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Some Singular Singularly Perturbed Problems

by

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Abstract

In 1994, Shi[16] studied a class of vector singular singularly perturbed boundary-value problems consisting of

$$P_\varepsilon \begin{cases} \dfrac{dx}{dt} = U(t,\ x,\ y,\ \varepsilon) \\ \varepsilon^2 \dfrac{dy}{dt} = V(t,\ x,\ \varepsilon^2 y,\ \varepsilon) \\ H_1(x(0,\varepsilon),\ y(0,\varepsilon),\ \varepsilon) = 0 \\ H_2(x(1,\varepsilon),\ y(1,\varepsilon),\ \varepsilon) = 0, \end{cases} \qquad (0 \le t \le 1)$$

where U, V, H_1 and H_2 are n-dimensional real-valued functions and infinitely differentiable with respect to their variables respectively and ε is a small positive parameter.

In this thesis, we study the following singular singularly perturbed boundary-value problems consisting of

$$P_\varepsilon \begin{cases} \dfrac{dx}{dt} = U(t,\ x,\ y,\ \varepsilon) \\ \varepsilon^2 \dfrac{dy}{dt} = V(t,\ x,\ \varepsilon y,\ \varepsilon) \\ H_1(x(0,\varepsilon),\ y(0,\varepsilon),\ \varepsilon) = 0 \\ H_2(x(1,\varepsilon),\ y(1,\varepsilon),\ \varepsilon) = 0, \end{cases} \qquad (0 \le t \le 1)$$

where U, V, H_1 and H_2 are scalar real-valued functions and infinitely differentiable with respect to their variables respectively. This problem extends Shi's problem for the scalar case. However, the vector case remains open.

Under appropriate assumptions and employing the method of matched asymptotic expansions, we construct an outer solution followed by appropriate left boundary layer corrections and right boundary layer corrections. Then for sufficiently small

$\varepsilon > 0$, we obtain a uniformly valid asymptotic solution, which consists of the outer solution and the left and right boundary layer corrections.

To illustrate our new results, we provide an example at the end of the thesis.

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Chapter 1

Introduction

In this thesis we study a class of singular pertubation problems. Some preliminary concepts and theorems are first given.

1.1 Singular perturbation problems

The term "*perturbation problem*" is generally used in mathematics when one deals with the following situation: There is a family of problems depending on a small parameter $\varepsilon > 0$, which we denote by P_ε. When $\varepsilon = 0$, we have the reduced problem P_0. We want to study the relationship between the solution of P_ε and the solution of P_0 under appropriate assumptions.

The perturbation problem P_ε may consist of an ordinary differential equation, or a system of differential equations, along with some given conditions, such as initial or boundary conditions. Thus, problem P_ε can, in general, be written in the form

$$P_\varepsilon \begin{cases} \dfrac{dy}{dt} = f(t, y, \varepsilon), \\ \text{appropriate initial or boundary or mixed conditions} \end{cases} \tag{1.1}$$

where y and f are $n-$dimensional vector functions, t is a scalar variable in a given interval.

A pertubation problem (1.1) is called a regular perturbation problem if, as $\varepsilon \to 0$, its solution $y_\varepsilon(t)$ converges to the solution $y_0(t)$ of the reduced problem uniformly with respect to the independent variable t in the entire interval.

We can call it a singular perturbation problem if $y_\varepsilon(t)$ converges to $y_0(t)$ only in some interval of t, but not throughout the entire interval, thus giving rise to an "*initial layer*" phenomenon at an initial point or "*boundary layers*" phenomena at both end-points.

We give three examples to illustrate the situation.

Example 1. Consider the perturbation problem

$$P_\varepsilon \begin{cases} \dfrac{dy}{dt} - \varepsilon y = 0 \\ y_\varepsilon(0) = 1, \qquad t \in [0,1] \end{cases} \tag{1.2}$$

where $\varepsilon > 0$ is a small parameter.

This problem has the unique solution

$$y_\varepsilon = e^{\varepsilon t}.$$

On the other hand, the reduced problem

$$P_0 \begin{cases} \dfrac{dy}{dt} = 0 \\ y_0(0) = 1, \qquad t \in [0,1] \end{cases}$$

has the solution $y_0(t) = 1$. Since

$$y_\varepsilon = e^{\varepsilon t} \to y_0(t) = 1 \qquad \text{as } \varepsilon \to 0$$

uniformly in [0,1], we conclude that this problem is a regular perturbation problem.

Example 2: The perturbation problem

$$P_\varepsilon \begin{cases} \varepsilon \dfrac{dy}{dt} + y = 0 \\ y_\varepsilon(0) = 1, \qquad t \in [0,1] \end{cases}$$

is a singular perturbation problem, for, as $\varepsilon \to 0$, the unique solution

$$y_\varepsilon = e^{-t/\varepsilon}$$

converges to $y_0 = 0$ in $(0, 1]$, but does not converge to $y_0 = 0$ uniformly in $[0, 1]$. The term $e^{-t/\varepsilon}$ is the "*initial layer*" at the initial point $t = 0$.

The following is an example of singular perturbed boundary-value problem.

Example 3: The perturbation problem

$$P_\varepsilon \begin{cases} \varepsilon^2 \dfrac{d^2y}{dt^2} - y + 1 = 0, & t \in [0,1], \\ y(0) = 0, \qquad y(1) = 2. \end{cases} \tag{1.3}$$

The unique solution is

$$y(t,\varepsilon) = 1 + \frac{-1 - e^{-1/\varepsilon}}{(1 - e^{-2/\varepsilon})} e^{-t/\varepsilon} + \frac{1 + e^{-1/\varepsilon}}{(1 - e^{-2/\varepsilon})} e^{-(1-t)/\varepsilon} \tag{1.4}$$

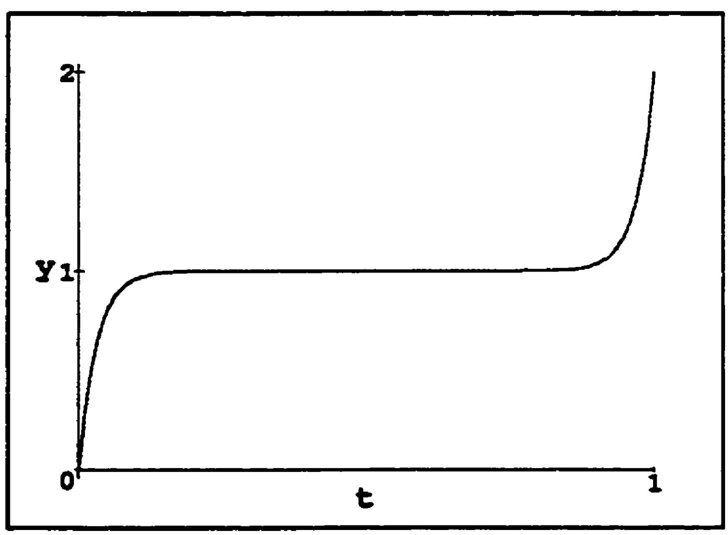

Figure 1.1: The solution $y(t, \varepsilon)$ for $\varepsilon = 0.03$

The solution of reduced problem is

$$y_0 = 1. \tag{1.5}$$

As $\varepsilon \to 0$, the solution (1.4) converges, uniformly on the interval $[\delta,1-\delta]$ ($0< \delta <1/2$), to the solution (1.5) but does not converge, uniformly on the interval [0,1], to the solution (1.5). The nonuniform convergence takes place near the two endpoints, $t=0$ and $t=1$[cf. Figure 1.1]. We call these two areas the left "*boundary layer*" and the right "*boundary layer*" respectively. The left boundary layer correction is

$$\frac{-1-e^{-1/\varepsilon}}{(1-e^{-2/\varepsilon})}e^{-t/\varepsilon},$$

and the right boundary layer correction is

$$\frac{1+e^{-1/\varepsilon}}{(1-e^{-2/\varepsilon})}e^{-(1-t)/\varepsilon}.$$

Singular perturbation problems P_ε involving the system

$$\begin{cases} \dfrac{dx}{dt} = U(t,\ x,\ y,\ \varepsilon) \\ \varepsilon\dfrac{dy}{dt} = V(t,\ x,\ y,\ \varepsilon) \end{cases} \tag{1.6}$$

and subject to given initial or boundary conditions have been studied extensively by many authors (cf. Hoppensteadt[10], O'Malley[14] and Smith[18]).

