Utility-Based Indifference Pricing in Regime Switching Models

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Abstract

In this paper we study utility-based indifference pricing and hedging of a contingent claim in a continuous-time, Markov, regime-switching model. The market in this model is incomplete, so there is more than one price kernel. We specify the parametric form of price kernels so that both market risk and economic risk are taken into account. The pricing and hedging problem is formulated as a stochastic optimal control problem and is discussed using the dynamic programming approach. A verification theorem for the Hamilton-Jacobi-Bellman (HJB) solution to the problem is given. An issuer's price kernel is obtained from a solution of a system of linear programming problems and an optimal hedged portfolio is determined.

Key words: Contingent Claim Valuation, Hedging, Regime Switching Risk, Utility Indifference, Product Price kernel, Dynamic Programming, Markov Regime Switching HJB equations, Exponential Utility, Linear Programming.
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1 Introduction

The work of Black and Scholes [1] and Merton [26] provided a solution in simple situations to derivative valuation and hedging. They considered the situation where the price dynamics of the underlying risky asset are described by a geometric Brownian motion (GBM). Together with the assumptions of a perfect market and the absence of arbitrage, they were able to perfectly replicate the payoff of a derivative security by continuously re-balancing a portfolio consisting of a bond and a share and so derived an explicit pricing formula for a standard European call option. The advantage of the Black-Scholes-Merton formula is that it is preference-free; that is, it does not depend on a choice of utility function and the appreciation rate of the underlying risky asset. Therefore, the subjective views of market participants about the appreciation rate and risk-preferences do not influence the option price in the Black-Scholes-Merton world. Despite its compact form and popularity, the assumptions underlying the Black-Scholes-Merton model are often questioned. The GBM cannot explain some observed important empirical features of asset price dynamics. In the past three decades a number of option pricing models based on more realistic price dynamics have been proposed. These include the jump-diffusion model, the stochastic volatility models, the GARCH option pricing model, and others. These models can provide a more realistic way to describe empirical features of share price data, such as heavy-tailedness, and the option price data, such as the implied volatility smile or smirk. However, they do not take into account structural changes in economic conditions when modeling asset price dynamics and valuing options. In practice, structural changes in economic conditions have significant impact on and economic implications for asset prices. For example, there have been substantial changes in asset prices before and after the global financial crisis of 2008. Structural changes in economic conditions represent an important risk factor that should be taken into account in asset pricing models, especially when one wishes to value long-dated option contracts. Failure to incorporate this risk factor appropriately may lead to incorrect assessment of the risk inherent from writing an option contract.

Regime-switching models have recently attracted serious attention among researchers and practitioners in economics, finance and actuarial science. They provide a natural and convenient way to incorporate structural changes in economic conditions when modeling asset prices movements. The idea of regime switching models may be traced back as early as the work of Quandt [30], where a two-state, regime-switching, regression model was developed. Goldfeld and Quandt [18] considered a regime-switching regression model for describing the nonlinearity and non-stationarity of economic data. Econometric applications of regime-switching models were pioneered in the original work of Hamilton [20], where a class of discrete-time, Markov-switching, autoregressive time series models was proposed. Empirically, this class of models can describe a number of important "stylised" facts of economic and financial time series, such as the heavy-tailedness of assets' returns, time-varying conditional volatility, volatility clustering, regime switchings, nonlinearity and nonstationarity. Economically, regime-switching models can describe structural changes in economic conditions and provide flexibility to describe stochastic evolution of investment opportunity sets over time. Applications of regime-switching models can be found in many fields of economics and finance. Some of these applications include Elliott and van der Hoek [7] for asset allocation, Pliska [29], Elliott et al. [8] and Elliott and Kopp [10] for short rate models, and Elliott and Hinz [9] for portfolio analysis and chart analysis.

Recently, attention has turned to the application of Markov regime-switching models to value derivative securities. The market in a regime-switching model is, in general, incomplete. Consequently, the standard Black-Scholes-Merton arguments cannot be applied and the option valuation problem becomes more challenging from both a mathematical and economic perspective. Different methods have been developed to value derivative securities in an incomplete market. Föllmer and Sondermann [15], Föllmer and Schweizer [16] and Schweizer [31] introduced the minimization of a quadratic function of hedging errors for valuation. Davis [3] used traditional economic equilibrium arguments to value options and formulated the problem as a utility maximization problem. Gerber and Shiu [17] pioneered the use of the Esscher transform, a well-known tool in actuarial science, to value options in an incomplete market. Hodges and Neuberger [25] developed a utility-based indifference pricing approach in an incomplete market. The idea of indifference pricing is to determine a seller's, (buyer's), price so that the seller, (the buyer), is indifferent to whether the claim is sold, (bought), or whether it is not sold, (bought). The utility-based approach has a solid economic foundation and is related to the concept of certainty equivalence, which has been applied in insurance economics for premium calculation. The idea of utility-based indifference pricing was then applied by Hobson and Henderson [24] for valuation of contingent claims in stochastic volatility models. Some existing works on option valuation in regime switching models include Naik [27], Guo [19], Buffington and Elliott [2], Elliott et al. [11], Elliott et al. [12], [13], Siu [32]. However, it seems that in these works, the regime-switching risk was not priced explicitly. In practice, this source of risk is important and should be priced appropriately.

In this paper, we study utility-based indifference pricing and hedging of a European-style contingent claim in a continuous-time, Markov, regime-switching model. The market interest rate, the appreciation rate and the volatility of a share are modulated by a continuous-time, finite-state, Markov chain whose states represent various states of an economy. There are two sources of risk in the regime-switching model. One source of risk is attributed to fluctuations of market prices, or rates. We refer to it as market risk and model this by a standard Brownian motion. The other source of risk is due to more long term changes in in economic conditions. This is referred to as economic risk and is modeled here by the Markov chain. In the context of utility-based indifference valuation, the market risk and the economic risk described here may be regarded as the tradeable and non-tradeable factors of risk. Since the market in this model is incomplete, there is more than one price kernel, and so more than one arbitrage-free price of the claim. Further, in an incomplete market, not all contingent claims can be perfectly hedged. Consequently, the selection of a price kernel and the determination of an optimal, (partially), hedged portfolio are two key issues. In this paper we first specify the parametric form of price kernels so that both market risk and economic risk are considered. This can be achieved by introducing a price kernel given by the product of two density processes for measure changes, one for the standard Brownian motion and another for the Markov chain. We determine an issuer's price and the optimal, (partially), hedged portfolio of the claim so that both the market risk and the economic risk are priced and hedged optimally. The valuation and hedging problem is then formulated as a stochastic optimal control problem. We use the dynamic programming approach to solve the problem. A verification theorem for the Hamilton-Jacobi-Bellman (HJB) equation is given. The local conditions of the theorem are then used to determine an issuer's price kernel and the optimal, (partially), hedged portfolio of the claim. In particular, the issuer's price kernel is given as the solution to a system of linear programming problems.

The paper is organized as follows. Section 2 gives the model dynamics and a parametric form of price kernels based on the product of two density processes. In Section 3, we present the utility indifference valuation method to select a price kernel and an optimal hedged portfolio for the issuer of the claim. Section 4 gives verification theorems for the HJB equations to the valuation problem. We also derive the local conditions for the issuer's utility-indifference price kernel and optimal hedged portfolio. A system of linear programming problems for determining the price kernel is obtained and is solved in the cases of two regimes and three regimes. The final section gives some concluding remarks.

