6.61257	6.67082	6.70118
0.6	5.90697	6.34099	6.54489	6.62289	6.66403
0.8	5.77535	6.25518	6.49221	6.58498	6.63451
1.0	5.68035	6.19057	6.45166	6.55558	6.61342
2.0	5.46050	6.03485	6.35165	6.48236	6.55380
4.0	5.36491	5.96745	6.30827	6.45056	6.52872
6.0	5.34536	5.95449	6.30016	6.44466	6.52410
8.0	5.33910	5.95055	6.29774	6.44292	6.52274
10.0	5.33653	5.94899	6.29680	6.44225	6.52221
$h_1 = 1.0$					
0.2	5.55732	5.62399	5.65068	5.66022	5.66512
0.4	5.48533	5.58415	5.62803	5.64441	5.65297
0.6	5.44070	5.55705	5.61196	5.63000	5.64414
0.8	5.41172	5.53864	5.60078	5.62500	5.63792
1.0	5.39216	5.52595	5.59299	5.61941	5.63355
2.0	5.35211	5.49978	5.57688	5.60781	5.62450
4.0	5.33756	5.49072	5.57141	5.60390	5.62146
6.0	5.33489	5.48919	5.57053	5.60328	5.62098
8.0	5.33407	5.48874	5.57028	5.60311	5.62089
10.0	5.33374	5.48857	5.57018	5.60304	5.62080
$h_1 = 2.0$					
0.2	5.36876	5.38128	5.38640	5.38824	5.38919
0.4	5.35622	5.37421	5.38234	5.38540	5.38701
0.6	5.34893	5.36976	5.37969	5.38352	5.38555
0.8	5.34442	5.36691	5.37797	5.38228	5.38459
1.0	5.34148	5.36503	5.37682	5.38146	5.38395
2.0	5.33457	5.36141	5.37462	5.37989	5.38272
4.0	5.33387	5.36025	5.37393	5.37946	5.38234
6.0	5.33353	5.36006	5.37383	5.37933	5.38229
8.0	5.33343	5.36001	5.37380	5.37931	5.38227
10.0	5.33339	5.35999	5.37379	5.37930	5.38226

$h_1 = 4.0$					
0.2	5.33810	5.33990	5.34064	5.34091	5.34105
0.4	5.33636	5.33891	5.34007	5.34051	5.34074
0.6	5.33537	5.33830	5.33971	5.34025	5.34060
0.8	5.33477	5.33792	5.33948	5.34009	5.34041
1.0	5.33402	5.33767	5.33933	5.33998	5.34033
2.0	5.33365	5.33721	5.33904	5.33978	5.34017
4.0	5.33340	5.33706	5.33896	5.33972	5.34012
6.0	5.33338	5.33704	5.33895	5.33971	5.34012
8.0	5.33335	5.33703	5.33894	5.33970	5.34012
10.0	5.33334	5.33702	5.33893	5.33970	5.34012

TABLE 2

Numerical Values of $T^* = T/\mu_1\omega$ for $\theta = 0$ ($h_2 = 0$)
for Various Values of h_1

h_1	0.25	0.50	1.0	2.0	4.0
T^*	9.81779	6.79768	5.68035	5.39216	5.34148

calculated from equation (4.2.60) by using the Simpson's rule by dividing the interval [0,1] in 24 equal parts. These numerical values of T^* are given in Tables 1 and 2 and are displayed graphically in Figures 5-13.

It may be noted that $\delta < 1$ (or $\delta > 1$) means that the modulus of rigidity of the upper layer is larger (or smaller) than the modulus of rigidity of the lower layer since $\delta = \mu_2/\mu_1$. Also it may be noted that $\theta < 1$ (or $\theta > 1$) means that the thickness of the upper layer is larger (or smaller) than the thickness of the lower layer since $\theta = h_2/h_1$.

Numerical values of $T^* = T/\mu_1\omega$ have been displayed against $\theta = h_2/h_1$, (θ = 0.2, 0.4, 0.6, 0.8, 1.0, 2.0, 4.0, 6.0, 8.0, 10.0) for h_1 = 0.5, 1.0, 2.0 and $\delta = \mu_2/\mu_1$ = 0.1, 0.5, 1.0, 5.0 in the following figures:

Fig. 5: h_1 = 0.5; δ = 0.1, 0.5, 1.0, 5.0; θ = 0.2, 0.4, 0.6, 0.8, 1.0

Fig. 6: h_1 = 0.5; δ = 0.1, 0.5; θ = 2.0, 4.0, 6.0, 8.0, 10.0

Fig. 7: h_1 = 0.5; δ = 1.0, 5.0; θ = 2.0, 4.0, 6.0, 8.0, 10.0

Fig. 8: h_1 = 1.0; δ = 0.1, 0.5, 1.0, 5.0; θ = 0.2, 0.4, 0.6, 0.8, 1.0

Fig. 9: h_1 = 1.0; δ = 0.1, 0.5; θ = 2.0, 4.0, 6.0, 8.0, 10.0

Fig. 10: h_1 = 1.0; δ = 1.0, 5.0; θ = 2.0, 4.0, 6.0, 8.0, 10.0

Fig. 11: h_1 = 2.0; δ = 0.1, 0.5, 1.0, 5.0; θ = 0.2, 0.4, 0.6, 0.8, 1.0

Fig. 12: h_1 = 2.0; δ = 0.1, 5.0; θ = 2.0, 4.0, 6.0, 8.0, 10.0

Fig. 13: h_1 = 2.0; δ = 1.0, 5.0; θ = 2.0, 4.0, 6.0, 8.0, 10.0

Figures 5, 6 and 7 show T^* against θ for h_1 = 0.5 i.e. when the thickness of the upper layer is half of the radius of the circular portion which is being rotated. In these figures it is clear that as the thickness of the lower layer increases the value of T^* decreases for all values of δ. Although for $\delta = 5$ (Fig. 5) i.e. as the ratio of the modulus of rigidities is increased there is not much change in the value of T^* with increase in $\theta = h_2/h_1$. From Figs. 6 and 7 we see that as θ increases from 2 to 6, the value of T^* decreases sharply but there is a slight decrease when the thickness increases from 6 to 10. When $\delta = 1$ i.e. both layers have same modulus of rigidity, the value of T^* tends to 5.333, which is equal to $T^{\infty}/\mu\omega = 16/3$, where T^{∞} is the torque for a semi-infinite space. So even if we increase the thickness beyond 10, the value remains close to $5.333\dot{3}$.