If the reduced system of (1.6)

$$\begin{cases} \dfrac{dx_0}{dt} = U(t,\ x_0,\ y_0,\ 0) \\ 0 = V(t,\ x_0, y_0,\ 0) \end{cases}$$

has a solution $(x_0(t), y_0(t))$ and if all eigenvalues of $V_y(t,\ x_0, y_0,\ 0)$ have either a positive or negative real part through the entire interval, we call the problem P_ε a "*regular singularly perturbed problem*". If the matrix $V_y(t,\ x_0(t), y_0(t),\ 0)$ is singular for some t, we call the problem P_ε a "*singular singularly perturbed problem*" (cf. O'Malley[14]).

Shi[16] studied the following singular singularly perturbed boundary-value problem with vector functions U, V, H_1 and H_2, which we will elaborate in Chapter 2.

$$P_\varepsilon \begin{cases} \dfrac{dx}{dt} = U(t,\ x,\ y,\ \varepsilon) \\ \varepsilon^2 \dfrac{dy}{dt} = V(t,\ x,\ \varepsilon^2 y,\ \varepsilon) \\ H_1(x(0,\varepsilon),\ y(0,\varepsilon),\ \varepsilon) = 0 \\ H_2(x(1,\varepsilon),\ y(1,\varepsilon),\ \varepsilon) = 0. \end{cases} \qquad (0 \le t \le 1) \qquad (1.7)$$

In Chapter 3, we will study the following singular singularly perturbed boundary-value problem with scalar functions.

$$P_\varepsilon \begin{cases} \dfrac{dx}{dt} = U(t,\ x,\ y,\ \varepsilon) \\ \varepsilon^2 \dfrac{dy}{dt} = V(t,\ x,\ \varepsilon y,\ \varepsilon) \\ H_1(x(0,\varepsilon),\ y(0,\varepsilon),\ \varepsilon) = 0 \\ H_2(x(1,\varepsilon),\ y(1,\varepsilon),\ \varepsilon) = 0. \end{cases} \qquad (0 \le t \le 1) \qquad (1.8)$$

The problem with vector functions is still open.

1.2 Asymptotic series solution

We will use the two ordering symbols o and O.

Let $f(\varepsilon)$ and $g(\varepsilon)$ be two scalar functions with a small parameter $\varepsilon > 0$. If $|f(\varepsilon)/g(\varepsilon)| \to 0$ as $\varepsilon \to 0$, we write

$$f = o(g).$$

If $|f(\varepsilon)/g(\varepsilon)|$ is bounded as $\varepsilon \to 0$, then we write

$$f = O(g).$$

In general, the exact solution for singular perturbation problems cannot be found, so, our main aim is to find an approximation solution with a certain accuracy for a singular perturbation problem. We seek an asymptotic power series solution for singular perturbation problem.

Definition. A function $f(\varepsilon)$ is said to have the asymptotic power series expansion

$$f(\varepsilon) \sim \sum_{i=0}^{\infty} f_i \varepsilon^i \quad \text{as } \varepsilon \to 0$$

if , for any integer $N \geq 0$,

$$\frac{1}{\varepsilon^N}(f(\varepsilon) - \sum_{i=0}^{N} f_i \varepsilon^i) \to 0 \qquad \text{as } \varepsilon \to 0.$$

If, we have the somewhat stronger result that

$$\frac{1}{\varepsilon^{N+1}}(f(\varepsilon) - \sum_{i=0}^{N} f_i \varepsilon^i)$$

is bounded as $\varepsilon \to 0$, then we will write this as

$$f(\varepsilon) = \sum_{i=0}^{N} f_i \varepsilon^i + O(\varepsilon^{N+1}) \qquad \text{as } \varepsilon \to 0.$$

Consider the function

$$f(\varepsilon) = \frac{1}{1-\varepsilon}$$

Clearly, it has the asymptotic power series expansion as $\varepsilon \to 0$

$$f(\varepsilon) \sim (1 + \varepsilon + \varepsilon^2 + \varepsilon^3 + \cdots).$$

Furthermore, we can write

$$f(\varepsilon) = \sum_{i=0}^{N} \varepsilon^i + O(\varepsilon^{N+1}) \qquad \text{as } \varepsilon \to 0.$$

A convergent power series should be an asymptotic power series, but in some cases, an asymptotic power series may not be a convergent power series.

Consider the exponential integral

$$E_i(x) = \int_{-\infty}^{x} e^t t^{-1} dt, \qquad x < 0.$$

Successive integrations by parts show

$$E_i(x) = e^x x^{-1}[1 + x^{-1} + 2!x^{-2} + \cdots + n!x^{-n} + R_n(x)]$$

where

$$R_n(x) = (n+1)!x \int_{-\infty}^{x} e^{t-x} t^{-n-2} dt$$

for any nonnegative n.

We have

$$\begin{aligned} |R_n(x)| &= |[(n+1)!xe^{t-x}t^{-n-2}]_{-\infty}^{x} + (n+2)!x \textstyle\int_{-\infty}^{x} e^{t-x}t^{-n-3}dt| \\ &\leq (n+1)!|x|^{-n-1} + (n+2)!|x| \times |\textstyle\int_{-\infty}^{x} t^{-n-3}dt| \\ &\leq 2(n+1)!|x|^{-n-1}, \qquad x < 0. \end{aligned}$$

So

$$R_n(x) = O(x^{-n-1})$$

Let

$$x = \frac{-1}{\varepsilon}, \qquad \varepsilon > 0.$$

Define

$$f(\varepsilon) = E_i(x)e^{-x}x.$$

Clearly

$$f(\varepsilon) \sim [1 - \varepsilon + 2!\varepsilon^2 + \cdots + (-1)^n n!\varepsilon^n + \cdots], \qquad as\ \varepsilon \to 0.$$

However, the series $[1-\varepsilon+2!\varepsilon^2+\cdots+(-1)^n n!\varepsilon^n+\cdots]$ is not convergent for $\varepsilon>0$.

