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# An Index Theory for Periodic Orbits in Hamiltonian Systems

by

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## ABSTRACT

For a Hamiltonian H of classical type,

$$H(x,y) \doteq \tfrac{1}{2}|y|^2 + W(x) \quad ,$$

we consider the problem of existence for a periodic solution of the associated second order Hamiltonian equations

$$\ddot{x} = -DW(x)$$

of specified energy h. The variational technique we use to obtain the existence of such an orbit involves minimizing the energy integral of the Jacobi metric relative to the class of arcs joining points on the boundary of the Hill's region N,

$$N = \{x \mid W(x) \leq h\} \quad .$$

Under assumptions on $H^{-1}(h)$ differing from those recently put forward in the literature, we obtain the existence of a brake orbit solution within N, that is, an orbit oscillating between two seperated points on the boundary of N but otherwise lying in the interior of N.

Due to the degenerate nature of the integrand the classical theory of the second variation is not immediately applicable. We extend this theory to cover the case at hand, and show that the second order critical point theory for brake orbits within N is at one with the classical second order theory for closed geodesics on a Riemannian manifold.

We generalize a theorem of Ambrose to obtain an index theory for brake orbits within N, and apply this to the study of brake orbits of minimal Jacobi arclength. We show that barring a degenerate parabolic case such orbits are hyperbolic, that is, all of their Lyapounov multipliers are off the unit circle. This confirms a conjecture of G.D. Birkhoff on hyperbolicity of periodic solutions of the second order Hamiltonian equations with more than two degrees of freedom.

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# INTRODUCTION

The purpose of this introduction is to provide background material for, and a summary of, results included in this thesis.

During the past decade there has been a surge of interest in the global problem of determining periodic solutions of Hamilton's equations

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}(x,y), \quad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}(x,y), \text{ for } i = 1,\dots,m,$$

on a given energy surface $H^{-1}(h) \subseteq \mathbb{R}^{2m}$. Indeed there are many results obtained recently for this problem, under various topological and geometric restrictions placed on $H^{-1}(h)$; see for example [R2], [R3], [BR], [C], [CF], [GZ] and [W]. These authors are concerned solely with the existence of periodic orbits, and use one form or another of variational analysis to obtain the periodic orbits as critical points of appropriate functionals, in the tradition of G.D. Birkhoff [B], and H. Seifert [S].

On the other hand, researchers in dynamical systems are interested in the existence of hyperbolic periodic orbits (i.e. closed orbits whose Lyapounov multipliers are off the unit circle) in Hamiltonian systems as a tool in understanding hyperbolic structures on more complicated invariant sets (see for example [Mos]). In this regard, there exist many classical mechanical systems of two degrees of freedom

(1) $$\ddot{x} = -DW(x), \; x \in \mathbb{R}^2,$$

where $DW(x)$ denote the gradient of the potential function $W$ at $x$, which exhibit some form of global hyperbolic behavior on specific energy surfaces (see [CPR] for a survey of some of these examples). For several such systems, methods have been developed which demonstrate the existence of hyperbolic periodic orbits on a prescribed energy surface $H^{-1}(h)$ (see [CPR2], [RPC]). However these methods seem to be mostly limited by the necessity of detailed knowledge of the Hamiltonian vector field on $H^{-1}(h)$.

In this thesis we adapt a variational approach, due mainly to Birkhoff [B], and Seifert [S], to obtain an existence theorem for hyperbolic periodic orbits on a prescribed energy surface of arbitrary dimension. We will confine our attention to the classical Hamiltonian

$$H(x,y) = \tfrac{1}{2}|y|^2 + W(x), \tag{2}$$

and the associated second order Hamiltonian equations

$$\ddot{x} = -DW(x), \; x \in \mathbb{R}^m. \tag{3}$$

In such a setting, and for prescribed energy $h$, the manifold $N$ with boundary

$$N = \{x \in \mathbb{R}^m \mid W(x) \le h\}$$

plays an important role, and indeed under the hypothesis (loosely stated) that $\partial N$ is disconnected and nonempty, while $N$ itself is connected but possibly not compact, we assert the existence of a hyperbolic periodic solution of (3) (barring a degenerate parabolic case) which joins separated points on $\partial N$ but otherwise runs through the interior of $N$ (such a solution if referred to by Weinstein [W] as a brake orbit); see Theorem 1.1, and Theorem 3.27.

In some sense, our result is foreshadowed by a result of Birkhoff [B], which states that in two degrees of freedom, closed orbits of (3) which arise as the minimum of the Jacobi arclength functional (see (1.10)) are hyperbolic, again barring the degenerate case of Lyapounov multipliers ±1. Birkhoff also conjectured that his result was true in dimensions higher than 2 (([B] p. 130), however he considered only those periodic solutions of (3) which did not intersect $\partial N$ where the Jacobi metric becomes degenerate. Our result confirms Birkhoff's conjecture for brake orbits of minimum type.

When considering the orbital stability of a periodic orbit arising as a critical point for a specific functional, one is led naturally to study the second order neighbourhood of the critical point in question. For example, the linearized equations of (3) along a periodic orbit $\pi(t)$ are

$$\ddot{\xi} = -D^2W(\pi(t))\xi \ , \tag{4}$$

where $D^2W(x)$ is the Hessian matrix of the function $W$. It is readily verified that (4) is also the Jacobi differential equation associated with the functional

$$\int \{\tfrac{1}{2}|\dot{x}|^2 - W(x)\}dt.$$

The fruitful interplay between these two ideas has been developed for closed geodesics on Riemannian manifolds (see [K]), and for periodic orbits arising from variational principles associated with convex Hamiltonians (see [E]). In this thesis we adopt the same approach, and develop a Morse index theory for arbitrary brake orbits of (3) whether or not these orbits originate in a variational principle.

In Chapter 1, we begin with a brief survey of results for the existence of periodic orbits of prescribed energy obtained by other authors using variational methods. We then formulate a variational principle of minimum type, using the functional

$$J(c) = \tfrac{1}{2}\int_0^1 2(h-W(c))\,|c'|^2 ds,$$

whose admissable curves $c(s)$ must satisfy

$$W(c) \leq h, \text{ and } c(0), c(1) \in \partial N.$$

These constraints on the admissable curves prevent the application of standard existence results from the calculus of variations. This problem is surmounted by considering a sequence of variational subproblems, governed by the restriction

$$W(c) \leq h-\delta_n, \quad \text{as } \delta_n \downarrow 0.$$

With this additional constraint, standard arguments are used to obtain a solution of the subproblem. Once the translation between extremal curves for $J(c)$ and the solutions of (3) has been specified, the solutions of the variational subproblems are shown to converge to a brake orbit solution of (3). This gives us the main result of Chapter 1 (Theorem 1.1).

In the second chapter, we begin by presenting the theory of the second variation on critical arcs $c$ of the functional $J$. Although our final theorem (Theorem 3.27) on the existence of hyperbolic periodic orbits is proven using the results of the second chapter, it does not require the full power of the methods developed. Perhaps the larger contribution will be the discovery of the fact that brake orbits within potential wells may be analyzed Morse theoretically in a fashion similar to that of closed geodesics on a Riemannian manifold.

From an arbitrary brake orbit solution of (3), by reparameterizing according to the formula provided in Chapter 1, we obtain a critical point of the functional $J$ relative to the class of arcs $c$, $c:[0,1] \to N$, joining separated points on $\partial N$. The second variation $J^{**}$ of $J$ along $c$ is shown to be a quadratic form on an appropriate Hilbert space of vector fields along $c$, and the index of this quadratic form is shown to be finite. The difficulty to overcome here, and the difference between the classical theorems for the theory of the second variation and our results, is the fact that the strong Legendre condition does not hold along the entire length of the critical arc. Once this difficulty has been surmounted, the theorem relating the index of $J^{**}$ to focal points of the endpoint submanifolds follows in broad outline the same result in the classical case (see Ambrose [A]).

One significant point upon which our treatment of the second variation differs from the usual treatment, is that we use a Hamiltonian rather than Lagrangian format. This pays off in several ways. Firstly, due to the failure of the strong Legendre condition at $s = 0,1$, of the critical arc $c$, the derivatives of the variation vector fields

$P(s)$, in general, do not satisfy

$$P'(s) \in L^2[0,1]$$

which normally is a requisite condition for an index theory. However by using the linearization of the Legendre transform, $L_*(c,c')$, it may be shown (Lemma 2.9) that if

$$(P,R)(s) = L_*(c,c')(P,P')(s),$$

then $R(s) \in L^2[0,1]$. Consequently, we may develop an index theory for $J_{**}$ defined on the fields $(P,R)(s)$ belonging to an appropriate Hilbert space. This leads to an analysis of the linearized Hamiltonian equations, which we dub the co-Jacobi equations, rather than the usual Jacobi equations.

Secondly, we obtain an explicit relation (Theorem 2.39) between the solutions of the linearized equations (4) along $\pi(t)$, which determine the linearized Poincaré mapping, and the solutions of the co-Jacobi equations. This becomes useful when we study the connection between the eigenvalues of $J_{**}$, in terms of Jacobi fields, and the stability properties of $\pi(t)$, in terms of the linearized Poincaré mapping. Thirdly, we may make explicit use of the symplectic structure inherited from the Hamiltonian system $(X_H,w)$, where

$$w = \sum_{i=1}^{m} dy_i \wedge dx_i.$$

It turns out that this structure enables us to give a rather nice treatment of the structure of Jacobi fields.

As a result of the failure of the strong Legendre condition, the co-Jacobi equations are singular at the endpoints. To overcome this it is necessary that we restrict ourselves to variation vector fields

(Definition 2.4) $P(s)$ so that, if

$$(P,R)(s) = L_*(c,c')(P,P')(s),$$

then $(P,R)(s)$ lies in the tangent space of the energy surface $H^{-1}(h)$. This restriction however is natural with regard to the study of negative eigenvalues of the form $J_{**}$ (see Proposition 2.14). The Jacobi fields $P(s)$ are then reparameterized, using the time variable $t$ borrowed from the brake orbit $\pi(t)$, and the new fields $U(t)$ are shown to be $C^2$ on the entire real line (Proposition 2.41). The energy constraint imposed on Jacobi fields forces them to remain orthogonal to the tangent direction along the brake orbit $\pi(t)$, and thus we are led to study the properties of the reparameterized orthogonal Jacobi fields $U(t)$ (Definition 2.33).

Having restricted ourselves to orthogonal Jacobi fields, we find that many of the classical constructions for Jacobi fields along closed geodesics have an immediate counterpart for reparameterized Jacobi fields along a brake orbit $\pi(t)$. For example, orthogonal wave front sets of trajectories nearby $\pi(t)$ may be analyzed using orthogonal Jacobi fields. Moreover, Fermi coordinates (see Remark 2.42) along $\pi(t)$ are developed. When the co-Jacobi equations are expressed in these coordinates, we obtain an $m$-1 dimensional system of second order equations which contain all the geometric information relevant to the study of stability properties of $\pi(t)$. This reduction of dimension may be of prime importance. For example, if $\pi(t)$ is a minimum distance line associated with a solution of the variational problem posed in Chapter 1, then the reduced co-Jacobi equations are shown to be disconjugate on $(-\infty,\infty)$ (Lemma 3.22). This is in contrast to the $m$ dimensional system of second order variational equations along $\pi(t)$ (4) , which always have an oscillatory solution, $\dot{\pi}(t)$. The vector space $J_{\pi}^{\perp}$ of reparameterized orthogonal Jacobi fields is a symplectic space (Proposition 2.47), and we conclude Chapter 2 with a look at the Lagrangian subspaces of $J_{\pi}^{\perp}$ which are closely associated with the moving wavefront sets of nearby trajectories to $\pi(t)$.

In Chapter 3, we continue to examine the geometric structures associated with Lagrangian subspaces of $J_{\pi}^{\perp}$; of particular importance to the index theorem are the Lagrangian subspaces associated with the wavefront sets originating on the boundary of $N$, near the endpoints $\pi(0)$, $\pi(T)$. The properties of the second fundamental form at $\pi(t)$, of the moving wavefront sets, and specifically the notion of relative convexity between two such wavefronts, are developed early in the chapter. We present several results (Propositions 3.9, 3.12, 3.13) relating the relative convexity of two wavefront sets and the evolution of focal points of these wavefront sets. Although we use these results in the main argument, they are given also with an eye to further applications.

An extension of Ambrose's index theorem [A], for an arbitrary brake orbit $\pi(t)$, follows as a corollary of Ambrose's result once a decomposition lemma for $J^{**}$ (Lemma 3.17) is proven. This theorem (Theorem 3.16) gives the index of $\pi(t)$ in terms of focal points along $\pi(t)$ between $\pi(0)$ and $\pi(T)$, and the relative convexity of the wavefront sets associated with $\partial N$.

The application of the index theorem to minimum distance lines leads to the result that the reduced co-Jacobi equations along $\pi(t)$ are disconjugate on $(-\infty,\infty)$ (Lemma 3.22). There is a large amount of literature on linear Hamiltonian systems which are disconjugate on an interval I, see for example [Co], [Ha] (p.384), and [Ha2]. In particular, in [Ha2], Hartman shows that in the case of second order equations arising as the Jacobi differential equations of the geodesic flow on a compact manifold with negative sectional curvature, disconjugacy on $(-\infty,\infty)$ implies the existence of a certain instability, namely that the geodesic flow is hyperbolic on $T_1^*M$. This suggests that, more generally, a similar type of instability must hold for an arbitrary periodic orbit when the linearized equations along the periodic orbit are disconjugate on $(-\infty,\infty)$. We are able to show that a pair of Floquet multipliers on the unit circle, under the presence of disconjugate co-Jacobi equations, implies a certain restriction on the eigenvectors associated with such multipliers (Proposition 3.25).

Under the assumption of non-degeneracy, no zero eigenvalues of $J^{**}$, we obtain a general form for the monodromy matrix $S$ of the reduced co-Jacobi equations along an arbitrary brake orbit $\pi(t)$. It is the particular form of $S$, deriving from the fact that $S$ is a symplectic matrix, which allows the application of Proposition 3.25 to verify Birkhoff's conjecture for brake orbits $\pi(t)$ of minimum type (Theorem 3.27). This representation of the monodromy matrix appears to be new.

# CHAPTER 1

## A VARIATIONAL PRINCIPLE FOR PERIODIC ORBITS

The purpose of this chapter is to demonstrate the existence, under certain assumptions to be detailed later, of free oscillations in Hamiltonian systems. Hamilton's equations are

$$(1.1)\qquad \frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}(x,y),\ \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}(x,y),\ \text{for } i = 1,\ldots,m,$$

and the question of the existence of periodic solutions to such a system has received renewed interest in the past decade due in part to the rapid progress of several authors on two closely related problems. These two problems are respectively, finding a periodic solution $(x(t),y(t))$ of (1.1) of specified period $T$, and finding a periodic orbit of specified energy $h$. The latter problem relies on the fact that for an autonomous Hamiltonian system, the energy $H$ is an integral of the motion, and therefore every nonempty energy surface $H^{-1}(h)$ is invariant under the flow of (1.1). Both of the problems mentioned above are global in nature and are not, for example, confined to the study of small oscillations around an equilibrium solution. By way of an introduction to the problem considered in this chapter, we will survey selected results obtained by different authors (see also the survey of Rabinowitz [R1] for a more complete bibliography).

An early global result by Seifert [S] shows the existence of a periodic orbit on a prescribed energy surface $H^{-1}(h)$ in case

$$(1.2) \qquad H(x,y) = \frac{1}{2} \sum_{i,j=1}^{m} a_{ij}(x) y_i y_j + W(x),$$

where $a_{ij}$ and $W$ are $C^2$ functions on $\mathbb{R}^m$, the symmetric matrix $[a_{ij}(x)]$ is positive definite, and the manifold $N$ with boundary,

$$N = \{x \mid W(x) \leq h\}$$

is diffeomorphic to the closed unit ball in $\mathbb{R}^m$.

Seifert's result is geometric in nature. He used the Jacobi metric (see 1.10) and adapted the Birkhoff curve shortening process (see [B]) in this metric so as to apply to curves which run between boundary points of $N$, but which otherwise lie in the interior. Invoking Birkhoff's minimax argument he produced a geodesic chord (in the Jacobi metric) which joins distinct points of the boundary $\partial N$. After a suitable reparameterization, this curve $x(t)$ together with

$$y(t) = [a_{ij}(x(t))]^{-1}\dot{x}(t)$$

forms a solution of (1.1) for the Hamiltonian (1.2), and this solution has rest points at the boundary

$$y(0) = y(T) = 0, \text{ where } x(0), x(T) \in \partial N.$$

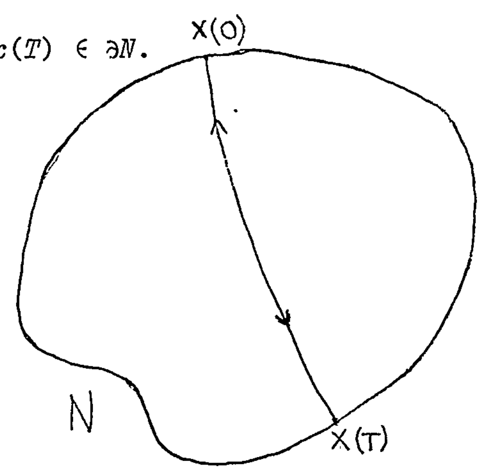


Fig 1

By conservation of energy, $x(t)$ may be continued as an even function around $t = 0,T$, and $y(t)$ may be continued as an odd function around $t = 0,T$ to obtain a periodic solution of (1.1) on $H^{-1}(h)$ (see Fig. 1). Such a to and fro motion within the potential well $N$ is referred to as a *brake orbit* by Weinstein [W] who generalized the theorem of Seifert in the following way. Weinstein showed that Seifert's construction still holds if instead of (1.2) we assume that $H$ is a $C^2$ function on $\mathbb{R}^{2m}$ which is even and strictly convex in $y$.

As an application of his result, Weinstein proved in [W] that if $H \varepsilon C^2(\mathbb{R}^{2m}, \mathbb{R})$, and $H^{-1}(h)$ bounds a compact convex region in $\mathbb{R}^{2m}$, then (1.1) admits a periodic orbit on $H^{-1}(h)$.

P. Rabinowitz [R2], [R3] using the calculus of variations and approximation techniques, and later Benci and Rabinowitz [BR] using direct variational methods, also obtained results on both the prescribed energy and the prescribed period problems. In particular Rabinowitz proved [R2] that if $H \varepsilon C^1(\mathbb{R}^{2m}, \mathbb{R})$, and $H^{-1}(h)$ is a manifold which bounds a compact star shaped region in $\mathbb{R}^{2m}$ then (1.1) admits a periodic solution on $H^{-1}(h)$.

F. Clarke [C] gave a much simpler proof of Weinstein's result when $H^{-1}(h)$ bounds a convex compact region, using techniques from convex analysis and the calculus of variations. Clarke and Ekėlund [CE] also proved the existence of a periodic solution of (1.1) having prescribed minimal period, when $H$ is convex with a global minimum at the origin of $\mathbb{R}^{2m}$, and $H$ satisfies certain subquadratic growth restrictions at 0 and $\infty$. As a corollary they obtain a prescribed energy result under the same assumptions on $H$. The approach taken by these latter authors uses a dual action principle introduced by F. Clarke, and is restricted to Hamiltonians which are convex in both arguments. However they are able to remove all smoothness assumptions on $H$, and consider Hamiltonian inclusions rather than (1.1).

Finally, Gluck and Ziller [GZ] have extended the approach of Seifert-Weinstein in the following result. Let $M$ be a smooth manifold and $H:T^*M \to \mathbb{R}$ a smooth function on the cotangent bundle of $M$. If $H$ is convex and even on each fibre $(T^*M)_q$, and $H^{-1}(h)$ is compact, nonempty and regular, then there is a periodic orbit of Hamilton's equations on $H^{-1}(h)$. Moreover, if $H$ is of the form (1.2), and the energy $h$ is chosen so that $\partial N \neq \emptyset$, then the periodic orbit arises from a brake orbit within $N$.

All of the results discussed so far have dealt exclusively with the existence problem. Only very recently has an attempt been made to furnish information on the second order neighbourhood of the periodic orbits arising as critical points of the various functionals. I. Ekelund [E] has produced a Morse theory for periodic orbits in convex Hamiltonian systems, and has used his results to obtain information on the orbital stability of those periodic orbits corresponding to critical points of the functional associated with Clarke's dual action principle. We will return to this point in the second chapter.

In this thesis, we will assume throughout that the Hamiltonian is of classical type

$$H = \text{kinetic energy} + \text{potential energy} \tag{1.3}$$
$$= \frac{1}{2}\,|y|^2 + W(x)\ ,$$

where $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{R}^m$, and $W(x)$ is a $C^3$ function on $\mathbb{R}^m$; however our results carry over easily to the case where $H$ is of the form (1.2). In case (1.3) obtains, the Hamiltonian equations (1.1) reduce to the familiar equations from classical mechanics of motion within a conservative force field

$$\ddot{x} = -DW(x)\,,\ \text{where}\ \ddot{x} = \frac{d^2}{dt^2}\,x(t)\,,\ \text{and} \tag{1.4}$$

$DW(x)$ denotes the gradient of $W$ at $x$.

This chapter is concerned with an existence theorem for periodic orbits of (1.4) in the case of prescribed energy. The approach and indeed the conclusions of our result most closely resemble that of Seifert's, but are based on a variational rather than geometric point of view. The manifold $N$ with boundary

$$N = \{x \in \mathbb{R}^m \mid W(x) \le h\} \tag{1.5}$$

plays an important role, and indeed under the assumption (loosely stated) that $\partial N$ is disconnected and nonempty while $N$ itself is connected but possibly not compact (see Fig. 2), we assert (Theorem 1.1) the existence of a periodic orbit of (1.4) on $H^{-1}(h)$ arising from a brake orbit within $N$. Notice that this assumption on $\partial N$

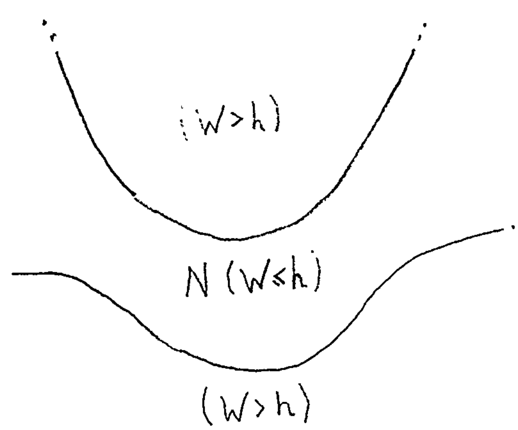



Fig 2

prevents the application of Seifert's result, and the potential $W(x)$ may fail to be convex so that the results of [CE] do not apply. All of the results surveyed above depend upon $H^{-1}(h)$ being compact, and are not applicable when $N$ is not compact.

The geometric assumptions which we place on $H^{-1}(h)$ will now be stated. Let $\Phi^t$ denote the flow in $\mathbb{R}^{2m}$ of the Hamiltonian vector field $X_H$ associated with (1.4),

$$X_H(x,y) = (y, -DW(x)), \tag{1.6}$$

and let $\varphi^t$ denote the projection of $\Phi^t$ into position or $x$-space. Let $B_r$ denote the open ball in $\mathbb{R}^m$ with radius $r > 0$, centered at the origin, and let $B^c$ denote the set theoretic complement of a subset $B \subset \mathbb{R}^m$.

(W1) The manifold $N$ with boundary

$$N = \{x \mid W(x) \le h\}$$

is connected and has nonempty interior $N^0$.

(W2) There exists $\delta^* > 0$ with the following properties:

(a) for $0 \le \delta \le \delta^*$, the equipotential surface

$$W_{h-\delta} = \{x \mid W(x) = h-\delta\}$$

is the union of two connected components (see Fig. 3)

$$W_{h-\delta} = W^0_{h-\delta} \cup W^1_{h-\delta}.$$

(b) for $0 \le \delta \le \delta^*$, if $x^0_n \in W^0_{h-\delta}$, $x^1_n \in W^1_{h-\delta}$, and either (or both) $|x^0_n| \to +\infty$, or $|x^1_n| \to +\infty$, then

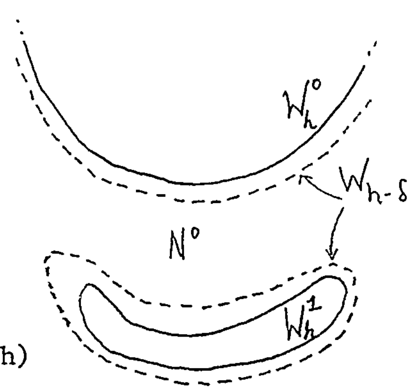



Fig 3

$|x_n^0 - x_n^1| \to +\infty$, as $n \to \infty$. This condition holds vacuously if $W_h$ is compact.

(W3) Let $R_\delta = \{x \mid h-\delta \le W(x) \le h\}$ for $\delta > 0$, and let $A_r = B_r^c$ for $r > 0$. If either $W_h^0$ or $W_h^1$ is not compact, there is a number $q > 0$ such that; if $r_0 > q$ then we can find $r_1 \ge r_0$, and $0 < \delta < \delta^*$ such that

$x \in A_{r_1} \cap R_\delta$, and $(x,y) \in H^{-1}(h)$ imply that $\varphi^t(x,y) \in A_{r_0}$ for all $t$ where defined (see Fig. 4).

(W4) $G = \inf\limits_{x \in W_h} |DW(x)| > 0$

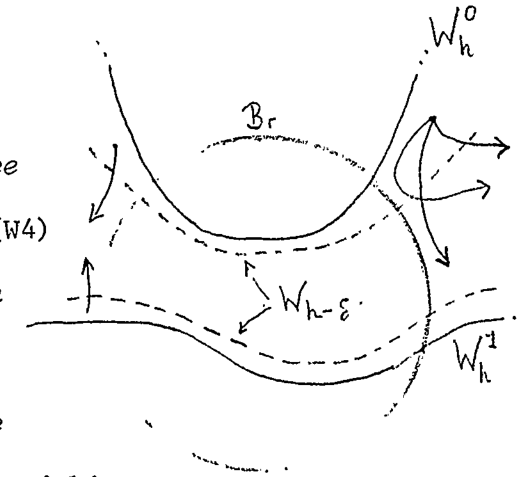


Fig 4

Theorem 1.1. *On any energy surface $H^{-1}(h)$ such that conditions* (W1)-(W4) *hold, the second order Hamiltonian equations* (1.4) *admit a periodic solution. Moreover, this periodic solution arises from a brake orbit within $N$.*

Remark 1.2. The assumption (W3) holds, for example, if there is an $r > 0$ with, $\langle -DW(x), x\rangle > 0$ for $x \in B_r^c \cap N$. The verification of this is carried out in the examples below. The condition (W4) guarantees that $W_h$ is a regular hypersurface in $\mathbb{R}^m$.

The brake orbit whose existence is asserted in Theorem 1.1 will correspond in a way to be made precise below, with a solution of the following variational problem. Consider an arc $c$,

$$
(1.7)\qquad \begin{cases} c:[0,1] \to N, \\ c \text{ is absolutely continuous on } [0,1], \text{ and} \\ c'(s) = \dfrac{d}{ds}\, c(s) \in L^2[0,1]. \end{cases}
$$

Denote the set of such arcs by $H^1$,

$$
(1.8)\qquad H^1 = \{\text{arcs } c \text{ which satisfy } (1.7)\},
$$

and consider the functional $J$ with domain $H^1$

$$
(1.9)\qquad J(c) = \frac{1}{2}\int_0^1 [2(h-W\circ c)]|c'|^2\, ds.
$$

We consider the variational problem

$$
(J)\quad \min J(c) \;:\; c \in H^1,\ c(0) \in W_h^0, \text{ and } c(1) \in W_h^1.
$$

We pause before proving that $(J)$ has a solution under the restrictions (W1)-(W4), to indicate the genesis of the functional $J$.

On the manifold $N$ we introduce the Jacobi metric $(d\tau)^2$ in terms of the Euclidean metric $(d\rho)^2$

$$
(1.10)\qquad (d\tau)^2 = 2(h-W(x))(d\rho)^2.
$$

The Jacobi arclength of an absolutely continuous curve

$$
c:[0,1] \to N
$$

is,

$$
L(c) = \int_0^1 \sqrt{2(h-W\circ c)}\ |c'|\, ds.
$$

Our functional $J(c)$ is the "energy integral" (see [Mi] p. 70) associated with the Jacobi metric. The Jacobi metric was introduced by Jacobi (naturally) and has been used as a geometrical tool in the analysis of classical mechanical systems by Birkhoff [B], Seifert [S], Weinstein [W], Gluck and Ziller [GZ] to name a few. Thus our functional $J$ is not new however the manner in which it is used, especially with regards the second variation in the second chapter and the applications of the second variation in the third chapter, do appear to be new. The main drawback to the use of the Jacobi metric is that it is singular on $\partial N$; thus Birkhoff [B] was led to consider only those periodic solutions of (1.4) which did not intersect $\partial N$. Similarly, Seifert [S] introduced the methodology used in this chapter (see the sequence of subproblems $(J_n)$ below) to produce a solution of $(J)$.

The following examples illustrate the verification of hypotheses (W1)-(W4).

Example 1: The potential function on $\mathbb{R}^2$, $x = (x_1, x_2) \in \mathbb{R}^2$,

$$W(x) = x_2^{2n} - x_1^{2n} \quad \text{with} \quad n \geq 1$$

satisfies the above hypotheses when $h > 0$. Indeed take $\delta^* = \frac{h}{2}$. Then

$$DW(x) = 2n(-x_1^{2n-1}, x_2^{2n-1}) \ ;$$

the only critical point of $W$ is $x = (0,0)$ and $|DW(x)| \to +\infty$ as $|x| \to +\infty$, implying (W4) is satisfied. To verify (W3), consider the following inequality valid for $(x_1, x_2) \in A_{r_0} \cap N$ with $r_0 > 0$,

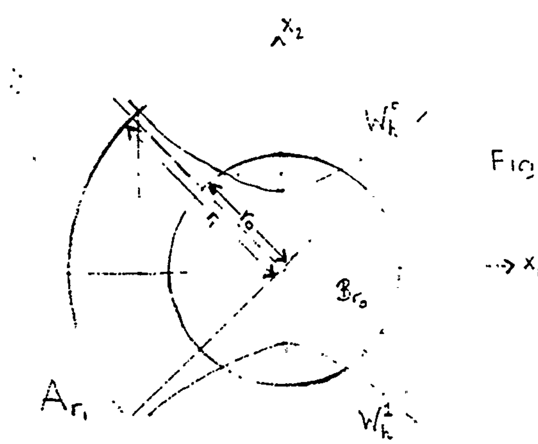


Fig 5

$$<-DW(x), \left(\frac{x_1}{|x_1|}, 0\right)> = 2n \cdot \frac{x_1^{2n}}{|x_1|} > 0 \ .$$

Let $r_1 > r_0$ be determined as in the diagram:

$$r_1 = \sqrt{r_0^2 + x_2^2} + \varepsilon \qquad \text{where } \varepsilon > 0 \text{ and}$$

$$W(r_0, x_2) = h \ .$$

We will find $\delta > 0$ so that $x = (x_1,x_2) \in R_\delta \cap A_{r_1}$ and $(x,y) \in H^{-1}(h)$ imply that $|x\ (t)| > r_0$ thereby showing that $\varphi^t(x,y) \in A_{r_0}$ for all $t$ where defined. Assuming that, $\mathrm{sgn}(x_1(t))$ is constant on $(0,t)$,

$$\frac{x_1}{|x_1|}(t)y_1(t) = \mathrm{sgn}(x_1)y_1 + \int_0^t <-DW(x(t')),\left(\frac{x_1}{|x_1|}(t'),0\right)>dt',$$

and provided that $(x,y) \in H^{-1}(h)$, $x \in A_{r_1} \cap R_\delta$

$$(1.11) \qquad \frac{x_1}{|x_1|}(t)y_1(t) \geq -\sqrt{2\delta} + \int_0^t <-DW\ ,\left(\frac{x_1}{|x_1|}(t'),0\right)>dt' .$$

Let $0 < t_0 = \inf\left\{\begin{array}{l}\text{time required for } \varphi^t(x,y) \text{ to reach } \{|x_1| = r_0\} \\ \qquad \text{for } (x,y) \in H^{-1}(h) \text{ and } x \in A_{r_1}\end{array}\right\}$

and

$$0 < c = \inf\left\{ <-DW(x),\left(\frac{x_1}{|x_1|},0\right)> \text{ for } x = (x_1x_2) \in A_{r_0}\right\} .$$

Then $\frac{x_1}{|x_1|}(t_0)y_1(t_0) \geq -\sqrt{2\delta} + t_0c$ from (1.11). Choose $\delta > 0$ so that $(x,y) \in H^{-1}(h)$, $x \in A_{r_1} \cap R_\delta$ imply that

$$\frac{x_1}{|x_1|}(t_0)y_1(t_0) \geq 0 \quad \text{and} \quad |x_1(t_0)| \geq r_0 .$$

Since $\left.\frac{d}{dt}\right|_{t_0} \frac{x_1}{|x_1|}(t)y_1(t) \geq c > 0$ it follows that on some nonempty interval $[t_0,t_0+a]$; $\frac{x_1}{|x_1|}(t)y_1(t) \geq 0$ and $|x_1(t)| \geq r_0$ .

Let $a_0 = \sup\left\{\alpha : \frac{x_1}{|x_1|}(t)y_1(t) \geq 0 \text{ and } |x_1(t)| \geq r_0 \text{ on } [t_0,t_0+a]\right\}$.

Then if $a_0$ is finite, $\left.\frac{d}{dt}\right|_{t_0+a_0} \frac{x_1}{|x_1|}(t)y_1(t) \geq c > 0$ and $0 < \left.\frac{d}{dt}\right|_{t_0+a_0} \frac{1}{2}\, x_1^2(t) = x_1(t_0)y_1(t_0)$ so either $X_H(\Phi^{t_0+a_0}(x,y))$ is not defined or $a_0 = +\infty$. Therefore (W3) holds for any $q > 0$.

<u>Example 2</u>: We refer the reader to [CPR] (p.119) for a survey of results on the following potential: let $x = (x_1, x_2) \in \mathbb{R}^2$, and

$$W(x) = \frac{1}{2}(x_1^2 + x_2^2) - x_1 x_2^2 .$$

Then $W(x)$ satisfies hypotheses (W1)-(W4) for $h > \frac{1}{8}$. Let $\delta^* = \frac{1}{2}[h - \frac{1}{8}]$,

$$DW(x) = (x_1 - x_2^2, x_2 - 2x_1 x_2) ;$$

the only critical points of $W$ occur at

$$(0,0) , \quad P\pm = \frac{1}{2}(1, \pm\sqrt{2})$$

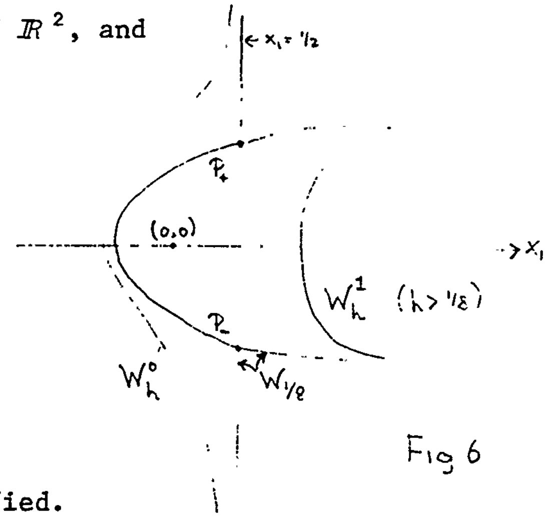



Fig 6

and $|DW(x)| \to +\infty$ as $|x| \to \infty$ so (W4) is satisfied.

To verify (W3) we need only modify the argument given in Example 1. Indeed, it is possible to choose $r_0 > 0$ so that $(x_1, x_2) \in A_{r_0} \cap N$ implies that

$$\begin{aligned} \langle -DW(x), \frac{x}{|x|} \rangle &= \frac{x_2^2 x_1 - x_1^2 + 2x_1 x_2^2 - x_2^2}{[x_1^2 + x_2^2]^{\frac{1}{2}}} \\ &= \frac{2[\frac{1}{2} x_1 x_2^2 - W(x)]}{[x_1^2 + x_2^2]^{\frac{1}{2}}} \\ &\geq \frac{2[\frac{1}{2} x_1 x_2^2 - h]}{[x_1^2 + x_2^2]^{\frac{1}{2}}} > 0 . \end{aligned}$$

Let $0 < t_0 = \inf\Big\{$time required for $\varphi^t(x,y)$ to reach $A_{r_0}$ for $(x,y) \in H^{-1}(h)$, $x \in A_{2r_0}\Big\}$, and

$$\theta = \inf\{ \langle -DW(x), \frac{x}{|x|} \rangle : x \in A_{r_0} \}.$$

Then given $(x,y) \in H^{-1}(h)$, $x \in A_{2r_0} \cap R_\delta$, we have

$$\langle \frac{x}{|x|}(t_0), y(t_0) \rangle = \langle \frac{x}{|x|}, y \rangle + \int_0^{t_0} \frac{|y(t')|^2}{|x(t')|} - \frac{\langle \frac{x}{|x|}(t'), y(t') \rangle^2}{|x(t')|} +$$

$$+ \int_0^{t_0} <-DW, \frac{x}{|x|}(t')>dt'$$

$$\geq - \sqrt{2\delta} + \int_0^{t_0} <-DW, \frac{x}{|x|}(t')>dt' \geq \sqrt{2\delta} + c_{t_0} .$$

Therefore we can find $\delta > 0$ so that $(x,y) \in H^{-1}(h)$, $x \in A_{2r_0} \cap R_\delta$ implies that

$$<\frac{x}{|x|}(t_0), y(t_0)> \geq 0 \quad \text{and} \quad |x(t_1)| \geq r_0 .$$

The remainder of the verification is analogous to Example 1 and will be omitted.

