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A Systematic Study of the Geeta Probability Model

by

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Abstract

My supervisor, Professor P.C. Consul found that there has been a need of a more versatile L-shaped discrete probability model in order to cope with the modern complex data sets. So, in 1990, he defined and introduced the Geeta Probability Model to the statistical literature. Therefore, I decided to have a systematic study of the Geeta Probability Model and this becomes the primary objective of this thesis.

An introduction of different types of discrete distributions has been given in Chapter 1. Major families of discrete distributions and their basic properties have also been presented there.

Chapter 2 has given the definition and the basic properties of the Geeta distribution which are essential for our further study.

A systmatic investigation of some important properties of the Geeta distribution has been done in Chapter 3.

Chapter 4 deals with the estimation of the parameters and functions of parameter of the Geeta probability model.

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Chapter 1

Introduction

1.1 Mathematical Formulae, Notations and Terminology

Some mathematical formulae, notations and terminology, which are used at various places in this thesis are defined as below:

1.1.1 Abbreviations

- pmf : probability mass function of a discrete distribution.
- pdf : probability density function of a continuous distribution.
- pgf : probability generating function.
- gf : generating function.
- mgf : moment generating function.
- cgf : cumulant generating function.

1.1.2 The Gamma Function

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx, a > 0.$$

Also,

$$a\Gamma(a) = \Gamma(a+1), a > 0,$$

and

$\Gamma(a+1) = a!$, when a is a non-negative integer.

In particular,

$n! = 1.2.3...(n-1).n$, if n is a positive integer.

$0! = 1$, and $\Gamma(1/2) = \sqrt{\pi}$.

1.1.3 Binomial Coeifficents

When n and x are two non-negative integers, $x \leq n$, then

$$\begin{pmatrix} n \\ x \end{pmatrix} = \frac{n!}{(n-x)!x!} = \begin{pmatrix} n \\ n-x \end{pmatrix}.$$

Also,

$$\begin{pmatrix} -n \\ x \end{pmatrix} = \frac{(-n).(-n-1)...(-n-x+1)}{x!} = \frac{(-1)^x.(n).(n+1)...(n+x-1)}{x!}.$$

When n and r are any real numbers, then

$$\begin{pmatrix} n \\ r \end{pmatrix} = \frac{\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)}.$$

1.1.4 Descending and Ascending Factorials

For all real values of a and positive integral values of r, the descending and ascending factorials are defined by,

$a^{(r)} = (a).(a-1).(a-2)...(a-r+1)$,

$a_{(r)} = (a).(a+1)).(a+2)...(a+r-1)$,

$a^{(0)} = 1$ and $a_{(0)} = 1$.

1.1.5 Feller(1968)'s Inequality :

(1.1.1) $$\sqrt{2\pi}y^{y+1/2}e^{-y}e^{\frac{1}{12y+1}} < \Gamma(y+1) < \sqrt{2\pi}y^{y+1/2}e^{-y}e^{\frac{1}{12y}},$$

where $y > 0$.

1.1.6 Lagrange Expansions

If $\Phi(x)$ is an analytic function then Taylor's theorem can be used to express it as a convergent power series in x, i.e.,

$\Phi(x) = \sum_{j=0}^{\infty} \frac{D^j\Phi(0)x^j}{j!}$,

where $D^j\Phi(0)$ is the jth derivative of the function Φ, evaluated at zero.

Then, the equation $x = y\Phi(x)$ can be easily written in the form

$y = x/\Phi(x) = \sum_{k=0}^{\infty} a_k x^k$

by the simple operation of division.

Lagrange(1770) was concerned with the inversion of the above series so that the variable x may be expressed as a power series in y when $\Phi(0) \neq 0$. Hence, any other analytic function f(x) can be explicitly given as a function of y. He obtained these in the form of a power series of y.

The Lagrange's expansion is usually defined for complex functions f(z) and g(z) which are analytic on and within a Contour C surrounding a point a. If u is another variable such that the inequality $|ug(z)| < |z - a|$ is satisfied at all points of z on the perimeter of Contour C, then the equation $z = a + ug(z)$ has one root in the interior of the Contour C, then by Lagrange's theorem any function f(z), which is analytic on and inside the Contour C, can be expanded as a power series in u by the formula,

$$f(z) = f(a) + \sum_{k=1}^{\infty} \frac{u^k}{k!} \frac{\partial^{k-1}}{\partial z^{k-1}} [(g(z))^k f'(z)]_{z=a}, \tag{1.1.2}$$

where $f'(z)$ is the derivative of f(z) with respect to z and $g(a) \neq 0$.

See [Whittaker & Watson,1972].

The inversion of the function z under $|ug(z)| \leq |z-a|$, follows directly from the above by taking $f(z) = z$ and becomes

$$z = \sum_{k=1}^{\infty} \frac{u^k}{k!} \frac{\partial^{k-1}}{\partial z^{k-1}} [(g(z))^k]_{z=a}. \tag{1.1.3}$$

1.1.7 Taylor Expansion for bivariate functions

[Burden & Faires,1985] have given the well-known Taylor's bivariate expansion as the following theorem.

Theorem : Suppose that f(t,y) and all its partial derivatives of order less than or equal to n+1 are continuous on D = {(t,y): $a \leq t \leq b$, $c \leq y \leq d$ }. Let $(t_o, y_o) \in D$. For every $(t, y) \in D$, there exists some numbers ϵ (between t and t_o) and η (between y and y_o) with

$$f(t,y) = P_n(t,y) + R_n(t,y),$$

where,

$$P_n(t,y) = f(t_o, y_o) + [(t-t_o)\frac{\partial f(t_o,y_o)}{\partial t} + (y-y_o)\frac{\partial f(t_o,y_o)}{\partial y}]$$

$$+[\frac{(t-t_o)^2}{2}\frac{\partial^2 f(t_o,y_o)}{\partial t^2} + (t-t_o)(y-y_o)\frac{\partial^2 f(t_o,y_o)}{\partial t \partial y}$$

$$+\frac{(y-y_o)^2}{2}\frac{\partial^2 f(t_o,y_o)}{\partial y^2}] + \ldots + [\frac{1}{n!}\sum_{j=0}^{n}\begin{pmatrix} n \\ j \end{pmatrix}(t-t_o)^{n-j}(y-y_o)^j\frac{\partial^n f(t_o,y_o)}{\partial t^{n-j}\partial y^j}]$$

and

$$R_n = \frac{1}{(n+1)!}\sum_{j=0}^{n+1}\begin{pmatrix} n+1 \\ j \end{pmatrix}(t-t_o)^{n+1-j}(y-y_o)^j\frac{\partial^{n+1} f(\epsilon,\eta)}{\partial t^{n+1-j}\partial y^j}.$$

1.2 Discrete Distributions

In many scientific investigations, observations are taken repeatedly under essentially the same set of conditions or with slight variations. Each of these observation taking procedures is called an 'experiment' and the corresponding observations are called the 'outcome' of the experiment. Clearly, we cannot predict the result or outcome of each experiment before the experiment is performed.

An arbitrary outcome is denoted by 'ω' which will refer to the observed member of all possible outcomes that could be realized. Such an experiment is a 'random experiment', a particular outcome ω is a 'sample point', and the set Ω of all possible outcomes is the 'sample space'.

A random variable (r.v.) X is a real-valued function X(ω) defined on the sample space Ω for the sample point ω.

A cumulative distribution function $F_X(x)$ for the r.v. X is, in general, a continuous function of x. When $F_X(x)$ is a step function with an enumerable number of steps, then it represents a discrete distribution.

By far, the most commonly used discrete distributions are those for which x's are the non-negative integers. They are used in models for "count data", which include variables representing the results of counts(of defective items, apples on a tree, etc.). However, it is not necessary that the random varible X takes only integer values (an observed proportion is a simple counter example).

Most of the discrete distributions commonly used in theoretical and applied statistics belong to a much narrower class of distributions called 'lattice distribution' which will be described in the following section.

1.2.1 Lattice Distributions

A discrete random variable X has a lattice distribution if there exits numbers a and $h > 0$ such that all possible values of X are representable in the form $a + kh$, where k may take on any integral values in $(-\infty < k < \infty)$. See [Gnedenko,1967].

Lemma: A necessary and sufficient condition for a probability distribution with characteristic function f(t) to be a lattice distribution is that there exists a real number $t_o \neq 0$ such that $|f(t_o)| = 1$.

Proof: If X has a lattice distribution and p_k is the probability of $X = a + kh$, then the characteristic function of the variable X is

$$f(t) = \sum_{k=-\infty}^{\infty} p_k e^{it(a+kh)} = e^{iat} \sum_{k=-\infty}^{\infty} p_k e^{itkh}.$$

From this, we have

$$f(\frac{2\pi}{h}) = e^{2\pi i \frac{a}{h}} \sum_{k=-\infty}^{\infty} p_k e^{2\pi ik} = e^{2\pi i \frac{a}{h}}.$$

Hence, we see that

$|f(\frac{2\pi}{h})| = 1$ for every lattice distribution.

Now, assume that for some $t_o \neq 0$, $|f(t_o)| = 1$, then X will be shown to have a lattice distribution.

The last equation implies that for some θ

$$f(t_o) = e^{i\theta}.$$

So,

$$\int e^{it_ox} dF(x) = e^{i\theta},$$

or,

$$\int e^{i(t_ox-\theta)} dF(x) = 1.$$

It follows that

$$\int \cos(t_ox - \theta) dF(x) = 1.$$

In order that the above relation be possible, it is necessary that the function F(x) increases only at those values of x for which

$$\cos(t_ox - \theta) = 1.$$

This implies that all the possible values of X must be of the form

$$x = \frac{\theta}{t_o} + k\frac{2\pi}{t_o}.$$

Hence, the proof is completed.

1.2.2 Inflated Distributions

When the probability of one value, say x_o, of the discrete random variable X is increased and the remaining probabilities are multiplied by an appropriate constant to keep the sum of probabilities equal to unity, the distribution of such modified probabilities is called an inflated distribution F^* of the original distribution F.

Denoting $P(X = r|F)$ by P_r, the inflated distribution F^* has

$$P^*(X = x_o) = 1 - \alpha + \alpha P_{x_o}, \quad 0 < \alpha < 1,$$
$$P^*(X = x) = \alpha P_x, \quad for\ all\ x \neq x_o.$$

In terms of the moments about zero $\{\mu_j'\}$ of F, the rth moment about zero of the inflated distribution F^* is

$$\mu_r^{*'} = (1-\alpha)x_o^r + \alpha\mu_r'.$$

It can be shown that if a recurrence relation $g(\mu_1', \mu_2', ...) = 0$ holds for F, the moments of the inflated distribution satisfy the recurrence relation,

$$g(\frac{\mu_1^{*'} - (1-\alpha)x_o}{\alpha}, \frac{\mu_2^{*'} - (1-\alpha)x_o^2}{\alpha}, ...) = 0.$$

In particular, if $\sum_j c_j\mu_j' = 0$, then
$\sum_j c_j\mu_j^{*'} = (1-\alpha)\sum_j c_j x_o^j$.

1.2.3 Decapitated Distributions

When a discrete probability distribution is defined over non-negative integers including $X = 0$ and if the probability mass at $X = 0$ is proportionately distributed at all other values of the r.v. X, the resultant distribution is termed as the 'zero-truncated' or 'decapitated' distribution of the original.

Thus, corresponding to the Poisson distribution

(1.2.4) $$P(X = x) = \frac{e^{-\theta}\theta^x}{x!}, (x = 0, 1, 2, ...),$$

the decapitated Poisson distribution is

(1.2.5) $$P^*(X = x) = (1 - e^{-\theta})^{-1}\frac{e^{-\theta}\theta^x}{x!}, (x = 1, 2, 3, ...).$$

Sometimes, this term is extended to include truncation by omission of more than one variable value. If value $x = 0$ and $x = 1$ are truncated, the term 'doubly decapitated' is used occassionally.

1.3 Classes of Discrete Distribution

The discrete probability distributions have been classified into many broad classes. Four of these are,

(i) Generalized Power series distributions(GPSD)

(ii) Modified Power series distributions(MPSD)

(iii)Factorial series distributions(FSD)

(iv) Lagrangian probability distributions(LPD)

(i) is a sub-class of (ii). Also, (ii) is a sub-class of (iv). Hence, the properties that are proven true for (iv) will be true for (i) and (ii) and the properties which are proven for (ii) will also be true for (i). The definitions of the probability distributions belonging to the above four classes and some general properties are given in the following sections.

1.3.1 Generalized Power Series Distributions(GPSD)

Let $f(\theta)$ be a positive analytic function such that

(1.3.6) $$f(\theta) = \sum_{x=0}^{\infty} a(x)\theta^x,$$

where $a(x) \geq 0$ for all integral values of x, then the Power Series Distribution(PSD) is defined by

$$(1.3.7) \qquad P(X=x) = \begin{cases} \frac{a(x)\theta^x}{f(\theta)}, & \text{x=0,1,2,...,}\theta > 0 \\ 0, & \text{otherwise} \end{cases}$$

The PSD was introduced by [Kosambi,1949] and [Noack,1950] independently.

Later, [Patil,1962] generalized the domain of the distribution 1.3.7 to be the set T which is a subset of the set of non-negative integers. The resultant family of distributions is said to be the Generalized Power Series Distribution(GPSD). Some general properties of the GPSD are :

1. A truncated distribution of a PSD is a GPSD. Probability distributions 1.2.4 and 1.2.5 are examples of this type. A truncated GPSD is also a GPSD. For example, distribution 1.2.5 is a GPSD, if it is truncated further for the value of x=1, then the resultant distribution is given by

   $$(1.3.8) \qquad P^{**}(X=x) = \frac{1}{1-e^{-\theta}(1+\theta)}\frac{e^{-\theta}\theta^x}{x!}, x = 2, 3, ...,$$

   which is still a GPSD.

2. The specific choices of $f(\theta)$ and T reduce the GPSD to many well known distributions like the Negative Binomial, Poisson and Logarithmic series distributions and their truncated forms.

3. Moments and generating functions :

   $$(1.3.9) \qquad mean\ \mu = \theta\frac{d\ln f(\theta)}{d\theta} = \theta\frac{f'(\theta)}{f(\theta)}$$

   $$(1.3.10) \quad variance\ \sigma^2 = \mu + \theta^2\frac{d^2\ln f(\theta)}{d\theta^2} = \mu - \mu^2 + \theta^2\frac{f''(\theta)}{f(\theta)} = \theta\frac{d\mu}{d\theta}$$

The pgf and mgf of GPSD are given by

$$\begin{aligned} g(t) &= \frac{f(t\theta)}{f(\theta)} \\ m(t) &= \frac{f(\theta e^t)}{f(\theta)}. \end{aligned} \tag{1.3.11}$$

1.3.2 Modified Power Series Distributions(MPSD)

A discrete random variable X is said to follow a Modified Power Series Distribution(MPSD), if its probability mass function (pmf) is given by

$$P(X = x) = \begin{cases} \frac{a(x)[G(\theta)]^x}{f(\theta)}, & x \in T \\ 0, & \text{otherwise} \end{cases} \tag{1.3.12}$$

where T is a subset of the set of non-negative integers; $a(x) > 0$; $G(\theta)$ and $f(\theta)$ are positive finite and successively differentiable functions of θ and if $f(\theta) = \sum_{x \in T} a(x)[G(\theta)]^x$. See [Gupta,1974].

If $G(\theta)$ equals θ or if $G(\theta)$ is invertible,1.3.12 reduces to the GPSD. In addition, if T is the set of all non-negative integers,1.3.12 becomes the PSD. Therefore, the PSD and GPSD are sub-families of the MPSD. Accordingly, all properties of the MPSD are also properties of the GPSD.

Similar to 1.3.9, a truncated MPSD is also a MPSD. Also, the MPSD family includes not only all the discrete probability distributions of the GPSD but also it contains the Generalized Negative Binomial distributions, the Generalized Poisson distributions, the Generalized Logarithmic Series distributions and their truncated forms.

Some general properties of the MPSD are :

1. mean μ

$$\mu = \frac{G(\theta) f'(\theta)}{f(\theta) G'(\theta)}. \tag{1.3.13}$$

By following [Gupta,1974]'s method, the proof of 1.3.13 is provided as follows :

Since $f(\theta) = \sum_x a(x)[G(\theta)]^x$,

differentiating both sides of the above with respect to θ, we have

$f'(\theta) = \sum_x xa(x)[G(\theta)]^{x-1}G'(\theta)$, or

$f'(\theta) = E(X)\frac{f(\theta)G'(\theta)}{G(\theta)}$.

Hence, $E(X) = \mu = \frac{G(\theta)f'(\theta)}{f(\theta)G'(\theta)}$.

2. <u>Variance σ^2</u>

(1.3.14)
$$\sigma^2 = \frac{G(\theta)}{G'(\theta)}\frac{d\mu}{d\theta}.$$

3. <u>Recurrence relation between the moments μ_r',</u>

(1.3.15)
$$\mu_{r+1}' = \frac{G(\theta)}{G'(\theta)}\frac{d\mu_r'}{d\theta} + \mu_r'\mu_1', r = 1, 2, ...$$

4. <u>Recurrence relation between the central moments μ_r,</u>

(1.3.16)
$$\mu_{r+1} = \frac{G(\theta)}{G'(\theta)}\frac{d\mu_r}{d\theta} + r\mu_2\mu_{r-1}, r = 1, 2, ...$$

Note that 1.3.9 and 1.3.10 are particular cases of 1.3.13 and 1.3.14 respectively.

1.3.3 Factorial Series Distribution(FSD)

The family of FSD defined and studied by [Berg,1974, Berg,1978].

Let f(N) be an analytic function of the integer-valued variable N.

Suppose that f(N) can be expanded in a factorial series in N with non-negative coefficients a_x. That is, we assume that

$$f(N) = \sum_x a_x N^{(x)},$$

where $N^{(x)}$ is given by section 1.1.5, and $a_x \geq 0$.

Based on this expansion, the probability mass function(pmf) for a FSD is given by

$$P(X = x) = \begin{cases} \frac{N^{(x)} a_x}{f(N)}, & \text{x=0,1,2,...,N} \\ 0, & \text{otherwise} \end{cases} \tag{1.3.17}$$

The set of values of x, for which $a_x > 0$, is called the range of the FSD.

It has been shown that $a_x = \frac{\Delta^x f(0)}{x!}$, where $\Delta^x f(0)$ is the xth forward difference of the function f(N), computed at zero.