2 The Model Dynamics and Price Kernels

Consider a continuous-time economy with two primitive assets, a bond and an ordinary share. We suppose that these assets are traded continuously over time on a finite time horizon $\mathcal{T} := [0, T]$, where $T \in (0, \infty)$. Fix a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\mathcal{P}$ is a real-world probability measure. We suppose that the probability space is rich enough to model both market risk and economic risk.

Firstly, we describe the model for transitions of states of an economy. Let $\mathbf{X} := \{\mathbf{X}(t) | t \in \mathcal{T}\}$ be a continuous-time, finite-state, Markov chain on $(\Omega, \mathcal{F}, \mathcal{P})$ whose states represent different states of the economy. Suppose the chain $\mathbf{X}$ takes values in a finite state space $\mathcal{S} := \{\mathbf{s}_1, \mathbf{s}_2 \ldots, \mathbf{s}_N\} \in \Re^N$. We assume that $\mathbf{X}$ is observable and that its states are proxies of some observable, (macro)-economic, indicators. For example, they may be interpreted as the credit ratings of a region, or sovereign credit ratings. They may also be interpreted as the gross domestic product (GDP) and retail price index (RPI). Following the convention in Elliott et al. [6], we identify, without loss of generality, the state space of the chain $\mathbf{X}$ with a finite set of basis vectors $\mathcal{E} := \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\} \in \Re^N$, where the j^{th} component of $\mathbf{e}_i$ is the Kronecker delta δ_{ij}, for each $i, j = 1, 2, \ldots, N$. The space $\mathcal{E}$ is called a canonical state space of the Markov chain.

Let $\mathbf{A}(t) := [a_{ij}(t)]_{i,j=1,2,\cdots,N}$, $t \in \mathcal{T}$, be a family of generators of the Markov chain $\mathbf{X}$ under $\mathcal{P}$. These generators are also called rate matrices, transitions intensity matrices, or Q-matrices. They specify the statistical properties of the chain $\mathbf{X}$ under $\mathcal{P}$. For each $i, j = 1, 2, \ldots, N$, $a_{ij}(t)$ is the instantaneous intensity of the transition of the chain $\mathbf{X}$ from state j to state i at time t. Note that for each $t \in \mathcal{T}$, $a_{ij}(t) \geq 0$, for $i \neq j$ and that $\sum_{i=1}^{N} a_{ij}(t) = 0$, so $a_{ii}(t) \leq 0$. For each $i, j = 1, 2, \ldots, N$ with $i \neq j$ and each $t \in \mathcal{T}$, we suppose that $a_{ij}(t) > 0$, so $a_{ii}(t) < 0$. For any such matrix $\mathbf{A}(t)$, write $\mathbf{a}(t) := (a_{11}(t), \ldots, a_{ii}(t), \ldots, a_{NN}(t))'$, where $\mathbf{y}'$ is the transpose of a matrix, or a vector, $\mathbf{y}$. With the canonical state space representation, Elliott et al. [6] gave the following semi-martingale dynamics for $\mathbf{X}$:

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \mathbf{A}(u)\mathbf{X}(u-)du + \mathbf{M}(t) \ ,$$

where $\{\mathbf{M}(t) | t \in \mathcal{T}\}$ is an $\Re^N$-valued martingale with respect to the right-continuous, $\mathcal{P}$-completed, filtration generated by $\mathbf{X}$, denoted as $F^{\mathbf{X}} := \{\mathcal{F}^{\mathbf{X}}(t) | t \in \mathcal{T}\}$, under $\mathcal{P}$. Since $\{\int_0^t \mathbf{A}(u)\mathbf{X}(u-)du | t \in \mathcal{T}\}$ is a predictable process of bounded variation, $\mathbf{X}$ is a special semi-martingale and the above semi-martingale decomposition is unique.

We are now ready to present the price dynamics of the bond and the share. Let $r(t)$ denote the instantaneous market interest rate of the bond at time t, $t \in \mathcal{T}$. We suppose that the chain $\mathbf{X}$ determines $r(t)$ as:

$$r(t) := \langle \mathbf{r}, \mathbf{X}(t) \rangle \ ,$$

where $\mathbf{r} := (r_1, r_2, \cdots, r_N)' \in \Re^N$ with $r_i > 0$, for each $i = 1, 2, \cdots, N$; r_i is the market interest rate of the bond when the economy is in the i^{th} state; the scalar product $\langle \cdot, \cdot \rangle$ selects the component of $\mathbf{r}$ that is in force at a particular time based on the state of the economy at that time.

The price process of the bond is then given by:

$$B(t) = \exp\Big(\int_0^t r(u) du \Big) \ , \quad t \in \mathcal{T} \ , \quad B(0) = 1 \ .$$

Let $W := \{W(t) | t \in \mathcal{T}\}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$ with respect to the $\mathcal{P}$-augmentation of its own natural filtration. To simplify the issue, we suppose that W and $\mathbf{X}$ are stochastically independent. For each $t \in \mathcal{T}$, let $\mu(t)$ and $\sigma(t)$ be the appreciation rate and the volatility of the share at time t. Again we assume that the chain determines $\mu(t)$ and $\sigma(t)$ as:

$$\mu(t) := \langle \boldsymbol{\mu}, \mathbf{X}(t) \rangle \ ,$$
$$\sigma(t) := \langle \boldsymbol{\sigma}, \mathbf{X}(t) \rangle \ .$$

Here $\boldsymbol{\mu} := (\mu_1, \mu_2, \cdots, \mu_N)' \in \Re^N$ with $\mu_i \in r_i$ and $\boldsymbol{\sigma} := (\sigma_1, \sigma_2, \cdots, \sigma_N)' \in \Re^N$ with $\sigma_i > 0$; μ_i and σ_i are the appreciation rate and the volatility of the share when the economy is in the i^{th} state, respectively, for each $i = 1, 2, \cdots, N$.

Then the share price process evolves over time according to the following Markov, regime-switching, geometric Brownian motion (GBM):

$$dS(t) = \mu(t) S(t) dt + \sigma(t) S(t) dW(t) \ ,$$
$$S(0) = s > 0 \ .$$

Since the economy considered here consists of two sources of random shocks and two primitive securities, the market in the economy is incomplete.

2.2 Price Kernels as a Product of Two Density Processes

The key to value a contingent claim is to determine a price kernel, or an equivalent martingale measure. The first step is to determine a parametric form for price kernels. [2] There are different ways to specify the parametric

[2] Indeed, one may also consider a non-parametric approach to specify a price kernel. However, the non-parametric approach may be subject to the problem of curse of dimensionality. To solve this problem, one may then consider a semi-parametric

form of price kernels. Here we specify a class of price kernels by a product of two density processes so that both the market risk and economic risk are taken into account in the parametric specification of the price kernels. One of the density processes is for a measure change for the standard Brownian motion W and the other one is for a measure change of the Markov chain $\mathbf{X}$. The product of two density processes and a Girsanov transform for the Markov chain were used in Elliott and Siu [14] for risk minimizing investment portfolios.