Since the exact solution for singular perturbation problems in general cannot be found, we try to seek an asymptotic power series solution of the form

$$y(t,\varepsilon) \sim \sum_{i=0}^{\infty} y_i(t)\varepsilon^i, \tag{1.9}$$

for a corresponding singular perturbation problem P_ε. However, (1.9) is not usually valid. The failure takes places where the solution of P_ε does not converge to the solution of P_0, as $\varepsilon \to 0$. We need to make some corrections. For (1.7) or (1.8), we anticipate the solution to have the form

$$\begin{cases} x(t,\varepsilon)=X(t,\varepsilon)+\varepsilon\overline{X}(\tau,\varepsilon)+\varepsilon\widehat{X}(\sigma,\varepsilon) \\ y(t,\varepsilon)=Y(t,\varepsilon)+\overline{Y}(\tau,\varepsilon)+\widehat{Y}(\sigma,\varepsilon) \end{cases}$$

where

$$\begin{pmatrix} X(t,\varepsilon) \\ Y(t,\varepsilon) \end{pmatrix} \sim \sum_{i=0}^{\infty} \begin{pmatrix} x_i(t) \\ y_i(t) \end{pmatrix} \varepsilon^i, \tag{1.10}$$

and

$$\begin{pmatrix} \overline{X}(\tau,\varepsilon) \\ \overline{Y}(\tau,\varepsilon) \end{pmatrix} \sim \sum_{i=0}^{\infty} \begin{pmatrix} \overline{x}_i(\tau) \\ \overline{y}_i(\tau) \end{pmatrix} \varepsilon^i, \tag{1.11}$$

and

$$\begin{pmatrix} \widehat{X}(\sigma,\varepsilon) \\ \widehat{Y}(\sigma,\varepsilon) \end{pmatrix} \sim \sum_{i=0}^{\infty} \begin{pmatrix} \widehat{x}_i(\sigma) \\ \widehat{y}_i(\sigma) \end{pmatrix} \varepsilon^i. \tag{1.12}$$

Here

$$\tau=\frac{t}{\varepsilon} \tag{1.13}$$

is the left "*stretched variable*" for the left "*boundary layer*" near $t=0$, and

$$\sigma=\frac{1-t}{\varepsilon}. \tag{1.14}$$

is the right "*stretched variable*" for the right "*boundary layer*" near $t = 1$.

Furthmore, we require that

$$\overline{X}(\tau, \varepsilon) \to 0, \overline{Y}(\tau, \varepsilon) \to 0, \qquad \text{as } \tau \to +\infty$$

and

$$\widehat{X}(\sigma, \varepsilon) \to 0, \widehat{Y}(\sigma, \varepsilon) \to 0, \qquad \text{as } \sigma \to +\infty.$$

We can use the method of matched asymptotic expansion to find asymptotic solutions (1.10), (1.11) and (1.12) for the problem (1.7) or (1.8). In Chapters 2 and 3, we will elaborate how to use this method to find the asymptotic solution under appropriate conditions.

We refer to O'Malley[13](pp15-17) for a brief history and references for the method of matched asymptotic expansion.

1.3 Preliminary Theorems

Let S be a Banach space, let k be a positive number, and let B_k denote the closed ball in S of radius k centered at the origin

$$B_k = \{s \in S \mid \|s\| \leq k\} \tag{1.15}$$

We have the following theorem. (cf. Smith[18])

Theorem 1.1 (Banach/Picard fixed-point theorem): *Let T map the closed ball B_k of* (1.15) *into itself; namely $Ts \in B_k$, for all $s \in B_k$. And let T be a contraction map on B_k*; namely, $\|Ts_1 - Ts_2\| \leq \gamma \|s_1 - s_2\|$ *for all $s_1, s_2 \in B_k$, for some fixed positive constant $\gamma < 1$. Then there exists a unique element s in B_k such that*

$$Ts = s.$$

This theorem will be used to prove the existence of the solution of singular perturbation problem.

We group together some fundamental results which we will use in Chapters 2 and 3.

Consider the nonhomogeneous differential equation

$$\frac{dx}{dt} = A(t)x + f(t) \qquad t \in [t_1, t_2] \tag{1.16}$$

and the boundary condition

$$Lx(t_1) + Rx(t_2) = \alpha \tag{1.17}$$

where $A(t)$ is a $n \times n$ matrix-valued function, L and R are given constant $n \times n$ matrices, x is a n-dimensional real vector-valued function, $f(t)$ is a known n-dimension real vector-valued function, α is a given constant vector of dimension n.

The homogeneous part of (1.16) is

$$\frac{dx}{dt} = A(t)x \qquad t \in [t_1, t_2] \tag{1.18}$$

A fundmental matrix solution $X(t)$ for (1.18) is a $n \times n$ order real nonsingular matrix-valued function satisfying

$$\frac{dX}{dt} = A(t)X \qquad t \in [t_1, t_2]. \tag{1.19}$$

Theorem 1.2: *Let $X(t)$ be a fundmental matrix solution for* (1.18), *then* the *problem* (1.16) *and* (1.17) *has the unique solution if and only if the matrix*

$$M = LX(t_1) + RX(t_2) \tag{1.20}$$

is nonsingular. The unique solution $x(t)$ of (1.16) is given by

$$x(t) = X(t)M^{-1}\alpha + \int_{t_1}^{t_2} G(t,s)f(s)ds \tag{1.21}$$

where the Green function $G = G(t,s)$ is the matrix-valued function

$$G(t,s) = \begin{cases} X(t)M^{-1}LX(t_1)X(s)^{-1} & \text{for } t > s \\ -X(t)M^{-1}RX(t_2)X(s)^{-1} & \text{for } t \leq s \end{cases} \tag{1.22}$$

Theorem 1.2 can be found in Smith[18](pp3-4).

Lemma 1.1: *Let $h(t,y,y')$ be a continuous n-dimensional real vector-valued function on $[0,+\infty) \times R^n \times R^n$ and let $\alpha \in R^n$ be a given vector. Assume that for the problem*

$$\begin{cases} \dfrac{d^2y}{dt^2} = h(t,y,y'), \ t \in [0,+\infty) \\ y(0) = \alpha, \qquad y(+\infty) = 0 \end{cases} \tag{1.23}$$

there exists a nonnegative function $r(t) \in C^2[0,+\infty]$ satisfying

(i) $r(0) \geq \|\alpha\|$, $r(+\infty) = 0$.

(ii) $r'' \leq y^T h(t,y,y')/\|y\|$ whenever $r(t) = \|y(t)\|$ and $r' = y^T y'/\|y\|$, $\|\cdot\|$ is the Euclidean Norm.

(iii) $h(t,y,y')$ satisfies the Nagumo condition on the domain $D = \{(t,y) |\ \|y\| \leq r(t), t \in [0,+\infty)\}$; in other words, there exists a positive nondecreasing and continous function φ on $[0,+\infty)$ such that

$$\|h(t,y,z)\| \leq \varphi(\|z\|), \ (t,y) \in D, \ z \in R^n$$

and

$$\lim_{s \to +\infty} \frac{s^2}{\varphi(s)} = +\infty.$$

Then problem (1.23) *has a solution* $y = y(t) \in C^2[0, +\infty)$ *such that*

$$\|y(t)\| \le r(t) \text{ and } \|y'(t)\| \le M, \ t \in [0, +\infty).$$

where M *is a certain positive constant depending only on* φ *and* r.

The proof for Lemma 1.1 can be found in Shi[16].

Lemma 1.2: *Let* $a(t), b(t)$ *and* $f(t)$ *be real-valued functions. If* $a(t)$ *satisfies*

$$a(t) \ge n_0^2, \qquad n_0 \text{ is a positive constant.}$$

for $t \in [0, +\infty)$ *and the problem*

$$\begin{cases} \dfrac{d^2y}{dt^2} = b(t)\dfrac{dy}{dt} + a(t)y + f(t) \\ y(0) = \alpha, \qquad y(+\infty) = \beta \end{cases} \tag{1.24}$$

has a solution $y = y(t) \in C^2[0, +\infty)$ *for constants* $\alpha, \beta \in R$, *then the solution* $y = y(t)$ *is the unique solution of problem (1.24).*

Proof: It is equivalent to showing that

$$\begin{cases} \dfrac{d^2y}{dt^2} = b(t)\dfrac{dy}{dt} + a(t)y \\ y(0) = 0, \qquad y(+\infty) = 0 \end{cases} \tag{1.25}$$

has only the zero solution. Suppose the contrary. Then the problem (1.25) has a non-zero solution $y = y(t) \in C^2[0, +\infty)$. Let $w(t) = y(t)^2$. Then $w(t) \in C^2[0, +\infty)$ and $w(t)$ is not everywhere equal to zero.