Accordingly, the FSD can be written in the form

$$P(X = x) = \begin{cases} \binom{n}{x} \frac{\Delta^x f(0)}{f(N)}, & \text{x=0,1,...,N} \\ 0, & \text{otherwise.} \end{cases} \tag{1.3.18}$$

The factorial moments of a random variable X having an FSD are given by

$$E(X^{(r)}) = \mu^{(r)} = N^{(r)} \frac{[\Delta^r f(N-r)]}{f(N)}. \tag{1.3.19}$$

The proof of 1.3.19 is given as follows :

$$
\begin{aligned}
E(X^{(r)}) &= \sum_{x=0}^{N} x^{(r)} P(X=x) \\
&= \sum_{x} \frac{x^{(r)} N^{(x)} \Delta^{x} f(0)}{x! f(N)} \\
&= \frac{N^{(r)}}{f(N)} \sum_{x} \frac{N^{(x)} \Delta^{x} f(0)}{N^{(r)} (x-r)!} \\
&= \frac{N^{(r)}}{f(N)} \sum_{x} \frac{(N-r)^{(x-r)}}{(x-r)!} \Delta^{r} (\Delta^{x-r} f(0)) \\
&= \frac{N^{(r)}}{f(N)} \Delta^{r} [\sum_{x-r} \frac{(N-r)^{(x-r)}}{(x-r)!} \Delta^{x-r} f(0)] \\
&= \frac{N^{(r)}}{f(N)} [\Delta^{r} f(N-r)].
\end{aligned}
$$

In particular, putting r=1 in 1.3.19, the mean of X is

$$E(X) = \mu = \frac{N}{f(N)} [f(N) - f(N-1)]. \tag{1.3.20}$$

The family of FSD introduced here is the discrete parameter analogue of the PSD. Note that the two families of discrete distributins have many properties in common. See [Berg,1974] and [Johnson & Kotz,1977]. Some of them are listed in the following table.

Table 1.3.1: Common Properties of PSD & FSD

PSD	FSD
$P(X=x)=\frac{\theta^x D^x f(0)}{x! f(\theta)}$, where x=0,1,2,... and $D^x f(0)$ is the xth derivative of $f(\theta)$ computed at zero.	$P(X=x)=\frac{N^{(x)}\Delta^x f(0)}{x! f(N)}$, where x=0,1,2,...and $\Delta^x f(0)$ is the xth forward difference of f(N) evaluated at zero.
$\mu^{(r)}=\frac{\theta^r D^r f(\theta)}{f(\theta)}$,	$\mu^{(r)}=\frac{N^{(r)}\Delta^r f(N-r)}{f(N)}$,
$E[\frac{D^x(h(0)f(0))}{D^x f(0)}]=h(\theta)$, provided that $\sum_x \frac{\theta^x}{x!}D^x(h(0)f(0))=h(\theta)f(\theta)$.	$E[\frac{\Delta^x(h(0)f(0))}{\Delta^x f(0)}]=h(N)$, provided that $\sum_x \frac{N^{(x)}}{x!}\Delta^x(h(0)f(0))=h(N)f(N)$.

The family of FSD includes members among others, such as the Binomial distribution (with $f(N)=(1+\theta)^N$), the classical occupancy distribution ($f(N)=N^n$) and the matching distribution (f(N)=N!).

1.3.4 Lagrangian Probability Distributions(LPD)

The wide class of discrete Lagrangian probability distributions, introduced and defined by [Consul & Shenton,1972], consists of many important families auch as the Generalized Poisson distribution, Generalized Negative Binomial distribution, Generalized Logarithmic distribution, Modified Power Series distribution. The LPD was named by the above authors on account of the fact that LPD was generated by the well known Lagrange expansion of a function f(x) as a power series in y when $y = \frac{x}{g(x)}$.

Long before the introduction of LPD into the statistical literature, [Otter,1949] pointed out the applicability of Lagrange expansion into branching process in univariate situation. Later, [Good,1965] extended it to the multivariate case.

Basic Lagrangian Probability Distributions(BLD) When f(t)=t and g(t) is an analytic function of t, such that

$$g(0) > 0, g(1) = 1 \quad and \quad (\frac{\partial}{\partial t})^{x-1}[g(t)]^x|_{t=0} \geq 0, x \geq 1, \tag{1.3.21}$$

then the Lagrangian expansion equation (1.1.3) with the transformation

$$t = ug(t) \tag{1.3.22}$$

gives

$$t = \phi(u) = \sum_{x=1}^{\infty} \frac{u^x}{x!}[(\frac{\partial}{\partial t})^{x-1}[g(t)]^x]_{t=0}. \tag{1.3.23}$$

Since 1.3.22 gives u=0 for t=0 and u=1 for t=1, also 1.3.23 satisfies 1.3.21 and the Lagrange's expansion conditions, thus the equation 1.3.23 satisfies the property of a pgf. Hence, the equation 1.3.23 can be written as

$$\phi(u) = E[u^x].$$

The expansion 1.3.23 is the basic Lagrangian pgf and the discrete distribution represented by it, i.e.,

$$P(X = x) = \frac{1}{x!}[(\frac{\partial}{\partial t})^{x-1}(g(t))^x]_{t=0}, x \in N. \tag{1.3.24}$$

The distribution 1.3.24 is the basic Lagrangian probability distribution (BLD) defined on N, a subset of the set of positive integers.

It is easily seen that numerous values of g(t) satisfying the conditons of 1.3.21 give paricular families of the BLD. Some members of the BLD are the Geometric distribution, the Haight distribution, the Consul distribution, the Geeta distribution and the Borel distribution.

[Consul & Shenton,1974] showed that all BLD's are closed under convolution and all BLD's are the probability distributions of the busy periods of a single server when the queue is initiated by a single customer and is served on the basis of first come first served.

The General LPD If f(t) is another analytic function such that

$$\begin{gathered} 0 \le f(0) \le 1, f(1) = 1, \\ and \\ \frac{\partial^{x-1}}{\partial t^{x-1}}[(g(t))^x \frac{\partial f(t)}{\partial t}]_{t=0} \ge 0, x \ge 1, \end{gathered} \tag{1.3.25}$$

then, by using 1.1.2 and the transformation 1.3.22 one gets the General Lagrangian probability generating fucntion which gives the General LPD as

$$P(X=x)=\begin{cases}\frac{1}{x!}[\frac{\partial^{x-1}}{\partial t^{x-1}}([g(t)]^x f'(t))]_{t=0}; & x\in N\\ f(0); & \text{x=0}\end{cases} \tag{1.3.26}$$

where N is a subset of the set of positive integers.

Note that in this case the pgf of the General LPD is a function of u given by f(t), where $t=ug(t)$.

Numerous families of distributions can be obtained by assigning different values of g(t) and f(t) satisfying the conditions of 1.3.21 and 1.3.25.

The Generalized Poisson distribution, the Generalized Negative Binomial distribution and the Generalized Logarthmic Series distribution , are some well known members of the General LPD.

It should be noted that [Consul & Shenton,1972] originally took f(t) and g(t) as pgf's defined on non-negative integers such that $g(0)\neq 0$. [Consul,1981] removed the restriction that f(t) and g(t) necessarily be pgf's and widened the class of the General LPD to include MPSD as a subclass.

Chapter 2

The Geeta Distribution(GD)

2.1 Definition of the Geeta probability model

[Consul,1990a] defined the Geeta probability model by a discrete random variable X, defined over the set of all positive integers, with the probability mass function,

$$(2.1.1) \qquad P(X=x) = \begin{cases} \frac{1}{\beta x-1}\binom{\beta x-1}{x}\theta^{x-1}(1-\theta)^{\beta x-x}, & x=1,2,3,\ldots \\ 0, & \text{otherwise} \end{cases}$$

where $0<\theta<1$ and $1<\beta\leq\theta^{-1}$.

The model exists for all values of β in the above range, However, the moments do not exist for all those values of β and θ where $\beta\theta=1$. When $\beta\to 1$, the model degenerates to a single point at $x=1$.

The model can also be expressed as a location-parameter probability distribution which is given by

$$(2.1.2) \quad P(X=x) = \begin{cases} \frac{1}{\beta x-1}\binom{\beta x-1}{x}\left[\frac{\mu-1}{\mu(\beta-1)}\right]^{x-1}\left[\frac{\mu(\beta-1)}{\beta\mu-1}\right]^{\beta x-1}, & x=1,2,3,\ldots \\ 0, & \text{otherwise} \end{cases}$$

where μ is the mean of the Geeta distribution and $\beta>1$.

Note that this form does not seem to have an upper limit on β, however, one should not conclude that it is superior to the previously defined Geeta probability model because the above model presupposes the existence of the mean μ which needs the condition $\beta\theta<1$.

The family of Geeta probability models belongs to the classes of the Modified Power Series distribution(MPSD), the exponential class and the Lagrangian Series distributions(LPSD).

Various properties of the Geeta probability model are described in the following sections.

2.1.1 Recurrence Formula for successive probabilities

The successive probabilities for different values of X can be computed from the values

$$P(X=1) = [\frac{(\beta-1)\mu}{\beta\mu-1}]^{\beta-1}, \tag{2.1.3}$$

$$P(X=2) = \frac{\mu-1}{\mu}[\frac{(\beta-1)\mu}{\beta\mu-1}]^{2\beta-1}, \tag{2.1.4}$$

and the recurrence relation,

$$P(X=k+1) = [\prod_{i=2}^{k}(1+\frac{\beta}{k\beta-i})]\frac{\mu-1}{\mu}[\frac{(\beta-1)\mu}{\beta\mu-1}]^{\beta}P(X=k), \tag{2.1.5}$$

for $k = 2, 3, 4, \ldots$

2.1.2 Graphical Representation of the GD

There are only three L-shaped discrete probability models, namely, the Logarithmic series distribution, the discrete Pareto distribution, and the Yule distribution [Johnson & Kotz,1969].

All of them have a single parameter, so they are not versatile enough to meet the needs of modern complex data sets. [Consul,1990a] defined the GD which has two parameters and is L-shaped, which is far more versatile than other L-shaped probability model. See [Consul,1991].

In order to study the behaviour of the family of Geeta models with varying values of β and μ, the probabilities for the model were computed for various values of x and different sets of values of β and μ. Thirty bar-diagrams were drawn on which the probabilities were ploted on various values of the two parameters.

For the study of the effect of changes in the value of β, Graph 2.1, 2.2,and 2.3 were plotted with fixed values of $\mu = 1.5, 3.0,$ *and* 6.0 respectively. Each of these three graphs contains six bar-diagrams of $\beta = 1.1, 1.6, 2.6, 4.6, 8.6$ *and* 16.6.

It is observed from these graphs that $P(X = 1)$ reduces as β increases and the probabilities for all other values of X increase but the model always remains L-shaped. Thus the tail becomes more and more heavy and long with the increase in the value of β. Also, it is observed that $P(X = 1)$ seems reduces faster at Graph 2.3 (with fixed $\mu = 6.0$) than that from Graph 2.1($\mu = 1.5$). This seems to imply that at higher value of μ, the influence of β to reduce $P(X = 1)$ and to increase other probabilities is more effective.

Similarly, in order to study the effect of changes in the value of μ, Graph 2.4 and 2.5 were drawn with fixed value of $\beta = 1.1,$ and 2.6 respectively. Each of these two graphs contains six bar-diagrams of various value of $\mu = 1.5, 3.0, 6.0, 9.0, 12.0$ and 15.0.

It is observed from these graphs that $P(X = 1)$ also reduces as μ increases.However, it is noticed that the effect of reducing $P(X = 1)$ and increasing the tail probabilities is more obvious in Graph 2.5(with $\beta = 2.6$) than that of Graph 2.4 ($\beta = 1.1$) with the same amount of increase of μ. But all the bar-diagrams still remain in L-shaped.

Moreover, through careful visual comparision between graph 2.1 and graph 2.4, one can observe that the changes for μ(with fixed β) are at a faster rate than those changes for β(with fixed μ).

It is also observed from all these bar-diagrams that all the corresponding values of $P(X=1)$ never drop below 0.4 no matter how much β and μ increase. This seems to suggest that there may exist a limiting value for $P(X=1)$ as β or μ increase.

To investigate further on the limiting value of $P(X=1)$, one can differentiate $P(X=1)$ or P_1 by μ and by β respectively. One can get

$$\begin{aligned}
\frac{dP_1}{d\mu} &= -\frac{(\beta-1)}{\mu(\beta\mu-1)}[\frac{(\beta-1)\mu}{\beta\mu-1}]^{(\beta-1)} < 0. \\
\frac{dP_1}{d\beta} &= \frac{1}{\mu}[\frac{(\beta-1)\mu}{\beta\mu-1}]^{(\beta-1)} \\
&\quad \times\{\ln[\frac{(\beta-1)\mu}{\beta\mu-1}]\beta\mu - \ln[\frac{(\beta-1)\mu}{\beta\mu-1}] - 1 + \mu\} \\
&< 0.
\end{aligned}$$

Therefore, $P(X=1)$ is a monotonically decreasing function for μ and β respectively. So, $P(X=1)$ achieves its minimum value at the largest possible value of μ and β. Note that μ is a monotonically increasing function of β and $\mu \to \infty$ as $\beta \to \infty$.(See section 2.2.1) Hence, taking limit of P_1 as $\mu \to \infty$ and as $\beta \to \infty$ respectively, one has

$$\begin{aligned}
\lim_{\mu\to\infty} P_1 &= \lim_{\mu\to\infty}[\frac{(\beta-1)\mu}{\beta\mu-1}]^{(\beta-1)} \\
&= (1-\frac{1}{\beta})^{(\beta-1)}.
\end{aligned}$$

However, the RHS of the above is a monotonically decreasing function of β. Hence, one can deduce that

$$\begin{aligned}\lim_{\mu\to\infty} P_1 &= (1-\frac{1}{\beta})^{(\beta-1)} \\ &> \lim_{\beta\to\infty}(1-\frac{1}{\beta})^{(\beta-1)} \\ &= e^{-1}.\end{aligned}$$

Similarly,

$$\begin{aligned}\lim_{\beta\to\infty} P_1 &= \lim_{\beta\to\infty}[\frac{(1-\frac{1}{\beta})\mu}{\mu-\frac{1}{\beta}}]^{\beta-1} \\ &= \lim_{\beta\to\infty}(1-\frac{1}{\beta})^{\beta-1}\lim_{\beta\to\infty}(\frac{\mu}{\mu-\frac{1}{\beta}})^{\beta-1} \\ &= e^{-1}.e^{0} \\ &= e^{-1}.\end{aligned}$$

Thus, the limiting value of $P(X=1)$ is found to be e^{-1}.

Also, it is clear from all these five graphs that none of the bar-diagrams has two humps, it seems to indicate that the GD models are all unimodal. Some of these properties will be verified in the following chapters.

2.2 Mean and Variance of the GD

2.2.1 Mean μ

Since the GD is a member of MPSD with $G(\theta)=\theta(1-\theta)^{\beta-1}$,

$G'(\theta)=(1-\beta\theta)(1-\theta)^{\beta-2}$, $f(\theta)=\theta$, and $f'(\theta)=1$.

Hence, by applying 1.3.13, the mean μ is given by

$$\begin{aligned} \mu &= \frac{G(\theta)f'(\theta)}{f(\theta)G'(\theta)} \\ &= \frac{(1-\theta)}{(1-\beta\theta)}. \end{aligned} \tag{2.2.6}$$

One point to note about the mean μ is that if one differentiates the μ with respect to θ and β respectively, one can get

$$\begin{aligned} \frac{d\mu}{d\theta} &= \frac{d}{d\theta}(1-\theta)(1-\beta\theta)^{-1} \\ &= \frac{(\beta-1)}{(1-\beta\theta)^2} > 0, \ \ for\ \beta > 1,\ \beta\theta < 1, \end{aligned} \tag{2.2.7}$$

and,

$$\begin{aligned} \frac{d\mu}{d\beta} &= \frac{d}{d\beta}(1-\theta)(1-\beta\theta)^{-1} \\ &= \frac{\theta(1-\theta)}{(1-\beta\theta)^2} > 0, \ \ as\ 0 < \theta < 1. \end{aligned} \tag{2.2.8}$$

Therefore, μ is a monotonically increasing function of θ and β respectively which is a useful fact in the last section of this chapter.

2.2.2 Variance σ^2

By applying the same $G(\theta)$ and $G'(\theta)$ of the above section and 2.2.7 in 1.3.14, then the variance σ^2 is given by

$$\begin{aligned} \sigma^2 &= \frac{G(\theta)}{G'(\theta)}\frac{d\mu}{d\theta} \\ &= (\beta-1)\theta(1-\theta)(1-\beta\theta)^{-3} \\ &= \mu(\mu-1)(\beta\mu-1)(\beta-1)^{-1}. \end{aligned} \tag{2.2.9}$$

Since

$$\frac{d\sigma^2}{d\mu} = [(\mu-1)(\beta\mu-1)+\mu(\beta\mu-1)+\beta\mu(\mu-1)](\beta-1)^{-1}$$

$$> 0, \; \textit{as } \mu > 1 \textit{ and } \beta\mu > 1, \tag{2.2.10}$$

so σ^2 increases monotonically as μ increases in value and that the smallest value of σ^2 is zero when $\mu = 1$, i.e. when the model reduces to a single point at $x = 1$. Also,

$$\frac{d\sigma^2}{d\beta} = \frac{-\mu(\mu-1)^2}{(\beta-1)^2} < 0.$$

Thus, the variance σ^2 decreases monotonically as β increases and the smallest value of σ^2, for the largest value of β, becomes $\mu^2(\mu-1)$. From this, one can conclude that when $\beta\mu - 1 \leq (\beta-1)(\mu-1)^{-1}$, the variance will be less than the mean μ and will have the range $\mu^2(\mu-1) < \sigma^2 \leq \mu$. If $\beta\mu - 1 > (\beta-1)(\mu-1)^{-1}$, the value of σ^2 will become larger than μ.

2.3 Properties of the Geeta distribution(GD)

2.3.1 Generation of the Geeta distribution

There are basically two ways of generating the GD, which are described in the following section.

Method I: By Basic Lagrangian Expansion

The pgf of the GD can be obtained by using the Basic Lagrangian Expansion 1.1.3 with $g(t) = (\frac{1-\theta}{1-\theta t})^{\beta-1}$,and $\beta > 1$. Details of this method will be described in the section of 'Generating Functions'.