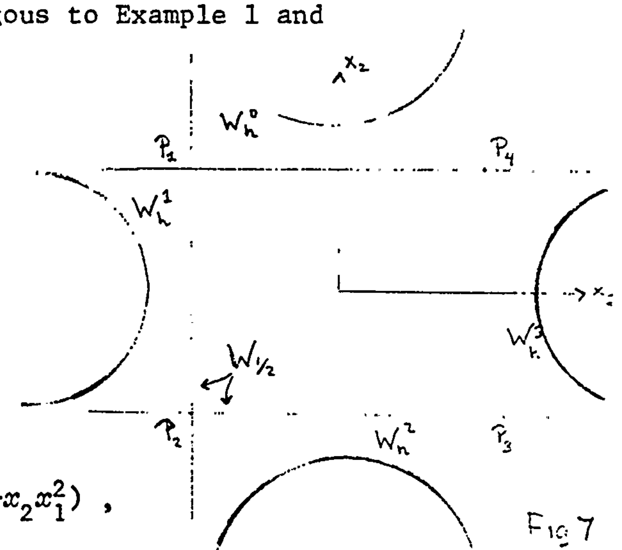


Fig 7

Example 3: For $x \in \mathbb{R}^2$, the potential

$$W(x) = \frac{1}{2}(x_1^2+x_2^2) - \frac{1}{2}x_1^2x_2^2$$

has been studied in [CPR] (see p. 120). $W$ satisfies all hypotheses except W2(a) with $\delta^* = \frac{1}{2}[h - \frac{1}{2}]$. $DW(x) = (x_1-x_1x_2^2, x_2-x_2x_1^2)$, and the only critical points of $W$ are $(0,0)$ and $p_i = (\pm1,\pm1)$ for $i = 1,\ldots,4$. (W4) is thereby satisfied since $|DW(x)| \to +\infty$ as $|x| \to +\infty$. (W3) holds due to the inequality

$$<-DW(x), \frac{x}{|x|}> = [x_1^2x_2^2-x_1^2+x_2^2x_1^2-x_2^2][x_1^2+x_2^2]^{-\frac{1}{2}}$$

$$= \frac{2[\frac{1}{2}x_1^2x_2^2-W(x)]}{[x_1^2+x_2^2]^{\frac{1}{2}}} \geq \frac{2[\frac{1}{2}x_1^2x_2^2-h]}{[x_1^2+x_2^2]^{\frac{1}{2}}}$$

$$> 0$$

provided that $r_0$ is sufficiently large and $(x_1,x_2) \in A_{r_0} \cap N$. The verification is identical to that of Example 2.

This example has some interesting features. In spite of the fact (W2(a)) does not hold for this potential, the existence of a smooth minimum distance line joining adjacent branches of $W_h$, $h > \frac{1}{2}$, $W_h^0$ and $W_h^1$ in Fig. 8 for example, may be demonstrated by modifying the results in this chapter. On the other hand, it follows from results in Chapter 3 that, if we take opposite pairs of branches, $W_h^0$ and $W_h^2$ for example, the corresponding variational problem $(J)$ will in general, have a non-smooth solution. In fact it is possible to show, using results obtained in [CPSR], for

$$\frac{1}{2} < h < h_1, \; h_1 \approx 1.15$$

$(J)$ has no smooth minimum distance line.

In the remainder of this chapter, we will demonstrate the existence of a solution to $(J)$, and show how such a solution corresponds to a brake orbit within $N$ to obtain Theorem 1.1.

For elements $c \in H^1$ (see (1.8)) we will at various times use the norms, $\|c\|_1$, $\|c\|_2$, $\|c\|_\infty$, $\|c\|_{H^1}$, where $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$ are the $L^1[0,1]$, $L^2[0,1]$, $L^\infty[0,1]$ norms respectively, and $\|c\|_{H^1} = |c(0)| + \|c'\|_2$, $c' = \frac{d}{ds} c$.

<u>Lemma 1.3</u>. *Let* $0 \le s_0 < s_1 \le 1$, *and* $g(s) = s_0 + s(s_1 - s_0)$. *If* $c \in H^1$, *then* $\tilde{c} = c \circ g : [0,1] \to N$ *also belongs to* $H^1$, *and* $J(\tilde{c}) \le (s_1 - s_0) J(c)$.

*Proof.*
$$\begin{aligned} J(\tilde{c}) &= \int_0^1 (h-Wo\tilde{c})\,|\tilde{c}'|^2\,ds \\ &= \int_0^1 (h-Woc(g))\,|c'(g)|^2|g'|^2\,ds \\ &= \int_{s_0}^{s_1} (h-Woc)\,|c'|^2|g'|\,ds \quad \text{(by change of variables)} \\ &\leq (s_1-s_0)J(c). \end{aligned}$$
□

The idea mentioned above of producing a solution of $(J)$ as the limiting solution of a subproblem will now be developed. For $\delta_n \downarrow 0$, and $0 < \delta_n \leq \delta^*$ (see (W2)), we consider the following family of variational subproblems

$$(J_n) \quad \min J(c) \,:\, c \in H^1,\ c(0) \in W^0_{h-\delta_n},\ c(1) \in W^1_{h-\delta_n},$$
$$\text{and} \qquad W(c(s)) \leq h-\delta_n.$$

We will denote by $F_n$ the subset of $H^1$ whose elements $c$ satisfy the above restrictions.

<u>Proposition 1.4</u>. *The variational problem* $(J_n)$ *has a smooth solution* $c^n$. *Furthermore there is a positive constant* $E_n$ *so that*

$$c^{n\prime}(0) = \frac{E_n}{\sqrt{2\delta_n}}\,\frac{[-DW(c^n(0))]}{|DW(c^n(0))|}\,,\quad c^{n\prime}(1) = \frac{E_n}{\sqrt{2\delta_n}}\,\frac{DW(c^n(1))}{|DW(c^n(1))|}\,.$$

Before proving Proposition 1.4 we will need the following lemma.

<u>Lemma 1.5</u>. *Let* $\{c_k\}_{k=1}^{\infty}$ *be a sequence of elements of* $H^1$ *which belong to the subset* $F_n$ *and such that* $J(c_k)$ *is uniformly bounded in* $k$. *Then* $\|c_k\|_\infty$ *is uniformly bounded in* $k$.

*Proof.* From the Cauchy-Schwartz inequality, for $c \in H^1$

$$\|c'\|_1 \le \|c'\|_2 .$$

Furthermore

$$|c(s)| \le |c(0)| + \|c'\|_1 \le |c(0)| + \|c'\|_2 = \|c\|_{H^1} < \infty . \tag{1.12}$$

Assume that $\|c_k\|_\infty \to +\infty$ as $k \to \infty$. We will show that in this case $\|c_k'\|_2 \to +\infty$ and that this yields a contradiction. Indeed, from (1.12), if $|c_k(0)|$ is bounded and $\|c_k\|_\infty \to +\infty$ then $\|c_k'\|_2 \to +\infty$. On the other hand if $|c_k(0)|$ is unbounded, since $|c_k(1)-c_k(0)| \le \|c_k'\|_1 \le \|c_k'\|_2$, condition W2(b) guarantees that $\|c_k'\|_2 \to +\infty$.

The following inequality holds since $c_k \in F_n$ implies that $0 < 2\delta_n \le 2(h-W\circ c_k)$:

$$\delta_n \|c_k'\|_2^2 = \int_0^1 \delta_n \cdot |c_k'|^2 ds \le \int_0^1 (h-W\circ c_k)\,|c_k'|^2 ds = J(c_k). \tag{1.13}$$

However we have assumed that $J(c_k)$ is uniformly bounded in $k$; therefore $\|c_k'\|_2$ must be uniformly bounded in $k$ which in turn implies that $\|c_k\|_\infty$ is uniformly bounded in $k$.

*Proof of Proposition 1.4.*

We will give a direct proof from the calculus of variations.

1. Since $J(c) \ge 0$, it follows that

$$\frac{1}{2} E_n^2 = \inf_{c \in F_n} J(c) \ge 0. \tag{1.14}$$

Furthermore $\frac{1}{2} E_n^2 < \infty$ : let $x_n(s)$ be the Euclidean minimum distance line segment joining $W_{h-\delta_n}^0$ to $W_{h-\delta_n}^1$ (see Fig. 8)

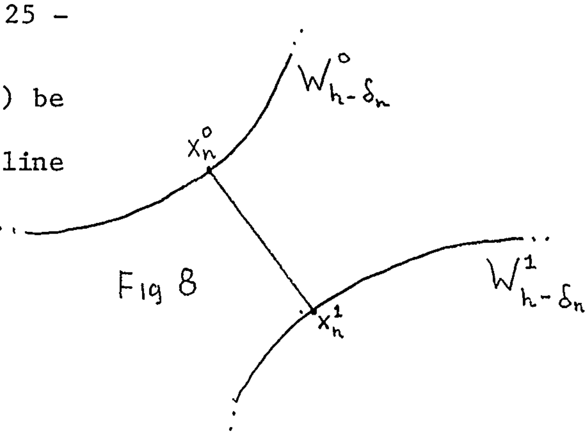


Fig 8

$$x_n(s) = x_n^0 + s(x_n^1 - x_n^0)$$

The fact that $W(x_n(s)) \le h-\delta_n$, for $s \in [0,1]$, follows since $x_n$ is a Euclidean minimum distance line.

Then $\frac{1}{2} E_n^2 \le J(x_n) = \int_0^1 (h - W \circ x_n(s)) \cdot |x_n^1 - x_n^0|^2 ds < \infty$, since $W$ is bounded on compact sets. Let $\{c_k\}_{k=1}^{\infty}$ be a minimizing sequence, i.e.

$$\lim_{k\to\infty} J(c_k) = \frac{1}{2} E_n^2 .$$

We may assume that $\{J(c_k)\}_{k=1}^{\infty}$ is uniformly bounded in $k$. We must show that there exists $c^* \in F_n$ so that $c_k \to c^*$ in some topology and that $J(c^*) = \frac{1}{2} E_n^2$.

2. From eq. (1.13) we know that $\|c_k'\|_2$ is uniformly bounded in $k$ since $J(c_k)$ is uniformly bounded. However since the unit ball in $L^2[0,1]$ is weakly sequentially compact and $|c_k(0)|$ is bounded (Lemma 1.5) we may assume that

$$c_k' \xrightarrow{\text{wkly}} f \in L^2[0,1]$$

and

$$c_k(0) \to c^* .$$

Define the arc $c^*(s) = c_0^* + \int_0^s f$, then $c_k(s) \to c^*(s)$ pointwise. Therefore $c^* \in F_n$. Furthermore we may show that

$$c_k \to c^* \text{ uniformly.}$$

To see this we notice that

$$|c_k(s_2)-c_k(s_1)| \leq \int_{s_1}^{s_2} |c_k'|ds \leq \left[\int_{s_1}^{s_2} 1\right]^{\frac{1}{2}} \cdot \left[\int_0^1 |c_k'|^2\right]^{\frac{1}{2}}$$

from the Cauchy-Schwartz inequality; therefore

$$|c_k(s_2)-c_k(s_1)| \leq |s_2-s_1|^{\frac{1}{2}} \cdot \|c_k'\|_2$$

so that the family $\{c_k\}_{k=1}^{\infty}$ is equicontinuous and uniformly bounded from Lemma 1.5. By the Arzela-Ascoli theorem this same family is sequentially compact in the topology of uniform convergence. Since $c_k(s) \to c^*(s)$ pointwise, this convergence is uniform.

3. We have shown that $c_k \xrightarrow{\text{unif}} c^*$ on $[0,1]$, and that $c^* \in F_n$. We recall that a smooth convex function $G$ satisfies

$$G(Y_1) - G(Y_2) \leq \langle G_Y(Y_1), Y_1-Y_2\rangle.$$

Set $G(Y) = \frac{1}{2}|Y|^2$, then

$$J(c^*)-J(c_k) = \int_0^1 [2(h-Woc^*)]\tfrac{1}{2}|c^{*\prime}|^2 ds - \int_0^1 [2(h-Woc_k)]\tfrac{1}{2}|c_k'|^2 ds$$

$$= \int_0^1 [2(h-Woc^*)-2(h-Woc_k)]\tfrac{1}{2}|c^{*\prime}|^2 ds + \int_0^1 [2(h-Woc_k)]\cdot[\tfrac{1}{2}|c^{*\prime}|^2 - \tfrac{1}{2}|c_k'|^2]ds$$

$$\leq \int_0^1 [2(h-Woc^*)-2(h-Woc_k)]\tfrac{1}{2}|c^{*\prime}|^2 ds + \int_0^1 [2(h-Woc_k)]\ \langle c^{*\prime}, c^{*\prime}-c_k'\rangle ds\ .$$

The first integral vanishes in the limit by uniform convergence, the second by weak convergence. Therefore

$$J(c^*) \leq \frac{1}{2}E_n^2 \quad \text{which implies that}$$

$$J(c^*) = \frac{1}{2}E_n^2 \quad \text{since } c^* \in F_n.$$

Therefore $c^*$ solves the variational problem $(J_n)$.

4. We must show that the solution $c^*$ satisfies the Euler-Lagrange equations for the Lagrangian $f$

(1.15) $$f(x,v) = (h-W(x))|v|^2,$$

and the first order transversality conditions at the endpoints $c^*(0)$, $c^*(1)$. This will follow from the observation that the minimizing curve $c^*$ intersects $W_{h-\delta_n}$ only at the points $c^*(0)$, $c^*(1)$, and hence the restriction

$$W(c^*) \leq h - \delta_n$$

does not bear upon the first order necessary conditions.

To see that the curve $s \longrightarrow (s,c^*(s)) \in \mathbb{R} \times \mathbb{R}^m$ intersects $\mathbb{R} \times W_{h-\delta_n}$ only at $(0,c^*(0))$, $(1,c^*(1))$, assume for the moment that $(s_0,c^*(s_0)) \in \mathbb{R} \times W_{h-\delta_n}$, with $0 < s_0 < 1$. For the sake of argument, suppose that $c^*(s_0) \in W^0_{h-\delta_n}$. Let

$$\tilde{c}(s) = c^* \circ g(s),\ g(s) = s_0 + s(1-s_0),\ \text{for } s \in [0,1].$$

Then $\tilde{c} \in H^1$, $W(\tilde{c}(s)) \leq h-\delta_n$, and $\tilde{c}(0) \in W^0_{h-\delta_n}$, $\tilde{c}(1) \in W^1_{h-\delta_n}$, so that $\tilde{c} \in F_n$. Invoking Lemma 1.3, we may conclude that

$$J(\tilde{c}) < J(c^*)$$

which contradicts the fact that $c^*$ affords a global minimum for $J$ with respect to $c \in F_n$.

Hence the usual first order necessary conditions (see [He]p.88) are in force for $c^*$. In particular $c^*$ is a $C^3[0,1]$ solution without corners of the Euler-Lagrange equations (see [He]p.60), and the first order transversality conditions hold:

$$c^{*\prime}(0) \text{ is orthogonal to } W^0_{h-\delta_n}, \text{ and}$$

$$c^{*\prime}(1) \text{ is orthogonal to } W^1_{h-\delta_n}.$$

The sign ± on $DW$ comes from the fact that at $s = 0$, $\frac{d}{ds} W \circ c^*(s) < 0$, while at $s = 1$, $\frac{d}{ds} W \circ c^*(s) > 0$. We observe that (speed in Jacobi metric) = $\sqrt{2(h - W \circ c^*(s))}\ |c^{*\prime}(s)|$ = constant (see [Mi]pp.55,72), and $J(c^*) = \frac{1}{2} E_n^2$ which accounts for $|c^{*\prime}(s)|$ at $s = 0,1$. For each $n$, we have constructed a smooth arc $c^n$ as asserted in the statement of the Proposition.

□

Before invoking for $c^n$ the necessary conditions of the calculus of variations as the next step in the analysis of the variational problem $(J)$, it is necessary that we first discuss the translation between solutions of (1.4) of energy $h$, and solutions of the Euler-Lagrange equations for the Lagrangian $f$ (see (1.15)). Rather than the Euler-Lagrange equations, we prefer to use instead the equivalent Hamiltonian equations associated with (1.15) via the Legendre transform (see [A & M]p.218)

(1.16) $$(x,y) = L(x,v) = (x,f_v) = (x,2(h-W(x))v),$$

and the Hamiltonian $F(x,y)$

(1.17) $$F(x,y) = \frac{\frac{1}{2}|y|^2}{2(h-W(x))}.$$

The Hamiltonian equations for $F$ (see (1.1)) are

(1.18) $$c'(s) = \frac{y(s)}{2(h-W \circ c(s))}, \quad y'(s) = \frac{-|y(s)|\ DW(c(s))}{[2(h-W \circ c(s))]^2},$$

and the solutions within $N^0$ (see (W1)) of the Euler-Lagrange equations for (1.15) are in one to one correspondence via (1.16) with solutions of (1.18). For an alternate version of this next lemma, see [A & M], (p.228).

Lemma 1.6. *An arbitrary non-trivial solution $c(s)$ through $s = 0$ of the Euler-Lagrange equations for $f$ (1.15) corresponds to a unique solution $x(t)$ of energy $h$ of (1.4), through the relation*

(1.19) $$c(s(t)) = x(t), \text{ with}$$

(1.20) $$s(t) = E^{-1}\int_0^t 2(h-W\circ x(t'))dt',$$

*where $E$ is a non zero positive constant determined by*

(1.21) $$E = \sqrt{2(h-W\circ c(s))}\ |c'(s)|, \text{ independent of } s.$$

*Proof.* An arbitrary non trivial solution $c(s)$ through $s = 0$ of the Euler-Lagrange equations for $f$ is a geodesic (in the Jacobi metric) parameterized proportional to (Jacobi) arclength (see [Mi] pp.69-72). Since $c(s)$ has constant speed $E > 0$ (in the Jacobi metric), then

$$E = \sqrt{2f(c,c')} = \sqrt{2(h-W(c))}\ |c'|,$$

independent of $s$. Let $x_0 = c(0)$, $y_0 = E^{-1}[2(h-W(x_0))]c'(0)$, and notice that, $H(x_0,y_0) = h$ (see (1.3)). Let $x(t)$ be the unique solution of (1.4) of energy $h$ with initial conditions

$$x(0) = x_0,\ \dot{x}(0) = y_0.$$

Let $$s(t) = E^{-1}\int_0^t 2(h-W\circ x(t'))dt',$$ and set

$$\bar{x}(t) = c(s(t)).$$ Then

$\bar{x}(0) = x_0$, $\dot{\bar{x}}(0) = y_0$, and by virtue of (1.18) and the fact that

$$\frac{|y(s)|^2}{[2(h-W(c(s)))]} = E^2,$$

$$\ddot{\bar{x}}(t) = -DW(\bar{x}(t)).$$

Since $\overline{x}(t)$ is a solution of (1.4) with the same initial conditions as $x(t)$, $\overline{x}(t) = x(t)$, and (1.19) follows.

□

The following three lemmas are in preparation for the result that $\{c^n\}_{n=1}^{\infty}$ converges in an appropriate topology to a solution of the variational problem ($J$).

<u>Lemma 1.7.</u> *The sequence* $\{E_n\}_{n=1}^{\infty}$ *(see equation* 1.14) *is monotonically increasing and bounded above*: $0 < E_n < E^* < \infty$ .

*Proof.* The fact that $\{E_n\}$ is an increasing sequence is intuitively obvious from the accompanying diagram (Fig. 9). To make this argument precise, we can find numbers $s_0$, $s_1$ so that:

$0 \le s_0 < s_1 \le 1$ and

$W(c^{n+1}(s)) \le h-\delta_n$, for $s \in [s_0,s_1]$,

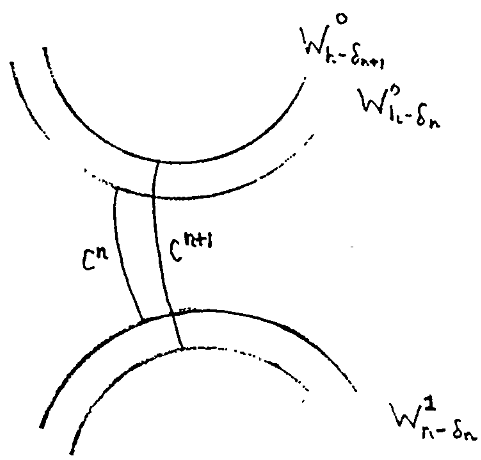



Fig 9

and

$$c^{n+1}(s_i) \in W^i_{h-\delta_n}, \text{ for } i = 0,1.$$

Therefore, if $\widetilde{c} = c^{n+1} \circ g$, $g(s) = s_0 + s(s_1-s_0)$ for $s \in [0,1]$, then $\widetilde{c} \in F_n$ (see ($J_n$) for notation). Moreover by virtue of Lemma 1.3, and the fact that $c^n$ solves ($J_n$),

$$\tfrac{1}{2}E_{n+1}^2 = J(c^{n+1}) \ge (s_1-s_0)J(c^{n+1}) \ge J(\widetilde{c}) \ge J(c^n) = \tfrac{1}{2}E_n^2.$$

To show that $\{E_n\}$ is bounded from above, let $\bar{x}^0 \in W_h^0$, $\bar{x}^1 \in W_h^1$ satisfy

$$|\bar{x}^1-\bar{x}^0| = \min_{x^0 \in W_h^0,\, x^1 \in W_h^1} |x^1-x^0| .$$

Let $\bar{x}(s) = \bar{x}^0 + s(\bar{x}^1-\bar{x}^0)$ for $s \in [0,1]$, and

$$\bar{x}_n^i = \bar{x}(s_n^i) \in W_{h-\delta_n}^i , \quad i = 0,1.$$

Then if $\bar{x}_n(s) = \bar{x}_n^0 + s(\bar{x}_n^1-\bar{x}_n^0)$, for $s \in [0,1]$, we have

$$\begin{aligned} J(\bar{x}_n) &= \int_0^1 (h-W\circ\bar{x}_n)|\bar{x}_n^1-\bar{x}_n^0|^2 ds \\ &= |\bar{x}^1-\bar{x}^0|^2 \cdot \int_0^1 (h-W\circ\bar{x}_n) ds \\ &\le |\bar{x}^1-\bar{x}^0|^2 \cdot \max_{[0,1]} (h-W\circ\bar{x}(s)) . \end{aligned}$$

Therefore, $\frac{1}{2} E_n^2 = J(c^n) \le J(\bar{x}_n)$ is bounded above. □

Recall that $\varphi^t(x,y)$ denotes the solution of (1.4) with initial conditions $(x,y)$ at $t = 0$.

<u>Lemma 1.8.</u> *$\|c^n\|_\infty$ is uniformly bounded in $n$.*

*Proof.* Let $\varphi^t(x_n,y_n) = c_n(s_n(t))$ (see Proposition 1.4, and (1.19)) with $x_n \in W_{h-\delta_n}^0$, $(x_n,y_n) \in H^{-1}(h)$, and (see (1.20))

$$(1.22) \qquad s_n(t) = \frac{1}{E_n} \int_0^t 2(h-W\circ\varphi^{t'}(x_n,y_n))dt' .$$

The existence of numbers $0 < s_0^n < s_1^n < 1$ with the following properties follows from the continuity of $c^n$ (see Fig. 10):

> $c^n(s_0^n)$ is the last point of intersection of the image of $c^n$ with $W_{h-\delta^*}^0$ (see W2 for $\delta^*$);

$c^n(s_1^n)$ is the first point of intersection after $c^n(s_0^n)$ of the intersection of the image of $c^n$ with $W^1_{h-\delta^*}$ (see Fig. 10).

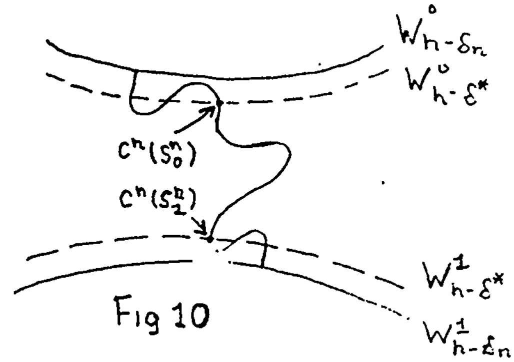


Fig 10

Since $c^n(s)$ has constant (Jacobi) speed $E_n'$ (1.21),

$$\sqrt{2(h-W(c^n(s)))}\ |c^{n\prime}(s)| = E_n,$$

we may conclude that

$$|c^n(s_1^n)-c^n(s_0^n)| \le \int_{s_0^n}^{s_1^n} |c^{n\prime}|ds = \int_{s_0^n}^{s_1^n} \frac{E_n}{\sqrt{2(h-W\circ c^n)}}\, ds \le \frac{E_n}{\sqrt{2\delta^*}} \int_{s_0^n}^{s_1^n} ds \le \frac{E_n}{\sqrt{2\delta^*}} .$$

Invoking condition W2(b) we deduce that $|c^n(s_0^n)|$ and $|c^n(s_1^n)|$ are uniformly bounded in $n$. Choose $r_0 > q$ (see W3) so that

$$c^n(s_0^n) \text{ and } c^n(s_1^n) \in B_{r_0}\ , \text{ for all } n.$$

From (W3) we obtain $r_1 \ge r_0$, $\delta \le \delta^*$ so that

$$x \in A_{r_1} \cap R_\delta \text{ and } (x,y) \in H^{-1}(h)$$

implies that

$$\varphi^t(x,y) \in A_{r_0} \text{ for all } t \text{ where defined.}$$

This has an immediate consequence: for $\delta_n < \delta$ ,

(1.23) $\qquad$ if $\varphi^t(x_n,y_n) \in R_\delta$ then $\varphi^t(x_n,y_n) \in B_{r_1}$ ;

otherwise $\varphi^t(x_n,y_n) \in A_{r_0}$ for all time, contradicting $c^n(s_1^n) \in B_{r_0}$ .

Let $D_n = \{s \in [0,1] \,|\, c^n(s) \in A_{r_1}\}$. Then $s \in D_n$ implies that $c^n(s) \notin R_\delta$ from (1.23) and

$$(1.24) \qquad \int_{D_n} |c^{n\prime}|ds = \int_{D_n} \frac{E_n}{\sqrt{2(h-W\circ c^n)}} \le \frac{E}{\sqrt{2\delta}} \int_{D_n} ds \le \frac{E_n}{\sqrt{2\delta}} .$$

Let $[s_0,s] \subset D_n$ with $|c^n(s_0)| = r_1$ (see Fig. 11). Then from (1.24) and Lemma 1.7

$$(1.25)\quad |c^n(s)| \le r_1 + \int_{s_0}^{s} |c^{n\prime}|\,ds \le r_1 + \frac{E^*}{\sqrt{2\delta}} < \infty .$$

Hence $|c^n(s)|$ is uniformly bounded from (1.23) and (1.25).

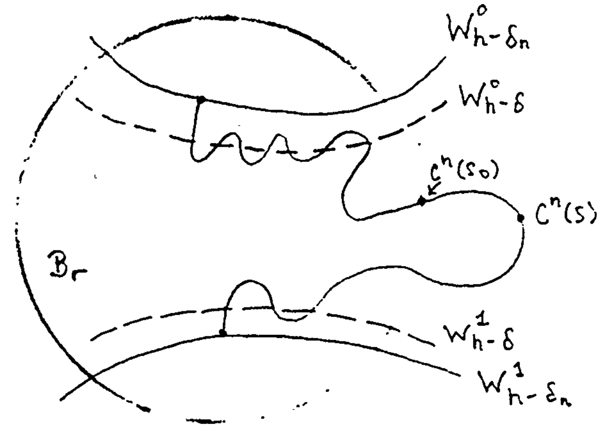



Fig 11. □

Remark 1.9. From (1.19), (1.20) we may deduce the existence of $T_n < \infty$, such that

$$c^n(s_n(T_n)) = \varphi^{T_n}(x_n,y_n) \in W^1_{h-\delta_n}, \text{ with}$$
$$s_n(T_n) = 1 \text{ (see (1.22))}.$$

Notice that by (1.21), the Jacobi arclength of $c^n$, satisfies

$$L(c^n) = E_n \le E^* \quad \text{(Lemma 1.7)}. \text{ Since}$$

$\|c^n\|_\infty < \alpha < \infty$ (Lemma 1.8) we may apply the following result from [W] (p.516).

Lemma 1.10. *There exist constants* $0 < t_0 < T^*$ *such that*

$$t_0 \le T_n \le T^* \quad \textit{for all } n \ \textit{(see Remark 1.9 for } T_n).$$
□

The fact that $\|c^n\|_\infty < \alpha$ for all $n$ means that we need only consider a compact subset $B$ of $x$-space containing the base integral curves $\varphi^t(x_n,y_n)$ for $0 \le t \le T_n$ and compact subsets of the branches $W^0_{h-\delta_n}$, $W^0_h$, $W^1_{h-\delta_n}$, $W^1_h$ for all $n$: Weinstein's result applies in this context.

We will now show that the trajectories $\varphi^t(x_n, y_n)$ (see Remark 1.9) converge to a brake orbit of energy $h$ of (1.4). Recall that, for $z \in \mathbb{R}^{2m}$, $\Phi^t z$ is the orbit in phase space corresponding to the solution $\varphi^t z$ of (1.4).

Proposition 1.11. *Let $z_n(t) = \Phi^t(x_n, y_n)$ as in Remark 1.9. There is a $C^2$ mapping*

$$z(t) = (\pi(t), \lambda(t)) \quad on \ [0, T^*]$$

*so that a subsequence of $\{z_n\}$, also denoted by $\{z_n\}$, has the following properties:* $\quad z_n \to z$ *uniformly on* $[0, T^*]$

$$\dot{z}_n = \frac{d}{dt} z_n \to \dot{z} \quad uniformly \ on \ [0, T^*].$$

*$z(\cdot)$ is an integral curve of $X_H$ (see (1.6)) on $H^{-1}(h)$ and there exists $0 < T = \lim T_n < \infty$ (see Remark 1.9), so that*

$$\pi(0) \in W_h^0 \quad and \quad \pi(T) \in W_h^1 .$$

*Proof.* Since $\varphi^t(x_n, y_n)$ and thereby $2(h - W \circ \varphi^t(x_n, y_n))$ are uniformly bounded on $[0, T^*]$, it follows that there is a compact convex set $B$ so that

$$z_n(t) \in B$$

and therefore $\dot{z}_n(t) = X_H(z_n(t))$ (1.6) are uniformly bounded in $n$ on $[0, T^*]$. We will show that $\dot{z}_n(t)$ is an equicontinuous family of mappings. Indeed let

$$|X_H(\bar{z})| \leq K_0 \quad \text{for} \quad \bar{z} \in B \text{ , and}$$

let $K_1$ be a uniform Lipschitz constant for $X_H$ in the compact convex set $B$; $z_1, z_2 \in B$ implies that

$$|X_H(z_2) - X_H(z_1)| \leq K_1 |z_2 - z_1| \quad . \tag{1.26}$$

The existence of $K_1$ follows from Theorem 9.19 of Rudin [R] since $X_H(\cdot)$ is $C^1$ on $B$. Given $0 < \varepsilon < T^* - t$,

$$\begin{aligned} |\dot{z}_n(t+\varepsilon) - \dot{z}_n(t)| &\leq K_1 |z_n(t+\varepsilon) - z_n(t)| \quad \text{from (1.26)} \\ &\leq K_1 \cdot K_0 \cdot \varepsilon \quad \text{from the mean value theorem.} \end{aligned}$$

Therefore $\{\dot{z}_n\}$ is an equi-Lipschitzian family and thereby equi-continuous. From the Arzelà-Ascoli theorem, we may select a subsequence of $\{\dot{z}_n\}$, also denoted by $\{\dot{z}_n\}$, so that

$$\dot{z}_n \text{ is uniformly convergent on } [0,T^*], \text{ and } z_n(0) \text{ converges.}$$

The conditions of Theorem 7.17 Rudin [R] are met and we may deduce the existence of a $C^1$ mapping $z$ so that

$$z_n \xrightarrow{\text{unfmly}} z \quad \text{on } [0,T^*],$$

and

$$\dot{z}_n \xrightarrow{\text{unfmly}} \dot{z} \quad \text{on } [0,T^*],$$

$z(\cdot)$ is an integral curve of $X_H$ since

$$\dot{z}(t) = \lim_{n\to\infty} \dot{z}_n(t) = \lim_{n\to\infty} X_H(z_n(t)) = X_H(z(t))$$

and $z(t) \in H^{-1}(h)$ since $z_n(t) \in H^{-1}(h)$ for all $n$. Finally we may select a subsequence of $T_n$, see Remark 1.9, also denoted by $T_n$, which converges $T_n \to T \leq T^*$ ; then if $z = (\pi,\lambda)$

$$W(\pi(T)) = \lim_{n\to\infty} W(\varphi^{T_n}(x_n,y_n)) = \lim_{n\to\infty} (h-\delta_n) = h,$$

and

$$W(\pi(0)) = \lim_{n\to\infty} W(x_n) = \lim_{n\to\infty} (h-\delta_n) = h \ . \qquad \square$$

*Proof of Theorem 1.1.* Under the conditions of Theorem 1.1, we constructed the sequence $c^n \in H^1$, and the associated sequence $\varphi^t(x_n, y_n)$ (see the proof of Lemma 1.8). The conclusions of Theorem 1.1 now follow immediately from the existence, asserted in Proposition 1.11, of the brake orbit $\pi(t)$.

Actually, we may deduce more about the brake orbit $\pi(t)$. It provides us with a solution of the variational problem (J).

Theorem 1.12. *Under the conditions of Theorem 1.1, there exists a solution* $\hat{c}$ *of the variational problem* (J). *Moreover, for*

$$s(t) = E^{-1} \int_0^t 2(h - W \circ \pi(t'))\,dt', \tag{1.27}$$

*with* $E = \lim E_n$ *(Lemma 1.7), and* $\pi(t)$ *as in Proposition 1.11,*

$$\hat{c}(s(t)) = \pi(t).$$

*We may assume that* $E = 1$.

*Proof.* Let $E = \lim_{n \to \infty} E_n$, and define

$$\hat{c}(s(t)) = \pi(t), \text{ see } (1.27). \tag{1.28}$$

For $T$ as in Proposition 1.11, it follows by uniform convergence of $\varphi^t(x_n, y_n)$, that

$$E^{-1} \int_0^T 2(h - W \circ \pi(t'))\,dt' = \lim_{n \to \infty} E_n^{-1} \int_0^{T_n} 2(h - W \circ \varphi^{t'}(x_n, y_n))\,dt'$$

$$= 1 \quad \text{(see Remark 1.9).}$$

Therefore $\hat{c}:[0,1] \to N$, and from (1.28) we may deduce that the (Jacobi) speed of $\hat{c}(s)$ is constant on $(0,1)$,

$$\sqrt{2(h - W \circ \hat{c}(s))}\;|\hat{c}'(s)| = E. \tag{1.29}$$

Therefore,

$$\int_0^1 |\hat{c}'|^2 ds = \int_0^1 \frac{E}{2(h-W\circ\hat{c})}\, ds = E\int_0^1 \frac{dt}{ds}\, ds = ET < \infty,$$

so that $\hat{c}' \in L^2[0,1]$, and consequently $\hat{c} \in H^1$. Moreover

$$\hat{c}(0) = \pi(0) \in W_h, \text{ and } \hat{c}(1) = \pi(T) \in W_h.$$

Since $\hat{c}$ is an admissible arc, we need only show that it is optimal. For an arbitrary admissable arc $c \in H^1$, we can determine $0 \le s_0^n < s_1^n \le 1$, so that

$c(s_0^n)$ is the last intersection with $W^0_{h-\delta_n}$, and

$c(s_1^n)$ is the first intersection after $c(s_0^n)$ with $W^1_{h-\delta_n}$.

Let $g^n(s) = s_0^n + s(s_1^n - s_0^n)$, $s \in [0,1]$, and $\tilde{c}^n = c\circ g^n$. Then $\tilde{c}^n \in F_n$, and invoking Lemma 1.3, we deduce that

$$J(c) \ge (s_1^n - s_0^n)J(c) \ge J(\tilde{c}^n) \ge J(c^n) = \tfrac{1}{2}E_n^2$$

where $c^n$ is the solution of the variational problem $(J_n)$. Therefore, $J(c) \ge \lim\limits_{n\to\infty} \tfrac{1}{2}E_n^2 = \tfrac{1}{2}E^2$. However, by virtue of (1.29)

$$J(\hat{c}) = \tfrac{1}{2}E^2,$$

and $\hat{c}$ thereby solves the variational problem $(J)$. We may assume that $E = 1$ in (1.27), by reparameterizing $\hat{c}$ so that $\hat{c}(s/E) = \pi(t)$.

□

<u>Remark 1.13</u>. We may assume that $E = 1$ (Lemma 1.6) by reparameterizing $c$, $ds'/ds = E$. Then the Hamiltonian vector field $X_F$ (see (1.18)) is

(1.30) $$X_F(x,y) = \left(\frac{y}{2(h-W(x))}, \frac{-DW(x)}{2(h-W(x))}\right) = \frac{X_F(x,y)}{2(h-W(x))} \quad (1.6).$$

If $\Psi^s$ is the flow of $X_F$, and $\Phi^t$ the flow of $X_H$ (1.6), then

(1.31) $$H^{-1}(h) = F^{-1}(\tfrac{1}{2}) \cup \{(x,0)\ |\ x \in \partial N\}, \text{ and}$$

(1.32) $$\Psi^{s(t)}z = \Phi^t z,\ z \in F^{-1}(\tfrac{1}{2}), \text{ where}$$

$s(t)$ is specified in (1.20) with $E = 1$.