Figure 8 shows T^* against θ for $h_1 = 1$ i.e. when the thickness of the layer is equal to the radius of the circular portion which is being rotated. In this case we see that as θ increases from .2 to 1 the value of T^* decreases, although there is not much change in the value of T^* when θ changes from 0.8 to 1. As δ increases i.e. the ratio of modulus of rigidities increases from .1 to 5, the value of T^* increases which is obvious since as δ increases the lower layer becomes stiffer relative to the upper layer, so it needs more force to rotate the circular portion. From Figure 9 and 10 we see that there is not much decrease in the value of T^* as θ increases from 6 to 10 especially when $\delta = 1$ and 5. Also when $\delta = 1$, the value of T^* tends to 5.333 (Fig. 10), which is same as shown by Gladwell [15].

When h_1 is equal to 2 i.e. the thickness of the upper layer is twice the radius of the circular portion which is being rotated one sees that

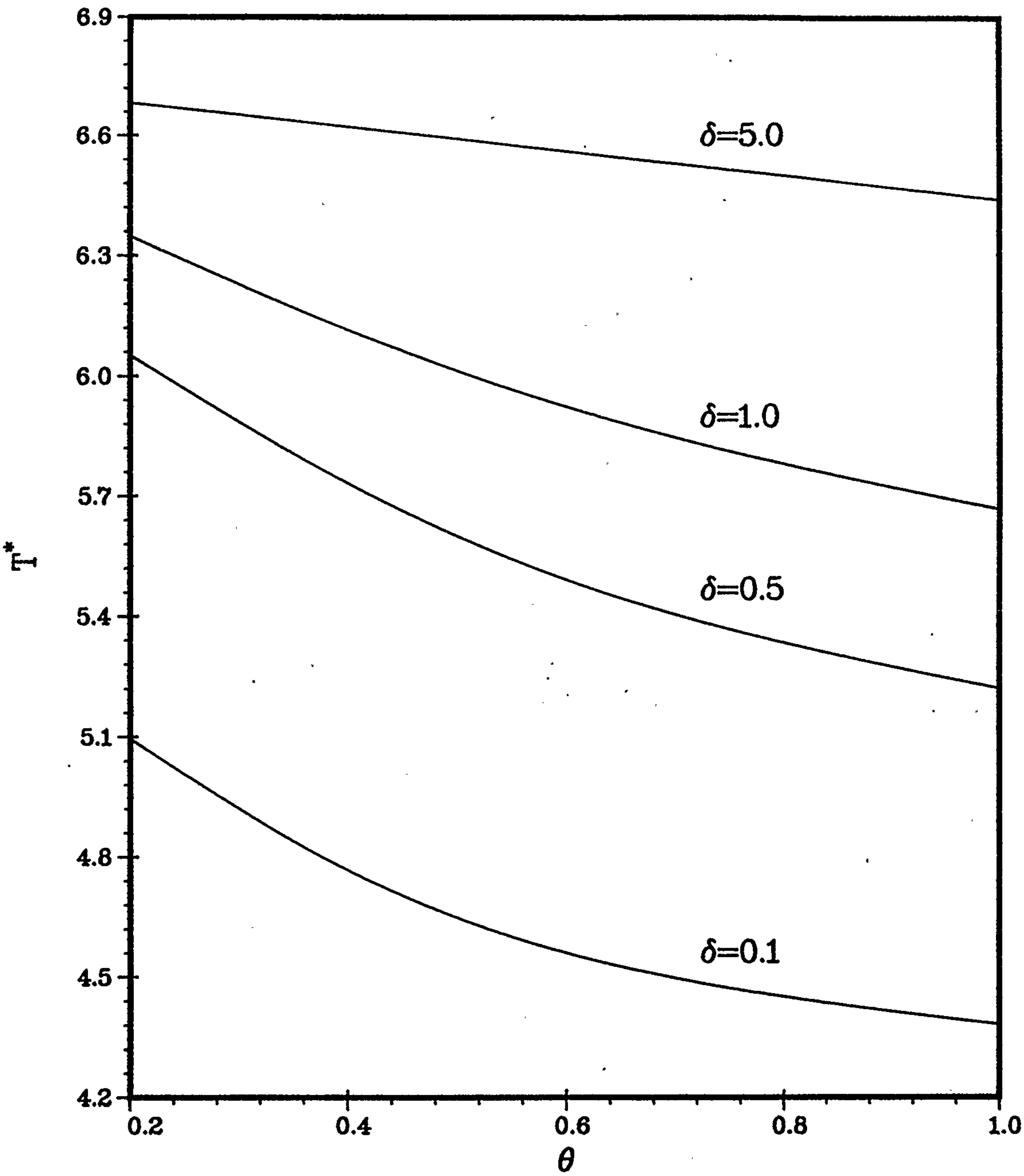

Figure 5:Numerical values of $T^*=T/\mu_1\omega$ against $\theta=h_2/h_1$=.2,.4,.6,.8,1.0 for h_1=0.5 and δ=.1,.5,1,5.