From

$$w(0) = 0, \qquad w(+\infty) = 0,$$

we know that $w(t)$ has a positive maximum at some point $t_0 \in (0, +\infty)$.

Thus

$$w(t_0) > 0, \qquad w'(t_0) = 0, \qquad w''(t_0) \le 0.$$

Since $w'(t_0) = 0$ and $y(t_0) \neq 0$, we have

$$y'(t_0) = 0.$$

However

$$\begin{aligned} w''(t_0) &= 2[y(t_0)y''(t_0) + y'(t_0)^2] \\ &= 2y(t_0)[b(t_0)y'(t_0) + a(t_0)y(t_0)] \\ &= 2a(t_0)y(t_0)^2 \\ &> 0. \end{aligned}$$

We reach a contradiction. Therefore, the problem (1.25) has only the zero solution.

Chapter 2

A singular singularly perturbed problem

In 1994, Shi[16] studied the following singular singularly perturbed boundary-value problem P_ε consisting of the differential equations

$$\begin{cases} \dfrac{dx}{dt} = U(t,x,y,\varepsilon) \\ \varepsilon^2 \dfrac{dy}{dt} = V(t,x,\varepsilon^2 y,\varepsilon) \end{cases} \tag{2.1}$$

and the boundary conditions

$$\begin{cases} H_1(x(0,\varepsilon), y(0,\varepsilon), \varepsilon) = 0 \\ H_2(x(1,\varepsilon), y(1,\varepsilon), \varepsilon) = 0 \end{cases} \tag{2.2}$$

where $t \in [0,1]$, $0 < \varepsilon \le \varepsilon_0$, U, V, H_1 and H_2 are n-dimensional real-valued functions and infinitely differentiable with respect to their variables respectively.

He made the following 4 assumptions.

Assumption 1: The reduced problem

$$\begin{cases} 0 = V(t, x_0(t), 0, 0) \\ x_0'(t) = U(t, x_0(t), y_0(t), 0) \end{cases} \tag{2.3}$$

has a solution $(x_0(t),\, y_0(t)) \in C^1[0,1]$

Assumption 2: The left boundary layer problem

$$\begin{cases} \dfrac{d^2\overline{y}_0}{d\tau^2} = V_x(0, x_0(0), 0, 0)[(U(0, x_0(0), y_0(0) + \overline{y}_0(\tau), 0) - U(0, x_0(0), y_0(0), 0)] \\ H_1(x_0(0), y_0(0) + \overline{y}_0(0), 0) = 0 \\ \overline{y}_0(+\infty) = 0 \end{cases} \tag{2.4}$$

has a solution $\overline{y}_0 = \overline{y}_0(\tau) \in C^2[0, +\infty)$.

The right boundary layer problem

$$\begin{cases} \dfrac{d^2\widehat{y}_0}{d\sigma^2} = V_x(1, x_0(1), 0, 0)[(U(1, x_0(1), y_0(1) + \widehat{y}_0(\sigma), 0) - U(1, x_0(1), y_0(1), 0)] \\ H_2(x_0(1), y_0(1) + \widehat{y}_0(0), 0) = 0 \\ \widehat{y}_0(+\infty) = 0 \end{cases} \tag{2.5}$$

has a solution $\widehat{y}_0 = \widehat{y}_0(\sigma) \in C^2[0, +\infty)$.

Assumption 3: Let

$$\begin{cases} B_0(t, \tau, \sigma) = U_y(t, x_0(t), y_0(t) + \overline{y}_0(\tau) + \widehat{y}_0(\sigma), 0) \\ C_0(t) = V_x(t, x_0(t), 0, 0) \end{cases} \tag{2.6}$$

where τ, σ are defined in (1.13) and (1.14) respectively. Let $C_0(t)B_0(t, \tau, \sigma)$ be positive definite uniformly on the region $0 \le t \le 1$, $0 \le \tau < +\infty, 0 \le \sigma < +\infty$; that is, let there exist a constant $\lambda_0 > 0$ such that

$$y^T C_0(t) B_0(t, \tau, \sigma) y \ge \lambda_0^2 \|y\|^2 \tag{2.7}$$

uniformly for $(t, \tau, \sigma) \in [0, 1] \times [0, +\infty) \times [0, +\infty)$, $y \in R^n$.

Furthmore, let $[C_0(t)B_0(t, \tau, \sigma)]^{1/2} \in C^1[0, 1]$ and $\dfrac{d}{dt}[C_0(t)B_0(t, \tau, \sigma)]^{1/2}$ be bounded uniformly on $[0, 1] \times (0, \varepsilon_0)$.

Assumption 4: Let the matrices of partial derivatives

$$H_{1,r}(x_0(0), y_0(0) + \overline{y}_0(0), 0) \tag{2.8}$$

and

$$H_{2,s}(x_0(1), y_0(1) + \widehat{y}_0(0), 0) \tag{2.9}$$

be non-singular, where

$$H_1(p,r,\varepsilon)=H_1(x(0,\varepsilon),y(0,\varepsilon),\varepsilon),\ H_2(q,s,\varepsilon)=H_2(x(1,\varepsilon),y(1,\varepsilon),\varepsilon)$$

Let problem (2.1)-(2.2) have the formal asymptotic solutions

$$\begin{cases} x(t,\varepsilon)=X(t,\varepsilon)+\varepsilon\overline{X}(\tau,\varepsilon)+\varepsilon\widehat{X}(\sigma,\varepsilon) \\ y(t,\varepsilon)=Y(t,\varepsilon)+\overline{Y}(\tau,\varepsilon)+\widehat{Y}(\sigma,\varepsilon) \end{cases} \tag{2.10}$$

where

$$\begin{pmatrix} X(t,\varepsilon) \\ Y(t,\varepsilon) \end{pmatrix} \sim \sum_{i=0}^{\infty} \begin{pmatrix} x_i(t) \\ y_i(t) \end{pmatrix} \varepsilon^i \tag{2.11}$$

with

$$\begin{pmatrix} \overline{X}(\tau,\varepsilon) \\ \overline{Y}(\tau,\varepsilon) \end{pmatrix} \sim \sum_{i=0}^{\infty} \begin{pmatrix} \overline{x}_i(\tau) \\ \overline{y}_i(\tau) \end{pmatrix} \varepsilon^i \tag{2.12}$$

and

$$\begin{pmatrix} \widehat{X}(\sigma,\varepsilon) \\ \widehat{Y}(\sigma,\varepsilon) \end{pmatrix} \sim \sum_{i=0}^{\infty} \begin{pmatrix} \widehat{x}_i(\sigma) \\ \widehat{y}_i(\sigma) \end{pmatrix} \varepsilon^i \tag{2.13}$$

Under these assumptions, Shi[16] constructed the outer solution (2.11) for (2.1)-(2.2), which can be determined by the following equations

$$\begin{cases} x_0'(t)=U(t,x_0(t),y_0(t),0) \\ 0=V(t,x_0(t),0,0) \end{cases} \tag{2.14}$$

and

$$\begin{cases} x_1'(t)=U_x(t,x_0,y_0,0)x_1(t)+U_y(t,x_0,y_0,0)y_1(t)+P_1(t) \\ 0=V_x(t,x_0,0,0)x_1(t)+Q_1(t) \end{cases} \tag{2.15}$$

and

$$\begin{cases} x_i'(t)=U_x(t,x_0,y_0,0)x_i(t)+U_y(t,x_0,y_0,0)y_i(t)+P_i(t) \\ y_{i-2}'(t)=V_x(t,x_0,0,0)x_i(t)+Q_i(t) \end{cases} \quad (i\geq 2) \tag{2.16}$$

where

$$\begin{cases} P_i(t) = P_i(x_0, x_1, \cdots, x_{i-1},\ y_0, y_1, \cdots, y_{i-1}) \\ Q_i(t) = Q_i(x_0, x_1, \cdots, x_{i-1},\ y_0, y_1, \cdots, y_{i-1}) \end{cases}$$

are infinitely differentiable with respect to their variables respectively for $i \geq 1$.