# CHAPTER 2

## THE SECOND VARIATION AND ORTHOGONAL JACOBI FIELDS

Introduction.

In this chapter and the next we wish to develop the results concerning the index and nullity for critical extremals $\hat{c}$ of the variational problem $(J)$ considered in the last chapter.

Recall that our solution arc $\hat{c}$ of the variational problem $(J)$ joined two points on the boundary $\partial N$ of $N = \{x \mid W(x) \leq h\}$ (see (1.5) for notation) where the Jacobi metric is degenerate, see (1.10). However, in order to use the index theorem of Ambrose [A] as stated, the Jacobi metric must be a bona fide metric in a neighbourhood of the image of $\hat{c}$. In order to relax this restriction on $\hat{c}$, we will develop a theory of the second variation of $J$, for critical arcs joining separated points on $\partial N$.

We introduce Jacobi fields along such extremals and by reparametrizing these fields, we will show how the index theorem of Ambrose may be extended to cover the situation where a geodesic chord $\hat{c}$ joins separated points $c_0$, $c_1$ on $\partial N$ and neighbourhoods $K$, $L$ within $\partial N$ of $c_0$, $c_1$ respectively replace the endpoint submanifolds in Ambrose's theorem.

Since we are ultimately interested in determining the stability of the associated periodic orbit $\sigma(t) = (\pi(t), \lambda(t))$ (see Proposition 1.11) of the Hamiltonian vector field $X_H$ (see (1.6)), we will develop a Hamiltonian rather than the usual Lagrangian formulation of the index theory. This will facilitate the comparison of the index of $\sigma$ as a periodic orbit with the index of $\hat{c}$ as a critical extremal of the functional $J$.

Let $M^0, M^1$ be codimension 1 hypersurfaces in $\mathbb{R}^m$ lying within $W_h^0, W_h^1$ respectively (see (W2)), and

(2.1) $\quad H^1_{M^0\times M^1} = \{c \in H^1 | c(0) \in M^0,\ c(1) \in M^1\}$ (see (1.8))

As in the previous chapter we will denote the derivative of an element $c$ in $H^1$ by $c' := \frac{d}{ds}c$. Consider the functional $J$ defined on $H^1_{M^0\times M^1}$,

$$J(c) = \int_0^1 (h - W\circ c)\,|c'|^2\, ds$$

<u>Remark 2.1</u>. Let $N$ be the manifold with boundary specified in (1.5). If $c\colon [0,1] \to N$ and $\tau_N\colon TN \to N$ is the tangent bundle of $N$ then we denote the vector bundle over $[0,1]$ induced by $\tau_N$ by $c^*\tau_N$

$c^*\tau_N\colon c^*(TN) \to [0,1]$, see Klingenberg [K] p.27,

for the case where $N$ is a manifold without boundary.

<u>Remark 2.2</u>. By a piecewise $C^3$ arc $c$ of the manifold $N$ with boundary conditions $M^0\times M^1$ we mean a continuous map $c:[0,1] \to N$ such that

(a) there exists a subdivision

$$0 = s_0 < s_1 \ldots < s_{n+1} = 1$$

of $[0,1]$ such that

(a) $c|_{(0,s_1]}$, $c|_{[s_{i-1},s_i]}$, $c|_{[s_n,1)}$ are $C^3$ $i = 2,\dots,n$.

(b) $c(0) \in M^0$ and $c(1) \in M^1$.

Remark 2.3. Intuitively, $H^1_{M^0\times M^1}$ can be interpreted as a smooth infinite dimensional manifold, see Klingenberg [K] p.158, where he gives details showing how this can be made precise if $N$ is a manifold without boundary. Carrying this analogy further, for an element $c \in H^1_{M^0\times M^1}$, the tangent space to $H^1_{M^0\times M^1}$ at $c$ may be identified with the space of absolutely continuous vector fields $P(s)$ along $c(s)$ such that

$$P(0) \in T_{c(0)}M^0 \quad \text{and} \quad P(1) \in T_{c(1)}M^1 ,$$

the tangent spaces of $M^0$ and $M^1$ at $c(0)$, $c(1)$ respectively.

We will present the main results concerning the first and second variations of $J$ relative to the boundary conditions $M^0\times M^1$, giving references for details not provided.

Let $c \in H^1_{M^0\times M^1}$ (see (2.1)) be a piecewise $C^3$ arc.

Definition 2.4. Let $c \in H^1_{M^0\times M^1}$ (see (2.1)) be a piecewise $C^3$ arc. A piecewise $C^3$ variation through $c$ is a continuous mapping

$$Q: (-\hat{\varepsilon}_0,\hat{\varepsilon}_0) \times (-\hat{\varepsilon}_1,\hat{\varepsilon}_1) \times [0,1] \to N \quad \text{so that}$$

$$\begin{aligned} &Q(0,0,s) = c(s) && \text{for} \quad s \in [0,1], \\ &Q(\varepsilon_0,\varepsilon_1,0) \in M^0 && \text{for} \quad (\varepsilon_0,\varepsilon_1) \in (-\hat{\varepsilon}_0,\hat{\varepsilon}_0) \times (-\hat{\varepsilon}_1,\hat{\varepsilon}_1) , \\ &Q(\varepsilon_0,\varepsilon_1,1) \in M^1 && \text{for} \quad (\varepsilon_0,\varepsilon_1) \in (-\hat{\varepsilon}_0,\hat{\varepsilon}_0) \times (-\hat{\varepsilon}_1,\hat{\varepsilon}_1) , \end{aligned}$$

for which there is a subdivision $0 = s_0 < s_1 < \dots < s_{n+1} = 1$

of $[0,1]$ such that $Q|_{(-\hat{\varepsilon}_0,\hat{\varepsilon}_0)\times(-\hat{\varepsilon}_1,\hat{\varepsilon}_1)\times(0,s_1]}$,

$Q|_{(-\hat{\varepsilon}_0,\hat{\varepsilon}_0)\times(-\hat{\varepsilon}_1,\hat{\varepsilon}_1)\times[s_{i-1},s_i]}$ and $Q|_{(-\hat{\varepsilon}_0,\hat{\varepsilon}_0)\times(-\hat{\varepsilon}_1,\hat{\varepsilon}_1)\times[s_n,1)}$

are $C^3$ for $i = 2,\dots,n$. For fixed $(\varepsilon_0,\varepsilon_1)$, let $c_{\varepsilon_0\varepsilon_1}(s) = Q(\varepsilon_0,\varepsilon_1,s)$ and let $\frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_0} c_{00}(s) \doteqdot \frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_0}\Big|_{\varepsilon_0=\varepsilon_1=0} Q(\varepsilon_0,\varepsilon_1,s)$.

We will assume that all first and second partial derivatives of $c_{\varepsilon_0\varepsilon_1}(s)$ with respect to $(\varepsilon_0,\varepsilon_1)$ are continuous on $(0,1)$ and converge uniformly in $(\varepsilon_0,\varepsilon_1)$ on compact neighbourhoods of $(\varepsilon_0,\varepsilon_1) = (0,0)$, as $s \downarrow 0$ or $s \uparrow 1$.

Finally, we will assume that

(2.2) $2(h\text{-}W\circ c_{\varepsilon_0\varepsilon_1}(s))\cdot \frac{\partial^2 Q}{\partial s\partial\varepsilon i}(\varepsilon_0,\varepsilon_1,s)$ is piecewise continuous on $[0,1]$, for $i = 0,1$.

The variation vector fields $P_0,P_1$ of $Q$ are

$$P_i(s) = \frac{\partial}{\partial\varepsilon_i}\Big|_{\varepsilon_0=\varepsilon_1=0} Q(\varepsilon_0,\varepsilon_1,s) \text{ for } i = 0,1, \quad\text{and}$$

$$\Delta s_i(P_j') \doteqdot \lim_{s\downarrow s_i} P_j'(s) - \lim_{s\uparrow s_i} P_j'(s) \text{ for } j = 0,1 \text{ and } i = 1,\dots,n.$$

□

Remark 2.5. One parameter variations $Q$ may be obtained by setting $\varepsilon_1 \equiv 0$ in Definition 2.4.

We give here a few remarks on the motivation for Definition 2.4. The differentiability properties of $Q$ listed in Definition 2.4 were suggested by examining one parameter families of broken extremals, i.e. one parameter families of broken base integral curves of $X_F$

(see (1.30)), joining $M^0$ and $M^1$. That such families do in fact have these differentiability properties may be deduced from Proposition 2.36 and (2.38).

The variation vector fields $P(s)$ are continuous sections of $c^*\tau_N$ with boundary conditions $(T_{c(0)}M^0) \times (T_{c(1)}M^1)$, which are piecewise $C^2$ on $(0,1)$. However, as we shall see, $P'(s) \notin L^2[0,1]$ in general. We have added condition (2.2) in order to obtain a workable index theory. Condition (2.2) is also met by one parameter families of broken extremals joining $M^0$ and $M^1$.

□

Definition 2.6. A piecewise $C^3$ arc $c$ is a critical arc of the functional $J$ with the boundary conditions $M^0 \times M^1$ if $c \in H^1_{M^0 \times M^1}$ and for every piecewise $C^3$ variation $Q$ through $c$,

$$\frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon = 0} J(c_\varepsilon) = 0 .$$

Recall the Legendre transform $L$, see (1.16), for the Lagrangian $f(x,v) = (h-W(x))|v|^2$ :

$$L(x,v) = (x, 2(h-W(x))v) = (x,y) .$$

Proposition 2.7. *Let $c$ be a critical arc of the functional $J$ subject to the boundary conditions $M^0 \times M^1$* (see Definition 2.6). *Then*

$$(c(s), y(s)) = L(c(s), c'(s)) \textit{ is continuous on } [0,1]$$

*and $C^2$ on $(0,1)$. $L(c(s),c'(s))$ is an integral curve of $X_F$ on $F^{-1}(1/2)$* (see (1.4)) *and*

$$y(0) = 0 \ , \ y(1) = 0.$$

*Furthermore,* $\lim_{s \downarrow 0} y(s)/|y(s)| = -DW(c(0))/|DW(c(0))|$,

$$\lim_{s \uparrow 1} y(s)/|y(s)| = DW(c(1))/|DW(c(1))|.$$

*Proof*: The usual first order conditions on a minimizing arc (see Hestenes [He] pp.58-60) require only that the arc be a critical arc. Therefore these conditions apply to the critical arc $c$. In particular $c(s)$ is a $C^2$ solution of the Euler-Lagrange equations on $(0,1)$, since $\det(f_{vv}(c(s),c'(s))) > 0$ for $s \in (0,1)$. Using the equivalence of the Euler vector field and the Hamiltonian vector field $X_F$ we conclude that $L(c(s),c'(s))$ is an integral curve of $X_F$ on $F^{-1}(E^2/2)$ for some constant $E$. We may assume without loss of generality that $E = 1$ (see Remark 1.13).

Let $(c(s),y(s)) = \Psi^{s-s_0}z$ for some $0 < s_0 < 1$, where $\Psi$ is the flow of $X_F$, see (1.30) and $z = (c(s_0),y(s_0)) \in H^{-1}(h)$. Let $t_0 > 0$ satisfy

$$s_0 = \int_{-t_0}^{0} 2(h - W\circ\varphi^{t'} z)\,dt'$$

and let $\sigma = \Phi^{-t_0} z$, where $\Phi^t$ denotes the flow of $X_H$ (1.6). By virtue of (1.32), we may conclude that

$$\Psi^{s-s_0} z = \Phi^{t-t_0} z \qquad \text{with}$$

$$s - s_0 = \int_0^{t-t_0} 2(h - W\circ\varphi^{t'} z)\,dt' .$$

However, $\Phi^{t-t_0} z = \Phi^t \sigma \doteq (x(t),\lambda(t))$ and

$$s = s_0 + \int_0^{t-t_0} 2(h - W\circ\varphi^{t'} z)\,dt' = \int_{-t_0}^{t-t_0} 2(h - W\circ\varphi^{t'} z)\,dt'$$

$$= \int_0^t 2(h - W\circ\varphi^{t'}\sigma)dt' \ .$$ Therefore

(2.3) $$(c(s), y(s)) = (x(t), \lambda(t))$$ with

$$s = s(t) = \int_0^t 2(h - W\circ\varphi^{t'}\sigma)dt' \ , \ \sigma = (x,\lambda)(0) \ .$$

Since $c' \in L^2[0,1]$ and $|c'|^2 = [2(h-W)]^{-1} = \left|\frac{ds}{dt}\right|^{-1}$ there exists $T < \infty$, $T = \int_0^1 \left|\frac{ds}{dt}\right|^{-1} ds = \int_0^1 |c'|^2 \, ds$, such that

$$1 = \int_0^T 2(h - W\circ\varphi^{t'}\sigma)dt'.$$

It follows from (2.3) that $(c(s), y(s))$ is continuous on $[0,1]$, since $(x(t), \lambda(t))$ is continuous on $[0,T]$. Furthermore, $x(0) = c(0) \in \partial N$ , $x(T) = c(1) \in \partial N$, which implies that $y(0) = \lambda(0) = 0$ and $y(1) = \lambda(T) = 0$. These endpoint conditions on $y(s)$ are referred to as first order transversality conditions on $c$ (see Hestenes [He] p.88). Since the orbit $(x,\lambda)(t)$ of $X_H$ has energy $h$, $x(t)$ approaches $\partial N$ orthogonally, that is

$$\lim_{s\downarrow 0} y(s)/|y(s)| = \lim_{t\downarrow 0} \lambda(t)/|\lambda(t)| = -DW(x(0))/|DW(x(0))| . \ \square$$

<u>Remark 2.8</u>. If $\hat{c} \in H^1_{M^0\times M^1}$ minimizes $J$ relative to $c \in H^1_{M^0\times M^1}$, then $\hat{c}$ is a critical arc. $\square$

The Legendre transform $L(x,v)$ (see (1.2)) and its linearization $L_*(x,v)$ will be used to describe the main results concerning the second variation of the functional $J$.

Given two vectors in $\mathbb{R}^m$, $a = (a_1,\ldots,a_m)$ and $b = (b_1,\ldots,b_m)$, we will denote their matrix tensor product by

$$a \otimes b \doteq [a_i b_j] \in L(\mathbb{R}^m, \mathbb{R}^m).$$

<u>Lemma 2.9</u>. *The Jacobian matrix* $L_*(x,v)$ *of the Legendre transform*

$$L(x,v) = (x, 2(h-W(x))\,v) = (x,y)$$

*is given by*

$$L_*(x,v) = \left|\begin{array}{c|c} I_m & 0 \\ \hline -2v \otimes DW(x) & 2(h-W(x))\, I_m \end{array}\right|$$

*If* $P(s)$, $s \in [0,1]$, *is the variation vector field along a critical arc* $c(s)$ *(see Definition 2.6) of a piecewise* $C^3$ *variation* $Q$ *(see Definition 2.4) such that (2.2) holds, then*

$$(P,R)(s) = L_*(c,c') \quad (P,P')(s)$$

*is piecewise continuous on* $[0,1]$, *and piecewise* $C^1$ *on* $(0,1)$.

*Proof.* The formula for $L_*(x,v)$ follows upon differentiating $L(x,v)$ with respect to $(x,v)$.

Now suppose that $P(s)$ is the variation vector field of $Q$ along $c(s)$ and that (2.2) holds. From Definition 2.4 we may deduce that $P(s)$ is continuous on $[0,1]$ and piecewise twice continuously differentiable on $(0,1)$. We need only show that $R(s)$ is piecewise continuous on $[0,1]$ and piecewise $C^1$ on $(0,1)$, where

$$(2.4) \qquad R(s) = -2\langle P(s), DW(c(s))\rangle \cdot c'(s) + 2(h-W\circ c(s)) \cdot P'(s).$$

Notice that $\langle P(0), DW(c(0))\rangle = 0$ and $\langle P(1), DW(c(1))\rangle = 0$ since $Q(\varepsilon,0) \in M^0$ and $Q(\varepsilon,1) \in M^1$ for $\varepsilon \in (-\hat{\varepsilon},\hat{\varepsilon})$.

Let $y^*(s) = y(s)/|y(s)|$ for $0 < s < 1$. Since $|c'(s)| = [2(h-W\circ c(s))]^{-\frac{1}{2}}$, see Proposition 2.7, we may use L'Hospital's rule to evaluate the following;

$$\lim_{s\downarrow 0} \langle P(s), DW(c(s))\rangle\cdot c'(s) = \lim_{s\downarrow 0} \frac{\langle P(s), DW(c(s))\rangle\cdot y^*(s)}{[2(h-W\circ c(s)]^{\frac{1}{2}}}$$

$$= \lim_{s\downarrow 0} \frac{\langle P', DW\circ c\rangle(s) + \langle P, (D^2W)\cdot c'\rangle(s)}{-[2(h-W\circ c(s))]^{-\frac{1}{2}}\langle DW\circ c, c'\rangle(s)} \cdot y^*(s)$$

$$= \lim_{s\downarrow 0} \left\{ \frac{\langle 2(h-W\circ c)\cdot P', DW\circ c\rangle(s)}{-\langle DW\circ c, y^*\rangle(s)} \cdot y^*(s) + \frac{[2(h-W\circ c)]^{\frac{1}{2}}\langle P, D^2W\cdot y^*\rangle(s)\cdot y^*(s)}{-\langle DW\circ c, y^*\rangle(s)} \right\}$$

$$= \frac{\langle \lim_{s\downarrow 0} 2(h-W\circ c)\cdot P', DW(x(o))\rangle (DW)^*(x(0))}{-|DW(x(0))|}$$ (see (2.2) and Proposition 2.7).

By a similar computation at $s = 1$, $R(s) - 2(h-W\circ c(s))\, P'(s)$ and hence $R(s)$ is piecewise continuous on $[0,1]$, see (2.2) and (2.4). Now $R'(s)$ exists and is continuous at $s$ whenever $P'(s)$, $P''(s)$ exist and are continuous at $s$. Therefore $R(s)$ is piecewise $C^1$ on $(0,1)$ as claimed.

Note that, by the computation above, and the fact that

$$2(h-W\circ c(s))\cdot P'(s) = \frac{2\langle DW\circ c, P\rangle(s)}{2(h-W\circ c(s))}\; y(s) + R(s)$$

$$(2.5)\quad \left( \frac{\langle P, DW\circ c\rangle(s)}{[2(h-W\circ c)]^{\frac{1}{2}}}\; y^*(s) + \frac{\langle R, DW\circ c\rangle(s)}{3\langle DW\circ c, y^*\rangle(s)}\; y^*(s) \right) \to 0 \quad \text{as } s\downarrow 0 \text{ or } s\uparrow 1$$

$\square$

<u>Definition 2.10</u>. Given a piecewise $C^3$ two parameter variation $Q$ through a critical arc $c$ of the functional $J$, with the boundary conditions $M^0 \times M^1$; the second variation of $J$ through $Q$ is

$$J^{**}(P_0, P_1) \doteqdot \frac{\partial^2}{\partial\varepsilon_1 \partial\varepsilon_0}\bigg|_{\varepsilon_1 = \varepsilon_0 = 0} J(c_{\varepsilon_0\varepsilon_1}) \ .$$

<u>Proposition 2.11</u>. *Let $P_i(s)$ $(i = 0,1)$ denote the respective variation vector fields of a piecewise $C^3$ variation $Q$ through the critical arc $c$. Let*

$$(c(s), y(s)) = L(c(s), c'(s))$$

*and*

$$(P_i(s), R_i(s)) = L^*(c, c') \cdot (P_i(s), P_i'(s)) .$$

*The second variation of $J$ through $Q$ is finite and depends only on the variation vector fields $P_i$, not on the underlying variation $Q$. Furthermore,*

$$J^{**}(P_0, P_1) = \int_0^1 \left\{ \frac{\langle R_1, R_0\rangle(s)}{2(h - W\circ c)} - 4\,\frac{\langle P_1, DW(c)\rangle\langle P_0, DW(c)\rangle(s)}{[2(h - W\circ c)]^2} - \frac{\langle D^2W(c)\cdot P_0, P_1\rangle(s)}{2(h - W\circ c)} \right\} ds$$

*Proof*. We will use the notational convention that

$$\frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_0} c_{00}(s) \doteqdot \frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_0}\bigg|_{\varepsilon_0 = \varepsilon_1 = 0} Q(\varepsilon_0, \varepsilon_1, s) .$$

$$\frac{\partial}{\partial\varepsilon_0} J(c_{\varepsilon_0\varepsilon_1}) = \int_0^1 -\langle DW(c_{\varepsilon_0\varepsilon_1}), \frac{\partial}{\partial\varepsilon_0} c_{\varepsilon_0\varepsilon_1}\rangle(s) \cdot |c'_{\varepsilon_0\varepsilon_1}|^2 \, ds$$

$$+ \int_0^1 2(h - W\circ c_{\varepsilon_0\varepsilon_1}(s)) \, \langle c'_{\varepsilon_0\varepsilon_1}, \frac{\partial}{\partial\varepsilon_0} c'_{\varepsilon_0\varepsilon_1}\rangle(s) \, ds .$$

Now $\frac{\partial}{\partial\varepsilon_1}\Big|_{\varepsilon_0 = \varepsilon_1 = 0} \{2(h-W\circ c_{\varepsilon_0\varepsilon_1}(s))\cdot c'_{\varepsilon_0\varepsilon_1}(s)\} = R_1(s)$.

Therefore $\frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_0}\Big|_{\varepsilon_0 = \varepsilon_1 = 0} J(c_{\varepsilon_0\varepsilon_1}) = \int_0^1 \{-2\langle c',P_1'\rangle\langle DW,P_0\rangle$

$$- \ |c'|^2\langle D^2W\cdot P_1,P_0\rangle\}ds + \int_0^1 -|c'|^2\langle DW, \frac{\partial^2}{\partial\varepsilon_0\partial\varepsilon_1} c_{00}(s)\rangle\, ds$$

$$+ \int_0^1 \{\langle R_1,P_0'\rangle + \langle y, \frac{\partial^2}{\partial\varepsilon_0\partial\varepsilon_1} c'_{00}(s)\rangle\}\ ds.$$

Notice from (2.4) that

$$\langle c',P_1'\rangle = \frac{2\langle P_1,DW\rangle + \langle R_1,y\rangle}{[2(h-W)]^2} \quad ,$$

and

$$\langle R_1,P_0'\rangle = \frac{2\langle P_0,DW\rangle}{[2(h-W)]^2}\langle y,R_1\rangle + \frac{\langle R_1,R_0\rangle}{[2(h-W)]} .$$ We deduce that

$$-2\langle c',P_1'\rangle\ \langle DW,P_0\rangle + \langle R_1,P_0'\rangle = \frac{-4\langle P_1,DW\rangle\ \langle P_0,DW\rangle}{[2(h-W)]^2} + \frac{\langle R_1,R_0\rangle}{[2(h-W)]} .$$

Furthermore, $\langle y(s), \frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_0} c_{00}(s)\rangle$ is continuous on $[0,1]$, piecewise $C^1$ on $(0,1)$ and vanishes as does $y(s)$ at $s = 0,1$, by the assumptions in Def. 2.4. Therefore,

$$0 = \int_0^1 \frac{d}{ds}\langle y(s),\frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_0}c_{00}(s)\rangle ds = \int_0^1 \langle -|c'|^2 DW\circ c,\frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_0}c_{00}(s)\rangle\, ds$$

$$+ \int_0^1 \langle y(s),\frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_0} c'_{00}(s)\rangle\, ds.$$ Therefore

$$\frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_0}\Big|_{\varepsilon_0=\varepsilon_1=0} J(c_{\varepsilon_0\varepsilon_1}) = \int_0^1\left\{\frac{\langle R_1,R_0\rangle}{2(h-W)} - 4\frac{\langle DW,P_0\rangle\ \langle DW,P_1\rangle}{[2(h-W)]^2}\right.$$

$$\left. - \frac{\langle D^2W\cdot P_1,P_0\rangle}{[2(h-W)]}\right\}\ ds,$$

to show finiteness of $J_{**}$, notice that $[2(h-W\circ c)]^{-1} \in L^1[0,1]$: from (2.3);

$$\int_0^1 [2(h-W\circ c)]^{-1}\, ds = \int_0^1 \frac{dt}{ds}\, ds = T < \infty .$$

Furthermore, see (2.5),

$$\frac{\langle DW,P_0\rangle(s) \cdot \langle DW,P_1\rangle(s)}{2(h-W\circ c(s))} \to \frac{\langle R_0,DW\rangle\langle R_1,DW\rangle}{9\;|DW|^2} \quad \text{as } s\downarrow 0 \text{ or } s\uparrow 1.$$

Since $(P_i,R_i)(s)$ is piecewise continuous on $[0,1]$, $J{*}{*}(P_0,P_1) < \infty$.

□

Remark 2.12. Later in Chapter 3, we will need a more general formula for the second variation. This formula is derived from variations identical to those specified in Definition 2.4 except that the endpoints of $c_{\varepsilon_0\varepsilon_1}(s)$ at $s = 0,1$, belong to hypersurfaces $K$ and $L$ of $\mathbb{R}^m \cap N^0$ (see (W1)).

The critical arc $c(s)$ would then satisfy

$$c(0) \in K,\ c(1) \in L\ , \quad \text{and}$$

$$y(0) \perp T_{c(0)} K\ ,\ y(1) \perp T_{c(1)} L.$$

In this case $y(0)$ and $y(1)$ do not vanish. We find that $J{*}{*}(P_0,P_1)$ is the sum of an expression like the integral expression given in Proposition 2.11 and the boundary terms

$$(2.6)\quad \langle y(s), \frac{\partial^2}{\partial\varepsilon_0\partial\varepsilon_1} c_{00}(s)\rangle\Big|_{s=0}^{s=1} = -|y(s)|\langle \Xi_s(P_1(s)),P_0(s)\rangle\Big|_{s=0}^{s=1}$$

where $\Xi_0$ is the Weingarten mapping of $K$ at $c(0)$ relative to the unit normal of $K$ compatible with $y(0)/|y(0)|$, and $\Xi_1$ is the Weingarten mapping of $L$ at $c(1)$ relative to $y(1)/|y(1)|$, see Hicks [Hi] p. 21.

To verify (2.6), we let $D$ denote the standard connection in

$\mathbb{R}^m$ (see Hicks [Hi], p. 18, where he uses $\overline{D}$ for the standard connection). Then $\frac{\partial}{\partial \varepsilon_0} c_{0\varepsilon_1}(s)$ is a vector field along the curve $\varepsilon_1 \to Q(0,\varepsilon_1,s)$, and $\frac{\partial}{\partial \varepsilon_1} c_{00}(s) = P_1(s)$. Therefore $\frac{\partial^2}{\partial \varepsilon_0 \partial \varepsilon_1} c_{00}(s)$ $= D_{P_1(s)} P_0$ (see Hicks [Hi], p. 19). The formula (2.6) now follows after an application of the Gauss equation (see Hicks [Hi], p.26).

□

Definition 2.13. Let $V$ denote the vector space of fields $P(s)$ along $c(s)$ such that

(1) $P(s)$ is continuous on $[0,1]$, with

(2) $(P,R)(s) = L_*(c,c')\cdot(P,P')(s)$ piecewise continuous on $[0,1]$ and piecewise $C^1$ on $(0,1)$, and

(3) $P(0) \in T_{c(0)}\, M^0$, $P(1) \in T_{c(1)}\, M^1$ .

$J_{**}(P_0,P_1)$ as given in Proposition 2.11 is a symmetric bilinear form on $V$. Our attention will now shift to the associated quadratic form $J_{**}(P) \doteq J_{**}(P,P)$ on $V$. We will not pause to ask whether every $P \in V$ can be realized as a variation vector field of some variation $Q$, but will intuitively consider elements $P \in V$ as having been so derived.

We remark here that, by virtue of (2.4), (2.5) holds for all $P \in V$. This fact may be used, along the lines of Proposition 2.11, to show that $J_{**}(P_0,P_1) < \infty$ for every $P_0,P_1 \in V$.

Notice that (2) above implies that $P$ is piecewise $C^2$ on $(0,1)$, again by virtue of (2.4).

Proposition 2.14.

(a) *Let $c(s)$ be a critical arc of $J$ subject to the boundary conditions $M^0 \times M^1$. Then $J_{**}(P_0,P_1)$ =*

$$- \int_0^1 \langle R_1' + \frac{D^2W \cdot P_1}{[2(h-W)]} + \left\{ \frac{4\langle P_1,DW\rangle + 2\langle y,R_1\rangle}{[2(h-W)]^2} \right\} DW, P_0\rangle \, ds$$

$$+ \sum_{i=1}^{n} \langle \Delta s_i R_1, P_0(s_i)\rangle + \langle R_1(s),P_0(s)\rangle \Big|_{s=0}^{s=1}$$

(b) *Recall the Hamiltonian $H(x,y)$ in (1.3), $H(x,y) = \frac{1}{2}|y|^2 + W(x)$. Let $H$ be the vector space of fields $P(s) \in V$ such that $(P,R)(s) \in T_{(c,y)(s)} H^{-1}(h)$ for every $s \in [0,1]$. Let $E$ be the subspace of $V$ containing fields $P(s)$ such that*

$$P(s) = g(s) \cdot (sc'(s))$$

*where (1.) $g$ is continuous on $[0,1]$, (2.) piecewise $C^2$ on $(0,1)$, (3.) $s \cdot [2(h-W \circ c(s))]^{\frac{1}{2}} g'(s)$ is piecewise continuous on $[0,1]$, and (4.) $g(1) = 0$ (see the comments following Definition 2.13) Then $V = H \oplus E$, and $J_{**}$ is positive definite on $E$.*

(c) *Index $J_{**}$ = Index $J_{**}|_H$, see Definition 2.15.*

*Proof.* (a) From (2.4), we may deduce that

$$\frac{\langle R_1,R_0\rangle}{2(h-W\circ c)} = \frac{d}{ds}\langle R_1,P_0\rangle - 2\,\frac{\langle DW,P_0\rangle\langle y,R_1\rangle}{[2(h-W\circ c)]^2} - \langle R_1',P_0\rangle ,$$

where the derivatives are interpreted as right and left derivatives when $s = s_i$, $i = 0,\ldots,n+1$. The formula for $J_{**}$ now follows from Proposition 2.11.

(b) Let $\overline{P}(s) = g(s)\cdot(s\cdot c'(s)) = \frac{g(s)sy(s)}{2(h-Woc)(s)}$ where $g$ has the four restrictions noted above. Notice that our conditions on $g'(1)$ and $g(1)$ together with L'Hospital's rule guarantee that $\overline{P}(0) = 0 = \overline{P}(1)$. Now

$$\overline{P}' = g'(sc') + g(sc')' \text{ , and from (2.4),}$$

$$\overline{R}(s) = g(s)[y(s) + \frac{s(-DWoc)}{2(h-W)}] + g'(s)\ sy(s).$$

Conditions (1) through (4) on $g$ guarantee that $\overline{R}(s)$ is piecewise continuous on $[0,1]$. Therefore

$$dH(\overline{P},\overline{R})(s) = g\,\frac{\langle sy,DW\rangle}{2(h-W)} + g|y|^2 + g\frac{\langle sy,-DW\rangle}{2(h-W)} + s\ g'\cdot|y|^2$$

$$= |y|^2\cdot(g + sg') \ .$$

It is now easy to see that $H \cap E = \{0\}$: indeed the only continuous solution of $g + sg' = 0$ such that $g(1) = 0$ is $g \equiv 0$.

Since $H^{-1}(h)$ has codimension one, $V = H + E$. Indeed, if $P \in V$, set $q(s) = dH(P,R)(s) = \langle DWoc,P\rangle(s) + \langle R,y\rangle(s)$. Then $q(s)$ satisfies the following three conditions: (a) $q(0) = q(1) = 0$ (Definition 2.13 (3)), (b) $q(s)\cdot[2(h-Woc(s))]^{-\frac{1}{2}}$ is piecewise continuous on $[0,1]$ (see (2.5)), and (c) $q(s)$ is piecewise $C^1$ on $(0,1)$. For such a $q(s)$, there is a unique solution of

$$g(s) + s\cdot g'(s) = q(s)\ [2(h-Woc(s)]^{-1}$$

on $[0,1]$, which satisfies the conditions (1) through (4) listed in the description of $E$. For this $g(s)$, let $\overline{P}(s) = g(s)\cdot(sc'(s))$. Then $dH(P,R)(s) - dH(\overline{P},\overline{R})(s) \equiv 0$ on $[0,1]$ and therefore $(P-\overline{P},R-\overline{R})(s) \in T_{(c,y)}H^{-1}(h)$ on $[0,1]$, which implies that $P \in \overline{P} + H$ as claimed.

To show that $J_{**}$ is positive definite on $E$ we consider $P \in E$, $P(s) = g(s)P_0(s)$, where $P_0(s) = sc'(s)$. If $R(s)$ and $R_0(s)$ are determined by (2.4) from $P(s)$, $P_0(s)$ respectively, then using the fact that

$$c''(s) = \frac{-DWoc}{[2(h-Woc)]^2}(s) + \frac{2\langle DWoc, c'\rangle}{[2(h-Woc)]^2}(s)y(s), \quad \text{we obtain}$$

$$R(s) = g(s)R_0(s) + g'(s)sy(s), \; R_0(s) = y(s) + \frac{s(-DW(c(s)))}{2(h-Woc(s))}, \text{ and}$$

$$R_0' + \frac{D^2W \cdot P_0}{2(h-Woc)} + \left\{ \frac{4\langle P_0, DW\rangle + 2\langle y, R_0\rangle}{[2(h-Woc)]^2} \right\} DW(c) = 0. \quad \text{Moreover}$$

$\langle R,P\rangle\Big|_0^1 = 0$, since $R$ is piecewise continuous, and $P(0) = P(1) = 0$. A straightforward computation using these facts and the formula from part (a) yields

$$J_{**}(P,P) = -\int_0^1 \left\{\langle \frac{d}{ds}(g'sy), P\rangle + g'\langle y + \frac{sDW}{2(h-W)}, P\rangle \right\} ds$$
$$+ \sum_{i=1}^{n} (\Delta_{s_i} g')\langle sy, P\rangle(s_i).$$

The integral part of this expression is reduced, after an integration by parts on the first term, to

$$-\int_0^1 \left\{ \frac{d}{ds}\langle g'sy, P\rangle - s^2|g'|^2 \right\} ds.$$

Therefore, $J_{**}(P,P) = \int_0^1 s^2|g'|^2 \, ds > 0$, if $P \neq 0$.

(c) To show that Index $J_{**}$ = Index $J_{**}\big|_H$, it suffices to show that if $\rho\colon V \to H$ is the projection associated with the splitting $V = H \oplus E$, then for $P \in V$, $J_{**}(\rho P, \rho P) \le J_{**}(P,P)$, with equality only if $P \in H$ (see Ambrose [A], p.64). By virtue of part (b), this will follow if we can show that for $P_0 \in H$, $P_1 \in E$,

$$J^{**}(P_1, P_0) = 0.$$

To this end, let $\dot{P}_0 \in H$, then we deduce that

$$0 = dH\,(P_0, R_0) = \langle P_0, DW\rangle + \langle R_0, y\rangle, \text{ and}$$

$$0 = \langle P_0', DW\rangle + \frac{\langle P_0, D^2W\cdot y\rangle}{2(h-Woc)} + \langle R_0', y\rangle - \frac{\langle R_0, DW\rangle}{2(h-Woc)} \quad ,$$

except at $s = s_i$, $i = 0,\ldots,n+1$. This latter expression reduces, upon using (2.4) and the fact that $0 = dH\,(P_0, R_0)$, to

$$0 = \left\langle \left\{ R_0' + \frac{D^2W\cdot P_0}{2(h-Woc)} + \left(\frac{4\langle P_0, DW\rangle + 2\langle R_0, y\rangle}{[2(h-Woc)]}\right) DW \right\}, y \right\rangle .$$

This implies, since $P_1(s) = g(s)sy(s)$, that

$$J^{**}(P_1, P_0) = \sum_{i=1}^{n} \langle \Delta_{s_i} R_0, sgy\rangle + \langle R_0, P_1\rangle \Big|_{s=0}^{s=1} .$$

All of these terms vanish since for any $s \in [0,1]$,

$$\langle R_0(s), y(s)\rangle = -\langle DWoc(s), P_0(s)\rangle \text{ , and}$$

$$P_1(0) = P_1(1) = 0.$$

Therefore Index $J^{**}$ = Index $J^{**}\,|_H$.

□

<u>Definition 2.15</u>. The index of $J^{**}$ is defined to be the dimension of the maximal subspace of $V$ such that $J^{**}$ restricted to this subspace is negative definite. The nullity of $J^{**}$ is the difference of the dimension of the maximal subspace upon which $J^{**}$ is negative semidefinite and the index of $J^{**}$.

□

Given a Lagrangian $f(x,v)$ such that $f_{vv}(x(s), x'(s)) > 0$ as a quadratic form for all $s \in [0,1]$, it is shown in classical variational analysis how this implies that the quadratic form associated with the second variation has finite index (see Duistermaat [D]). For our functional $J$,

$$f(x,v) = (h-W(x))|v|^2$$

and $f_{vv} = 0$ at the endpoints of the critical arc $c(s)$. Nevertheless we may still prove Theorem 2.16 below.

For $P \in V$, $(P,R)(s) = L_*(c,c')\cdot(P,P')(s)$, let

$$I(P,P) = \int_0^1 \frac{|P|^2}{[2(h-W\circ c)]}\, ds$$

and

$$M(P,P) = \int_0^1 \frac{|P|^2 + |R|^2}{[2(h-W\circ c)]}\, ds \quad .$$

The above integrals are convergent since $\int_0^1 \frac{1}{2(h-W\circ c)}\, ds = \int_0^1 \frac{dt}{ds}\, ds = T < \infty$, see the discussion following (2.3). Let $s(t)$ be specified as in (2.3), and

$$(U,V)(t) = (P,R)(s(t)).$$

The norm associated with $I(P,P)$ induces the topology on $V$ of $L^2[0,T]$ convergence of $U(t)$; $M(P,P)$ induces the $H^1$ topology of $L^2[0,T]$ convergence of the fields $U(t)$ and their derivatives $\frac{d}{dt}U(t)$ on $[0,T]$.

