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On Some Gauge-Invariant Perturbation Methods and Their Application to Gravitational Waves

by

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ABSTRACT

We review the gauge-invariant formalism of Gerlach and Sengupta for the perturbations away from spherically symmetric spacetimes. This formalism is then applied to a cosmological model containing a perfect fluid. In particular, we obtain the propagation equations for the gauge-invariant perturbation quantities representing pure gravitational radiation. From these we derive a master equation for the odd-parity waves, and show that in the case of a Friedmann-Robertson-Walker background the same master equation governs the even-parity waves. Lastly, we compare the results obtained in the Gerlach-Sengupta and Newman-Penrose formalism for the odd-parity gravitational waves when the background is Minkowski. We show that in this situation the formalisms are equivalent, and exhibit a simple relation between the gauge-invariants of each of these formalisms.

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TABLE OF CONTENTS

page

APPROVAL PAGE ii

ABSTRACT iii

ACKNOWLEDGEMENTS iv

TABLE OF CONTENTS v

1. Introduction 1

§1.1 - Gauge Freedom in Gravitational Perturbations 1

§1.2 - Plan of the Thesis 3

§1.3 - Basic Notation 4

2. Gauge-Invariant Formalism 6

§2.1 - Background Geometry 7

§2.2 - Harmonic Expansion of Perturbations 10

§2.3 - Gauge-Invariants 14

§2.4 - Linearized Field Equations 18

3. Pure Gravitational Radiation for Perfect Fluid 22

§3.1 - Perfect Fluid 23

§3.2 - Gravitational Waves 25

§3.3 - Conditions for Pure Gravitational Radiation 27

§3.4 - Odd-Parity Perturbations 31

§3.5 - Even-Parity Perturbations 32

4. A Comparison with the Newman-Penrose Formalism 39

§4.1 - Null-Tetrad Formalism 40

§4.2 - Perturbation Methods 47

§4.3 - Gauge Fixing 52

§4.4 - Interpretation of the Results 56

5. Conclusion 60

REFERENCES 63

APPENDIX 66

CHAPTER 1

Introduction

§1.1 Gauge Freedom in General Relativity

The general covariance principle in gravitation theory requires that all physical results be preserved under general coordinate transformations. A coordinate system represents no more than a choice of label for points in spacetime, and so a change in the choice of coordinates should not alter the physical predictions of the theory. Accordingly, we say that two metrics represent the same physical spacetime if they are related by a coordinate transformation. This freedom in the metric implies that all metrics related by the action of the group of diffeomorphisms are equivalent.[1] Given a specific solution of the Einstein field equations, we can pick a unique representation if we impose a set of coordinate conditions that fixes the coordinate freedom contained in the metric.

Due to the non-linearity of the field equations, the search for physically realistic metrics within general relativity is extremely complicated, and so, in practice we are often forced to use various approximation schemes. However, in order for the approximation to be physically meaningful, it must also pass the test of general covariance. In perturbation theory, we start with an idealized model from which we obtain a solution for the field equations. We then inject some realism into our model by allowing small fluctuations in the geometry and matter content of this spacetime. The perturbations we wish to solve for correspond to the difference between the quantities in the real perturbed model and the ficticious background. In this context, general covariance means that two metric perturbations represent the same physical perturbation if they differ by the action of an infinitesimal diffeomorphism.

Conceptually, the simplest way to view these transformations is to regard them as a one-to-one identification between points in the full perturbed spacetime and those in the ficticious background. A *gauge transformation* is a change in the correspondence between the points of these two spacetimes and does not involve a coordinate transformation in the background.[2–4] Because of this gauge freedom, a general solution for the perturbation equations will contain some physically meaningless degrees of freedom. Unless we know how to work with this gauge freedom, the perturbations we obtain could very well correspond to pure coordinate variations. Moreover, it can be shown that in some cases, a perturbed quantity which is gauge-dependent can be made to vanish by the appropriate gauge condition. This was demontrated by Ellis and Bruni for the density perturbations of a Friedmann-Robertson-Walker model.[3] Since we usually select a gauge to gain information about a system, not only can our solution contain residual gauge modes, but the interpretation of a gauge-dependent quantity will be different for different choices of gauge, and we may therefore not be able to agree on the meaning of the solution.

Some gauge-invariant approaches have recently been formulated.[2–5] The motivating idea is to define objects, appearing in the perturbation analysis, that are independent of the choice of gauge. Hopefully one can find enough gauge-invariants to remove all of the terms that are sensitive to gauge from the linearized field equations. In this framework, the solution for the perturbation equations possesses an inherent meaning since it will not vary under gauge transformations.

§1.2 Plan of the Thesis

We begin in chapter 2 by reviewing the gauge-invariant formalism of Gerlach and Sengupta.[6] Their method not only facilitates the study of perturbation analysis by eliminating the gauge freedom, but also yields an elegant formulation that is particularly flexible since the only *a priori* restriction on the background is that of spherical symmetry.

General relativity predicts the existence of gravitational waves.[7–10] These correspond to variations in the gravitational field that can be viewed as *ripples* in the curvature of spacetime. Therefore, in many problems, we can think of a gravitational field as being made of two parts: a smooth background and a small fluctuation in the curvature. Although gravitational waves have been studied extensively, from both a theoretical and an experimental point of view, there is still much to be understood. For instance, the mathematical description of a gravitational wave is still unclear; many definitions have been proposed, any one of which has only a limited range of applications.

We can gain considerable insight by looking at the propagation of small gravitational waves against a background of spherically symmetric perfect fluids. In chapter 3, we propose a mathematical description for gravitational waves radiating in a perfect fluid. We then apply the Gerlach-Sengupta gauge-invariant formalism to purely gravitational radiation in a spherically symmetric perfect fluid. We derive a master equation for the odd-parity waves, and we show that in the special case of a Friedmann-Robertson-Walker background the same master equation governs the even-parity waves. These results are not new, but since they were obtained from a different approach, the material in this chapter provides an independent verification for the work of Couch and Torrence.[11]

Gravitational radiation has been the subject of several investigations.[12–14] In particular, since the waves travel at the speed of light, some interesting results have been obtained using the null-tetrad formalism of Newman and Penrose.[15] In the case of a conformally flat background, some of the fundamental quantities in the Newman-Penrose formalism are gauge-invariant. It is therefore a natural question to ask whether the gauge-invariants in these two formalisms can be related. In chapter 4, we compare the results obtained in the Gerlach-Sengupta and Newman-Penrose formalisms for the odd-parity perturbations when the background is Minkowski space. We show that in this situation the formalisms are equivalent, and exhibit a simple relation between the gauge-invariants of each of these formalisms.

§1.3 Basic Notation

Our basic geometrical notation is as follows:

Tensor Indices

greek indices $(\alpha, \beta, \gamma, \ldots)$ run from 0 to 3

Metric

total spacetime $g_{\mu\nu}$

two-sphere γ_{ab}

Christoffel Symbols

$$\Gamma^{\sigma}_{\mu\nu} = \tfrac{1}{2} g^{\sigma\rho}(g_{\mu\rho,\nu} + g_{\nu\rho,\mu} - g_{\mu\nu,\rho})$$

Riemann Tensor

$$R^{\mu}{}_{\nu\alpha\beta} = \Gamma^{\mu}_{\nu\beta,\alpha} - \Gamma^{\mu}_{\nu\alpha,\beta} + \Gamma^{\mu}_{\rho\alpha}\Gamma^{\rho}_{\nu\beta} - \Gamma^{\mu}_{\rho\beta}\Gamma^{\rho}_{\nu\alpha}$$

$$R_{\mu\nu} = R^{\alpha}{}_{\nu\alpha\beta}$$

$$R = R^{\alpha}{}_{\alpha}$$

Einstein Tensor

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R$$

Symmetrization and Antisymmetrization

$$M_{(\mu\nu)} = \tfrac{1}{2}(M_{\mu\nu} + M_{\nu\mu}),$$

$$M_{[\mu\nu]} = \tfrac{1}{2}(M_{\mu\nu} - M_{\nu\mu}).$$

Numeration

Whenever we refer to an equation derived from a chapter other than that of the reference, we indicate such by preceding the number of the chapter from which it derives; otherwise, for the sake of convenience we omit the chapter number and provide only the number of the section followed by the number of the equation.

CHAPTER 2

Gauge-Invariant Formalism

In this chapter, we review the gauge-invariant formalism of Gerlach and Sengupta[6] for the perturbations away from spherically symmetric spacetimes. First, we use the spherical symmetry to cast the background geometry in a $2 \oplus 2$ form. This allows us to separate the angular dependence, and effectively reduce the problem to a two-dimensional one. In §2.2, the perturbations are expressed as geometrical objects on the two-dimensional submanifold spanned by some *as-yet-unspecified* time and radial coordinates. From these objects we obtain new gauge-independent quantities, which we describe in §2.3. In the last section, we substitute these gauge-invariant quantities in the linearized Einstein field equations, and obtain a set of coupled partial differential equations for the gauge-invariants.

Our notation follows closely that of reference [6]. In particular, greek indices run from 0 to 3. Capital latin indices, A, B, C, etc., run from 0 to 1 and represent the time and radial coordinates, whereas lower-case latin indices, a, b, c, etc., run from 2 to 3 and denote the usual spherical angles θ and ϕ.

§2.1 Background Geometry

The structure of spacetime is given by specifying a Lorentz metric $g_{\mu\nu}$ on a four-dimensional manifold. The metric determines the curvature of the spacetime manifold, which in turn is related to the matter distribution in the universe by the Einstein field equations

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = \tfrac{1}{2} \kappa t_{\mu\nu}, \tag{1.1}$$

here the gravitational coupling constant κ is given by

$$\kappa = 16\pi G c^{-4}. \tag{1.2}$$

The explicit form of these equations can be greatly simplified if one imposes some symmetry on the metric. With this in mind, we restrict ourselves to the study of spherically symmetric spacetime. It should be noted that, while this facilitates our approach to perturbation analysis, it does not in itself constitute an unreasonable restriction. Included in the class of spherically symmetric spacetimes are some very important cases, such as Minkowski, Robertson-Walker, Schwarzschild, Kantowski-Sachs, etc.[16]

The distinguishing feature of a spherically symmetric spacetime is that it admits the rotation group $SO(3)$ as a group of isometries.[1] That is, since all radial directions are equivalent, the metric must be invariant under rotations. It is natural to use the group structure to simplify the form of the metric. Here, the orbits of the rotation group, corresponding to the action of the group on a given point, are the two-dimensional spheres S^2. Let M^4 be an arbitrary spherically symmetric Lorentzian manifold with signature $(-,+,+,+)$. The space of orbits $M^2 \equiv M^4/S^2$ is a two-dimensional manifold spanned by the time and radial coordinates x^C. With every point x^C in M^2, we associate a scalar function $R(x^C)$, which characterizes the concentric spheres in the sense that the area of the two-sphere $\widetilde{S}$ associated with the point $\tilde{x}$ in M^2 is given by $4\pi R^2(\tilde{x}^C)$.[12] The geometry on this two-surface is described by the metric

$$ds^2 = R^2 \gamma_{ab}\, dx^a dx^b, \tag{1.3}$$

where γ_{ab} is the unit-sphere metric given by

$$\gamma_{ab} dx^a dx^b = d\theta^2 + \sin^2\theta d\phi^2. \tag{1.4}$$

We can now embed this two-dimensional geometry in the four-dimensional manifold M^4. First, we choose a set of coordinates θ and ϕ on one of the two-spheres; then we extend this definition to the other two-spheres by means of geodesics. Recall that no vector field tangent to the sphere can be made everywhere non-zero. Therefore, to preserve rotational symmetry, these geodesics must be orthogonal to each two-sphere. With this definition, M^2 is a totally geodesic submanifold,[17] and the metric tensor on M^4 has the form

$$ds^2 = g_{AB}(x^C)\, dx^A dx^B + R^2(x^C)\gamma_{ab}\, dx^a dx^b. \tag{1.5}$$

If Eq. (1.1) is satisfied, then the components G_{aA} of the Einstein tensor obtained using the form of the metric, Eq. (1.5), vanish. It follows that a $2 \oplus 2$ split can also be performed on the matter tensor. Accordingly, we write the components of the energy momentum tensor corresponding to a spherically symmetric metric as

$$t_{\mu\nu} dx^\mu dx^\nu = t_{AB}(x^C) dx^A dx^B + \tfrac{1}{2} t_a{}^a R^2(x^C)\gamma_{ab} dx^a dx^b. \tag{1.6}$$

Here, because of the symmetry, the partial trace $\frac{1}{2}t_a{}^a$ is a scalar field on M^2 independent of the spherical angles θ and ϕ.

The formulation above is strictly geometrical; it allows us to take advantage of the symmetry of the spacetime with no further assumption on the background or coordinate system on M^2. Furthermore, the coefficients of the metric and the energy-momentum tensor, Eqs. (1.5) and (1.6), are given entirely in terms of geometrical objects on M^2: two tensor fields g_{AB} and t_{AB} and two scalar fields R and $\frac{1}{2}t_a{}^a$. We can now cast the background Einstein field equations, Eq. (1.1), in a reduced form on a two-dimensional manifold. We obtain one tensor and one scalar equation on M^2:

$$\frac{\kappa}{2}t_{AB} = -2(v_{A|B} + v_A v_B) + (2v_C{}^{|C} + 3v_C v^C - R^{-2})g_{AB} \equiv G_{AB}, \quad (1.7)$$

$$\frac{\kappa}{2}(\tfrac{1}{2}t_a{}^a) = v_C{}^{|C} + v_C v^C - \mathcal{R} \equiv \tfrac{1}{2}G_a{}^a. \quad (1.8)$$

Here, the vertical bars refer to covariant derivatives on M^2, the vector field v_A is constructed from the scalar field R on M^2

$$v_A = \frac{R_{,A}}{R}, \quad (1.9)$$

and $\mathcal{R}$ is the Gaussian curvature[18] on the submanifold M^2 defined by

$$R_{ABCD} = \mathcal{R}(g_{AC}g_{BD} - g_{AD}g_{BC}) \quad (1.10)$$

The conservation law $t_{\mu\nu}{}^{;\nu} = 0$, implied by equations (1.7) and (1.8) is

$$R^{-2}(R^2 t_A{}^B)_{|B} - v_A t_a{}^a = 0. \quad (1.11)$$

We give the background expressions for the connection coefficients, the Ricci curvature, the scalar curvature, and the Gaussian curvature, explicitely in the Appendix.

§2.2 Harmonic Expansion of Perturbations

We consider perturbation fields away from a spherically symmetric background. These are described by the changes in the metric tensor and the energy-momentum tensor, which we denote $h_{\mu\nu}$ and $\Delta t_{\mu\nu}$. The harmonic functions used in the expansion of the perturbations are the scalar, vector, and tensorial eigenfunctions of the Laplace operator on S^2.[19] They are classified into even or odd parity according to how they transform under the parity operator

$$P : \{\, \theta \to \pi - \theta, \ \phi \to \pi + \phi \,\}. \tag{2.1}$$

By convention, we call even a parity of $(-1)^\ell$ and odd a parity of $(-1)^{\ell+1}$.[20] The coefficients in the expansions, free of angular dependence, are geometrical objects on M^2. They represent the changes under rotation of the actual metric $g_{\mu\nu}$ and energy-momentum tensor $t_{\mu\nu}$. Accordingly, we can start by classifying the components of an arbitrary symmetric tensor on M^4 with respect to how they transform under a rotation. This will provide the necessary prescription for the harmonic decomposition on M^4.