Firstly, we define a density process for a measure change for the Brownian motion W. Let $F^W := \{\mathcal{F}^W(t)|t \in \mathcal{T}\}$ be the right-continuous, complete, filtration generated by W. For each $i = 1, 2, \ldots, N$, let $\{\theta_i(t)|t \in \mathcal{T}\}$ be a real-valued, $\mathcal{F}^W$-progressive measurable, stochastic process such that

(1) for each $i = 1, 2, \cdots, N$, $|\theta_i(t)| = |\theta_i(t, \omega)| < K < \infty$, $\forall (t, \omega) \in \mathcal{T} \times \Omega$;
(2)

$$\int_0^T |\theta_i(t)|^2 dt < \infty \ , \quad \mathcal{P}\text{-a.s.} \ , \quad i = 1, 2, \ldots, N \ .$$

Consider a Markov, regime-switching, process $\theta := \{\theta(t)|t \in \mathcal{T}\}$ as:

$$\theta(t) = \langle \boldsymbol{\theta}(t), \mathbf{X}(t) \rangle \ , \quad t \in \mathcal{T} \ ,$$

where $\boldsymbol{\theta}(t) := (\theta_1(t), \theta_2(t), \ldots, \theta_N(t))' \in \Re^N$.

Suppose $F^S := \{\mathcal{F}^S(t)|t \in \mathcal{T}\}$ and $F^{\mathbf{X}} := \{\mathcal{F}^{\mathbf{X}}(t)|t \in \mathcal{T}\}$ are the right-continuous, $\mathcal{P}$-completions of the filtrations generated by the price process of the share S and the Markov chain $\mathbf{X}$, respectively. Define, for each $t \in \mathcal{T}$, $\mathcal{G}(t) := \mathcal{F}^S(t) \vee \mathcal{F}^{\mathbf{X}}(t)$ to be the enlarged σ-algebra generated by both $\mathcal{F}^S(t)$ and $\mathcal{F}^{\mathbf{X}}(t)$. Consequently, for each $t \in \mathcal{T}$, $\mathcal{G}(t)$ contains information generated by the share price and observable economic information up to and including time t. Write $G := \{\mathcal{G}(t)|t \in \mathcal{T}\}$. This represents the flow of public information over time.

Define a G-adapted process $\Lambda^\theta := \{\Lambda^\theta(t)|t \in \mathcal{T}\}$ associated with θ by putting

$$\Lambda^\theta(t) := \exp\Big(- \int_0^t \theta(s) dW(s) - \frac{1}{2} \int_0^t \theta^2(s) ds \Big) \ . \tag{1}$$

approach to select a price kernel. This may represent an interesting topic for further research. Both the semi-parametric and non-parametric approaches are more complicated than a parametric approach. For simplicity, we consider a parametric approach here.

Since $|\theta_i(t,\omega)| < K < \infty$, $\forall (t,\omega) \in \mathcal{T} \times \Omega$, it is not difficult to check that the Novikov condition is satisfied. That is, $\{\theta(t)|t \in \mathcal{T}\}$ satisfies the following condition:

$$\mathrm{E}\Big[\exp\Big(\frac{1}{2} \int_0^T |\theta(t)|^2 dt \Big) \Big] < \infty \ .$$

For detail about the Novikov condition, interested reader may refer to Elliott [5], (Chapter 13, therein).

Consequently, Λ^θ is a $(G, \mathcal{P})$-martingale, and

$$\mathrm{E}[\Lambda^\theta(T)] = 1 \ .$$

We now define a density process for a measure change for the Markov chain $\mathbf{X}$. For each $i, j, k = 1, 2, \cdots, N$, we consider a real-valued, F^W-predictable, bounded stochastic process $\{c^k_{ij}(t)|t \in \mathcal{T}\}$ on $(\Omega, \mathcal{F}, \mathcal{P})$ such that for each $t \in \mathcal{T}$,

(1) $c^k_{ij}(t) \geq 0$, for $i \neq j$;
(2) $\sum_{i=1}^N c^k_{ij}(t) = 0$, so $c^k_{ii}(t) \leq 0$.

Let $\mathbf{C}(t) := \{c_{ij}(t)\}_{i,j=1,2,\cdots,N}$, $t \in \mathcal{T}$, be a second family of rate matrices of the chain $\mathbf{X}$ under a new probability measure such that for each $i, j = 1, 2, \cdots, N$,

$$c_{ij}(t) := \langle \mathbf{c}_{ij}(t), \mathbf{X}(t-) \rangle \ , \quad t \in \mathcal{T} \ . \tag{2}$$

Here $\mathbf{c}_{ij}(t) := (c^1_{ij}(t), c^2_{ij}(t), \cdots, c^N_{ij}(t))' \in \Re^N$, where $c^k_{ij}(t)$ is the instantaneous intensity of transition of the chain $\mathbf{X}$ from state j to state i at time t when $\mathbf{X}(t-) = \mathbf{e}_k$. Consequently, $c_{ij}(t)$ depends on $\mathbf{X}(t-)$ only through the scalar product $\langle \cdot, \cdot \rangle$.

Let $\mathbf{C}^k(t) := [c^k_{ij}(t)]_{i,j=1,2,\cdots,N}$, for each $k = 1, 2, \cdots, N$ and each $t \in \mathcal{T}$, where $\mathbf{C}^k(t)$ is the rate matrix of the chain $\mathbf{X}$ at time t when $\mathbf{X}(t-) = \mathbf{e}_k$. Consequently, we can write

$$\mathbf{C}(t) := \sum_{k=1}^N \mathbf{C}^k(t) \, \langle \mathbf{X}(t-), \mathbf{e}_k \rangle \ , \quad t \in \mathcal{T} \ . \tag{3}$$

Now we wish to introduce a new probability measure under which $\mathbf{C}$ is a family of rate matrices of the chain $\mathbf{X}$. The development here follows that of

Dufour and Elliott [4], where a version of Girsanov transform for the Markov chain was adopted.

Define, for each $t \in \mathcal{T}$, the following matrix:

$$\mathbf{D^C}(t) := [c_{ij}(t)/a_{ij}(t)]_{i,j=1,2,\ldots,N} \ .$$

Note that $a_{ij}(t) > 0$, for each $t \in \mathcal{T}$, so $\mathbf{D}(t)$ is well-defined.

For each $t \in \mathcal{T}$, let

$$\mathbf{d^C}(t) := (d^{\mathbf{C}}_{11}(t), d^{\mathbf{C}}_{22}(t), \cdots, d^{\mathbf{C}}_{NN}(t))' \in \Re^N \ .$$

Write, for each $t \in \mathcal{T}$,

$$\mathbf{D^C_0}(t) := \mathbf{D^C}(t) - \mathbf{diag}(\mathbf{d^C}(t)) \ ,$$

where $\mathbf{diag}(\mathbf{y})$ is a diagonal matrix with diagonal elements given by the vector $\mathbf{y}$.

Consider the vector-valued counting process, $\mathbf{N} := \{\mathbf{N}(t) | t \in \mathcal{T}\}$, on $(\Omega, \mathcal{F}, \mathcal{P})$, where for each $t \in \mathcal{T}$, $\mathbf{N}(t) := (N_1(t), N_2(t), \cdots, N_N(t))' \in \Re^N$ and $N_j(t)$ counts the number of jumps of the chain $\mathbf{X}$ to state j up to time t, for each $j = 1, 2, \cdots, N$. Then it is not difficult to check that $\mathbf{N}$ admits the following semi-martingale representation: (See Dufour and Elliott [4])

$$\begin{aligned} \mathbf{N}(t) &= \int_0^t (\mathbf{I} - \mathbf{diag}(\mathbf{X}(u-)))' d\mathbf{X}(u) \\ &= \mathbf{N}(0) + \int_0^t (\mathbf{I} - \mathbf{diag}(\mathbf{X}(u-)))' \mathbf{A}(t)\mathbf{X}(t) dt \\ &\quad + \int_0^t (\mathbf{I} - \mathbf{diag}(\mathbf{X}(u-)))' d\mathbf{M}(t) \ , \quad t \in \mathcal{T} \ . \end{aligned} \tag{4}$$

Here $\mathbf{N}(0) = \mathbf{0}$, the zero vector in $\Re^N$.