By Assumption 1, 2.14 has a solution $x_0(t), y_0(t)$.

He obtained the following lemma.

Lemma 2.1.1: *Let Assumptions 1 and 3 hold, then the problem* (2.15)-(2.16) *have unique solutions* $x_i(t), y_i(t) \in C^\infty[0,1]$ *for* $i \geq 1$.

Employing the matched asymptotic method, he used the following equations

$$\begin{cases} \dfrac{d\overline{x}_0}{d\tau} = U(0, x_0(0), y_0(0) + \overline{y}_0(\tau), 0) - U(0, x_0(0), y_0(0), 0) \\ \dfrac{d\overline{y}_0}{d\tau} = V_x(0, x_0(0), 0, 0)\overline{x}_0(\tau) \\ H_1(x_0(0), y_0(0) + \overline{y}_0(0), 0) = 0 \end{cases} \tag{2.17}$$

and

$$\begin{cases} \dfrac{d\overline{x}_i}{d\tau} = U_y(0, x_0(0), y_0(0) + \overline{y}_0(\tau), 0)\overline{y}_i(\tau) + \overline{P}_i(\tau) \\ \dfrac{d\overline{y}_i}{d\tau} = V_x(0, x_0(0), 0, 0)\overline{x}_i(\tau) + \overline{Q}_i(\tau) \qquad (i \geq 1) \\ H_{1,\ r}(x_0(0), y_0(0) + \overline{y}_0(0), 0)\overline{y}_i(0) + \overline{M}_i = 0 \end{cases} \tag{2.18}$$

to find the left boundary layer correction, and the following equations

$$\begin{cases} \dfrac{d\widehat{x}_0}{d\sigma} = -[U(1, x_0(1), y_0(1) + \widehat{y}_0(\sigma), 0) - U(1, x_0(1), y_0(1), 0)] \\ \dfrac{d\widehat{y}_0}{d\sigma} = -[V_x(1, x_0(1), 0, 0)\widehat{x}_0(\sigma) + V_{\epsilon y}(1, x_0(1), 0, 0)\widehat{y}_0(\sigma)] \\ H_2(x_0(1), y_0(1) + \widehat{y}_0(0), 0) = 0 \end{cases} \tag{2.19}$$

and

$$\begin{cases} \dfrac{d\widehat{x}_i}{d\sigma} = -U_y(1, x_0(1), y_0(1) + \widehat{y}_0(\sigma), 0)\widehat{y}_i(\sigma) + \widehat{P}_i(\sigma) \\ \dfrac{d\widehat{y}_i}{d\sigma} = -[V_x(1, x_0(1), 0, 0)\widehat{x}_i(\sigma) + V_{xy}(1, x_0(1), 0, 0)\widehat{y}_i(\sigma)] + \widehat{Q}_i(\sigma) \qquad (i \geq 1) \\ H_{2,\,s}(x_0(1), y_0(1) + \widehat{y}_0(0), 0)\widehat{y}_i(0) + \widehat{M}_i = 0 \end{cases} \tag{2.20}$$

to find the right boundary layer correction. Here

$$\begin{cases} \overline{P}_i(\tau) = \overline{P}_i(\tau, x_0, x_1, \cdots x_i;\ y_0, y_1, \cdots y_i;\ \overline{x}_0, \overline{x}_1, \cdots \overline{x}_{i-1};\ \overline{y}_0, \overline{y}_1, \cdots \overline{y}_{i-1}) \\ \widehat{P}_i(\sigma) = \widehat{P}_i(\sigma, x_0, x_1, \cdots x_i;\ y_0, y_1, \cdots y_i;\ \widehat{x}_0, \widehat{x}_1, \cdots \widehat{x}_{i-1};\ \widehat{y}_0, \widehat{y}_1, \cdots \widehat{y}_{i-1}) \\ \overline{Q}_i(\tau) = \overline{Q}_i(\tau, x_0, x_1, \cdots x_i;\ y_0, y_1, \cdots y_i;\ \overline{x}_0, \overline{x}_1, \cdots \overline{x}_{i-1};\ \overline{y}_0, \overline{y}_1, \cdots \overline{y}_{i-1}) \\ \widehat{Q}_i(\sigma) = \widehat{Q}_i(\sigma, x_0, x_1, \cdots x_i;\ y_0, y_1, \cdots y_i;\ \widehat{x}_0, \widehat{x}_1, \cdots \widehat{x}_{i-1};\ \widehat{y}_0, \widehat{y}_1, \cdots \widehat{y}_{i-1}) \end{cases}$$

are infinitely differentiable with respect to their variables respectively, and

$$\begin{cases} \overline{M}_i = \overline{M}_i(\tau, x_0, x_1, \cdots x_i;\ y_0, y_1, \cdots y_i;\ \overline{x}_0, \overline{x}_1, \cdots \overline{x}_{i-1};\ \overline{y}_0, \overline{y}_1, \cdots \overline{y}_{i-1})|_{t=0,\tau=0} \\ \widehat{M}_i = \widehat{M}_i(\sigma, x_0, x_1, \cdots x_i;\ y_0, y_1, \cdots y_i; \widehat{x}_0, \widehat{x}_1, \cdots \widehat{x}_{i-1};\ \widehat{y}_0, \widehat{y}_1, \cdots \widehat{y}_{i-1})|_{t=1,\sigma=0} \end{cases}$$

are known values.

Finally, he obtained the following lemma and theorem.

Lemma 2.1.2: *Let Assumptions 1-4 hold. Then the problems* (2.17) *and* (2.19) *have solutions*

$$\overline{x}_0(\tau),\ \overline{y}_0(\tau) \text{ and } \widehat{x}_0(\sigma),\ \widehat{y}_0(\sigma) \in C^\infty[0, +\infty)$$

such that

$$\frac{d^j}{d\tau^j}\begin{pmatrix} \overline{x}_0(\tau) \\ \overline{y}_0(\tau) \end{pmatrix} = O(e^{-\lambda_0 \tau}),$$

and

$$\frac{d^j}{d\sigma^j}\begin{pmatrix} \widehat{x}_0(\sigma) \\ \widehat{y}_0(\sigma) \end{pmatrix} = O(e^{-\lambda_0 \sigma}),$$

furthermore, the problem (2.18) and the problem (2.20) have unique solutions

$$\overline{x}_i(\tau),\ \overline{y}_i(\tau) \text{ and } \widehat{x}_i(\sigma),\ \widehat{y}_i(\sigma) \in C^\infty[0,+\infty\ )$$

such that

$$\frac{d^j}{d\tau^j}\begin{pmatrix} \overline{x}_i(\tau) \\ \overline{y}_i(\tau) \end{pmatrix} = O(e^{-(1-\mu_i)\lambda_0\tau}),$$

and

$$\frac{d^j}{d\sigma^j}\begin{pmatrix} \widehat{x}_i(\sigma) \\ \widehat{y}_i(\sigma) \end{pmatrix} = O(e^{-(1-\mu_i)\lambda_0\sigma}),$$

for any integer $j \geq 0$, τ and $\sigma \in [0,+\infty)$, *and*

$$0 < \mu_1 < \mu_2 < \cdots < 1.$$

Theorem 2.1.1: *Let Assumptions 1-4 hold, then,*when $N \geq 0$ is an integer and ε *is sufficiently small, the problem (2.1)-(2.2) has a unique solution* $x(t,\varepsilon)$ *and* $y(t,\varepsilon)$ $\in C^\infty[0,1]$ *satisfying*