Let us consider an arbitrary tensor on the manifold M^4

$$M_{\mu\nu} = \begin{bmatrix} M_{AB} & M_{Ab} \\ M_{aB} & M_{ab} \end{bmatrix}. \tag{2.2}$$

Under the transformation $(\theta, \phi) \to (\tilde\theta, \tilde\phi)$, we have

$$\tilde M_{\mu\nu} = \frac{\partial x^\alpha}{\partial \tilde x^\mu} \frac{\partial x^\beta}{\partial \tilde x^\nu} M_{\alpha\beta}. \tag{2.3}$$

For convenience, we shall consider each of the 2×2 components M_{AB}, M_{Ab}, M_{aB}, and M_{ab} separately. Eliminating the vanishing terms, we obtain

$$\tilde M_{AB} = \frac{\partial x^I}{\partial \tilde x^A} \frac{\partial x^J}{\partial \tilde x^B} M_{IJ} = \delta^I_A \delta^J_B M_{IJ} = M_{AB}, \tag{2.4}$$

$$\tilde{M}_{Ab} = \frac{\partial x^I}{\partial \tilde{x}^A}\frac{\partial x^j}{\partial \tilde{x}^b}M_{Ij} = \frac{\partial x^j}{\partial \tilde{x}^b}\delta^I_A M_{Ij} = \frac{\partial x^j}{\partial \tilde{x}^b}M_{Aj}, \tag{2.5}$$

$$\tilde{M}_{aB} = \frac{\partial x^i}{\partial \tilde{x}^a}\frac{\partial x^J}{\partial \tilde{x}^B}M_{iJ} = \frac{\partial x^i}{\partial \tilde{x}^a}\delta^J_B M_{iJ} = \frac{\partial x^i}{\partial \tilde{x}^a}M_{iB}, \tag{2.6}$$

$$\tilde{M}_{ab} = \frac{\partial x^i}{\partial \tilde{x}^a}\frac{\partial x^j}{\partial \tilde{x}^b}M_{ij}. \tag{2.7}$$

It follows from equation (2.4) that, under rotation, M_{AB} transforms like a scalar. A suitable basis for the expansion of M_{AB} can be constructed if we use the scalar spherical harmonics $Y_{\ell,m}(\theta,\phi)$, which are known to form a complete set over the sphere. They are given by[21]

$$Y_{\ell,m} = \frac{(-1)^\ell}{(2\ell)!!}\left[\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}\right]^{\frac{1}{2}}(\sin\theta)^m\left[\frac{d^{\ell+m}}{(d\cos\theta)^{\ell+m}}[(\sin\theta)^{2\ell}]\right]\exp(im\phi), \tag{2.8}$$

where

$$n!! = \begin{cases} 2\times4\times6\times\ldots\times n & n \text{ even} \\ 1\times3\times5\times\ldots\times n & n \text{ odd.}\end{cases} \tag{2.9}$$

To simplify the notation, we omit the angular indices ℓ and m, since they will not be used explicitly in our formalism other than to evaluate the parity of the harmonic functions.The three independent components of a symmetric tensor, (M_{00}, M_{01}, M_{11}), can be expanded if we use the following tensors:

$$\hat{s}_1 = \begin{bmatrix} Y & 0 & & \\ 0 & 0 & \multicolumn{2}{c}{0} \\ & & 0 & 0 \\ \multicolumn{2}{c}{0} & 0 & 0 \end{bmatrix},\quad \hat{s}_2 = \begin{bmatrix} 0 & Y & & \\ * & 0 & \multicolumn{2}{c}{0} \\ & & 0 & 0 \\ \multicolumn{2}{c}{0} & 0 & 0 \end{bmatrix},\quad \hat{s}_3 = \begin{bmatrix} 0 & 0 & & \\ 0 & Y & \multicolumn{2}{c}{0} \\ & & 0 & 0 \\ \multicolumn{2}{c}{0} & 0 & 0 \end{bmatrix}. \tag{2.10}$$

The $(*)$ symbol indicates that $M_{AB} = M_{BA}$.

Under the parity operation, Eq. (2.1), the scalar harmonics Y satisfy

$$\hat{P}(Y) = (-1)^\ell Y. \tag{2.11}$$

Hence, the tensors $\hat{s}_1, \hat{s}_2$, and $\hat{s}_3$ defined in Eq. (2.10) are even-parity tensors. The coefficients in the representation of a 2-scalar are

$$M_{AB} = M_{AB}^{even}(x^C) Y(\theta, \phi). \tag{2.12}$$

For a symmetric tensor, $M_{Aa} = M_{aA}$, and so we can consider these components simultaneously. It is evident, from Eqs. (2.5) – (2.6), that they transform like two-dimensional covectors under rotation. Therefore we need a basis for the cotangent space at each point of S^2. We take the gradient of Y, and the dual of the gradient

$$S_a = \epsilon_a{}^b Y_{,b}. \tag{2.13}$$

Here, $\epsilon_a{}^b$ is the antisymmetric tensor on S^2 given by

$$\epsilon_2{}^3 = -\frac{1}{\sin\theta}, \qquad \epsilon_3{}^2 = \sin\theta, \qquad \epsilon_2{}^2 = \epsilon_3{}^3 = 0. \tag{2.14}$$

From direct calculation, we see that the vector $Y_{,a}$ has even parity, and since taking the dual changes the parity, the vector S_a has the opposite parity. The four independent components $(M_{02}, M_{03}, M_{12}, M_{13})$ can be expanded if we use the following pairs of odd-parity and even-parity tensors:

$$\hat{v}_1^{odd} = \begin{bmatrix} 0 & \begin{matrix} S_2 & S_3 \\ 0 & 0 \end{matrix} \\ \begin{matrix} * & 0 \\ * & 0 \end{matrix} & 0 \end{bmatrix}, \qquad \hat{v}_2^{odd} = \begin{bmatrix} 0 & \begin{matrix} 0 & 0 \\ S_2 & S_3 \end{matrix} \\ \begin{matrix} 0 & * \\ 0 & * \end{matrix} & 0 \end{bmatrix},$$

$$\hat{v}_3^{even} = \begin{bmatrix} 0 & \begin{matrix} Y_{,2} & Y_{,3} \\ 0 & 0 \end{matrix} \\ \begin{matrix} * & 0 \\ * & 0 \end{matrix} & 0 \end{bmatrix}, \qquad \hat{v}_4^{even} = \begin{bmatrix} 0 & \begin{matrix} 0 & 0 \\ Y_{,2} & Y_{,3} \end{matrix} \\ \begin{matrix} 0 & * \\ 0 & * \end{matrix} & 0 \end{bmatrix}. \tag{2.15}$$

The expansion of a 2-vector in M^4 splits into an odd and an even part whose respective coefficients are

$$\begin{aligned} M_{Aa}^{odd} &= M_A^{odd}(x^C)\, S_a(\theta, \phi), \\ M_{Aa}^{even} &= M_A^{even}(x^C)\, Y_{,a}(\theta, \phi). \end{aligned} \tag{2.16}$$

To complete our basis, we need a representation for a second-rank two-dimensional symmetric tensor, Eq. (2.7). The tensors used in the expansion of the last three independent components (M_{22}, M_{23}, M_{33}) are obtained by taking covariant derivatives on $Y_{,a}$ and S_a. They are

$$\hat{t}_1^{odd} = \begin{bmatrix} \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 2S_{2:2} & S_{2:3}+S_{3:2} \\ * & 2S_{3:3} \end{matrix} \end{bmatrix},$$

$$\hat{t}_2^{even} = \begin{bmatrix} \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} \gamma_{22} & 0 \\ 0 & \gamma_{33} \end{matrix} \end{bmatrix}, \qquad \hat{t}_3^{even} = \begin{bmatrix} \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} Y_{,2:2} & Y_{,2:3} \\ * & Y_{,3:3} \end{matrix} \end{bmatrix}. \tag{2.17}$$

Here, γ_{ab} is the metric on the unit-sphere given by Eq. (1.4), and the colon denotes covariant derivatives on the manifold S^2 . The tensors $\hat{t}_1^{odd}, \hat{t}_2^{even}$, and $\hat{t}_3^{even}$ are linearly independent for the radiative modes $\ell \geq 2$.[22] We do not consider the modes with $\ell = 0$ and $\ell = 1$, since they are non-radiatives.[10] Using Eq. (2.17), we can write the coefficients for the odd and even components of a 2-tensor on M^4 as

$$\begin{aligned} M_{ab}^{odd} &= 2M(x^C)\, S_{(a:b)}(\theta, \phi), \\ M_{ab}^{even} &= M_1(x^C)\, \gamma_{ab} Y(\theta, \phi) + M_2(x^C)\, Y_{,a:b}(\theta, \phi). \end{aligned} \tag{2.18}$$

The tensors given in Eqs. (2.10), (2.15), and (2.17) provide a covariant basis for symmetric tensors on M^4. In the Gerlach-Sengupta formalism, we use this basis to expand the perturbations of the metric and energy-momentum tensor. The coefficients in the expansions are modeled after Eqs. (2.12), (2.16), and (2.18). We give the harmonic decomposition for the odd and even parities separately. For the odd-parity perturbations we have

$$h_{\mu\nu}^{odd} dx^\mu dx^\nu = 2h_A^{odd}(x^C) S_a(\theta, \phi) dx^{(A} dx^{a)} + 2h(x^C) S_{(a:b)}(\theta, \phi) dx^a dx^b, \tag{2.19a}$$

and

$$\Delta t_{\mu\nu}^{odd} dx^\mu dx^\nu = 2\Delta t_A^{odd} S_a(\theta, \phi) dx^{(A} dx^{a)} + 2\Delta t(x^C) S_{(a:b)}(\theta, \phi) dx^a dx^b, \tag{2.19b}$$

The coefficients in these expansions consist of two vector fields h_A^{odd} and Δt_A^{odd}, and two scalar fields h and Δt on M^2.

Similarly, we can write the even-parity perturbations as

$$h_{\mu\nu}^{even}dx^\mu dx^\nu = h_{AB}(x^C)\,Y(\theta,\phi)dx^A dx^B + 2h_A^{even}(x^C)Y_{,a}(\theta,\phi)dx^{(A}dx^{a)}+ \\ + R^2(x^C)[K(x^C)\gamma_{ab}Y(\theta,\phi) + G(x^C)Y_{,a:b}(\theta,\phi)]dx^a dx^b, \tag{2.20a}$$

and

$$\Delta t_{\mu\nu}^{even}dx^\mu dx^\nu = \Delta t_{AB}(x^C)Y(\theta,\phi)dx^A dx^B + 2\Delta t_A^{even}(x^C)Y_{,a}(\theta,\phi)dx^{(A}dx^{a)}+ \\ + [R^2(x^C)\Delta t^1(x^C)\gamma_{ab}Y(\theta,\phi) + \Delta t^2(x^C)Y_{,a:b}(\theta,\phi)]dx^a dx^b. \tag{2.20b}$$

They are characterized by two symmetric tensor fields h_{AB} and Δt_{AB}, two vector fields h_A^{even} and Δt_A^{even}, and four scalar fields K, G, Δt^1,and Δt^2 on M^2.

§2.3 Gauge-Invariants

In the linearized theory, two perturbations p and $\bar{p}$ represent the same physical perturbation if they differ by the action of an infinitesimal diffeomorphism. The gauge freedom corresponds to the Lie derivative of the background quantity with respect to an infinitesimal vector field X_μ. Under a gauge transformation

$$p \rightarrow \bar{p} = p + \pounds_X \overset{0}{p}, \tag{3.1}$$

and ambiguities arise, for we now have two equivalent descriptions for a perturbed quantity

$$\overset{0}{p} + p \iff \overset{0}{p} + \bar{p}. \tag{3.2}$$

To counter this difficulty, we adopt a new representation for the perturbation fields, based on Moncrief's definition of gauge-invariance.[5]

A perturbation p is said to be gauge-invariant if the gauge changes

$$\delta p \equiv \bar{p} - p = 0. \tag{3.3}$$

For example, it can be seen from Eq. (3.1), that any quantity for which the zeroth-order part vanishes identically is a gauge-invariant quantity. The metric perturbations $h_{\mu\nu}$ and the matter perturbations $\Delta t_{\mu\nu}$ are position-dependent in the background, and hence are gauge-dependent.[2] However, we can construct gauge-invariant quantities from these by taking appropriate linear combinations of the coefficients introduced in their harmonic expansions, Eqs. (2.19) and (2.20). We start by calculating the changes due to an infinitesimal gauge transformation in the metric and energy-momentum tensor perturbations.

As shown in §2.2, an infinitesimal 4-vector field can be expanded by means of spherical harmonics. The odd and even components are

$$X_\mu^{odd} dx^\mu = M(x^C) S_a dx^a, \tag{3.4}$$

and

$$X_\mu^{even} dx^\mu = \xi_A(x^C) Y dx^A + \xi(x^C) Y_{,a} dx^a, \tag{3.5}$$

for some scalar fields M and ξ, and vector field ξ_A on the submanifold M^2.

The Lie derivatives of the background metric and energy-momentum tensor with respect to X_μ are

$$\begin{aligned}\pounds_X \overset{0}{g}_{\mu\nu}\, dx^\mu dx^\nu &= -2X_{(\mu;\nu)} dx^\mu dx^\nu \\ &= -2\left[X_{(A|B)} dx^A dx^B + (X_{A:a} + X_{a|A}) dx^{(A} dx^{a)} + X_{(a:b)} dx^a dx^b\right],\end{aligned} \tag{3.6a}$$

$$\pounds_X \overset{0}{t}_{\mu\nu}\, dx^\mu dx^\nu = -(t_{\mu\nu;\sigma}X^\sigma + t_\mu{}^\sigma X_{\sigma;\nu} + t_\nu{}^\sigma X_{\sigma;\mu})dx^\mu dx^\nu$$
$$= -(t_{AB|C}X^C + t_{AC}X^C{}_{|B} + t_{CB}X^C{}_{|A})dx^A dx^B +$$
$$- 2(t_{AC}X^C{}_{:a} + \tfrac{1}{2}t_a{}^a X^C{}_{|A})dx^{(A}dx^{a)} +$$
$$- [(t^c{}_c R^2)_{,A}\gamma_{ab}X^A + t^c{}_c X_{(a:b)}]dx^a dx^b. \tag{3.6b}$$

For the odd-parity, we substitute in Eqs. (3.6a) and (3.6b) the odd component X_μ^{odd}, Eq. (3.4), of the infinitesimal vector field X_μ, and obtain

$$\pounds_X \overset{0}{g}_{\mu\nu}\, dx^\mu dx^\nu = -2\left[R^2(R^{-2}M)_{,A}S_a dx^{(A}dx^{a)} + MS_{(a:b)}dx^a dx^b\right], \tag{3.7a}$$

$$\pounds_X \overset{0}{t}_{\mu\nu}\, dx^\mu dx^\nu = -t^c{}_c\left[R^2(R^{-2}M)_{,A}S_a dx^{(A}dx^{a)} + MS_{(a:b)}dx^a dx^b\right]. \tag{3.7b}$$

By comparing the coefficients in Eqs. (3.7a) and (3.7b) with those appearing in the odd expansions of the metric and energy-momentum tensor perturbations, Eqs. (2.19a) and (2.19b), we obtain the gauge changes of the odd-parity metric coefficients

$$\delta h_A = -R^2(R^{-2}M)_{,A}\,,$$
$$\delta h = -M, \tag{3.8}$$

and the gauge changes of the odd-parity matter coefficients

$$\delta\Delta t_A = -\tfrac{1}{2}t_a{}^a R^2(R^{-2}M)_{,A}\,,$$
$$\delta\Delta t = -\tfrac{1}{2}t_a{}^a M\,. \tag{3.9}$$

Similarly, we substitute the even-parity component X_μ^{even}, Eq. (3.5), into the Lie derivative of the background metric and energy-momentum tensor, Eqs.