A field $\hat{P} \in H$ together with $\mu \in \mathbb{R}$ is an eigenpair for $J_{**}$ relative to $I$ if

$$J_{**}(\hat{P},P) - \mu I(\hat{P},P) = 0$$

for every $P \in H$.

It is not difficult to see that (see Hestenes [He], p.111, for the statement and proof in "Lagrangian" format) $(\mu,\hat{P})$ is an eigenpair for $J_{**}$ relative to $I$ if and only if the following conditions hold (see Prop. 2.14(a)):

$$(\hat{P},\hat{R})(s) \text{ is } C^1 \text{ on } (0,1), \text{ and}$$

$$\text{(2.7)(a)} \quad \hat{R}' + \frac{\mu\hat{P}}{[2(h-W)]} + \frac{D^2W\cdot\hat{P}}{[2(h-W)]} + 2\left\{2\frac{\langle\hat{P},DW\rangle}{[2(h-W)]^2}DW + \frac{\langle\hat{R},y\rangle}{[2(h-W)]^2}DW\right\} = 0 \text{ on } (0,1),$$

$$\text{(2.7)(b)} \quad \hat{R}(s) \perp T_{c(s)} M^s \text{ for } s = 0,1.$$

For $\mu = 0$, (2.7)(a) together with (2.4) are just the linearized Hamiltonian equations (see (2.12)) for $X_F$ (see (1.30)) along $(c(s),y(s))$:

$$\text{(2.8)(a)} \qquad P' = 2\frac{\langle DW,P\rangle}{[2(h-W)]^2}y + \frac{R}{[2(h-W)]}$$

$$\text{(2.8)(b)} \qquad R' = -\frac{D^2W\cdot P}{[2(h-W)]} - 2\left\{\frac{2\langle P,DW\rangle + \langle R,y\rangle}{[2(h-W)]^2}\right\} DW$$

The equations (2.8) are equivalent, via the linearized Legendre transform $L_*(c(s),c'(s))$ (Lemma 2.9), to the linearized Euler equations about $(c(s),c'(s))$, also known as the Jacobi differential equations (see Hestenes [He], pp. 122,123). We will refer to (2.8) as the "co-Jacobi" equations.

<u>Theorem 2.16</u>. *The index of $J_{**}$ is equal to the number of negative eigenvalues of $J_{**}$ relative to $I$, counted with their algebraic multiplicity, and this number is finite. The nullity of $J_{**}$ is also finite.*

*Proof.* By virtue of the fact that $\exists\, a > 0$ with

$$M(P) \geq a\cdot(\|P\|_2^2 + \|R\|_2^2) \qquad (\|\cdot\|_2 \text{ denotes the usual } L^2 \text{ norm})$$

we may follow the argument given in Duistermaat [D], pp. 176-177, provided that we can find $\mu^* \leq 0$ and $m > 0$ so that, for every $P \in H$,

$$J^{**}(P) - \mu^*\cdot I(P) \geq m\cdot M(P) \ .$$

The proof of this inequality uses a standard argument (see Hestenes [He], p.110) which we will repeat here.

Let (see formula for $J^{**}$ in Proposition 2.11)

$$(2.9) \quad j(s,P,R) = \begin{cases} |R|^2 - \dfrac{4\langle P,DW\rangle^2}{[2(h-W)]} - \langle D^2W\cdot P,P\rangle & \text{if } s \in (0,1) \\ |R|^2 - \dfrac{4\langle R,DW\rangle^2}{9|DW|^2} - \langle D^2W\cdot P,P\rangle & \text{if } s = 0,1 \end{cases}$$

Then $j(s,P,R)$ is a quadratic form in $(P,R)$ continuous on $[0,1]$, since $\dfrac{\langle P,DW\rangle^2}{[2(h-W)]} \to \dfrac{\langle R,DW\rangle^2}{9|DW|^2}$ as $s\downarrow 0$ or $s\uparrow 1$, see (2.5).

To show that $J^{**}(P) - \mu^* I(P) \geq mM(P)$, it suffices to show that for all $(s,P,R) \in [0,1] \times \mathbb{R}^m \times \mathbb{R}^m$,

$$j(s,P,R) - \mu^*|P|^2 > m\cdot[|P|^2 + |R|^2] \ .$$

To prove the existence of $\mu^* \leq 0$ and $m > 0$ which satisfy the above inequality, we will argue by contradiction.

Assume that for every integer $n > 0$ we can find

$$(s_n,P_n,R_n) \in [0,1] \times \mathbb{R}^m \times \mathbb{R}^m$$

so that the inequality above is not satisfied with

$$-\mu^* = n,\ m = 1/n \text{ and } (s,P,R) = (s_n,P_n,R_n) \ .$$

Since the expressions on both sides of the inequality are quadratic in $(P,R)$, we may assume that

$$|P_n|^2 + |R_n|^2 = 1.$$

Therefore, by compactness, we may choose a subsequence $n_k$ with

$$(s_{n_k}, P_{n_k}, R_{n_k}) \to (s_0, P_0, R_0) \text{ with } |P_0|^2 + |R_0|^2 = 1.$$

Therefore $j(s_0,P_0,R_0) + \limsup n \cdot |P_0|^2 \leq 0$. Since $|P_0|^2 \geq 0$, $P_0 = 0$. It follows that $|R_0|^2 = 1$, and $j(s_0,0,R_0) \leq 0$. However if $s_0 \in (0,1)$ , $j(s_0,0,R_0) = |R_0|^2 = 1$ while if $s_0 = 0,1$ then

$$j(s_0,0,R_0) = |R_0|^2 - 4/9 \frac{\langle R_0, DW\rangle^2}{|DW|^2} \geq 5/9\ |R_0|^2 = 5/9.$$

In both cases we obtain a contradiction. The result now follows along the lines given in Duistermaat [D], pp. 176-177.

□

The index theorem that we wish to use relates the index of $J^{**}$ to certain geometrical quantities which we now describe. Recall that $\Psi^s$ denotes the co-geodesic flow of the Hamiltonian vector field $X_F$ (see (1.30)). Let $\tau_N^*: T^*N \to N$ be the canonical projection.

<u>Definition 2.17</u>. Let $M$ denote a submanifold of $N$ with $M \cap \partial N = \emptyset$, (see (W1) for notation). Define the $m$-dimensional submanifold of $T^*N \approx N \times \mathbb{R}^m$

$$\perp(M) = \{(x,y)\,|\,x \in M,\ \langle y,P\rangle = 0 \text{ for every } P \in T_xM\}.$$

For a relatively compact open set $B$ of $\perp(M)$ and an appropriate interval $I_B$ containing $s_0$, define the mapping

$$\chi: B \times I_B \to T^*N \text{ by } \chi(z,s) = \Psi^{s-s_0}z,\ z = (x,y) \in B \text{ and } s \in I_B.$$

Denote the projection $\tau_N^* \circ \chi(z,s)$ by $\psi^{s-s_0}z$. Let $p = \psi^{s_1-s_0}z_0$, then $(p,s_1)$ is a focal point of $z_0 = (x,y) \in \perp(M)$ if $T_{z_0}\psi^{s_1-s_0}$ is singular as a map from $T_{z_0}\perp(M)$ to $T_pN$. The order of the focal point at $(p,s_1)$ is equal to the number $\dim(\mathrm{Kernel}(T_{z_0}\psi^{s_1-s_0}))$. We will also refer to $s_1$ as being a focal point of $x \in M$ if $z_0$ and $p$ are clear from the context. □

Notice that we have also denoted the projection of $\Psi^s z$ for $z \in T^*N$ into $x$-space by $\psi^s z$. However, this will cause no confusion since in the context of focal points of a submanifold $M$ the restriction of $\Psi^s$ to $\perp(M)$ will always be implicit.

Remark 2.18. Geometrically a focal point in $x$-space is the limiting point of intersection with $\psi^{s-s_0}z$ of a one parameter family of trajectories $\psi^{s-s_0}z_\varepsilon$ through $\psi^{s-s_0}z$ where $z_\varepsilon \in \perp(M)$ and $z_0 = z$ (see Figure 13). Analytically we look at vector field solutions $(P(s),R(s))$ of the linearized equations of $X_F$ about $\Psi^{s-s_0}z$, $z \in \perp(M)$, which satisfy the boundary condition

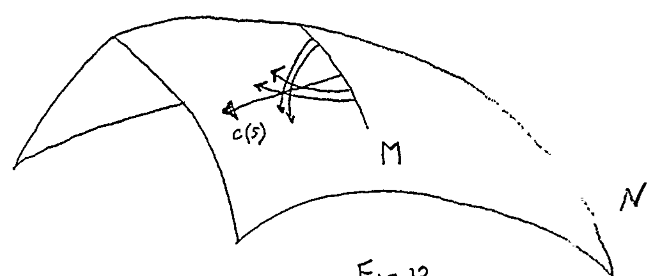



Fig 13

(2.10) $$(P(s_0),R(s_0)) \in T_z\perp(M).$$

It turns out that this boundary condition determines an $m$-dimensional subspace of solutions $J_{c,M}$ of the $2m$-dimensional vector space of solutions of the co-Jacobi equations (2.8) about $(c(s),y(s)) = \Psi^{s-s_0}z$ (see Proposition 2.20). If $\{P_1(s),\ldots,P_m(s)\}$ represents a basis of solutions for $J_{c,M}$ then $(\psi^{s_1-s_0}z, s_1)$ is a focal point of $z \in \perp(M)$ provided that

$$\det[P_1,\ldots,P_m](s_1) = 0 \qquad \text{(see Morse [Mo], p. 125).}$$

To complete our preliminary discussion of focal points we include the following standard result. The proof is an adaption of that found in Morse [Mo], p. 125 and refers to an arbitrary Hamiltonian $F(x,y)$ on $T^*N \approx TN$ . We recall the standard symplectic form $\omega = \sum_i dx_i \wedge dy_i$ on $T^*N \approx N \times \mathbb{R}^m$ ,

$$\text{(2.11)} \qquad \omega((P,R),(U,V)) = \langle P,V\rangle - \langle U,R\rangle \ .$$

Proposition 2.19. *Let $M$ be a submanifold of $N$ such that $M \cap \partial N = \emptyset$ . Let $z = (x,y) \in \perp(M)$ (see Definition 2.17). Then the focal points of $z$ along $c(s) = \psi^s z$ are isolated provided that $F_{yy}(\Psi^s z)$ is definite (either positive or negative) along $\Psi^s z$.*

*Proof.* The variational equations of $X_F = (F_y, -F_x)$ along $\Psi^s z$ are

$$\text{(2.12)} \qquad \begin{aligned} P'(s) &= F_{yx}(\Psi^s z)\cdot P(s) + F_{yy}(\Psi^s z)\cdot R(s) \\ R'(s) &= -F_{xx}(\Psi^s z)\cdot P(s) - F_{xy}(\Psi^s z)\cdot R(s). \end{aligned}$$

Since $\perp(M)$ is a Lagrangian submanifold of $T^*N$ (see Proposition 2.20 below for a proof when $M$ is a hypersurface) if

$$\left\{\begin{pmatrix} P_1 \\ R_1 \end{pmatrix},\ldots,\begin{pmatrix} P_m \\ R_m \end{pmatrix}\right\} \text{ is a basis of } T_z \perp(M)$$

then $\omega((P_i,R_i),(P_j,R_j)) = 0$ for $i,j = 1,\ldots,m$. Let $(P_i(s),R_i(s))$ be the solution of (2.12) with initial conditions

$$(P_i(s_0),R_i(s_0)) = (P_i,R_i) \qquad i = 1,\ldots,m.$$

Suppose that $\det[P_1,\ldots,P_m](s)$ vanishes on a set of points $s_j$ accumulating at $s^*<\infty$. Then by continuity $\det[P_1,\ldots,P_m](s^*) = 0$ and since $s_j \to s^*$, $\left.\frac{d}{ds}\right|_{s=s^*} \det[P_1,\ldots,P_m](s) = 0$. We may assume without loss of generality that $P_1(s^*) = 0$. Therefore, $0 = \frac{d}{ds}\det[P_1,\ldots,P_m](s^*) = \det[P_1',P_2,\ldots,P_m](s^*)$. Since $F_{yy}(\Psi^s z) \neq 0$ by assumption, if $P_1'(s^*) = 0$ then $R_1(s^*) = 0$ and consequently $P_1(s) \equiv 0$, see (2.12). Since $P_1(s_0) \neq 0$ by construction, there exists numbers $c_2,\ldots,c_m$ not all zero such that

$$P_1'(s^*) = c_2P_2(s^*) + \ldots + c_mP_m(s^*). \tag{2.13}$$

Let $P(s) = c_2P_2(s) + \ldots + c_mP_m(s)$, $R(s) = c_2R_2(s) + \ldots + c_mR_m(s)$. Then $(P(s),R(s))$ is a solution of (2.12) and $(P(s_0),R(s_0)) \in T_z{}^{\perp}(M)$. Therefore $0 = \omega\left(\begin{pmatrix}P_1(s)\\R_1(s)\end{pmatrix},\begin{pmatrix}P(s)\\R(s)\end{pmatrix}\right)$ (simply differentiate the right hand side of this equation in $s$ using (2.12) and the fact that $\omega\left(\begin{pmatrix}P_1(0)\\R_1(0)\end{pmatrix},\begin{pmatrix}P(0)\\R(0)\end{pmatrix}\right) = 0$). However

$P_1'(s^*) = F_{yx}\cdot P_1(s^*) + F_{yy}\cdot R_1(s^*) = F_{yy}\cdot R_1(s^*)$ since $P_1(s^*) = 0$.

$$\begin{aligned}\text{Therefore } 0 = \omega\left(\begin{pmatrix}P_1(s^*)\\R_1(s^*)\end{pmatrix},\begin{pmatrix}P(s^*)\\R(s^*)\end{pmatrix}\right) &= -\langle P(s^*),R_1(s^*)\rangle \text{ (from (2.11))}\\ &= -\langle P(s^*),F_{yy}^{-1}\cdot P_1'(s^*)\rangle \ (F_{yy} \neq 0)\\ &= -\langle P_1'(s^*),F_{yy}^{-1}\cdot P_1'(s^*)\rangle.\end{aligned}$$

Since $F_{yy}^{-1}$ is also definite, this contradicts our previous observation that $P_1'(s^*) \neq 0$. □

Let $K,L$ be hypersurfaces of $\mathbb{R}^m \cap N^0$ (see (W1)). The index theorem of Ambrose [A] applies to a critical arc $c$ of the functional $J$ with boundary conditions $K \times L$. We are interested in extending the index theorem so that it applies to our boundary conditions $M^0 \times M^1$. To carry out this extension, and in order to generalize the notions of focal point and convexity (see Ambrose [A]), we must study the co-Jacobi equations (2.8) along a critical arc $c$. We will find that solutions of the co-Jacobi equations may be continued to the boundaries and indeed beyond in a $C^1$ way (see Proposition 2.41). The key to this result lies in the connection between solutions of the co-Jacobi equations along $c(s)$ and solutions of the linearized Hamiltonian equations of $X_H$ (see (1.6)) along the corresponding periodic orbit $\sigma(t) = (\pi(t), \lambda(t))$ of $X_H$ on $H^{-1}(h)$ (see Lemma 1.6 and Remark 1.13).

To begin our study of the co-Jacobi equations and in light of the discussion in Remark 2.18 (see (2.10)) we present the following result which in part characterizes the tangent space of $\perp(M)$ at $z = (x,y) \in \perp(M)$. In the following results we will identify $(x,y) \in T^*N$ with $(x,y) \in TN$ as usual on $\mathbb{R}^{2m}$. If $M$ is an oriented hypersurface in $N^0$ we denote the Weingarten map of $M$ at $(x,y) \in \perp(M)$ by $\Xi$: $T_xM \to T_xM \subset T_xN$ (see Hicks [Hi], p. 21, and also Remark 2.12 for notation concerning the standard connection $D$).

Proposition 2.20. *Let $M$ be an oriented hypersurface in $N^0$. Recall that $F(x,y) = \frac{1}{2}|y|^2 \cdot [2(h-W(x))]^{-1}$. Let $z = (x,y) \in \perp(M)$, $y \neq 0$, and denote the Weingarten map of $M$ at $(x,y)$ by $\Xi$. Then $\perp(M)$ is a Lagrangian submanifold of $T^*N$, and*

(2.14) $(P,R) \in T_z\perp(M) \Leftrightarrow P \in T_xM$, *and* $R-|y|\cdot\Xi(P) \perp T_xM$ (compare (2.7)b).

*Moreover, we may choose a basis $\{(P_i,R_i)\}_{i=1}^{m}$ of $T_z\perp(M)$, so that $dF(P_i,R_i) = 0$, $i = 1,\dots,m-1$ and $dF(P_m,R_m) \neq 0$.*

*Proof.* Let $z_\varepsilon = (x_\varepsilon, y_\varepsilon)$ be a curve in $\perp(M)$ through $z = z_0$ with $\left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0} z_\varepsilon = (P,R)$. Let $\{e_\varepsilon^i\}_{i=1}^{m-1}$ be a basis of tangent vectors for $T_{x_\varepsilon}M$. Then

$$\langle y_\varepsilon, e_\varepsilon^i\rangle = 0, i = 1,\ldots,m-1, \tag{2.15}$$

and we may assume that $y_\varepsilon \neq 0$ for $\varepsilon$ sufficiently small since $y \neq 0$ by assumption. Let $\{e^i\}_{i=1}^{m-1}$ be vector fields on a neighbourhood within $N^0$ of $x \in N^0$, compatible with $\{e_\varepsilon^i\}_{i=1}^{m-1}$; that is $e^i(x_\varepsilon) = e_\varepsilon^i$. Let $n(q)$ denote the unit normal at $q \in M$ compatible with $y/|y|$, that is $n(x) = y/|y|$. Then (see Remark 2.12 for notation) $\left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0} n(x_\varepsilon) = D_P n = \Xi(P)$ (see Hicks [Hi], pp. 19, 21). Furthermore, $\left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0} e_\varepsilon^i = D_P\, e^i$ $i = 1,\ldots,m-1$. By the Gauss equation, see Hicks [Hi], p. 26, $\langle y, D_P\, e^i\rangle = -|y|\ \langle \Xi(P), e_0^i\rangle, i = 1,\ldots,m-1$. Therefore, from (2.15)

$$\begin{aligned} 0 = \left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0} \langle y_\varepsilon, e_\varepsilon^i\rangle &= \langle R, e_0^i\rangle + \langle y, D_P e^i\rangle \\ &= \langle R, e_0^i\rangle - |y|\ \langle \Xi(P), e_0^i\rangle \\ &= \langle R - |y|\ \Xi(P), e_0^i\rangle, i = 1,\ldots,m-1, \end{aligned}$$

that is, $R - |y|\ \Xi(P) \perp T_x M$. Moreover, $P \in T_x M$ since $x_\varepsilon \in M$ and $P = \left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0} x_\varepsilon$.

If $(P_1,R_1)$, $(P_2,R_2) \in T_z \perp(M)$, by virtue of (2.14), it follows that

$$\begin{aligned} \omega((P_1,R_1), (P_2,R_2)) &= \langle P_1, R_2\rangle - \langle P_2, R_1\rangle \quad \text{(see (2.11))} \\ &= |y|\ \langle P_1, \Xi(P_2)\rangle - |y|\ \langle P_2, \Xi(P_1)\rangle \qquad (2.16) \\ &= 0 \quad \text{(since } \Xi \text{ is symmetric, see} \end{aligned}$$

Hicks [Hi], p. 23).

We will now show $\dim T_z \perp(M) \geq m$ and conclude that $\dim(T_z \perp(M)) = m$ since (2.16) shows that $\omega|T_z \perp(M) = 0$, see [A & M], p. 404.

Let $x_\varepsilon^i$ be curves in $M$ through $x$ which satisfy

$$\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} x_\varepsilon^i = e_0^i, \quad i = 1,\dots,m-1.$$

Choose $y_\varepsilon^i$ so that $(x_\varepsilon^i, y_\varepsilon^i) \in \perp(M)$, $y_0^i = y$ and

$$F(x_\varepsilon^i, y_\varepsilon^i) = \tfrac{1}{2}|y_\varepsilon^i|^2/2(h-W(x_\varepsilon^i)) = F(x,y), \quad i = 1,\dots,m-1.$$

Let $(P_i, R_i) = \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} (x_\varepsilon^i, y_\varepsilon^i) \in T_z \perp(M)$; then $dF(P_i,R_i) = 0$ and $(P_i,R_i)$, $i = 1,\dots,m-1$ are linearly independent since the $P_i$ are. Let $(P_m,R_m) = (0,y) = \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} (x, (1+\varepsilon)y) \in T_z \perp(M)$.
Then, $dF(P_m,R_m) = \langle \frac{\partial F}{\partial y}(x,y), y \rangle = \frac{|y|^2}{2(h-W(x))} \neq 0$ and $(P_m,R_m)$ is linearly independent of $(P_i,R_i)$, $i = 1,\dots,m-1$. □

<u>Corollary 2.21</u>. *Let $h$ be a regular value of the Hamiltonian $H(x,y) = \tfrac{1}{2}|y|^2 + W(x)$, that is, $dH \neq 0$ on $H^{-1}(h)$. Let $M$ be a hypersurface as specified in Proposition 2.20. Then $\perp^h(M) = \perp(M) \cap H^{-1}(h)$ is a submanifold of $T^*N$ with dimension $m-1$. For $z \in \perp^h(M)$, the vectors $\{(P_i,R_i)\}_{i=1}^{m-1}$ as constructed in Proposition 2.20 form a basis for $T_z \perp^h(M)$.*

*Proof.* $\perp^h(M) = \tilde{H}^{-1}(h)$ where $\tilde{H} = H|\perp(M)$. Therefore $\perp^h(M)$ is a submanifold of $T^*N$ provided we can show that $h$ is a regular value of $\tilde{H}$. However if $(x,y) \in \tilde{H}^{-1}(h)$ then $y \neq 0$ since $x \notin \partial N$ and $dH_{(x,y)}(0,y) = |y|^2 \neq 0$. The dimension of $\perp^h(M)$ is $m-1$ since the codimension of $\tilde{H}^{-1}(h)$ in $\perp(M)$ is 1. To verify the last statement concerning the vectors $(P_i,R_i) \in T_z\perp(M)$, $i = 1,\dots,m-1$, we need only verify that $(P_i,R_i) \in T_z \perp^h(M)$, but this follows since $0 = dF(P_i,R_i) = 2(h-W(x))^{-1} \cdot dH(P_i,R_i)$. □

Recall that the base integral curves $c(s)$ of $X_F$ (see (1.30)) through $z \in T^*N$ will also be denoted by

$$c(s) = \psi^{s-s_0}z \text{ , where } (c(s_0),y(s_0)) = z. \tag{2.17}$$

Let $M$ be a hypersurface in $\mathbb{R}^m \cap N^0$. We will now begin to collect some results concerning the "wavefront set of $M$",

$$\psi^{s-s_0}(\perp^h(M)) \text{ ,} \tag{2.18}$$

and the related notion of orthogonal Jacobi fields (see Definition 2.28).

<u>Lemma 2.22</u>. *Let $M$ be a hypersurface as in Proposition 2.20 and let $z = (x,y) \in \perp^h(M)$ (see Corollary 2.21). If $(\psi^{s_1-s_0}z,s_1)$ is not a focal point of $x \in M$ then there is a neighbourhood $B \subset \perp^h(M)$ of $z \in \perp^h(M)$ such that*

$$\psi^{s_1-s_0}(B) \textit{ is a hypersurface in } \mathbb{R}^m \cap N^0.$$

*Proof.* If $T_z\, \psi^{s_1-s_0}|\, T_z \perp^h(M)$ is nonsingular (see Definition 2.17) then there is a neighbourhood $B$ of $z$ in $\perp^h(M)$ such that $\psi^{s_1-s_0}|B$ is locally a diffeomorphism onto its image $\psi^{s_1-s_0}(B)$. The result follows. □

Notice that the tangent space of $\psi^{s_1-s_0} \perp^h(M)$ at $\psi^{s_1-s_0}z$ is spanned by

$$P_i(s_1) = T_z\, \psi^{s_1-s_0}(P_i,R_i),\ i = 1,\ldots,m-1 \tag{2.19}$$

where $(P_i,R_i)$ are a basis of $T_z \perp^h(M)$ as specified in Corollary 2.21.

It is convenient at this time to amplify the observation (2.19) and introduce Jacobi fields along the base integral curve $c(s)$ of $X_F$ on $F^{-1}(\frac{1}{2})$. We recall that $H^{-1}(h) = F^{-1}(\frac{1}{2}) \cup \{(x,0) \mid x \in \partial N\}$ (see Remark 1.13).

<u>Definition 2.23</u>. Let $(x_\varepsilon, y_\varepsilon) = z_\varepsilon$ be a curve through $z_0$ in $H^{-1}(h)$, $x_\varepsilon \notin \partial N$ with

$$(2.20) \qquad (P,R) = \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} z_\varepsilon \in T_{z_0} H^{-1}(h).$$

For $(P,R)$ as in (2.20) we define the Jacobi field $P(s)$ along $c(s) = \psi^{s-s_0} z_0$ by (see (2.19))

$$(2.21) \qquad \begin{aligned} P(s) &= T_{z_0} \psi^{s-s_0} \cdot (P,R) \\ &= \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} \psi^{s-s_0} z_\varepsilon \ . \end{aligned}$$

□

<u>Proposition 2.24</u>. *Let $P(s)$ be a Jacobi field along $c(s) = \psi^{s-s_0} z_0$. Let*

$$(c,y)(s) = L(c,c')(s), \qquad \textit{and}$$

$$(P,R)(s) = L_*(c,c') \cdot (P,P')(s) \ .$$

*Then $(P,R)(s)$ is a solution of the co-Jacobi equations (2.8) along $(c,y)(s)$ and $P \in H$ (see Proposition 2.14 (b)). Moreover,*

$$(2.22) \qquad \langle DW \circ c, P\rangle(s) + \langle y, R\rangle(s) \equiv 0 \ .$$

*Proof.* Let $c_\varepsilon(s) = \psi^{s-s_0} z_\varepsilon$ be the base integral curve (see (2.17)) of the orbit $\Psi^{s-s_0} z_\varepsilon$ of $X_F$ (see (1.30)) through $z_\varepsilon \in F^{-1}(\frac{1}{2}) \subset H^{-1}(h)$. Let

$$P(s) = T_{z_0} \psi^{s-s_0} \cdot (P,R)(s_0) \qquad \text{(see (2.21))}.$$

Then

$$(c_\varepsilon, y_\varepsilon)(s) = L(c_\varepsilon, c_\varepsilon')(s) = \Psi^{s-s_0} z_\varepsilon$$

and

(2.23) $\quad (P,R)(s) = L_*(c,c')\cdot(P,P')(s) = T_{z_0}\Psi^{s-s_0}\cdot(P,R)(s_0)$ .

The linear mapping $T_{z_0}\Psi^{s-s_0}\colon T_{z_0}T^*N \to T(T^*N)$ is a fundamental (matrix) solution of the co-Jacobi equations (2.8) (see also (2.12)), see Hartman [Ha], pp. 95, 252. In particular, $(P,R)(s)$ is a solution of the co-Jacobi equations along $(c(s),y(s))$ (see (2.23)).

Now $T_{z_0}\Psi^{s-s_0}$ leaves $T(H^{-1}(h))$ invariant. Indeed

$$T_{z_0}\Psi^{s-s_0}\cdot X_F(z_0) = X_F(\Psi^{s-s_0}z_0) \qquad \text{and}$$

$$(\Psi^{s-s_0})^*\omega = \omega$$

(that is $\Psi^{s-s_0}$ is a symplectic transformation, see (2.11) and [A & M], p. 188). Therefore (see [A & M], p. 187),

$$\begin{aligned} dF_{(c,y)(s)}\,(P,R)(s) &= \omega(X_F(\Psi^{s-s_0}z_0),\ (P,R)(s)) \\ &= (\Psi^{s-s_0})^*\omega(X_F(z_0),\ (P,R)(s_0)) \\ &= dF_{z_0}(P,R)(s_0) = 0 \quad \text{(see (2.20)).} \end{aligned}$$

Consequently, $(P,R)(s) \in T_{(c,y)(s)}\,H^{-1}(h)$ and $P \in \mathcal{H}$. Furthermore, $0 = dH_{(c,y)(s)}(P,R)(s) = \langle DW\circ c,P\rangle(s) + \langle y,R\rangle(s)$. □

Remark 2.25. The usual definition for Jacobi fields is that $P(s)$ is a Jacobi field along $c(s)$ if $P(s)$ satisfies the Jacobi differential equations along $c(s)$ (equivalent to the co-Jacobi equations via $L_*(c,c')$), see Hestenes [He], p. 123. However in light of Proposition 2.14, we are mainly concerned with those Jacobi fields $P \in \mathcal{H}$, hence our Definition 2.23.

Note that (2.22) allows us to simplify equation (2.8)(b): for a Jacobi field $P$ along $c(s)$,

$$\text{(a)} \quad P'(s) = \frac{2\langle DWoc,P\rangle(s)}{[2(h-Woc(s))]^2}\, y(s) + \frac{R(s)}{2(h-Woc(s))}$$

(2.24)

$$\text{(b)} \quad R'(s) = \frac{-(D^2Woc)\cdot P(s)}{2(h-Woc(s))} - \frac{2\langle DWoc,P\rangle(s)}{[2(h-Woc(s))]^2}\, DWoc(s).$$

Lemma 2.26. *(Gauss) Let $P(s)$ be a Jacobi field (see Definition 2.23) along $c(s)$ such that $\langle P(s_0), y(s_0)\rangle = 0$ where*

$$(c,y)(s) = L(c,c')(s) \ .$$

*Then*

$$\langle P(s), y(s)\rangle \equiv 0$$

*for all $s$ where $X_F(c,y)(s)$ is defined (see (1.30)).*

*Proof.* $\frac{d}{ds}\langle P(s),y(s)\rangle = \langle P',y\rangle(s) - \frac{\langle P,DWoc\rangle(s)}{2(h-Woc(s))}$ (see (1.18))

$$= \frac{2\langle P,DWoc\rangle(s)}{2(h-Woc(s))} + \frac{\langle R,y\rangle(s)}{2(h-Woc(s))} - \frac{\langle P,DWoc\rangle(s)}{2(h-Woc(s))}$$

$= 0$ (see (2.24) and (2.22)). □

Remark 2.27. Let $\psi^{s-s_0}(\perp^h(M))$ be the wavefront set of $M\subset N^0$, and let

$$z = (x,y) \in \perp^h(M).$$

Let $(c,y)(s) = \Psi^{s-s_0}z$ and assume that $(\psi^{s-s_0}z,s)$ is not a focal point of $z \in \perp^h(M)$ (see Definition 2.17). Select a relatively compact neighbourhood $B_z$ of $z$ in $\perp^h(M)$ satisfying the conclusions of Lemma 2.22. It follows from (2.18) and Lemma 2.26 that

(2.25) $$(c,y)(s) \in \perp^h(\psi^{s-s_0}(B_z)).$$

Indeed, $c(s) \in \psi^{s-s_0}\perp^h(M)$, and $y(s) \perp T_{c(s)}\psi^{s-s_0}\perp^h(M)$, since

(2.14) and Lemma 2.26 imply that

$$\langle y(s), P_i(s)\rangle = 0 \text{ for } i = 1,\ldots,m-1,$$

where $P_i(s)$ and $P_i(s_0)$ are given by (2.19).

We will refer to (2.25) with the expression "the wavefront set of $M$ remains orthogonal to the trajectory $c(s)$". The focal points of $z \in \perp(M)$ along $c(s)$ correspond to points $(c(s),s)$ where a singularity develops in the wavefront set (see Figure 14).

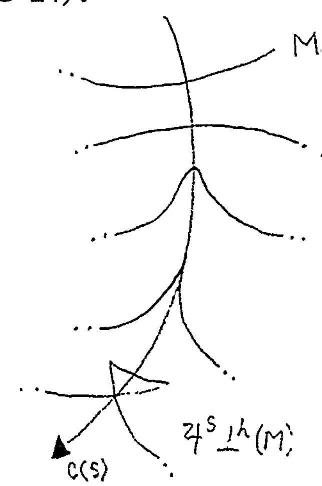



Fig 14

Definition 2.28. An orthogonal Jacobi field $P(s)$ along $c(s)$ (see (2.17)) is a Jacobi field (see Definition 2.23) such that

$$\langle P(s_0), y(s_0)\rangle = 0$$

for some $s_0$.

Definition 2.29. Let $M$ be a hypersurface of $\mathbb{R}^m \cap N^0$ as specified in Proposition 2.20. Let $(c(s),y(s)) = \Psi^{s-s_0}z$ with $z \in \perp^h(M)$. An $M$-orthogonal Jacobi field $P(s)$ along $c(s)$ is a Jacobi field with initial conditions

$$(P,R) \in T_z \perp^h(M)$$

(see (2.20) and (2.21)). Note that by virtue of (2.14), the conclusion of Lemma 2.26 is in force, and $P(s)$ is an orthogonal Jacobi field. □

Remark 2.30. By virtue of Proposition 2.24, the space of Jacobi fields $P(s)$ along $c(s)$ is isomorphic to the vector space of initial conditions $(P,R)(s_0)$. This latter space has dimension $(2m-1)$ (see (2.20)).

The space of orthogonal Jacobi fields $P(s)$ along $c(s)$ is $2m-2$ dimensional since

$$\{(P,R)(s_0) \mid \langle P,y\rangle(s_0) = 0\} \cap T_{(c,y)(s_0)}H^{-1}(h)$$

is a $2m-2$ dimensional vector space. Moreover, the space of $M$-orthogonal Jacobi fields (see Definition 2.29) is $m-1$ dimensional by virtue of Corollary 2.21.

As we shall see shortly, the space of orthogonal Jacobi fields along $c(s)$ is a symplectic space. The space of $M$-orthogonal Jacobi fields along $c(s)$ is a Lagrangian subspace of the space of orthogonal Jacobi fields along $c(s)$.

Remark 2.31. (a) Up until now we have considered Jacobi fields along a base integral curve $c(s)$ of $X_F$ (see (2.17)) on $F^{-1}(\frac{1}{2})$ for some parameter interval $A$ containing $s_0$, such that

$$\{c(s) \mid s \in A\} \cap \partial N = \emptyset \ .$$

We will now want to consider the special case where $c(s)$ is a critical arc of the functional $J$ with boundary conditions $M^0 \times M^1$ (see Definition 2.6). In particular

$$c(i) \in M^i \subset \partial N, \text{ for } i = 0,1 \ .$$

In this case, for $0 < s_0 < 1$ and $A = (0,1)$ our previous results in this chapter hold. We would like to extend these results to include $\overline{A} = [0,1]$. It turns out that the orthogonal Jacobi fields along $c(s)$ (see Definition 2.28) have a $C^1$ extension to the entire real line $(-\infty,\infty)$ (see Proposition 2.41). However, the linear vector field (2.24) blows up at the boundary where $W(x) = h$. We must therefore reparameterize the Jacobi fields in order to remove these singularities. The original time parameter $t$ (see (2.3) and the discussion following this equation) will be used to reparameterize the moving wavefront sets $\psi^s(B)$ (see Lemma 2.22) and the orthogonal Jacobi fields $P(s)$ along $c(s)$ which span the tangent space of $\psi^s(B)$ at $c(s)$ (see (2.19)). This will allow us to consider the wavefront sets associated with (subsets with compact closure of) the boundary branches $W_h^i$ ($i = 0,1$).

(b) Let $c(s)$ be a critical arc of the functional $J$ subject to the boundary conditions $M^0 \times M^1$ (see Definition 2.6). By our results in Proposition 2.7 and especially the discussion following (2.3), we make the following observations. Let

$$(c,y)(s) = L(c,c')(s), \text{ for } s \in [0,1], \text{ and}$$

let $\sigma(t) = (\pi,\lambda)(t)$ be the associated integral curve of $X_H$ on $H^{-1}(h)$ (see (2.3)):

$$(c,y)(s(t)) = (\pi,\lambda)(t) \qquad \text{with}$$

$$s(t) = \int_0^t 2(h - W\circ\pi(t'))\,dt' \,. \tag{2.26}$$

Then there is a smallest $T > 0$ such that

$$\pi(0) \in M^0 \text{ and } \pi(T) \in M^1$$

and we may assume that $1 = \int_0^T 2(h-W\circ\pi(t'))dt'$ (see Remark 1.13). If $(P(s),R(s))$ is the solution of (2.24) (a) and (b) associated with an orthogonal Jacobi field $P(s)$ with initial conditions $(P,R)$, i.e. $(P(s_0),R(s_0)) = (P,R)$ for some $0 < s_0 < 1$ (see (2.20)), we consider the reparameterized vector fields along $\pi(t)$

(2.27) $\quad \{U(t),V(t)\} = \{P(s(t)),R(s(t))\} \quad (0 < t < T)$

where $s(t)$ is given in (2.26). □

Lemma 2.32. *Given $(U,V)(t)$ as specified in (2.27), for $t \in (0,T)$*

(2.28)
$$\dot{U}(t) = \frac{d}{dt}U(t) = \frac{2\langle DW(\pi(t)),U(t)\rangle}{2(h-W(\pi(t)))}\lambda(t) + V(t)$$
$$\dot{V}(t) = \frac{d}{dt}V(t) = -D^2W(\pi(t))\cdot U(t) - \frac{2\langle DW(\pi(t)),U(t)\rangle}{2(h-W(\pi(t)))}DW(\pi(t))$$

*Proof.* Immediate from (2.27), (2.26) and (2.24). □

Definition 2.33. Let $P(s)$ be an orthogonal Jacobi field along $c(s)$ and let $U(t) = P(s(t))$, $s(t)$ as specified in (2.26). We will refer to such a vector field $U(t)$ along $\pi(t)$ as a (reparameterized) orthogonal Jacobi field along $\pi(t)$ and denote the $(2m-2)$-dimensional vector space of such fields by $J_\pi^\perp$ (see Remark 2.30).