$$\begin{cases} x(t,\varepsilon) = x^N(t,\varepsilon) + O(\varepsilon^{N+1}), \\ y(t,\varepsilon) = y^N(t,\varepsilon) + O(\varepsilon^{N+1}). \end{cases}$$

where

$$\begin{cases} x^N(t,\varepsilon) = \sum_{i=0}^{N}(x_i(t) + \overline{x}_{i-1}(\tau) + \widehat{x}_{i-1}(\sigma))\varepsilon^i \\ y^N(t,\varepsilon) = \sum_{i=0}^{N}(y_i(t) + \overline{y}_i(\tau) + \widehat{y}_i(\sigma))\varepsilon^i \end{cases}$$

and $\overline{x}_{-1}(\tau) = \widehat{x}_{-1}(\sigma) = 0$

Chapter 3

A new singular singularly perturbed problem

In this last and somewhat long chapter, we consider a new singular singularly perturbed boundary-value problem P_ε consisting of differential equations

$$\begin{cases} \dfrac{dx}{dt} = U(t,x,y,\varepsilon) \\ \varepsilon^2 \dfrac{dy}{dt} = V(t,x,\varepsilon y,\varepsilon) \end{cases} \tag{3.1}$$

subject to the boundary conditions

$$\begin{cases} H_1(x(0,\varepsilon),y(0,\varepsilon),\varepsilon) = 0 \\ H_2(x(1,\varepsilon),y(1,\varepsilon),\varepsilon) = 0 \end{cases} \tag{3.2}$$

where $t \in [0,1]$, $0 < \varepsilon \leq \varepsilon_0$. U, V, H_1 and H_2 are scalar functions and infinitely differentiable with respect to their variables respectively. For examples, the function $V(t,x,\varepsilon y,\varepsilon) = t + x^2 + (\varepsilon y)^3 + \varepsilon$ is infinitely differentiable with respect to each of variables t, x, (εy) and ε, the function $H_1(x(0,\varepsilon),y(0,\varepsilon),\varepsilon) = x(0,\varepsilon)^2 + y(0,\varepsilon) + \varepsilon$ is also infinitely differentiable with respect to each of variables $x(0,\varepsilon)$, $y(0,\varepsilon)$ and ε.

Basically we employ the same approach as used by Shi[16] and Smith[18].

3.1 Assumptions

The singular perturbation problem (3.1) and (3.2) will be studied under the following four assumptions.

Assumption 1: The reduced problem

$$\begin{cases} 0 = V(t, x_0(t), 0, 0) \\ x_0'(t) = U(t, x_0(t), y_0(t), 0) \end{cases} \tag{3.3}$$

has a solution $(x_0(t),\ y_0(t)) \in C^1[0,1]$

Assumption 2: The left boundary layer problem

$$\begin{cases} \dfrac{d^2\overline{y}_0}{d\tau^2} = V_{\epsilon y}(0, x_0(0), 0, 0)\dfrac{d\overline{y}_0}{d\tau} + V_x(0, x_0(0), 0, 0) \\ \qquad\qquad [(U(0, x_0(0), y_0(0) + \overline{y}_0(\tau), 0) - U(0, x_0(0), y_0(0), 0)] \\ H_1(x_0(0), y_0(0) + \overline{y}_0(0), 0) = 0 \\ \overline{y}_0(+\infty) = 0 \end{cases} \tag{3.4}$$

has a solution $\overline{y}_0 = \overline{y}_0(\tau) \in C^2[0, +\infty)$.

The right boundary layer problem

$$\begin{cases} \dfrac{d^2\widehat{y}_0}{d\sigma^2} = -V_{\epsilon y}(1, x_0(1), 0, 0)\dfrac{d\widehat{y}_0}{d\sigma} + V_x(1, x_0(1), 0, 0) \\ \qquad\qquad [(U(1, x_0(1), y_0(1) + \widehat{y}_0(\sigma), 0) - U(1, x_0(1), y_0(1), 0)] \\ H_2(x_0(1), y_0(1) + \widehat{y}_0(0), 0) = 0 \\ \widehat{y}_0(+\infty) = 0 \end{cases} \tag{3.5}$$

has a solution $\widehat{y}_0 = \widehat{y}_0(\sigma) \in C^2[0, +\infty)$.

Assumption 3: Define

$$\begin{cases} B_0(t, \tau, \sigma) = U_y(t, x_0(t), y_0(t) + \overline{y}_0(\tau) + \widehat{y}_0(\sigma), 0), \\ C_0(t) = V_x(t, x_0(t), 0, 0). \\ D_0(t) = V_{\epsilon y}(t, x_0(t), 0, 0). \end{cases} \tag{3.6}$$

where τ and σ are defined in (1.13) and (1.14) respectively.

Let

$$C_0(t)B_0(t,\tau,\sigma) \geq \lambda_0^2, \qquad \lambda_0 \text{ is a positive constant.} \tag{3.7}$$

uniformly for $0 \leq t \leq 1$, $0 \leq \tau < +\infty, 0 \leq \sigma < +\infty$, and

$$D_0(0) \leq 0, \qquad D_0(1) \geq 0 \tag{3.8}$$

Assumption 4: Let partial derivatives

$$H_{1,r}(x_0(0), y_0(0) + \overline{y}_0(0), 0) \tag{3.9}$$

and

$$H_{2,s}(x_0(1), y_0(1) + \widehat{y}_0(0), 0) \tag{3.10}$$

be nonzero, where

$$H_1(p,r,\varepsilon) = H_1(x(0,\varepsilon), y(0,\varepsilon), \varepsilon),\ H_2(q,s,\varepsilon) = H_2(x(1,\varepsilon), y(1,\varepsilon), \varepsilon)$$

Lemma 3.1.1: *Let Assumptions 1-3 hold, then in Assumption 1, the solution $x_0(t), y_0(t) \in C^\infty[0,1]$ and in Assumption 2, the solution $\overline{y}_0(\tau)$, $\widehat{y}_0(\sigma) \in C^\infty[0,+\infty)$.*

Proof: we only show that in Assumption 1, the solution $x_0(t), y_0(t) \in C^\infty[0,1]$. The method can be applied to show that in Assumption 3, the solution $\overline{y}_0(\tau), \widehat{y}_0(\sigma)$ $\in C^\infty[0,+\infty)$.

By Assumption 1, we have

$$0 = V(t, x_0(t), 0, 0) \tag{3.11}$$

and

$$x_0(t) \in C^1[0,1].$$

We can differentiate both sides of (3.11) with respect to t to obtain

$$0 = V_t(t, x_0(t), 0, 0) + V_x(t, x_0(t), 0, 0)x_0'(t). \tag{3.12}$$

By Assumption 3

$$C_0(t) = V_x(t, x_0(t), 0, 0)$$

is nonzero.

Thus

$$x_0'(t) = -C_0(t)^{-1} V_t(t, x_0(t), 0, 0) \equiv F_1(t, x_0(t)).$$

$F_1(t, x_0(t))$ is infinitely differentiable with respect to its variables.

So

$$x_0'(t) \in C^1[0,1],$$

or

$$x_0(t) \in C^2[0,1].$$

In fact

$$x_0''(t) = \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial x} x_0'(t) \equiv F_2(t, x_0(t), x_0'(t)).$$

$F_2(t, x_0(t), x_0'(t))$ is infinitely differentiable with respect to its variables.

Then

$$x_0''(t) \in C^1[0,1],$$

or

$$x_0(t) \in C^3[0,1].$$

In fact

$$x_0'''(t) = \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial u} x_0'(t) + \frac{\partial F_2}{\partial v} x_0''(t).$$

where $F_2 = F_2(t, u, v)$..