(3.6a) and (3.6b), and obtain for the even-parity

$$\pounds_X \overset{0}{g}_{\mu\nu}\, dx^\mu dx^\nu = -2\left[\xi_{(A|B)} Y dx^A dx^B + [\xi_A + R^2(R^{-2}\xi)_{,A}] Y_{,A} dx^{(A} dx^{a)}\right] + \\ -2\left[\xi Y_{,a:b} + R^2 v^A \xi_A \gamma_{ab} Y\right] dx^a dx^b, \tag{3.10a}$$

$$\pounds_X \overset{0}{t}_{\mu\nu}\, dx^\mu dx^\nu = -[t_{AB|C} + t_{AC}\xi^C{}_{|B} + t_{CB}\xi^C{}_{|A}] Y dx^A dx^B + \\ -2[t_{AC}\xi^C + \tfrac{1}{2} t_a{}^a R^2 (R^{-2}\xi)_{,A}] Y_{,a} dx^{(A} dx^{a)} + \\ -[\xi^C (t^b{}_b R^2)_{,C} \gamma_{ab} Y + t^b{}_b \xi Y_{,a:b}] dx^a dx^b. \tag{3.10b}$$

Comparing the coefficients in Eqs. (3.10a) and (3.10b) with those in the even-parity expansions of the metric and energy-momentum tensor perturbations, Eqs. (2.20a) and (2.20b), we obtain the gauge changes for the even-parity metric coefficients

$$\begin{aligned} \delta h_{AB} &= -2\xi_{(A|B)}, \\ \delta h_A &= -\xi_A - R^2(R^{-2}\xi)_{,A}, \\ \delta K &= -2v^A \xi_A, \\ \delta G &= -2R^{-2}\xi, \end{aligned} \tag{3.11}$$

and the gauge changes for the even-parity matter coefficients

$$\begin{aligned} \delta\Delta t_{AB} &= -(t_{AB|C}\xi^C + t_{AC}\xi^C{}_{|B} + t_{CB}\xi^C{}_{|A}), \\ \delta\Delta t_A &= -t_{AB}\xi^B - \tfrac{1}{2} t_a{}^a R^2 (R^{-2}\xi)_{,A}, \\ \delta\Delta t^1 &= -\tfrac{1}{2} R^2 (t^b{}_b R^2)_{,C} \xi^C, \\ \delta\Delta t^2 &= -t^b{}_b \xi. \end{aligned} \tag{3.12}$$

We now define gauge-invariant geometrical objects on M^2. These will be used to produce an entirely gauge-invariant formulation of the linearized Einstein field equations. For the odd-parity, they are

$$\text{(metric)} \qquad k_A = h_A - R^2(R^{-2}h)_{,A}\,, \tag{3.13a}$$

$$\text{(matter)} \qquad L_A = \Delta t_A - \tfrac{1}{2}t_a{}^a h_A\,,$$

$$L = \Delta t - \tfrac{1}{2}t_a{}^a h\,. \tag{3.13b}$$

For the even-parity, we have

$$\text{(metric)} \qquad k_{AB} = h_{AB} - 2p_{(A|B)},$$

$$k = K - 2v^A p_A\,, \tag{3.14a}$$

$$\text{(matter)} \qquad T_{AB} = \Delta t_{AB} - t_{AB}{}^{|C}p_C - (t_{CA}p^C{}_{|B} + t_{CB}p^C{}_{|A}),$$

$$T_A = \Delta t_A - t_A{}^C p_C - \tfrac{1}{4}t^a{}_a R^2 G_{,A}\,,$$

$$T^1 = \Delta t^1 - R^{-2}p^C(\tfrac{1}{2}t_a{}^a R^2)_{,C}\,,$$

$$T^2 = \Delta t^2 - \tfrac{1}{2}t_a{}^a R^2 G, \tag{3.14b}$$

where

$$p_C = h_C - \tfrac{1}{2}R^2 G_{,C}\,. \tag{3.15}$$

It is easily verified, from Eqs. (3.8) – (3.9), and (3.11) – (3.12), that the gauge changes under the gauge transformation given in Eqs. (3.4) – (3.5) vanish for all the above quantities.

§2.4 Linearized Field Equations

The perturbed physical spacetime is approximated to first-order in the usual manner[12]

$$g_{\mu\nu} = \overset{0}{g}_{\mu\nu} + h_{\mu\nu}, \tag{4.1}$$

$$t_{\mu\nu} = \overset{0}{t}_{\mu\nu} + \Delta t_{\mu\nu}. \tag{4.2}$$

The zeroth-order parts, labelled by 0, are the background quantities described in §2.1, Eqs. (1.5) and (1.6). In the linearized theory, we use the background metric $\overset{0}{g}_{\mu\nu}$ to raise or lower the indices of the first-order metric perturbation $h_{\mu\nu}$. We can calculate the contravariant form of the actual metric $g_{\mu\nu}$, using the identity

$$g_{\mu\sigma}g^{\sigma\nu} = \delta_\mu{}^\nu. \tag{4.3}$$

Expanding this identity, we have

$$\begin{aligned}\delta_\mu{}^\nu &= (\overset{0}{g}_{\mu\sigma} + h_{\mu\sigma})(\overset{0}{g}{}^{\sigma\nu} + \tilde{h}^{\sigma\nu}) \\ &= \delta_\mu{}^\nu + \overset{0}{g}_{\mu\sigma}\,\tilde{h}^{\sigma\nu} + h_{\mu\sigma}\,\overset{0}{g}{}^{\sigma\nu} + 2nd \;\text{ order terms.}\end{aligned} \tag{4.4}$$

Thus, to first-order,

$$\overset{0}{g}_{\mu\sigma}\,\tilde{h}^{\sigma\nu} = -h_{\mu\sigma}\,\overset{0}{g}{}^{\sigma\nu}, \tag{4.5}$$

which implies

$$\tilde{h}^{\alpha\nu} = -h^{\alpha\nu}. \tag{4.6}$$

Therefore, the contravariant form of the perturbed metric is

$$g^{\mu\nu} = \overset{0}{g}{}^{\mu\nu} - h^{\mu\nu}. \tag{4.7}$$

To obtain the linearized Einstein field equations, we must first expand the Christoffel connections, the curvature tensors, and the Einstein tensor in $h_{\mu\nu}$ using Eqs. (4.1) and (4.7). Neglecting higher order terms, we can write each of the perturbed geometrical quantities in the form: $\Gamma^\sigma_{\mu\nu} = \overset{0}{\Gamma}{}^\sigma_{\mu\nu} + \overset{1}{\Gamma}{}^\sigma_{\mu\nu}$, $R_{\mu\nu} = \overset{0}{R}_{\mu\nu} + \overset{1}{R}_{\mu\nu}$, and so on. The first-order corrections, labelled by 1, are

$$\overset{1}{\Gamma}{}^\sigma_{\mu\nu} = \tfrac{1}{2}(h_\mu{}^\sigma{}_{;\nu} + h_\nu{}^\sigma{}_{;\mu} - h_{\mu\nu;}{}^\sigma), \tag{4.8}$$

$$\overset{1}{R}_{\mu\nu} = -\tfrac{1}{2}(h_{\mu\nu;\alpha}{}^{;\alpha} - h_{\mu\alpha;\nu}{}^{;\alpha} - h_{\alpha\nu;\mu}{}^{;\alpha} + h_\alpha{}^\alpha{}_{;\mu\nu}), \tag{4.9}$$

$$\overset{1}{R} = -(h^\beta{}_{\beta;\alpha}{}^{;\alpha} - h_{\beta\alpha}{}^{;\beta\alpha} + h^{\alpha\beta}\,\overset{0}{R}_{\alpha\beta}), \tag{4.10}$$

$$\overset{1}{G}_{\mu\nu} = \overset{1}{R}_{\mu\nu} - \tfrac{1}{2}\,(\overset{0}{g}_{\mu\nu}\overset{1}{R} + h_{\mu\nu}\,\overset{0}{R}). \tag{4.11}$$

Here, the semi-colon denotes covariant derivative with respect to the background metric $\overset{0}{g}_{\mu\nu}$.

The linearized Einstein field equations are the first-order part of the full perturbed equations

$$\overset{0}{G}_{\mu\nu} + \overset{1}{G}_{\mu\nu} = \tfrac{1}{2}\kappa(\overset{0}{t}_{\mu\nu} + \Delta t_{\mu\nu}). \tag{4.12}$$

Using Eqs. (4.8) – (4.11), we can write them in terms of the perturbations of the metric tensor:

$$\begin{aligned}(h_{\mu\nu;\alpha}{}^{;\alpha} - h_{\mu\alpha;\nu}{}^{;\alpha} - h_{\alpha\nu;\mu}{}^{;\alpha} + h_{\alpha}{}^{\alpha}{}_{;\mu\nu}) + \\ - g_{\mu\nu}(h^{\beta}{}_{\beta;\alpha}{}^{;\alpha} - h_{\beta\alpha}{}^{;\beta\alpha} + h^{\alpha\beta}\overset{0}{R}_{\alpha\beta}) + h_{\mu\nu}\overset{0}{R} = -\kappa\Delta t_{\mu\nu}.\end{aligned} \tag{4.13}$$

This can now be translated into a gauge-invariant formulation. The procedure is straight-forward but lengthy, and, that being the case, we do not reproduce it here. Instead, we give a brief overview of the method, and state the results obtained by Gerlach and Sengupta.[22]

First, the harmonic expansions of the perturbations of the metric and matter tensors, Eqs. (2.19)–(2.20), are substituted in the linearized equations Eq. (4.13). Collecting the coefficients of the linearly-independent harmonics, we obtain two odd-parity equations corresponding to S_a and $S_{(a:b)}$, and four even-parity field equations corresponding to Y, $Y_{,a}$, $\gamma_{ab}Y$, and $Y_{,a:b}$. We then introduce the gauge-invariant perturbation objects defined in §2.3, Eqs. (3.13) – (3.14), by forming the appropriate linear combinations. The result is a set of partial differential equations on M^2.

For the odd-parity, we have a scalar and a vector equation. They are

$$k_A{}^{|A} = \kappa L, \tag{4.14a}$$

$$-[R^4(R^{-2}k^A)^{|C} - R^4(R^{-2}k^C)^{|A}]_{|C} + (\ell-1)(\ell+2)k^A = \kappa R^2 L^A. \tag{4.14b}$$

The set of even-parity equations consists of a tensor, a vector, and two scalar equations. They are

$$
\begin{aligned}
&2v^C(k_{AB|C} - k_{CA|B} - k_{CB|A}) - [\ell(\ell+1)R^{-2} + \overset{0}{G}_C{}^C + \overset{0}{G}_a{}^a + 2\overset{0}{\mathcal{R}}]k_{AB} + \\
&\quad - 2g_{AB}v^C(k_{ED|C} - k_{CE|D} - k_{CD|E})g^{ED} + g_{AB}(2v^{C|D} + 4v^Cv^D - \overset{0}{G}{}^{CD})k_{CD} + \\
&\quad + g_{AB}[\ell(\ell+1)R^{-2} + \tfrac{1}{2}(\overset{0}{G}_C{}^C + \overset{0}{G}_a{}^a) + \overset{0}{\mathcal{R}}]k_D{}^D + 2(v_A k_{,B} + v_B k_{,A} + k_{,A|B}) + \\
&\quad - g_{AB}[2k_{,C}{}^{|C} + 6v^C k_{,C} - (\ell-1)(\ell+2)R^{-2}k] = -\kappa T_{AB},
\end{aligned} \tag{4.15a}
$$

$$
k_{,A} - k_{AC}{}^{|C} + k_C{}^C{}_{|A} - v_A k_C{}^C = -\kappa T_A, \tag{4.15b}
$$

$$
\begin{aligned}
&(k_{,C}{}^{|C} + 2v^C k_{,C} + \overset{0}{G}_a{}^a k) + [k_{CD}{}^{|C|D} + 2v^C k_{CD}{}^{|D} + 2(v^{C|D} + v^Cv^D)k_{CD}] + \\
&\qquad + [k_C{}^C{}_{|D}{}^{|D} + v^C k_D{}^D{}_{|C} + \overset{0}{\mathcal{R}}\, k_C{}^C - \tfrac{1}{2}\ell(\ell+1)R^{-2}k_C{}^C] = \kappa T^1,
\end{aligned} \tag{4.15c}
$$

$$
k_C{}^C = -\kappa T^2. \tag{4.15d}
$$

We have now obtained linearized propagation equations for the metric gauge-invariant quantities k, k_A, and k_{AB}. The formalism is free of the traditional gauge problem, in the sense that the solutions to these equations cannot be transformed by means of gauge transformation.

CHAPTER 3

Pure Gravitational Radiation for Perfect Fluid

In this chapter, we apply the formalism developed in the preceding chapter to a cosmological model containing a perfect fluid. In particular, we obtain second-order propagation equations for the gauge-invariant perturbation quantities k, k_A, and k_{AB}, representing pure gravitational radiation. First, we introduce the background line element and the energy-monentum tensor describing a spherically symmetric perfect fluid. In §3.2, we propose a physical and mathematical description of pure gravitational waves. From this we deduce, in §3.3, a prescription for gravitational waves in the Gerlach and Sengupta formalism, and a set of gauge conditions for their existence. These enable us to partially decouple the linearized field equations for the gauge-invariants, and derive a master equation that governs the perturbations. The result for the odd-parity perturbations is given in §3.4. In §3.5, after imposing some restrictions on the line element, we obtain a similar result for the even-parity perturbations. Most of the material in this chapter provides an independent verification for the work of Couch and Torrence.[11]

§3.1 Perfect Fluid

We consider the spherically symmetric line element

$$ds^2 = g_{AB}dx^A dx^B + R^2(x^C)\gamma_{ab}dx^a dx^b, \tag{1.1}$$

derived in chapter 2. Locally, the two-dimensional Lorentz metric g_{AB} can always be put in a diagonal form

$$^{(2)}ds^2 = A^2(t,r)[-dt^2 + dr^2], \tag{1.2}$$

where the chosen coordinates $x^0 = t$ and $x^1 = r$ are solutions of the harmonic equation $x^{|C}{}_{|C} = 0$ on the submanifold M^2.[1]

Using Eq. (1.2), we can rewrite the arbitrary spherically symmetric line element, Eq. (1.1), as

$$ds^2 = A^2(t,r)[-dt^2 + dr^2 + F^2(t,r)d\Omega^2], \tag{1.3}$$

where

$$d\Omega^2 \equiv \gamma_{ab}dx^a dx^b. \tag{1.4}$$

A is a cosmic scale factor, and the function F is defined by $R = AF$. The specific expressions for the scalars A and F can be obtained, when the matter content of the universe is known, by solving the Einstein field equations [see §2.1]. For simplicity, we restrict ourselves to cosmological models containing a perfect fluid. These are described by the fluid energy ρ and pressure p, and the 4-velocity vector field V_μ tangent to the flow lines of the fluid. In terms of these, the energy-momentum tensor becomes

$$t_{\mu\nu} = (\rho + \mathrm{p})\mathrm{V}_\mu \mathrm{V}_\nu + \mathrm{p}\mathrm{g}_{\mu\nu}. \tag{1.5}$$

Because of rotational invariance, the energy density and the pressure of a spherically symmetric perfect fluid have no angular dependence:

$$\rho = \rho(x^C) \qquad \text{and} \qquad \mathrm{p} = \mathrm{p}(\mathrm{x}^\mathrm{C}). \tag{1.6}$$

Furthermore, the 4-velocity vector field V_μ is orthogonal to the two-spheres, and

$$V_\mu = V_\mu(x^C). \tag{1.7}$$

We are free to choose a comoving coordinate system , $\vec{V} = A\partial_t$, while preserving the form of the metric Eq. (1.3).[16] In this system of reference, the 4-velocity vector field is a timelike unit-length vector

$$V^\alpha V_\alpha = -1, \tag{1.8}$$

given by

$$V_\mu = A\delta^0_\mu. \tag{1.9}$$

Together, the form of the metric and matter tensors, Eqs. (1.3) and (1.5), and the velocity of the fluid, Eq. (1.9), require that the Einstein tensor $G_{\mu\nu}$ be diagonal. If one calculates the components of the Einstein tensor using the line element described by Eq. (1.3), all, with the exception of G_{01}, off-diagonal components of the Einstein tensor, vanish. The additional condition $G_{01} = 0$ is achieved whenever the scalars A and F are chosen to be functions of one variable only. Accordingly, we can subdivide the allowed line elements into four classes:

$$\begin{aligned} I.\ A &= A(t) \ \text{ and } \ F = F(t), \\ II.\ A &= A(t) \ \text{ and } \ F = F(r), \\ III.\ A &= A(r) \ \text{ and } \ F = F(r), \\ IV.\ A &= A(r) \ \text{ and } \ F = F(t). \end{aligned} \tag{1.10}$$

Among these classes we find both the static and time-dependent solutions. In particular, the first class represents the Kantowski-Sachs line element, and the second class includes the Friedmann-Robertson-Walker universes.