Our immediate task is to show that (reparameterized) orthogonal Jacobi fields are not restricted to the interval $(0,T)$ (see (2.27)) but in fact have well defined extensions to the entire real line. To carry out this extension, we must introduce the linearized equations of $X_H$ along $(\pi,\lambda)(t)$.

Remark 2.34. We recall some basic facts. Let $z = (x,y) \in H^{-1}(h)$ and $X_H(x,y) = (y,-DW(x))$. The linearized equations of $X_H$ about $\Phi^t z$ are

(2.29) $\quad \dot{\xi}(t) = \eta(t),\ \dot{\eta}(t) = -D^2W(\varphi^t z)\cdot\xi(t).$

The solutions of (2.29) are defined on $(-\infty,\infty)$ provided that $\Phi^t z$ is (see Hartman [Ha], p. 45). The linearized flow mapping $T_z\,\Phi^t$ is a fundamental (matrix) solution of (2.29) (Hartman [Ha], pp. 95, 252). Furthermore, $T_z\Phi^t$ leaves $T(H^{-1}(h))$ invariant (see Proposition 2.24, in particular the proof that $P \in H$ if $P$ is a Jacobi field). □

Remark 2.35. At a point $z = (x,0) \in H^{-1}(h)$, $x \in \partial N$, the co-geodesic flow $\Psi^s$ is not properly defined since $X_F(z)$ is not defined (see (1.30)). Nevertheless we have been able to make sense of the expression

$$\Psi^{s(t)}(x,0) = \Phi^t(x,0)$$

(see (2.3) where Lemma 1.6 was adapted to cover this case). In order to give an extension of $U \in J_\pi^\perp$ to $(-\infty,\infty)$, we will introduce a "time function" $g$ (see (2.31) below) which will, among other things, allow us to properly use the expression $\Psi^s z$, $z = (x,0) \in H^{-1}(h)$, $x \in \partial N$.

For Proposition 2.36 we will temporarily relax our convention of associating $s = 0$ along a critical arc $c(s)$ with a boundary point $x \in \partial N$.

By Remark 1.13 and (2.3) (see above), the flow $\Phi^t$ (of $X_H$) and the co-geodesic flow $\Psi^s z$ (see (1.5)) are related by

$$\Psi^{s(t,z)}z = \Phi^t z, \quad \text{where}$$

$$s(t,z) = \int_0^t 2(h - W\circ\varphi^{t'}z)\,dt' , \tag{2.30}$$

(recall that $\varphi^t z$ denotes the base integral curve for $\Phi^t z$). Our next result establishes the existence and smoothness properties of the inverse function $g(s,z)$,

$$(2.31) \qquad s = \int_0^{g(s,z)} 2(h-W\circ\varphi^{t'}z)\,dt' \ .$$

Proposition 2.36. *Let $z = (x,y) \in H^{-1}(h)$. The function $g(s,z) : \mathbb{R} \times H^{-1}(h) \to \mathbb{R}$ in (2.31) is well defined and continuous in $(s,z)$. Furthermore, $g$ is $C^3$ in a neighbourhood of $(s,z)$ provided that $2(h-W\circ\varphi^{g(s,z)}z) \neq 0$. If $2(h-W\circ\varphi^{g(s_0,z_0)}z_0) = 0$ for some $s_0$ and $z_0 = (x_0,y_0)$, then the following limits of directional derivatives of $g$ in $z$ exist and have the stated value. Let*

$$(2.32) \qquad \begin{gathered} t_0 = -g(s_0,z_0),\ (x(t),\lambda(t)) = \Phi^{t-t_0}z_0 \text{ and} \\ (\xi,\eta)(t) = T_{z_0}\Phi^{t-t_0}\cdot(\xi_0,\eta_0), \\ \text{for } \zeta = (\xi_0,\eta_0) \in T_{z_0}H^{-1}(h) \qquad (\text{see } (2.29)). \end{gathered}$$

*Then provided that $\langle \xi_0, y_0\rangle = 0$,*

$$\lim_{s\to s_0} \langle \frac{\partial g}{\partial z}(s,z_0),\zeta\rangle = \frac{\langle DW(x(0)),\eta(0)\rangle}{|DW(x(0))|^2}$$

*( $\langle\cdot,\cdot\rangle$ will also denote the inner product in $\mathbb{R}^{2m}$).*

*Proof.* To show the existence of $g(s,z)$ satisfying (2.31), we observe that for fixed $z \in H^{-1}(h)$, $t \to s(t,z)$ (see (2.30)) is one to one. Indeed, if $2(h-W\circ\varphi^t z) \neq 0$, then $\frac{\partial s}{\partial t}(t,z) > 0$; while if $2(h-W\circ\varphi^t z) = 0$, then $\frac{\partial^3 s}{\partial t^3}(t,z) > 0$ and $\frac{\partial^2 s}{\partial t^2}(t,z) = \frac{\partial s}{\partial t}(t,z) = 0$. Therefore for each $z \in H^{-1}(h)$, an inverse $g(s,z)$ of $s(t,z)$ exists. Continuity in $(s,z)$ now follows from properties of $s(t,z)$. Indeed,

$$s(g(s,z),z) = s,$$

$s(\cdot,\cdot)$ is continuous in $(t,z)$ and $s(\cdot,z)$ is one to one. Consequently

$g(s,z)$ is continuous.

Consider the $C^3$ function

$$r(t,s,z) = s - \int_0^t 2(h-Wo\varphi^{t'}z)\, dt'.$$

Then $r(g(s,z),s,z) = 0$. By the implicit function theorem $g$ is $C^3$ is $(s,z)$ provided that $0 \neq \frac{\partial r}{\partial t}(g(s,z),s,z) = 2(h-Wo\varphi^{g(s,z)}z)$. Now let $2(h-Wo\varphi^{g(s_0,z_0)}z_0) = 0$, then $2(h-Wox(0)) = 0$ (see (2.32)). Since $\left.\frac{d^2}{dt^2}\right|_{t=0} 2(h-Wox(t)) > 0$, there is a neighbourhood $G$ of 0 so that $t \in G\backslash\{0\}$ implies that $2(h-Wox(t)) > 0$. Therefore, for $t \in G\backslash\{0\}$, $t-t_0 = g(s(t-t_0,z_0),z_0)$ and

$$0 = \frac{\partial g}{\partial s}(s(t-t_0,z_0),z_0)\cdot\frac{\partial s}{\partial z}(t-t_0,z_0) + \frac{\partial g}{\partial z}(s(t-t_0,z_0),z_0).$$

Since $\frac{\partial g}{\partial s}(s(t-t_0,z_0),z_0) = [2(h-Wox(t))]^{-1}$, for $\zeta \in T_{z_0} H^{-1}(h)$,

$$\langle\frac{\partial g}{\partial z}(s(t-t_0,z_0),z_0),\zeta\rangle = -[2(h-Wox(t))]^{-1}\cdot\langle\frac{\partial s}{\partial z}(t-t_0,z_0),\zeta\rangle .$$

By virtue of the fact that

$$\langle\frac{\partial s}{\partial z}(t-t_0,z_0),\zeta\rangle = -\int_0^{t-t_0} 2\langle DW(\varphi^{t'}z_0),T_{z_0}\varphi^{t'}\cdot\zeta\rangle\, dt',$$

$$= -\int_{t_0}^{t} 2\langle DW(\varphi^{t'-t_0}z_0),T_{z_0}\varphi^{t'-t_0}\cdot\zeta\rangle\, dt',$$

where $T_{z_0}\varphi^{t'-t_0}\cdot\zeta = \xi(t')$ (see (2.32)), it follows that

$$(2.33)\quad \langle\frac{\partial g}{\partial z}(s(t-t_0,z_0),z_0),\zeta\rangle = \frac{2\int_{t_0}^{t}\langle DWox,\xi\rangle(t')\, dt'}{2(h-Wox(t))} .$$

The following lemma will be proven at the conclusion of Proposition 2.36.

Lemma 2.37. *Let* $(s_0, z_0) \in \mathbb{R} \times H^{-1}(h)$ *be as specified in Proposition 2.36. Then using the notation of (2.32), and assuming that* $\langle \xi_0, y_0 \rangle = 0$ *, we have*

$$\int_{t_0}^{0} \langle DW(s(t')), \xi(t') \rangle dt' = 0. \qquad \square$$

To conclude our proof of Proposition 2.36, notice that

$$(2.34) \qquad \lambda(0) = 0 \quad (\text{see } (2.32)) \ .$$

Since $(\xi, \eta)(0) \in T_{(x,\lambda)(0)} H^{-1}(h)$ (see Remark 2.34), from (2.34) we have

$$(2.35) \qquad 0 = dH_{(x,\lambda)(0)}(\xi(0), \eta(0)) = \langle DW(x(0)), \xi(0) \rangle \ .$$

Therefore, from (2.33) and Lemma 2.37, by applying L'Hospital's rule (twice) we obtain

$$\lim_{t \to 0} \langle \frac{\partial g}{\partial z}(s(t-t_0, z_0), z_0), \zeta \rangle = \lim_{t \to 0} \frac{\langle DW \circ x, \xi \rangle (t)}{-\langle DW \circ x, \lambda \rangle (t)} \quad (\text{see } (2.35))$$

$$= \lim_{t \to 0} \frac{\langle (D^2 W \circ x) \cdot \lambda, \xi \rangle (t) + \langle DW \circ x, \eta \rangle (t)}{-\langle (D^2 W \circ x) \cdot \lambda, \lambda \rangle (t) + |DW(x(t))|^2}$$

$$= \langle DW(x(0)), \eta(0) \rangle \cdot |DW(x(0))|^{-2} \quad (\text{see } (2.34)).$$

The conclusion of Proposition 2.36 now follows since

$$s(-t_0, z_0) = s_0 \quad \text{and}$$

$$t \to s(t - t_0, z_0) \quad \text{is one to one.} \qquad \square$$

*Proof of Lemma 2.37.* Using our notation in (2.32),

$$\langle \lambda(t), \xi(t) \rangle = \langle y_0, \xi_0 \rangle + \int_{t_0}^{t} \{ \langle -DW(x(t')), \xi(t') \rangle + \langle \lambda(t'), \eta(t') \rangle \} dt'$$

However, $(\xi(t), \eta(t)) \in T_{(x,\lambda)(t)} H^{-1}(h)$ so that

$$0 = dH(\xi(t),\eta(t)) = \langle DW(x(t)),\xi(t)\rangle + \langle\lambda(t),\eta(t)\rangle \ .$$

Therefore, since $\langle y_0,\xi_0\rangle = 0$ by hypothesis,

$$\langle\lambda(t),\xi(t)\rangle = -2\int_{t_0}^{t} \langle DW(x(t')),\xi(t')\rangle dt' \ . \tag{2.36}$$

In particular $2\int_{t_0}^{0} \langle DW\circ x,\xi\rangle dt' = 0$ (see (2.34)). □

Corollary 2.38. *Let $c(s)$ be a critical arc of the functional $J$ subject to the boundary conditions $M^0 \times M^1$. Let*

$$z_0 = L(c,c')(s_0) \quad \text{for some} \quad 0 < s_0 < 1.$$

*Then for $s \in (-\infty, \infty)$*

$$\Psi^{s-s_0}z_0 = \Phi^{g(s-s_0,z_0)}z_0 \tag{2.37}$$

*provides a continuous extension of $(c,y)(s) = L(c,c')(s)$ to $(-\infty,\infty)$. For $z_0 = (x,y)$, $\zeta = (\xi_0,\eta_0) \in T_{z_0}H^{-1}(h)$, the directional derivative of (2.37)*

$$T_{z_0}\Psi^{s-s_0}\cdot\zeta$$

*is defined and continuous for $s \in (-\infty,\infty)$ provided that*

$$\langle\xi_0,y\rangle = 0 \ .$$

*Proof.* The conclusions all follow from Proposition 2.36 once we verify that for $0 < s < 1$,

$$(c,y)(s) = \Phi^{g(s-s_0,z_0)}z_0.$$

However, as we have seen in (2.3),

$$(c,y)(s(t)) = (\pi,\lambda)(t) = \Phi^{t-t_0}z_0$$

where $0 < t_0 < T$ satisfies $s_0 = s(t_0)$ (see (2.26)). Now (see (2.30)), $t-t_0 = g(s(t-t_0,z_0),z_0) = g(s(t)-s(t_0),z_0) = g(s(t)-s_0,z_0)$, since $s(t-t_0,z_0) = s(t) - s(t_0)$. Therefore, for $0 < t < T$,

$$(c,y)(s(t)) = \Phi^{g(s(t)-s_0,z_0)} z_0.$$

This is exactly what we wanted to show since $0 < s(t) < 1$ for $0 < t < T$. □

Theorem 2.39. *Let $c(s)$ be a critical arc of the functional $J$ subject to the boundary conditions $M^0 \times M^1$. Let*

$$(c,y)(s) = L(c,c')(s) \text{ , and}$$
$$(c,y)(s(t)) = (\pi,\lambda)(t) \quad \text{(see 2.26))}$$

*with $\pi(0)$, $\pi(T) \in \partial N$ as in (2.3) (see also Remark 2.31 (b)). For $0 < t_0 < T$, let $z_0 = (\pi,\lambda)(t_0)$ and let $\zeta = (\xi_0,\eta_0) \in T_{z_0} H^{-1}(h)$ with $\langle \xi_0, \lambda(t_0)\rangle = 0$. Let $P(s)$, $0 < s < 1$ , be the Jacobi field along $c(s)$ so that,*

$$P(s) = T_{z_0} \psi^{s-s(t_0)}\cdot\zeta \quad \text{(see (2.21)).}$$

*Let $(U,V)(t) = (P,R)(s(t))$ (see (2.27)), and*

$$(\xi,\eta)(t) = T_{z_0} \Phi^{t-t_0}\cdot\zeta \text{ (see (2.29)).}$$

*Then $(U,V)(t)$ has a continuous extension to $(-\infty,\infty)$, and*

$$(2.38) \qquad (U,V)(t) = (\xi,\eta)(t) - \frac{\langle\lambda,\xi\rangle(t)}{2(h-W\circ\pi(t))}\cdot X_H(\pi,\lambda)(t).$$

*Proof.* For $0 < t < T$, we have by virtue of (2.27),

$$(2.39) \qquad (U,V)(t) = T_{z_0} \Psi^{s(t)-s(t_0)}\cdot\zeta \quad \text{(see (2.23)).}$$

Since $\langle \xi_0, \lambda(t_0)\rangle = 0$ by assumption, it follows from Corollary 2.38

that $(U,V)(t)$ has a continuous extension to $(-\infty,\infty)$. Moreover, (2.39) together with (2.37) implies that

$$(U,V)(t) = T_{z_0}\Phi^{g(s(t)-s(t_0),z_0)}\cdot\zeta + \langle\frac{\partial g}{\partial z}(s(t)-s(t_0),z_0),\zeta\rangle\cdot X_H(\pi,\lambda)(t)$$

and the right hand side of this may be extended continuously to $(-\infty,\infty)$ (see Proposition 2.36).

Now $s(t)-s(t_0) = s(t-t_0,z_0)$ (see (2.30)) so that

$$g(s(t) - s(t_0),z_0) = t-t_0 .$$

Moreover, by (2.33) we have

$$(U,V)(t) = T_{z_0}\Phi^{t-t_0}\cdot\zeta + \frac{2\int_{t_0}^{t} \langle DW\circ\pi,\xi\rangle(t')dt'}{2(h-W\circ\pi(t))}\cdot X_H(\pi,\lambda)(t).$$

The conclusion of our theorem now follows. Indeed,

$$T_{z_0}\Phi^{t-t_0}\cdot\zeta = (\xi,\eta)(t) \text{ and by virtue of (2.36),}$$

$$\frac{2\int_{t_0}^{t} \langle DW\circ\pi,\xi\rangle(t')dt'}{2(h-W\circ\pi(t))}\cdot X_H(\pi,\lambda)(t) = \frac{-\langle\dot{\lambda},\xi\rangle(t)}{2(h-W\circ\pi(t))}\cdot X_H(\pi,\lambda)(t).$$

□

Preparatory to showing that (2.38) gives us in fact a $C^1$ extension of $(U,V)(t)$ to $(-\infty,\infty)$, we need the following result.

Lemma 2.40. *Let $U \in J_\pi^\perp$ (see Definition 2.33), where $\pi(kT) \in \partial N$, $k \in \mathbb{Z}$. For $(U,V)(t)$ as in (2.38), let*

$$Z(t) = \frac{\langle U,DW\circ\pi\rangle(t)}{2(h-W\circ\pi(t))}\lambda(t) + V(t). \tag{2.40}$$

*The following four limits, as $t \to kT$, exist and have the stated values*

(a) $\dfrac{\langle\lambda, DW\rangle\cdot DW - \lambda\cdot|DW|^2}{2(h-W)\cdot\langle DW,\lambda\rangle} \to \dfrac{-\langle DW,(D^2W)\cdot DW\rangle\cdot DW + |DW|^2\cdot(D^2W)\cdot DW}{-3|DW|^4}(\pi(kT))$

(b) $\dfrac{[2(h-W)]\cdot DW - \langle\lambda, DW\rangle\lambda}{[2(h-W)]^{3/2}} \to 0$

(c) $\langle U, DW\rangle\cdot[2(h-W)]^{-1} \to \dfrac{-\langle U,(D^2W)\cdot DW(\pi)\rangle}{3|DW(\pi)|^2}(kT)$

(d) $\langle Z,\lambda\rangle(t) \equiv 0,$ *and* $\langle Z, DW\rangle\cdot[2(h-W)]^{-\frac{1}{2}} \to 0.$

*Proof.* (a) Let

$$(2.41)\qquad \lambda^*(t) = \begin{cases} \lambda(t)/|\lambda(t)| & \text{if } t \notin T\cdot\mathbb{Z}. \\ \lim\limits_{t'\downarrow kT} \lambda^*(t') & \text{if } t = kT,\ k\in\mathbb{Z}. \end{cases}$$

Applying L'Hospital's rule to the expression on the left in part (a) above, we may deduce that it has the same limit, as $t \to kT$, as does the expression,

$$\frac{\{\langle\lambda, D^2W\cdot\lambda\rangle DW(\pi) + \langle\lambda, DW(\pi)\rangle D^2W\cdot\lambda - 2\langle D^2W\cdot\lambda, DW(\pi)\rangle\lambda\}(t)}{-2(\langle DW(\pi),\lambda\rangle(t))^2 - 2(h-W\circ\pi)\cdot|DW(\pi)|^2(t)},$$

which in turn, since $|\lambda(t)| = [2(h-W\circ\pi(t))]^{\frac{1}{2}}$ and $\lambda^*(kT) = \dfrac{\pm DW(\pi(kT))}{|DW(\pi(kT))|}$, tends to the right hand side of (a) above.

(b) Applying L'Hospital's rule to the expression on the left in part (b) above, we find that it has the same limit as does the expression,

$$\frac{\langle\lambda, DW(\pi)\rangle DW(\pi) - |DW(\pi)|^2\cdot\lambda(t)}{3[2(h-W\circ\pi)]^{\frac{1}{2}}\langle DW(\pi),\lambda\rangle(t)} + \frac{\langle\lambda^*, D^2W\cdot\lambda^*\rangle\lambda(t) - D^2W\cdot\lambda(t)}{3\langle DW(\pi),\lambda^*\rangle(t)}.$$

The first term in the expression above vanishes by virtue of part (a), the second term vanishes since $\lambda(kT) = 0$.

(c) For $(\xi,\eta)(t)$ as specified in Proposition 2.39, since $(\xi,\eta)(t) \in T_{(\pi,\lambda)(t)}H^{-1}(h)$,

$$(2.42) \qquad 0 = dH_{(\pi,\lambda)(t)}(\xi,\eta)(t) = \langle DW(\pi),\xi\rangle(t) + \langle\lambda,\eta\rangle(t).$$

We apply L'Hospital's rule to the expression (see (2.38)),

$$\frac{\langle U,DW\rangle}{2(h-W)} = \frac{-\langle\xi,\lambda\rangle\langle\lambda,DW\rangle}{[2(h-W)]^2} + \frac{[2(h-W)]\langle\xi,DW\rangle}{[2(h-W)]^2},$$

and we find that it has the same limit as does the sum of the following four expressions (see (2.29)):

$$-\left(\frac{\langle\eta,\lambda\rangle\langle\lambda,DW\rangle + \langle\xi,DW\rangle\langle\lambda,DW\rangle}{-4[2(h-W)]\langle\lambda,DW\rangle}\right) - \frac{\langle\xi,\lambda\rangle\langle\lambda,D^2W\cdot\lambda\rangle}{-4[2(h-W)]\langle\lambda,DW\rangle}$$

$$+ \frac{[2(h-W)]\langle\eta,DW\rangle + \langle\xi,\lambda\rangle|DW|^2}{-4[2(h-W)]\langle\lambda,DW\rangle} + \frac{\langle\xi,D^2W\cdot\lambda\rangle}{-4\langle\lambda,DW\rangle}.$$

The first term in the above expression vanishes identically by virtue of (2.42). The second term vanishes since $|\lambda(t)| = [2(h-W\circ\pi)]^{\frac{1}{2}}$ and

$$\langle\xi,DW(\pi)\rangle(t) \to 0 \quad \text{as } t \to kT \quad \text{(see (2.35))}.$$

The remaining two terms we will rewrite as

$$(2.43) \qquad \frac{[2(h-W)]\langle\eta,DW\rangle + \langle\lambda,DW\rangle\langle\xi,DW\rangle}{-4[2(h-W)]\langle\lambda,DW\rangle} + \frac{\langle\xi,\langle\lambda,DW\rangle DW-|DW|^2\lambda\rangle}{4[2(h-W)]\langle\lambda,DW\rangle}$$

$$+ \frac{\langle\xi,D^2W\cdot\lambda\rangle}{-4\langle\lambda,DW\rangle}.$$

The second term in (2.43), by virtue of (a) above and the fact that $\langle\xi,DW(\pi)\rangle \to 0$ (see (2.35)), tends to

$$\frac{\langle \xi, D^2W \cdot DW(\pi) \rangle}{-12\,|DW(\pi)|^2} \quad (kT) \ .$$

The first term in (2.43) vanishes in the limit as $t \to kT$. Indeed

$$\frac{2(h-W)\langle \eta, DW\rangle + \langle \lambda, DW\rangle\langle \xi, DW\rangle}{-4[2(h-W)]\langle \lambda, DW\rangle} = \frac{\langle \lambda, DW\rangle\langle \eta, \lambda\rangle + \langle \lambda, DW\rangle\langle \xi, DW\rangle}{-4[2(h-W)]\langle \lambda, DW\rangle}$$

$$+ \ \frac{2(h-W)\langle \eta, DW\rangle - \langle \lambda, DW\rangle\langle \eta, \lambda\rangle}{-4[2(h-W)]\langle \lambda, DW\rangle} \ .$$

The first term vanishes identically (see (2.42)) and the second term vanishes by virtue of (b) above. Collecting the nonzero terms from (2.43), we have, since $\xi(kT) = U(kT)$ (see (2.38) and (2.35)),

$$\langle U, DW\rangle \cdot [2(h-W)]^{-1} \to \frac{\langle U, D^2W \cdot DW(\pi)\rangle(kT)}{-12\,|DW(\pi(kT))|^2} + \frac{\langle U, D^2W \cdot DW(\pi)\rangle(kT)}{-4\,|DW(\pi(kT))|^2} \ .$$

(d) From (2.40) we find that (see (2.22) and (2.27))

$$\langle Z, \lambda\rangle(t) = \langle U, DW(\pi)\rangle(t) + \langle V, \lambda\rangle(t) = 0.$$

Moreover, from (2.40) and (2.38),

$$\frac{\langle Z, DW\rangle}{[2(h-W)]^{\frac{1}{2}}} = \frac{\langle \eta, DW\rangle}{[2(h-W)]^{\frac{1}{2}}} + \frac{\langle \xi, \lambda\rangle |DW|^2}{[2(h-W)]^{3/2}} + \frac{\langle \xi, DW\rangle\langle \lambda, DW\rangle}{[2(h-W)]^{3/2}} - \frac{\langle \xi, \lambda\rangle\langle \lambda, DW\rangle^2}{[2(h-W)]^{5/2}}$$

$$= \frac{[2(h-W)]\langle \eta, DW\rangle + \langle \xi, DW\rangle\langle \lambda, DW\rangle}{[2(h-W)]^{3/2}} + \left(\frac{[2(h-W)]\,|DW|^2 - \langle \lambda, DW\rangle^2}{[2(h-W)]^{3/2}}\right)\frac{\langle \xi, \lambda\rangle}{[2(h-W)]}$$

The first term in the above expression is the same as the first term in (2.43), which was shown to vanish in the limit $t \to kT$. The second term vanishes in the limit by virtue of (b) above and the fact that, by (2.36) and (2.33),

$$\frac{\langle \xi, \lambda\rangle}{2(h-W)}(t) = -\langle \frac{\partial g}{\partial z}(s(t) - s(t_0), z_0), \zeta\rangle.$$

Therefore, by Proposition 2.36, $\langle \xi, \lambda\rangle \cdot [2(h-W)]^{-1}$ is bounded in the limit, since $U \in J_\pi^\perp$ implies that $\langle \xi_0, \lambda(t_0)\rangle = 0$. □

In Lemma 2.32, we showed that $(U,V)(t)$ was a solution of (2.28) for $0 < t < T$. However, by virtue of Lemma 2.40 (c), the right hand side of (2.28) is continuous on $(-\infty,\infty)$. We are thereby led to ask whether $(U,V)(t)$ as given in (2.38) is in fact a $C^1$ solution of (2.28) for $t \in (-\infty,\infty)$.

<u>Proposition 2.41</u>. *For $U \in J_\pi^\perp$ (see Definition 2.33), the extension of $(U,V)(t)$ to $(-\infty,\infty)$ given by (2.38) is $C^1$, and $(U,V)(t)$ satisfies (2.28) for $t \in (-\infty,\infty)$.*

*Proof.* If $t \notin T \cdot \mathbb{Z}$, then by (2.38), $(U,V)(t)$ is $C^1$ at $t$, and it is easily checked that $(U,V)(t)$ satisfies (2.28) at $t$ (see (2.39) and (2.8)). Therefore, to verify the statement of the proposition, we need only check that $(U,V)(t)$ is $C^1$ at $t = kT$, $k \in \mathbb{Z}$.

Notice that for $(\xi,\eta)(t)$ as specified in Theorem 2.39,

$$(2.44)\qquad \lim_{t\to kT} \frac{\langle\lambda,\xi\rangle(t)}{2(h-W\circ\pi(t))} = \frac{-\langle DW(\pi),\eta\rangle(kT)}{|DW(\pi(kT))|^2}.$$

Indeed (see (2.36)), $\langle\lambda,\xi\rangle\cdot[2(h-W)]^{-1}$ has the same limit as $\frac{-2\langle DW(\pi),\xi\rangle(t)}{-2\langle\lambda,DW(\pi)\rangle(t)}$, which tends in the limit (see (2.35)), to the right hand side of (2.44). We may deduce from (2.38), (2.44) that,

$$(2.45)\qquad (U,V)(kT) = \left(\xi,\eta - \frac{\langle DW(\pi),\eta\rangle}{|DW(\pi)|^2} DW(\pi)\right)(kT).$$

Consider the quotient (see (2.38)),

$$\frac{\{U(t) - \xi(kT) - (t-kT)V(kT)\}}{t-kT} = \frac{-\langle\lambda,\xi\rangle(t)}{2(h-W\circ\pi(t))}\cdot\frac{\lambda(t)}{t-kT} + \frac{\xi(t)-\xi(kT)}{t-kT} - V(kT)$$

$$= \frac{+\ \langle\lambda,\xi\rangle(t)}{2(h-W\circ\pi(t))}\cdot DW(\pi(t)) + \eta(kT) - V(kT) + \frac{O(t-kT)}{t-kT}\to 0, \text{ as } t\to kT, \text{ see}$$

(2.38). Therefore from (2.28) and Lemma 2.40 (c), $U$ is $C^1$ on $(-\infty,\infty)$.

Now consider the expression (see (2.38), (2.45)),

$$(t-kT)^{-1}\,\{V(t) - V(kT)\}$$

(2.46)

$$= \frac{1}{t-kT}\,\{\eta(t) - \eta(kT)\} + \frac{1}{t-kT}\left\{\frac{\langle\xi,\lambda\rangle(t)}{2(h-W\circ\pi(t))}\,DW(\pi(t)) + \frac{\langle DW(\pi),\eta\rangle(kT)}{|DW(\pi(kT))|^2}\,DW(\pi(kT))\right\} .$$

Applying L'Hospital's rule to the second term on the right in (2.46), we find it has the same limit as

$$\frac{\langle\eta,\lambda\rangle + \langle\xi,DW\rangle - 2\cdot\langle\xi,DW\rangle}{2(h-W\circ\pi(t))}\,DW(\pi(t)) + \frac{\langle\xi,\lambda\rangle(t)}{2(h-W\circ\pi(t))}\,D^2W\cdot\lambda(t)$$

$$+ \frac{2\langle\xi,\lambda\rangle\langle\lambda,DW\rangle}{[2(h-W\circ\pi(t))]^2}\,DW(\pi(t)) .$$

By virtue of (2.42) and (2.44), we need only consider the following expression:

$$\frac{-2[2(h-W\circ\pi(t))]\ \xi,DW(\pi)\ (t) + 2\langle\lambda,DW(\pi)\rangle\langle\xi,\lambda\rangle(t)}{[2(h-W\circ\pi(t)]^2}\,DW(\pi(t)) .$$

Applying L'Hospital's rule to this expression, we find the coefficient multiplying $DW(\pi(t))$ has the same limit as the following sum:

$$\frac{2\langle\eta,\langle\lambda,DW\rangle\lambda - 2(h-W)DW\rangle}{-4[2(h-W)]\langle\lambda,DW\rangle} + \frac{2\langle\xi,\langle\lambda,DW\rangle DW - |DW|^2\lambda\rangle}{-4[2(h-W)]\langle\lambda,DW\rangle}$$

$$+ \frac{2\langle\xi,\lambda\rangle\cdot\langle\lambda,D^2W\cdot\lambda\rangle}{-4[2(h-W)]\langle\lambda,DW\rangle} + \frac{2\langle\xi,D^2W\cdot\lambda\rangle}{4\langle\lambda,DW\rangle} .$$

The first term in this last expression vanishes in the limit (see Lemma 2.40 (b)), as does the third term (see (2.44)). The second term is the same (except for constants) as the second term in (2.43), and therefore tends to $\dfrac{\langle \xi, D^2W\cdot DW(\pi)\rangle(kT)}{6|DW(\pi(kT))|^2}$, while the last term tends to $\dfrac{\langle \xi, D^2W\cdot DW(\pi)\rangle(kT)}{2|DW(\pi(kT))|^2}$. Collecting these latter two terms with the nonzero term from the first expression on the right hand side of (2.46) (see (2.29)), we find that the derivative of $V(t)$ at $t = kT$ is

$$\frac{2\langle \xi, D^2W\cdot DW(\pi)\rangle(kT)}{3|DW(\pi(kT))|^2}\, DW(\pi(kT)) - D^2W\cdot\xi(kT) ,$$

which is exactly the expression $\lim_{t\to kT} \dot{V}(t)$, obtained from (2.28) and Lemma 2.40 (c). Therefore $(U,V)(t)$ is $C^1$ for all $t$ and satisfies (2.28) on $(-\infty,\infty)$. □

Remark 2.42. We will now motivate our introduction of the vector field $Z(t)$ along $\pi(t)$ (see (2.40)). Let $c(s)$ be a critical arc of the functional $J$ subject to boundary conditions $M^0 \times M^1$, with (see Remark 2.31 (b))

$$(c,y)(s) = L(c,c')(s) \in H^{-1}(h) \text{ for } s \in [0,1].$$

Let $P(s)$ be an orthogonal Jacobi field along $c(s)$ and $U(t)$ the associated reparametrized orthogonal Jacobi field along $\pi(t)$ (see (2.27)). On the tangent bundle $TN^0$ of $N^0$, denote the Levi-Civita connection of the Jacobi metric $(d\tau)^2$ (see discussion after Remark 1.1 and Remark 1.2) along $c(s)$ by $\nabla_{c'}$ (see Klingenberg [K] p. 74). It is not difficult to see that (see Lemma 2.43 below),

$$\nabla_{c'}P = P'(s) - \frac{\langle P(s), DW(c(s))\rangle}{2(h-W\circ c(s))}\, c'(s) .$$

Since $y(s) = 2(h-W\circ c(s))\, c'(s)$, for $0 < t < T$,

$$\nabla_y P = 2(h-W\circ c(s))\ P'(s) - \frac{\langle P(s), DW(c(s))\rangle}{2(h-W\circ c(s))}\, y(s)$$

$$= \dot{U}(t) - \frac{\langle U(t), DW(\pi(t))\rangle}{2(h-W\circ\pi(t))}\,\lambda(t) \quad = \quad Z(t)$$

see (2.24),(2.28) and (2.40). Now the restriction $0 < t < T$ may be considered to reflect the fact that the above construction makes sense only on $TN^0$. However, by virtue of Proposition 2.41 and (2.40), $Z(t)$ is $C^1$ on $(-\infty,\infty)$. Indeed, it is easily verified from (2.40) and (2.28), that for $U \in J_\pi^\perp$, $(U,Z)(t)$ is a $C^1$ solution of the following differential equation on $(-\infty,\infty)$.

(2.47)
$$\dot{U}(t) = \frac{\langle U, DW(\pi)\rangle}{2(h-W\circ\pi(t))}\,\lambda(t) + Z(t)$$

$$\dot{Z}(t) = -D^2W\cdot U(t) - \frac{3\langle U, DW(\pi)\rangle(t)}{2(h-W\circ\pi(t))}\cdot\frac{[2(h-W\circ\pi)\cdot DW(\pi) - \langle\lambda, DW(\pi)\rangle\cdot\lambda](t)}{2(h-W\circ\pi(t))}$$
$$+ \frac{\langle U, D^2W\cdot\lambda\rangle(t)}{2(h-W\circ\pi(t))}\,\lambda(t) + \frac{\langle Z, DW(\pi)\rangle(t)}{2(h-W\circ\pi(t))}\,\lambda(t).$$

Alternatively, by restricting the initial conditions $(U,Z)(t_0)$ so that

(2.48) $$\langle U,\lambda\rangle(t_0) = \langle Z,\lambda\rangle(t_0) = 0 \text{ (see Lemma 2.40 (d))},$$

we may consider (2.47) as a $2m-2$ dimensional system of equations. This is formally carried out by introducing "Fermi coordinates" along $\pi(t)$ (see Klingenberg [K], pp. 110, 111), see Lemma 2.43 below.

Let

(2.49) $$\Sigma_t = \{X \in T_{\pi(t)}N \mid \langle X, \lambda^*(t)\rangle = 0\} \quad \text{(see (2.41))}.$$

Let $\{E_i\}_{i=1}^{m-1}$ be an orthonormal basis for $\Sigma_{t_0}$, $t_0 \notin T\cdot\mathbb{Z}$. Consider the initial value problem, for $i = 1,\ldots,m-1$,

$$(2.50) \qquad \frac{d}{dt} E_i(t) = \frac{\langle E_i, DW(\pi)\rangle(t)}{2(h-W\circ\pi(t))} \lambda(t), E_i(t_0) = E_i.$$

Lemma 2.43. *Let $E_i(t)$ ($i = 1,\dots,m-1$) be the solution on the maximal interval of existence $I$ of (2.50). Then $I = (-\infty,\infty)$, moreover the following is true.*

*(a) The differential equation in (2.50) is the equation of parallel translation along $\pi(t)$, that is (see Remark 2.42),*

$$\nabla_{\lambda(t)} = \left(\frac{d}{dt}\right) - \frac{\langle \cdot, DW(\pi)\rangle(t)}{2(h-W\circ\pi(t))} \lambda(t).$$

*(b) $\langle E_i, DW(\pi)\rangle(t)\ [2(h-W\circ\pi(t))]^{-\frac{1}{2}} \to 0$ as $t \to kT$, $k \in \mathbb{Z}$.*

*(c) The vector fields $E_i(t)$ are $2T$-periodic.*

*Proof.* The fact that $I = (-\infty,\infty)$ will follow from (b) below. To prove (a), we notice that, for $i,j = 1,\dots,m-1$,

$$\frac{d}{dt} \langle E_i, \lambda\rangle(t) = \langle E_i, DW(\pi)\rangle(t) + \langle E_i, -DW(\pi)\rangle(t) = 0, \text{ and}$$

$$\frac{d}{dt}\langle E_i, E_j\rangle(t) = \frac{\langle E_i, DW(\pi)\rangle(t)}{2(h-W\circ\pi(t))}\cdot\langle\lambda, E_j\rangle(t) + \frac{\langle E_j, DW(\pi)\rangle(t)}{2(h-W\circ\pi(t))}\cdot\langle\lambda, E_i\rangle(t) = 0.$$

It follows that (2.50) determines parallel translation along $\pi(t)$.

$$\text{(b)} \qquad \frac{\langle E_i, DW(\pi)\rangle(t)}{[2(h-W\circ\pi(t)]^{\frac{1}{2}}} = \frac{\langle E_i, \left(2(h-W\circ\pi)DW(\pi) - \langle\lambda, DW(\pi)\rangle\lambda\right)\rangle(t)}{[2(h-W\circ\pi(t)]^{3/2}}$$
$$+ \frac{\langle E_i, \lambda\rangle\ \langle\lambda, DW(\pi)\rangle(t)}{[2(h-W\circ\pi(t))]^{3/2}}.$$

The first term on the right vanishes in the limit (use the Cauchy-Schwartz inequality, part (a) above and Lemma (2.40)(b). The second term vanishes identically by part (a).