We can repeat the above process for infinitely many times to conclude

$$x_0(t) \in C^\infty[0,1].$$

On the other hand

$$x_0'(t) = U(t, x_0(t), y_0(t), 0),$$

then

$$x_0''(t) = U_t + U_x x_0'(t) + U_y y_0'(t). \tag{3.13}$$

By Assumption 3

$$U_y = U_y(t, x_0(t), y_0(t), 0)$$

is nonzero.

From (3.13), we have

$$y_0'(t) = -U_y^{-1}[U_t + U_x x_0'(t) - x_0''(t)]. \tag{3.14}$$

We can use the method for showing $x_0(t) \in C^\infty[0\ ,1]$ to prove

$$y_0(t) \in C^\infty[0, 1].$$

Finally the same method can be adopted to show

$$\overline{y}_0(\tau),\ \widehat{y}_0(\sigma) \in C^\infty[0, +\infty)$$

where $\overline{y}_0(\tau)$ and $\widehat{y}_0(\sigma)$ are the solutions in Assumption 2.

This completes the proof.

We note that $C_0(t) = V_x(t, x_0(t), 0, 0)$ is bounded on the interval $t \in [0, 1]$.

Lemma 3.1.2: *Let Assumptions 1 and 2 hold, then*

$$\overline{y}_0(\tau) \text{ and } \widehat{y}_0(\sigma)$$

are bounded on the interval $[0, +\infty)$. *Furthermore in Assumption 3*

$$B_0(t, \tau, \sigma) \text{ and hence } C_0(t)B_0(t, \tau, \sigma)$$

are bounded for $t \in [0, 1]$, $\tau \in [0, +\infty)$, $\sigma \in [0, +\infty)$.

Proof. We consider $\overline{y}_0(\tau)$ in Assumption 2. Since

$$\overline{y}_0(+\infty) = 0.$$

So, there exists a constant positive δ such that

$$|\overline{y}_0(\tau)| < 1.$$

for $\tau \geq \delta$.

Since $\overline{y}_0(\tau)$ is continuous, there exists a constant $k > 0$ such that

$$|\overline{y}_0(\tau)| < k.$$

Thus

$$|\overline{y}_0(\tau)| < \max(1, k)$$

on the whole interval $[0, +\infty)$.

Similarly, $\widehat{y}_0(\sigma)$ is bounded on the interval $[0, +\infty)$.

Since

$$B_0(t, \tau, \sigma) = U_y(t, x_0(t), y_0(t) + \overline{y}_0(\tau) + \widehat{y}(\sigma), 0)$$

is a continuous function, where $x_0(t)$, $y_0(t)$, $\overline{y}(\tau)$ and $\widehat{y}(\sigma)$ are all bounded for $t \in [0, 1], \tau \in [0, +\infty)$ and $\sigma \in [0, +\infty)$.

Thus

$$B_0(t, \tau, \sigma) \text{ and hence } C_0(t)B_0(t, \tau, \sigma)$$

are bounded for $t \in [0, 1]$, $\tau \in [0, +\infty)$ and $\sigma \in [0, +\infty)$.

This completes the proof.

3.2 The formal asymptotic solution

We obtain the solution of (3.1)-(3.2) in the following form

$$\begin{cases} x(t,\varepsilon) = X(t,\varepsilon) + \varepsilon\overline{X}(\tau,\varepsilon) + \varepsilon\widehat{X}(\sigma,\varepsilon) \\ y(t,\varepsilon) = Y(t,\varepsilon) + \overline{Y}(\tau,\varepsilon) + \widehat{Y}(\sigma,\varepsilon) \end{cases} \tag{3.15}$$

where τ and σ are defined in (1.13) and (1.14) respectively.

Assume

$$\begin{pmatrix} X(t,\varepsilon) \\ Y(t,\varepsilon) \end{pmatrix} \sim \sum_{i=0}^{\infty} \begin{pmatrix} x_i(t) \\ y_i(t) \end{pmatrix} \varepsilon^i \tag{3.16}$$

with

$$\begin{pmatrix} \overline{X}(\tau,\varepsilon) \\ \overline{Y}(\tau,\varepsilon) \end{pmatrix} \sim \sum_{i=0}^{\infty} \begin{pmatrix} \overline{x}_i(\tau) \\ \overline{y}_i(\tau) \end{pmatrix} \varepsilon^i \tag{3.17}$$

and

$$\begin{pmatrix} \widehat{X}(\sigma,\varepsilon) \\ \widehat{Y}(\sigma,\varepsilon) \end{pmatrix} \sim \sum_{i=0}^{\infty} \begin{pmatrix} \widehat{x}_i(\sigma) \\ \widehat{y}_i(\sigma) \end{pmatrix} \varepsilon^i \tag{3.18}$$

In other words

$$x(t,\varepsilon) = x_0(t) + (x_1(t) + \overline{x}_0(\tau) + \widehat{x}_0(\sigma))\varepsilon + \cdots \tag{3.19}$$

and

$$y(t,\varepsilon) = y_0(t) + \overline{y}_0(\tau) + \widehat{y}_0(\sigma) + (y_1(t) + \overline{y}_1(\tau) + \widehat{y}_1(\sigma))\varepsilon + \cdots \tag{3.20}$$

We use the method of matched asymptotic expansion to first find $(x_0(t), y_0(t))$, then $(x_1(t), y_1(t))$, and $(x_i(t), y_i(t))$ for any $i \geq 2$.

Later we can see that, based on our Assumptions, $(x_0(t), y_0(t))$ and the boundary conditions, we can find $(\overline{x}_0(\tau), \overline{y}_0(\tau))$ and $(\widehat{x}_0(\sigma), \widehat{y}_0(\sigma))$. Generally speaking, in order to find $(\overline{x}_i(\tau), \overline{y}_i(\tau))$ and $(\widehat{x}_i(\sigma), \widehat{y}_i(\sigma))$ for $i \geq 1$, we need to find $(\overline{x}_k(\tau), \overline{y}_k(\tau))$ and $(\widehat{x}_k(\sigma), \widehat{y}_k(\sigma))$ [for all $0 \leq k \leq (i-1)$] and $(x_k(t), y_k(t))$ [for all $0 \leq k \leq i$].

3.2.1 Construction of the outer solution

We first substitute (3.16) formally into (3.1) and (3.2) to obtain

$$\sum_{i=0}^{\infty} x_i'(t)\varepsilon^i = U(t, \sum_{i=0}^{\infty} x_i(t)\varepsilon^i, \sum_{i=0}^{\infty} y_i(t)\varepsilon^i, \varepsilon) \tag{3.21}$$

and

$$\sum_{i=0}^{\infty} y_i'(t)\varepsilon^{i+2} = V(t, \sum_{i=0}^{\infty} x_i(t)\varepsilon^i, \sum_{i=0}^{\infty} y_i(t)\varepsilon^{i+1}, \varepsilon) \tag{3.22}$$