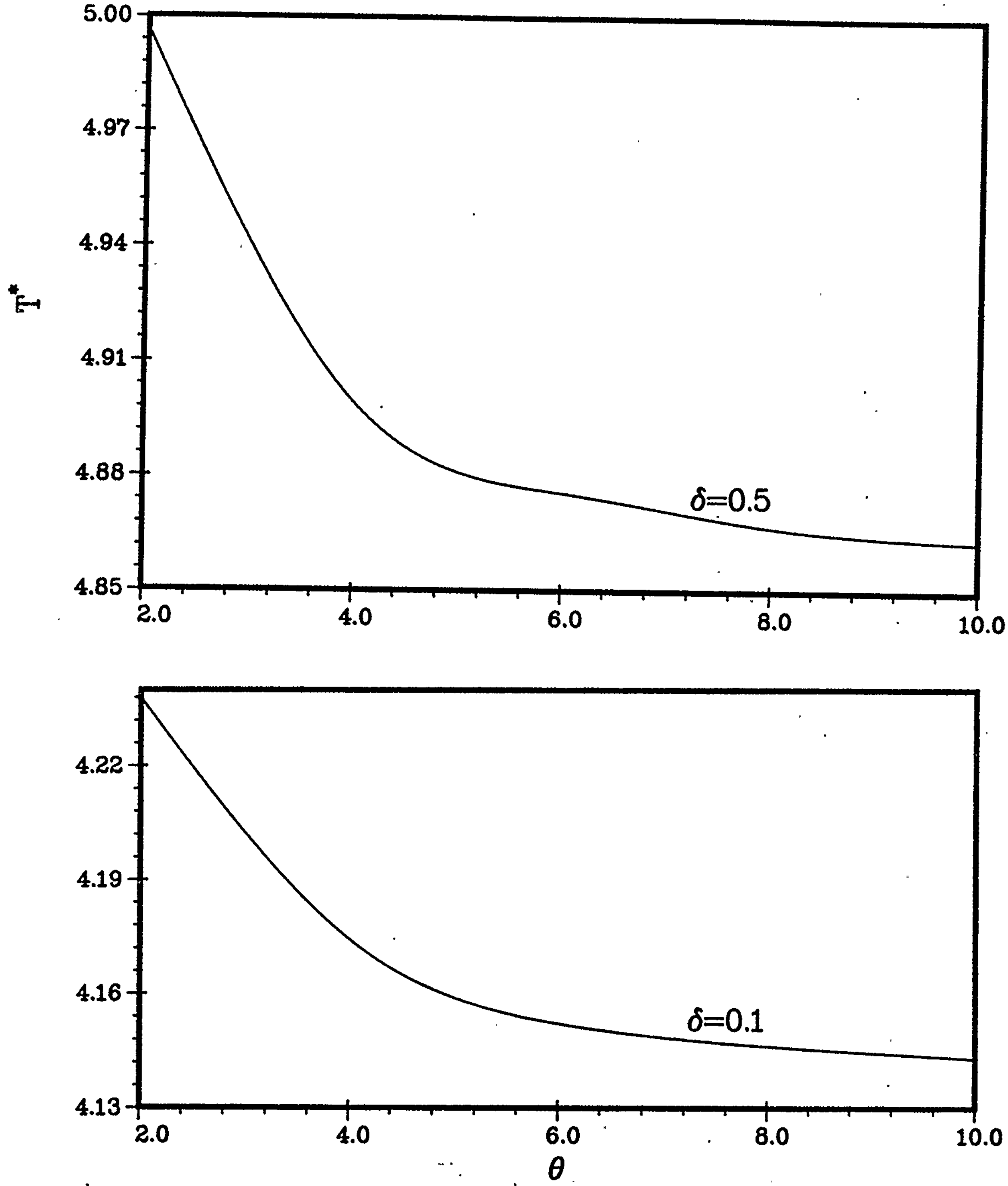

Figure 6:Numerical values of $T^*=T/\mu_1\omega$ against $\theta=h_2/h_1=2,4,6,8,10$ for $h_1=0.5$ and $\delta=\mu_2/\mu_1=0.1,0.5$

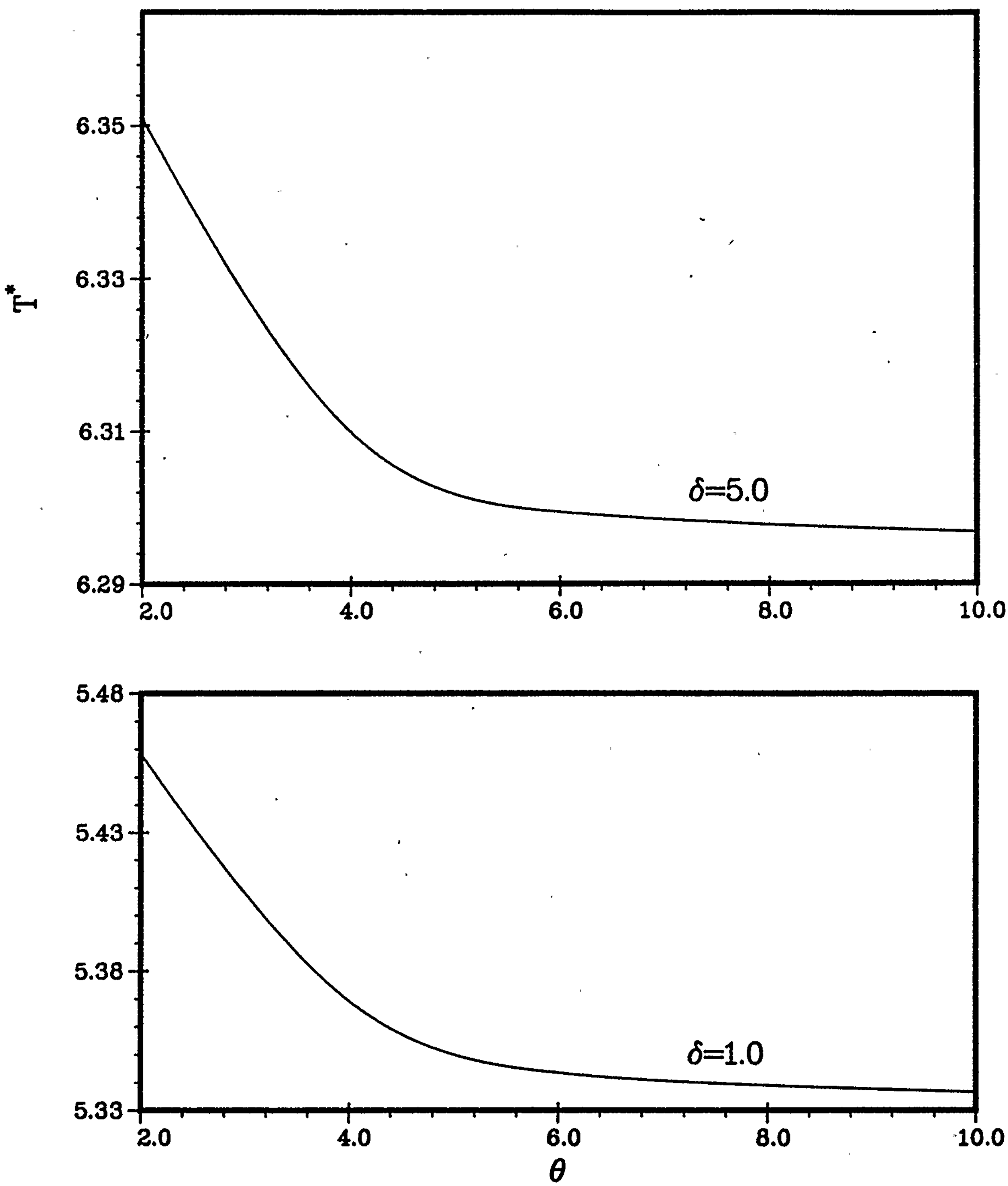

Figure 7:Numerical values of $T^*=T/\mu_1\omega$ against $\theta=h_2/h_1=2,4,6,8,10$ for $h_1=0.5$ and $\delta=\mu_2/\mu_1=1.0,5.0$