§3.2 Gravitational Waves

There are three types of perturbations associated with a perfect fluid. They represent gravitational waves and changes in the fluid density and velocity.[23] Gravitational waves correspond to fluctuations in the geometry; in the case of pure gravitational radiation, the matter content of the perturbed universe is assumed to remain identical to that of the background. There are several mathematical descriptions for gravitational waves in the literature;[12,23–25] in each case the description is tailored to suit a particular formalism.

The definition we propose for purely gravitational perturbations in a spherically symmetric model with perfect fluid is

$$\Delta t_\mu{}^\nu = 0, \tag{2.1}$$

$$h_{\mu\nu}\overset{0}{V}{}^\mu\overset{0}{V}{}^\nu = 0. \tag{2.2}$$

This definition is a natural one since, as demonstrated by Niedra,[24] in the case of a Friedmann background, the vanishing of the perturbations of the energy momentum with mixed indices Eq. (2.1), together with the condition (2.2), imply that there is no change in the matter content:

$$\overset{1}{\rho} = \overset{1}{\mathrm{p}} = 0 \qquad \text{and} \qquad \overset{1}{V}_\mu = 0. \tag{2.3}$$

It turns out that the specialization to a Friedmann background is not necessary. In fact, we can extend this result to the case of a general spherically symmetric model filled with a perfect fluid.

Given a perfect fluid, we can write the first-order correction of the energy-momentum tensor with mixed indices as

$$\Delta t_\mu{}^\nu = (\overset{0}{\rho} + \overset{0}{\mathrm{p}})\left[\overset{1}{V}_\mu\overset{0}{V}{}^\nu - \overset{0}{V}_\mu\overset{1}{V}{}^\nu\right] + (\overset{1}{\rho} + \overset{1}{\mathrm{p}})\,\overset{0}{V}_\mu\overset{0}{V}{}^\nu + \overset{1}{\mathrm{p}}\delta_\mu{}^\nu. \tag{2.4}$$

If we impose condition (2.1), and substitute for the zeroth-order velocity vector the definition given in §3.1, we obtain

$$\overset{1}{\mathrm{p}} = 0, \tag{2.5}$$

$$\overset{1}{\rho} = -(\overset{0}{\rho} + \overset{0}{\mathrm{p}}) \left[A^{-1} \overset{1}{V}_0 + A \overset{1}{V}{}^0 \right], \tag{2.6}$$

$$(\overset{0}{\rho} + \overset{0}{\mathrm{p}}) \left[A \overset{1}{V}{}^i \right] = 0, \tag{2.7}$$

$$(\overset{0}{\rho} + \overset{0}{\mathrm{p}}) \left[A^{-1} \overset{1}{V}_i \right] = 0, \tag{2.8}$$

where $i = 1, 2$, and 3.

We then expand to first-order the contraction of the perturbed velocity vector field V_μ with itself, Eq. (1.8), and obtain the relation

$$2 \overset{0}{V}_\beta \overset{1}{V}{}^\beta = h_{\alpha\beta} \overset{0}{V}{}^\alpha \overset{0}{V}{}^\beta. \tag{2.9}$$

The condition that $h_{\mu\nu} \overset{0}{V}{}^\mu \overset{0}{V}{}^\nu = 0$, Eq. (2.2), yields

$$\overset{1}{V}_0 = \overset{1}{V}{}^0 = 0; \tag{2.10}$$

substituting in Eq. (2.6), we have

$$\overset{1}{\rho} = 0. \tag{2.11}$$

The result is that, provided $\overset{0}{\rho} + \overset{0}{\mathrm{p}} \neq 0$, the conditions

$$\Delta t_\mu{}^\nu = 0 \qquad \text{and} \qquad h_{\mu\nu} \overset{0}{V}{}^\mu \overset{0}{V}{}^\nu = 0, \tag{2.12}$$

imply no perturbations of the matter:

$$\overset{1}{\rho} = \overset{1}{\mathrm{p}} = 0 \qquad \text{and} \qquad \overset{1}{V}_\mu = 0. \tag{2.13}$$

In the special case of a *false vacuum*,[26] characterized by $\overset{0}{\rho} + \overset{0}{\mathrm{p}} = 0$, we cannot obtain, at this stage, the full set of conditions corresponding to no perturbations of matter. However, we show in the next section that, in this case, the conditions given in Eq. (2.12) do not utilize all of the gauge freedom, and so we are free to use three of the gauge functions to satisfy the required condition

$$\overset{1}{V}_i = 0, \tag{2.14}$$

where $i = 1, 2$, and 3. To simplify the notation in the remainder of this chapter, we omit the first-order superscript on the perturbed quantities.

§3.3 Conditions for Pure Gravitational Radiation

Gauge transformations do not affect the perturbation quantities we wish to solve for. They do, however, play an important role in determining the linearized field equations for these gauge-invariants. Since the conditions defining pure gravitational perturbations, Eq. (2.12), are gauge-dependent statements, we can use them to reduce the gauge freedom in the metric and energy-momentum tensor perturbations.

The full energy-momentum tensor $t_{\mu\nu}$ can be obtained by contracting $t_\mu{}^\nu$ with the perturbed metric. To first-order, we have

$$t_{\mu\nu} = \overset{0}{g}_{\mu\alpha} \overset{0}{t}{}^\alpha{}_\nu + \overset{0}{g}_{\mu\alpha} \Delta t^\alpha{}_\nu + h_{\mu\alpha} \overset{0}{t}{}^\alpha{}_\nu. \tag{3.1}$$

By applying the condition for pure gravitational radiation, $\Delta t_\mu{}^\nu = 0$, we obtain the first-order correction for the matter tensor

$$\Delta t_{\mu\nu} = h_{\mu\alpha} \overset{0}{t}{}^\alpha{}_\nu. \tag{3.2}$$

Next, using the definition for the zeroth-order velocity vector field, Eq. (1.9), we obtain from the condition $h_{\mu\nu} \overset{0}{V}{}^\mu \overset{0}{V}{}^\nu = 0$

$$h_{00} = 0. \tag{3.3}$$

It follows that the perturbations represent pure gravitational radiation if there exists a gauge such that the condition

$$\Delta t_{\mu\nu} dx^\mu dx^\nu = h_{\mu\alpha} \overset{0}{t}{}^{\alpha}{}_{\nu} dx^\mu dx^\nu \tag{3.4}$$

is achieved, and

$$h_{00} = 0. \tag{3.5}$$

By comparing the coefficients in both the odd and even harmonic expansions of the energy-momentum tensor perturbations, Eqs. (2.2.19) and (2.2.20), with those of the expression on the right-hand side of Eq. (3.4), we obtain a set of conditions equivalent to $\Delta t_\mu{}^\nu = 0$. Because expressions such as (3.4) and (3.5) differ for opposite parity, we proceed by analyzing the odd and even parities separately.

For the odd-parity, since the perturbations do not have scalar components, the condition $h_{00} = 0$ is trivially satisfied. The requirement that the energy-momentum vanishes, Eq. (3.4), gives

$$\Delta t_A^{odd} S_a dx^A dx^a = h_A^{odd} S_b \overset{0}{t}{}^b{}_a dx^A dx^a, \tag{3.6a}$$

$$\Delta t_A^{odd} S_a dx^A dx^a = h_D^{odd} S_a \overset{0}{t}{}^D{}_A dx^a dx^A, \tag{3.6b}$$

$$\Delta t S_{(a:b)} dx^a dx^b = h S_{(a:d)} \overset{0}{t}{}^d{}_b dx^a dx^b. \tag{3.6c}$$

For a perfect fluid, this implies

$$\Delta t_0^{odd} = h_0^{odd} \overset{0}{\mathrm{p}} = -h_0^{odd} \overset{0}{\rho}, \tag{3.7a}$$

$$\Delta t_1^{odd} = h_1^{odd} \overset{0}{\mathrm{p}}, \tag{3.7b}$$

$$\Delta t = h \overset{0}{\mathrm{p}}. \tag{3.7c}$$

The gauge freedom in the odd-parity perturbations consists of one scalar function on M^2 [see §2.3]. In the case where $\overset{0}{\rho} + \overset{0}{\mathrm{p}} = 0$, there is no inconsistency in Eq. (3.7), and the gauge function $M(x^C)$ is arbitrary. If $\overset{0}{\rho} + \overset{0}{\mathrm{p}} \neq 0$, we can use M to set $h_0^{odd} = 0$. In this gauge, pure gravitational radiation implies

$$\Delta t_0^{odd} = h_0^{odd} = 0, \tag{3.8a}$$

$$\Delta t_1^{odd} = h_1^{odd} \overset{0}{\mathrm{p}}, \tag{3.8b}$$

$$\Delta t = h \overset{0}{\mathrm{p}}. \tag{3.8c}$$

For the even-parity perturbations, condition (3.4) yields the following equations:

$$\Delta t_{AB} Y dx^A dx^B = h_{AD} \overset{0}{t}{}^D{}_B Y dx^A dx^B, \tag{3.9a}$$

$$\Delta t_A^{even} Y_{,A} dx^A dx^a = h_A^{even} Y_{,b} \overset{0}{t}{}^b{}_a dx^A dx^a, \tag{3.9b}$$

$$\Delta t_A^{even} Y_{,a} dx^a dx^A = h_D^{even} Y_{,a} \overset{0}{t}{}^D{}_A dx^a dx^A, \tag{3.9c}$$

$$R^2 \Delta t^1 Y \gamma_{ab} dx^a dx^b = R^2 K Y (\gamma_{ad} \overset{0}{t}{}^d{}_b + \gamma_{bd} \overset{0}{t}{}^d{}_a) dx^a dx^b, \tag{3.9d}$$

$$\Delta t^2 Y_{,a:b} dx^a dx^b = R^2 G Y (Y_{,a:d} \overset{0}{t}{}^d{}_b + Y_{,b:d} \overset{0}{t}{}^d{}_a) dx^a dx^b. \tag{3.9e}$$

And the full set of equations for a perfect fluid is given by

$$\Delta t_{00} = -h_{00} \overset{0}{\rho}, \tag{3.10a}$$

$$\Delta t_{01} = h_{01} \overset{0}{\mathrm{p}} = -h_{01} \overset{0}{\rho}, \tag{3.10b}$$

$$\Delta t_{11} = h_{11} \overset{0}{\mathrm{p}}, \tag{3.10c}$$

$$\Delta t_0^{even} = h_0^{even} \overset{0}{\mathrm{p}} = -h_0^{even} \overset{0}{\rho}, \tag{3.10d}$$

$$\Delta t_1^{even} = h_1^{even} \overset{0}{\mathrm{p}}, \tag{3.10e}$$

$$\Delta t^1 = \overset{0}{\mathrm{p}} K, \tag{3.10f}$$

$$\Delta t^2 = \overset{0}{\mathrm{p}} R^2 G, \tag{3.10g}$$

together with

$$h_{00} = 0. \tag{3.10h}$$

The gauge freedom in the even-parity perturbations is contained in the scalar field $\xi(x^C)$ and the 2-vector field $\xi_A(x^C)$ [see §2.3]. When $\overset{0}{\rho} + \overset{0}{\mathrm{p}} = 0$, we can use one function worth of gauge freedom to satisfy $h_{00} = 0$, and the remaining two gauge functions are arbitrary. Otherwise, we select a gauge such that $h_{00} = h_{01} = h_0^{even} = 0$. In this gauge, the conditions implied by pure gravitational radiation are

$$\Delta t_{00} = h_{00} = 0, \tag{3.11a}$$

$$\Delta t_{01} = h_{01} = 0, \tag{3.11b}$$

$$\Delta t_{11} = h_{11}\overset{0}{\mathrm{p}}, \tag{3.11c}$$

$$\Delta t_0^{even} = h_0^{even} = 0, \tag{3.11d}$$

$$\Delta t_1^{even} = h_1^{even}\overset{0}{\mathrm{p}}, \tag{3.11e}$$

$$\Delta t^1 = \overset{0}{\mathrm{p}}K, \tag{3.11f}$$

$$\Delta t^2 = \overset{0}{\mathrm{p}}R^2 G. \tag{3.11g}$$

In summary, for the case where $\overset{0}{\rho} + \overset{0}{\mathrm{p}} \neq 0$, our choice of gauge is equivalent to the synchronous gauge

$$h_{\alpha 0} = 0, \tag{3.12}$$

widely used in perturbation theory.[23,27,28] For the *false vacuum* case, we choose $h_{00} = 0$ and require that the three arbitrary gauge functions be chosen so that the condition

$$\overset{1}{V}_i = 0 \tag{3.13}$$

is satisfied.

§3.4 Odd-Parity Perturbations

The odd-parity linearized field equations for the gauge invariants were introduced in §2.4. From these, we can obtain propagation equations for odd gravitational waves by imposing the pure gravitational radiation conditions, Eq. (3.7) for the *false vacuum* or Eq. (3.8) for the generic case $\overset{0}{\rho} + \overset{0}{\mathrm{p}} \neq 0$, derived in the previous section. It is easy to show that, under either of these conditions, the matter gauge-invariant quantities L_A and L vanish identically [see §2.3].The relevant field equations are therefore homogeneous. They can be expressed in the following form:

$$k_{0,0} = k_{1,1}, \tag{4.1}$$

and

$$\left(R^4\left[(R^{-2}k_0)_{,1} - (R^{-2}k_1)_{,0}\right]\right)_{|A} = A^2(\ell-1)(\ell+2)k_{1-A}, \tag{4.2}$$

where the angular index $\ell \geq 2$.

The explicit form of Eq. (4.2) is such that one can achieve a partial decoupling for the type of background under consideration, class I-IV [see §3.1]. With the help of Eq. (4.1), we can decouple Eq. (4.2) into an homogeneous equation on the one hand, and on the other an inhomogeneous equation driven by the solution to the homogeneous equation. In particular, k_0 satisfies an homogeneous equation if the scalar F is chosen to be a function of the time coordinate t (class I and IV), and k_1 satisfies an homogeneous equation if F is chosen to be a function of the radial coordinate r (class II and III).

But the perturbation equations receive their simplest treatment if one introduces a potential $\mathcal{V}$, which satisfies

$$R^4\left[(R^{-2}\mathcal{V}_{,1})_{,1} - (R^{-2}\mathcal{V}_{,0})_{,0}\right] = A^2(\ell-1)(\ell+2)\mathcal{V}, \tag{4.3}$$

related to the gauge-invariant quantities k_A by

$$k_{1-A} = \mathcal{V}_{,A}. \tag{4.4}$$

From Eq. (4.4), we can see that the integrability condition for the existence of the potential

$$\mathcal{V}_{,01} = \mathcal{V}_{,10}, \tag{4.5}$$

is equivalent to Eq. (4.1). By differentiating Eq. (4.3) with respect to the time coordinate t, and substituting the defining equation (4.4), we recover Eq. (4.2) with $A = 1$. We obtain Eq. (4.2) with $A = 0$ in turn from Eq. (4.3) by differentiating with respect to the radial coordinate r and substituting Eq. (4.4). It follows that the solutions of the odd-parity perturbations are most easily obtained from Eq. (4.3), and we call this equation a *master equation*, for it single-handedly governs the odd-parity perturbations.