The following lemma gives a compensated version of $\mathbf{N}$ under $\mathcal{P}$, which is a martingale associated with $\mathbf{N}$. This result is due to Dufour and Elliott [4] and we recall it here.

Lemma 2.1. *Let* $\mathbf{A}_0(t) := \mathbf{A}(t) - \mathbf{diag}(\mathbf{a}(t))$*, where* $\mathbf{a}(t) := (a_{11}(t), a_{22}(t), \cdots, a_{NN}(t))' \in \Re^N$*, for each* $t \in \mathcal{T}$*. Then the process* $\tilde{\mathbf{N}} := \{\tilde{\mathbf{N}}(t) | t \in \mathcal{T}\}$ *defined by putting*

$$\tilde{\mathbf{N}}(t) := \mathbf{N}(t) - \int_0^t \mathbf{A}_0(u)\mathbf{X}(u-)du \ , \quad t \in \mathcal{T} \ , \tag{5}$$

is an $\Re^N$-valued, $(F^{\mathbf{X}}, \mathcal{P})$-martingale.

Consider the $F^{\mathbf{X}}$-adapted process $\Lambda^{\mathbf{C}} := \{\Lambda^{\mathbf{C}}(t)|t \in \mathcal{T}\}$ on $(\Omega, \mathcal{F}, \mathcal{P})$ associated with $\mathbf{C}$ defined by setting

$$\Lambda^{\mathbf{C}}(t) = 1 + \int_0^t \Lambda^{\mathbf{C}}(u-)[\mathbf{D}_0^{\mathbf{C}}(u)\mathbf{X}(u-) - \mathbf{1}]' d\tilde{\mathbf{N}}(u) \ .$$

Here $\mathbf{1} := (1, 1, \ldots, 1)' \in \Re^N$.

Then the following result is an immediate consequence of Lemma 2.1 and the boundedness of $c_{ij}(t)$, for each $i, j = 1, 2, \cdots, N$ and each $t \in \mathcal{T}$.
Lemma 2.2. $\Lambda^{\mathbf{C}}$ *is an $(F^{\mathbf{X}}, \mathcal{P})$-martingale.*

Let $\mathcal{K}$ be a subspace of the space of rate matrices of the chain $\mathbf{X}$ defined by:

$$\mathcal{K} := \{\mathbf{C}|\Lambda^{\mathbf{C}} \text{ is an } (F^{\mathbf{X}}, \mathcal{P})\text{-martingale}\} \ .$$

Then for each $\mathbf{C} \in \mathcal{K}$, $\Lambda^{\mathbf{C}}$ is used as a density process for a measure change for the chain $\mathbf{X}$.

Consider a G-adapted process $\Lambda^{\theta,\mathbf{C}} := \{\Lambda^{\theta,\mathbf{C}}(t)|t \in \mathcal{T}\}$ on $(\Omega, \mathcal{F}, \mathcal{P})$ defined by:

$$\Lambda^{\theta,\mathbf{C}}(t) := \Lambda^{\theta}(t) \cdot \Lambda^{\mathbf{C}}(t) \ , \quad t \in \mathcal{T} \ .$$

Our assumptions ensure that $\Lambda^{\theta,\mathbf{C}}$ is a $(G, \mathcal{P})$-martingale.

We now define a probability measure $\mathcal{Q}^{\theta,\mathbf{C}}$ absolutely continuous with respect to $\mathcal{P}$ on $\mathcal{G}(T)$ as:

$$\left.\frac{d\mathcal{Q}^{\theta,\mathbf{C}}}{d\mathcal{P}}\right|_{\mathcal{G}(T)} := \Lambda^{\theta,\mathbf{C}}(T) \ . \tag{6}$$

This is a density process for a measure change for both the standard Brownian motion W and the Markov chain $\mathbf{X}$.

The following theorem gives the probability laws of the Brownian motion W and the chain $\mathbf{X}$ under the new measure $\mathcal{Q}^{\theta,\mathbf{C}}$.

Theorem: 2.1. *The process defined by*

$$W^{\theta}(t) := W(t) - \int_{0}^{t} \theta(u)du \ , \quad t \in \mathcal{T} \ ,$$

is a $(G, \mathcal{Q}^{\theta,\mathbf{C}})$*-standard Brownian motion. Under* $\mathcal{Q}^{\theta,\mathbf{C}}$*, the chain* $\mathbf{X}$ *has a family of rate matrices* $\mathbf{C}$ *and can be represented as:*

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_{0}^{t} \mathbf{C}(u)\mathbf{X}(u-)du + \mathbf{M}^{\mathbf{C}}(t) \ ,$$

where $\mathbf{M}^{\mathbf{C}} := \{\mathbf{M}^{\mathbf{C}}(t) | t \in \mathcal{T}\}$ *is an* $\Re^N$*-valued,* $(G, \mathcal{Q}^{\theta,\mathbf{C}})$*-martingale.*

Furthermore, under $\mathcal{Q}^{\theta,\mathbf{C}}$*, the share price process evolves over time as:*

$$\begin{aligned} dS(t) &= (\mu(t) + \sigma(t)\theta(t))S(t)dt + \sigma(t)S(t)dW^{\theta}(t) \ , \quad t \in \mathcal{T} \ , \\ S(0) &= s \ . \end{aligned}$$

Proof. The results follow directly from Girsanov's theorem for the Brownian motion and a Girsanov transform for the Markov chain, (see, Elliott [5] and Dufour and Elliott [4]) □

3 Selection of a Price Kernel and Utility Indifference Approach

In this section, we present a utility indifference approach for the selection of a price kernel and the optimal hedged portfolio for a European-style contingent claim from an issuer's perspective. We first demonstrate that the martingale condition is not sufficient to fix a price kernel if one wishes to price both market risk and economic risk. We then illustrate the use of the utility indifference approach to solve the problem. The central idea of the utility indifference approach is to determine an issuer's price kernel, or price, so that from a perspective of utility maximization, the issuer is indifferent to whether the claim is sold, or not sold. Here we use this approach to determine an issuer's price of the claim. A buyer's price of the claim can be determined in the same way.

Harrison and Kreps [21] and Harrison and Pliska [22], [23] established a fundamental relationship between the absence of arbitrage and the existence of an equivalent martingale measure. This is known as the fundamental theorem of asset pricing. A version of this theorem states that the absence of arbitrage is "essentially" equivalent to the existence of an equivalent martingale measure under which all discounted price processes are martingales. We refer the latter condition to as a martingale condition. In our current context, the martingale condition implies that the discounted share price process $\tilde{S}(t) := \exp(-\int_0^t r(u)du)S(t)$, $t \in \mathcal{T}$, is a martingale $(G, \mathcal{Q}^{\theta,\mathbf{C}})$. That is,

$$\tilde{S}(u) = \mathrm{E}^{\theta,\mathbf{C}}[\tilde{S}(t)|\mathcal{G}(u)] \ , \quad t, u \in \mathcal{T} \ , \quad t \geq u \ . \tag{7}$$

Here $\mathrm{E}^{\theta,\mathbf{C}}$ is expectation under $\mathcal{Q}^{\theta,\mathbf{C}}$.