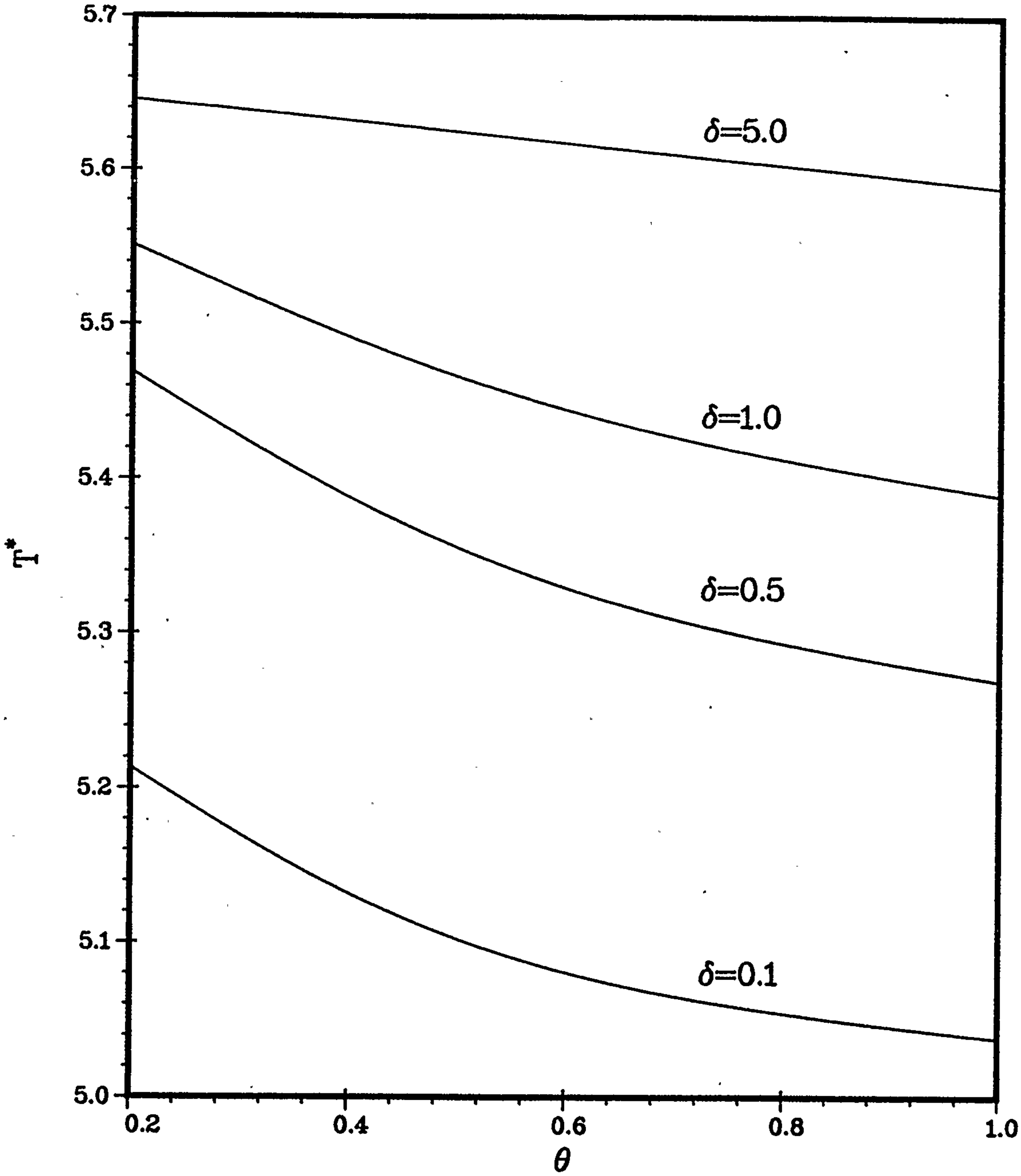

Figure 8:Numerical values of $T^*=T/\mu_1\omega$ against $\theta=h_2/h_1=.2,.4,.6,.8,1.0$ for $h_1=1.0$ and $\delta=.1,.5,1,5$.

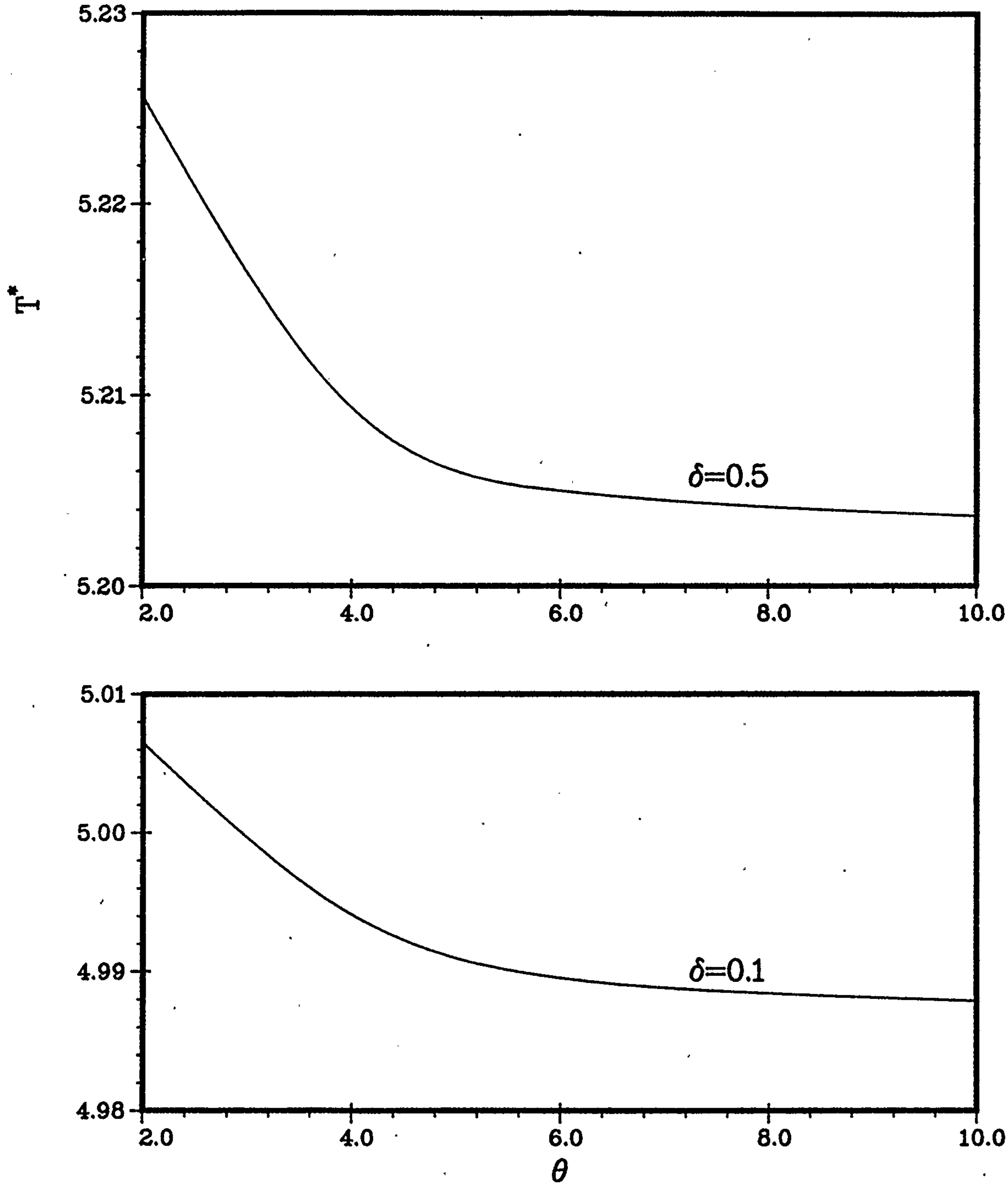

Figure 9:Numerical values of $T^{*}=T/\mu_1\omega$ against $\theta=h_2/h_1=2,4,6,8,10$ for $h_1=1$ and $\delta=\mu_2/\mu_1=0.1,0.5$