§3.5 Even-Parity Perturbations

In this section, we discuss the propagation equations for the even-parity gravitational waves in backgrounds that are filled with perfect fluid, but whose line element are restricted to those of class II [see §3.1]

$$ds^2 = A^2(t)\left[-dt^2 + dr^2 + F^2(r)\, d\Omega^2\right]. \tag{5.1}$$

It is well known[1] that the zeroth-order solution for the field equations, corresponding to the line element (5.1) and a perfect fluid, is given by the Friedmann-Robertson-Walker cosmological models. These models describe homogeneous and isotropic spacetimes with perfect fluid. The hypersurfaces of homogeneity, $\sum_t$, are three-dimensional surfaces of constant curvature. Since it is always possible to obtain a normalized curvature ϵ by rescaling the function $A(t)$, the 3 admissible geometries for $\sum_t$, corresponding to a positive , negative, and zero curvature, are first, the unit three-sphere ($\epsilon = 1$), second, the three-dimensional flat space ($\epsilon = 0$), and third, the three-dimensional hyperboloid ($\epsilon = -1$). The Robertson-Walker line element is characterized by

$$F = \begin{cases} \sin r & \text{if } \epsilon = 1, \\ r & \text{if } \epsilon = 0, \\ \sinh r & \text{if } \epsilon = -1. \end{cases} \tag{5.2}$$

The explicit forms for the background density and pressure can be obtained from the tensorial field equation for the background Eq. (2.1.7). They are

$$\tfrac{1}{2}A^2\kappa \overset{0}{\mathrm{p}} = -\left[2(\tfrac{A_{,0}}{A})_{,0} + (\tfrac{A_{,0}}{A})^2 + \epsilon\right], \tag{5.3}$$

and

$$\tfrac{1}{2}A^2\kappa \overset{0}{\rho} = 3\left[(\tfrac{A_{,0}}{A})^2 + \epsilon\right]. \tag{5.4}$$

It follows from these expressions, that the restriction to the Robertson-Walker line element implies

$$\overset{0}{\rho} = \overset{0}{\rho}(t) \qquad \text{and} \qquad \overset{0}{\mathrm{p}} = \overset{0}{\mathrm{p}}(t). \tag{5.5}$$

When Eq. (5.5) and the condition for pure gravitational radiation Eq. (3.11) are taken into considerations, the matter gauge-invariants can be written as

$$\begin{aligned}
T^1 &= A^{-2}\overset{0}{\mathrm{p}}_{,0}p_0 + \overset{0}{\mathrm{p}}k, \\
T^2 &= 0, \\
T_1 &= 0, \\
T_0 &= (\overset{0}{\rho} + \overset{0}{\mathrm{p}})p_0, \\
T_{00} &= \overset{0}{\rho}_{,0}\, p_0 - \overset{0}{\rho}\, k_{00}, \\
T_{01} &= (\overset{0}{\rho} + \overset{0}{\mathrm{p}})p_{0,1} + \overset{0}{\mathrm{p}}k_{01}, \\
T_{11} &= \overset{0}{\mathrm{p}}_{,0}p_0 + \overset{0}{\mathrm{p}}k_{11}.
\end{aligned} \tag{5.6}$$

First, we consider the generic case $\overset{0}{\rho} + \overset{0}{\mathrm{p}} \neq 0$. The results for the *false vacuum* are similar, and will be outlined at the end of this section. If we use the matter gauge-invariants given in Eq. (5.6), the scalar field equation corresponding to T^2, Eq. (2.4.15d), yields that k_{AB} is traceless

$$k^c{}_c = 0, \tag{5.7}$$

or simply

$$k_{00} = k_{11}. \tag{5.8}$$

The differential equations for the gauge-invariant metric quantities k, k_{00}, and k_{01}, are obtained by substituting Eq. (5.8) and the matter gauge-invariants, Eq. (5.6), into the remaining linearized field equations (2.4.15). The resulting equations are

$$k_{00,1} - k_{01,0} - S_{,1} = 0, \tag{5.9a}$$

$$k_{00,0} - k_{01,1} + S_{,0} - 2v_0 S = -A^2\kappa(\overset{0}{\rho} + \overset{0}{\mathrm{p}})p_0, \tag{5.9b}$$

$$\begin{aligned} &k_{00,00} + k_{00,11} - 2k_{01,01} + 2v_1(k_{00,1} - k_{01,0}) - 2(v_0^2 - v_{0,0} + \epsilon)k_{00} + \\ &\quad + S_{,00} - S_{,11} - 2v_0 S_{,0} - 2v_1 S_{,1} - 2v_{0,0} S = -A^2\kappa\overset{0}{\mathrm{p}}_{,0} p_0, \end{aligned} \tag{5.9c}$$

$$\begin{aligned} &2\left[v_0 k_{00,0} + v_1 k_{00,1} - 2v_0 k_{01,1} - 4v_0 v_1 k_{01} + (v_0^2 + 2v_1^2 - \epsilon)k_{00}\right] + \\ &\quad + \frac{(\ell-1)(\ell+2)}{F^2}k_{00} - 2\left[S_{,11} - 2v_0 S_{,0} + 3v_1 S_{,1} + 4v_0^2 S\right] + \\ &\qquad + \frac{(\ell-1)(\ell+2)}{F^2}S = A^2\kappa\, \overset{0}{\rho}_{,0}\, p_0, \end{aligned} \tag{5.9d}$$

$$\begin{aligned} &2\left[v_0 k_{00,0} + v_1 k_{00,1} - 2v_1 k_{01,0} + (2v_{0,0} - v_0^2 - \epsilon)k_{00}\right] - \frac{(\ell-1)(\ell+2)}{F^2}k_{00} + \\ &\quad + 2\left[S_{,00} - v_1 S_{,1} - 2v_0 S_{,0} - 2v_{0,0} S\right] + \frac{(\ell-1)(\ell+2)}{F^2}S = -A^2\kappa\overset{0}{\mathrm{p}}_{,0} p_0, \end{aligned} \tag{5.9e}$$

$$\begin{aligned} &2\left[v_0 k_{00,1} - v_1 k_{00,0} + 2v_1 v_0 k_{00} + (v_{0,0} - v_0^2 - v_1^2)k_{01}\right] - \frac{(\ell-1)(\ell+2)}{F^2}k_{01} + \\ &\quad + 2\left[S_{,01} - 2v_0 S_{,1} + v_1 S_{,0} - 2v_0 v_1 S\right] = -A^2\kappa(\overset{0}{\rho} + \overset{0}{\mathrm{p}})\left[\tfrac{1}{2}k_{01} + p_{0,1}\right]. \end{aligned} \tag{5.9f}$$

Here, the function S is defined by

$$S \equiv A^2 k, \tag{5.10}$$

and the components of the vector field v_A are

$$v_0 = \frac{A_{,0}}{A},$$
$$v_1 = \frac{F_{,1}}{F}. \tag{5.11}$$

In the above, we have used the following identities:

$$\frac{1}{F^2} = -\left(\frac{F_{,1}}{F}\right)_{,1}, \tag{5.12}$$
$$\epsilon = \frac{1}{F^2} - \left(\frac{F_{,1}}{F}\right)^2. \tag{5.13}$$

We can use equation (5.9b) and our knowledge of the background matter fields to eliminate the gauge-dependent quantities from the remaining propagation equations. The background conservation law Eq. (2.1.11) yields

$$\overset{0}{\rho}_{,0} = -3v_0(\overset{0}{\rho} + \overset{0}{\mathrm{p}}). \tag{5.14}$$

Therefore, by using (5.9b), we obtain for the right-hand side of Eq. (5.9d)

$$A^2\kappa\, \overset{0}{\rho}_{,0}\, p_0 = 3v_0[k_{00,0} - k_{01,1} + S_{,0} - 2v_0 S]. \tag{5.15}$$

Next, if we take the partial derivative with respect to time of (5.9b), we get

$$- A^2\kappa \left[(\overset{0}{\rho}_{,0} + \overset{0}{\mathrm{p}}_{,0})p_0 + (\overset{0}{\rho} + \overset{0}{\mathrm{p}})p_{0,0} + 2v_0(\overset{0}{\rho} + \overset{0}{\mathrm{p}})p_0\right]$$
$$= k_{00,00} - k_{01,10} + S_{,00} - 2v_{0,0}S - 2v_0 S_{,0}. \tag{5.16}$$

Using Eq. (5.14) and the gauge-invariant metric quantity

$$k_{00} = -2(p_{0,0} - v_0 p_0), \tag{5.17}$$

we obtain for the right-hand sides of Eqs. (5.9c) and (5.9e)

$$-A^2\kappa\overset{0}{\mathrm{p}}_{,0}p_0 = k_{00,00} - k_{01,10} + S_{,00} - 2v_0 S_{,0} - 2(v_0^2 - v_{0,0} + \epsilon)k_{00}. \tag{5.18}$$

Finally, we use the partial derivative with respect to the radial coordinate r of (5.9b), and the background matter fields Eqs. (5.3) and (5.4), and obtain for the right-hand side of Eq. (5.9f)

$$-A^2\kappa(\overset{0}{\rho}+\overset{0}{\mathrm{p}})\left[\tfrac{1}{2}k_{01}+p_{0,1}\right]=k_{00,01}-k_{01,11}+S_{,01}-2v_0S_{,1}+2(v_0^2-v_{0,0}+\epsilon)k_{01}. \tag{5.19}$$

In order to decouple the linearized field equations, we introduce a new gauge-invariant quantity Ψ, defined by

$$\Psi=k_{00}-S. \tag{5.20}$$

In terms of Ψ, equation (5.9a) reduces to

$$\Psi_{,1}=k_{01,0}. \tag{5.21}$$

We obtain an homogeneous partial differential equation for Ψ by substituting this result and Eq. (5.18) in (5.9e):

$$\Psi_{,11}-\Psi_{,00}-2v_1\Psi_{,1}+2v_{0,0}\Psi_{,0}-\frac{(\ell-1)(\ell+2)}{F^2}\Psi=0. \tag{5.22}$$

Likewise, we substitute Eqs. (5.19) and (5.21) in (5.9f). The result is an inhomogeneous partial diffential equation for the gauge-invariant quantity k_{01} that is driven by a solution of Eq. (5.22):

$$k_{01,11}-k_{01,00}+2v_0k_{01,0}+2(\epsilon-v_1^2)k_{01}-\frac{(\ell-1)(\ell+2)}{F^2}k_{01}=2v_1\Psi_{,0}-4v_1v_0\Psi. \tag{5.23}$$

As in the odd-parity perturbations, the propagation equations for Ψ and k_{01}, Eqs. (5.22) and (5.23), are best described if we introduce a potential $\mathcal{V}$ [see §3.4], satisfying

$$R^4\left[(R^{-2}\mathcal{V}_{,1})_{,1}-(R^{-2}\mathcal{V}_{,0})_{,0}\right]=A^2(\ell-1)(\ell+2)\mathcal{V}. \tag{5.24}$$

Here, the potential $\mathcal{V}$ is related to the gauge-invariant quantities by

$$\Psi = \mathcal{V}_{,0}, \tag{5.25}$$

and

$$k_{01} = \mathcal{V}_{,1}. \tag{5.26}$$

Eq. (5.21) is equivalent to the integrability condition for $\mathcal{V}$; if we take the partial derivative of (5.24) with respect to time we have Eq. (5.22). On the other hand, if we take the partial derivative of (5.24) with respect to r we get Eq. (5.23). This similarity between the two parities was first pointed out by Couch and Torrence[11].

We can determine the gauge-invariant quantity k from the remaining field equations (5.9c) and (5.9d). First we substitute Eq. (5.15) in Eq. (5.9d). The result is

$$\begin{aligned}2v_1\Psi_{,1} - v_0(\Psi_{,0} + k_{01,1}) - 8v_0v_1k_{01} + 2(v_0^2 + 2v_1^2 - \epsilon)\Psi + \frac{(\ell-1)(\ell+2)}{F^2}\Psi \\ = 2A^2\left[k_{,11} + 2v_1k_{,1} - (2v_1^2 - \epsilon)k - \frac{(\ell-1)(\ell+2)}{F^2}k\right].\end{aligned} \tag{5.27}$$

Next, we define

$$B \equiv A^2(\overset{0}{\rho} + \overset{0}{\mathrm{p}}). \tag{5.28}$$

Using this notation, we can write

$$A^2\kappa\overset{0}{\mathrm{p}}_{,0} = B_{,0} + v_0B. \tag{5.29}$$

From this and Eq. (5.9b), we obtain for the right-hand side of Eq. (5.9c)

$$\left(\tfrac{B_{,0}}{B} + v_0\right)[k_{00,0} - k_{01,1} + S_{,0} - 2v_0S]. \tag{5.30}$$

Substitution in Eq. (5.9c) yields

$$(\Psi_{,0} - k_{01,1})_{,0} - (\tfrac{B_{,0}}{B} + v_0)(\Psi_{,0} - k_{01,1}) - \tfrac{B}{2}\Psi$$
$$= -2A^2 \left[k_{,00} + 2v_0 k_{,0} + (2v_{0,0} - \tfrac{B_{,0}}{B} v_0 - \epsilon)k\right]. \quad (5.31)$$

It follows that a solution for k must be of the form

$$k = k_p + F_0(r)\left[c_0 G_0(t) + c_1 G_1(t)\right] + F_1(r)\left[c_2 G_0(t) + c_3 G_1(t)\right], \quad (5.32)$$

where k_p is a particular solution of equation (5.27) determined by the master equation (5.24). F_A satisfies the 2nd-order homogeneous differential equation

$$\frac{d^2k}{dr^2} + 2v_1 \frac{dk}{dr} - (2{v_1}^2 - \epsilon)k - \frac{(\ell-1)(\ell+2)}{F^2} k = 0, \quad (5.33)$$

and G_A satisfies

$$\frac{d^2k}{dt^2} + 2v_0 \frac{dk}{dt} + \left(2v_{0,0} - \tfrac{B_{,0}}{B} v_0 - \epsilon\right) k = 0. \quad (5.34)$$

Accordingly, the master equation (5.24) governs all of the even-parity perturbations except for the freedom of four arbitrary constants contained in k. The issue of consistency was treated in reference[11].

We now consider the case of the *false vacuum*. When $\overset{0}{\rho} + \overset{0}{\mathrm{p}} = 0$, it follows from Eq. (5.14) that $\overset{0}{\rho}$ and $\overset{0}{\mathrm{p}}$ are both constant. The linearized field equations (5.9) are homogeneous and the propagation equations (5.8). (5.22), (5.23), (5.26), and (5.27) are unchanged. This time Eq. (5.9c) is empty, and Eq. (5.9b) yields

$$\Psi_{,0} - k_{01,1} = -2A^2[k_{,0} + v_0 k]. \quad (5.35)$$

A solution for k must be of the form

$$k = k_p + G(t)\left[c_0 F_0(r) + c_1 F_1(r)\right], \quad (5.36)$$

where k_p is a particular solution of Eq. (5.27), F_A satisfies the differential equation (5.33), and $G(t)$ satisfies

$$\frac{dk}{dt} + v_0 k = 0. \quad (5.37)$$

The freedom in k is reduced to two arbitrary constants.

CHAPTER 4

A Comparison with the Newman-Penrose Formalism

Thus far, we have used, as a mathematical basis for perturbation theory, a tensorial description of the Einstein field theory. An alternative treatment is provided by the null-tetrad formalism of Newman and Penrose.[15] The Newman-Penrose formalism is especially useful in applications in which null vectors are geometrically preferred. Since gravitational waves have null trajectories, a large body of work on pure gravitational radiation has been done in this formalism. In this chapter, we explore the relation between the formalism of Gerlach and Sengupta and that of Newman and Penrose. In particular, we consider the odd-parity perturbations representing pure gravitational radiation propagating on a flat background, under the simplifying assumption of axial symmmetry.

First, we give a brief geometrical introduction of the Newman-Penrose formalism. In §4.2, we describe the null-tetrad formulation of perturbation theory, specialize the background to that of Minkowski space, and supply the equations governing the perturbations. The preliminary work required to compare the two formalisms is presented in §4.3, along with the gauge transformation that allows us to express the components of the Gerlach-Sengupta metric perturbations in terms of the Newman-Penrose quantities. In the last section, we establish the relationships between the gauge invariant quantities of each of these formalisms, and discuss their implications.