Method II: Expansion of θ in powers of $\theta(1-\theta)^{\beta-1}$

The GD can also be generated by expanding the parameter θ $(0 < \theta < 1)$ by the basic Lagrange expansion 1.1.3 under the transformation of $\theta = u(1-\theta)^{1-\beta}$, $\beta > 1$ as follows:

$$
\begin{aligned}
\theta &= \sum_{x=1}^{\infty} \frac{u^x}{x!} \frac{\partial^{x-1}}{\partial \theta^{x-1}} [(1-\theta)^{x-x\beta}]_{\theta=0} \\
&= \sum_{x=1}^{\infty} \frac{[\theta(1-\theta)^{\beta-1}]^x}{x!} (\beta x - x)(\beta x - x + 1)\dots(\beta x - 2) \\
&= \sum_{x=1}^{\infty} \frac{1}{\beta x - 1} \binom{\beta x - 1}{x} \theta^x (1-\theta)^{\beta x - x}.
\end{aligned} \tag{2.3.11}
$$

On division by θ on both sides of 2.3.11, one can immediately obtain two kinds of result, namely, the GD probability distribution is obtained by the above expansion of θ and also one has already proved that the sum of the probabilities of the GD 2.1.1 over the domain of X is unity.

2.3.2 Generating Functions

The probability generating function of the Geeta distribution 2.1.1 is given by

$$f(u) = t(u), where\ t = u(1-\theta)^{\beta-1}(1-\theta t)^{-\beta+1}, \beta > 1. \tag{2.3.12}$$

It is clear from 2.3.12 that $u = 1$ when $t = 1$. The Lagrange expansion of t in terms of u, under the given transformation, becomes

$$
\begin{aligned}
f(u) &= t(u) \\
\text{(2.3.13)} \qquad &= \sum_{x=1}^{\infty} \frac{u^x}{x!} \frac{\partial^{x-1}}{\partial t^{x-1}} [(1-\theta)^{\beta x - x}(1-\theta t)^{-\beta x + x}]_{t=0} \\
&= \sum_{x=1}^{\infty} \frac{u^x}{x!} (1-\theta)^{\beta x - x} \theta^{x-1} (\beta x - x)(\beta x - x + 1) \ldots (\beta x - 2) \\
&= \sum_{x=1}^{\infty} u^x \frac{1}{\beta x - 1} \binom{\beta x - 1}{x} \theta^{x-1} (1-\theta)^{\beta x - x},
\end{aligned}
$$

which establishes the result.

The properties of the generating function can be studied by the implicit function t(u) defined by

$$
t(u) = u(1-\theta)^{\beta-1}(1-\theta t(u))^{-\beta+1}
$$

or

$$
\text{(2.3.14)} \qquad t(u)(1-\theta t(u))^{\beta-1} = u(1-\theta)^{\beta-1}.
$$

On differentiation of 2.3.14 with respect to u, one can get

$$
(1-\theta)^{\beta-1} = t'(u)(1-\theta t(u))^{\beta-1} + (\beta-1)(1-\theta t(u))^{\beta-2}(-\theta t'(u))t(u),
$$

or

$$
(1-\theta)^{\beta-1} = [1-\theta t(u)]^{\beta-2} t'(u)[1-\theta t(u)\beta].
$$

After rearranging terms, one has

$$
\text{(2.3.15)} \qquad \frac{dt(u)}{du} = \frac{(1-\theta)^{\beta-1}(1-\theta t(u))^{-\beta+2}}{(1-\beta\theta t(u))}
$$

By comparing coefficients of 2.3.13, one obtains

$$P(X=1) = \frac{dt(u)}{du}|_{u=0} = (1-\theta)^{\beta-1}. \tag{2.3.16}$$

Similarly, on further differentiation of 2.3.15 and comparing coefficients of 2.3.13, one has

$$\begin{aligned} P(X=2) &= \frac{1}{2!}\frac{d^2t(u)}{du^2}|_{u=0} = (\beta-1)\theta(1-\theta)^{2(\beta-1)} \\ &\vdots \\ P(X=k) &= \frac{1}{k!}\frac{d^kt(u)}{du^k}|_{u=0} \\ &\vdots \end{aligned} \tag{2.3.17}$$

2.3.3 Recurrence Relations for Central Moments

The k-th central moment μ_k of the GD 2.1.1 is

$$\mu_k = \sum_{x=1}^{\infty} \frac{1}{\beta x-1}\binom{\beta x-1}{x}\theta^{x-1}(1-\theta)^{\beta x-x}(x-\mu)^k$$

Multiplying the above by θ on both sides and differentiating with respect to θ, one has

$$\mu_k + \theta\frac{d\mu_k}{d\theta} = -k\theta\mu_{k-1}\frac{d\mu}{d\theta} + \sum_{x=1}^{\infty}(x-\mu)^k\frac{1}{\beta x-1}\binom{\beta x-1}{x}x\theta^{x-1}(1-\theta)^{\beta x-x-1}(1-\beta\theta),$$

which can be written in the form

$$\mu_k + \theta\frac{d\mu_k}{d\theta} = -k\theta\mu_{k-1}\frac{d\mu}{d\theta} + \frac{1-\beta\theta}{1-\theta}[\mu_{k+1} + \mu\mu_k]$$

.

The above expression gives the recurrence formula [Consul,1990a],

$$\mu_{k+1} = \mu\theta[\frac{d\mu_k}{d\theta} + \frac{k(\beta-1)}{(1-\beta\theta)^2}\mu_{k-1}],\ for\ k = 2,3,4,\ldots. \tag{2.3.18}$$

It may be noted that $\mu_1 = 0$ and $\mu_2 = \sigma^2$ whose value is given in 2.2.9. By using the formula 2.3.18 successively for $k = 2, 3, 4$ and 5, one can get

$$\mu_3 = (\beta-1)\theta(1-\theta)[1-2\theta+2\beta\theta-\beta\theta^2](1-\beta\theta)^5,$$

$$\begin{aligned}\mu_4 = {} & 3\frac{(\beta-1)^2\theta^2(1-\theta)^2}{(1-\beta\theta)^6} + \frac{(\beta-1)\theta(1-\theta)}{(1-\beta\theta)^7}[1-6\theta+6\theta^2 \\ & +\beta\theta(8-18\theta+8\theta^2)+\beta^2\theta^2(6-6\theta+\theta^2)],\end{aligned}$$

$$\begin{aligned}\mu_5 = {} & 10\frac{(\beta-1)^2\theta^2(1-\theta)^2}{(1-\beta\theta)^8} + \frac{(\beta-1)\theta(1-\theta)}{(1-\beta\theta)^9}[1-14\theta+36\theta^2-24\theta^3 \\ & +\beta\theta(16-113\theta+152\theta^2-58\theta^3) \\ & +\beta^2\theta^2(58-134\theta+91\theta^2-18\theta^3) \\ & +\beta^3\theta^3(24-36\theta+24\theta^2+\theta^3)],\end{aligned} \tag{2.3.19}$$

$$\begin{aligned}\mu_6 = {} & 15\frac{(\beta-1)^3\theta^3(1-\theta)^3}{(1-\beta\theta)^9} + 10\frac{(\beta-1)^2\theta^2(1-\theta)^2}{(1-\beta\theta)^{10}}[1-2\theta+2\beta\theta-\beta\theta^2]^2 \\ & +15\frac{(\beta-1)^2\theta^2(1-\theta)^2}{(1-\beta\theta)^{10}}[1-6\theta+6\theta^2+\beta\theta(8-18\theta+8\theta^2)+\beta^2\theta^2(6-6\theta+\theta^2)] \\ & +\frac{(\beta-1)\theta(1-\theta)}{(1-\beta\theta)^{11}}[1-30\theta+150\theta^2-240\theta^3+24\theta^4\end{aligned}$$

$$+\beta\theta(40 - 520\theta + 1360\theta^2 - 1350\theta^3 + 444\theta^4)$$
$$+\beta^2\theta^2(286 - 1484\theta + 2450\theta^2 + 1494\theta^3 + 300\theta^4)$$
$$+\beta^3\theta^3(444 - 1260\theta + 1260\theta^2 - 488\theta^3 + 28\theta^4)$$
$$+\beta^4\theta^4(120 - 240\theta + 180\theta^2 - 46\theta^3 - \theta^4)].$$

The recurrence formula 2.3.18 gives the values of the higher moments in terms of the parameter θ and β. If the values of μ_k are needed in terms of the mean μ and the parameter β, one can obtain another recurrence formula by using form 2.1.2 of the GD and by differentiating

$$\mu_k = \sum_{x=1}^{\infty} \frac{1}{\beta x - 1}\binom{\beta x - 1}{x}[\frac{\mu - 1}{\mu(\beta - 1)}]^{x-1}[\frac{\mu(\beta - 1)}{\mu\beta - 1}]^{\beta x - 1}(x - \mu)^k$$

with respect to μ. On rearrangement and simplification, one can get the recurrence formula

(2.3.20) $$\mu_{k+1} = \sigma^2[\frac{d\mu_k}{d\mu} + k\mu_{k-1}],\ for\ k = 2, 3, 4, \ldots.$$

The above formula gives

(2.3.21) $$\mu_3 = \sigma^2(3\beta\mu^2 - 2\beta\mu - 2\mu + 1)(\beta - 1)^{-1},$$

and

(2.3.22) $$\begin{aligned}\mu_4 &= 3\sigma^4 + \sigma^2[(3\beta\mu^2 - 2\beta\mu + 2\mu + 1)^2(\beta - 1)^{-2} \\ &\quad +2\sigma^2(3\beta\mu - \beta + 1)(\beta - 1)^{-1}] \\ &= 3\sigma^4 + \frac{\sigma^2}{(\beta - 1)^2}[6\mu^2 + \beta\mu(15\mu - 20\mu^2) \\ &\quad +\beta^2\mu^2(6 - 20\mu + 15\mu^2)]\end{aligned}$$

2.3.4 Convolution Theorem

If X_i, $i = 1, 2, \ldots, n$ are n independent and identically distributed random variables having the Geeta distribution 2.1.1 then it can be shown that the probability distribution of the sum $Y = X_1 + X_2 + \ldots + X_n$ is the following Geeta-n distribution(GND).

$$P(Y = y) = \begin{cases} \frac{n}{y}\frac{(\beta y - y)_{(y-n)}}{(y-n)!}\theta^{y-n}(1-\theta)^{\beta y - y}, & y = n, n+1, \ldots. \\ 0, & \text{otherwise} \end{cases} \tag{2.3.23}$$

where $(a)_{(x)} = a(a+1)\ldots(a+x-1)$.

The proof of 2.3.23 is given as follows:

Since the pgf for the Geeta distribution is

$$\begin{aligned} f(u) &= t(u) \\ &= \sum_{x=1}^{\infty} u^x \frac{1}{\beta x - 1}\binom{\beta x - 1}{x}\theta^{x-1}(1-\theta)^{\beta x - x}, \; where\ u = 1\ when\ t = 1 \\ &= \sum_{x=1}^{\infty} u^x \frac{1}{x}\frac{(\beta x - x)_{(x-1)}}{(x-1)!}\theta^{x-1}(1-\theta)^{\beta x - x} \end{aligned}$$

So,

$$\begin{aligned} [t(u)]^2 &= [\sum_{x_1=1}^{\infty} u^{x_1}\frac{1}{x_1}\frac{(\beta x_1 - x_1)_{(x_1-1)}}{(x_1-1)!}\theta^{x_1-1}(1-\theta)^{\beta x_1 - x_1}] \\ &\qquad \times[\sum_{x_2=1}^{\infty} u^{x_2}\frac{1}{x_2}\frac{(\beta x_2 - x_2)_{(x_2-1)}}{(x_2-1)!}\theta^{x_2-1}(1-\theta)^{\beta x_2 - x_2}] \end{aligned}$$

Since both series are absolutely convergent for $0 < u \leq 1$, they can be multiplied and rearranged into power series of u as

$$\begin{aligned} [t(u)]^2 &= \sum_{y=2}^{\infty} u^y [\sum_{x_1=1}^{y-1}\frac{1}{x_1}\frac{(\beta x_1 - x_1)_{(x_1-1)}}{(x_1-1)!}\frac{1}{y-x_1}\frac{(\beta(y-x_1) - y + x_1)_{(y-x_1-1)}}{(y-x_1-1)!}] \\ &\qquad \times\theta^{y-2}(1-\theta)^{(\beta-1)y} \end{aligned} \tag{2.3.24}$$

where $x_1 + x_2 = y$.

Also, for the Lagrangian expansion of $[t(u)]^2$ under the transformation $t = u(1-\theta)^{\beta-1}(1-\theta t)^{-\beta+1}$, $\beta > 1$, one gets

$$
\begin{aligned}
[t(u)]^2 &= \sum_{y=2}^{\infty} u^y \frac{1}{y!}\frac{\partial^{y-1}}{\partial t^{y-1}}[(1-\theta)^{\beta y-y}(1-\theta t)^{-\beta y+y}2t]_{t=0} \\
&= \sum_{y=2}^{\infty} u^y (1-\theta)^{\beta y-y}\frac{2}{y!}\binom{y-1}{1}(\beta y-y)_{(y-2)}\theta^{y-2} \qquad (2.3.25)\\
&= \sum_{y=2}^{\infty} u^y \theta^{y-2}(1-\theta)^{\beta y-y}\frac{2}{y}\frac{(\beta y-y)_{(y-2)}}{(y-2)!}
\end{aligned}
$$

By equating the coefficients of $u^y\theta^{y-2}(1-\theta)^{(\beta-1)y}$ in 2.3.24 and 2.3.25, one can get

$$\sum_{x_1=1}^{y-1}\frac{1}{x_1}\frac{(\beta x_1-x_1)_{(x_1-1)}}{(x_1-1)!}\frac{1}{y-x_1}\frac{(\beta(y-x_1)-y+x_1)_{(y-x_1-1)}}{(y-x_1-1)!} = \frac{2}{y}\frac{(\beta y-y)_{(y-2)}}{(y-2)!}$$

which proves the convolution theorem for $Y = X_1 + X_2$.

As a result, in general, for the convolution theorem of $Y = X_1 + X_2 + \ldots + X_n$, one can use the expansion of t^n on one side and the product of the n expansions of t on the other side. By doing so, one has

$$(2.3.26) \qquad [t(u)]^n = \prod_{i=1}^{n}[\sum_{x_i=1}^{\infty} u^{x_i}\frac{1}{x_i}\frac{(\beta x_i-x_i)_{(x_i-1)}}{(x_i-1)!}\theta^{x_i-1}(1-\theta)^{\beta x_i-x_i}].$$

Since all the n series are absolutely convergent for $0 < u \leq 1$, they can be multiplied and rearranged into power series of u as

$$
\begin{aligned}
[t(u)]^n = \sum_{y=n}^{\infty} u^y [\sum_{y_{n-1}=n-1}^{y-1} & \frac{(n-1)}{y_{n-1}}\frac{(\beta y_{n-1}-y_{n-1})_{(y_{n-1}-n+1)}}{(y_{n-1}-n+1)!} \\
& \times\frac{1}{y-y_{n-1}}\frac{(\beta(y-y_{n-1})-y+y_{n-1})_{(y-y_{n-1}-1)}}{(y-y_{n-1}-1)!}] \qquad (2.3.27)\\
& \times\theta^{y-n}(1-\theta)^{(\beta-1)y}
\end{aligned}
$$

where $y_{n-1}+x_n=y$, and $y_{n-1}=x_1+x_2+\ldots+x_{n-1}$.

Also, for the Lagrangian expansion of $[t(u)]^n$ under the transformation $t=u(1-\theta)^{\beta-1}(1-\theta t)^{-\beta+1}$, $\beta>1$, one gets

$$
\begin{aligned}
[t(u)]^n &= \sum_{y=n}^{\infty} u^y \frac{1}{y!}\frac{\partial^{y-1}}{\partial t^{y-1}}[(1-\theta)^{\beta y-y}(1-\theta t)^{-\beta y+y} n t^{n-1}]_{t=0} \\
&= \sum_{y=n}^{\infty} u^y (1-\theta)^{\beta y-y}\frac{n!}{y!}\binom{y-1}{n-1}(\beta y-y)_{(y-n)}\theta^{y-n} \\
&= \sum_{y=n}^{\infty} u^y\theta^{y-n}(1-\theta)^{\beta y-y}\frac{n}{y}\frac{(\beta y-y)_{(y-n)}}{(y-n)!}
\end{aligned}
\tag{2.3.28}
$$

By equating the coefficients of $u^y\theta^{y-n}(1-\theta)^{(\beta-1)y}$ in 2.3.27 and 2.3.28, one can get

$$
\begin{aligned}
&\sum_{y_{n-1}=n-1}^{y-1}\frac{(n-1)}{y_{n-1}}\frac{(\beta y_{n-1}-y_{n-1})_{(y_{n-1}-n+1)}}{(y_{n-1}-n+1)!}\frac{1}{y-y_{n-1}}\frac{(\beta(y-y_{n-1})-y+y_{n-1})_{(y-y_{n-1}-1)}}{(y-y_{n-1}-1)!} \\
=\;& \frac{n}{y}\frac{(\beta y-y)_{(y-n)}}{(y-n)!},
\end{aligned}
$$

which proves the convolution theorem of $Y=X_1+X_2+\ldots+X_n$.

The location parameter form of 2.3.23, where μ_n is the mean of the random variable Y is given by

$$
P(Y=y)=\begin{cases}\frac{n}{y}\frac{(\beta y-y)_{(y-n)}}{(y-n)!}\left[\frac{\mu_n-n}{\beta\mu_n-n}\right]^{y-n}\left[\frac{(\beta-1)\mu_n}{\beta\mu_n-n}\right]^{\beta y-y}, & y=n,n+1,\ldots. \\ 0, & \text{otherwise}\end{cases}
\tag{2.3.29}
$$

The Haight distribution is a particular case of the GND 2.3.23 which is given by $\beta=2$ and $\theta=\phi(1+\phi)^{-1}$.

Note that the probability distribution of 2.3.23 is also a Modified Power Series distribution(MPSD) which will be used to study the negative moments and cumulants in the following chapters.

2.4 Model leading to the GD based upon a Differential Difference Equation

[Consul,1990b] considered a regenerative process which is initiated by a single microbe, bacteria or cell and which may grow into any number. The resulting Differential Difference equation model is contained in the following theorem.

Let $P_x(\theta)$ denote the probability of x microbes or cell in a location and let the mean μ of the distribution of X be a function of two parameters θ and β.

Theorem: If the mean μ for the distribution of the microbes is increased by changing θ to $\theta + \Delta\theta$ in such a manner that

$$\frac{dP_x(\theta)}{d\theta} + \frac{x(\beta-1)}{1-\theta}P_x(\theta) = \frac{x-1}{x} \frac{(\beta x - x)_{(x-1)}}{\cdot\, (\beta x - x - \beta + 1)_{(x-2)}}(1-\theta)^{\beta-1}P_{x-1}(\theta), \tag{2.4.30}$$

for all integral values $x \geq 1$ with the initial condition $P_1(0) = 1$ and $P_x(\theta) = 0$ for $x \geq 2$, then show that the probability model P_x is the GD, where $(a)_{(k)} = a(a+1)\ldots(a+k-1)$.