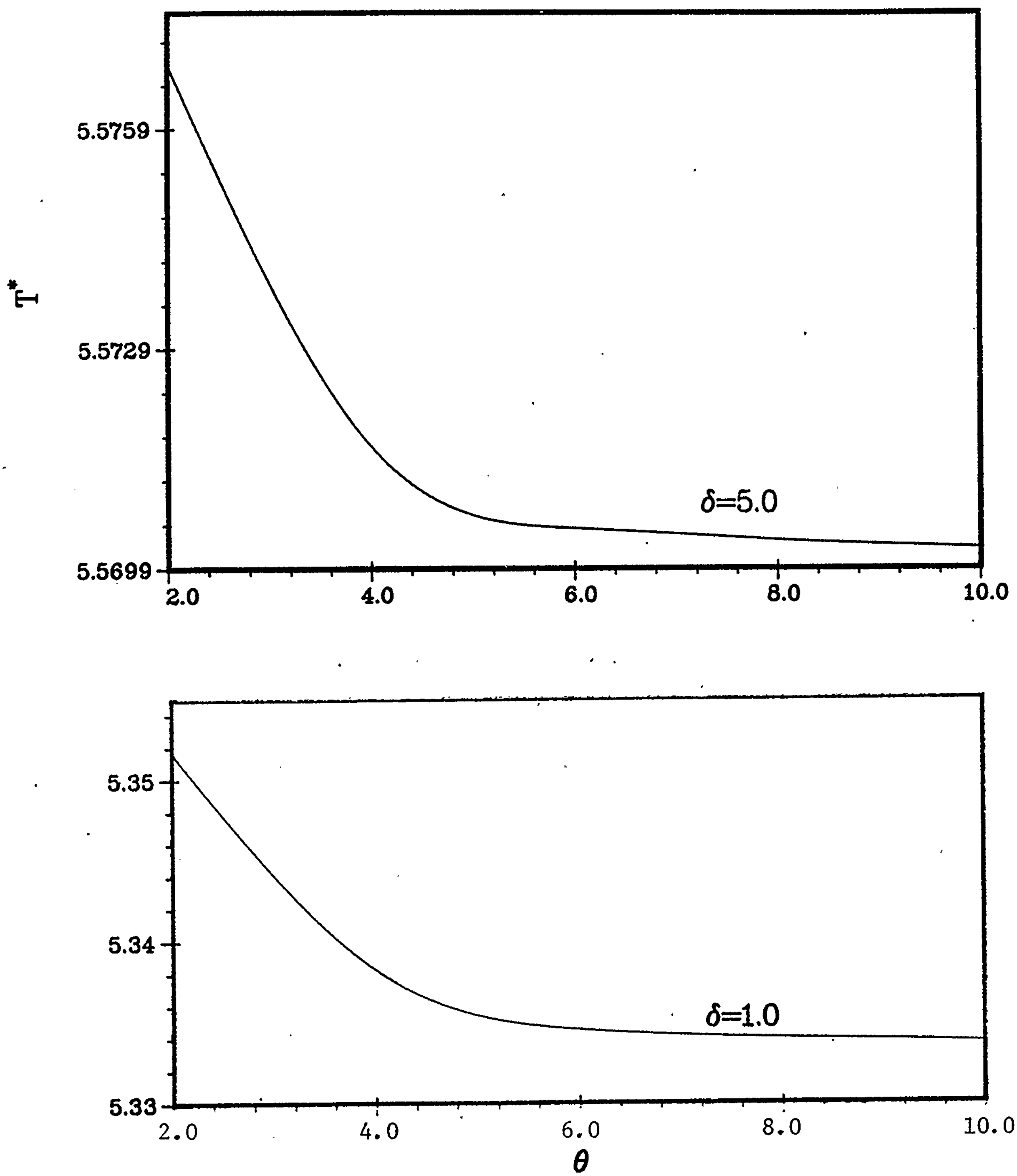

Figure 10:Numerical values of $T^*=T/\mu_1\omega$ against $\theta=h_2/h_1=2,4,6,8,10$ for $h_1=1$ and $\delta=\mu_2/\mu_1=1.0,5.0$