§4.1 Null-Tetrad Formalism

In the tetrad formulation of general relativity, we begin with a four dimensional Riemannian manifold with signature $(+,-,-,-)$, but rather than introduce a coordinate basis, we introduce a tetrad. A tetrad, or moving frame, is a *noncoordinate* basis of smooth vector fields.[1] The Newman-Penrose formalism[15] consists of choosing a null-tetrad made up of two real null vectors, l^μ and n^μ, and a pair of complex null vectors, m^μ and its complex conjugate $\overline{m}^\mu$, satisfying the normalization conditions

$$l^\mu n_\mu = -m^\mu \overline{m}_\mu = 1, \tag{1.1}$$

all other products being zero. In terms of the tetrad vectors, the metric is given by

$$g_{\mu\nu} = l_\mu n_\nu + l_\mu n_\nu - m_\mu \overline{m}_\nu - \overline{m}_\mu m_\nu. \tag{1.2}$$

The Christoffel symbols are replaced by twelve complex functions called the spin coefficients. They measure the rotation of the tetrad under parallel transport, and are obtained by projecting the covariant derivatives of the tetrad vectors onto the null tetrad. They are given by

$$\kappa = l_{\mu;\nu} m^\mu l^\nu, \tag{1.3a}$$

$$\pi = -n_{\mu;\nu} \overline{m}^\mu l^\nu, \tag{1.3b}$$

$$\epsilon = \tfrac{1}{2}\left(l_{\mu;\nu} n^\mu l^\nu - m_{\mu;\nu} \overline{m}^\mu l^\nu\right), \tag{1.3c}$$

$$\rho = l_{\mu;\nu} m^\mu \overline{m}^\nu, \tag{1.3d}$$

$$\lambda = -n_{\mu;\nu}\overline{m}^{\mu}\overline{m}^{\nu}, \tag{1.3e}$$

$$\alpha = \tfrac{1}{2}\left(l_{\mu;\nu}n^{\mu}\overline{m}^{\nu} - m_{\mu;\nu}\overline{m}^{\mu}\overline{m}^{\nu}\right), \tag{1.3f}$$

$$\sigma = l_{\mu;\nu}m^{\mu}m^{\nu}, \tag{1.3g}$$

$$\mu = -n_{\mu;\nu}\overline{m}^{\mu}m^{\nu}, \tag{1.3h}$$

$$\beta = \tfrac{1}{2}\left(l_{\mu;\nu}n^{\mu}m^{\nu} - m_{\mu;\nu}\overline{m}^{\mu}m^{\nu}\right), \tag{1.3i}$$

$$\nu = -n_{\mu;\nu}\overline{m}^{\mu}n^{\nu}, \tag{1.3j}$$

$$\gamma = \tfrac{1}{2}\left(l_{\mu;\nu}n^{\mu}n^{\nu} - m_{\mu;\nu}\overline{m}^{\mu}n^{\nu}\right), \tag{1.3k}$$

$$\tau = l_{\mu;\nu}m^{\mu}n^{\nu}. \tag{1.3l}$$

Using the tetrad, we can express the various components of the curvature —the Weyl tensor, Ricci tensor, and scalar curvature— as follows:

$$\Psi_0 = -C_{\alpha\beta\gamma\delta}\, l^{\alpha}m^{\beta}l^{\gamma}m^{\delta}, \tag{1.4a}$$

$$\Psi_1 = -C_{\alpha\beta\gamma\delta}\, l^{\alpha}n^{\beta}l^{\gamma}m^{\delta}, \tag{1.4b}$$

$$\Psi_2 = -C_{\alpha\beta\gamma\delta}\, \overline{m}^{\alpha}n^{\beta}l^{\gamma}m^{\delta}, \tag{1.4c}$$

$$\Psi_3 = -C_{\alpha\beta\gamma\delta}\, \overline{m}^{\alpha}n^{\beta}l^{\gamma}n^{\delta}, \tag{1.4d}$$

$$\Psi_4 = -C_{\alpha\beta\gamma\delta}\, \overline{m}^{\alpha}n^{\beta}\overline{m}^{\gamma}n^{\delta}, \tag{1.4e}$$

$$\Phi_{00} = -\tfrac{1}{2}\, R_{\mu\nu}l^{\mu}l^{\nu}, \tag{1.5a}$$

$$\Phi_{01} = -\tfrac{1}{2}\, R_{\mu\nu}l^{\mu}m^{\nu}, \tag{1.5b}$$

$$\Phi_{02} = -\tfrac{1}{2}\, R_{\mu\nu}m^{\mu}m^{\nu}, \tag{1.5c}$$

$$\Phi_{10} = -\tfrac{1}{2}\, R_{\mu\nu}l^{\mu}\overline{m}^{\nu}, \tag{1.5d}$$

$$\Phi_{11} = -\tfrac{1}{4}\, R_{\mu\nu}(l^{\mu}n^{\nu} + m^{\mu}\overline{m}^{\nu}), \tag{1.5e}$$

$$\Phi_{12} = -\tfrac{1}{2}\, R_{\mu\nu}n^{\mu}m^{\nu}, \tag{1.5f}$$

$$\Phi_{20} = -\tfrac{1}{2}\, R_{\mu\nu}\overline{m}^{\mu}\overline{m}^{\nu}, \tag{1.5g}$$

$$\Phi_{21} = -\tfrac{1}{2}\, R_{\mu\nu}n^{\mu}\overline{m}^{\nu}, \tag{1.5h}$$

$$\Phi_{22} = -\tfrac{1}{2}\, R_{\mu\nu} n^\mu n^\nu, \tag{1.5i}$$

$$\Lambda = \tfrac{1}{24}\, R. \tag{1.6}$$

We also introduce a set of intrinsic differential operators:

$$\begin{aligned} D\varphi &\equiv \varphi_{;\mu} l^\mu, \\ \Delta\varphi &\equiv \varphi_{;\mu} n^\mu, \\ \delta\varphi &\equiv \varphi_{;\mu} m^\mu, \\ \overline{\delta}\varphi &\equiv \varphi_{;\mu} \overline{m}^\mu. \end{aligned} \tag{1.7}$$

The Newman-Penrose equations consist of a system of coupled first-order differential equations in the variables Ψ_A, Φ_{AB}, Λ, and the spin coefficients, which arises from the definition of the Riemann tensor in terms of the Newman-Penrose quantities.[29] Because all of the information contained in the Einstein field equations can be obtained from a solution to the Newman-Penrose equations, they are said to form an *equivalent* set.[30] The Newman-Penrose equations can be found in their most general form in reference [15]. At this stage, they are very long and somewhat complicated, but, as we shall see, we can simplify them considerably by choosing one of the tetrad vectors to be hypersurface orthogonal. This is precisely what happens in the case of pure gravitational radiation. Since gravitational waves are propagated at the speed of light, their wave fronts are null hypersurfaces, and we can take full advantage of the structure of the formalism by tailoring the tetrad to match the wave fronts.

Just as in the case of the tensorial description of the field equations, it is convenient to choose a coordinate system in order to write down specific information about a solution to the Newman-Penrose equations. For problems involving gravitational radiation, an appropriate choice of coordinates is the Bondi coordinate system,[15] constructed in the following manner.

First, we introduce a family of null hypersurfaces designated by u, representing a gravitational wave front, and take the vector l_μ orthogonal to the hypersurfaces u = constant. This means that l_μ is proportional to a gradient field, and we set

$$l_\mu = u_{,\mu}. \tag{1.8}$$

If we choose $x^0 = u$ as our first coordinate, then its covariant and contravariant components are simply

$$l_\mu = \delta^0_\mu \tag{1.9}$$

and

$$l^\mu = \delta^\mu_1. \tag{1.10}$$

The hypersurface u = constant is generated by a two-parameter family of null geodesics; we choose the two parameters labelling the geodesics as our coordinates x^2 and x^3. The coordinate $x^1 = r$ can be determined by taking an affine parameter[31] associated with the congruence of null geodesics at each point u, x^2, and x^3 constant.[1] Because affine parameters are not unique, different ones being related by linear transformations,[1] the coordinate x^1 is not uniquely determined. An appropriate selection is usually made at a later time to match the geometry of a specific application. With this choice of coordinate system, the metric takes the form

$$g^{\mu\nu} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & & & \\ 0 & & g^{ij} & \\ 0 & & & \end{bmatrix}, \tag{1.11}$$

$(i, j = 1, 2, 3)$.

Applying the normalization conditions Eq. (1.1), we obtain that the remaining tetrad vectors are of the form

$$n^\mu = \delta^\mu_0 + U\delta^\mu_1 + X^i\delta^\mu_i, \tag{1.12}$$

$$m^\mu = \omega\delta^\mu_1 + \xi^i\delta^\mu_i, \tag{1.13}$$

where U, X^i, ω and ξ^i are arbitrary functions of the coordinates x^μ, and $i = 2, 3$.

Equation (1.2) tells us that the metric components are related to the tetrad components, Eqs. (1.10), (1.12), and (1.13) by

$$
\begin{aligned}
g^{11} &= 2(U - \omega\overline{\omega}), \\
g^{1i} &= X^i - (\xi^i\overline{\omega} + \overline{\xi}^i\omega), \\
g^{ij} &= -(\xi^i\overline{\xi}^j + \overline{\xi}^i\xi^j),
\end{aligned}
\tag{1.14}
$$

$(i, j = 2, 3)$.

The tetrad is still not fully determined, for there remains the freedom of rotating the tetrad about l^μ. One way of eliminating most of this freedom is to require that the vectors n^μ and m^μ be parallelly propagated along l^μ. This, together with the choice of l^μ, Eq. (1.8), yields the following simplifications for the spin coefficients:

$$
\begin{aligned}
&\kappa = \pi = \epsilon = 0, \\
&\rho = \overline{\rho}, \\
&\tau = \overline{\alpha} + \beta.
\end{aligned}
\tag{1.15}
$$

In terms of the Bondi coordinate system, the differential operators D, Δ, and δ are given as follows:

$$
\begin{aligned}
D &= \frac{\partial}{\partial r}, \\
\Delta &= U\frac{\partial}{\partial r} + \frac{\partial}{\partial u} + X^i\frac{\partial}{\partial x^i}, \\
\delta &= \omega\frac{\partial}{\partial r} + \xi^i\frac{\partial}{\partial x^i},
\end{aligned}
\tag{1.16}
$$

here $i = 2, 3$.

Finally, with each of the Newman-Penrose quantities, we associate a spin weight according to how it transforms under a rotation of the tetrad legs m^μ and

$\overline{m}^{\mu}$.[32] We say that a quantity η, defined on the two-sphere, has spin weight s if, under the transformation

$$m^{\mu} \rightarrow \exp{(i\lambda)} m^{\mu}, \qquad \lambda \text{ real}, \tag{1.17}$$

it transforms as

$$\eta \rightarrow \exp{(is\lambda)}\, \eta. \tag{1.18}$$

Let η be a quantity with spin weight s. We define the spin-weighted differential operator $\eth$,[32] pronounced *edth*, as follows:

$$\eth\eta = -(\sin\theta)^{s} \left\{ \frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi} \right\} \left[(\sin\theta)^{-s}\eta \right]. \tag{1.19}$$

The concept of spin weight will play an important role in the spherical harmonic decomposition of the Newman-Penrose quantities.

To conclude this section, we give, written in the Bondi coordinate system, the Newman and Penrose field equations and the Bianchi identities in empty space. These will be the starting point of the perturbative scheme. They are obtained by inserting the proper expression for the stress energy tensor; for vacuum we set $\Lambda = 0$ and $\Phi_{AB} = 0$. The Newman-Penrose equations are

$$D\xi^{i} = \rho\xi^{i} + \sigma\overline{\xi}^{i}, \tag{1.20a}$$

$$D\omega = \rho\omega + \sigma\overline{\omega} - \tau, \tag{1.20b}$$

$$DX^{i} = \overline{\tau}\xi^{i} + \tau\overline{\xi}^{i}, \tag{1.20c}$$

$$DU = \overline{\tau}\omega + \tau\overline{\omega} - (\gamma + \overline{\gamma}), \tag{1.20d}$$

$$D\rho = \rho^{2} + \sigma\overline{\sigma}, \tag{1.20e}$$

$$D\sigma = 2\rho\sigma + \Psi_{0}, \tag{1.20f}$$

$$D\tau = \rho\tau + \sigma\overline{\tau} + \Psi_{1}, \tag{1.20g}$$

$$D\alpha = \rho\alpha + \overline{\sigma}\beta, \tag{1.20h}$$

$$D\beta = \rho\beta + \sigma\alpha + \Psi_1, \tag{1.20i}$$

$$D\gamma = \tau\alpha + \overline{\tau}\beta + \Psi_2, \tag{1.20j}$$

$$D\lambda = \rho\lambda + \overline{\sigma}\mu, \tag{1.20k}$$

$$D\mu = \rho\mu + \sigma\lambda + \Psi_2, \tag{1.20l}$$

$$D\nu = \tau\lambda + \overline{\tau}\mu + \Psi_3, \tag{1.20m}$$

$$\delta X^i - \Delta\xi^i = (\mu + \overline{\gamma} - \gamma)\xi^i + \overline{\lambda}\,\overline{\xi}^i, \tag{1.21a}$$

$$\delta\overline{\xi}^i - \overline{\delta}\xi^i = (\overline{\beta} - \alpha)\xi^i - (\beta - \overline{\alpha})\overline{\xi}^i, \tag{1.21b}$$

$$\delta\overline{\omega} - \overline{\delta}\omega = (\overline{\beta} - \alpha)\overline{\omega} + (\overline{\alpha} - \beta)\overline{\omega} + \mu - \overline{\mu}, \tag{1.21c}$$

$$\delta U - \Delta\omega = (\mu + \overline{\gamma} - \gamma)\omega + \overline{\lambda}\overline{\omega} - \overline{\nu}, \tag{1.21d}$$

$$\Delta\lambda - \overline{\delta}\nu = 2\alpha\nu + (\overline{\gamma} - 3\gamma - \mu - \overline{\mu})\lambda - \Psi_4, \tag{1.21e}$$

$$\delta\alpha - \overline{\delta}\beta = \mu\rho - \lambda\sigma - 2\alpha\beta + \alpha\overline{\alpha} + \beta\overline{\beta} - \Psi_2, \tag{1.21f}$$

$$\delta\rho - \overline{\delta}\sigma = \tau\rho + (\overline{\beta} - 3\alpha)\sigma - \Psi_1, \tag{1.21g}$$

$$\delta\lambda - \overline{\delta}\mu = \overline{\tau}\mu + (\overline{\alpha} - 3\beta)\lambda - \Psi_3, \tag{1.21h}$$

$$\delta\nu - \Delta\mu = \gamma\mu - 2\beta\nu + \overline{\gamma}\mu + \mu^2 + \lambda\overline{\lambda}, \tag{1.21i}$$

$$\delta\gamma - \Delta\beta = \tau\mu - \sigma\nu + (\mu - \gamma + \overline{\gamma})\beta + \alpha\overline{\lambda}, \tag{1.21j}$$

$$\delta\tau - \Delta\sigma = 2\tau\beta + (\overline{\gamma} - 3\gamma + \mu)\sigma + \overline{\lambda}\rho, \tag{1.21k}$$

$$\Delta\rho - \overline{\delta}\tau = (\gamma + \overline{\gamma} - \overline{\mu})\rho - 2\alpha\tau - \lambda\sigma - \Psi_2, \tag{1.21l}$$

$$\Delta\alpha - \overline{\delta}\gamma = \rho\nu - \tau\lambda - \beta\lambda + (\overline{\gamma} - \gamma - \overline{\mu})\alpha - \Psi_3. \tag{1.21m}$$

The Bianchi identities,

$$R_{\mu\nu[\rho\sigma;\lambda]} = 0, \tag{1.22}$$

take the following form:

$$D\Psi_{A+1} - \overline{\delta}\Psi_A = (4 - A)\rho\Psi_{A+1} - 2(2 - A)\alpha\Psi_A - A\lambda\Psi_{A-1}, \tag{1.23a}$$

$$\Delta\Psi_A - \delta\Psi_{A+1} = A\nu\Psi_{A-1} - 2(A-2)\gamma\Psi_A + (A-4)\tau\Psi_{A+1} + \\ - (A+1)\mu\Psi_A + 2(A-1)\beta\Psi_{A+1} - (A-3)\sigma\Psi_{A+2}, \tag{1.23b}$$

where $(A = 0, 1, 2, 3)$.