(c) The periodicity of $E_i(t)$ follows since integration of (2.50) on the interval $[0,2T]$ amounts to parallel translation up to time $T$, and then back along the same path $\pi(t)$.

The fact that $I = (-\infty,\infty)$ now follows since the vector field in (2.50) is continuous on $(-\infty,\infty)$. □

The moving frame $\{E_i(t)\}_{i=1}^{m-1}$ allows us to simplify the linear differential equations for $(U,Z)(t)$ (see (2.47)).

<u>Proposition 2.44</u>. *Let $U \in J_\pi^\perp$ and $Z(t)$ be given by (2.40) on $(-\infty,\infty)$. Then $(U,Z)(t) \in \Sigma_t \times \Sigma_t$ on $(-\infty,\infty)$ (see (2.49)). Let $U(t) = a_i(t)\, E_i(t)$, $Z(t) = b_j(t)\, E_j(t)$ (repeated indices indicates summation) where $E_i(t), i = 1,\dots,m-1$ are the vector field solutions of (2.50) on $(-\infty,\infty)$. Then,*

$$\dot{a}_i(t) = b_i(t), \qquad \textit{and}$$

(2.51)

$$\dot{b}_j(t) = -a_i(t)\langle D^2W\cdot E_i, E_j\rangle(t) - 3a_i(t)\langle DW,E_j\rangle(t)\cdot\langle DW,E_i\rangle(t)\cdot[2(h-W\circ\pi(t)]^{-1}.$$

*Proof.* By virtue of the Gauss Lemma (Lemma 2.26) and Lemma 2.40 (d), $(U,Z)(t) \in \Sigma_t \times \Sigma_t$. From (2.47) we have,

$$\dot{U} = \dot{a}_iE_i + a_i\dot{E}_i = a_i(t)\cdot\frac{\langle E_i,DW(\pi)\rangle(t)}{2(h-W\circ\pi(t))}\lambda(t) + b_i(t)\cdot E_i(t).$$

Therefore $(\dot{a}_i-b_i)E_i(t) = 0$ from (2.50). Since $E_i(t)$ are linearly independent for all $t$, $\dot{a}_i = b_i, i = 1,\dots,m-1$. Similarly (see (2.47)),

$$\dot{Z} = \dot{b}_iE_i(t) + b_i\frac{\langle E_i,DW\rangle(t)}{2(h-W\circ\pi(t))}\lambda(t) = -a_iD^2W\cdot E_i(t) +$$

$$a_i\langle\lambda^*,D^2W\cdot E_i\rangle(t)\cdot\lambda^*(t) + 3a_i\frac{\langle E_i,DW\rangle(t)}{2(h-W\circ\pi(t))}[\langle\lambda^*,DW\rangle(t)\cdot\lambda^*(t)-DW(\pi(t))]$$

$$+ b_i\cdot\frac{\langle E_i,DW\rangle(t)}{2(h-W\circ\pi(t))}\cdot\lambda(t).$$

Therefore, taking the dot product of this last expression with $E_i(t)$,

$$\dot{b}_i(t) = -a_j(t)\cdot\langle D^2W\cdot E_i, E_j\rangle(t) - 3a_j(t)\cdot\langle DW,E_i\rangle(t)\langle DW,E_j\rangle(t)\,[2(h-W\circ\pi(t))]^{-1}$$

for $i = 1,\ldots,m-1$. □

Proposition 2.44 will be used in Chapter 3 to prove that minimum distance lines correspond to hyperbolic periodic orbits on $H^{-1}(h)$.

Definition 2.45. Let $M$ be a hypersurface in $N^0$ (see Remark 1.1) with $(\pi,\lambda)(t_0) \in \perp^h(M)$ for some $t_0$. A (reparameterized) $M$-orthogonal Jacobi field $U$ is an element of $J^\perp_\pi$ (see Definition 2.33) such that for $(U,V)(t)$ as specified in (2.38),

$$(U,V)(t_0) \in T_{(\pi,\lambda)(t_0)} \perp^h(M).$$

The $(m-1)$-dimensional subspace of such vector fields $U(t)$ along $\pi(t)$ will be denoted by $J^\perp_{\pi,M}$ (see Remark 2.30).

Lemma 2.46. *Let* $z_0 = (\pi,\lambda)(t_0) \in H^{-1}(h)$. *The linear mapping*

$$\Lambda(t_0)\colon T_{z_0} H^{-1}(h) \to T_{z_0} H^{-1}(h)$$

*defined by (compare with (2.38) and (2.44))*

$$(2.52)\qquad \Lambda(t_0)\cdot(\xi,\eta) = \begin{cases} (\xi,\eta) - \dfrac{\langle\xi,\lambda(t_0)\rangle}{2(h-W\circ\pi(t_0))}X_H(z_0) & \text{if } t_0 \notin T\cdot\mathbb{Z} \\[2ex] (\xi,\eta) + \dfrac{\langle\eta, DW(\pi(kT))\rangle}{|DW(\pi(kT))|^2}X_H(z_0) & \text{if } t_0 = kT,\ k\in\mathbb{Z} \end{cases}$$

*has the following properties*

*(a)* $\ker\Lambda = \langle X_H(z_0)\rangle$, *(b)* $\Lambda^2 = \Lambda$, *(c)* $E(t_0) \doteq \Lambda(t_0)\cdot(T_{z_0}H^{-1}(h))$ *is a symplectic subspace with symplectic form* $\omega_E = \omega\big|_{E(t_0)}$ *(see (2.11)). (d) For* $\zeta_1,\zeta_2 \notin \ker\Lambda$,

$$\omega_E(\Lambda\zeta_1,\Lambda\zeta_2) = \omega(\zeta_1,\zeta_2).$$

*Proof.* We will prove (a) - (d) assuming that $t_0 \notin T\cdot\mathbb{Z}$, the verification for $t_0 = kT$ being analagous. We recall that

$$X_H(z_0) = (\lambda, -DW(\pi))(t_0)$$

(a) $\Lambda\cdot(X_H) = (\lambda,-DW(\pi))(t_0) - \dfrac{|\lambda(t_0)|^2}{2(h-W\circ\pi(t_0))} X_H(z_0) = 0.$

Moreover, $\Lambda\cdot(\xi,\eta) \neq 0$ if $(\xi,\eta)$ is not linearly dependent on $X_H(z_0)$.

(b) For $(\xi,\eta) \in T_{z_0} H^{-1}(h)$ (see (2.52)),

$$\Lambda^2(\xi,\eta) = \Lambda(\xi,\eta) - \frac{\langle\lambda(t_0),\xi\rangle}{2(h-W\circ\pi(t_0))}\Lambda(X_H) = \Lambda(\xi,\eta) \text{ (see part (a)).}$$

Therefore $\Lambda$ is a projection mapping. Let $E(t_0) \doteqdot \Lambda(T_{z_0} H^{-1}(h))$.

(c) We will denote the $\omega$-orthogonal complement of a subspace $G$ of $\mathbb{R}^{2m}$ by

$$G^\perp = \{\zeta \in \mathbb{R}^{2m} \mid \omega(\zeta,q) = 0 \ \forall q \in G\} .$$

We recall that,

(2.53) $$\dim G + \dim G^\perp = 2m \quad \text{(see [A\&M], p. 403).}$$

Let $\omega_E \doteqdot \omega\big|_{E(t_0)}$ (see (2.11)). Then $\omega_E$ is an antisymmetric bilinear form on $E(t_0)$. We need only show that $\omega_E$ is non degenerate. To this end, let $\zeta \in E(t_0)$, $\omega_E(\zeta,\cdot) = 0$ on $E(t_0)$. Then $\zeta = 0$. Indeed, since $0 = dH(\zeta) = \omega(X_H,\zeta)$ (see [A&M], p.187) and

(2.54) $$T_{z_0} H^{-1}(h) = E(t_0) \oplus \langle X_H\rangle ,$$

it follows that $\zeta \in [T_{z_0} H^{-1}(h)]^\perp$. Now $[T_{z_0} H^{-1}(h)]^\perp$ is one dimensional (see (2.53)) and $X_H \in [T_{z_0} H^{-1}(h)]^\perp$. Therefore, $\zeta$ is linearly dependent on $X_H$ which means that $\zeta = 0$ (see (2.54)).

(d) This follows from the definitions of $E(t_0)$ and $\omega_E$. $\square$

<u>Proposition 2.47</u>. *There is a natural symplectic structure on* $J^{\perp}_{\pi}$ *(see Definition 2.33). For* $U_1, U_2 \in J^{\perp}_{\pi}$, $(U_i, V_i)(t)$ $(i = 1,2)$ *as specified by (2.38),*

(2.55) $\omega_{\pi}(U_1, U_2) \doteq \langle U_1, V_2\rangle(t) - \langle U_2, V_1\rangle(t)$, *for arbitrary* $t$.

*In particular, the right hand side of (2.55) is independent of* $t$. *Given* $t_0$ *and the projection* $\Lambda(t_0)$ *(see (2.52)),* $J^{\perp}_{\pi}$ *is symplectomorphic to* $E(t_0)$ *(see Lemma 2.46 (c)). For any hypersurface* $M$ *in* $N^0$ *(see Remark 1.1) with* $(\pi,\lambda)(t_0) \in \perp^h(M)$, *the subspace* $J^{\perp}_{\pi,M}$ *(see Definition 2.45) is a Lagrangian subspace of* $J^{\perp}_{\pi}$.

*Proof.* The fact that the right hand side of (2.55) is independent of $t$ follows directly from (2.28), therefore $\omega_{\pi}$ is well defined on $J^{\perp}_{\pi}$. Consider the isomorphism $Q\colon J^{\perp}_{\pi} \to E(t_0)$, $U \to (U,V)(t_0) \in E(t_0)$ (see (2.38) and (2.52)). Then (see Lemma 2.46 (c) and (2.11)),

$$Q^{*}\omega_{E} = \omega_{\pi} \,, \tag{2.56}$$

and $J^{\perp}_{\pi}$ is thereby symplectomorphic to $E(t_0)$.

Suppose that $(\pi,\lambda)(t_0) \in \perp^h(M)$ (see Corollary 2.21). Notice that $Q(J^{\perp}_{\pi,M}) = T_{(\pi,\lambda)(t_0)} \perp^h(M)$ (see Definition 2.45). The final conclusion of the Proposition will follow from (2.56), once we establish *that* $T_{(\pi,\lambda)(t_0)} \perp^h(M)$ is a Lagrangian subspace of $E(t_0)$.

Let $(\xi,\eta) \in T_{(\pi,\lambda)(t_0)} \perp^h(M)$. Then

$$\Lambda(t_0)(\xi,\eta) = (\xi,\eta) \in E(t_0),$$

since $\langle \xi, \lambda(t_0)\rangle = 0$ (see (2.14), (2.52)). Moreover,

$$\omega_E | T_{(\pi,\lambda)(t_0)} \perp^h(M) = 0 \quad \text{(see (2.16)), and}$$

$$\dim T_{(\pi,\lambda)(t_0)} \perp^h(M) = m-1 \quad \text{(see Corollary 2.21).}$$

Therefore $T_{(\pi,\lambda)(t_0)} \perp^h(M)$ is indeed a Lagrangian subspace of $E(t_0)$, and $J^{\perp}_{\pi,M}$ is thereby (from (2.56)) a Lagrangian subspace of $J^{\perp}_{\pi}$.

□

Remark 2.48. Let $J$ be a Lagrangian subspace of $J^{\perp}_{\pi}$. We will refer to $J$ as a Lagrange family. If $U \in J$, then for $(U,V)(t)$ as specified in (2.38),

(2.57) $$(U,V)(t) \in T_{(\pi,\lambda)(t)} H^{-1}(h) \;\; \forall \; t \in (-\infty,\infty).$$

Therefore we may unambiguously refer to the subspace of $\mathbb{R}^m$

(2.58) $$J(t) = \{U(t) \mid U \in J\}.$$

Notice that $\dim J(t) \leq m-1$, by virtue of the Gauss Lemma 2.26. We will refer to $(\pi(t),t)$ or just simply $t$ as a focal point of $J$ if

$$\dim J(t) < m-1.$$

In this case,

(2.59) order of the focal point at $t = (m-1) - \dim J(t)$.

# CHAPTER 3

## THE INDEX OF $\mathcal{J}_{**}(M^0 \times M^1)$ AND MINIMUM DISTANCE LINES

Introduction.

In the last chapter we saw how it was possible to continuously extend the co-geodesic flow $\Psi^s$ along orbits $\sigma(t)$ of $X_H$ on $H^{-1}(h)$. In fact this flow has continuous directional derivatives $T_{\sigma(t_0)}\, \Psi^s \cdot \zeta$, for $s \in (-\infty,\infty)$, whenever $\zeta = (\xi_0,\eta_0)$, $\sigma(t_0) = (\pi(t_0),\lambda(t_0))$, and

$$(3.1) \qquad \langle \xi_0,\lambda(t_0)\rangle = 0 \quad \text{(see Corollary 2.38).}$$

By abuse of notation, we will use the tangent functor $T_z\, \Psi^s$, keeping in mind we only operate on those $\zeta$ such that (3.1) is satisfied. In particular, if $z = (x,0) \in H^{-1}(h)$, with $x \in \partial N$, then by $T_z\, \Psi^s \cdot \zeta$ we will always mean

$$\langle \frac{\partial}{\partial z}\, \Phi^{g(s,z)}{}_z, \zeta \rangle ,$$

where $\Phi$ is the flow of $X_H$ on $\mathbb{R}^{2m}$ (see Proposition 2.36).

In this chapter we will first consider the wavefront sets associated with relatively open subsets $M^i$ of $W_h^i$ ($i = 0,1$, see (1.8)). The focal points of these wavefront sets correspond to parameter values $t$ so that a nonzero subspace of reparameterized $M^i$-orthogonal Jacobi fields vanishes at $\pi(t)$ (see Remark 2.18 and (2.27)). Secondly we generalize the notion of convexity, as presented in Ambrose [A], for these wavefront sets. This allows us to generalize the index theorem of Ambrose [A], so as to apply to the variational problem ($\mathcal{J}$) considered in Chapter 1.

Finally we turn to the study of the stability properties of minimum distance lines.

Let $s(t)$ be given by (2.26). Recall that $s(t) = s(t,\sigma(0))$ (see (2.30)), therefore $s(t)-s(T) = s(t-T, \sigma(T))$. Notice also that, for submanifolds $M$ of $\mathbb{R}^m$ lying within $\partial N$ :

(3.2) $$\perp^h(M) = \{(x,0) \in \mathbb{R}^{2m} \mid x \in M\}$$

Definition 3.1. Let $\sigma(t) = (\pi,\lambda)(t)$ be the orbit of $X_H$ corresponding to a critical arc $c$ (see Definition 2.6). Let $M^i \subset W_h^i$ $(i = 0,1)$ be an open neighbourhood of $\pi(t_0)$ $(t_0 = 0,T)$ within $W_h^i$ $(i = 0,1)$ respectively. Define the wavefront sets

$$M^0(t) = \psi^{s(t)}\perp^h(M^0) = \{\varphi^{g(s(t),z)}z \mid z \in \perp^h(M^0)\}$$

and

$$M^1(t) = \psi^{s(t)-s(T)}\perp^h(M^1) = \{\varphi^{g(s(t-T,\sigma(T)),z)}z \mid z \in \perp^h(M^1)\}.$$

□

Notice that we have adjusted the "start time" $t = 0$ so that it corresponds for $M^0(t)$ and $M^1(t)$ to $\pi(0)$.

In Proposition 2.20, we investigated the boundary conditions (see (2.14)) satisfied by $U \in J^{\perp}_{\pi,M}$ (see Definition 2.45) for hypersurfaces $M$ within $N^0$. We will now consider the case where $M \cap \partial N \neq \emptyset$.

Recall that the reversing symmetry

(3.3) $$R: \mathbb{R}^{2m} \to \mathbb{R}^{2m},\ R(x,y) = (x,-y),$$

in conjunction with the Hamiltonian $H(x,y) = \frac{1}{2}|y|^2 + W(x)$, and the flow $\Phi^t$ of the Hamiltonian vector field $X_H$, has the following properties

$$\text{(3.4)}\quad \begin{array}{lll} \text{(a)} & H\circ R & = H, \\ \text{(b)} & R\circ\Phi^{t} & = \Phi^{-t}\circ R\ , \text{ and} \\ \text{(c)} & R^{*}\omega = -\omega & \text{(see (2.11)).} \end{array}$$

We let *Fix* $R$ denote the fixed point set of $R$.

Proposition 3.2. *The (reparameterized) orthogonal Jacobi fields* $U \in J^{\perp}_{\pi,M^0}$ *(see Definition 2.45) are characterized by* $(U,V)(0) \in Fix\ R \cap T_{\sigma(0)}\ H^{-1}(h)$; *those belonging to* $J^{\perp}_{\pi,M^1}$ *by* $(U,V)(T) \in Fix\ R \cap T_{\sigma(T)}\ H^{-1}(h)$. *In particular* $J^{\perp}_{\pi,M^i}$ *are Lagrangian subspaces of* $J^{\perp}_{\pi}$ *for* $i = 0,1$.

*Proof.* By virtue of (3.2) we may deduce that

$$Fix\ R \cap H^{-1}(h) = \{(x,0)\,|\,x \in \partial N\} \supset \perp^{h}(M^{i}) \quad (i = 0,1), \text{ and}$$

$$T_{\sigma(0)} \perp^{h}(M^0) = \{(P,0)\,|\,P \in T_{\pi(0)}M^0\} = Fix\ R \cap T_{\sigma(0)}\ H^{-1}(h).$$

Indeed, the tangent spaces coincide since $M^0$ is a relatively open subset of $W^0_h$ containing $\pi(0)$. Similarly we find

$$T_{\sigma(T)} \perp^{h}(M^1) = \{(P,0)\,|\,P \in T_{\pi(T)}M^1\} = Fix\ R \cap T_{\sigma(T)}\ H^{-1}(h).$$

To conclude that $J^{\perp}_{\pi,M^i}$ are Lagrange spaces of $J^{\perp}_{\pi}$, which has dimension $2m-2$, we need only observe that (3.4)(c) together with $R^{*}\omega\big|_{Fix\ R} = \omega$ implies that $\omega\big|_{Fix\ R} = 0$. □

The following result gives a geometrical picture tying these ideas together.

<u>Proposition 3.3</u>. *If $(\pi(t),t)$ with $t \notin T\,\mathbb{Z}$, is not a focal point of $\sigma(0) \in \perp^h(M^0)$, and $(\pi(t),t-T)$ is not a focal point of $\sigma(T) \in \perp^h(M^1)$, then the wavefront sets $M^0(t)$, $M^1(t)$ are local hypersurfaces tangent to each other at $\pi(t)$. Their common tangent space at $\pi(t)$ is*

$$J_0(t) = J_1(t) = \Sigma_t \quad \textit{(see (2.49)), where}$$

(3.5) $\quad J_0(t) = \{U(t)\,|\,U \in J^{\perp}_{\pi,M^0}\},\ J_1(t) = \{U(t)\,|\,U \in J^{\perp}_{\pi,M^1}\}$

*(see (2.58)).*

*Proof.* Notice that $\pi(t) \in M_0(t) \cap M_1(t)$. Indeed,

$$\psi^{s(t)}\sigma(0) = \varphi^{g(s(t),\sigma(0))}\sigma(0) = \varphi^{t}\sigma(0) = \pi(t), \text{ and}$$

$$\psi^{s(t)-s(T)}\sigma(T) = \psi^{s(t-T,\sigma(T))}\sigma(T) = \varphi^{t-T}\sigma(T) = \pi(t).$$

Since $\pi(t)$ is not a focal point of either $\sigma(0)$ or $\sigma(T)$, arguing along the lines of Lemma 2.22, there are neighbourhoods $B_0, B_1$ of $\sigma(0) \in \perp^h(M^0)$, $\sigma(T) \in \perp^h(M^1)$ respectively, so that

$$\psi^{s(t)}(B_0) \text{ and } \psi^{s(t)-s(T)}(B_1)$$

are hypersurfaces in $N^0$.

Now $T_{\pi(t)} M^0(t)$ is spanned by (see (2.19)) $\{T_{\sigma(0)}\psi^{s(t)}\cdot\zeta\,|\,\zeta \in T_{\sigma(0)} \perp^h(M^0)\} = J_0(t)$ (see (2.21) and the introduction for remarks concerning $T_{\sigma(0)}\Psi^s$), and $T_{\pi(t)}M^1(t)$ is spanned by

$$\{T_{\sigma(T)}\psi^{s(t)-s(T)}\cdot\zeta\,|\,\zeta \in T_{\sigma(T)} \perp^h(M^1)\} = J_1(t).$$

Finally, $J_0(t), J_1(t) \subset \Sigma_t$ (Lemma 2.26), and by virtue of the assumption that $\{t\}$ is not a focal point of either $J^{\perp}_{\pi,M^0}$ or $J^{\perp}_{\pi,M^1}$,

$$J_0(t) = J_1(t) = \Sigma_t \quad \text{(see (2.59))}.$$

□

Lemma 3.4. *If* $U \in J_\pi^\perp$, *and* $\hat{U}(t) = U(t + 2kT)$, $k \in \mathbb{Z}$, *then* $\hat{U} \in J_\pi^\perp$. *In particular, if* $(U,V)(t)$ *is given by (2.38), then* $(\hat{U},\hat{V})(t) = (U,V)(t + 2kT)$ *is a solution of (2.28) on* $(-\infty,\infty)$.

*Proof.* Given $U \in J_\pi^\perp$, and $(U,V)(t)$ as in (2.38),

$$(\hat{U},\hat{V})(t) = (U,V)(t+2kT) = (\xi,\eta)(t+2kT) - \frac{\langle\lambda,\xi\rangle(t+2kT)}{2(h-W\circ\pi(t+2kT))} X_H(\sigma(t+2kT)).$$

By periodicity of $\sigma(t)$, $(\hat{U},\hat{V})(t)$ has a representation as in (2.38) and $\hat{U}$ thereby belongs to $J_\pi^\perp$. The final statement now follows from Proposition 2.41.

□

Remark 3.5. We have seen that the Hamiltonian flow $\Phi^t$ is time reversible (see (3.4)(b) with respect to the symmetry $R$ (see (3.3)). It may be shown directly from (2.31), (3.4)(b) that

(3.6) $$-g(s,z) = g(-s,Rz), \ (s,z) \in \mathbb{R} \times H^{-1}(h).$$

By virtue of (2.37) and (3.6), the co-geodesic flow $\Psi^{s-s_0}$ is also time reversible with respect to $R$,

(3.7) $$R\circ\Psi^{s-s_0}z = \Psi^{s_0-s}\circ Rz, \ (s,z) \in \mathbb{R} \times H^{-1}(h).$$

Moreover, the linearized flow $T_z\Psi^{s-s_0}$ along a critical arc $c(s)$ is time reversible. If

(3.8) $$\langle\xi,y(s_0)\rangle = 0, \ \zeta = (\xi,\eta) \in T_{(c,y)(s_0)}H^{-1}(h),$$

then

(3.9) $$R\circ T_{(c,y)(s_0)}\Psi^{s-s_0}\cdot\zeta = T_{R(c,y)(s_0)}\Psi^{s_0-s}\circ R\zeta$$

(see Corollary 2.38). In particular, (3.8) and (3.9) hold for any orthogonal Jacobi field $P(s)$ along $c(s)$ (see Definition 2.28).

Proposition 3.6. *An orthogonal Jacobi field $U \in J^{\perp}_{\pi,M^0}$ is $(2T)$-periodic if and only if $U \in J^{\perp}_{\pi,M^0} \cap J^{\perp}_{\pi,M^1}$.*

*Proof.* (a) Suppose that $U \in J^{\perp}_{\pi,M^0} \cap J^{\perp}_{\pi,M^1}$. Then by Proposition 3.2

$$\begin{aligned}(U,V)(0) &= R\cdot(U,V)(0)\\ &= R\cdot T_{\sigma(T)}\Psi^{-s(T)}(U,V)(T) \quad \text{(see (2.39) with } t = T,\ z = \sigma(T))\\ &= T_{R\sigma(T)}\Psi^{s(T)}R\cdot(U,V)(T)\\ &= T_{\sigma(T)}\Psi^{s(T)}(U,V)(T) \text{ (since } R\sigma(T)=\sigma(T), R\cdot(U,V)(T) = (U,V)(T))\\ &= (U,V)(2T).\end{aligned}$$

The conclusion now follows from Lemma 3.4 and uniqueness of the initial value problem for equations (2.28).

(b) Suppose that $U \in J^{\perp}_{\pi,M^0}$ is $(2T)$-periodic. Then $(U,V)(0) = T_{\sigma(T)}\Psi^{s(T)}R\cdot(U,V)(T)$ as in (a) above. On the other hand, $(U,V)(0) = (U,V)(2T) = T_{\sigma(T)}\Psi^{s(T)}\cdot(U,V)(T)$. Therefore $R\cdot(U,V)(T) = (U,V)(T)$ since $T_{\sigma(T)}\Psi^{s(T)}$ is one to one as a map $T_{\sigma(T)}H^{-1}(h) \to T_{\sigma(0)}H^{-1}(h)$ (keeping in mind the restriction (3.1) imposed on $T_{\sigma(T)}\Psi^{s(T)}$). The conclusion now follows from Proposition 3.2 which implies that $U \in J^{\perp}_{\pi,M^1}$. $\square$

We refer the reader for notation to the preamble to Proposition 2.20. In the absence of focal points of $M^0$ and $M^1$, the hypersurfaces $M^0(t)$ and $M^1(t)$ have associated Weingarten mappings $\Xi^0$, $\Xi^1$, at $(\pi,\lambda)(t) \in \perp^h(M^i(t))$ for $i = 0,1$ (see (2.3), (2.25)).

These mappings characterize the orthogonal Jacobi fields $U \in J^{\perp}_{\pi,M^i}$ ($i = 0,1$), see Proposition 2.20.

Following Ambrose [A] (p. 53), we will define operators $\Gamma^i(t)$ ($i = 0,1$) on $T_{\pi(t)}M^i(t)$ ($i = 0,1$) which are essentially the Weingarten mappings discussed above. The mappings $\Gamma^i(t)$ will allow us to generalize the convexity term used in Ambrose's index theorem.

Let $\Sigma_t$ be specified as in (2.49), and $J, J^*$ be Lagrange families (see Remark 2.48) of $J^{\perp}_{\pi}$.

Notice that, for any Lagrange family $J$ of $J^{\perp}_{\pi}$ (see (2.58))

(3.10) $\qquad J(t) = \Sigma_t$ provided $(\pi(t),t)$ is not a focal point of $J$,

(see for example Proposition 3.3 where this is shown for $J_0(t)$, $J_1(t)$).

If $J(t) = \{U(t) \mid U \in J\}$, $J^*(t) = \{U^*(t) \mid U^* \in J^*\}$, then

(3.11) $\qquad \Delta_t\colon \Sigma_t \to J(t)$, $\Delta^*_t\colon \Sigma_t \to J^*(t)$ denotes the orthogonal projections of $\Sigma_t$ onto $J(t)$, $J^*(t)$ respectively.

<u>Definition 3.7</u>. For Lagrange families $J, J^*$ of $J^{\perp}_{\pi}$, $U \in J$ and $U^* \in J^*$, let $Z(t)$, $Z^*(t)$ be given as in (2.40). Then

$$\Gamma(t)\colon J(t) \to J(t) \text{ and } \Gamma^*(t)\colon J^*(t) \to J^*(t)$$

are defined as follows:

$\Gamma(t) \cdot U(t) = \Delta_t \cdot Z(t)$ and $\Gamma^*(t) \cdot U^*(t) = \Delta^*_t \cdot Z^*(t)$ (see (3.11)) where $U(t) \in J(t)$, $U^*(t) \in J^*(t)$. $\square$

The mappings $\Gamma(t)$, $\Gamma^*(t)$ are well defined: indeed if $U_1, U_2 \in J$ with $U_1(t) = U_2(t)$, that is $(\pi(t),t)$ is a focal point for $J$ (see (2.59)), then for any $U \in J$, since $J$ is a Lagrange family,

$0 = \omega_\pi(U_1 - U_2, U) = \langle U_1 - U_2, V\rangle(t) - \langle U, V_1 - V_2\rangle(t)$ (see (2.55))

$= - \langle U, V_1 - V_2\rangle(t) = - \langle U, Z_1 - Z_2\rangle(t)$ (see (2.40)).

Therefore, $Z_1(t) - Z_2(t) \perp J(t)$, so that $\Delta_t \cdot Z_1(t) = \Delta_t \cdot Z_2(t)$.

The following four Propositions outline the important properties of the mappings $\Gamma(t)$, $\Gamma^*(t)$.

Proposition 3.8. *Let $J$ be a Lagrange family of $J_\pi^\perp$. $\Gamma(t)$ is a linear, symmetric operator on $J(t)$ (see Definition 3.7). This operator is continuous in $t$ on any interval $(a,b)$ not containing any focal points of $J$. That is, if $(a,b)$ contains no focal point of $J$ then $\Gamma(t) \cdot U(t)$ is a continuous vector field along $\pi(t)$ whenever $U(t) \in J(t)$ is a continuous vector field along $\pi(t)$, $t \in (a,b)$.*

*Proof.* We have seen that $Z(t) \in \Sigma_t$ on $(-\infty,\infty)$ (see Proposition 2.44). Since $Z$ is linear with respect to $U \in J$ (see (2.40)), it follows from Definition 3.7 that $\Gamma(t)$ is a linear mapping with domain and range as indicated. Since $J$ is a Lagrange family, $U_1, U_2 \in J$ implies that

$$0 = \omega_\pi(U_1, U_2) = \langle U_1, V_2\rangle - \langle U_2, V_1\rangle.$$

Therefore (see (2.40)),

$$\langle \Gamma(t) \cdot U_1, U_2\rangle = \langle V_1, U_2\rangle = \langle U_1, V_2\rangle = \langle U_1, \Gamma(t) \cdot U_2\rangle.$$

It remains only to show that $\Gamma(t) \cdot U(t)$ is continuous on $(a,b)$ when $U(t)$ is continuous on $(a,b)$, $U(t) \in J(t)$ and $(a,b)$ contains no focal points of $J$. In this case, $J(t) = \Sigma_t$ and $\Delta_t =$ identity on $\Sigma_t$ (see (3.10)). Let $U \in J$, and $Z(t)$ be specified as in (2.40). Then $\Gamma(t) \cdot U(t) = Z(t)$ is $C^1$ on $(a,b)$ (see Remark 2.42). An arbitrary continuous field $U(t) \in \Sigma_t$ along $\pi$ may be represented by, $U(t) = \sum_j \alpha_j(t)U_j(t)$ where $\alpha_j \in C^0(a,b)$ and $U_j \in J$. The final conclusion now follows by linearity of $\Gamma(t)$. □

Proposition 3.9. *Let $J, J^*$ be Lagrange families of $J^\perp_\pi$, and suppose that $\{t_0\}$ is not a focal point of either $J$ or $J^*$. Let*

$$U \in J \text{ and } U^* \in J^*.$$

*Then*

$$U(t_0) = U^*(t_0) \text{ and } \Gamma(t_0) \cdot U(t_0) = \Gamma^*(t_0) \cdot U^*(t_0)$$

*if and only if $U \equiv U^*$ and consequently $U \in J \cap J^*$.*

*Proof.* If $\{t_0\}$ is not a focal point of either $J$ or $J^*$ then $J(t_0) = J^*(t_0) = \Sigma_{t_0}$ and therefore $\Gamma(t_0) \cdot U(t_0) = Z(t_0)$ and $\Gamma^*(t_0) \cdot U^*(t_0) = Z^*(t_0)$. However, $U(t_0) = U^*(t_0)$ and $Z(t_0) = Z^*(t_0)$ if and only if $U \equiv U^*$ (uniqueness of the initial value problem for equations (2.47)). □

Remark 3.10. In an interval $(a,b)$ not containing focal points of either $J^\perp_{\pi,M^0}$ or $J^\perp_{\pi,M^1}$, the operators $\Gamma^i(t)$, $i = 0,1$, coincide with the Weingarten mappings $\Xi^i$ of the wavefront sets $M^i(t)$ at $\sigma(t) \in \perp(M^i(t))$ $(i = 0,1)$: indeed for any $U \in \Sigma_t$, $U^i \in J^\perp_{\pi,M^i}$,

$$\langle \Gamma^i(t) \cdot U^i(t), U \rangle = \langle V^i(t), U \rangle = |\lambda(t)| \cdot \langle \Xi^i \cdot U^i(t), U \rangle \tag{3.12}$$

(see Proposition 2.20 with $M^i(t)$ replacing $M$ and $(U^i, V^i)(t)$ replacing $(P,R)$ in (2.14)). The operators $\Gamma^i(t)$ thereby characterize the local relative convexity of $M^0(t)$, $M^1(t)$. In this sense, Proposition 3.9 characterizes the space $J^\perp_{\pi,M^0} \cap J^\perp_{\pi,M^1}$ as the set of independent directions in the common tangent space of $M^0(t), M^1(t)$ at $\pi(t)$ such that $M^0(t)$ matches $M^1(t)$ to second order along these directions (see also Ambrose [A], p.54). □

The following result is useful in determining the relative convexity of the wavefront sets $M^0(t)$, $M^1(t)$ in terms of the families $J^{\perp}_{\pi,M^0}$, $J^{\perp}_{\pi,M^1}$.

Lemma 3.11. *Let $J, J^*$ be Lagrange families of $J^{\perp}_{\pi}$, and suppose that $\{t_0\}$ is not a focal point of either $J$ or $J^*$. Let $U \in J$, $U^* \in J^*$. Then*

$$J(t_0) \cap J^*(t_0) = \Sigma_{t_0}, \qquad \text{and} \tag{3.13}$$

$$\omega_{\pi}(U^*, U) = \langle U^*(t_0), (\Gamma(t_0) - \Gamma^*(t_0)) \cdot U(t_0) \rangle. \tag{3.14}$$

*In particular, for the Lagrange families $J^{\perp}_{\pi,M^0}$ and $J^{\perp}_{\pi,M^1}$, if $U^i \in J^{\perp}_{\pi,M^i}$ and $U^0(t_0) = a \cdot U^1(t_0)$ with $a > 0$, then the sign of $\omega_{\pi}(U^1, U^0)$ determines the relative convexity of $M^0(t_0)$ and $M^1(t_0)$ in the common direction of $U^0(t_0)$.*

*Proof.* From (3.10) we may deduce that $J(t_0) \cap J^*(t_0) = \Sigma_{t_0}$. The following computation yields (3.14):

$$\omega_{\pi}(U^*, U) = \langle U^*, V \rangle (t_0) - \langle U, V^* \rangle (t_0)$$

$= \langle U^*, \Gamma \cdot U \rangle (t_0) - \langle U, \Gamma^* \cdot U^* \rangle (t_0)$ (since $\{t_0\}$ is not a focal point of $J, J^*$)

$= \langle U^*, \Gamma \cdot U \rangle (t_0) - \langle \Gamma^* \cdot U, U^* \rangle (t_0)$ (since $\Gamma, \Gamma^*$ are symmetric)

$= \langle U^*, (\Gamma - \Gamma^*) \cdot U \rangle (t_0)$.

Now suppose that $U^i \in J^{\perp}_{\pi,M^i}$ and $U^0(t_0) = aU^1(t_0)$ $(a > 0)$.

Then by virtue of (3.14), the sign of $\omega_\pi(U^1, U^0)$ is the same as the sign of $\langle U^0, (\Gamma^0 - \Gamma^1)\cdot U^0\rangle(t_0)$. Now $\langle U^0, (\Gamma^0 - \Gamma^1)\cdot U^0\rangle(t_0)$ is positive (negative) if and only if the normal curvature of $M^0(t_0)$ at $\pi(t_0)$ in the direction of $U^0(t_0)$ (the normal curvature of $M^0(t_0)$ in the direction of $U^0(t_0)$ is

$|U^0|^{-2}\langle U^0, \Xi^0(U^0)\rangle(t_0)$, see Hicks [Hi], p. 52, where we have chosen the sign of curvature so that the standard sphere with outward pointing normal has positive curvature) is greater (less) than that of $M^1(t_0)$ at $\pi(t_0)$ in the same direction (see (3.12) and Figures 15 and 16). □

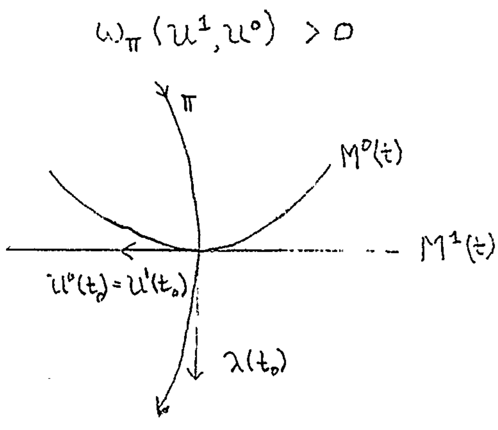



Fig 15

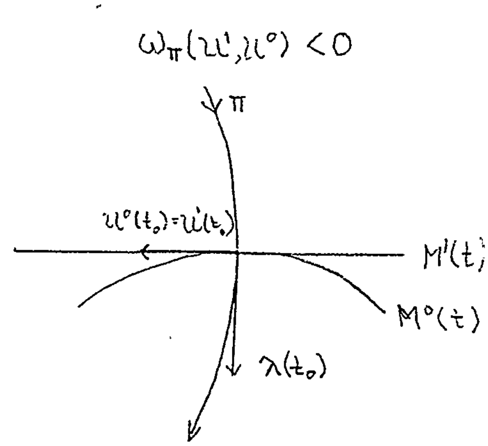



Fig 16

<u>Proposition 3.12</u>. *Let $J, J^*$ be Lagrange families of $J_\pi^\perp$. Suppose that $(a,b)$ contains no focal points of either $J$ or $J^*$. Then for $t \in (a,b)$*

(a) *Nullity* $(\Gamma - \Gamma^*)(t) = \dim\,(J \cap J^*)$

(b) *Index* $(\Gamma - \Gamma^*)(t) =$ *constant throughout* $(a,b)$.