Proof: For $x = 1$, the equation 2.4.30 becomes

$$\frac{dP_1(\theta)}{d\theta} + \frac{\beta-1}{1-\theta}P_1(\theta) = 0$$

which is a simple differential equation with the general solution

$$P_1(\theta) = C_1(1-\theta)^{\beta-1}. \tag{2.4.31}$$

By the initial condtion $P_1(0) = 1$, the constant $C_1 = 1$.

For $x = 2$, the equation 2.4.30, on using 2.4.31 gives

$$\frac{dP_2(\theta)}{d\theta} + \frac{2(\beta-1)}{1-\theta}P_2(\theta) = (\beta-1)(1-\theta)^{2\beta-2},$$

which is a linear differential equation with the integrating factor $(1-\theta)^{-2\beta+2}$. Thus, the solution of the differential equation becomes

$$\begin{aligned} P_2(\theta) &= (1-\theta)^{2\beta-2}\int(1-\theta)^{-2\beta+2}(\beta-1)(1-\theta)^{2\beta-2}d\theta \\ &= (\beta-1)\theta(1-\theta)^{2\beta-2} + C_2(1-\theta)^{2\beta-2}. \end{aligned}$$

Since $P_2(0) = 0$, the constant $C_2 = 0$. Therefore,

$$\begin{aligned} P_2(\theta) &= (\beta-1)\theta(1-\theta)^{2\beta-2} \\ &= (2\beta-2)_{(1)}\frac{\theta(1-\theta)^{2\beta-2}}{2!}. \end{aligned} \tag{2.4.32}$$

By putting $x = 3$ in 2.4.30 and by using 2.4.32, one can get

$$\frac{dP_3(\theta)}{d\theta} + 3\frac{(\beta-1)}{1-\theta}P_3(\theta) = \frac{1}{3}(3\beta-3)(3\beta-2)(1-\theta)^{3\beta-3}\theta.$$

On integration, the solution of the above linear differential equation is

$$\begin{aligned} P_3(\theta) &= (1-\theta)^{3\beta-3}\int(3\beta-3)_{(2)}\frac{\theta}{3}d\theta \\ &= \frac{(3\beta-3)_{(2)}}{3!}\theta^2(1-\theta)^{3\beta-3} + C_3(1-\theta)^{3\beta-3}. \end{aligned}$$

By the initial condition $P_3(0) = 0$, the constant $C_3 = 0$ and hence

$$P_3(\theta) = \frac{(3\beta - 3)_{(2)}}{3!}\theta^2(1-\theta)^{3\beta-3}. \tag{2.4.33}$$

Now, assuming the value of $P_k(\theta)$ as above

$$P_k(\theta) = \frac{(k\beta - k)_{(k-1)}}{k!}\theta^{k-1}(1-\theta)^{k\beta-k}, \tag{2.4.34}$$

Putting $x = k+1$ in 2.4.30 and use 2.4.34 to get

$$\frac{dP_{k+1}(\theta)}{d\theta} + \frac{(k+1)(\beta-1)}{1-\theta}P_{k+1}(\theta) = \frac{k}{k+1}\frac{(\beta k + \beta - k - 1)_{(k)}}{k!}\theta^{k-1}(1-\theta)^{k\beta+\beta-k-1}.$$

On integration of the above linear differential equation,

$$\begin{aligned} P_{k+1}(\theta) &= (1-\theta)^{(k+1)(\beta-1)}\frac{(\beta k + \beta - k - 1)_{(k)}}{(k+1)!}\int k\theta^{k-1}d\theta \\ &= \frac{(\beta k + \beta - k - 1)_{(k)}}{(k+1)!}\theta^k(1-\theta)^{(k+1)(\beta-1)}, \end{aligned} \tag{2.4.35}$$

as the constant of integration vanishes by the initial condition $P_{k+1}(0) = 0$. Hence, the result is true for all integral values of k. This completes the proof of the theorem.

Chapter 3

Other Important Properties of the GD

In this chaper, other important properties of the GD models will be investigated with reference to some of the fundamental results described in the last chapter.

3.1 Unimodality of GD models

[Wegman,1972] and [Barndorff-Nielsen,1976] have shown that the property of unimodality plays an important role in the problem of density estimation. [Keilson & Gerber,1971] as well as [Steutel & Van Harn,1979] have presented some interesting results on the unimodality of discrete distributions.

A discrete probability distribution $\{P_x\}$ is said to be unimodal if there exits at least one integer M such that

$$P_x \geq P_{x-1} \quad for\ all \quad x \leq M$$

$$and$$

$$P_{x+1} \leq P_x \quad for\ all \quad x \geq M.$$

Theorem: The Geeta distribution(GD) in (2.1) is unimodal for all values of θ and β in $0 < \theta < 1$ and $1 < \beta < \theta^{-1}$ respectively, and the mode is at point x=1.

Proof: Consider the unimodality of the GD for $1 < \beta < \theta^{-1}$ only, since when $\beta \to 1$, the model degenerates to a single point at x=1.

Let the mode be at the point x=M.

For the mode of GD to be at point M=1, one has to show that

$$P_{x+1} < P_x \quad for\ all \quad x = 1, 2, 3, ...$$

Now,

$$\begin{aligned}
\frac{P_{x+1}}{P_x} &= \frac{\frac{1}{\beta(x+1)-1}\binom{\beta(x+1)-1}{x+1}\theta^{(x+1)-1}(1-\theta)^{\beta(x+1)-(x+1)}}{\frac{1}{\beta x-1}\binom{\beta x-1}{x}\theta^{x-1}(1-\theta)^{\beta x-x}} \\
&= \frac{(\beta x-1)\Gamma(\beta(x+1))x!\Gamma(\beta x-x)\theta(1-\theta)^{\beta-1}}{(\beta(x+1)-1)(x+1)!\Gamma(\beta(x+1)-(x+1))\Gamma(\beta x)} \\
&= \frac{(\beta x-1)}{(\beta(x+1)-1)}\frac{1}{(x+1)}\frac{\Gamma(\beta(x+1))\Gamma(\beta x-x)}{\Gamma(\beta(x+1)-(x+1))\Gamma(\beta x)}\theta(1-\theta)^{\beta-1} \\
(3.1.1) \qquad &= \frac{(\beta x-1)}{(\beta(x+1)-1)}\frac{1}{x}\frac{\Gamma(\beta(x+1))\Gamma(\beta x-x+1)}{\Gamma(\beta(x+1)-x)\Gamma(\beta x)}\theta(1-\theta)^{\beta-1} \\
& \quad for\ x = 1, 2, 3, \ldots .
\end{aligned}$$

Since $\theta(1-\theta)^{\beta-1}$ is an increasing fucntion of θ and $1 < \beta < \theta^{-1}$, i.e. $\theta < \frac{1}{\beta}$, and $0 < \theta < 1$,

so, when $x = 1$,

$$\begin{aligned}
\frac{P_2}{P_1} &< \frac{1}{2}\frac{(\beta-1)\Gamma(2\beta)\Gamma(\beta-1)}{(2\beta-1)\Gamma(2\beta-2)\Gamma(\beta)}\frac{1}{\beta}(1-\frac{1}{\beta})^{\beta-1} \\
&= (1-\frac{1}{\beta})^{\beta} \\
&< 1,\ for\ all\ \beta > 1.
\end{aligned}$$

When, $x = 2$,

$$\frac{P_3}{P_2} = \frac{1}{2}(3\beta-2)\theta(1-\theta)^{\beta-1}$$

$$< \frac{(3\beta-2)}{2\beta}(1-\frac{1}{\beta})^{\beta-1}$$

By differentiation, one can show that the right hand side of the above is an increasing function of β. Hence, it achieves its maximum value at the largest possible value of β. Therefore, one has

$$\lim_{\beta\to\infty}\frac{(3\beta-2)}{2\beta}(1-\frac{1}{\beta})^{\beta-1} = \frac{3}{2}e^{-1} < 1.$$

So, $\frac{P_3}{P_2} < 1$, for $1 < \beta < \theta^{-1}$, and $0 < \theta < 1$.

When $x = 3$, one has,

$$\begin{aligned} \frac{P_4}{P_3} &= \frac{2}{3}\frac{(2\beta-1)(4\beta-3)}{(3\beta-2)}\theta(1-\theta)^{\beta-1} \\ &< \frac{2}{3}\frac{(2\beta-1)(4\beta-3)}{(3\beta-2)}\frac{1}{\beta}(1-\frac{1}{\beta})^{\beta-1} \end{aligned}$$

Similarly, the right hand side of the above is an increasing function of β and ,

$$\lim_{\beta\to\infty}\frac{2}{3}\frac{(2\beta-1)(4\beta-3)}{(3\beta-2)}\frac{1}{\beta}(1-\frac{1}{\beta})^{\beta-1} = \frac{16}{9}e^{-1} < 1.$$

So, $\frac{P_4}{P_3} < 1$, for $1 < \beta < \theta^{-1}$, and $0 < \theta < 1$.

For $x \geq 4$, one can use the Feller's inequality 1.1.1 on 3.1.1.

When applying the inquality, one has

$$\frac{\Gamma(\beta x+\beta)\Gamma(\beta x-x+1)}{\Gamma(\beta x+\beta-x)\Gamma(\beta x)} < \frac{(\beta x+\beta-1)^{\beta x+\beta-1+\frac{1}{2}}(\beta x-x)^{\beta x-x+\frac{1}{2}}}{(\beta x+\beta-x-1)^{\beta x+\beta-x-1+\frac{1}{2}}(\beta x-1)^{\beta x-1+\frac{1}{2}}} \times \frac{e^{\frac{1}{12(\beta x+\beta-1)}}e^{\frac{1}{12(\beta x-x)}}}{e^{\frac{1}{12(\beta x+\beta-x-1)+1}}e^{\frac{1}{12(\beta x-1)+1}}}.$$

Therefore,

$$\begin{aligned}\frac{P_{x+1}}{P_x} &< \frac{(\beta x-1)}{(\beta(x+1)-1)}\frac{1}{x}\theta(1-\theta)^{\beta-1}\frac{(\beta x+\beta-1)^{\beta x+\beta-1+\frac{1}{2}}(\beta x-x)^{\beta x-x+\frac{1}{2}}}{(\beta x+\beta-x-1)^{\beta x+\beta-x-1+\frac{1}{2}}(\beta x-1)^{\beta x-1+\frac{1}{2}}} \\ &\quad\times\frac{e^{\frac{1}{12(\beta x+\beta-1)}}e^{\frac{1}{12(\beta x-x)}}}{e^{\frac{1}{12(\beta x+\beta-x-1)+1}}e^{\frac{1}{12(\beta x-1)+1}}} \\ &= \frac{(\beta x-1)}{(\beta(x+1)-1)}\frac{1}{x}\theta(1-\theta)^{\beta-1}\frac{(\beta-1)x}{e}[\frac{\beta x+\beta-1}{\beta x-x+\beta-1}]^{\beta} \\ &\quad\times[\frac{(x+1)(\beta x-1)}{x(\beta x+\beta-1)}]^{\frac{1}{2}}[\frac{\beta x+\beta-1}{\beta x-1}]^{\beta x}[\frac{x}{x+1}]^{\beta x-x} \\ &\quad\times\exp\{\frac{12\beta-11}{12(\beta x-x)[12(\beta x-x+\beta-1)+1]}-\frac{12\beta-1}{12(\beta x+\beta-1)[12(\beta x-1)+1]}\}.\end{aligned}$$

Further, by applying $\theta < \frac{1}{\beta}$ and also since the exponent part of the right hand side of the above is a decreasing function of both β and x, and it approaches one as either β or x increases. Hence, on further simplification, one has

$$\begin{aligned}\frac{P_{x+1}}{P_x} &\leq \frac{(\beta x-1)}{(\beta(x+1)-1)}\frac{1}{\beta}[\frac{\beta-1}{\beta}]^{\beta-1} \\ &\quad\times\frac{\beta-1}{e}[\frac{\beta x+\beta-1}{(\beta-1)(x+1)}]^{\beta}[\frac{x+1}{x}]^{\frac{1}{2}}[\frac{\beta x+\beta-1}{\beta x-1}]^{\beta x-\frac{1}{2}}[\frac{x}{x+1}]^{\beta x-x}\end{aligned}$$

$$\begin{aligned} &= \frac{(\beta x-1)}{(\beta(x+1)-1)}[\frac{\beta x+\beta-1}{\beta x+\beta}]^{\beta}\frac{1}{e}(1+\frac{1}{x})^{x}[\frac{\beta x^2+\beta x-x}{\beta x^2+\beta x-x-1}]^{\beta x-\frac{1}{2}} \\ &\leq [\frac{\beta x+\beta-1}{\beta x+\beta}]^{\beta} < 1. \end{aligned}$$

So, $\frac{P_{x+1}}{P_x} < 1$ for $x = 1, 2, 3, \ldots$ and it follows that the GD is unimodal for all values of θ and β in $0 < \theta < 1$ and $1 < \beta < \theta^{-1}$ respectively, and the mode is at the point $x = 1$. Hence, the proof is completed.

3.2 Recursive relation between negative moments of the Geeta distribution(GD)

The negative moments are found to be useful in applied statistics, especially in Estimation, Life testing and in survey sampling, where ratio estimates are used. See [Deming,1950].

Let X be a random variable(r.v.) having GD defined by 2.1.1.

Let k be a non-negative number. For a positive integer r, the rth negative moment, M(r,k) about the point $-k$ is defined as

$$M(r,k) = E[(X+k)^{-r}] \tag{3.2.2}$$

$$= \sum_{x=1}^{\infty} \frac{a(x)[\theta(1-\theta)^{\beta-1}]^{x}}{(x+k)^{r}\theta}, \tag{3.2.3}$$

where

$$a(x) = \frac{1}{\beta x - 1}\binom{\beta x - 1}{x}.$$

Obviously, $M(0,k) = 1$,

and

$$\frac{d}{d\theta}[\theta(1-\theta)^{\beta-1}] = (1-\beta\theta)(1-\theta)^{\beta-2}. \tag{3.2.4}$$

On differentiation of 3.2.3 with respect to θ, one get

$$M'(r,k) = \sum_{x=1}^{\infty} \frac{a(x)}{(x+k)^{r}} \frac{[\theta(1-\theta)^{\beta-1}]^{x}}{\theta}[\frac{x(1-\beta\theta)(1-\theta)^{\beta-2}}{\theta(1-\theta)^{\beta-1}} - \frac{1}{\theta}],$$

$$(3.2.5) \quad = \sum_{x=1}^{\infty} \frac{a(x)}{(x+k)^r} \frac{[\theta(1-\theta)^{\beta-1}]^x}{\theta} [\frac{x(1-\beta\theta)(1-\theta)^{\beta-2}}{\theta(1-\theta)^{\beta-1}} + \frac{k(1-\beta\theta)(1-\theta)^{\beta-2}}{\theta(1-\theta)^{\beta-1}} - \frac{k(1-\beta\theta)(1-\theta)^{\beta-2}}{\theta(1-\theta)^{\beta-1}} - \frac{1}{\theta}]$$

or equivalently,

$$(3.2.6) \qquad M'(r,k) = \frac{(1-\beta\theta)}{\theta(1-\theta)} M(r-1,k) - [\frac{k(1-\beta\theta)}{\theta(1-\theta)} + \frac{1}{\theta}] M(r,k).$$

Hence, 3.2.6 provides us a simple linear differential equation,

$$(3.2.7) \qquad M'(r,k) + [\frac{k(1-\beta\theta)}{\theta(1-\theta)} + \frac{1}{\theta}] M(r,k) = \frac{(1-\beta\theta)}{\theta(1-\theta)} M(r-1,k).$$

Multiplying 3.2.7 by the integrating factor $[\theta(1-\theta)^{\beta-1}]^k\theta$ and integrating from 0 to θ, one can obtain

$$(3.2.8) \; M(r,k) = \theta^{-k-1}(1-\theta)^{-(\beta-1)k} \times \int_0^\theta M(r-1,k)\theta(1-\beta\theta)(1-\theta)^{\beta-2}[\theta(1-\theta)^{\beta-1}]^{k-1} d\theta.$$

So, 3.2.8 provides us a recursive relation between the negative moments about the point $-k$ for the GD and is given by

$$(3.2.9) \qquad M(r,k) = \frac{1}{\theta^{k+1}(1-\theta)^{\beta k-k}} \int_0^\theta M(r-1,k) I(k) d\theta,$$

where

$$\begin{aligned} I(k) &= \theta(1-\beta\theta)(1-\theta)^{\beta-2}[\theta(1-\theta)^{\beta-1}]^{k-1} \\ &= (1-\beta\theta)\theta^k(1-\theta)^{\beta k-k-1}. \end{aligned}$$

3.2.1 Negative moments for the Geeta distribution

The first negative moment about the point $-k$ $(k > 0)$ for th random variable having the Geeta distribution 2.1.1 is given by the recursive relation 3.2.9 for $r = 1$ as

$$E[\frac{1}{X+k}] = \frac{1}{\theta^{k+1}(1-\theta)^{\beta k-k}} \int_0^\theta (1-\beta\theta)\theta^k(1-\theta)^{\beta k-k-1}d\theta. \tag{3.2.10}$$

For $\beta k - k = \ell > 0$, we have

$$\begin{aligned} E[\frac{1}{X+k}] &= \frac{1}{\theta^{k+1}(1-\theta)^\ell}\int_0^\theta (\theta^k - \beta\theta^{k+1})(1-\theta)^{\ell-1}d\theta \\ &= \frac{1}{k\theta^{k+1}(1-\theta)^\ell}\int_0^\theta \frac{k(1-\beta\theta)}{\theta(1-\theta)}\theta^{k+1}(1-\theta)^{\beta k-k}d\theta, \\ &= \frac{1}{k\theta^{k+1}(1-\theta)^\ell}\int_0^\theta [\frac{k(1-\beta\theta)}{\theta(1-\theta)} + \frac{1}{\theta} - \frac{1}{\theta}]\theta^{k+1}(1-\theta)^{\beta k-k}d\theta, \\ &= \frac{1}{\theta^{k+1}(1-\theta)^{\beta k-k}}[\frac{1}{k}\theta^{k+1}(1-\theta)^{\beta k-k} - \frac{1}{k}\int_0^\theta \theta^k(1-\theta)^{\beta k-k}d\theta] \\ &= \frac{1}{k} - \frac{1}{k}\frac{1}{\theta^{k+1}(1-\theta)^{\beta k-k}}B_\theta(k+1,\ell+1) \end{aligned} \tag{3.2.11}$$

where $B_\theta(p,q)$ is the incomplete beta function

$B_\theta(p,q) = \int_0^\theta x^{p-1}(1-x)^{q-1}dx;\ \ 0 < \theta < 1,\ \ p > 0, q > 0.$

The first negative moment (about k=0) of the Geeta distribution is given by the recursive formula 3.2.9 for r=1 and k=0 as

$$\begin{aligned} E[\frac{1}{X}] &= \frac{1}{\theta}\int_0^\theta (1-\beta\theta)(1-\theta)^{-1}d\theta \\ &= \frac{1}{\theta}\int_0^\theta \frac{1-\beta+\beta(1-\theta)}{1-\theta}d\theta \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\theta}\int_0^{\theta}[\frac{1-\beta}{1-\theta}+\beta]d\theta \\
&= \frac{1}{\theta}[(\beta-1)\ln(1-\theta)+\beta\theta]_0^{\theta} \\
(3.2.12) \qquad &= \frac{1}{\theta}[(\beta-1)\ln(1-\theta)+\beta\theta] \\
&= \beta+(\beta-1)\theta^{-1}\ln(1-\theta).
\end{aligned}
$$

On substituting the result 3.2.12 in 3.2.9 for $r=2$ and $k=0$, one can get the second negative moment about $k=0$ of the Geeta distribution as

$$
\begin{aligned}
E[\frac{1}{X^2}] &= \frac{1}{\theta}\int_0^{\theta}\{(\beta-1)\ln(1-\theta)+\beta\theta\}\frac{(1-\beta\theta)}{(1-\theta)}d\theta \\
&= \frac{1}{\theta}\int_0^{\theta}\{(\beta-1)\ln(1-\theta)+\beta\theta\}\{\beta+\frac{1-\beta}{1-\theta}\}d\theta \\
&= \frac{1}{\theta}\int_0^{\theta}\{\beta(\beta-1)\ln(1-\theta)+\beta^2\theta-(\beta-1)^2\frac{\ln(1-\theta)}{(1-\theta)}-\frac{(\beta-1)\beta\theta}{(1-\theta)}\}d\theta \\
&= \frac{1}{\theta}[\beta(\beta-1)\{\theta\ln(1-\theta)\}+\frac{1}{2}(\beta-1)^2[\ln(1-\theta)]^2+\frac{1}{2}\beta^2\theta^2]_0^{\theta} \\
(3.2.13) \qquad &= \beta(\beta-1)\ln(1-\theta)+\frac{1}{2}\beta^2\theta+\frac{1}{2\theta}(\beta-1)^2[\ln(1-\theta)]^2.
\end{aligned}
$$

3.3 Cumulants of the GD

The expression of the cumulants can be calculated from the central moments. See [Kendal,1987]. However, this is a painful and time-consuming task. So, in the later part of this section, a recursive relation between cumulants of MPSD wil be developed and the recursive formula of the GD will be obtained as a particular case of this recursive relation.