§4.2 Perturbation Method

A well-known application of the Newman-Penrose formalism is the study of the asymptotic behavior of the Riemann tensor and the metric tensor in empty space.[16] The motivations arise from the need to study a model that represents an *isolated* source.[33] The requirement that a solution be asymptotically flat reflects the fact that, as one recedes from an *isolated* source of radiation, the geometry of the spacetime should approach that of flat space. In what follows we make use of a program of study for gravitational radiation in asymptotically flat spaces proposed by Janis and Torrence.[34] The procedure, based on the Newman-Penrose formalism, applies a small-parameter perturbation approximation to a flat background. The specialization to pure gravitational radiation is achieved by the requirement that the first-order corrections satisfy the linearized field equations corresponding to empty space.

In general, we expect to have two types of gravitational radiation: a retarded solution, which represents outgoing radiation emitted by the source itself, and an advanced solution coming in from infinity, which will affect the source in the future. To facilitate their analysis, it is customary to isolate each type of radiation by means of boundary conditions. In this section, we present the first-order solutions for retarded asymptotic gravitational radiation fields in Minkowski as given by Couch *et al.*,[35] with a view to comparing these results with those obtained in the Gerlach-Sengupta formalism. We start by giving the null-tetrad formulation for Minkowski space used in reference [35].

The geometrical meaning of the null tetrad is best understood if we first consider the Minkowski line element in its usual spherical form,

$$ds^2 = dt^2 - dr^2 - r^2\, d\Omega^2. \tag{2.1}$$

Here, an obvious choice for the family of null hypersurfaces is the future light cones at $r = 0$ and t =constant. We label each of these null cones with the parameter $u = x^0$, and choose the unique affine parameter $r = x^1$ such that r is the proper radius of the two-spheres defined by u and r constant. Then, l^μ, as defined in the previous section, is tangent to the future light cone, n^μ is tangent to the past light cone, and m^μ and its complex conjugate $\overline{m}^\mu$ are vectors tangent to the two-spheres.

Also, since the null coordinate u designates the future light cone, $u = t - r$. Therefore, the Minkowski line element expressed in the null coordinates u and r becomes

$$ds^2 = du^2 + 2du dr - r^2\, d\Omega^2. \tag{2.2}$$

The explicit relationship between this choice of coordinates and the tetrad vectors is[36]

$$\begin{aligned} l^\mu &= \delta_1^\mu, \\ n^\mu &= \delta_0^\mu - \frac{1}{2}\delta_1^\mu, \\ m^\mu &= \frac{1}{\sqrt{2}r}(\delta_2^\mu + \frac{i}{\sin\theta}\delta_3^\mu). \end{aligned} \tag{2.3}$$

It follows from the definitions, Eqs. (1.3), (1.4), (1.12), and (1.13), that the non-zero Newman-Penrose quantities are

$$r\mu = U = -\frac{1}{2},$$

$$\rho = -\frac{1}{r},$$
$$\alpha = -\beta = -\frac{1}{2\sqrt{2}r}\cot\theta,$$
$$\xi^i = \frac{1}{\sqrt{2}r}(1, \frac{i}{\sin\theta}). \tag{2.4}$$

The perturbed geometry is approximated to first-order in the usual manner: we assume that each of the Newman-Penrose quantities can be written in the form $\rho = \overset{0}{\rho} + \overset{1}{\rho}$, $\Psi_A = \overset{0}{\Psi}_A + \overset{1}{\Psi}_A$, $U = \overset{0}{U} + \overset{1}{U}$, and so on. (To simplify the notation in the remainder of this chapter, we omit the first-order superscript on the perturbed quantities.) Each of the first-order corrections is approximated by means of harmonic functions according to its spin weight. This time, the appropriate harmonic functions are the *spin-weighted spherical harmonics*,[32,37] which we denote by ${}_sY_{\ell,m}(\theta,\phi)$. They are constructed from the usual spherical harmonics as follows:

$${}_sY_{\ell,m} = \begin{cases} P_{-s}(\ell)\eth^{\,s}Y_{\ell,m}, & (0 \le s \le \ell), \\ (-1)^s P_s(\ell)\bar{\eth}^{-s}Y_{\ell,m}, & (-\ell \le s \le 0), \end{cases} \tag{2.5}$$

where

$$P_s(\ell) \equiv \left(\frac{(\ell+s)!}{(\ell-s)!}\right)^{\frac{1}{2}}. \tag{2.6}$$

The differential operator $\eth$ was defined in §4.1.

In the null-tetrad perturbation method, we consider small variations in the Weyl tensor rather than in the metric tensor. The idea is to regard Ψ_A, which encodes all the information about the Weyl tensor, as a field in Minkowski space. When written in terms of the Newman-Penrose quantities, the Bianchi identities decouple and provide a set of equations for that field.

If we use the angular differential operator $\eth$, the linearized Bianchi identities reduce to

$$\dot{\Psi}_A - \frac{1}{2}D\Psi_A - \frac{A+1}{2r}\Psi_A + \frac{\sqrt{2}}{2r}\eth\,\Psi_{A+1} = 0, \tag{2.7}$$
$$D\Psi_{A+1} + \frac{4-A}{r}\Psi_{A+1} + \frac{\sqrt{2}}{2r}\bar{\eth}\,\Psi_A = 0, \tag{2.8}$$

where $A = 0, 1, 2, 3$ and $\dot{}$ denotes $\partial/\partial u$.

If we impose the condition that the solution be asymptotically flat at null infinity,[38] a solution to Eq. (2.8) is given by

$$\Psi_{A+1} = \frac{\Psi^o_{A+1}}{r^{4-A}} - \frac{\sqrt{2}}{2r^{4-A}} \int_\infty^r r'^{3-A} \overline{\eth} \Psi_A dr'. \tag{2.9}$$

Here the Ψ^o_A are arbitrary functions of the variables u, θ, and ϕ. Substituting this result in Eq. (2.7), we obtain

$$\dot{\Psi}^o_i = -\frac{\sqrt{2}}{2} \eth \Psi^o_{i+1}, \tag{2.10}$$

where $i = 1, 2, 3$, and

$$\dot{\Psi}_0 - \frac{1}{2} D\Psi_0 - \frac{1}{2r}\Psi_0 - \frac{1}{2r^5} \int_\infty^r r'^3 \eth \overline{\eth} \Psi_0 dr' + \frac{\sqrt{2}}{2} \frac{\eth \Psi^o_1}{r^5} = 0 \tag{2.11}$$

Given initial conditions, and a solution for Ψ_0, Eq. (2.11), the solutions for the other Ψ_A are immediately obtained from Eqs. (2.9) and (2.10). We derive the first-order corrections for the remaining Newman-Penrose quantities by linearizing and integrating the field equations (1.20). The results, as given in reference [35], are:

$$\rho = 0, \tag{2.12a}$$

$$\sigma = \frac{\sigma^o}{r^2} + \frac{1}{r^2} \int_\infty^r r'^2 \Psi_0 dr', \tag{2.12b}$$

$$\alpha = - \overset{0}{\alpha} \int_\infty^r \overline{\sigma} dr', \tag{2.12c}$$

$$\beta = -\overline{\alpha} + \frac{1}{r} \int_\infty^r r' \Psi_1 dr' \tag{2.12d}$$

$$\tau = \overline{\alpha} + \beta, \tag{2.12e}$$

$$\gamma = -r \overset{0}{\alpha} \int_\infty^r \frac{1}{r'} (\overline{\tau} - \tau) dr' + \int_\infty^r \Psi_2 dr', \tag{2.12f}$$

$$\mu = \frac{1}{r} \int_\infty^r r' \Psi_2 dr', \tag{2.12g}$$

$$\lambda = -\frac{1}{2r}\int_\infty^r \overline{\sigma} dr' + \frac{\lambda^o}{r}, \tag{2.12h}$$

$$\nu = -\int_\infty^r \frac{\overline{\tau}}{2r'} dr' + \int_\infty^r \Psi_3 dr', \tag{2.12i}$$

$$U = -\int_\infty^r (\gamma + \overline{\gamma}) dr', \tag{2.12j}$$

$$\omega = -\frac{1}{r}\int_\infty^r r'\tau dr' + \frac{\omega^o}{r}, \tag{2.12k}$$

$$X^i = r\,\overset{0}{\overline{\xi}}{}^i \int_\infty^r \frac{\tau}{r'} dr' + r\,\overset{0}{\xi}{}^i \int_\infty^r \frac{\overline{\tau}}{r'} dr', \tag{2.12l}$$

$$\xi^i = \overset{0}{\overline{\xi}}{}^i \int_\infty^r \sigma dr', \tag{2.12m}$$

where σ^o, λ^o, and ω^o are arbitrary functions of the variables u, θ, and ϕ. Substituting Eq. (2.12) in the linearized remaining field equations (1.21), we obtain a set of equations relating the integration functions:

$$\lambda^o = \dot{\overline{\sigma}}^o, \tag{2.13a}$$

$$\omega^o = -\frac{\sqrt{2}}{2}\overline{\eth}\sigma^o, \tag{2.13b}$$

$$\Psi_2^o - \overline{\Psi}_2^o = \tfrac{1}{2}(\overline{\eth}^2\sigma^o - \eth^2\overline{\sigma}^o), \tag{2.13c}$$

$$\Psi_3^o = \frac{\sqrt{2}}{2}\eth\dot{\overline{\sigma}}^o, \tag{2.13d}$$

$$\Psi_4^o = -\ddot{\overline{\sigma}}^o. \tag{2.13e}$$

Essentially, this constitutes a complete solution to the first-order problem. But, in the present application, to compare the two formalisms, we also need to calculate the first-order metric components. This is done by linearizing the tetrad-formalism expressions for the metric components Eq. (1.14), and substituting in the Minkowski solutions Eq. (2.4) for the background quantities. The results are:

$$h^{11} = -2U, \tag{2.14a}$$

$$h^{1i} = -X^i + (\overset{0}{\xi}{}^i\overline{\omega} + \overset{0}{\overline{\xi}}{}^i\omega), \tag{2.14b}$$

$$h^{ij} = (\overset{0}{\xi}{}^i \bar{\xi}^j + \overset{0}{\bar{\xi}}{}^i \xi^j) + (\xi^i \overset{0}{\bar{\xi}}{}^j + \bar{\xi}^i \overset{0}{\xi}{}^j), \tag{2.14c}$$

$(i, j = 2, 3)$

§4.3 Gauge fixing

If the solutions obtained in the Gerlach-Sengupta formalism represent pure gravitational radiation, then they are effectively equivalent to those obtained using the Newman-Penrose formalism, and so there must be a gauge in which the forms of the metric perturbations in both formalisms are identical. In this section, we show that this is indeed the case, at least for the odd-parity perturbations, by deriving the Gerlach-Sengupta gauge transformation that will allow us to identify the odd-parity metric perturbations in each of the formalisms. To match the results presented in the previous section, we take Minkowski space as background for the perturbations, and for simplicity we assume axial symmetry: the perturbed quantities are assumed to be independent of ϕ.

We start with the Newman-Penrose approach to perturbation theory. First, it is necessary to isolate the portion of the Newman-Penrose metric that corresponds to the odd-parity perturbations. It was shown by Janis and Newman[36] that the real part of Ψ_A is the *electric/even* part of the perturbations, and that the imaginary part of Ψ_A is the *magnetic/odd* part of the perturbations. Here, the terminology is borrowed from electrodynamics,[20] in which the parity of an electric multipole is $(-1)^\ell$ (even) and that of a magnetic multipole is $(-1)^{\ell+1}$ (odd). To incorporate this in the metric tensor, we need to calculate the metric components in terms of the Ψ_A.

Using (2.12), we obtain for the first-order correction of the metric variables

$$
\begin{aligned}
U &= -2\int_\infty^r \int_\infty^{r'} Re\Psi_2 dr'dr, \\
\omega &= -\frac{1}{r}\left[\int_\infty^r r'\Psi_1 dr'dr - \omega^o\right], \\
X^2 &= \sqrt{2}\int_\infty^r \frac{1}{r'^2}\int_\infty^{r'} r'' Re\Psi_1 dr'dr, \\
X^3 &= \frac{\sqrt{2}}{\sin\theta}\int_\infty^r \frac{1}{r'^2}\int_\infty^{r'} r'' Im\Psi_1 dr'dr, \\
\xi^2 &= \frac{1}{\sqrt{2}r}\left[\int_\infty^r \frac{\sigma^o}{r'^2}dr + \int_\infty^r \frac{1}{r'^2}\int_\infty^{r'} r''^2\Psi_0 dr'dr\right], \\
\xi^3 &= -\frac{i}{\sqrt{2}r\sin\theta}\left[\int_\infty^r \frac{\sigma^o}{r'^2}dr + \int_\infty^r \frac{1}{r'^2}\int_\infty^{r'} r''^2\Psi_0 dr'dr\right].
\end{aligned}
\tag{3.1}
$$

By substituting these into Eq.(2.14), we get the components of the metric perturbations in terms of the Weyl tensor:

$$
\begin{aligned}
h^{11} &= 4\int_\infty^r \int_\infty^{r'} Re\Psi_2 dr'dr, \\
h^{12} &= -\frac{\sqrt{2}}{r^2}\left[r^2\int_\infty^r \frac{1}{r'^2}\int_\infty^{r'} r'' Re\Psi_1 dr'dr + \int_\infty^r\int_\infty^{r'} r'' Re\Psi_1 dr'dr - Re\,\omega^o\right], \\
h^{13} &= -\frac{\sqrt{2}}{r^2\sin\theta}\left[r^2\int_\infty^r \frac{1}{r'^2}\int_\infty^{r'} r'' Im\Psi_1 dr'dr + \int_\infty^r\int_\infty^{r'} r'' Im\Psi_1 dr'dr - Im\,\omega^o\right], \\
h^{22} &= \frac{2}{r^2}\left[\int_\infty^r \frac{Re\sigma^o}{r'^2}dr + \int_\infty^r \frac{1}{r'^2}\int_\infty^{r'} r''^2 Re\Psi_0 dr'dr\right], \\
h^{23} &= \frac{2}{r^2\sin\theta}\left[\int_\infty^r \frac{Im\sigma^o}{r'^2}dr + \int_\infty^r \frac{1}{r'^2}\int_\infty^{r'} r''^2 Im\Psi_0 dr'dr\right], \\
h^{33} &= -\frac{2}{r^2\sin\theta}\left[\int_\infty^r \frac{Re\sigma^o}{r'^2}dr + \int_\infty^r \frac{1}{r'^2}\int_\infty^{r'} r''^2 Re\Psi_0 dr'dr\right].
\end{aligned}
\tag{3.2}
$$

We arrive at the odd-parity metric perturbations by setting

$$Re(\Psi_A) = 0, \tag{3.3}$$

and

$$Re(\sigma^o) = 0. \tag{3.4}$$

The latter condition is a consequence of (3.3), and is obtained by substituting it into Eqs. (2.9) and (2.13c).

To facilitate the comparison between the formalisms, we depart from the usual Newman and Penrose notation, and explicitly write the harmonic functions used in the perturbative expansions. For the case of axial symmetry, we set $m = 0$. The spin-weighted spherical harmonics ${}_sY_{\ell,0}$ are then real, and we denote them by ${Y^s}_\ell$. The spin weight of each of the Newman-Penrose quantities can be directly calculated from their definitions Eqs. (1.3) – (1.4). For the quantities appearing in the odd-parity metric perturbations, we have

$$\begin{aligned}
&\Psi_A \longrightarrow \psi_A(u,r)\, P_{-A}(\ell){Y^{2-A}}_\ell, \\
&\sigma^o \longrightarrow \widetilde{\sigma}^o(u,r)\, P_{-2}(\ell){Y^2}_\ell, \\
&\omega^o \longrightarrow \widetilde{\omega}^o(u,r)\, P_{-1}(\ell){Y^1}_\ell,
\end{aligned} \tag{3.5}$$

$A = 1, 2$.