*Proof.* (a) Since $J(t) \cap J^*(t) = \Sigma_t$ (see (3.12)), this is an immediate corollary of Proposition 3.9.

(b) Suppose that for some $a < t_0 < t_1 < b$, $k = \mathrm{Index}(\Gamma - \Gamma^*)(t_0)$, $0 \le k \le m-1$ and $\mathrm{Index}(\Gamma - \Gamma^*)(t_1) \ne k$. Since $(\Gamma - \Gamma^*)(t)$ has continuous eigenvalues throughout $(a,b)$ (see Proposition 3.8) there is some point $t_0 < \hat{t} < t_1$ so that

$$\mathrm{Nullity}(\Gamma - \Gamma^*)(\hat{t}) \ne \mathrm{Nullity}(\Gamma - \Gamma^*)(t_0),$$

which contradicts part (a). □

Proposition 3.13. *Let $J, J^*$ be Lagrange families of $J_\pi^\perp$. Suppose that $t_0$ is not a focal point of $J$ or $J^*$ and let*

$$\{U_i\}_{i=1}^{m-1} \quad \textit{and} \quad \{U_i^*\}_{i=1}^{m-1}$$

*be bases of $J, J^*$ respectively, such that*

$$U_i(t_0) = U_i^*(t_0) \textit{ for } i = 1,\ldots,m-1.$$

*If there is an $(m-1) \times (m-1)$ positive semi-definite diagonal matrix $\gamma = [\gamma_{ij}]$ such that*

$$(3.15) \qquad \omega_\pi(U_i^*, U_j) = \gamma_{ij} \textit{ for } i,j = 1,\ldots,m-1,$$

*then $J$ cannot have a focal point before $J^*$. That is, if $J^*$ does not have a focal point in $[t_0,t_1]$, then $J$ does not either.*

*Proof.* Let $[t_0,t_1]$ contain no focal points of $J^*$ and suppose that $t_0 < t' \le t_1$ and $\pi(t')$ is the first focal point for $J$ in $[t_0,t_1]$. We shall derive a contradiction from this assumption. We may assume without loss of generality that

$$(3.16) \qquad U_1(t') = 0.$$

Since $\{U_j^*(t)\}_{i=1}^{m-1}$ spans $\Sigma_t$ for $t \in [t_0,t_1]$, it follows that there are $C^1$ functions $\alpha_j(t)$ for $j = 1,\ldots,m-1$, so that

$$(3.17) \qquad U_1(t) = \sum_{j=1}^{m-1} \alpha_j(t)\, U_j^*(t) \text{ for } t \in [t_0,t_1].$$

Notice that, by (3.16) and (3.17), for $i = 1,\ldots,m-1$,

$$(3.18) \qquad \alpha_i(t') = 0.$$

From (3.17) it follows that, $\dot{U}_1 = \sum_j \{\dot{\alpha}_j(t)U_j^*(t) + \alpha_j(t)\dot{U}_j^*(t)\}$. By virtue of (3.16), $V_1(t') = \dot{U}_1(t')$ (see (2.28)), and (3.18) allows us to conclude that

$$(3.19) \qquad V_1(t') = \sum_j \dot{\alpha}_j(t')U_j^*(t') \ .$$

From this we may deduce that $\dot{\alpha}_1(t') > 0$. Indeed, from (3.16),

$$(3.20) \qquad \langle U_j^*,V_1\rangle(t') = \omega_\pi(U_j^*,U_1) = \begin{cases} \gamma_{11} \text{ if } j = 1 \\ 0 \text{ otherwise.} \end{cases}$$

Notice that $\gamma_{11} > 0$. Otherwise, $\omega_\pi(U_1,U^*) = 0 \ \forall U^* \in J^*$ (see (3.15)), and therefore $U_1 \in J^*$ which thereby has a focal point at $t'$ (see (3.16)). However this contradicts our assumption regarding the interval $[t_0,t_1]$. Upon taking the inner product of (3.19) with $V_1(t')$, and using (3.20), we find

$$|V_1(t')|^2 = \dot{\alpha}_1(t') \cdot \gamma_{11} \ ,$$

and hence $\dot{\alpha}_1(t') > 0$ since $U_1 \neq 0$ by assumption.

Since $\alpha_1(t_0) = 1$, $\alpha_1(t') = 0$ and $\dot{\alpha}_1(t') > 0$, it follows that there is some $t_0 < t'' < t' \leq t_1$ with $\alpha_1(t'') = 0$. Let

$$U^*(t) = \alpha_2(t'')U_2^*(t) + \ldots + \alpha_{m-1}(t'')U_{m-1}^*(t).$$

Then $U^* \in J^*$ since $U_i^* \in J^*$ for $i = 2,\ldots,m-1$, and $U^* \neq 0$ since

$$(3.21) \qquad U^*(t'') = U_1(t'') \neq 0 \quad (t' \text{ is the first focal point of in } [t_0,t_1]).$$

Now, $(\Gamma - \Gamma^*)(t_0)$ (see Definition 3.7) is positive semi-definite. This follows since by assumption $U_j^*(t_0) = U_j(t_0)$, and $0 \leq \gamma_{jj} = \omega_\pi(U_j^*,U_j) = \langle U_j^*,(\Gamma - \Gamma^*)U_j\rangle(t_0)$ (see (3.15)). By virtue of the fact that $t'$ is the first focal point of $J$ in $[t_0,t_1]$, from Proposition 3.12 (b) we may conclude that

$$(\Gamma - \Gamma^*)(t'') \geq 0.$$

Now $\omega_\pi(U_j^*,U_1) = 0$ for $j = 2,\ldots,m-1$, therefore

$$0 = \sum_{j=2}^{m-1} \alpha_j(t'')\omega_\pi(U_j^*,U_1) = \omega_\pi(U^*,U_1) = \langle U^*,(\Gamma - \Gamma^*)U_1\rangle(t'').$$

Since $(\Gamma - \Gamma^*)(t'')$ is positive semi-definite, we may conclude from (3.21),

$$U^*(t'') = U_1(t'') \text{ and } \Gamma^* \cdot U^*(t'') = \Gamma \cdot U_1(t'').$$

By virtue of Proposition 3.9, $U^* \equiv U_1 \in J \cap J^*$. From (3.16) we may thereby deduce that $J^*$ has a focal point at $t' \in [t_0,t_1]$. This is the desired contradiction. □

Having obtained these preliminary results on Lagrange families and their associated mappings $\Gamma$, we will digress temporarily to give an extension of Ambrose's index theorem (Ambrose [A]), promised from Chapter 2. Preparatory to stating the generalization of Ambrose's index theorem, we prove the following two results.

<u>Lemma 3.14</u>. *There exists* $t^* > 0$ *so that* $(0,t^*)$ *contains no focal points of* $J^{\perp}_{\pi,M^0}$ *or* $J^{\perp}_{\pi,M^1}$.

*Proof*. This is an immediate consequence of the fact that focal points of any Lagrange family are isolated (see Proposition 2.19). □

Remark 3.15. From Lemma 3.14, it follows that we may choose $0 < t^* < 1$ so that:

$$(3.22)\quad \begin{array}{l} (0,t^*) \text{ does not contain any focal points of } J^{\perp}_{\pi,M^0} \text{ or } J^{\perp}_{\pi,M^1} \\ (T-t^*,T) \text{ does not contain any focal points of } J^{\perp}_{\pi,M^1}. \end{array}$$

For $t_0 \in (0,t^*)$, we may assume that, by suitably restricting $M^0,M^1$,

$$M^0(t_0) \cap \partial N = \emptyset \text{ and } M^1(T-t_0) \cap \partial N = \emptyset.$$

Recall that the Weingarten mappings $\Xi_0$, $\Xi_1$ of $M^0(t_0)$, $M^1(T-t_0)$ at $\sigma(t_0) \in \perp(M^0(t_0))$, $\sigma(T-t_0) \in \perp(M^1(T-t_0))$ respectively, are essentially the same as $\Gamma^0(t_0)$ and $\Gamma^1(T-t_0)$ (see (3.12));

$$(3.23)\quad \begin{array}{l} \Gamma^0(t_0) = |\lambda(t_0)| \cdot \Xi_0 \\ \Gamma^1(T-t_0) = |\lambda(T-t_0)| \cdot \Xi_1 . \end{array}$$

If $s_0 = s(t_0)$, $s_1 = s(T-t_0)$ (see (2.26)), let

$$H|_{[s_0,s_1]} = \{P|_{[s_0,s_1]} : P \in H\},$$

the vector space of fields $P \in H$ restricted to $[s_0,s_1]$.

Let $(c,y)(s(t)) = (\pi,\lambda)(t)$. As we noted in Remark 2.12, the second variation of the functional $J$ subject to the boundary conditions

$$(3.24)\qquad K \times L = M^0(t_0) \times M^1(T-t_0)$$

evaluated at the critical arc $c(s)|_{[s_0,s_1]}$ subject to the same boundary conditions (replace $M^0 \times M^1$ by $K \times L$ and $[0,1]$ by $[s_0,s_1]$ in Definition 2.6), leads to the index form (see (2.6)),

$$J^{**}(K \times L)\cdot(P,P) = |y(s_0)|\langle\Xi_0\cdot P,P\rangle(s_0) - |y(s_1)|\langle\Xi_1\cdot P,P\rangle(s_1) + \int_{s_0}^{s_1} \frac{j(s,P,R)}{2(h-W\circ c)}\, ds$$

where $j(s,P,R)$ is given in (2.9). According to (3.23) this is the same as

$$J**(K\times L)\cdot(P,P) = \langle\Gamma^0\cdot P,P\rangle(s_0) - \langle\Gamma^1\cdot P,P\rangle(s_1) + \int_{s_0}^{s_1}\frac{j(s,P,R)}{2(h-Woc)}\,ds \tag{3.25}$$

where $\Gamma^0(s_0) \doteq \Gamma^0(t_0)$ and $\Gamma^1(s_1) \doteq \Gamma^1(T-t_0)$.

Since the Jacobi metric (see (1.10)) is a bona fide metric in a neighbourhood of the geodesic segment $c|_{[s_0,s_1]}$, Ambrose's index theorem (see Ambrose [A] and Appendix A) for separated endpoint hypersurfaces applies:

$$\text{Index } J**(K\times L) = \sum_{t_0<t'<T-t_0}(\text{focal points of } J^{\perp}_{\pi,M^1} \text{ at } t') + \text{Index}(\Gamma^0-\Gamma^1)(t_0)$$

$$\text{Nullity } J**(K\times L) = \dim\left(J^{\perp}_{\pi,M^0}\cap J^{\perp}_{\pi,M^1}\right) = \text{Nullity}(\Gamma^0-\Gamma^1)(t_0).$$

□

Theorem 3.16. *Let $t^*$ be specified as in (3.22), and choose $0<t_0<t^*$. Let $J**$ be the index form on the vector space $H$ (see Proposition 2.14) defined as the second variation of the functional $J$ subject to the boundary conditions $M^0\times M^1$ (see Proposition 2.11). Then*

$$\textit{Index } J** = \textit{Index}(\Gamma^0-\Gamma^1)(t_0) + \sum_{0<t'<T}(\textit{focal points of } J^{\perp}_{\pi,M^1} \textit{ at } t')$$

$$\textit{Nullity } J** = \dim(J^{\perp}_{\pi,M^0}\cap J^{\perp}_{\pi,M^1}) = \textit{Nullity }(\Gamma^0-\Gamma^1)(t_0).$$

Preparatory to the proof of this result we need the following Lemma. If $s_0 = s(t_0)$ and $s_1 = s(T-t_0)$, let $H'\subset H$ be the subspace

$$H' = \{P\in H : \exists\ U^i \in J^{\perp}_{\pi,M^i}\ (i=0,1) \text{ so that for } P^i(s(t)) = U^i(t)\ (i=0,1)$$
$$P|_{[0,s_0]} = P^0|_{[0,s_0]} \text{ and } P|_{[s_1,1]} = P^1|_{[s_1,1]}. \tag{3.26}$$

Lemma 3.17. *If $J_{**}$ $(K \times L)$ (see (3.24)) is defined as in (3.25) then there is a positive semidefinite quadratic form $A$ defined on $H$ such that for $P \in H$,*

$$J_{**} \cdot (P,P) = J_{**}\ (K \times L) \cdot (P,P) + A(P,P).$$

*Moreover, $A$ vanishes when restricted to $H'$ (see (3.26)).*

*Proof.* Let $P \in H$, $(P,R)(s) = L_*(c,c') \cdot (P,P')(s)$, $s \in [0,1]$. We recall that (see (2.8)(a), (1.18))

$$\frac{d}{ds} \langle P,y \rangle (s) = \frac{\langle DW(c),P\rangle(s) + \langle y,R\rangle(s)}{2(h-Woc(s))} \in L^1[0,1], \text{ and}$$

by virtue of (2.22), $\frac{d}{ds} \langle P,y \rangle (s) = 0$ for $s \in (0,1)$. Since $\langle P,y \rangle (0) = 0$,

$$\text{(3.27)} \qquad \langle P,y \rangle (s) \equiv 0 \text{ on } [0,1].$$

Let $\{U^i_j\}_{j=1}^{m-1}$ be a basis for $J^{\perp}_{\pi,M^i}$ $(i = 0,1)$, and $P^i_j(s(t)) = U^i_j(t)$, $j = 1,\ldots,m-1$, $i = 0,1$. Then according to (3.22) and (3.10), $\{U_j(t)\}_{j=1}^{m-1}$ and $\{U_j(t)\}_{j=1}^{m-1}$ span $\Sigma_t$ on $[0,t_0]$ and $[T-t_0,T]$ respectively. By virtue of (3.27), $P(s(t)) \in \Sigma_t$ for $t \in [0,T]$. Therefore, there are continuous functions $\alpha_i(s)$ on $[0,s_0]$ such that $\dot{\alpha}_i(s) \doteq 2(h-Woc(s)) \cdot \alpha_i'(s)$ is piecewise differentiable on $[0,s_0]$ (see Definition 2.13), and using our summation convention

$$\text{(3.28)} \qquad P(s) = \alpha_i(s) P^0_i(s) \text{ for } s \in [0,s_0].$$

Similarly, there are $\beta_j(s)$ on $[s_1,1]$ so that

$$P(s) = \beta_j(s) P^1_j(s) \text{ for } s \in [s_1,1].$$

Notice that, from (2.4) and (3.28), for $s \in [0,s_0]$,

$$\text{(3.29)} \qquad R(s) = \alpha_i(s) R^0_i(s) + \dot{\alpha}_i(s) P^0_i(s),$$

with an analogous expression in $\beta_j$ for $s \in [s_1,1]$. Now

$$J_{**}(P,P) = \int_0^{s_0} \frac{j(s,P,R)}{2(h-Woc)}ds + \int_{s_0}^{s_1} \frac{j(s,P,R)}{2(h-Woc)}ds + \int_{s_1}^{1} \frac{j(s,P,R)}{2(h-Woc)}ds .$$

We will analyse the first term in this expression, the last being similar. From (3.29), it follows that, for $s \in (0,s_0]$

$$R'(s) = \alpha_i(s)R_i^{0\prime}(s) + \alpha_i'(s)R_i^0(s) + (\dot{\alpha}_i(s)P_i^0(s))'.$$

After integrating by parts (as in Proposition 2.14 (a)), and recalling that $(P_i^0,R_i^0)(s)$ satisfy the co-Jacobi equations (2.8), we find

$$\int_0^{s_0} \frac{j(s,P,R)}{2(h-Woc)}ds = \langle R,P\rangle\Big|_0^{s_0} + \sum_{i=1}^{k} \Delta_{s_i}\dot{\alpha}_j \langle P_j^0,P\rangle - \int_0^{s_0} \langle(\dot{\alpha}_jP_j^0)',P\rangle ds$$

$$- \int_0^{s_0} \langle \alpha_i'R_i^0,P\rangle \, ds.$$

Another integration by parts yields,

$$- \int_0^{s_0} \langle(\dot{\alpha}_jP_j^0)',P\rangle ds = -\langle\dot{\alpha}_jP_j^0,P\rangle\Big|_0^{s_0} - \sum_{i=1}^{k} \Delta_{s_i}\dot{\alpha}_j \langle P_j^0,P\rangle$$

$$+ \int_0^{s_0} \langle\dot{\alpha}_jP_j^0,P'\rangle \, ds.$$

Using this and (3.29), we compute

$$\int_0^{s_0} \frac{j(s,P,R)}{2(h-Woc)}ds = \langle\alpha_iR_i^0 + \dot{\alpha}_iP_i^0,P\rangle\Big|_0^{s_0} - \langle\dot{\alpha}_jP_j^0,P\rangle\Big|_0^{s_0}$$

$$+ \int_0^{s_0} \langle\dot{\alpha}_jP_j^0,\alpha_i'P_i^0 + \alpha_iP_i^{0\prime}\rangle ds - \int_0^{s_0} \langle\alpha_i'R_i^0,P\rangle ds$$

$$= \langle\alpha_iR_i^0,P\rangle\Big|_0^{s_0} + \int_0^{s_0} \frac{|\dot{\alpha}_jP_j^0|^2}{2(h-Woc)}ds + \int_0^{s_0} \{-\alpha_i'\alpha_j\langle R_i^0,P_j^0\rangle + \alpha_i'\alpha_j\langle P_i^0,R_j^0\rangle\}ds$$

$$= \langle\Gamma^0 \cdot P,P\rangle(s_0) + \int_0^{s_0} \frac{|\dot{\alpha}_jP_j^0|^2}{2(h-W)} \, ds.$$

The last equality follows since, $U_i^0 \in J^{\perp}_{\pi,M^0}$ for $i = 1,\dots,m-1$, implies that

$$\omega(P_i^0,P_j^0) = \langle P_i^0,R_j^0\rangle(s) - \langle R_i^0,P_j^0\rangle(s) \equiv 0, \text{ and}$$

$$R_i^0(0) = 0. \quad i = 1,\dots,m-1 \text{ (see Proposition 3.2).}$$

We obtain an analogous expression for

$$\int_{s_1}^{1} \frac{j(s,P,R)}{2(h-W)}ds = -\langle\Gamma^1 \cdot P,P\rangle(s_1) + \int_{s_1}^{1} \frac{|\dot{\beta}_j P_j^1|^2}{2(h-W)}\,ds.$$

Now set $A(P,P) = \displaystyle\int_0^{s_0} \frac{|\dot{\alpha}_j P_j^0|^2}{2(h-Woc)}ds + \int_{s_1}^{1} \frac{|\dot{\beta}_j P_j^1|^2}{2(h-Woc)}\,ds.$

It follows that $A(P,P) = 0$ when $P \in H'$, since in that case, $\dot{\alpha}_j = 0 = \dot{\beta}_j$ for $j = 1,\dots,m-1$ (see (3.28)). □

*Proof* of Theorem 3.16.

First, we observe that by virtue of (3.22), given $P \in H$ there is a unique $U^i \in J^{\perp}_{\pi,M^i}$ $(i = 0,1)$ such that

$$P(s(t_0)) = U^0(t_0) \text{ and } P(s(T-t_0)) = U^1(T-t_0).$$

Let $P^i(s(t)) = U^i(t)$, and

$\beta : H \to H'$ be the linear operator defined by

$$\beta P(s) = \begin{cases} P^0(s) \ , \ s \in [0,s_0] \\ P\,(s) \ , \ s \in [s_0,s_1] \\ P^1(s) \ , \ s \in [s_1,1] \ . \end{cases}$$

From Lemma 3.17, we may deduce that

$$\begin{aligned} J{**}(\beta P,\beta P) &= J{**}\ (K\times L)(\beta P,\beta P) \\ &= J{**}\ (K\times L)(P,P) \quad \text{(since } \beta P \text{ agrees with } P \text{ on } [s_0,s_1]) \\ &\le J{**}\ (P,P) \qquad \text{(since } A \ge 0). \end{aligned}$$

Moreover, $J_{**}(\beta P,\beta P) = J_{**}(P,P)$ if and only if $P \in H'$. Notice that $H = H' \oplus \{P - \beta P \mid P \in H\}$, and therefore, a result of Ambrose (see Ambrose [A], p.64) applies and we may conclude that the index and nullity of $J_{**}$ are the same as the index and nullity of $J_{**}$ restricted to $H'$. However, for $P \in H'$ (see Lemma 3.17)

$$A(P,P) = 0, \text{ and therefore}$$

$$J_{**}(P,P) = J_{**}(K \times L)(P,P).$$

Hence, from the discussion following (3.25),

$$\begin{aligned} \text{Index } J_{**} &= \text{Index of } J_{**}(K \times L) \text{ on } H' \\ &= \sum_{t_0 < t' < T - t_0} (\text{focal points of } J^{\perp}_{\pi,M^1} \text{ at } t') \\ &\quad + \text{Index } (\Gamma^0 - \Gamma^1)(t_0), \text{ and} \end{aligned}$$

$$\begin{aligned} \text{Nullity } J_{**} &= \text{Nullity of } J_{**}(K \times L) \text{ on } H' \\ &= \dim (J^{\perp}_{\pi,M^0} \cap J^{\perp}_{\pi,M^1}) \; . \end{aligned}$$

Since $t_0$ was chosen arbitrarily from $(0,t^*)$, let $t_0 \downarrow 0$ to obtain the statement of the theorem (see Proposition 3.12 (b)). □

Our main theorem in this chapter concerns the stability type of minimum distance lines whose existence under general assumptions on $W(x)$, see (W1) through (W4), was studied in Chapter 1. The characterizing features of minimum distance lines are given in the following result.

Corollary 3.18. *Let $\hat{c}(s)$ for $s \in [0,1]$ be a minimum distance line, and $\pi(t)$ for $t \in (-\infty,\infty)$ the associated brake orbit (see Theorem 1.12). Then $J^{\perp}_{\pi,M^1}$ has no focal points in $(0,T]$ and $(\Gamma^0 - \Gamma^1)(t)$ is positive semidefinite in $(0,t^*)$ where $t^*$ is specified in (3.22).*

*Proof.* $\hat{c}(s)$, $s \in [0,1]$, is a critical arc, see Remark 2.8, and $J_{**}(P,P) \geq 0$ for every variation vector field $P(s)$. □

Lemma 3.19. *If $(\Gamma^0 - \Gamma^1)(t)$ is positive semidefinite in some interval $(0,t^*)$ not containing focal points of either $J^{\perp}_{\pi,M^0}$ or $J^{\perp}_{\pi,M^1}$, then $[0,t^*)$ does not contain any focal points of either $J^{\perp}_{\pi,M^0}$ or $J^{\perp}_{\pi,M^1}$. In particular, if $U^1 \in J^{\perp}_{\pi,M^1}$, and $U^1(0) = 0$ then $U^1 \equiv 0$.*

*Proof.* Let $U^1 \in J^{\perp}_{\pi,M^1}$ satisfy $U^1(0) = 0$. We will assume that $U^1 \neq 0$, so that

$$\dot{U}^1(0) = Z^1(0) \neq 0. \tag{3.30}$$

Notice that (3.30) together with $U^1(0) = 0$ imply that

$$\lim_{t \downarrow 0} \frac{U^1(t)}{t} = Z^1(0). \tag{3.31}$$

We also observe that,

$$\lim_{t \downarrow 0} \Gamma^0 U^1(t) = 0. \tag{3.32}$$

Indeed, by virtue of Proposition 3.2,

$$\Gamma^0(0) = 0, \quad \text{and} \tag{3.33}$$

$\Gamma^0(t)$ is continuous at $t = 0$ by Proposition 3.8. By virtue of Lemma 3.14 and Definition 3.7, there is a $\delta > 0$ so that,

$$\Gamma^1 U^1(t) = Z^1(t), \text{ for } t \in (0,\delta),$$

and therefore from (3.31), (3.22) and (3.30),

$$\lim_{t \downarrow 0} \left\langle \frac{U^1(t)}{t}, (\Gamma^0 - \Gamma^1) U^1(t) \right\rangle = -|Z^1(0)|^2 < 0.$$

This gives us the desired contradiction since by assumption,

$$(\Gamma^0-\Gamma^1)(t) \geq 0, \text{ for } t \in (0,t^*), \text{ and}$$

$$\langle \frac{U^1(t)}{t}, (\Gamma^0-\Gamma^1)U^1(t)\rangle \geq 0.$$

□

The following result significantly strengthens Corollary 3.18.

Proposition 3.20. *Let $\hat{c}(s)$, $s \in [0,1]$, be a minimum distance line, and $\pi(t)$, $t \in (-\infty,\infty)$, the associated brake orbit of $X_H$ (see (1.1)). Then $J^{\perp}_{\pi,M^1}$ has no focal points in $(-\infty,\infty)$.*

*Proof.* By virtue of (2.39) (set $t_0 = T$, $z_0 = \sigma(T)$, $\zeta = (U^1,V^1)(T)$) and (3.9) (set $s_0 = s(T) = 1$), for $U^1 \in J^{\perp}_{\pi,M^1}$

$$(U^1,V^1)(T-t) = T_{\sigma(T)}\Psi^{s(T-t)\,-\,s(T)}\cdot\zeta = T_{\sigma(T)}\Psi^{-s(t)}\cdot\zeta$$

$$= R\cdot T_{R\sigma(T)}\Psi^{s(t)}\cdot R\zeta = R\cdot T_{\sigma(T)}\Psi^{s(t)}\cdot\zeta \text{ (see (3.3),}$$

(3.9) and Proposition 3.2). Since this last expression is equal to $R(U^1,V^1)(T+t)$ (see (2.39)),

(3.34) $\quad (U^1,V^1)(T-t) = R(U^1,V^1)(T+t)$ for $t \in (-\infty,\infty)$.

A similar computation for $U^0 \in J^{\perp}_{\pi,M^0}$ yields

(3.35) $\qquad (U^0,V^0)(-t) = R(U^0,V^0)(t)$ for $t \in (-\infty,\infty)$.

As a consequence of (3.34), we observe that

(3.36) $\qquad J^{\perp}_{\pi,M^1}$ has no focal points in $[0,2T]$.

Indeed, from Corollary 3.18 and Lemma 3.19, it follows that $J^{\perp}_{\pi,M^1}$ has no focal points in $[0,T]$, and according to (3.34) (see (3.2))

(3.37) $\quad U^1(T-t) = U^1(T+t)$, for $t \in (-\infty,\infty)$, $U^1 \in J^{\perp}_{\pi,M^1}$.

In particular (3.36) follows upon taking $t \in [0,T]$ in (3.37).

Next we observe that $J^{\perp}_{\pi,M^0}$ has no focal points before $J^{\perp}_{\pi,M^1}$. To see this, we notice that since $t = 0$ is not a focal point of either $J^{\perp}_{\pi,M^0}$ or $J^{\perp}_{\pi,M^1}$, there is a $t^* > 0$ such that (see Corollary 3.18 and Proposition 3.12)

(3.38) $\qquad (\Gamma^0 - \Gamma^1)(t) \geq 0$ for $t \in [0,t^*)$.

Recalling that for any Jacobi field $U^0 \in J^{\perp}_{\pi,M^0}$,

(3.39) $\quad 0 = V^0(0) = Z^0(0)$ (see (2.40), and Proposition 3.2),

we may conclude from Definition 3.7 that $\Gamma^0(0) = 0$, and

$$\Gamma^1(0) \leq 0 \quad \text{(see (3.38))}.$$

Choose the unique $U^1_i \in J^{\perp}_{\pi,M^1}$ for $i = 1,\ldots,m-1$ whose initial values are equal to some set of orthonormal eigenvectors corresponding to the $(m-1)$ nonpositive eigenvalues of $\Gamma^1(0)$, $\beta_1,\ldots,\beta_{m-1}$;

(3.40) $\quad \Gamma^1 \cdot U^1_i(0) = \beta_i U^1_i(0), \beta_i \leq 0, \ |U^1_i(0)| = 1$, for $i = 1,\ldots,m-1$.

Let $$\gamma_{ij} = \begin{cases} -\beta_i & \text{if } i = j \\ 0 & \text{otherwise, and} \end{cases}$$

choose the unique (see (3.39)) $U_i^0 \in J^{\perp}_{\pi,M^0}$, such that,

(3.41) $$U_i^0(0) = U_i^1(0) \text{ for } i = 1,\dots,m-1.$$

By virtue of (3.40), we find that

$$\begin{aligned}\omega_\pi(U_i^1,U_j^0) &= \langle U_i^1,V_j^0\rangle(0) - \langle U_j^0,V_i^1\rangle(0) \quad \text{(see (2.55))}\\ &= -\langle U_j^0,V_i^1\rangle(0) \quad \text{(see (3.39))}\\ &= -\langle U_j^0,\Gamma^1\cdot U_i^1\rangle(0) = -\langle U_j^1,\Gamma^1\cdot U_i^1\rangle(0) \quad \text{(see (3.41))}\\ &= \gamma_{ij}\ .\end{aligned}$$

Thus, the condition (3.15) is met for $J^* = J^{\perp}_{\pi,M^1}$, $J = J^{\perp}_{\pi,M^0}$ and $t_0 = T$. Hence, we may conclude from Proposition 3.13 that

(3.42) $J^{\perp}_{\pi,M^0}$ does not have a focal point in $[0,\infty)$ before $J^{\perp}_{\pi,M^1}$.

Let $U_j(t) = U_j^0(t-2T)$ for $j = 1,\dots,m-1$, and denote by $\hat{J}$ the Lagrange family of $J^{\perp}_{\pi}$ spanned by $\{U_j\}_{j=1}^{m-1}$ (see Lemma 3.4), then

$$\begin{aligned}\omega_\pi(U_i,U_j^1) &= \langle U_i^0(t-2T),V_j^1(t)\rangle - \langle U_j^1(t),V_i^0(t-2T)\rangle\\ &= \langle U_i^0(2T-t),V_j^1(t)\rangle + \langle U_j^1(t),V_i^0(2T-t)\rangle \quad \text{(see (3.35)}\\ &= -\langle U_i^0(2T-t),V_j^1(2T-t)\rangle + \langle U_j^1(2T-t),V_i^0(2T-t)\rangle \text{(see (3.34))}\\ &= \omega_\pi(U_j^1,U_i^0) = \gamma_{ji}\ .\end{aligned}$$

Moreover, $$\begin{aligned}U_j^1(2T) &= U_j^1(0) \quad \text{(see (3.37))}\\ &= U_j^0(0) = U_j(2T).\end{aligned}$$

For $J = J^{\perp}_{\pi,M^1}$, $J^* = \hat{J}$ and $t_0 = 2T$, the conditions necessary to apply Proposition 3.13 are met and we thereby conclude that

(3.43) $J^{\perp}_{\pi,M^1}$ does not have a focal point in $[2T,\infty)$ before $\hat{J}$.

Our conclusion may now be verified by showing that $J^{\perp}_{\pi,M^1}$ has no focal points in $[0,2kT]$ for $k \in \mathbb{Z}^+$, and invoking (3.37). We will use induction on $k$ to show this, the case $k = 1$ being covered by (3.36). If $J^{\perp}_{\pi,M^1}$ has no focal points in $[0,2kT]$, then $J^{\perp}_{\pi,M^0}$ has no focal points in $[0,2kT]$ (see (3.42)), and $\hat{J}$ thereby has no focal points in $[2T,2kT + 2T]$. We may conclude, by invoking (3.43), that $J^{\perp}_{\pi,M^1}$ has no focal points in $[0,2(k+1)T]$.

Therefore $J^{\perp}_{\pi,M^1}$ has no focal points in $[0,\infty)$, and as a consequence of (3.37), no focal points in $(-\infty,\infty)$. □

In order to determine the location in the complex plane of the characteristic multipliers of the periodic orbit $\sigma = (\pi,\lambda)(t)$, we will now turn to the linear system of equations along $\sigma$ which were derived in Proposition 2.44. Recall that this derivation was executed by means of the following representation:

(3.44)
$$\text{if } U \in J^{\perp}_{\pi}, \text{ then}$$
$$U(t) = \sum_{i=1}^{m-1} a_i(t)\, E_i(t),$$

where $\{E_i(t)\}_{i=1}^{m-1}$ are the parallel translates of an orthonormal frame field $\{E_i\}_{i=1}^{m-1}$ for $T_{\pi(0)}\, M^0$. In all that follows we will take the basis $\{E_i\}_{i=1}^{m-1}$ equal to the $m-1$ mutually orthogonal eigen directions of the symmetric operator

$$\Gamma^1(0) : T_{\pi(0)}\, M^1(0) \to T_{\pi(0)}\, M^1(0) \quad \text{(see (3.40))}.$$

We will write the linear system of differential equations (2.51) in the form

(3.45) $$\ddot{u} = Q(t)u, \quad Q(t + 2T) = Q(t),$$

where $u(t) = [a_1(t),\ldots,a_{m-1}(t)]^* \in \mathbb{R}^{m-1}$, and * denotes matrix or vector transposition, and $Q(t)$ is the $(m-1)$ square symmetric matrix (see (2.51)) determined by

(3.46) $$\begin{aligned} Q_{ij}(t) = & -\langle D^2W(\pi)\cdot E_i, E_j\rangle(t) \\ & - 3\langle DW(\pi), E_i\rangle(t)\cdot\langle DW(\pi), E_j\rangle(t)[2(h - W\circ\pi(t))]^{-1} \end{aligned}$$

for $i,j = 1,\ldots,m-1$.

We recall some basic facts concerning the system (3.45), and refer the reader to [Hal] (pp. 117 - 136) for proofs. We will have need to consider an associated system

(3.47) $$\ddot{A} = Q(t)A,$$

where $A(t)$ is an $(m-1)$ square matrix whose columns are solution vectors of (3.45). For such a matrix $A(t)$, we let

$$B(t) = \dot{A}(t),$$

then any two solutions $A_1(t)$, $A_2(t)$ of (3.47) must satisfy

$$A_1^*(t)B_2(t) - B_1^*(t)A_2(t) = K$$

where $K$ is a constant $(m-1)$ square matrix. A solution $A(t)$ of (3.47) such that

$$A^*B - B^*A = 0,$$

will be called a Lagrange solution. Such a solution is associated through (3.44) with a unique Lagrange family $\mathcal{J}$ (see Remark 2.48).

Let $Y(t)$ be a fundamental matrix solution of (3.45): $Y(t)$ is a $2(m-1)$ square matrix whose determinant never vanishes, and whose columns $(a_1,\ldots,a_{m-1},b_1,\ldots,b_{m-1})(t)^*$ satisfy the vector differential equation (2.51). There is a unique fundamental matrix solution $Y(t)$ such that

$$Y(0) = I_{2(m-1)}.$$

For this solution, $S = Y(2T)$ is called the monodromy or resolvant matrix of the system (3.45). The monodromy matrix $S$ is symplectic, or $J$-unitary, where $J = \begin{bmatrix} 0 & I_{m-1} \\ -I_{m-1} & 0 \end{bmatrix}$ :

$$S^* J S = J . \tag{3.48}$$

If $S$ is symplectic, then $S^*$ and $S^{-1}$ are also symplectic, and the eigenvalues of $S$ come in quadruples $\{\nu, \bar{\nu}, \nu^{-1}, \bar{\nu}^{-1}\}$.

The system (3.45) is said to be disconjugate on $(-\infty, \infty)$ if every non-trivial solution $u(t)$ has at most one zero in $(-\infty, \infty)$.

Definition 3.21. A critical point $c$ of the functional $J$ is non-degenerate if Nullity $J_{**} = 0$ (see Definition 2.15).