3.3.1 Recursive relation between cumulants of MPSD

Let X be a MPSD random variable given by 1.3.12.

Then, the moment generating function $\phi(t)$ is given by

$$\begin{aligned} \phi(t) &= E[e^{tX}] \\ &= \sum_{x\in T} \frac{a(x)[G(\theta)e^t]^x}{f(\theta)} \\ &= \frac{F(\theta, e^t)}{f(\theta)}. \end{aligned} \tag{3.3.14}$$

where $F(\theta, e^t) = \sum_{x\in T} a(x)[G(\theta)e^t]^x$.

Then,

$$\begin{aligned} F_t'(\theta, e^t) &= \frac{\partial}{\partial t} F(\theta, e^t) \\ &= \frac{\partial}{\partial t} \sum_{x\in T} a(x)[G(\theta)e^t]^x \\ &= \sum_{x\in T} a(x)x[G(\theta)e^t]^x, \end{aligned} \tag{3.3.15}$$

$$\begin{aligned} F_{tt}''(\theta, e^t) &= \frac{\partial}{\partial t} F_t'(\theta, e^t) \\ &= \sum_{x\in T} a(x)x^2[G(\theta)e^t]^x, \end{aligned} \tag{3.3.16}$$

$$F''_{t\theta}(\theta, e^t) = \frac{\partial}{\partial\theta} F'_t(\theta, e^t)$$
$$= \sum_x a(x)x^2[G(\theta)e^t]^x \frac{G'(\theta)}{G(\theta)}$$
$$(3.3.17) \qquad = \frac{G'(\theta)}{G(\theta)} F''_{tt}(\theta, e^t),$$

$$F'_\theta(\theta, e^t) = \frac{\partial}{\partial\theta} \sum_{x\in T} a(x)[G(\theta)e^t]^x$$
$$= \sum_x a(x)x[G(\theta)e^t]^x \frac{G'(\theta)}{G(\theta)}$$
$$(3.3.18) \qquad = \frac{G'(\theta)}{G(\theta)} F'_t(\theta, e^t).$$

So, the cumulant generating function $\psi(t)$ is given by 3.3.14 as

$$(3.3.19) \qquad \psi(t) = \log \phi(t)$$
$$= \log F(\theta, e^t) - \log f(\theta).$$

Differentiating 3.3.19 with respect to t, one has

$$(3.3.20) \qquad \frac{\partial\psi(t)}{\partial t} = \frac{F'_t(\theta, e^t)}{F(\theta, e^t)},$$

and differentiating 3.3.20 with respect to t again, one has

$$(3.3.21) \qquad \frac{\partial^2\psi(t)}{\partial t^2} = \{\frac{F''_{tt}(\theta, e^t)}{F(\theta, e^t)} - [\frac{F'_t(\theta, e^t)}{F(\theta, e^t)}]^2\}.$$

Now, differentiating 3.3.20 with respect to θ and substituting in 3.3.17 and 3.3.18, one has

$$\frac{\partial^2\psi(t)}{\partial\theta\partial t} = \{\frac{F''_{t\theta}(\theta, e^t)}{F(\theta, e^t)} - \frac{F'_t(\theta, e^t)F'_\theta(\theta, e^t)}{[F(\theta, e^t)]^2}\}$$
$$(3.3.22) \qquad = \frac{G'(\theta)}{G(\theta)}\{\frac{F''_{tt}(\theta, e^t)}{F(\theta, e^t)} - [\frac{F'_t(\theta, e^t)}{F(\theta, e^t)}]^2\}.$$

Comparing 3.3.21 and 3.3.22, one can deduce the following relationship for the MPSD,

$$\frac{\partial^2\psi(t)}{\partial t^2} = \frac{G(\theta)}{G'(\theta)}\frac{\partial^2\psi(t)}{\partial\theta\partial t}. \tag{3.3.23}$$

Since by defintion,

$$\psi(t) = \sum_{r=1}^{\infty} K_r \frac{t^r}{r!}, \quad r = 1, 2, 3, \ldots, \tag{3.3.24}$$

where K_r is the rth cumulant.

Hence, by comparing the coefficient of $\frac{t^r}{r!}$ for both sides of 3.3.23, one can deduce the following recursive relation between cumulants of MPSD,

$$K_{r+1} = \frac{G(\theta)}{G'(\theta)}\frac{\partial K_r}{\partial\theta}, \quad r = 1, 2, 3, \ldots. \tag{3.3.25}$$

3.3.2 Recursive formula between cumulants of the GD

Note that when $G(\theta) = \theta(1-\theta)^{\beta-1}$, $f(\theta) = \theta$ and $a(x) = \frac{1}{\beta x - 1}\binom{\beta x - 1}{x}$ in 1.3.12, the coresponding MPSD is the GD 2.1.1. So, by substituting $G(\theta)$ and $G'(\theta)$ into 3.3.25 the recursive formula between cumulants of the GD is given by

$$K_{r+1} = \frac{\theta(1-\theta)}{(1-\beta\theta)}\frac{\partial K_r}{\partial\theta}, \; for \; r = 1, 2, 3, \ldots, \tag{3.3.26}$$

where $K_1 = (1-\theta)(1-\beta\theta)^{-1}$.

By using the formula 3.3.26 for $r = 2, 3, 4$ and 5, one can get

$$K_2 = (\beta-1)\theta(1-\theta)(1-\beta\theta)^{-3}, \tag{3.3.27}$$

$$K_3 = (\beta-1)\theta(1-\theta)[1-2\theta+2\beta\theta-\beta\theta^2](1-\beta\theta)^{-5}, \tag{3.3.28}$$

$$\begin{aligned} K_4 &= \frac{(\beta-1)\theta(1-\theta)}{(1-\beta\theta)^7}[1-6\theta+6\theta^2 \\ &\quad +\beta\theta(8-18\theta+8\theta^2)+\beta^2\theta^2(6-6\theta+\theta^2)], \end{aligned} \tag{3.3.29}$$

$$\begin{aligned} K_5 &= \frac{(\beta-1)\theta(1-\theta)}{(1-\beta\theta)^9}[1-14\theta+36\theta^2-24\theta^3 \\ &\quad +\beta\theta(16-113\theta+152\theta^2-58\theta^3) \\ &\quad +\beta^2\theta^2(58-134\theta+91\theta^2-18\theta^3) \\ &\quad +\beta^3\theta^3(24-36\theta+24\theta^2+\theta^3)], \end{aligned} \tag{3.3.30}$$

$$(3.3.31) \qquad K_6 = \frac{(\beta-1)\theta(1-\theta)}{(1-\beta\theta)^{11}}[1 - 30\theta + 150\theta^2 - 240\theta^3 + 24\theta^4$$
$$+\beta\theta(40 - 520\theta + 1360\theta^2 - 1350\theta^3 + 444\theta^4)$$
$$+\beta^2\theta^2(286 - 1484\theta + 2450\theta^2 + 1494\theta^3 + 300\theta^4)$$
$$+\beta^3\theta^3(444 - 1260\theta + 1260\theta^2 - 488\theta^3 + 28\theta^4)$$
$$+\beta^4\theta^4(120 - 240\theta + 180\theta^2 - 46\theta^3 - \theta^4)].$$

3.4 Factorial Moments of the GD

The factorial moments are found to be useful in evaluating moments of some discrete distributions. In this section, a recursive relation between factorial moments of the MPSD will be developed and the recursive relation between factorial moments of the GD will be obtained as a particular case of this relation.

3.4.1 Recursive relation between factorial moments of the MPSD

Let X be a random variable having MPSD given by 1.3.12. Then, the rth factorial moments is defined by

$$\mu^{(r)} = E[X(X-1)\ldots(X-r+1)],\; r = 1,2,3,\ldots. \tag{3.4.32}$$

It is easily seen that when $r=1$,

$$\begin{aligned} \mu^{(1)} &= \mu \\ &= E(X) \\ &= \frac{f'_\theta(\theta)G(\theta)}{f(\theta)G'_\theta(\theta)}, \end{aligned} \tag{3.4.33}$$

where $f'_\theta(\theta) = \frac{\partial f(\theta)}{\partial \theta}$.

Let the pgf of this random variable X(defined at the begining of this section) be g(t), then the factorial moment generating function of X is g(1+t), such that

$$g(1+t) = \sum_{r=1}^{\infty} \mu^{(r)} \frac{t^r}{r!}, \tag{3.4.34}$$

where $\mu^{(r)}$ is the rth factorial moment.

However, g(1+t) can also be expressed as

$$g(1+t) = \sum_{x \in T} \frac{a(x)[G(\theta)]^x (1+t)^x}{f(\theta)}. \tag{3.4.35}$$

By differentiating 3.4.35 with respect to t, r and r+1 times respectively, one can have

$$\frac{\partial^r g(1+t)}{\partial t^r} = \sum_x \frac{a(x)[G(\theta)]^x}{f(\theta)} x(x-1)\ldots(x-r+1)(1+t)^{x-r}, \tag{3.4.36}$$

$$\begin{aligned} \frac{\partial^{r+1} g(1+t)}{\partial t^{r+1}} &= \sum_x \frac{a(x)[G(\theta)]^x}{f(\theta)} x(x-1)\ldots(x-r+1)(x-r)(1+t)^{x-r-1} \\ &= \sum_x \frac{a(x)[G(\theta)]^x}{f(\theta)} x^2(x-1)\ldots(x-r+1)(1+t)^{x-r-1} \\ &\quad -r\sum_x \frac{a(x)[G(\theta)]^x}{f(\theta)} x(x-1)\ldots(x-r+1)(1+t)^{x-r-1}. \end{aligned} \tag{3.4.37}$$

Then by differentiating 3.4.36 with respect to θ, one has

$$\begin{aligned} \frac{\partial^{r+1} g(1+t)}{\partial\theta\partial t^r} &= \sum_x a(x)x(x-1)\ldots(x-r+1)(1+t)^{x-r} \frac{\partial}{\partial\theta}\frac{[G(\theta)]^x}{f(\theta)} \\ &= \sum_x \frac{a(x)[G(\theta)]^x}{f(\theta)} x(x-1)\ldots(x-r+1)(1+t)^{x-r}\left[\frac{xG'_\theta(\theta)}{G(\theta)} - \frac{f'(\theta)}{f(\theta)}\right]. \end{aligned} \tag{3.4.38}$$

Multiplying 3.4.38 by $\frac{G(\theta)}{G'(\theta)}$ and substituting 3.4.33, one has

$$\begin{aligned} \frac{G(\theta)}{G'(\theta)}\frac{\partial^{r+1} g(1+t)}{\partial\theta\partial t^r} &= \sum_x \frac{a(x)[G(\theta)]^x}{f(\theta)} x^2(x-1)\ldots(x-r+1)(1+t)^{x-r} \\ &\quad -\frac{f'_\theta(\theta)G(\theta)}{f(\theta)G'_\theta(\theta)}\sum_x \frac{a(x)[G(\theta)]^x}{f(\theta)} x(x-1)\ldots(x-r+1)(1+t)^{x-r} \\ &= \sum_x \frac{a(x)[G(\theta)]^x}{f(\theta)} x^2(x-1)\ldots(x-r+1)(1+t)^{x-r} \\ &\quad -\mu^{(1)}\frac{\partial^r g(1+t)}{\partial t^r}. \end{aligned} \tag{3.4.39}$$

By evaluating the derivatives of 3.4.37 and 3.4.39 at $t = 0$, and comparing the result after evaluation, one can deduce the following recursive relation,

$$
\begin{aligned}
\frac{\partial^{r+1} g(1+t)}{\partial t^{r+1}}\Big|_{t=0} &= \frac{G(\theta)}{G'(\theta)} \frac{\partial^{r+1} g(1+t)}{\partial\theta\partial t^{r}}\Big|_{t=0} + \mu^{(1)} \frac{\partial^{r} g(1+t)}{\partial t^{r}}\Big|_{t=0} \\
&\quad - r\frac{\partial^{r} g(1+t)}{\partial t^{r}}\Big|_{t=0} \\
&= \frac{G(\theta)}{G'(\theta)} \frac{\partial^{r+1} g(1+t)}{\partial\theta\partial t^{r}}\Big|_{t=0} + (\mu^{(1)} - r) \frac{\partial^{r} g(1+t)}{\partial t^{r}}\Big|_{t=0}
\end{aligned}
\tag{3.4.40}
$$

From the relation of 3.4.34, if one compare the coefficient of $\frac{t^r}{r!}$ for the both sides of 3.4.40, one can deduce the following recursive relation between factorial moments of MPSD,

$$
\mu^{(r+1)} = \frac{G(\theta)}{G'(\theta)} \frac{\partial \mu^{(r)}}{\partial\theta} + (\mu^{(1)} - r)\mu^{(r)}, \quad r = 2, 3, 4, \ldots . \tag{3.4.41}
$$

Note that the above result is same as that obtained by [Gupta,1974] but in a different approach.

3.4.2 Recursive formula between factorial moments of the GD

Since GD is a member of MPSD and its corresponding $G(\theta)$ and $G'(\theta)$ is given by

$$
\begin{aligned}
G(\theta) &= \theta(1-\theta)^{\beta-1}, \\
G'(\theta) &= \frac{dG(\theta)}{d\theta} \\
&= (1-\beta\theta)(1-\theta)^{\beta-2}.
\end{aligned}
\tag{3.4.42}
$$

Substituting 3.4.42 in 3.4.41, the recursive formula between factorial moments of the GD is given by

$$
\mu^{(r+1)} = \frac{\theta(1-\theta)}{(1-\beta\theta)}\frac{\partial\mu^{(r)}}{\partial\theta} + (\mu^{(1)} - r)\mu^{(r)}, \; r = 2,3,4,\ldots, \tag{3.4.43}
$$

where $\mu^{(1)} = \frac{(1-\theta)}{(1-\beta\theta)}$.

By using the formula 3.4.43 successively, one can obtain

$$
\begin{aligned}
\mu^{(2)} &= \frac{(\beta-1)\theta(1-\theta)(2-\beta\theta)}{(1-\beta\theta)^3}, \\
\mu^{(3)} &= \frac{(\beta-1)\theta(1-\theta)}{(1-\beta\theta)^5}[\beta\theta(11+6\theta) \\
&\quad -\beta^2\theta^2(15+4\theta) + \beta^3\theta^3(9+\theta) - 2\beta^4\theta^4 - 6\theta], \\
\mu^{(4)} &= \frac{(\beta-1)\theta(1-\theta)}{(1-\beta\theta)^7}[\beta\theta(20-74\theta-24\theta^2) \\
&\quad +\beta^2\theta^2(73+116\theta+84\theta^2) - \beta^3\theta^3(120+93\theta+11\theta^2) \\
&\quad +\beta^4\theta^4(63+40\theta+2\theta^2) - \beta^5\theta^5(37+7\theta) \\
&\quad +6\beta^6\theta^6 + 24\theta^2].
\end{aligned}
\tag{3.4.44}
$$

3.5 The GD as a stochastic model of Epidemics

The problem of finding the probability distributions of the total size of an epidemics started by a single infectious has been considered by [Neyman & Scott, 1963]. This has been an important issue in the study of Epidemics. In this section, it will be shown that the GD model is a possible stochastic model of Epidemics by applying the Galton-Watson branching process.

Without loss of generality, one may assume that an epidemic in a particular habitat is started by a single individual who contracted the epidemic outside that particular habitat. This individual forms the zero-th generation of the branching process so that $X_0 = 1$.

Then, after a fixed incubation period T,this individual becomes infectious with sufficiently close contact. As a result, some of his/her immediate family members and other friends most probably will be infected and will become the first generation of the infected X_1, which is a random variable.

Similarly, after a constant incubation period T, each individual of X_1 becomes infectious and some of their classmates, colleagues, friends and relatives may also be infected and become the second generation of the infected X_2 under the same conditions as described above, and so on in successive generations of infected. Therefore, our task is to find the probability distribution of the total number of infected in that particular habitat.