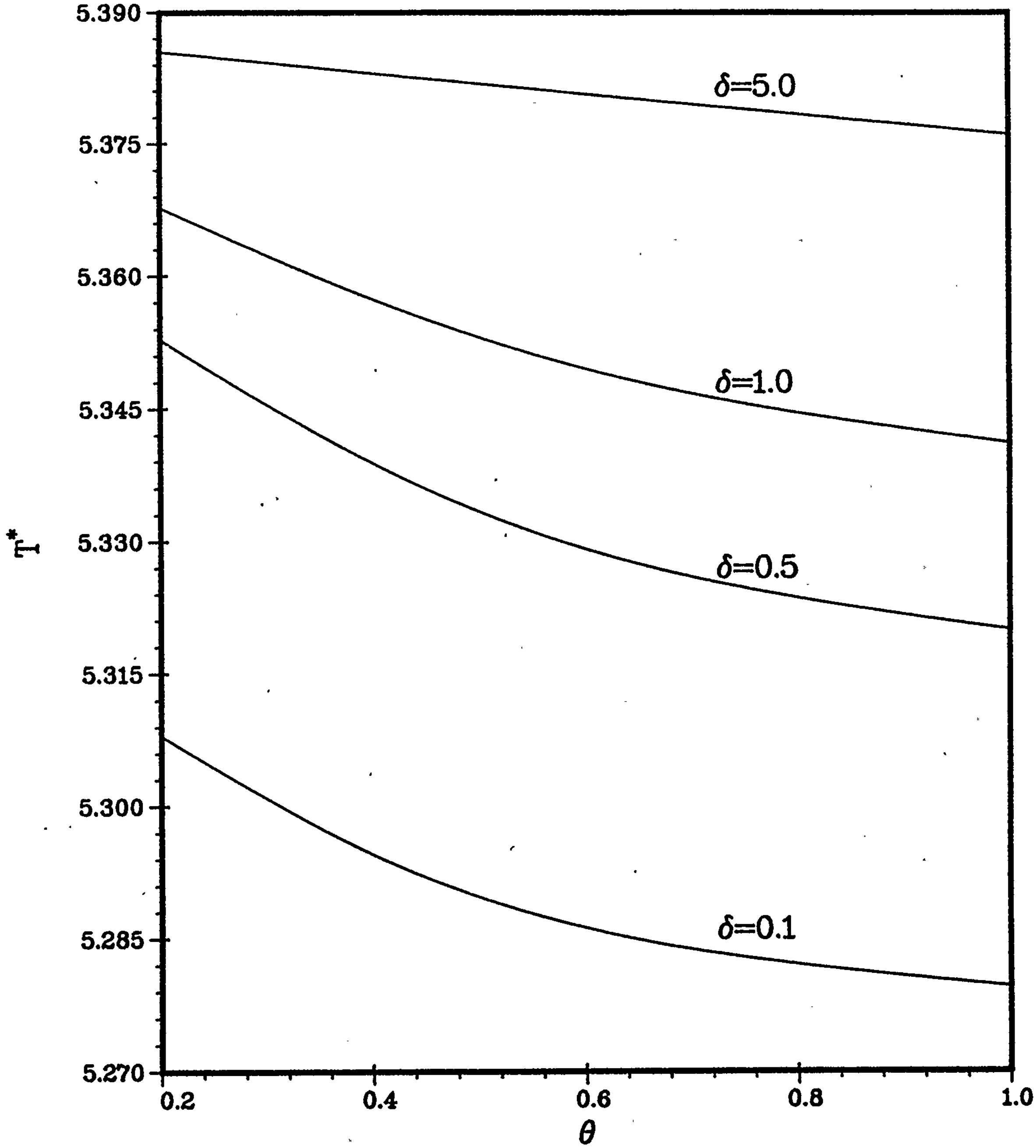

Figure 11:Numerical values of $T^{*}=T/\mu_1\omega$ against $\theta=h_2/h_1=.2,.4,.6,.8,1.0$ for $h_1=2.0$ and $\delta=.1,.5,1,5$.

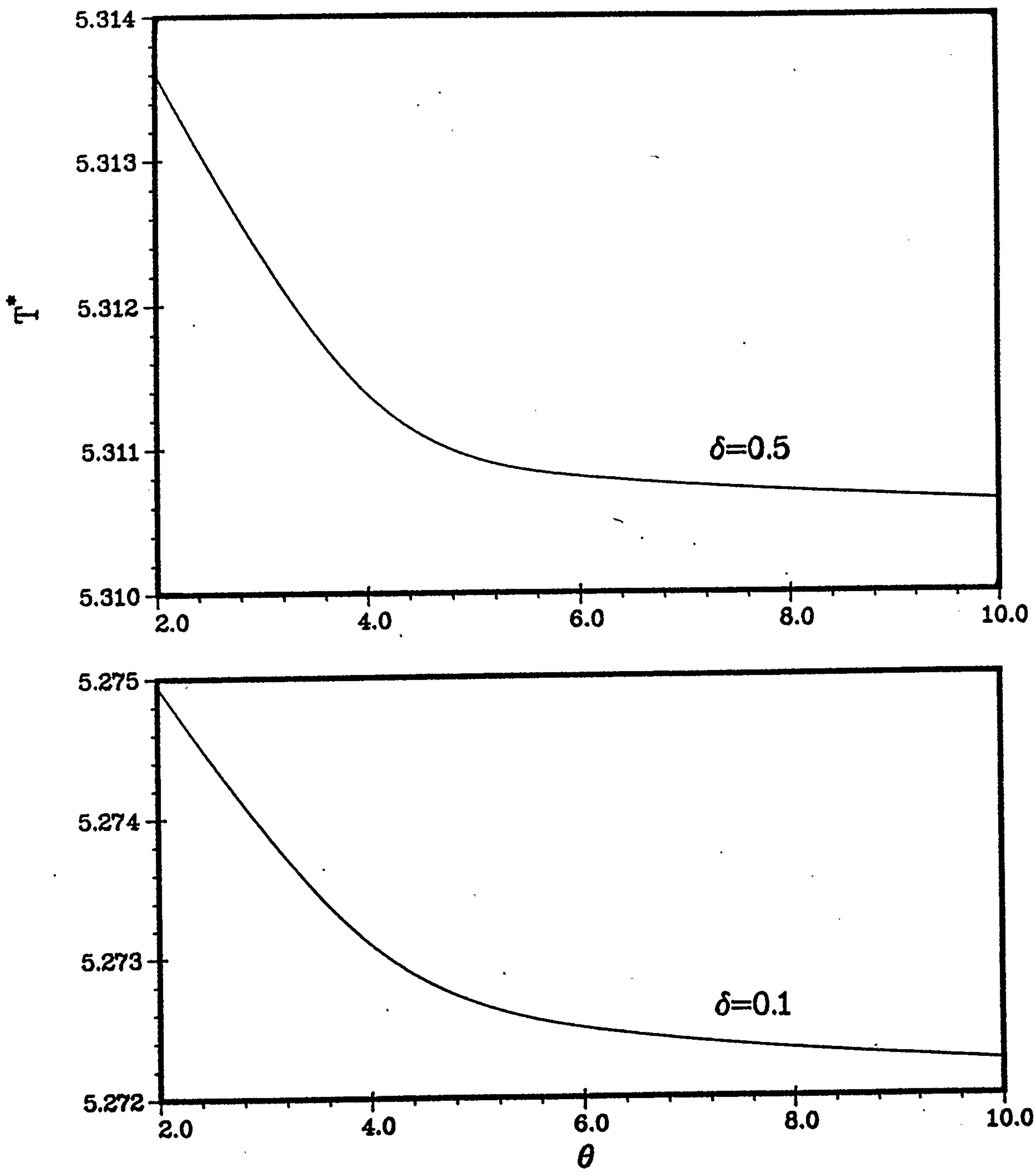

Figure 12:Numerical values of $T^*=T/\mu_1\omega$ against $\theta=h_2/h_1$=2,4,6,8,10 for h_1=2 and $\delta=\mu_2/\mu_1$=0.1,0.5