Lemma 3.22. *Let $\sigma = (\pi, \lambda)(t)$, $t \in (-\infty, \infty)$ be the periodic orbit of $X_H$ associated with a minimum distance line $\hat{c}(s)$, $s \in [0,1]$. The differential equations (3.45), where $Q(t)$ is specified in (3.46), are disconjugate on $(-\infty, \infty)$. Let*

$$\gamma = [-\Gamma^1(0)]_\beta , \tag{3.49}$$

*be the matrix of $-\Gamma^1(0)$ with respect to the basis $\beta = \{E_i\}_{i=1}^{m-1}$ of orthonormal eigenvectors. If the critical point $\hat{c}$ of $J$ is nondegenerate, then $\gamma > 0$ as a quadratic form, and the solution $A_0(t)$ of (3.47) associated with the Lagrange family $J^\perp_{\pi, M^0}$, $A_0(0) = I_{m-1}$, may be* *used to construct the monodromy matrix $S$ of (3.45),*

$$(3.50)\qquad S = \begin{bmatrix} A & (A-I)\gamma^{-1} \\ \gamma(I+A) & A^* \end{bmatrix}, \text{ with } \gamma A = A^*\gamma, \text{ and } A = A_0(2T).$$

<u>Proof</u>. To prove that the equations (3.45) are disconjugate on $(-\infty,\infty)$, we observe that the solution $A_1(t)$ of (3.47) associated with the Lagrange family $J^{\perp}_{\pi,M^1}$, such that $A_1(0) = I_{m-1}$, satisfies $\det A_1(t) \neq 0$ on $(-\infty,\infty)$, by virtue of Proposition 3.20. The result now follows since the existence of a Lagrange solution with this property is equivalent to disconjugacy on $(-\infty,\infty)$ (see [Ha] p.388). Now assume that $\hat{c}$ is nondegenerate, then as a consequence of Corollary 3.18, Lemma 3.19, and Proposition 3.12 (b) and (3.33),

$$\gamma = [(\Gamma^0 - \Gamma^1)(0)]_\beta > 0 .$$

Let $A_0(t)$ be the Lagrange solution of (3.47), associated with the Lagrange family $J^{\perp}_{\pi,M^0}$, such that $A_0(0) = I_{m-1}$. Then consider the fundamental matrix solution

$$Y(t) = \begin{bmatrix} A_0 & (A_0-A_1)\gamma^{-1} \\ B_0 & (B_0-B_1)\gamma^{-1} \end{bmatrix} (t) .$$

It follows from (3.49), that $B_1(0) = -\gamma$. This fact together with, $B_0(0) = 0$, and $A_0(0) = A_1(0) = I_{m-1}$, implies that $Y(0) = I_{2m-2}$. Moreover, as a result of (3.34), $A_1(2T) = I_{m-1}$, and $B_1(2T) = \gamma$. Therefore

$$S = \begin{bmatrix} A_0 & (A_0-I)\gamma^{-1} \\ B_0 & (B_0-\gamma)\gamma^{-1} \end{bmatrix} (2T).$$

The relation (3.48) implies that

(3.51) $$A_0^* B_0 - B_0^* A_0 = 0 \text{ , and}$$
$$A_0^*(B_0-\Upsilon)\Upsilon^{-1}(2T) - B_0^*(A_0-I)\Upsilon^{-1}(2T) = I.$$

This latter expression reduces upon using (3.51) and $\gamma^* = \gamma$, to

$$B_0(2T) = \Upsilon(I+A_0(2T)).$$

The fact that $\Upsilon A_0(2T)$ is a symmetric matrix now follows from a further application of (3.51). Finally

$$S = \begin{bmatrix} A_0 & (A_0-I)\Upsilon^{-1} \\ \Upsilon(I+A_0) & \Upsilon A_0\Upsilon^{-1} \end{bmatrix} (2T) = \begin{bmatrix} A_0 & (A_0-I)\Upsilon^{-1} \\ \Upsilon(I+A_0) & A_0^* \end{bmatrix} (2T),$$

since $\Upsilon A_0(2T) = A_0^*(2T)\Upsilon$.

□

Lemma 3.23. *Suppose that the minimum distance line $\hat{c}$ is a nondegenerate critical point. Then the monodromy matrix $S$ (see (3.50)) has no eigenvalues equal to* $\pm 1$. *Let*

$$\zeta = (\xi,\eta)^* \in \mathbb{C}^{2m-2},$$

*then* $(\nu,\zeta)$ *is an eigenpair for $S$ if and only if*

(3.52) $$A\xi = \tfrac{1}{2}(\nu+\nu^{-1})\xi \text{ , and } \eta = ((\nu+1)/(\nu-1))\Upsilon\xi.$$

Proof. If $(\zeta,\nu)$ is an eigenpair for $S$, by virtue of (3.50), then the following pair of equations must hold, and conversely:

(3.53) $$\text{(i)} \quad A\xi + (A-I)\gamma^{-1}\eta = \nu\xi$$
$$\text{(ii)} \quad \gamma(I+A)\xi + A^*\eta = \nu\eta .$$

Using the fact that $\gamma A$ is symmetric, (3.53) reduces to

(3.54) $$(1+\nu)\gamma\xi + (1-\nu)\eta = 0 .$$

Suppose that $\nu = +1$. Then according to (3.54), $\xi = 0$, and

$$A^*\eta = \eta.$$

Let $u(t)$ denote the solution of (3.45) such that

$$u(0) = \xi = 0 \ , \ \dot{u}(0) = \eta \ .$$

Then since $S\zeta = \zeta$, $u(2T) = 0$, $\dot{u}(2T) = \eta$. However, this contradicts the conclusion of Lemma 3.22 which states that (3.45) is disconjugate on $(-\infty,\infty)$.

Next, assume that $\nu = -1$. Then by (3.54),

$$\eta = 0 \ , \ A\xi = -\xi \ .$$

We will obtain a contradiction by showing that $\gamma A^{-1}$ is a symmetric positive definite matrix. The symmetry of $A\gamma^{-1}(A^* - I)$, and hence $\gamma A^{-1}$ follows from the fact that $S^*$ is symplectic.

We will compute $[\Gamma^0(2T)]_\beta$, where $\beta = \{E_i\}_{i=1}^{m-1}$ is the same basis as in Lemma 3.22. We draw the reader's attention to the fact that $E_i(2T) = E_i(0) = E_i$ (see Lemma 2.43 (c)), and therefore (see (3.44)), for $U_j^0 \in J^\perp_{\pi,M^0}$, (using our summation convention) with

$$U_j^0(2T) = a_{ij}(2T)E_i,$$
$$[a_{ij}(2T)] = A_0(2T) = A.$$

It follows that, for $[\alpha_{kj}] = A^{-1}$,

$$\Gamma^0(2T)E_i = \Gamma^0(2T)\alpha_{ki}U_k^0(2T) = \alpha_{ki}\Gamma^0(2T)U_k^0(2T).$$

Recall that (see Proposition 2.44),

$$\Gamma^0(2T)U_k^0(2T) = Z_k^0(2T) = b_{lk}(2T)E_l, \text{ where}$$
$$[b_{lk}(2T)] = B_0(2T) = \gamma(I + A) \quad \text{(see (3.50))}.$$

Therefore,

$$\Gamma^0(2T)E_i = b_{lk}\ \alpha_{ki}\ E_l, \text{ and}$$

$$[\Gamma^0(2T)]_\beta = B_0(2T)\cdot A^{-1} = \gamma(A^{-1} + I).$$

Finally to show that $\Upsilon A^{-1}$ is a positive definite symmetric matrix, we observe that

$$0 < [(\Gamma^0-\Gamma^1)(2T)]_\beta = \Upsilon(A^{-1} + I) - \Upsilon = \Upsilon A^{-1},$$

since $-Z_i^1(0) = Z_i^1(2T)$ (see (3.34) and (2.40)), and hence

$$-\Gamma^1(0) = \Gamma^1(2T).$$

If $S$ has an eigenvalue $\nu = -1$, then $A\xi = -\xi$, and since $\gamma > 0$ (Lemma 3.22) this would contradict the fact that $\Upsilon A^{-1}$ is positive.

Finally, (3.54) and (3.53)(i) imply that

$$A\xi + \left(\frac{\nu+1}{\nu-1}\right)(A-I)\xi = \nu\xi,$$

and consequently that,

$$\{A - \left(\frac{\nu^2+1}{2\nu}\right)I\}\xi = 0 \ ,$$

$$\eta = \left(\frac{\nu+1}{\nu-1}\right) \ \gamma\xi$$

□

The following result, of a more general nature, explores the relationship between disconjugacy of the differential equation (3.45), and the position in the complex plane of the characteristic multipliers of this system. First we recall an alternate characterization of disconjugacy.

Let $C[0,\alpha]$ denote the linear space of mappings

$$C[0,\alpha] = \{u:[0,\alpha] \to \mathbb{R}^{m-1} \mid u \text{ is absolutely continuous on } [0,\alpha],$$
$$\dot{u} \in L^2[0,\alpha], \text{ and } u(0) = 0 = u(\alpha)\},$$

and let $I(u;0,\alpha)$ denote the quadratic form on $C[0,\alpha]$

$$I(u;0,\alpha) = \int_0^\alpha \{|\dot{u}|^2 + u^* Q(t)u\}dt,$$

where $Q(t)$ is the $m$-1 square symmetric matrix of (3.46).

Lemma 3.24. *The differential equation (3.45) is disconjugate on* $[0,\infty)$, *if and only if, for every compact interval*

$$[0,\alpha] \subset [0,\infty),$$

$I(u;0,\alpha)$ *is positive definite on* $C[0,\alpha]$; *in particular,*

$$I(u;0,\alpha) = 0 \Leftrightarrow u = 0.$$

We refer the reader to [Ha] (p.390) for a proof of this result.

Proposition 3.25. *Let* $(\nu,\zeta)$ *be an eigenpair for the monodromy matrix* $S$ *of the equations (3.45). Suppose that (3.45) is disconjugate on* $[0,\infty)$. *If* $\nu \in$ *unit circle,* $\nu \neq \pm 1$, *and* $\zeta = (\xi,\eta)^* \in \mathbb{C}^{2m-2}$, *then* $\xi \notin \mathbb{R}^{m-1}$. *That is,* $\xi$ *must have a nonzero complex component.*

Proof. Suppose that $\nu$ is an eigenvalue for the monodromy matrix $S$ which lies on the unit circle, $\nu \neq \pm 1$. Since $S$ is real, we can without loss of generality assume that

$$\nu = e^{i\alpha 2T}, \text{ with } 0 < \alpha 2T < \pi.$$

We will assume that the eigenvector $\zeta$ of $S$ corresponding to $\nu$, $\zeta = (\xi,\eta)^*$, satisfies $\xi \in \mathbb{R}^{m-1}$, and obtain a contradiction.

Let the monodromy matrix $S$ satisfy

$$S = Y(2T), \text{ where } Y(0) = I_{2m-2},$$

$Y(t) = \begin{vmatrix} U_0 & U_1 \\ Z_0 & Z_1 \end{vmatrix}(t)$, and $U_0, U_1$ are Lagrange solutions of (3.47)

such that (see (3.48)),

$$(3.55) \qquad \text{(i)} \quad U_i^* Z_i - Z_i^* U_i = 0 \ (i = 0,1),$$
$$\text{(ii)} \quad U_0^* Z_1 - Z_0^* U_1 = I_{m-1} \ .$$

The theorem of Floquet (see [Ha] p.60) implies that, for $n \in \mathbb{Z}^+$

$$U_0(2nT)\xi + U_1(2nT)\eta = e^{i\alpha 2nT}\ \xi.$$

If we assume that $\xi \in \mathbb{R}^{m-1}$, $\eta = \eta_1 + i\eta_2 \in \mathbb{C}^{m-1}$, then rewriting our last equation in real and imaginary terms, we find

$$(3.56) \qquad \text{(i)} \quad U_0(2nT)\xi + U_1(2nT)\eta_1 = \cos\alpha 2nT\ \xi$$
$$\text{(ii)} \quad U_1(2nT)\eta_2 = \sin\alpha 2nT\ \xi.$$

We observe that, if $\alpha 2T/\pi$ is rational, then either $u(t) = U_0(t)\xi + U_1(t)\eta_1$, or $\hat{u}(t) = U_1(t)\eta_2$, will have infinitely many zeroes on $[0,\infty)$, thereby contradicting our assumption of disconjugacy in the system (3.45). We may therefore assume that $\alpha 2T/\pi$ is irrational. In this case, neither $\cos\alpha 2nT$ nor $\sin\alpha 2nT$ vanish for any integer $n$. For a temporarily unspecified positive integer $l$, consider the solution of (3.45),

$$\overline{u}(t) = \frac{U_0(t)}{\cos\alpha 2lT}\,\xi + \frac{U_1(t)}{\cos\alpha 2lT}\,\eta_1 - \frac{U_1(t)}{\sin\alpha 2lT}\,\eta_2 \ .$$

Then, for any integer $k$, $0 < k < l$, by virtue of (3.56)

$$\overline{u}(2lT) = 0, \ \overline{u}(2kT) = \left\{\frac{\cos\alpha 2kT}{\cos\alpha 2lT} - \frac{\sin\alpha 2kT}{\sin\alpha 2lT}\right\}\xi \ .$$

Denote the expression in the braces by $\delta = \delta(k)$. Then,

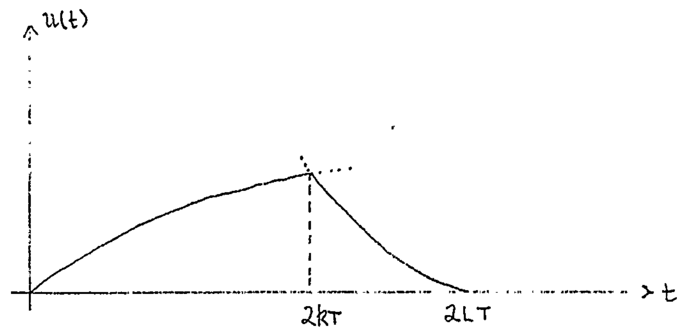
$$\delta = \frac{\sin\alpha 2T(l-k)}{\cos\alpha 2lT\ \sin\alpha 2lT} \neq 0, \text{ since } \alpha 2T/\pi \text{ is irrational.}$$

Let

$$u(t) = \begin{cases} \dfrac{U_1(t)}{\sin\alpha 2kT}\eta_2 , & t \in [0,2kT] \\[2ex] \dfrac{U_0(t)}{\delta\ \cos\alpha 2lT}\xi + \dfrac{U_1(t)}{\delta\ \cos\alpha 2lT}\eta_1 - \dfrac{U_1(t)}{\delta\ \sin\alpha 2lT}\eta_2, & t \in (2kT,2lT] \end{cases}$$

then the properties of $u(t)$ are summarized in Figure 17.

Fig 17




In particular, $u \in C[0,2lT]$, and $u(t)$ is piecewise $C^2$ on $[0,2lT]$. Therefore, after an integration by parts, we find

$$I(u;0,2lT) = +u^*(2kT)\Delta_{2kT}\dot{u} - \int_0^{2lT} u^*(\ddot{u}-Qu)\,dt,$$

where $\Delta_{2kT}\dot{u} = (\dot{u}^+)(2kT) - (\dot{u}^-)(2kT)$. Since $u(t)$ is a broken solution of (3.45), and $u(2kT) = \xi$,

$$I(u;0,2lT) = -(\dot{u}^-)^*(2kT)\xi + \xi^*(\dot{u}^+)(2kT).$$

Since $u$ is a broken continuous arc at $2kT$ we may evaluate $u(2kT) = \xi$ in two ways. Denoting $U_0(2kT)$ by $U_0$ etc., we find

$$I(u;0,2lT) = -\eta_2^* \frac{Z_1^*}{\sin\alpha 2kT}\left(\frac{U_0}{\delta\ \cos\alpha 2lT}\xi + \frac{U_1}{\delta\ \cos\alpha 2lT}\eta_1 - \frac{U_1}{\delta\ \sin\alpha 2lT}\eta_2\right)$$

$$+ \eta_2^* \frac{U_1^*}{\sin\alpha 2kT}\left(\frac{Z_0}{\delta\ \cos\alpha 2lT}\xi + \frac{Z_1}{\delta\ \cos\alpha 2lT}\eta_1 - \frac{Z_1}{\delta\ \sin\alpha 2lT}\eta_2\right)$$

$$= -\eta_2^* \frac{(Z_1^*U_0 - U_1^*Z_0)}{\delta \sin\alpha 2kT \cos\alpha 2lT}\xi - \eta_2^* \frac{(Z_1^*U_1 - U_1^*Z_1)}{\delta \cos\alpha 2lT \sin\alpha 2kT}\eta_1 + \eta_2^* \frac{(Z_1^*U_1 - U_1^*Z_1)}{\delta \sin\alpha 2kT \sin\alpha 2lT}\eta_2$$

$$= \frac{-\eta_2^*\xi}{\delta \sin\alpha 2kT \cos\alpha 2lT}, \text{ by virtue of (3.55) (i), (ii).}$$

We will now show that it is possible to choose $k, l$, $0 < k < l$, such that $I(u;0,2lT) \le 0$. Indeed, by assumption $\alpha 2T/\pi$ is irrational, and Jacobi's theorem tells us that

$$\{P_n = e^{i\alpha 2nT} \mid n \in \mathbb{Z}^+\}$$

is dense on the unit circle. There are three possible situations, which we summarize below:

(a) $\eta_2^*\xi < 0$, choose $P_k, P_l$ in the configuration of Figure 18;

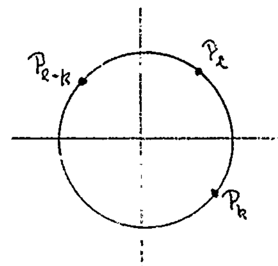



Fig 18

(b) $\eta_2^*\xi = 0 \Rightarrow I(u;0,2lT) = 0$;

(c) $\eta_2^*\xi > 0$, choose $P_k, P_l$ in the configuration of Figure 19.

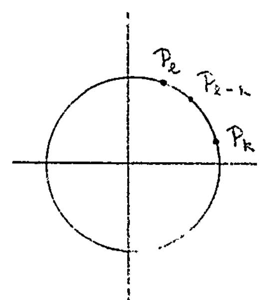



Fig 19

In all three cases, $I(u;0,2lT) \le 0$, and $u \not\equiv 0$. Therefore, invoking Lemma 3.24, we may conclude that (3.45) is not disconjugate on $[0,\infty)$. This is the desired contradiction, and completes the proof. □

Preparatory to the main result of Chapter 3, we recall our previous notation and some standard terminology.

We have used the notation $\Phi^t$ for the flow of the Hamiltonian vector field

$$X_H(x,y) = (y,-DW(x)).$$

Let $\sigma = (\pi,\lambda)(t)$ be a periodic orbit of $X_H$, and consider the linearized Poincare mapping at a point $z_0 = (\pi,\lambda)(t_0)$,

$$\widetilde{P}_{t_0} = T_{z_0}\Phi^{2T} : \mathbb{R}^{2m} \to \mathbb{R}^{2m}.$$

We recall that $\widetilde{P}_{t_0}$ is a symplectic mapping, and observe that $T_{z_0}H^{-1}(h)$ is invariant under $\widetilde{P}_{t_0}$. Denote the restriction by $P_{t_0}$,

$$P_{t_0} = \widetilde{P}_{t_0} \mid T_{z_0}H^{-1}(h).$$

The characteristic multipliers of $\sigma$ are the $2m-2$ eigenvalues of $P_{t_0}$ other than the eigenvalue +1 corresponding to the eigenvector $X_H(z_0)$, and these multipliers are independent of our choice of $t_0$. The orbit $\sigma$ is said to be a hyperbolic periodic orbit if all of its characteristic multipliers are off the unit circle.

The following result is used to show that the multipliers of $\sigma(t)$ are the same as the characteristic multipliers of the differential equation (3.45) with $Q(t)$ specified in (3.46). We recall our notation from Lemma 2.46. The symplectic subspace $E(t_0)$ of $T_{z_0}H^{-1}(h)$, and the symplectic form $\omega_E$ are defined respectively as

$$E(t_0) = \Lambda(t_0)\cdot(T_{z_0}H^{-1}(h)) \quad \text{(see (2.52) for } \Lambda(t_0)),$$

$$\omega = \sum_{i=1}^{m} dx_i \wedge dy_i, \quad \omega_E = \omega|_{E(t_0)}.$$

<u>Lemma 3.26</u>. *Let* $z_0 = (\pi,\lambda)(t_0)$*, and* $\check{P}_{t_0} : T_{z_0}H^{-1}(h) \to T_{z_0}H^{-1}(h)$ *be the mapping*

$$\check{P}_{t_0}\cdot\zeta = \begin{cases} P_{t_0}\zeta - \dfrac{\langle\xi,\lambda\rangle}{2(h-W\circ\pi)}(t_0+2T)X_H(z_0), & \text{if } t_0 \notin T\cdot\mathbb{Z} \\ P_{t_0}\zeta + \dfrac{\langle\eta,DW(\pi)\rangle}{|DW(\pi)|^2}(t_0+2T)X_H(z_0), & \text{if } t_0 = kT,\ k \in \mathbb{Z}. \end{cases}$$

*where* $(\xi,\eta)(t)$ *is the solution of the linearized Hamiltonian equations,*

$$\dot{\xi} = \eta,\ \dot{\eta} = -D^2W(\pi)\xi,$$

*with* $\zeta = (\xi,\eta)(t_0)$*. Let* $\Lambda(t_0)$*,* $E(t_0)$ *respectively denote the projection operator, and the symplectic subspace of* $T_{z_0}H^{-1}(h)$ *specified in Lemma 2.46. Then,*

(a) $\check{P}_{t_0}\cdot\Lambda(t_0) = \Lambda(t_0)\cdot P_{t_0}$;

(b) $E(t_0)$ *is invariant under* $\check{P}_{t_0}$;

(c) $\check{P}_{t_0}$ *is symplectic with respect to the form* $\omega_E$.

*Proof.* The proof of this result will be for the case $t_0 \notin T\cdot\mathbb{Z}$, the other case being analogous.

(a) Recall that (see Remark 2.34),

$$P_{t_0}\zeta = (\xi,\eta)(t_0+2T), \text{ and in particular}$$
$$P_{t_0}X_H(z_0) = X_H(z_0).$$

Therefore, using the expression for $\Lambda(t_0)$ from (2.52),

$$\check{P}_{t_0}\cdot\Lambda(t_0)\zeta = P_{t_0}\left\{\zeta - \frac{\langle\xi,\lambda\rangle(t_0)}{2(h-W\circ\pi)}X_H(z_0)\right\} - \frac{\left\langle \left\{\xi - \frac{\langle\xi,\lambda\rangle(t_0)}{2(h-W\circ\pi)(t_0)}\lambda\right\}, \lambda\right\rangle(t_0+2T)}{2(h-W\circ\pi)}X_H(z_0)$$

$$= P_{t_0}\zeta - \frac{\langle\xi,\lambda\rangle(t_0)}{2(h-W\circ\pi)(t_0)}X_H(z_0) + \frac{\langle\xi,\lambda\rangle(t_0)}{2(h-W\circ\pi)(t_0)}X_H(z_0) - \frac{\langle\xi,\lambda\rangle(t_0+2T)}{2(h-W\circ\pi)(t_0+2T)}X_H(z_0)$$

$= \Lambda(t_0) \cdot P_{t_0} \zeta.$

(b) This follows directly from the projection properties of $\Lambda(t_0)$, and part (a).

(c) For $\zeta_1, \zeta_2 \in E(t_0)$, $\alpha_i = -\dfrac{\langle \xi_i, \lambda \rangle (t_0+2T)}{2(h-W\circ\pi)}$ ,

$$\begin{aligned}\omega_E(\check{P}_{t_0}\zeta_1, \check{P}_{t_0}\zeta_2) &= \omega(P_{t_0}\zeta_1+\alpha_1 X_H, P_{t_0}\zeta_2+\alpha_2 X_H) \\ &= \omega(P_{t_0}\zeta_1, P_{t_0}\zeta_2) \\ &= \omega_E(\zeta_1,\zeta_2), \text{ since } P_{t_0} \text{ is symplectic.}\end{aligned}$$

□

Theorem 3.27. *If the minimum distance line $\hat{c}(s)$, $s \in [0,1]$, is a non-degenerate critical point of the functional $J$, then the associated periodic orbit of $X_H$, $\sigma(t) = (\pi,\lambda)(t)$, $t \in (-\infty,\infty)$, is a hyperbolic periodic orbit.*

*Proof.* By virtue of Lemma 3.26(c), there is a polynomial $Q(\lambda)$ such that $\det(\lambda - P_{t_0}) = (\lambda-1)Q(\lambda)$, and $\det(\lambda - \check{P}_{t_0}) = \lambda Q(\lambda)$. Therefore the characteristic multipliers of $\sigma$ are the same as the eigenvalues of $\check{P}_{t_0}$ restricted to $E(t_0)$. Now by virtue of (2.38), $\check{P}_{t_0}\big|_{E(t_0)}$ is the $2T$-period mapping of the reparameterized co-Jacobi equations (see (2.28)): if $U \in J_\pi^\perp$, then $(U,V)(t_0) \in E(t_0)$ (see (2.52)), and

$$\check{P}_{t_0}(U,V)(t_0) = (U,V)(t_0+2T) \in E(t_0).$$

On the other hand, the relationship between the vector field $(U,V)(t)$, and the vector field $(U,Z)(t)$ is provided by the linear mapping $\Omega(t_0)$:

$$\Omega(t_0) = \begin{bmatrix} I_m & 0 \\ \dfrac{\lambda\otimes DW(\pi)}{2(h-W\circ\pi)}(t_0) & I_m \end{bmatrix} \text{ if } t_0 \notin T\cdot\mathbb{Z},$$

$$\Omega(t_0) = \begin{bmatrix} I_m & 0 \\ 0 & I_m \end{bmatrix} \quad \text{if } t_0 = kT,\ k \in \mathbb{Z};$$

$$\begin{bmatrix} U \\ V \end{bmatrix}(t_0) = \Omega(t_0) \begin{bmatrix} U \\ Z \end{bmatrix}(t_0) \qquad \text{(see (2.40))}.$$

Since $\Omega(t_0)$ is a nonsingular transformation, $\check{P}_{t_0}$ is conjugate to the $2T$-period mapping of the vector fields $(U,Z)$. This latter mapping, when expressed in appropriate coordinates, is the monodromy matrix $S$ (see (3.50)) of the differential equation (3.45) with $Q(t)$ specified in (3.46). Therefore, to show that $\sigma$ is hyperbolic we need only show that $S$ has no eigenvalues on the unit circle.

To this end, let $(\nu,\zeta)$ be an eigenpair for $S$, where $\zeta = (\xi,\eta)^* \in \mathbb{C}^{2m-2}$. By invoking Lemma 3.23, we may conclude that $\nu \neq \pm 1$, and if

$$\nu = e^{i\alpha 2T}, \text{ then } A\xi = \cos\alpha 2T\ \xi \text{ (see (3.52))}.$$

In this case, we may choose $\xi \in \mathbb{R}^{m-1}$, $\xi \neq 0$. However, this contradicts the conclusion of Proposition 3.25 which is in force since (3.45) is disconjugate on $(-\infty,\infty)$ by virtue of Lemma 3.22. Therefore, we may conclude that $|\nu| \neq 1$ for any characteristic multiplier $\nu$, and this completes the proof.

□

This last result, answers in the affirmative a conjecture of Birkhoff [B] (p. 130) which states that in dimensions higher than $m = 2$, periodic orbits of minimum type are of unstable (hyperbolic) type.

## Appendix A: Ambrose's Index Theorem

In this appendix only, we will use the notation of Ambrose [A]. We first recall this notation and the principal ideas. The Riemannian manifold $M$ of dimension $d$ is given, and we consider $c$, a fixed (parameterized) geodesic of unit speed joining orthogonally the endpoint submanifolds of $M$: $K$ at $c(s_0)$, and $L$ at $c(s_1)$, $s_0 < s_1$. The boundary condition $S$ at $c(s_0)$ is defined as

$$S = (S_1, S_2),$$

a pair of linear transformations of the subspace

$$M_{s_0} = \{X \in T_{c(s_0)}M \mid X \perp c_*(s_0)\}$$

into itself, such that

$$S_1 = \text{orthogonal projection of } M_{s_0} \text{ into } T_{c(s_0)}K,$$

$$S_2 = \begin{cases} \Xi_z & \text{on } T_{c(s_0)}K \\ id & \text{on } [T_{c(s_0)}K]^\perp, \end{cases}$$

$$S_1 S_2 = S_2 S_1,$$

where $\Xi_z$ is the second fundamental form of the submanifold $K$ at $c(s_0)$ relative to $z = c_*(s_0)$, and $[T_{c(s_0)}K]^\perp$ is the orthogonal complement of $T_{c(s_0)}M$ relative to the inner product on $T_{c(s_0)}M$ induced by the Riemannian metric. The same definition holds for the boundary condition $T$ at $c(s_1)$ with $c(s_1)$ replacing $c(s_0)$.

We will prove that Ambrose's index theorem [A] (p. 86) simplifies in case the boundary condition $T$ at $c(s_1)$ arises from an hypersurface $L$ of $M$. In this case,

(1) $$T_1 = id \text{ on } M_{s_1}.$$

Ambrose uses the notation $J$ to denote the $2(d-1)$-dimensional space of Jacobi fields $X$ along $c$, orthogonal to $c_*(s)$ $\forall\, s$, and $\dot{X} = \nabla_{c_*(s)}X$ denotes the covariant derivative of $X$ in the direction of $c_*(s)$.

To an arbitrary boundary condition $S = (S_1, S_2)$ at $s = s'$, we have the associated subspaces of $J$:

$$J_S^* = \{X \in J \mid X(s') \in S_1,\ \dot{X}(s') - S_2X(s') \perp S_1\};$$

$$J_S = \{X \in J \mid X(s') \in S_1,\ \dot{X}(s') = S_2X(s')\}.$$

We make one observation concerning the subspaces $J_S, J_S^*$: if (1) holds,

$$\dot{X}(s_1) \in T_1,\ T_2X(s_1) \in T_1, \text{ and therefore}$$

(2) $$J_T = J_T^*.$$

From a given boundary condition $T$ at $s = s'$, Ambrose introduces two boundary conditions $T^*(s)$, $T(s)$ at each $s$. We repeat his definition of $T^*(s)$ because we wish to relate this to our operator $\Gamma^1(s)$ (see Definition 3.7).

$$T_1^*(s) = \text{projection of } M_s \text{ onto } \{Y(s) \mid Y \in J_T^*\}$$

$$T_2^*(s)y = \begin{cases} T_1^*(s)\dot{Y}(s) & \text{if } y = Y(s),\ Y \in J_T^*; \\ y & \text{if } y \perp T_1^*(s) \end{cases}$$

$$T_1(s) = \text{projection of } M_s \text{ onto } \{Y(s) \mid Y \in J_T\}$$

$$T_2(s) = \begin{cases} T_1(s)T_2^*(s)T_1(s) & \text{on } T_1(s) \\ id & \text{on } T_1(s)^\perp. \end{cases}$$

<u>Lemma A1</u>. *If* (1) *holds*, $T(s) = T^*(s)$, *and* $J_T = J_{T(s)}^*$.

*Proof.* From (2) we may deduce that $T_1^*(s) = T_1(s)$, and consequently

$$T_2(s) = \begin{cases} T_2^*(s)T_1^*(s) & \text{on } T_1(s) \\ id & \text{on } T_1(s)^{\perp}, \end{cases}$$

since $T_1^*(s)$, $T_2^*(s)$ commute and $T_1^*(s)$ is idempotent. Our first conclusion now follows since $T_1^*(s) = id$ on $T_1(s)$. To establish the second conclusion, we will invoke Lemma 1.4 of Ambrose [A] which states that

$$J_T^* = J_{T^*(s)}^*.$$

Now $J_T = J_T^*$ from (2), and $J_{T^*(s)}^* = J_{T(s)}^*$, therefore the lemma is proved.

□

Ambrose defines three types of conjugate points for the pair $S,T$: strong focal points, where

$$\dim \{X(s) \mid X \in J_T\} < d-1;$$

pure conjugate points, where

$$\dim (J_S^* \cap J_{T(s)}^*)/(J_S^* \cap J_T) > 0;$$

and conjugate points of mixed type where both conditions above are met.

Lemma A2. *If* (1) *holds, there are no pure conjugate points or conjugate points of mixed type.*

*Proof.* This follows immediately from Lemma A1, since

$$(J_S^* \cap J_{T(s)}^*) = (J_S^* \cap J_T).$$

□

The integer valued function $\overline{n}(s)$ is defined by Ambrose to be the order of the conjugate points $s$, and according to (1.1) of [A], and Lemma A2,

(3) $$\overline{n}(s) = \dim \{X \in J_T \mid X(s) = 0\}.$$

Now we turn our attention to the convexity term $C_{ST}$ introduced by Ambrose. He defines this as

(4) $$C_{ST} = \text{index } (S_2^*(s) - (T(t)_2^*(s)))$$

where the time values $s,t$ are specified in his Definition on p. 85.

Lemma A3. *If* (1) *holds, then* $T(t)^*(s) = T^*(s) \ \forall\, s,t$.

*Proof*. First we will show that $T^*(t)^*(s) = T^*(s)$. To this end we observe that

$$J_T^* = J^*_{T^*(t)^*(s)}.$$

Indeed, $J_T^* = J^*_{T^*(t)}$ (from (2), and Lemma A1), and $J^*_{T^*(t)} = J^*_{T^*(t)^*(s)}$ (Lemma 1.4 [A]). We may conclude that

$$T^*(t)_1^*(s) = T_1^*(s) \quad \forall\, s,t.$$

Moreover, by using this equality in the definition of $T^*(t)_2^*(s)$, we find

$$T^*(t)_2^*(s)y = \begin{cases} T_1^*(s)\dot{Y}(s) & \text{if } \ y = Y(s),\ Y \in J^*_{T^*(s)} \\ y & \text{if } \ y \perp T^*(t)_1^*(s). \end{cases}$$

As we have seen, $J_T^* = J^*_{T^*(t)}$, and therefore

$$T^*(t)_2^*(s) = T_2^*(s) \quad \forall\, s,t.$$

This proves that $T^*(t)^*(s) = T^*(s) \ \ \forall\, s,t$. Finally, we invoke Lemma A1, and conclude that, $T^*(t)^*(s) = T(t)^*(s)$. This completes the proof of the Lemma.

□

We may now restate the index theorem of Ambrose for the index form $I_{ST}$, in case (1) holds (see (2.1) [A] for $I_{ST}$, and Theorem 5, p.86):

(5) $\quad$ Index $I_{ST} = \sum_{s_0 < s < s_1} \bar{n}(s) + C_{ST}$

$= \sum_{s_0 < s < s_1}$ (focal points with multiplicity of $J_T$ at $s$)

$+ \text{index } (S_2^*(\hat{s}) - T_2^*(\hat{s}))$.

(6) $\quad$ Nullity $I_{ST} = \dim (J_S^* \cap J_T)$,

where $\hat{s} > s_0$ is chosen sufficiently close to $s_0$ so that there are no focal points of $S$ or $T$ in $(s_0, \hat{s}]$, no conjugate points of $c(s_0)$ in $(s_0, \hat{s}]$, and no conjugate points of $c(\hat{s})$ in $[s_0, \hat{s})$.

The proof of this statement is immediate from (3), (4) and Lemma A3.

To relate the second fundamental forms $S_2^*(s)$, $T_2^*(s)$ to the operators $\Gamma^0(t)$, $\Gamma^1(t)$ (see Definition 3.7) we let $\nabla$ denote the Levi-Civita connection of the Jacobi metric $(d\tau)^2$. In Remark 2.42, we introduced the vector field $Z(s)$ for an arbitrary Jacobi field $X(s)$ along $c(s)$,

$$Z(s) = \nabla_{y(s)} X,$$

where $y(s) = 2(h - W \circ c(s))\, c_*(s)$. Therefore

$$Z(s)^{\cdot} = 2(h - W \circ c(s))\, \dot{X}(s).$$

We make the observation that

$$\Gamma^0(t) = 2(h - W \circ c(s(t)))\, S_2^*(s(t)), \tag{7}$$

$$\Gamma^1(t) = 2(h - W \circ c(s(t)))\, T_2^*(s(t)),$$

with $s(t)$ specified in (2.26), and boundary conditions $K \times L$ (see 3.24)). Indeed, for $X \in J_S^*$

$$\Gamma^0(t) X(s(t)) = S_1^*(s(t)) Z(s(t)) = 2(h - W \circ c(s(t))) S_1^*(s(t)) \dot{X}(s(t))$$

$$= 2(h - W \circ c(s(t))) S_2^*(s(t)) X(s(t)),$$

and similarly for $\Gamma^1(t)$.

From (7) we may infer that $J_S^*$ is isomorphic, via a reparameterization of the fields, to $J^{\perp}_{\pi,K}$, and similarly for $J_T^*$ and $J^{\perp}_{\pi,L}$. Indeed, if $s_0 = s(t_0)$,

$$\dot{X}(s_0) - S_2X(s_0) \perp S_1 \Leftrightarrow Z(s_0) - \Gamma^0(t_0)X(s_0) \perp S_1$$

(multiply both sides by $2(h\text{-}Woc(s_0))$).

The second relation is easily seen to hold $\Leftrightarrow$

(8) $\quad U \in J^{\perp}_{\pi,K}$, with $U(t) = X(s(t))$.

Moreover, by virtue of (7), if $s_0 = s(t_0)$

(9) $\quad \text{index } (S_2^*(s_0) - T_2^*(s_0)) = \text{index } (\Gamma^0(t_0) - \Gamma^1(t_0))$.

The index theorem may be restated, with the help of (8), (9),

$$\text{Index } I_{ST} = \sum_{t_0 < t' < T-t_0} (\text{focal points of } J^{\perp}_{\pi,L} \text{ with multiplicities at } t') + \text{index } (\Gamma^0(t_0) - \Gamma^1(t_0)).$$

$$\text{Nullity } I_{ST} = \dim (J^{\perp}_{\pi,K} \cap J^{\perp}_{\pi,L}).$$

To complete our restatement of Ambrose's index theorem in the Hamiltonian format needed in Chapter 3, we need only verify that

(10) $\quad \text{Index } J_{**}\ (K \times L) = \text{Index } I_{ST}$

$\quad\quad\ \ \text{Nullity } J_{**}\ (K \times L) = \text{Nullity } I_{ST}$.

However these statements are a consequence of Theorem 2.16, and the representation of eigenvalues given in (2.7). First we replace (2.7)(b) with the boundary relations

(11) $\quad \hat{R}(s_0) - \Gamma^0\hat{P}(s_0) \perp T_{C(s_0)}K$, and $\hat{R}(s_1) - \Gamma^1\hat{P}(s_1) \perp T_{C(s_1)}L$

to account for the nonzero boundary terms in $J_{**}\ (K \times L)$ (see (3.25)). Next, it is well known that (see Hestenes [He] p.111), Index $I_{ST}$ is equal to the number of negative eigenvalues of $I_{ST}$, this value being the number of linearly independent solutions of (2.7)(a) subject to the boundary conditions (11) (use the linearized Legendre transform, Lemma 2.9, to obtain solutions of (2.7)(a) from the usual Jacobi equations). The same principle holds for zero eigenvalues, and the relations (10) are established. This completes the restatement of Ambrose's index theorem.

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