Given $X_0 = 1$, let $X_1, X_2, \ldots, X_n, \ldots$ represent the number of infected in the $1st, 2nd, \ldots, nth, \ldots$ generations. Note that the only assumption made here is that the probability distribution of the number of infected, generated by each infectious individual, remains the same over all generations.

Since all the infected persons in this particular habitat contracted the disease through the direct or indirect close contact with the single individual X_0, so it is reasonable to assume that X_1 has a contagious probability distribution whose pgf is g(t).

Let $g_n(t) = E[t^{X_n}]$ and $g_0(t) = t$, $g_1(t) = g(t)$.

For $n = 2, 3, 4, \ldots$,

$$\begin{aligned} g_{n+1}(t) &= \sum_{k=0}^{\infty} P(X_{n+1} = k) t^k \\ &= \sum_{k=0}^{\infty} t^k \sum_{j=0}^{\infty} P(X_{n+1} = k | X_n = j) P(X_n = j) \\ &= \sum_{j=0}^{\infty} P(X_n = j) \sum_{k=0}^{\infty} P(X_{n+1} = k | X_n = j) t^k \\ &= \sum_{j=0}^{\infty} P(X_n = j) [g(t)]^j \\ &= g_n(g(t)). \end{aligned} \tag{3.5.45}$$

Also, $g_2(t) = g_1(g(t)) = g(g(t)) = g(g_1(t))$,

and $g_3(t) = g_2(g(t)) = g(g_1(g(t))) = g(g_2(t))$.

Similarly,

$$g_{n+1}(t) = g(g_n(t)). \tag{3.5.46}$$

Now, assume that the increase of the number of infected stops and reaches a steady state after the nth generation.

Define $Z_n = X_1 + X_2 + X_3 + \ldots + X_n$ with its pgf being $G_n(t)$,

and $N_n = X_0 + Z_n = 1 + Z_n$, with its pgf being $R_n(t)$.

Since $Z_1 = X_1$, $G_1(t) = g_1(t) = g(t)$, and

$$
\begin{aligned}
R_1(t) &= E[t^{N_1}] \\
&= E[t^{1+X_1}] \\
&= tE[t^{X_1}] \qquad (3.5.47) \\
&= tG(t) \\
&= tg(t).
\end{aligned}
$$

Since each infected individual of X_1 will start a new generation, the pgf of Z_2 becomes

$$
\begin{aligned}
G_2(t) &= E[t^{Z_2}] \\
&= E[t^{X_1+X_2}] \\
&= \sum_{x_1=0}^{\infty}\sum_{x_2=0}^{\infty} t^{X_1+X_2} P(X_2 = x_2 | X_1 = x_1) P(X_1 = x_1) \\
&= \sum_{x_1=0}^{\infty} t^{x_1} P(X_1 = x_1) \sum_{x_2=0}^{\infty} t^{x_2} P(X_2 = x_2 | X_1 = x_1) \\
&= \sum_{x_1=0}^{\infty} P(X_1 = x_1) t^{x_1} (g(t))^{x_1} \\
&= g(tg(t)) \\
&= g(tG_1(t)) \\
&= g(R_1(t)). \qquad (3.5.48)
\end{aligned}
$$

Similarly, the pgf of $Z_3 = X_1 + X_2 + X_3$ is given by

$$\begin{aligned}
G_3(t) &= E[t^{X_1+X_2+X_3}] \\
&= \sum_{x_1=0}^{\infty}\sum_{x_2=0}^{\infty}\sum_{x_3=0}^{\infty} t^{x_1+x_2+x_3} P(X_3 = x_3 | X_2 = x_2, X_1 = x_1) \\
&\quad \times P(X_2 = x_2 | X_1 = x_1) P(X_1 = x_1) \\
&= \sum_{x_1=0}^{\infty}\sum_{x_2=0}^{\infty} t^{x_1+x_2} P(X_2 = x_2 | X_1 = x_1) P(X_1 = x_1) \\
&\quad \times \sum_{x_3=0}^{\infty} t^{x_3} P(X_3 = x_3 | X_2 = x_2, X_1 = x_1) \\
&= \sum_{x_1=0}^{\infty}\sum_{x_2=0}^{\infty} t^{x_1+x_2} P(X_2 = x_2 | X_1 = x_1) P(X_1 = x_1) [g(t)]^{x_2} \\
&= \sum_{x_1=0}^{\infty} t^{x_1} P(X_1 = x_1) \sum_{x_2=0}^{\infty} P(X_2 = x_2 | X_1 = x_1) [tg(t)]^{x_2} \\
&= \sum_{x_1=0}^{\infty} P(X_1 = x_1) t^{x_1} [g(tg(t))]^{x_1} \\
&= g(tg(tg(t))) \qquad (3.5.49) \\
&= g(tG_2(t)) \\
&= g(R_2(t)).
\end{aligned}$$

In general, it can be shown that,

$$\begin{aligned}
G_{n+1}(t) &= g(tG_n(t)) \\
&= g(R_n(t)), for\ n = 1, 2, 3, \ldots, \qquad (3.5.50)
\end{aligned}$$

or equivalently,

$$\begin{aligned} tG_n(t) &= R_n(t) \\ &= tg(R_{n-1}(t)), \end{aligned} \tag{3.5.51}$$

which gives the limiting form (as n increases)

$$G(t) = g(tG(t)), \tag{3.5.52}$$

and

$$R(t) = tg(R(t)), respectively. \tag{3.5.53}$$

Since our objective is to find the probability distribution of the total number of infected N, so by putting $R(t) = s$ in 3.5.53, one can obtain the Lagrange transformation,

$$s = tg(s). \tag{3.5.54}$$

Also, as the author pointed out earlier, g(s) requires to be the pgf of a contagious probability distribution, so it is not unreasonable to put $g(s) = (1-\theta)^{\beta-1}(1-\theta s)^{-\beta+1}$, $\beta > 1$, which is the pgf of the negative binomial distribution(a well known contagion model).

Hence, using the Lagrange expansion 1.1.3 to expand s in power of t under the transformation of 3.5.54, thus the required pgf of N can be obtained by putting $z = s$, $u = t$, $k = x$ and $a = 0$ in the formula 1.1.3. Therefore, one can get

$$s = \sum_{x=1}^{\infty} \frac{t^x}{x!} \frac{\partial^{x-1}}{\partial s^{x-1}} \{[g(s)]^x\}_{s=0}. \tag{3.5.55}$$

As a result, 3.5.55 gives the probability distribution of the total number of infected N(starting with a single infected individual) as

$$P(N=i) = \frac{1}{\beta i - 1}\binom{\beta i - 1}{i}\theta^{i-1}(1-\theta)^{\beta i - i},\ i\ =\ 1,2,3,\ldots, \tag{3.5.56}$$

which is the GD 2.1.1.

Chapter 4

Estimation of the Geeta Distribution(GD)

4.1 Estimation by method of Moments

Let X be a discrete R.V. having the GD 2.1.1, then the mean μ and the variance σ^2 is given by 2.2.6 and 2.2.9 respectively.

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n taken from the GD 2.1.1. Then, the sample mean and the sample variance are defined as

$$\begin{aligned} m_1 &= \tfrac{1}{n}\textstyle\sum_{i=1}^{n} x_i, \\ m_2 &= \tfrac{1}{n}\textstyle\sum_{i=1}^{n}(x_i - m_1)^2. \end{aligned} \tag{4.1.1}$$

Equating the sample mean m_1 with the population mean μ and the sample variance m_2 with the population variance σ^2, we obtain the moment estimators $\hat{\mu}$, $\hat{\beta}$ and $\hat{\theta}$ of the parametes μ, β and θ in the GD models 2.1.1 and 2.1.2 respectively, which are given by

$$\begin{aligned} \hat{\mu} = \quad m_1 &= \frac{1-\theta}{1-\beta\theta} \ \mathit{and} \\ m_2 &= \tfrac{(\beta-1)\theta(1-\theta)}{(1-\beta\theta)^3} \end{aligned} \tag{4.1.2}$$

Solving the first relation of 4.1.2 for θ and substituting that value in the second relation, we get

$$\begin{aligned} \hat{\beta} &= \tfrac{m_2 - m_1(m_1-1)}{m_2 - m_1^2(m_1-1)}, \\ \hat{\theta} &= 1 - \tfrac{m_1^2}{m_2}(m_1 - 1). \end{aligned} \tag{4.1.3}$$

4.1.1 Asymptotic Bias, Variance and Covariance of the Moment Estimators

Since the moment estimator $\hat{\mu}$ of the population mean μ is the sample mean m_1, so, we have

$$E(\hat{\mu}) = E(m_1) = \mu. \tag{4.1.4}$$

Hence, $\hat{\mu}$ is an unbiased estimator of μ.

The moment estimators $\hat{\theta}$ and $\hat{\beta}$ of θ and β respectively, are functions of the sample mean m_1 and the sample variance m_2.

Let

$$\begin{aligned} \hat{\theta} = \quad & F(m_1, m_2) = \quad 1 - \frac{m_1^2}{m_2}(m_1 - 1), \\ \textit{and} & \\ \hat{\beta} = \quad & G(m_1, m_2) = \quad \frac{m_2 - m_1(m_1 - 1)}{m_2 - m_1^2(m_1 - 1)}. \end{aligned}$$

Expanding $F(m_1, m_2)$ in a bivariate Taylor series as in section 1.1.8, we obtain the first six terms as

$$\begin{aligned} F(m_1, m_2) = & F_{00} + (m_1 - \mu_1')F_{10} + (m_2 - \mu_2)F_{01} + \frac{1}{2}(m_1 - \mu_1')^2 F_{20} \\ & + (m_1 - \mu_1')(m_2 - \mu_2)F_{11} + \frac{1}{2}(m_2 - \mu_2)^2 F_{02} + \dots, \end{aligned} \tag{4.1.5}$$

where

$$F_{ij} = \frac{\partial^{i+j} F(m_1, m_2)}{\partial m_1^i \partial m_2^j}\Bigg|\begin{array}{l} m_1 = \mu_1' \\ m_2 = \mu_2. \end{array}$$

For example, F_{01} is obtained by firstly differentiating $F(m_1, m_2)$ with respect to m_2 once. We get

$$\frac{\partial F(m_1,m_2)}{\partial m_2} = \frac{m_1^2(m_1-1)}{m_2^2}.$$

Secondly, the above expression is evaluated at $m_1 = \mu_1'$ and $m_2 = \mu_2$ respectively and simplified. Hence, we have

$$F_{01} = \frac{\mu_1'^2(\mu_1'-1)}{\mu_2^2} = \frac{\mu^2(\mu-1)}{\sigma^4}.$$

Similarly, F_{02} can be obtained by differentiating $F(m_1, m_2)$ with respect to m_2 twice we get

$$\frac{\partial^2 f(m_1,m_2)}{\partial m_2^2} = -2\frac{m_1^2(m_1-1)}{m_2^3}.$$

Then, the above expression is evaluated at $m_1 = \mu_1'$ and $m_2 = \mu_2$ respectively and after simplification, we have

$$F_{02} = -2\frac{\mu^2(\mu-1)}{\sigma^6}.$$

Following the above preocedure, we obtain F_{ij} for i,j = 1, 2 and $0 \le i+j \le 2$ as follows:

$$\begin{aligned} &F_{00} = \theta, && F_{11} = \tfrac{\mu(3\mu-2)}{\sigma^4} \\ (4.1.6) \qquad &F_{01} = \tfrac{\mu^2(\mu-1)}{\sigma^4}, && F_{02} = -2\tfrac{\mu^2(\mu-1)}{\sigma^6} \\ &F_{10} = -\tfrac{\mu(3\mu-2)}{\sigma^2}, && F_{20} = -\tfrac{2(3\mu-1)}{\sigma^2}. \end{aligned}$$

A similar expansion can be written for $G(m_1, m_2)$ by replacing F by G in 4.1.5 that is,

$$
\begin{aligned}
G(m_1, m_2) = {} & G_{00} + (m_1 - \mu_1')G_{10} + (m_2 - \mu_2)G_{01} + \frac{1}{2}(m_1 - \mu_1')^2 G_{20} \\
& + (m_1 - \mu_1')(m_2 - \mu_2)G_{11} + \frac{1}{2}(m_2 - \mu_2)^2 G_{02} + \ldots,
\end{aligned} \tag{4.1.7}
$$

where

$$G_{ij} = \left. \frac{\partial^{i+j} G(m_1, m_2)}{\partial m_1^i \partial m_2^j} \right| \begin{array}{l} m_1 = \mu_1' \\ m_2 = \mu_2. \end{array}$$

Similarly, G_{01} is obtained by firstly differentiating $G(m_1, m_2)$ with respect to m_2 once. We get

$$\frac{\partial G(m_1, m_2)}{\partial m_2} = -\frac{m_1(m_1 - 1)^2}{(m_2 - m_1^2(m_1 - 1))^2}.$$

Then, the above expression is evaluated at $m_1 = \mu_1'$ and $m_2 = \mu_2$ respectively. Hence, we have

$$G_{01} = -\frac{(\beta - 1)^2}{\mu(\mu - 1)^2}.$$

Also, G_{02} can be obtained by differentiating $G(m_1, m_2)$ with respect to m_2 twice we get

$$\frac{\partial^2 G(m_1, m_2)}{\partial m_2^2} = -2\frac{m_1(m_1-1)^2}{(-m_2+m_1^3-m_1^2)^3}.$$

Then, the above expression is evaluated at $m_1 = \mu_1'$ and $m_2 = \mu_2$ respectively and after simplification, we have

$$G_{02} = 2\frac{(\beta-1)^3}{\mu^2(\mu-1)^4}.$$

Following the above preocedure, we obtain G_{ij} for i,j = 1, 2 and $0 \leq i+j \leq 2$ as follows:

$$
\begin{aligned}
G_{00} &= \beta, \\
G_{11} &= -\frac{(\beta-1)^2}{\mu^2(\mu-1)^4}(1-3\mu^2-4\beta\mu+6\beta\mu^2), \\
G_{01} &= -\frac{(\beta-1)^2}{\mu(\mu-1)^2}, \\
G_{02} &= 2\frac{(\beta-1)^3}{\mu^2(\mu-1)^4}, \\
G_{10} &= \frac{(\beta-1)}{\mu(\mu-1)^2}(1-2\mu-2\beta\mu+3\beta\mu^2), \\
G_{20} &= 2\frac{(\beta-1)}{\mu(\mu-1)^4}(1-5\mu+5\mu^2-3\beta+8\beta\mu-\beta\mu^2-6\beta\mu^3 \\
&\qquad +4\beta^2\mu-12\beta^2\mu^2+9\beta^2\mu^3).
\end{aligned}
\tag{4.1.8}
$$

And, in general, we know from [Cramer,1946],

$$E[m_1 - \mu_1'] = 0,$$

$$
\begin{aligned}
E[m_2 - \mu_2] &= -\frac{\mu_2}{n}, \\
Var(m_1) &= E(m_1 - \mu_1')^2 = \frac{\mu_2}{n}, \\
Var(m_2) &= E(m_2 - \mu_2)^2 = \frac{(\mu_4 - \mu_2^2)}{n} + O(n^{-2}), \\
Cov(m_1, m_2) &= E[(m_1 - \mu_1')(m_2 - \mu_2)] = \frac{\mu_3}{n} + O(n^{-2}).
\end{aligned}
\tag{4.1.9}
$$

On taking expectation on both sides of 4.1.5, one can show that

$$
\begin{aligned}
E[F(m_1, m_2)] = F_{00} + E(m_2 - \mu_2)F_{01} + \frac{1}{2}Var(m_1)F_{20} \\
+Cov(m_1, m_2)F_{11} + \frac{1}{2}Var(m_2)F_{02} + O(n^{-2}),
\end{aligned}
\tag{4.1.10}
$$

Since

$$
Var[F(m_1, m_2)] = E[(F(m_1, m_2))^2] - (E[F(m_1, m_2)])^2,
$$

so, squaring and taking expectation of 4.1.5 and squaring 4.1.10, we have respectively,

$$
\begin{aligned}
E[(F(m_1, m_2))^2] = F_{00}^2 + 2Cov(m_1, m_2)F_{11}F_{00} + 2E(m_2 - \mu_2)F_{01}F_{00} \\
+E(m_1 - \mu_1')^2 F_{20}F_{00} + E(m_2 - \mu_2)^2 F_{02}F_{00} + E(m_1 - \mu_1')^2 F_{10}^2 \\
+2Cov(m_1, m_2)F_{01}F_{10} + E(m_2 - \mu_2)^2 F_{01}^2 + O(n^{-2}),
\end{aligned}
\tag{4.1.11}
$$

and

$$
\begin{aligned}
[E(F(m_1, m_2))]^2 = F_{00}^2 + 2Cov(m_1, m_2)F_{11}F_{00} + 2E(m_2 - \mu_2)F_{01}F_{00} \\
+E(m_1 - \mu_1')^2 F_{20}F_{00} + E(m_2 - \mu_2)^2 F_{02}F_{00} + O(n^{-2}).
\end{aligned}
\tag{4.1.12}
$$

Then, by subtracting 4.1.12 from 4.1.11, we get

$$Var[F(m_1,m_2)] = Var(m_1)F_{10}^2 + 2Cov(m_1,m_2)F_{01}F_{10} + Var(m_2)F_{01}^2$$
$$+O(n^{-2}). \tag{4.1.13}$$

In order to simplify the symbol, let $F = F(m_1, m_2)$ and $G = G(m_1, m_2)$.