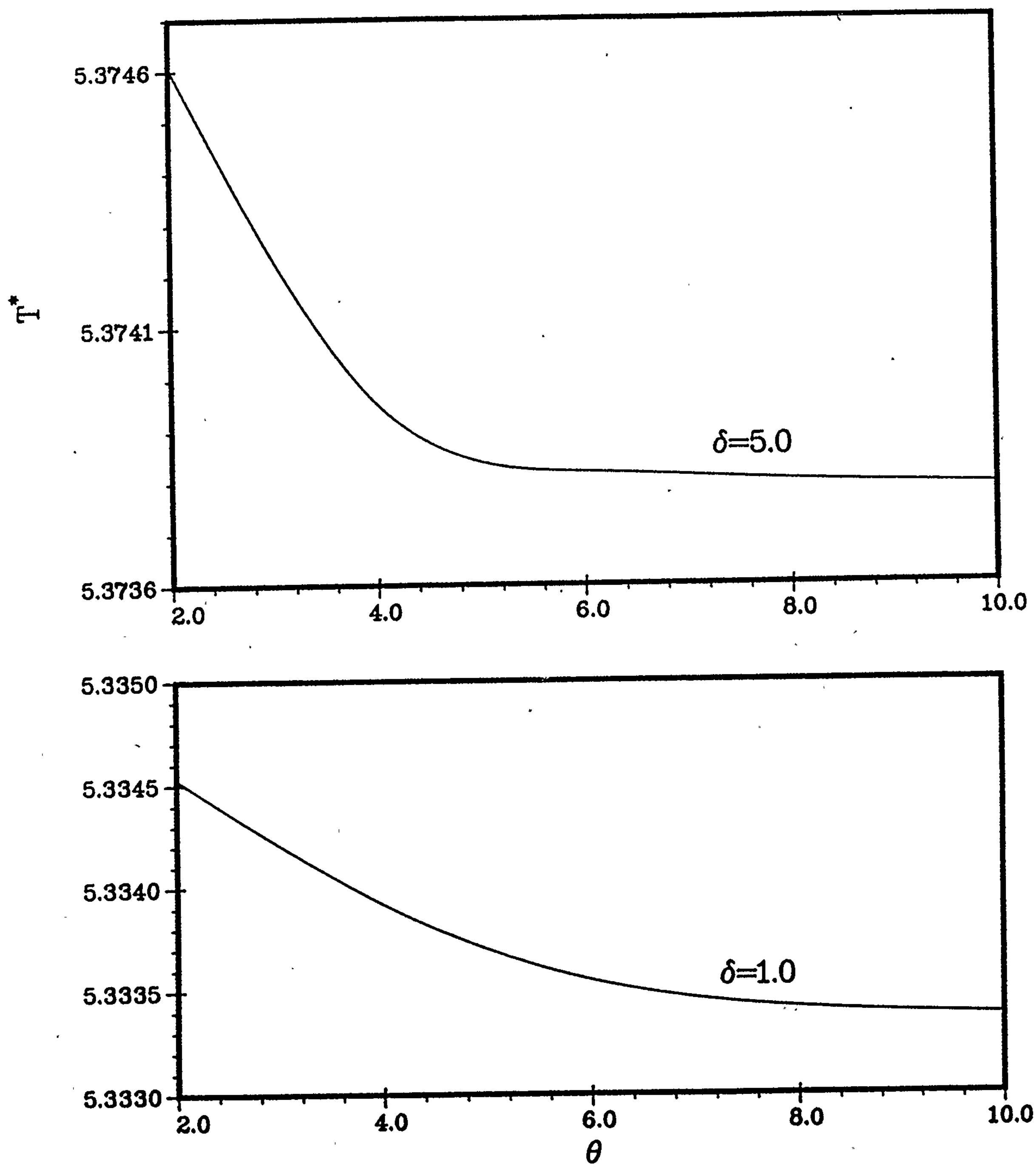

Figure 13:Numerical values of $T^*=T/\mu_1\omega$ against $\theta=h_2/h_1=2,4,6,8,10$ for $h_1=2$ and $\delta=\mu_2/\mu_1=1.0,5.0$

(Fig. 11, 12, 13) there is not much variation in the values of T^* as θ increases from 0.2 to 10. In this case too when δ is equal to one, T^* is approaching to 5.333 as θ increases (Fig. 13). Also when δ is equal to 5 there is not much change in the value of T^* as θ increases from 0.2 to 1 (Fig. 11). In this case we also see that the graphs are almost parallel in all the cases which is understandable since the thickness of the upper layer has been increased compared to the cases when h_1 is equal to 1 and 0.5 and hence there is a lesser effect of the thickness h_2 of the lower layer.

In conclusion we notice that when δ increases (i.e. when μ_2 increases for a fixed μ_1) T^* increases for any fixed value of $\theta = h_2/h_1$, but when θ increases (i.e. when h_2 increases for a fixed h_1) T^* decreases for any fixed value of $\delta = \mu_2/\mu_1$.

§4.4 Particular cases

The results for the following particular problems have been derived by assigning particular values to μ_1, μ_2, h_1 and h_2 in the results of section 2:

CASE I: *Reissner-Sagoci problem for the half-space (Fig. 14)*

To derive the results for this case, we let

$$h_1 \to \infty, \quad \mu_1 = \mu \tag{4.4.1}$$

By taking $h_1 \to \infty$ in equation (4.2.22) we get

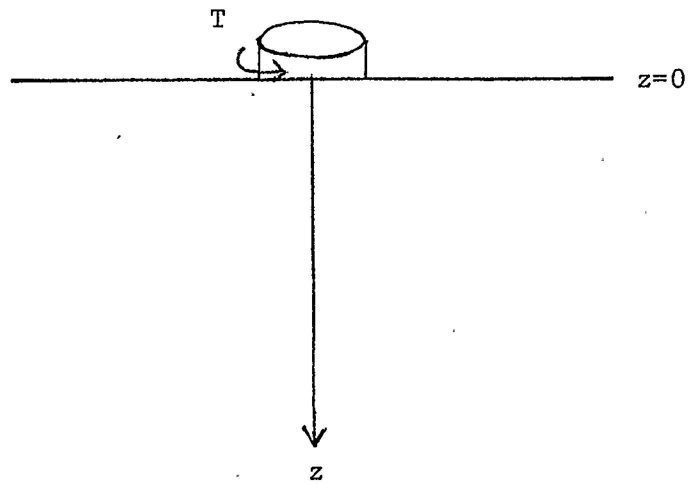

Figure 14: Semi-Infinite Space $z \geq 0$.