The following theorem gives a necessary and sufficient condition for the martingale condition.

Theorem: 3.1. *The martingale condition holds if, and only if*

$$\theta(t) = \frac{r(t) - \mu(t)}{\sigma(t)} \ , \quad \forall t \in \mathcal{T} \ . \tag{8}$$

Proof. By Corollary 2.1, under $\mathcal{Q}^{\theta,\mathbf{C}}$,

$$dS(t) = (\mu(t) + \sigma(t)\theta(t))S(t)dt + \sigma(t)S(t)dW^{\theta}(t) \ .$$

Applying Itô's differentiation rule to $\exp(-\int_0^t r(u)du)S(t)$ gives:

$$d\tilde{S}(t) = (\mu(t) - r(t) + \sigma(t)\theta(t))\tilde{S}(t)dt + \sigma(t)\tilde{S}(t)dW^{\theta}(t) \ .$$

This is a $(G, \mathcal{Q}^{\theta,\mathbf{C}})$-martingale if, and only if the drift term vanishes. The result follows. □

From Theorem 2.1 and Corollary 2.1, we see that the martingale condition is not sufficient to determine the family of rate matrices $\mathbf{C}$ under the new probability measure $\mathcal{Q}^{\theta,\mathbf{C}}$. We need an additional condition to determine $\mathbf{C}$. Here we adopt the utility-based indifference approach to select $\mathbf{C}$. From now on, we assume that $\{\theta(t)|t \in \mathcal{T}\}$ satisfies the martingale condition (8).

We consider a situation where the issuer invests his/her wealth in the bond and the share so as to maximize the expected utility of terminal wealth at time T. Firstly, we define a portfolio process of the issuer. For each $t \in \mathcal{T}$, let $\pi(t)$ be the proportion of wealth invested in the share at time t. We assume that $\pi := \{\pi(t)|t \in \mathcal{T}\}$ is G-progressively measurable and cádlág, (i.e., right continuous with left limits). This means that the issuer decides the number of units invested in the share at each instant according to information generated by the share and observable (macro)-economic states just prior that instant. We also assume that π is self-financing, (i.e., there is no income or consumption).

Let $V^{\pi} := \{V^{\pi}(t)|t \in \mathcal{T}\}$ be the wealth process of the issuer who invests according to the portfolio process π. In the following, to simplify the notation, we suppress the superscript π and write $V(t) := V^{\pi}(t)$, for each $t \in \mathcal{T}$, unless stated otherwise. Then it is not difficult to show that under $\mathcal{P}$, the wealth process of the issuer evolves over time as:

$$\begin{aligned} dV(t) &= [r(t) + (\mu(t) - r(t))\pi(t)]V(t)dt + \pi(t)\sigma(t)V(t)dW(t) \ , \\ V(0) &= v \ . \end{aligned} \tag{9}$$

We suppose that the portfolio process π is such that the stochastic differential equation (9) for the wealth process V has a unique strong solution and that

$$\int_0^T \Big[|V(t)||r(t) + (\mu(t) - r(t))\pi(t)| + \sigma^2(t)\pi^2(t)V^2(t)\Big]dt < \infty \ , \quad \mathcal{P}\text{-a.s.}$$

We also impose the technical condition that π satisfies

$$\int_0^T \pi^2(t)dt < \infty \ , \quad \mathcal{P}\text{-a.s.}$$

To preclude "doubling strategies", we may require that the wealth process V is uniformly bounded from below. For details, interested readers may refer to Harrison and Pliska [22]. We write $\mathcal{A}$ for the space of all such admissible portfolio strategies.

Corollary 3.1. *Suppose $\theta(t) = \frac{r(t)-\mu(t)}{\sigma(t)}$, $\forall t \in \mathcal{T}$, (i.e. $\{\theta(t)|t \in \mathcal{T}\}$ satisfies the martingale condition (8)). Then under $\mathcal{Q}^{\theta,\mathcal{C}}$, the dynamics of the share price process, the wealth process of the issuer and the Markov chain are given by:*

$$
\begin{aligned}
dS(t) &= r(t)S(t)dt + \sigma(t)S(t)dW^{\theta}(t)\ , \\
dV(t) &= r(t)V(t)dt + \pi(t)\sigma(t)V(t)dW^{\theta}(t)\ , \\
d\mathbf{X}(t) &= \mathbf{C}(t)\mathbf{X}(t)dt + d\mathbf{M}^{\mathbf{C}}(t)\ , \quad t \in \mathcal{T}\ .
\end{aligned}
$$

Proof. The result follows from Theorem 3.1 and Corollary 2.1. □

For valuing contingent claims, we only need to consider equivalent martingale measures. Consequently, we shall consider a family of probability measures $\mathcal{Q}^{\theta,\mathbf{C}}$ such that θ satisfies the martingale condition (8) and $\mathbf{C} \in \mathcal{K}$. In the sequel, we adopt the utility indifference pricing approach to determine a $\mathbf{C} \in \mathcal{K}$ and a $\pi \in \mathcal{A}$.

Let $U : \Re \to \bar{\Re}$ be a concave, strictly increasing and differentiable utility function, where $\bar{\Re}$ is the extended real line, (i.e. $\bar{\Re} := \Re \cup \{\infty\}$). Consider a general contingent claim with payoff of the form $F(T) := F(S(T), \mathbf{X}(T))$, for some Borel-measurable function $F : \Re_{+}^{2} \to \Re$; that is, the payoff of the claim at the terminal time T depends on both the share price and the economic state at that time.

Suppose the issuer of the claim F has a utility function U. Then conditional on $S(t) = s$, $\mathbf{X}(t) = \mathbf{x}$ and $V(t) = v$, the issuer charges a premium of $P^F := P^F(t, s, v, \mathbf{x})$ at time t for writing the claim F with maturity at time T. We consider the following two scenarios.

Scenario I: Suppose the claim F is sold at time t. Then conditional on $S(t) = s$, $V(t) = v$ and $\mathbf{X}(t) = \mathbf{x}$, the issuer faces the following optimization problem.

$$
\Phi(t, s, v + P^F, \mathbf{x}) := \sup_{\pi \in \mathcal{A}, \mathbf{C} \in \mathcal{K}} \mathrm{E}^{\theta,\mathbf{C}}_{t,s,v,\mathbf{x}}[U(V^{\pi}(T) - F(S(T), \mathbf{X}(T)))]\ . \quad (10)
$$

Here $\mathrm{E}^{\theta,\mathbf{C}}_{t,s,v,\mathbf{x}}$ is a conditional expectation under $\mathcal{Q}^{\theta,\mathbf{C}}$ given $S(t) = s$, $V(t) = v$ and $\mathbf{X}(t) = \mathbf{x}$. Φ is the value function of the optimization problem. In this problem, the objective of the issuer is to choose an optimal hedged strategy and a price kernel so as to maximize the expected utility on terminal wealth.

Scenario II: Suppose the claim F is not sold at time t. Then conditional on $S(t) = s$, $V(t) = v$ and $\mathbf{X}(t) = \mathbf{x}$, the issuer faces the optimization problem.

$$
\bar{\Phi}(t, s, v, \mathbf{x}) := \sup_{\pi \in \mathcal{A}, \mathbf{C} \in \mathcal{K}} \mathrm{E}^{\theta,\mathbf{C}}_{t,s,v,\mathbf{x}}[U(V^{\pi}(T))]\ . \quad (11)
$$

Similarly, the goal of the issuer is then to select an optimal portfolio strategy $\pi \in \mathcal{A}$ that maximizes the expected utility on terminal wealth.