Also,

$$Cov(F,G) = E(F.G) - E(F).E(G).$$

Hence , multiplying 4.1.5 by 4.1.7 and taking expectation, one can get

$$\begin{aligned} E(F.G) = {} & G_{00}F_{00} + E(m_2 - \mu_2)G_{01}F_{00} + \frac{1}{2}E(m_1 - \mu_1')^2G_{20}F_{00} \\ & + \frac{1}{2}E(m_2 - \mu_2)^2G_{02}F_{00} + Cov(m_1,m_2)G_{11}F_{00} + E(m_1 - \mu_1')^2F_{10}G_{10} \\ & + Cov(m_1,m_2)G_{01}F_{10} + E(m_2 - \mu_2)F_{01}G_{00} + Cov(m_1,m_2)F_{01}G_{10} \\ & + E(m_2 - \mu_2)^2F_{01}G_{01} + \frac{1}{2}E(m_1 - \mu_1')^2F_{20}G_{00} + \frac{1}{2}E(m_2 - \mu_2)^2F_{02}G_{00} \\ & + Cov(m_1,m_2)F_{11}G_{00} + O(n^{-2}). \end{aligned} \tag{4.1.14}$$

Also multiplying 4.1.10 with its counterpart of G (just by replacing F by G in

4.1.10), one can obtain

$$E(F).E(G) = G_{00}F_{00} + E(m_2 - \mu_2)G_{01}F_{00} + \frac{1}{2}E(m_1 - \mu_1')^2 G_{20}F_{00}$$
$$+\frac{1}{2}E(m_2 - \mu_2)^2 G_{02}F_{00} + Cov(m_1, m_2)G_{11}F_{00} + E(m_2 - \mu_2)F_{01}G_{00} \tag{4.1.15}$$
$$+\frac{1}{2}E(m_1 - \mu_1')^2 F_{20}G_{00} + \frac{1}{2}E(m_2 - \mu_2)^2 F_{02}G_{00} + Cov(m_1, m_2)F_{11}G_{00}$$
$$+O(n^{-2}).$$

Then, subtracting 4.1.15 from 4.1.14, one can get

$$Cov(F(m_1, m_2), G(m_1, m_2)) = Var(m_1)F_{10}G_{10} + Cov(m_1, m_2)[F_{10}G_{01} + F_{01}G_{10}]$$
$$+Var(m_2)F_{01}G_{01} + O(n^{-2}). \tag{4.1.16}$$

Now, since $F_{00} = \theta$ and $Bias[\hat{\theta}] = E[\hat{\theta}] - \theta$, so by subtracting F_{00} from both sides of 4.1.10, one can have

$$
\begin{aligned}
Bias[\hat{\theta}] &= E[F(m_1, m_2)] - \theta \\
&= E(m_2 - \mu_2)F_{01} + \frac{1}{2}Var(m_1)F_{20} + Cov(m_1, m_2)F_{11} \\
&\quad + \frac{1}{2}Var(m_2)F_{02} + O(n^{-2}).
\end{aligned} \tag{4.1.17}
$$

By substituting 4.1.6 and 4.1.9 into 4.1.17, we have the asymptotic bias of $\hat{\theta}$ as

$$
\begin{aligned}
Bias[\hat{\theta}] &\approx n^{-1}\{\frac{-1}{(\mu-1)(\beta\mu-1)^2} \\
&\quad \times[6\mu^2 - 1 + \beta\mu(7 + 4\mu - 14\mu^2) + 9\beta^2\mu^3(\mu-1)]\}.
\end{aligned} \tag{4.1.18}
$$

Also, by substituting 4.1.6 and 4.1.9 into 4.1.13, after collecting terms and simplification, one can obtain the asymptotic variance of $\hat{\theta}$,

$$
\begin{aligned}
Var[\hat{\theta}] &\approx n^{-1}\{\frac{\mu(\beta-1)}{(\mu-1)(\beta\mu-1)^3} \\
&\quad \times[5\mu^2 + \beta\mu(6 + \mu - 10\mu^2) + 6\beta^2\mu^3(\mu-1)]\}.
\end{aligned} \tag{4.1.19}
$$

Since $G_{00} = \beta$, so, similarly, one can have

$$
\begin{aligned}
Bias[\hat{\beta}] &= E[G(m_1, m_2)] - \beta \\
&= E(m_2 - \mu_2)G_{01} + \frac{1}{2}Var(m_1)G_{20} + Cov(m_1, m_2)G_{11} \\
&\quad + \frac{1}{2}Var(m_2)G_{02} + O(n^{-2}).
\end{aligned} \tag{4.1.20}
$$

And, by replacing F by G in 4.1.13, one can get

$$
\begin{aligned}
Var[\hat{\beta}] &= Var[G(m_1, m_2)] \\
&= Var(m_1)G_{10}^2 + 2Cov(m_1, m_2)G_{01}G_{10} + Var(m_2)G_{01}^2 \\
&\quad + O(n^{-2}).
\end{aligned} \tag{4.1.21}
$$

Substituting 4.1.8 and 4.1.9 in 4.1.20 and 4.1.21 respectively, one can obtain the asymptotic bias and variance of $\hat{\beta}$ as

$$
\begin{aligned}
Bias[\hat{\beta}] &\approx n^{-1}\{\frac{(\beta\mu - 1)}{\mu(\mu-1)^3} \\
&\quad \times[-1 + 8\mu^2 - 2\mu^3 + \beta\mu(6 + 4\mu - 16\mu^2 + 3\mu^3) + 6\beta^2\mu^3(\mu - 1)]\}.
\end{aligned} \tag{4.1.22}
$$

and,

$$
\begin{aligned}
Var[\hat{\beta}] &\approx n^{-1}\{\frac{(\beta - 1)(\beta\mu - 1)}{\mu(\mu-1)^3} \\
&\quad \times[-1 + 2\mu + 4\mu^2 + \beta\mu(6 + \mu - 10\mu^2) + 6\beta^2\mu^3(\mu - 1)]\}.
\end{aligned} \tag{4.1.23}
$$

Finally, by substituting 4.1.6, 4.1.8 and 4.1.9 into 4.1.16, and after simplification, one can obtain the asymptotic covariance of θ and β as

$$
\begin{aligned}
Cov[\hat{\beta}, \hat{\theta}] &\approx -n^{-1}\{\frac{(\beta - 1)}{(\mu-1)^2(\beta\mu - 1)} \\
&\quad \times[-1 + 2\mu + 4\mu^2 + \beta\mu(6 + \mu - 10\mu^2) + 6\beta^2\mu^3(\mu - 1)]\}.
\end{aligned} \tag{4.1.24}
$$

4.2 Maximum Likelihood Estimation

Let a random sample of size n, taken from the Geeta distribution 2.1.1, consists of the observations 1, 2, 3, ..., k with frequencies f_1, f_2, ..., f_k respectively, where $f_1 + f_2 + \ldots + f_k = n$. Also, let $\bar{x}$ and s^2 be the sample mean and sample variance respectively, given by

$$\bar{x} = n^{-1}[1.f_1 + 2.f_2 + 3.f_3 + \ldots + k.f_k]$$

and,

$$s^2 = (n-1)^{-1}\sum_{i=1}^{k}(i-\bar{x})^2.f_i.$$

It was shown in Chapter 2 that the minimum value of σ^2, for a maximum value of β, is $\mu^2(\mu - 1)$. Accordingly, a necessary condition for the applicability of the Geeta model to a given data set becomes

$$s^2 > \bar{x}^2(\bar{x} - 1). \tag{4.2.25}$$

The likelihood function L can be written with the help of individual probabilities in 2.1.2 in the form

$$\begin{aligned} L &= [\frac{(\beta-1)\mu}{\beta\mu-1}]^{\beta f_1 - f_1}[\frac{\mu-1}{\mu}]^{f_2}[\frac{(\beta-1)\mu}{\beta\mu-1}]^{2\beta f_2 - 2f_2} \\ &\quad \times \prod_{i=3}^{k}[\frac{1}{\beta i-1}\binom{\beta i-1}{i}]^{f_i}[\frac{\mu-1}{\beta\mu-1}]^{if_i - f_i}[\frac{\mu(\beta-1)}{\beta\mu-1}]^{i\beta f_i - if_i} \end{aligned}$$

which gives the log-likelihood function as

$$\ln L = n\bar{x}\{(\beta-1)\ln\mu(\beta-1) - \beta\ln(\beta\mu-1) + \ln(\mu-1)\} + n\{\ln(\beta\mu-1) -$$

$$\ln(\mu-1)\}+\sum_{2}^{k} f_i \sum_{j=2}^{i} \ln(\beta i-j)-\sum_{i=2}^{k} \ln(i!)^{f_i}. \tag{4.2.26}$$

On partial differentiation of $\ln L$ w.r.t. μ and on equating to zero,

$$\frac{\partial \ln L}{\partial \mu}=\frac{(\beta-1)\bar{x}}{\mu}-\frac{\beta^2\bar{x}}{\beta\mu-1}+\frac{\bar{x}-1}{\mu-1}+\frac{\beta}{\beta\mu-1}=0,$$

which gives, on simplification, the ML estimate $\hat{\mu}$ of μ as [Consul,1990a]

$$\hat{\mu}=\bar{x}. \tag{4.2.27}$$

On partial differentiation of 4.2.26 w.r.t. β and on substitution of $\mu=\bar{x}$, we get

$$\begin{aligned}\frac{\partial \ln L}{\partial \beta} &= n\bar{x}[\ln(\beta-1)+\ln\bar{x}-\ln(\beta\bar{x}-1)]+\sum_{i=2}^{k}\sum_{j=2}^{i}\frac{if_i}{i\beta-j}\\ &= 0.\end{aligned}$$

On simplification the above equation gives

$$\begin{aligned}\frac{(\beta-1)\bar{x}}{\beta\bar{x}-1} &= e^{-H(\beta)}\\ &= G(\beta),\end{aligned} \tag{4.2.28}$$

where

$$H(\beta)=\frac{1}{n\bar{x}}\sum_{i=2}^{k}\sum_{j=2}^{i}\frac{if_i}{i\beta-j}. \tag{4.2.29}$$

The equation 4.2.28 and 4.2.29 cannot specifically solved for the ML estimate $\hat{\beta}$. However, the function

$$G(\beta) = \frac{(\beta - 1)\bar{x}}{\beta\bar{x} - 1} \tag{4.2.30}$$

monotonically increases form 0^+ to 1^- as β increases from 1^+ to ∞. One can easily draw the graph of the curve 4.2.30 by a computer.

Also, the function $H(\beta)$, defined in 4.2.29, is a sum of a finite number of functions. Each one of these functions monotonically decreases as β increases from 1^+ to ∞. Therefore, the function $H(\beta)$ monotonically decreases with the increase in the value of β. Thus, the function

$$G(\beta) = e^{-H(\beta)} \tag{4.2.31}$$

represents a monotonically increasing curve, with a maximum value of 1^-. The curve 4.2.31 can also be drawn with the help of a computer. Since both curves, 4.2.30 and 4.2.31, are monotonically increasing with almost the same domain, they can have, at most, a single point of intersection. The value of β given by this unique point of intersection, if it exits, is the ML estimate of $\hat{\beta}$ [Consul,1990a].

Our conjecture is that if $s^2 > \bar{x}^2(\bar{x} - 1)$, the two curves 4.2.30 and 4.2.31 will always intersect. We have not been able to prove it though we have verified this conjecture in many examples.

4.3 Minimum Variance Unbiased Estimation

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the Geeta probability model 2.1.1. It can be shown by factorization theorem that the statistic $Y = X_1 + X_2 + \ldots + X_n$ is a sufficient statistic for the parameter θ in the model 2.1.1. Also, since the model 2.1.1 belongs to the exponential family, it is a complete family. Thus, Y is a complete and sufficient statistic for the parameter θ.

The probability distribution of Y is a Lagrangian power series distribution (LPSD), given by 2.3.23, and is of the form

$$P(Y = y) = b(n, y)\frac{\{\theta/g(\theta)\}}{(f(\theta))^n}, \quad for\ y = n, n+1, n+2, \ldots \tag{4.3.32}$$

where

$$\begin{aligned} f(\theta) &= \theta, \\ g(\theta) &= (1-\theta)^{1-\beta} \\ &\quad and \\ b(n,y) &= \frac{n}{y}\binom{\beta y - n - 1}{y - n}. \end{aligned}$$

[Kumar & Consul,1980] has given following theorem on the minimum variance unbiased (MVU) estimation of parametric function $\ell(\theta)$ for the Geeta distribution.

Theorem: A parametric function of the parameter θ in the LPSD is MVU estimable iff the function $\varphi(\theta) = \ell(\theta).(f(\theta))^n$ admits an expansion in a absolutely convergent Lagrange series.

$$\varphi(\theta) = \sum_{i \in E_n} c(n, i)(\theta/g(\theta))^i \quad where\ E_n \subseteq \{0, 1, 2, \ldots\}$$

and

$$(4.3.33) \qquad c(n,i) = \begin{cases} 0, & \text{where b(n,i)=0} \\ \frac{1}{i!}[\frac{d}{d\theta}]^{i-1}[(g(\theta))^i\psi'(\theta)]_{\theta=0}, & \text{otherwise.} \end{cases}$$

Then, the MVU estimator h(y) of $\ell(\theta)$ is given by

$$(4.3.34) \qquad h(y) = \begin{cases} \frac{c(n,y)}{b(n,y)}, & \text{if b(n,y)} \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

We denote the MVU estimator of $\ell(\theta)$ by $\langle \ell(\theta) \rangle$.

We shall obtain the MVU esitmator for $\frac{(1-\theta)^2}{(1-\beta\theta)}, \frac{\theta(1-\theta)}{(1-\beta\theta)^2}$, $\theta^{k-1}(1-\theta)^{\beta k-k}$ and P(X=k). The proofs are given for the MVU estimation of $\frac{(1-\theta)^2}{(1-\beta\theta)}$ and $\theta^{k-1}(1-\theta)^{\beta k-k}$ respectively.

Firstly, for $\langle (1-\theta)^2/(1-\beta\theta) \rangle$, let

$$(4.3.35) \qquad \varphi(\theta) = \ (1-\theta)^2(1-\beta\theta)^{-1}(f(\theta))^n = \ \theta^n(1-\theta)^2(1-\beta\theta)^{-1}.$$

By expanding the function $\varphi(\theta)$ into a series by Lagrange expansion,under the transformation $\theta = u(1-\theta)^{1-\beta}$, we get

$$\varphi(\theta) = \sum_{i=n}^{\infty} \frac{[\theta(1-\theta)^{\beta-1}]^i}{i!}[\frac{d}{d\theta}]^{i-1}[(1-\theta)^{i-i\beta}\frac{\partial}{\partial\theta}\{\theta^n(1-\theta)^2(1-\beta\theta)^{-1}\}]_{\theta=0}.$$

The expression for the (i-1)th derivative is

$$= [\frac{d}{d\theta}]^{i-1}[(1-\theta)^{-(\beta i-i-1)}\{n\theta^{n-1}+\sum_{j=1}^{\infty}\beta^{j-1}\theta^{n+j-1}[\beta(n+j)-(n+j+1)]\}]_{\theta=0}$$

$$= \binom{i-1}{n-1}n!(i\beta-i-1)_{(i-n)}+\sum_{j=1}^{i-n}\{\beta(n+j)-(n+j+1)\}\beta^{j-1}\binom{i-1}{n+j-1}(n+j-1)! \times(i\beta-i-1)_{(i-n-j)}.$$

On simplifying the above and using it with the value of $\varphi(\theta)$,we get

$$\theta^n(1-\theta)^2(1-\beta\theta)^{-1}=\sum_{i=n}^{\infty}\frac{[\theta(1-\theta)^{\beta-1}]^i}{i}[\frac{n}{(i-n)!}(i\beta-i-1)_{(i-n)}+\sum_{j=1}^{i-n}\frac{\beta(n+j)-(n+j+1)}{(i-n-j)!} \times\beta^{j-1}(i\beta-i-1)_{(i-n-j)}]$$

where $(a)_{(i)}=a(a+1)(a+2)\dots(a+i-1)$.

Thus, the function c(n,y) in 4.3.33 becomes

$$(4.3.36)\qquad \begin{cases} 0, & y<n \\ 1, & y=n \\ \frac{n}{y(y-n)!}(\beta y-y-1)_{(y-n)} & \\ +\frac{1}{y}\sum_{j=1}^{y-n}\frac{\beta(n+j)-(n+j+1)}{(y-n-j)!} & \\ \qquad\times\beta^{j-1}(\beta y-y-1)_{(y-n-j)}, & y=n+1,n+2,\dots \end{cases}$$

Hence, the MVU estimator of $\frac{(1-\theta)^2}{(1-\beta\theta)}$ pf the Geeta distribution becomes

$$h(y)=\frac{c(n,y)}{b(n,y)}=$$

$$(4.3.37)\qquad \begin{cases} \frac{(\beta y-y-1)}{(\beta y-n-1)}\sum_{j=1}^{y-n}\frac{(\beta y-y-1)}{n}\frac{(y-n-j+1)_{(j)}}{(\beta y-n-j-1)_{(j)}} & \\ \times[\beta(n+j)-(n+j+1)]\beta^{j-1}, & y=n+1,n+2,\ldots \\ 1, & y=n \\ 0, & \text{otherwise} \end{cases}$$

Secondly, for $\langle\theta^{k-1}(1-\theta)^{\beta k-k}\rangle$, let

$$(4.3.38)\qquad \varphi(\theta)=\theta^{n+k-1}(1-\theta)^{\beta k-k}.$$

Similarly, we have

$$\varphi(\theta)=\sum_{i=n+k-1}^{\infty}\frac{[\theta(1-\theta)^{\beta-1}]^i}{i!}[\frac{d}{d\theta}]^{i-1}[(1-\theta)^{i-i\beta}\frac{\partial}{\partial\theta}\{\theta^{n+k-1}(1-\theta)^{\beta k-k}\}]_{\theta=0}.$$

The expression for the corresponding (i-1)th derivative is

$$\begin{aligned}
&= [\frac{d}{d\theta}]^{i-1}[(1-\theta)^{-(\beta i-i)}\{\sum_{j=0}^{\infty}\binom{\beta k-k}{j}(-1)^j(n+k+j-1)\theta^{n+k+j-2}\}]_{\theta=0}\\
&= [\frac{d}{d\theta}]^{i-1}[(1-\theta)^{-(\beta i-i)}\{(n+k-1)\theta^{n+k-2}+\sum_{j=1}^{\infty}\binom{\beta k-k}{j}(-1)^j\\
&\qquad\qquad \times(n+k+j-1)\theta^{n+k+j-2}\}]_{\theta=0}\\
&= \binom{i-1}{n+k-2}(n+k-1)!(i\beta-i)_{(i-n-k+1)}+\sum_{j=1}^{i-n-k+1}\binom{\beta k-k}{j}(-1)^j\binom{i-1}{n+k+j-2}\\
&\qquad\qquad \times(n+k+j-1)!(i\beta-i)_{(i-n-k-j+1)}
\end{aligned}$$

On simplifying the above with $\varphi(\theta)$, we have

$$\theta^{n+k-1}(1-\theta)^{\beta k-k} = \sum_{i=n+k-1}^{\infty} \frac{[\theta(1-\theta)^{\beta-1}]^i}{i}\Big[\frac{(n+k-1)}{(i-n-k+1)!}(i\beta-i)_{(i-n-k+1)}$$
$$+\sum_{j=1}^{i-n-k+1}\binom{\beta k-k}{j}(-1)^j\frac{(n+k+j-1)}{(i-n-k-j+1)!}(i\beta-i)_{(i-n-k-j+1)}\Big].$$

Hence, the function c(n,y) for $<\theta^{k-1}(1-\theta)^{\beta k-k}>$ in 4.3.33 becomes

$$(4.3.39)\quad \begin{cases} 0, & y < n+k-1 \\ 1, & y = n+k-1 \\ \frac{(n+k-1)}{y(y-n-k+1)!}(\beta y-y)_{(y-n-k+1)} & \\ +\frac{1}{y}\sum_{j=1}^{y-n-k+1}\binom{\beta k-k}{j}(-1)^j & \\ \times\frac{(n+k+j-1)}{(y-n-k-j+1)!}(\beta y-y)_{(y-n-k-j+1)}, & y = n+k, n+k+1, \ldots \end{cases}$$

Therefore, the MVU estimator of $\theta^{k-1}(1-\theta)^{\beta k-k}$ of the Geeta distribution is

$$h(y) = \frac{c(n,y)}{b(n,y)} =$$

$$(4.3.40)\quad \begin{cases} \frac{(n+k-1)}{n}\frac{(y-n-k+2)_{(k-1)}}{(\beta y-n-k+1)_{(k-1)}} & \\ +\sum_{j=1}^{y-n-k+1}\binom{\beta k-k}{j}(-1)^j\frac{(n+k+j-1)}{n} & \\ \qquad\times\frac{(y-n-k-j+2)_{(k+j-1)}}{(\beta y-n-k-j+1)_{(k+j-1)}}, & y = n+k, n+k+1, \ldots \\ \frac{n+k-1}{n}\frac{(k-1)!}{(\beta(n+k-1)-(n+k-1))_{(k-1)}}, & y = n+k-1 \\ 0, & \text{otherwise} \end{cases}$$

The MVU estimators of $\frac{\theta(1-\theta)}{(1-\beta\theta)^2}$ and $P(X = k)$ are derived in the same manner. The corresponding results are provided in the following Table.