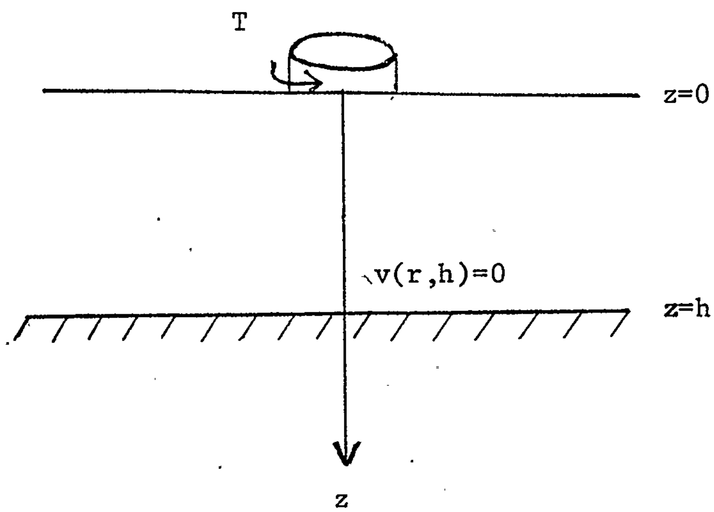

Figure 15: A Layer $0 \leq z \leq h$ Bonded to a Rigid Foundation.

$$H(\xi) = 0 \tag{4.4.2}$$

and hence from equation (4.2.51) and (4.2.60) we get

$$\Phi(t) = -\frac{4\omega}{\pi}\, t \,, \tag{4.4.3}$$

$$T = -4\mu\pi \int_0^1 t\, \Phi(t)dt$$

$$= \frac{16}{3}\,\omega\mu \,,$$

which agrees with the already known results (e.g. equation (3.4.30) gives the same value of T as above by taking $r_0 = 1$).

<u>CASE II</u>: *Reissner-Sagoci problem for a single layer of thickness* h *with its lower face rigidly fixed* (*Fig.* 15)

To obtain the results for this case, we let

$$h_2 \to 0,\ h_1 = h,\ \mu_1 = \mu \,. \tag{4.4.5}$$

By taking $h_2 \to 0$ and $h_1 = h$ in equation (4.2.22) we find

$$H(\xi) = \frac{2}{1 + e^{2\xi h}} \tag{4.4.6}$$

and hence from equation (4.2.45) we obtain

$$K(u,t) = -\frac{4}{\pi}\int_0^\infty \frac{1}{1+e^{2\xi h}}\sin(\xi u)\sin(tu)d\xi$$

$$= \frac{2}{\pi} \int_0^\infty \frac{1}{1+e^{2\xi h}} [\cos(u+t) - \cos(t-u)\xi] d\xi \ . \qquad (4.4.7)$$

Now $\phi(t)$ and the torque T are given by equations (4.2.51) and (4.2.60).

The above results are in full agreement with the results obtained by Gladwell [15] by noting the change in notation.

<u>CASE III</u>: *Reissner-Sagoci problem for a layer of thickness* h *with its lower face stress-free* (*Fig.* 16)

For this case we let

$$\mu_2 \to 0,\ h_1 = h,\ \mu_1 = \mu \ . \qquad (4.4.8)$$

For this case when $\mu_2 = 0$, we have $\delta = 0$ and hence $\bar{\mu} = -1$ and from (4.2.22) we obtain

$$H(\xi) = -\frac{2\ e^{-2\xi h}}{1 - e^{-2\xi h}} \ . \qquad (4.4.9)$$

Then from (4.2.44), we obtain

$$K(u,t) = \frac{4}{\pi} \int_0^\infty \frac{e^{-2\xi h}}{1-e^{-2\xi h}} \sin(\xi u)\sin(tu) d\xi$$

$$= \frac{2}{\pi} \int_0^\infty [\coth(\xi h) - 1][\cos(t-u)\xi - \cos(t+u)\xi] d\xi \qquad (4.4.10)$$

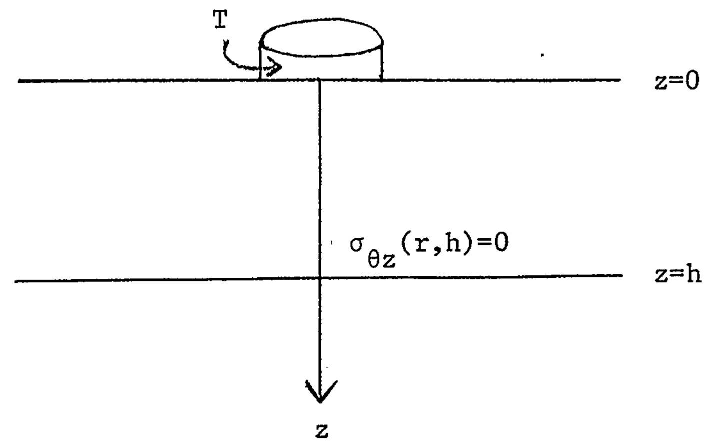

Figure 16: A Layer $0 \leq z \leq h$ With Lower Face Stress-Free.

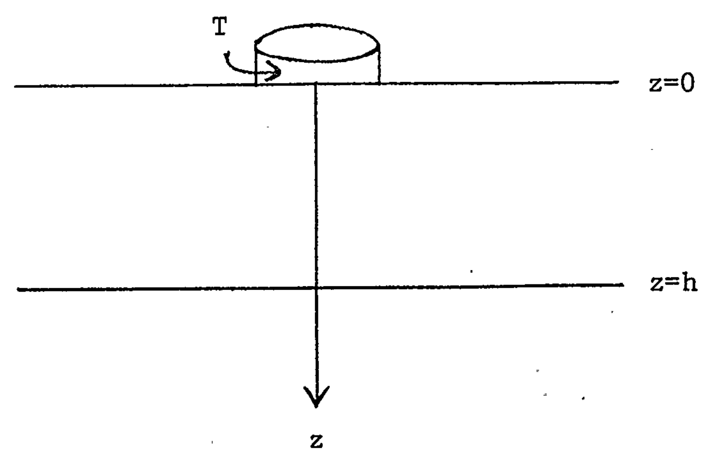

Figure 17: A Layer $0 \leq z \leq h$ Bonded to a Semi-Infinite Space $z \geq h$.

which is in agreement with that of Gladwell [15] by noting the change of notation.

CASE IV: *Reissner-Sagoci problem for a layer of thickness* h *bonded to an elastic half-space (Fig.* 17)

The results for this case may be derived by letting

$$h_2 \to \infty, \; h_1 = h. \tag{4.4.11}$$

For this case, we find from equation (4.2.22) and (4.2.45) that

$$H(\xi) = \frac{2}{1 + \bar{\mu}\, e^{2\xi h}}, \tag{4.4.12}$$

$$K(u,t) = \frac{2}{\pi}\int_0^\infty [1+\bar{\mu}e^{2\xi h}]^{-1}[\cos(t+u)\xi - \cos(t-u)\xi]d\xi \tag{4.4.13}$$

and $\Phi(t)$ and the torque T are given by equations (4.2.51) and (4.2.60) respectively. The above results are in agreement with those obtained by Jabali [16] by noting the change in notation.

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