Conditional on $S(t) = s$, $V(t) = v$ and $\mathbf{X}(t) = \mathbf{x}$, the utility indifference price of the issuer for the claim F at time t is defined as the premium P^F, which is a solution, (if it exists), to the following equation:

$$\Phi(t, s, v + P^F, \mathbf{x}) = \bar{\Phi}(t, s, v, \mathbf{x}) \ .$$

To determine a price kernel and an optimal hedging portfolio, we must solve the optimization problem in Scenario I. To find an issuer's utility indifference price of the claim F, we need to find the value functions in Scenario I and Scenario II. The existence of the solutions in the two scenarios depends on (1) the type of utility function U used, (2) the appropriate admissible strategies $\mathcal{A}$ and $\mathcal{K}$, and (3) the integrability conditions on the claim F.

4 A Verification Theorem for the Hamilton-Jacobi-Bellman (HJB) Solution

In this section we first give a verification theorem for the Hamilton-Jacobi-Bellman (HJB) solution of each of the optimization problems which arise in the utility-based indifference valuation. We then derive local conditions for the issuer's utility-based indifference price kernel and optimal hedged portfolio.

4.1 Verification Theorem

Firstly, we need some assumptions for the control processes, namely, the optimal portfolio strategy π and the family of the rate matrices $\mathbf{C}$. Note that the vector-valued process $\{(S(t), V(t), \mathbf{X}(t)) | t \in \mathcal{T}\}$ is Markov with respect to the enlarged filtration G. Under mild conditions, the class of Markov controls will perform as well as the larger class of adapted controls, (see, for example, Øksendal [28]). Elliott [5] noted that if the state processes are Markov, it is not unreasonable to assume that the optimal controls are Markov, so here we consider Markov controls.

Let $\mathcal{O} := (0, T) \times (0, \infty) \times (0, \infty)$ be the solvency region. Let $K_1 \subset \Re$ and $\mathbf{K}_2 \subset \Re^N \otimes \Re^N$ such that $\pi \in K_1$ and $\mathbf{C} \in \mathbf{K}_2$, where $\Re^N \otimes \Re^N$ is the space of all $(N \times N)$ matrices. Here we consider the case that $\mathbf{K}_2$ is a "rectangular" region in the sense that

$$c_{ij}(t) \in [c^-(i,j), c^+(i,j)] \ , \quad i, j = 1, 2, \cdots, N \ , \quad t \in \mathcal{T} \ ,$$

for some given real constants $c^-(i,j)$ and $c^+(i,j)$.

In addition, we impose the following constraints on the lower and upper bounds $c^-(i,j)$ and $c^+(i,j)$.

$$\sum_{i=1}^{N} c^-(i,j) = 0 \ , \quad \sum_{i=1}^{N} c^+(i,j) = 0 \ , \quad i,j = 1,2,\cdots,N \ .$$

These constraints are the same as that for the rate matrix.

Let $\bar{\pi} : \mathcal{O} \times \mathcal{E} \to K_1$ and $\bar{\mathbf{C}} : \mathcal{O} \times \mathcal{E} \to \mathbf{K}_2$ be two functions such that

$$\begin{aligned} \pi(t) &= \bar{\pi}(t, S(t), V(t), \mathbf{X}(t)) \ , \\ \mathbf{C}(t) &= \bar{\mathbf{C}}(t, S(t), V(t), \mathbf{X}(t)) \ . \end{aligned}$$

To simplify, with a slight abuse of notation, hereafter we do not distinguish between π and $\bar{\pi}$, and between $\mathbf{C}$ and $\bar{\mathbf{C}}$. Consequently, we can simplify the control processes with deterministic functions $\pi(t,s,v,\mathbf{x})$ and $\mathbf{C}(t,s,v,\mathbf{x})$, for each $(t,s,v,\mathbf{x}) \in \mathcal{O} \times \mathcal{E}$. We call them feedback controls. Define the following spaces of Markov controls:

$$\begin{aligned} \mathcal{A}_M &:= \{\pi \in \mathcal{A} | \pi \text{ is Markov}\} \ , \\ \mathcal{K}_M &:= \{\mathbf{C} \in \mathcal{K} | \mathbf{C} \text{ is Markov}\} \ . \end{aligned}$$

Suppose $h \in \mathcal{C}^{1,2}(\mathcal{T} \times (\Re^+)^2 \times \mathcal{E})$. Let $h_i(t,s,v) := h(t,s,v,\mathbf{e}_i)$, for each $(t,s,v) \in \mathcal{T} \times (\Re^+)^2$ and each $i = 1,2,\cdots,N$. To simplify the notation, let $h_i := h_i(t,s,v)$, for each $i = 1,2,\cdots,N$, and $h = h(t,s,v,\mathbf{x})$. Write $\mathbf{h} := (h_1, h_2, \cdots, h_N)' \in \Re^N$. Then, for any $(\pi, \mathbf{C}) \in \mathcal{A}_M \times \mathcal{K}_M$, the vector-valued, controlled, state process $(S, V^\pi, \mathbf{X}^{\mathbf{C}})$ is Markov with respect to the enlarged filtration G with generator $\mathcal{L}^{\pi,\mathbf{C}}$ on the function space $\mathcal{C}^{1,2}(\mathcal{T} \times (\Re^+)^2 \times \mathcal{E})$ under the measure $\mathcal{Q}^{\theta,\mathbf{C}}$ defined by setting:

$$\begin{aligned} &\mathcal{L}^{\pi,\mathbf{C}}[h(t,s,v,\mathbf{x})] \\ =& \frac{\partial h}{\partial t} + r(t)s\frac{\partial h}{\partial s} + r(t)v\frac{\partial h}{\partial v} + \frac{1}{2}\sigma^2(t)s^2\frac{\partial^2 h}{\partial s^2} + \frac{1}{2}\sigma^2(t)\pi^2(t)v^2\frac{\partial^2 h}{\partial v^2} \\ &+\sigma^2(t)\pi(t)sv\frac{\partial^2 h}{\partial s \partial v} + \langle \mathbf{h}, \mathbf{C}(t)\mathbf{x}_- \rangle \ , \quad \forall h \in \mathcal{C}^{1,2}(\mathcal{T} \times (\Re^+)^2 \times \mathcal{E}) \ , \end{aligned}$$

where $\mathbf{x}_- := \mathbf{X}(t-)$, for each $t \in \mathcal{T}$.

The following lemma is useful for the verification theorem.

Lemma 4.1. *Suppose τ is a stopping time, where $\tau < \infty$ $\mathcal{P}$-almost surely. Assume further that $h(t,s,v,\mathbf{x})$ and $\mathcal{L}^{\pi,\mathbf{C}}[h(t,s,v,\mathbf{x})]$ are bounded on $t \in [0,t]$. Then*

$$
\begin{aligned}
&E^{\theta,\mathbf{C}}[h(\tau, S(\tau), V(\tau), \mathbf{X}(\tau))] \\
&= h(0, s, v, \mathbf{x}) + E^{\theta,\mathbf{C}}\Big(\int_0^{\tau} \mathcal{L}^{\pi,\mathbf{C}}[h(t, S(t), V(t), \mathbf{X}(t))]dt\Big) \ .
\end{aligned}
$$

Proof. The result follows by applying Itô's differentiation rule to $h(t, S(t), V(t), \mathbf{X}(t))$ and conditioning on $(S(0), V(0), \mathbf{X}(0)) = (s, v, \mathbf{x})$. □

The following theorem gives the verification theorem for the HJB solution to the optimization problem in Scenario I and provides a sufficient condition for the optimality.