In terms of these, the non-zero components for the odd-parity metric perturbations h^{13} and h^{23} are, respectively,

$$-\frac{\sqrt{2}P_{-1}(\ell){Y^1}_\ell}{r^2\sin\theta}\left[r^2\int_\infty^r \frac{1}{r'^2}\int_\infty^{r'} r''Im\psi_1 dr'dr + \int_\infty^r\int_\infty^{r'} r''Im\psi_1 dr'dr - Im\widetilde{\omega}^o\right], \tag{3.6}$$

and

$$\frac{2P_{-2}(\ell){Y^2}_\ell}{r^2\sin\theta}\left[\int_\infty^r \frac{1}{r'^2}\int_\infty^{r'} r''^2 Im\psi_0 dr'dr + \int_\infty^r \frac{Im\widetilde{\sigma}^o}{r'^2}dr\right]. \tag{3.7}$$

We now turn our attention to the Gerlach-Sengupta formalism introduced in chapter 2. The odd-parity metric perturbations were written in §2.2 as

$$h_{\mu\nu} = \begin{bmatrix} 0 & & & \\ 0 & 0 & & \\ h_0S_2 & h_1S_2 & 2hS_{2:2} & \\ h_0S_3 & h_1S_3 & h(S_{2:3}+S_{3:2}) & 2hS_{3:3} \end{bmatrix}, \tag{3.8}$$

Several steps must be taken before we are able to compare this representation directly with that given by Eqs. (3.6) – (3.7). For a start, it should be noted that the two formalisms employ different signature conventions. Under a change of spacetime signature, the coordinate components of the metric tensor change sign, and so we have $h_{\mu\nu} \longrightarrow -h_{\mu\nu}$. Next, if we raise the indices using the background metric, and apply the coordinate transformation $t = u+r$, we obtain, for the contravariant form of the metric perturbations in the null-coordinate $\hat{x}^\mu = (u, r, \theta, \phi)$,

$$\widehat{h}^{\mu\nu} = \begin{bmatrix} 0 & & & \\ 0 & 0 & & \\ \frac{(h_0+h_1)S_2}{r^2} & -\frac{h_1 S_2}{r^2} & -\frac{2hS_{2:2}}{r^4} & \\ \frac{(h_0+h_1)S_3}{r^2\sin^2\theta} & -\frac{h_1 S_3}{r^2\sin^2\theta} & -\frac{h(S_{2:3}+S_{3:2})}{r^4\sin^2\theta} & -\frac{2hS_{3:3}}{r^4\sin^4\theta} \end{bmatrix}. \tag{3.9}$$

Here the hat indicates that the quantity has been converted to a null coordinates representation. The spherical harmonic vector S_a was defined in §2.2. Taking axial symmetry into account, the explicit expressions for the harmonic functions appearing in Eq. (3.9) are

$$\begin{aligned} S_2 &= 0, \\ S_3 &= \sin\theta \frac{\partial}{\partial\theta} Y, \\ S_{2:2} &= S_{3:3} = 0, \\ S_{2:3} + S_{3:2} &= \sin\theta [\frac{\partial^2}{\partial\theta^2} Y - \cot\theta \frac{\partial}{\partial\theta} Y]. \end{aligned} \tag{3.10}$$

To establish the connection between the harmonic functions, we calculate the spin-weighted spherical harmonics appearing in equations (3.6) and (3.7). By definition Eq. (2.5), they are

$$\begin{aligned} {Y^1}_\ell &= -P_{-1}(\ell) \frac{\partial}{\partial\theta} {Y^0}_\ell, \\ {Y^2}_\ell &= P_{-2}(\ell) [\frac{\partial^2}{\partial\theta^2} {Y^0}_\ell - \cot\theta \frac{\partial}{\partial\theta} {Y^0}_\ell]. \end{aligned} \tag{3.11}$$

Here, $Y^0{}_\ell \equiv Y$, the usual spherical harmonics used in the Gerlach-Sengupta formalism.

Using Eq. (3.11), we can write the odd-parity metric perturbations, Eq. (3.9), in terms of the spin-weighted spherical harmonics. The result is

$$\widehat{h}^{\mu\nu} = \begin{bmatrix} 0 & & & \\ 0 & 0 & & \\ 0 & 0 & 0 & \\ -\frac{(\bar{h}_0+\bar{h}_1)P_1Y^1{}_\ell}{r^2\sin\theta} & \frac{\bar{h}_1P_1Y^1{}_\ell}{r^2\sin\theta} & -\frac{\bar{h}P_2Y^2{}_\ell}{r^4\sin\theta} & 0 \end{bmatrix}, \tag{3.12}$$

A direct comparison between the two forms of the metric perturbations is now possible; consider the Newman-Penrose odd-parity metric perturbations

$$h^{\mu\nu} = \begin{bmatrix} 0 & & & \\ 0 & 0 & & \\ 0 & 0 & 0 & \\ 0 & \tilde{h}^{13}P_{-1}Y^1{}_\ell & \tilde{h}^{23}P_{-2}Y^2{}_\ell & 0 \end{bmatrix}. \tag{3.13}$$

The requirement that the metric perturbations $\widehat{h}^{\mu\nu}$ be put into an identical form is equivalent to the existence of a gauge in which

$$\bar{h}_0 + \bar{h}_1 = 0. \tag{3.14}$$

This can always be achieved if we choose the appropriate gauge function [see §2.3]. In this gauge, we obtain the following identification between the Gerlach-Sengupta metric components and the Newman-Penrose quantities,

$$\bar{h}_1 = -\sqrt{2}P_{-1}{}^2\left[r^2\int_\infty^r \frac{1}{r'^2}\int_\infty^{r'} r'' Im\psi_1 dr'dr + \int_\infty^r\int_\infty^{r'} r'' Im\psi_1 dr'dr - Im\widetilde{\omega}^o\right],$$

$$\bar{h} = -2P_{-2}{}^2r^2\left[\int_\infty^r \frac{1}{r'^2}\int_\infty^{r'} r''^2 Im\psi_0 dr'dr + \int_\infty^r \frac{Im\widetilde{\sigma}^o}{r''^2}dr\right]. \tag{3.15}$$

§4.4 Interpretation of the Results

Since we now have a means to go from one formalism to the other, we can compare the results obtained in each formalism to derive a relation between the gauge-invariants. Before turning to the explicit equations, let us briefly point out the conceptual differences in the respective perturbation methods. The formalism of Gerlach and Sengupta is characterized by gauge-invariant quantities. Because these quantities are defined in terms of the components of the metric perturbations, their perturbative approach is essentially a perturbation of the metric. Yet, we do not have a *gauge-independent* physical interpretation for the gauge-invariants. This information, because it is contained in the metric, depends on a choice of gauge. By contrast, the fundamental quantities in the null-tetrad perturbation analysis are the Ψ_A; these are the Newman-Penrose representation of the Weyl tensor. Because both the Ψ_A and the Weyl tensor vanish identically in vacuum, they are gauge-invariant [see §2.3], and there is no ambiguity in their physical interpretation.

We start by calculating the Gerlach-Sengupta odd-parity gauge-invariants k_A in terms of the Newman-Penrose equations. By definition, we have

$$k_A \equiv h_A - r^2 \left(\frac{h}{r^2}\right)_{,A} . \tag{4.1}$$

This definition can be converted into the null coordinates $\hat{x}^\mu = (u, r, \theta, \phi)$ by means of the coordinate transformation $t \to u + r$.

$$\begin{aligned} \hat{k}_0 &= \hat{h}_0 - r^2 \left(\frac{\hat{h}}{r^2}\right)_{,u} , \\ \hat{k}_1 &= \hat{h}_1 - r^2 \left[\left(\frac{\hat{h}}{r^2}\right)_{,r} - \left(\frac{\hat{h}}{r^2}\right)_{,u}\right] . \end{aligned} \tag{4.2}$$

To calculate their Newman-Penrose representations we choose a gauge in which $\hat{h}_0 = -\hat{h}_1$. This allows us to use the results from the previous section. We obtain

$$\begin{aligned} \hat{k}_0 = \sqrt{2}P^2_{-1}(\ell) &\left[\int_\infty^r \int_\infty^{r'} Im\psi_1 dr' dr - \frac{\sqrt{2}}{2} Im\hat{\sigma}^o\right] + \\ &- P^2_{-2}(\ell)\left[2r Im\dot{\hat{\sigma}}^o - r^2 \int_\infty^r Im\psi_0 dr + r^2 \int_\infty^r \frac{1}{r'^2} \int_\infty^{r'} r'' Im\psi_0 dr' dr\right] , \end{aligned}$$

$$\hat{k}_1 = -\sqrt{2}P^2_{-1}(\ell)\left[\int_\infty^r \int_\infty^{r'} Im\psi_1 dr' dr - \frac{\sqrt{2}}{2} Im\widehat{\sigma}^o\right] + P^2_{-2}(\ell)\left[2Im\widehat{\sigma}^o + 2r Im\dot{\widehat{\sigma}}^o\right] +$$
$$+ P^2_{-2}(\ell)\left[\int_\infty^r r'^2 Im\psi_0 dr - r^2 \int_\infty^r Im\psi_0 dr + r^2 \int_\infty^r \frac{1}{r'^2}\int_\infty^{r'} r'' Im\psi_0 dr' dr\right]. \tag{4.3}$$

In the above calculations, we have used the following identities:

$$\widehat{\omega}^o = \frac{\sqrt{2}}{2} P_{-1}(\ell) Y^1{}_\ell\, \widehat{\sigma}^o,$$
$$\dot{\psi}_0 P^2_{-2}(\ell) = \frac{1}{2r}\left[\frac{\partial}{\partial r}(r\psi_0)P^2_{-2}(\ell) - \sqrt{2}\,\psi_1 P^2_{-1}(\ell)\right]. \tag{4.4}$$

These were obtained from Eq. (2.13*b*) and from the linearized Bianchi identities, Eq. (2.7).

It is easy to verify that the two methods are compatible, for, if we substitute the exact retarded solutions given by Couch and Torrence[39] in equation (4.3), the resulting expressions satisfy the propagation equations derived in chapter 3. Moreover, this gives us a check on the correctness of the applied procedure.

Also, notice that the gauge-invariants themselves do not have a well defined spin weight. However, if we evaluate the potential $\mathcal{V}$ introduced in §3.4,

$$(\ell-1)(\ell+2)\mathcal{V} = r^4\left[(\frac{k_0}{r^2})_{,1} - (\frac{k_1}{r^2})_{,0}\right]$$
$$= 2\sqrt{2}r^2 P^2_{-1}(\ell)\left[\int_\infty^r r' Im\psi_1 dr - \frac{1}{r}\int_\infty^r \int_\infty^{r'} r'' Im\psi_1 dr' dr + \frac{Im\widehat{\omega}^o}{r}\right], \tag{4.5}$$

we find that it has a spin weight of 1. This could explain why it was necessary to take a particular combination of the gauge-invariants to derive a master equation.

Finally, we now have a readily available *gauge-independent* physical interpretation, for if we add the expressions for the Gerlach-Sengupta gauge-invariants, Eq. (4.3), and take a partial derivative with respect to r, the result is

$$(\hat{k}_0 + \hat{k}_1)_{,r} = 2P^2_{-2}(\ell) r^2 Im\psi_0. \tag{4.6}$$

By taking a particular combination of the k_A, we have obtained a potential for the imaginary part of Ψ_0. If we know k_0 and k_1, we effectively have a solution for the imaginary part of Ψ_A. The advantage is that we now have a means to classify the solution[29] without introducing a gauge.

CHAPTER 5

Conclusion

The results obtained in chapter 2 indicate that the Gerlach-Sengupta formalism is well suited to perturbation analysis. Since the gauge freedom is treated explicitly, we can impose some gauge conditions that simplify the calculations without disturbing the outcome. Although the fact that the same master equation governs both parities is quite a simplification, it is not surprising. In a related line of work Janis[40] also found a connection between the parities. There, it was shown that if the background is a Robertson-Walker spacetime, the even and odd gravitational waves share the same propagation properties. In the Gerlach-Sengupta formalism, this connection is expressed in a remarkably simpler form. Furthermore, the formalism is flexible and broadly based; it seems not unreasonable to anticipate that these results could be extended beyond the Friedmann-Robertson-Walker class of spacetimes. Another point worthy of mention is that in the work of Janis[40] the metric isn't easily recuperable: more calculations are required and the gauge issue is totally clouded. In the Gerlach and Sengupta framework, we can readily obtain the metric perturbations in the gauge of our choice.

Where do we go from here? There are two obvious extensions to the work presented here. First, our understanding of the odd-parity perturbations is much greater than that of the even-parity. The even-parity perturbations are inherently more complex. There are more gauge-invariants and, even in a simple case like Minkowski space, the equations that relate them are intricate. In spite of this, it would be useful to duplicate the results, obtained in §4.3 for the odd-parity, linking the Gerlach-Sengupta and Newman-Penrose formalisms. The comparison between the two formalisms may provide an explanation for the additional constants generated in the solution for the system.

Secondly, since the even-parity problem has been solved for only one of the classes of spacetimes introduced in §3.1, it is somewhat tempting to try to solve it for other classes. In particular, since there are known exact solutions for the Kantowski-Sachs spacetimes with perfect fluid,[41] we can utilize the background information to simplify the linearized field equations and attempt to reproduce the result obtained in §3.5. To our knowledge this hasn't been done. The situation is not so simple if we consider arbitrary stress-energy momentum tensors. During our investigation of gravitational waves it became apparent that $\Delta t_\mu{}^\nu = 0$ may not be an adequate mathematical description for purely gravitational radiation [see §3.2]. More intensive research is required in this direction.

Finally, we would like to point out that, since this work was started, two more gauge-invariant formulations have been proposed. The first one, by Ellis and Bruni,[3] offers a covariant framework that by-passes the need for an awkward three-dimensional harmonic decomposition. More recently, Stewart[4] has presented a formulation that allows a further breakdown in the perturbative analysis. In his approach, there are six independent sets of equations corresponding to scalar, vector, and tensorial equations for the odd and even parity perturbations. Each of these types of perturbation can be analyzed independently of the others, and this fact could potentially facilitate the study of the kind of questions considered in this thesis.

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31 An affine parameter is a parameter along the geodesic, such that the equation for the geodesic takes the standard form $\frac{d^2\xi^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{d\xi^\alpha}{ds}\frac{d\xi^\beta}{ds} = 0$.

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APPENDIX

In this appendix, we give the geometrical quantities obtained using the form of the metric derived in §2.1:

$$ds^2 = g_{AB}dx^A dx^B + R^2(x^C)\gamma_{ab}dx^a dx^b,$$

where

$$\gamma_{ab}dx^a dx^b = d\theta^2 + \sin^2\theta d\phi^2.$$

The non-vanishing Christoffel symbols are $\Gamma^{\sigma}_{\mu\nu}$:

$$\begin{aligned}
&\Gamma^C_{AB},\\
&\Gamma^b_{Aa} = v_A\delta^b_a,\\
&\Gamma^A_{ab} = -v^A g_{ab},\\
&\Gamma^2_{33} = -\sin\theta\cos\theta,\\
&\Gamma^3_{32} = \Gamma^3_{23} = \cot\theta.
\end{aligned}$$

Defining the Riemann tensor, the Ricci curvature tensor, and the scalar curvature as in §1.3, we obtain

$$\begin{aligned}
&R_{AB} = \Gamma^D_{AB,D} - \Gamma^D_{AD,B} + \Gamma^D_{FD}\Gamma^F_{AB} - \Gamma^D_{FB}\Gamma^F_{AD} - 2(v_{A|B} + v_A v_B),\\
&R_{Aa} = R_{aA} = 0,\\
&R_{ab} = \Gamma^d_{ab,d} - \Gamma^d_{ad,b} + \Gamma^c_{dc}\Gamma^d_{ab} - \Gamma^c_{db}\Gamma^d_{ac} - g_{ab}(v^A{}_{|A} + 2v^A v_A),\\
&R = -2(2v^A{}_{|A} + 3v^A v_A - R^{-2} - \mathcal{R}),
\end{aligned}$$

where $\mathcal{R}$ is the Gaussian curvature on M^2, given by

$$\begin{aligned}
\mathcal{R} &= \frac{{}^{(2)}R}{2}\\
&= \frac{g_{AB}}{2}\left[\Gamma^D_{AB,D} - \Gamma^D_{AD,B} + \Gamma^D_{FD}\Gamma^F_{AB} - \Gamma^D_{FB}\Gamma^F_{AD}\right],
\end{aligned}$$

and ${}^{(2)}R$ is the scalar curvature on M^2.