Table 4.3.1: MVU estimators of $\theta(1 - \theta)/(1 - \beta\theta)^2$ and $P(X = k)$ of the Geeta distribution

$\ell(\theta)$	MVU estimator $\langle \ell(\theta) \rangle$
$\frac{\theta(1-\theta)}{(1-\beta\theta)^2}$	$\begin{cases} 0, & y \le n \\ \frac{n+1}{n}\frac{1}{\beta(n+1)-(n+1)}, & y = n+1 \\ \frac{(n+1)(y-n)}{n(\beta y-n-1)} + \sum_{j=1}^{y-n-1} \frac{(y-n-j)_{(j+1)}}{(\beta y-n-j-1)_{(j+1)}} & \\ \qquad \times \frac{(\beta j+\beta-j)(n+j+1)\beta^{j-1}}{n}, & y = n+2, n+3, \ldots \end{cases}$
$P(X = k)$	$\begin{cases} 0, & y < n+k-1 \\ \frac{n+k-1}{nk}\frac{(\beta k-k)_{(k-1)}}{(\beta(n+k-1)-(n+k-1))_{(k-1)}}, & y = n+k-1 \\ \frac{(n+k-1)}{nk!}\frac{(\beta k-k)_{(k-1)}(y-n-k+2)_{(k-1)}}{(\beta y-n-k+1)_{(k-1)}} & \\ + \sum_{j=1}^{y-n-k+1} (-1)^j \frac{(n+k+j-1)}{nj!k!} (\beta k - k - j + 1)_{(j)} & \\ \qquad \times (\beta k - k)_{(k-1)} \frac{(y-n-k-j+2)_{(k+j-1)}}{(\beta y-n-k-j+1)_{(k+j-1)}}, & y = n+k, n+k+1, \ldots \end{cases}$

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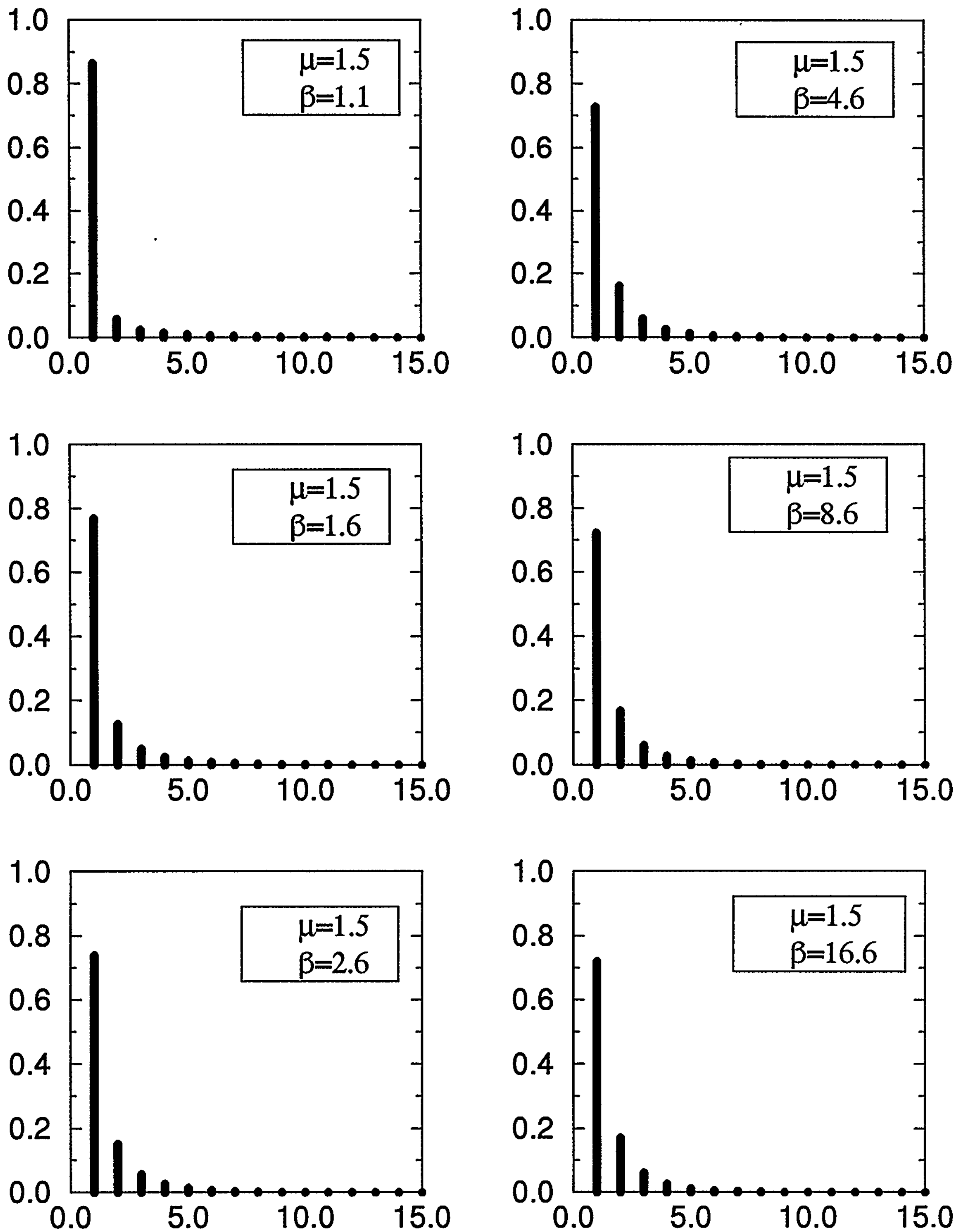
Appendix A

Graphs

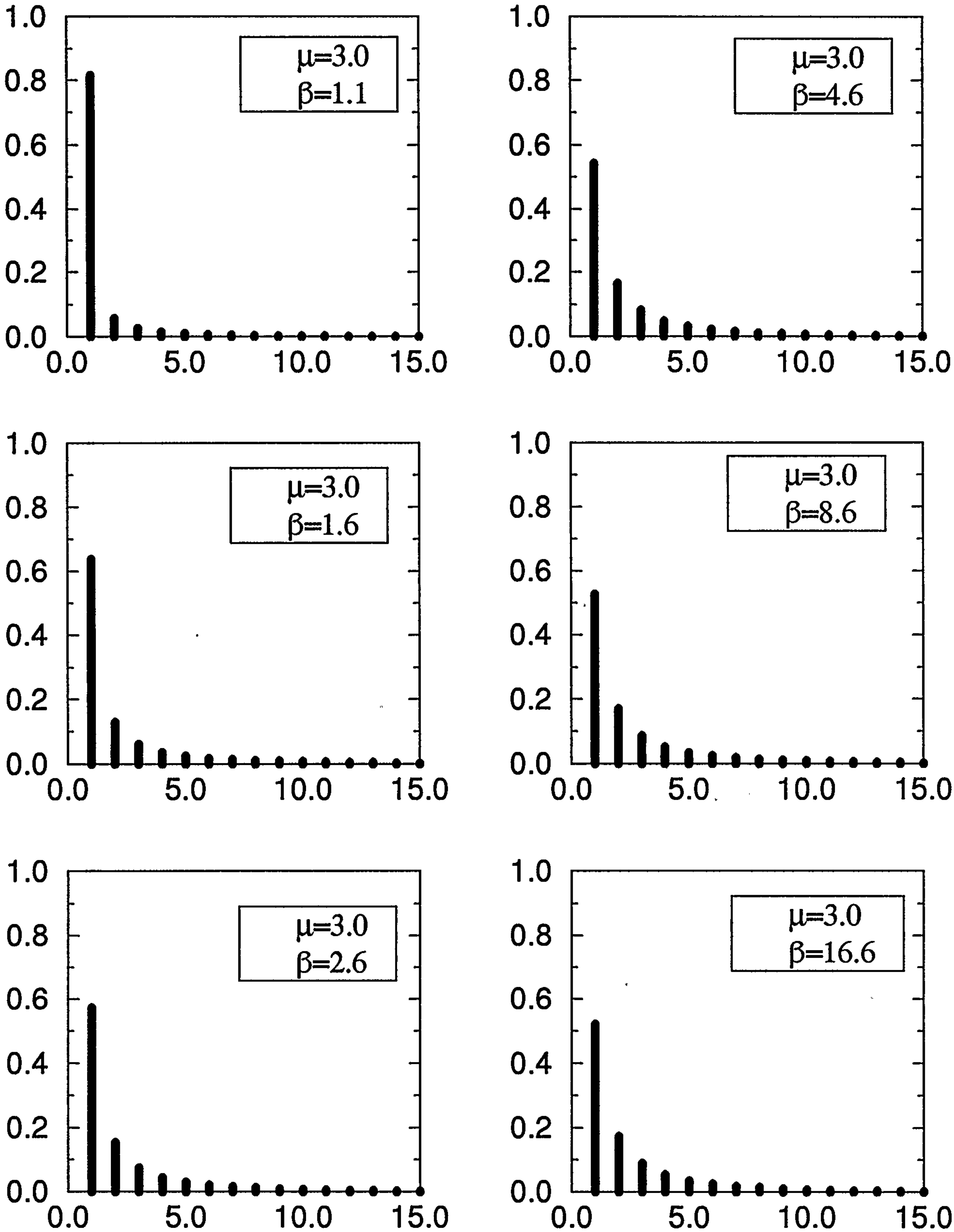
Graph 2.1

Bar-diagrams of the Geeta Distribution

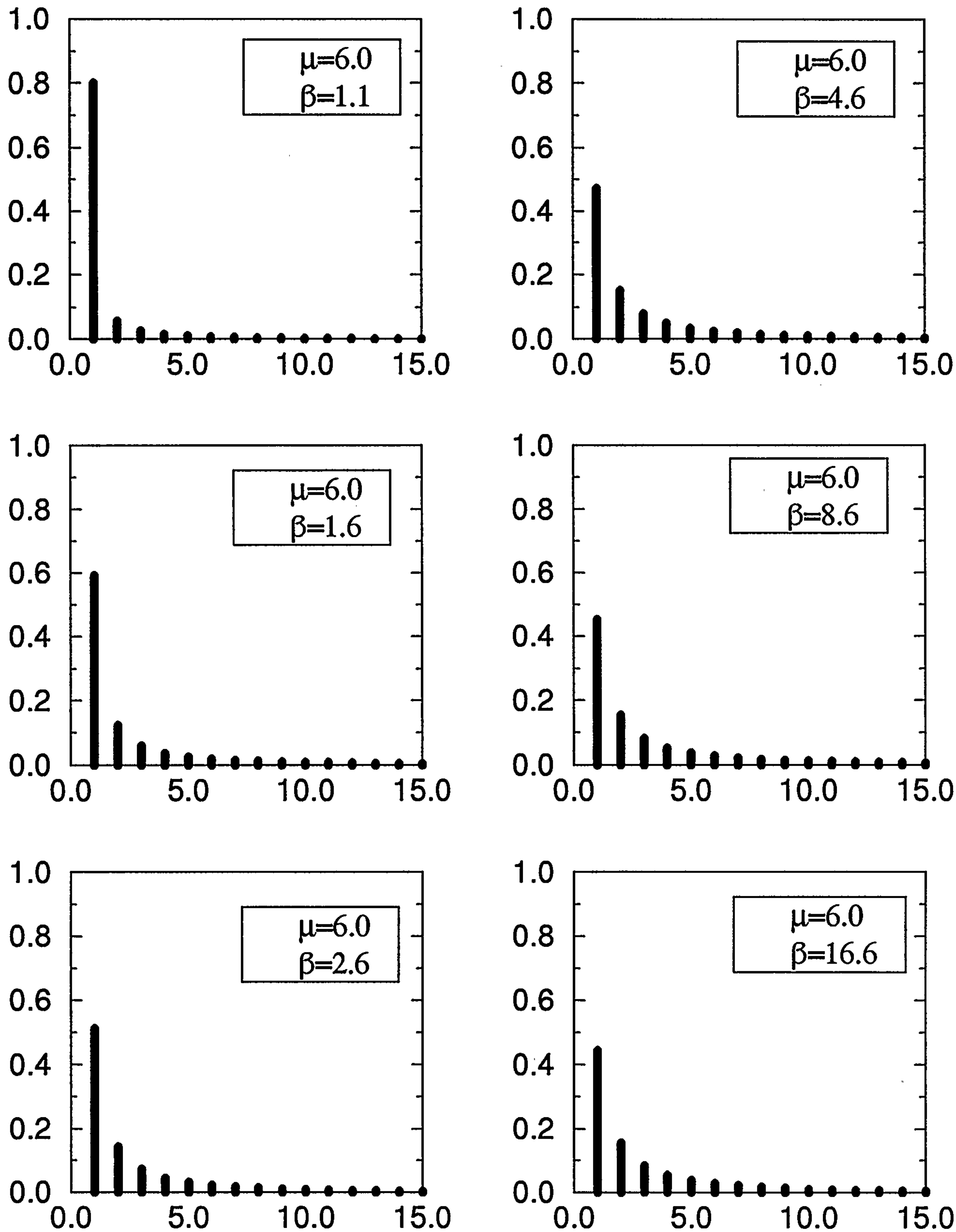


Graph 2.2

Bar-diagrams of the Geeta Distribution

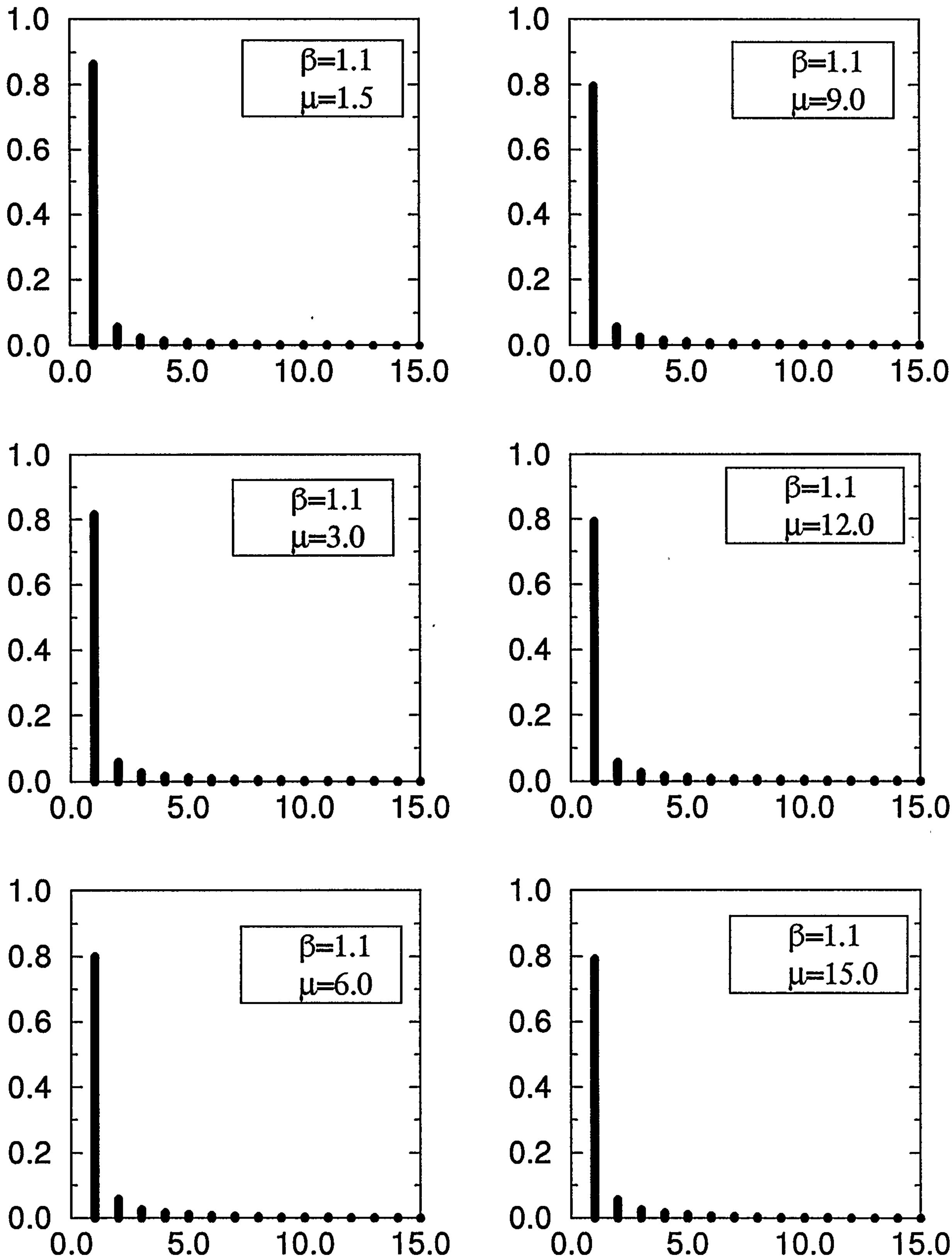


Graph 2.3
Bar-diagrams of the Geeta Distribution



Graph 2.4

Bar-diagrams of the Geeta Distribution



Graph 2.5

Bar-diagrams of the Geeta Distribution