Theorem: 4.1. *Let $\bar{\mathcal{O}}$ be the closure of $\mathcal{O}$. Suppose there is a function h such that for each $\mathbf{x} \in \mathcal{E}$, $h(\cdot,\cdot,\cdot,\mathbf{x}) \in \mathcal{C}^2(\mathcal{O}) \cap \mathcal{C}(\mathcal{O})$ and a Markov control $(\hat{\pi}(t), \hat{\mathbf{C}}(t)) \in \mathcal{A}_M \times \mathcal{K}_M$ such that:*

(1) $\mathcal{L}^{\hat{\pi},\hat{\mathbf{C}}}[h(t, s, v, \mathbf{x})] = \sup_{\pi\in K_1,\mathbf{C}\in\mathbf{K_2}} \mathcal{L}^{\pi,\mathbf{C}}[h(t, s, v, \mathbf{x})] = 0$, $\forall (t, s, v, \mathbf{x}) \in \mathcal{O}\times \mathcal{E}$;

(2) for all $(\pi, \mathbf{C}) \in \mathcal{A}_M \times \mathcal{K}_M$,

$$
\lim_{t\to T-} h(t, S(t), V(t), \mathbf{X}(t)) = U(V(T) - F(S(T), \mathbf{X}(T))) \ .
$$

(3) let $\mathcal{S}_T$ be the set of stopping times $\tau \le T$. The family $\{h(\tau, S(\tau), V(\tau), \mathbf{X}(\tau))|\tau \in \mathcal{S}_T\}$ is uniformly integrable.

Write, for each $(\pi, \mathbf{C}) \in \mathcal{A}_M \times \mathcal{K}_M$,

$$
J^{\pi,\mathbf{C}}(t, s, v, \mathbf{x}) = E^{\theta,\mathbf{C}}[U(V(T) - F(S(T), \mathbf{X}(T)))] \ .
$$

Then

$$
h(t, s, v, \mathbf{x}) = \Phi(t, s, v, \mathbf{x}) = J^{\hat{\pi},\hat{\mathbf{C}}}(t, s, v, \mathbf{x}) \ ,
$$

and $(\hat{\pi}, \hat{\mathbf{C}})$ is an optimal Markov control.

Proof. The proof of Theorem 4.1 is adapted from the proof of the verification theorem to the HJB solution to a standard stochastic optimal control problem in Øksendal [28]. However, the partial differential operator used in Øksendal [28] is for a diffusion process. Here we replace the partial differential operator in Øksendal [28] by the one in Lemma 4.1, and all of the steps in the proof of the verification theorem in Øksendal [28] can be followed exactly to prove Theorem 4.1 here. □

The verification theorem for the HJB solution of the optimization problem in Scenario II is the same as that of the Scenario I presented in Theorem 4.1, except that they have different terminal conditions. The terminal condition of the value function of the optimization problem in Scenario II is:

$$\bar{\Phi}(t, s, v, \mathbf{x}) = U(V(T)) \ .$$

4.2 The Local Conditions

We now give the local conditions that characterize a price kernel and an optimal hedged portfolio of the issuer. To determine the price kernel and the optimal hedged portfolio, we must determine $\pi_j(t) := \pi(t, s, v, \mathbf{e}_j)$ and $c_{ij}(t) := c_{ij}(t, s, v, \mathbf{e}_j)$, $i, j = 1, 2, \cdots, N$, for each $t \in \mathcal{T}$. We only need to solve the optimization problem in Scenario I to determine the price kernel and the optimal hedged portfolio. To find an issuer's utility indifference price of the claim, we have to solve the optimization problem in Scenario II as well and the issuer's price is then determined by the value functions of the optimization problems in the two scenarios.

Firstly, we fix some notation. For each $i = 1, 2, \cdots, N$, let $\Phi(i) := \Phi(t, s, v, \mathbf{e}_i)$ and $\bar{\Phi}(i) := \bar{\Phi}(t, s, v, \mathbf{e}_i)$. Write $\mathbf{\Phi} := (\Phi(1), \Phi(2), \cdots, \Phi(N))' \in \Re^N$ and $\bar{\mathbf{\Phi}} := (\bar{\Phi}(1), \bar{\Phi}(2), \cdots, \bar{\Phi}(N))' \in \Re^N$. For each $i = 1, 2, \cdots, N$, let

$$\begin{aligned} &\Phi_t(i) := \frac{\partial \Phi(i)}{\partial t} \ , \quad \Phi_s(i) := \frac{\partial \Phi(i)}{\partial s} \ , \quad \Phi_v(i) := \frac{\partial \Phi(i)}{\partial v} \\ &\Phi_{ss}(i) := \frac{\partial^2 \Phi(i)}{\partial s^2} \ , \quad \Phi_{sv}(i) := \frac{\partial^2 \Phi(i)}{\partial s \partial v} \ , \quad \Phi_{vv}(i) := \frac{\partial^2 \Phi(i)}{\partial v^2} \ . \end{aligned}$$

Similarly, we define the corresponding notation for the first-order and the second-order partial derivatives of $\bar{\Phi}(i)$, $i = 1, 2, \cdots, N$, with respect to the variables t, s and v.

The following theorem gives a system of equations which are satisfied by the unknown variables $\pi_j(t)$ and $c_{ij}(t)$, for $i, j = 1, 2, \cdots, N$.

Theorem: 4.2. *For each $j = 1, 2, \cdots, N$, the optimal hedged portfolio is given by:*

$$\hat{\pi}_j(t) = -\frac{\Phi_{sv}(j)}{\Phi_{vv}(j)} \ , \quad t \in \mathcal{T} \ .$$

The price kernel is determined by solving the following system of N linear dynamic programming problems indexed by $j = 1, 2, \cdots, N$:

$$\max_{c_{1j}(t),c_{2j}(t),\cdots,c_{Nj}(t)} \sum_{i=1}^{N} \Phi(i)c_{ij}(t) \ ,$$

subject to the linear constraint:

$$\sum_{i=1}^{N} c_{ij}(t) = 0 \ ,$$

and the inequality constraints:

$$\begin{aligned} c_{1j}(t) &\in [c^-(1,j), c^+(1,j)] \ , \\ c_{2j}(t) &\in [c^-(2,j), c^+(2,j)] \ , \\ &\vdots \\ c_{Nj}(t) &\in [c^-(N,j), c^+(N,j)] \ , \end{aligned}$$

for $j = 1, 2, \cdots, N$.

Proof. Firstly, note that the partial differential operator $\mathcal{L}^{\pi,\mathbf{C}}$ acting on Φ is equivalent to the following system of partial differential operators $\mathcal{L}_j^{\pi,\mathbf{C}}$ acting on $\Phi(j)$.

$$\begin{aligned} &\mathcal{L}_j^{\hat{\pi},\hat{\mathbf{C}}}[\Phi(j)] \\ &= \Phi_t(j) + r_j s \Phi_s(j) + r_j v \Phi_v(j) + \frac{1}{2}\sigma_j^2 s^2 \Phi_{ss}(j) + \frac{1}{2}\sigma_j^2 \hat{\pi}_j^2 v^2 \Phi_{vv}(j) \\ &\quad + \sigma_j^2 \hat{\pi}_j s v \Phi_{sv}(j) + \sum_{i=1}^{N} \Phi(j)\hat{c}_{ij}(t) \ , \quad j = 1, 2, \cdots, N \ . \end{aligned}$$

The first-order condition of maximizing $\mathcal{L}_j^{\pi,\mathbf{C}}[h(j)]$ with respect to π in Theorem 4.1 gives:

$$\hat{\pi}_j(t) = -\frac{\Phi_{sv}(j)s}{\Phi_{vv}(j)v} \ , \quad t \in \mathcal{T} \ .$$

We also note that the maximization of $\mathcal{L}_j^{\pi,\mathbf{C}}[h(j)]$ with respect to $\mathbf{C}$ is equivalent to the following system of maximization problems:

$$\max_{c_{1j}(t),c_{2j}(t),\cdots,c_{Nj}(t)} \sum_{i=1}^{N} \Phi(i)c_{ij}(t) \ , \quad j = 1, 2, \cdots, N \ ,$$

since the sum $\sum_{i=1}^{N} \Phi(i) c_{ij}(t)$ is the only part of $\mathcal{L}_j^{\pi,\mathbf{C}}[h(j)]$ that depends on $\mathbf{C}$.

The linear constraint

$$\sum_{i=1}^{N} c_{ij}(t) = 0 \ ,$$

comes from the property of a rate matrix and the "interval" constraints are attributed to the rectangularity of $\mathbf{K}_2$. $\square$

The optimization problem in Scenario II can be solved similarly. Then, the issuer's utility indifference price is given by the solution to the following equation:

$$\Phi(t, s, v + P^F, \mathbf{x}) = \bar{\Phi}(t, s, v, \mathbf{x}) \ .$$

Suppose the Markov chain $\mathbf{X}$ has two states, (i.e. $N = 2$). State $\mathbf{e}_1$ and State "$\mathbf{e}_2$" represent a "Good" economy and a "Bad" economy, respectively. In this case, we have the following two linear programming problems indexed by $j = 1, 2$:

$$\max_{c_{1j}(t), c_{2j}(t)} [\Phi(1) c_{1j}(t) + \Phi(2) c_{2j}(t)] \ ,$$

subject to the constraints:

$$\begin{aligned} & c_{1j}(t) + c_{2j}(t) = 0 \ , \\ & c_{1j}(t) \in [c^-(1, j), c^+(1, j)] \ . \end{aligned}$$

Solving then gives:

$$\begin{aligned} \widehat{c}_{1j}(t) &= c^+(1, j) I_{\{\Phi(1) - \Phi(2) > 0\}} + c^-(1, j) I_{\{\Phi(1) - \Phi(2) < 0\}} \\ \widehat{c}_{2j}(t) &= -\widehat{c}_{1j}(t) \ , \quad t \in \mathcal{T} \ , \quad j = 1, 2 \ . \end{aligned}$$

In the case when the Markov chain has three states, (i.e., "Good", "Medium" and "Bad" economic situations), we have the following three linear programming problems indexed by $j = 1, 2, 3$:

$$\max_{c_{1j}(t), c_{2j}(t), c_{3j}(t)} [\Phi(1) c_{1j}(t) + \Phi(2) c_{2j}(t) + \Phi(3) c_{3j}(t)] \ ,$$

subject to the constraints:

$$
\begin{aligned}
&c_{1j}(t) + c_{2j}(t) + c_{3j}(t) = 0 \ , \\
&c_{1j}(t) \in [c^-(1,j), c^+(1,j)] \ , \\
&c_{2j}(t) \in [c^-(2,j), c^+(2,j)] \ .
\end{aligned}
$$

These three programming problems can be simplified to the following three linear programming problems indexed by $j = 1, 2, 3$:

$$
\max_{c_{1j}(t), c_{2j}(t)} [(\Phi(1) - \Phi(3))c_{1j}(t) + (\Phi(2) - \Phi(3))c_{2j}(t)] \ ,
$$

subject to the constraints:

$$
\begin{aligned}
&c_{1j}(t) \in [c^-(1,j), c^+(1,j)] \ , \\
&c_{2j}(t) \in [c^-(2,j), c^+(2,j)] \ .
\end{aligned}
$$

Solving the three linear programming problems gives:

$$
\begin{aligned}
\widehat{c}_{1j}(t) = {} & c^+(1,j) I_{\{\Phi(1)>\Phi(3), \Phi(2)>\Phi(3)\}} + c^+(1,j) I_{\{\Phi(1)>\Phi(3), \Phi(2)<\Phi(3)\}} \\
& + c^-(1,j) I_{\{\Phi(1)<\Phi(3), \Phi(2)>\Phi(3)\}} + c^-(1,j) I_{\{\Phi(1)<\Phi(3), \Phi(2)<\Phi(3)\}} \ , \\
\widehat{c}_{2j}(t) = {} & c^+(2,j) I_{\{\Phi(1)>\Phi(3), \Phi(2)>\Phi(3)\}} + c^-(2,j) I_{\{\Phi(1)>\Phi(3), \Phi(2)<\Phi(3)\}} \\
& + c^+(2,j) I_{\{\Phi(1)<\Phi(3), \Phi(2)>\Phi(3)\}} + c^-(2,j) I_{\{\Phi(1)<\Phi(3), \Phi(2)<\Phi(3)\}} \ , \\
\widehat{c}_{3j}(t) = {} & -(c^+(1,j) + c^+(2,j)) I_{\{\Phi(1)>\Phi(3), \Phi(2)>\Phi(3)\}} \\
& -(c^+(1,j) + c^-(2,j)) I_{\{\Phi(1)>\Phi(3), \Phi(2)<\Phi(3)\}} \\
& -(c^-(1,j) + c^+(2,j)) I_{\{\Phi(1)<\Phi(3), \Phi(2)>\Phi(3)\}} \\
& -(c^-(1,j) + c^-(2,j)) I_{\{\Phi(1)<\Phi(3), \Phi(2)<\Phi(3)\}} \ , \quad t \in \mathcal{T} \ , \quad j = 1, 2, 3 \ .
\end{aligned}
$$

To find the price kernel, the optimal hedging strategy and the utility indifference price, we must determine the value functions $\Phi(i)$ and $\bar{\Phi}(i)$, $i = 1, 2, \cdots, N$, in Scenario I and Scenario II. However, even for some paramatric forms of the utility function, say a power utility and an exponential utility, it is very difficult, if not impossible, to obtain analytical forms for $\Phi(i)$ and $\bar{\Phi}(i)$. Consequently, one may resort to some numerical approximation methods such as finite difference method to approximate the solutions of the partial differential equations for the value functions $\Phi(i)$ and $\bar{\Phi}(i)$.

5 Conclusion

We investigated utility-based indifference pricing and hedging of contingent claims in continuous-time, Markov, regime-switching models from an issuer's perspective. A parametric form of the price kernels was introduced based on the product of two density processes so that both market risk and economic risk are priced. We illustrated the use of a Girsanov transform for the Markov chain to price economic risk, or regime-switching risk. The valuation and hedging problem was formulated as a stochastic optimal control problem. A verification theorem for the HJB solution to the problem was provided. It was shown that the determination of a price kernel can be formulated as linear programming problems. We solve the linear programming problems and give the price kernel in the case that the economy has two regimes or three regimes.

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