Next, we formally obtain Taylor series expansion for the right side of (3.21) about $\varepsilon = 0$, that is

$$\begin{aligned}
&U(t, \sum_{i=0}^{\infty} x_i(t)\varepsilon^i, \sum_{i=0}^{\infty} y_i(t)\varepsilon^i, \varepsilon) \\
&= U(t, x_0(t), y_0(t), 0) + \frac{1}{i!}\varepsilon^i \sum_{i=1}^{\infty}[(x_1 + x_2\varepsilon + \cdots)\frac{\partial}{\partial x} + \\
&\quad (y_1 + y_2\varepsilon + \cdots)\frac{\partial}{\partial y} + \frac{\partial}{\partial \varepsilon}]^i U(t, x_0(t), y_0(t), 0) \\
&= U(t, x_0(t), y_0(t), 0) + \varepsilon[x_1\frac{\partial}{\partial x} + y_1\frac{\partial}{\partial y} + \frac{\partial}{\partial \varepsilon}]U(t, x_0(t), y_0(t), 0) + \varepsilon^2 \\
&\quad [x_2\frac{\partial}{\partial x} + y_2\frac{\partial}{\partial y} + \frac{1}{2}x_1\frac{\partial^2}{\partial x^2} + \frac{1}{2}y_1\frac{\partial^2}{\partial y^2} + \frac{1}{2}\frac{\partial^2}{\partial \varepsilon^2} + x_1\frac{\partial^2}{\partial x \partial \varepsilon} + \\
&\quad y_1\frac{\partial^2}{\partial y \partial \varepsilon} + x_1 y_1 \frac{\partial^2}{\partial x \partial y}]U(t, x_0(t), y_0(t), 0) + \cdots
\end{aligned} \tag{3.23}$$

From (3.21) and (3.23), we equate the coefficients of like powers of ε to obtain

$$\left\{\begin{array}{l} x_0'(t) = U(t, x_0(t), y_0(t), 0) \\ x_1'(t) = [x_1\frac{\partial}{\partial x} + y_1\frac{\partial}{\partial y} + \frac{\partial}{\partial \varepsilon}]U(t, x_0(t), y_0(t), 0) \\ x_2'(t) = [x_2\frac{\partial}{\partial x} + y_2\frac{\partial}{\partial y} + \frac{1}{2}x_1\frac{\partial^2}{\partial x^2} + \frac{1}{2}y_1\frac{\partial^2}{\partial y^2} + \frac{1}{2}\frac{\partial^2}{\partial \varepsilon^2} + x_1\frac{\partial^2}{\partial x \partial \varepsilon} + \\ \qquad y_1\frac{\partial^2}{\partial y \partial \varepsilon} + x_1 y_1\frac{\partial^2}{\partial x \partial y}]U(t, x_0(t), y_0(t), 0) \end{array}\right. \tag{3.24}$$

and generally

$$x_i'(t) = [x_i\frac{\partial}{\partial x} + y_i\frac{\partial}{\partial y}]U(t, x_0(t), y_0(t), 0) + P_i(t) \qquad i \geq 1 \tag{3.25}$$

where

$$P_i(t) = P_i(x_0, x_1, \cdots, x_{i-1},\ y_0, y_1, \cdots, y_{i-1}).$$

is infinitely differentiable with respect to its variables.

With the same analysis for (3.22), we can obtain

$$\left\{\begin{array}{l} 0 = V(t, x_0(t), 0, 0) \\ 0 = [x_1\frac{\partial}{\partial x} + \frac{\partial}{\partial \varepsilon}]V(t, x_0(t), 0, 0) + y_0(t)V_{\varepsilon y}(t, x_0(t), 0, 0) \\ y_0'(t) = [x_2\frac{\partial}{\partial x} + y_1\frac{\partial}{\partial(\varepsilon y)} + \frac{1}{2}x_1^2\frac{\partial^2}{\partial x^2} + \frac{1}{2}\frac{\partial^2}{\partial \varepsilon^2} + x_1\frac{\partial^2}{\partial x \partial \varepsilon} + \\ \qquad y_0\frac{\partial^2}{\partial(\varepsilon y)\partial \varepsilon} + \frac{1}{2}y_0^2\frac{\partial^2}{\partial(\varepsilon y)^2}]V(t, x_0(t), 0, 0) \end{array}\right. \tag{3.26}$$

and generally

$$y_{i-2}'(t) = \frac{\partial}{\partial x}V(t, x_0(t), 0, 0)x_i(t) + Q_i(t) \qquad i \geq 2 \tag{3.27}$$

where

$$Q_i(t) = Q_i(x_0, x_1, \cdots, x_{i-1},\ y_0, y_1, \cdots, y_{i-1}).$$

is infinitely differentiable with respect to its variables.

From (3.24)-(3.27), we have

$$\begin{cases} x_0'(t) = U(t, x_0(t), y_0(t), 0) \\ 0 = V(t, x_0(t), 0, 0) \end{cases} \tag{3.28}$$

and

$$\begin{cases} x_1'(t) = U_x(t, x_0(t), y_0(t), 0)x_1(t) + U_y(t, x_0(t), y_0(t), 0)y_1(t) + P_1(t) \\ 0 = V_x(t, x_0(t), 0, 0)x_1 + V_\varepsilon(t, x_0(t), 0, 0) + V_{\varepsilon y}(t, x_0(t), 0, 0)y_0(t) \end{cases} \tag{3.29}$$

and specially

$$\begin{cases} x_i'(t) = U_x(t, x_0(t), y_0(t), 0)x_i(t) + U_y(t, x_0(t), y_0(t), 0)y_i(t) + P_i(t) \\ y_{i-2}'(t) = V_x(t, x_0(t), 0, 0)x_i + Q_i(t)\,. \end{cases} \quad (i \geq 2) \tag{3.30}$$

By Assumption 1, problem (3.28) has a solution $x_0(t)$, $y_0(t)$.

Lemma 3.2.1: *Let Assumptions 1 and 3 hold, then the problems (3.29) and (3.30) have unique solutions*

$$x_i(t), \quad y_i(t) \in C^\infty[0,1] \qquad \textit{for } i \geq 1.$$

Proof: First we consider the problem (3.29), where $x_0(t), y_0(t)$ is the solution of (3.28).

By Theorem 3.1.1

$$x_0(t),\ y_0(t) \in C^\infty[0,1]. \tag{3.31}$$

By Assumption 3

$$V_x(t, x_0(t), 0, 0) \neq 0.$$

From the second equation of (3.29), we have

$$x_1(t) = -V_x(t, x_0(t), 0, 0)^{-1}Q_1(t). \tag{3.32}$$

Substitute $x_1(t)$ into the first equation of (3.29) to obtain

$$-[V_x(t,x_0(t),0,0)^{-1}Q_1(t)]' = -U_x(t,x_0(t),y_0(t),0)V_x(t,x_0(t),0,0)^{-1}Q_1(t)$$
$$+U_y(t,x_0(t),y_0(t),0)y_1(t) + P_1(t)$$

Since

$$U_y(t,x_0(t),y_0(t),0) \neq 0.$$

We obtain

$$\begin{aligned} y_1(t) = U_y(t,x_0(t),y_0(t),0)^{-1}\{&U_x(t,x_0(t),y_0(t),0)V_x(t,x_0(t),0,0)^{-1}Q_1(t) \\ &-P_1(t) - [V_x(t,x_0,0,0)^{-1}Q_1(t)]'\}. \end{aligned} \tag{3.33}$$

From (3.29), (3.31), (3.32) and (3.33), we can adopt the method for proving Lemma 3.1.1 to prove

$$x_1(t),\ y_1(t) \in C^\infty[0,1]. \tag{3.34}$$

Similarly we can use the above method to prove that (3.30) has the unique solution such that

$$x_i(t),\ y_i(t) \in C^\infty[0,1] \qquad \text{for } i \geq 2. \tag{3.35}$$

3.2.2 Construction of the boundary layer corrections

We now use the method of matched asymptotic expansion to find

$$\overline{X}(\tau,\varepsilon),\ \overline{Y}(\tau,\varepsilon),\ \widehat{X}(\sigma,\varepsilon),\ \widehat{Y}(\sigma,\varepsilon).$$

First we consider the left boundary layer correction near $t = 0$.

Substitute

$$\begin{cases} x = X(t,\varepsilon) + \varepsilon\overline{X}(\tau,\varepsilon) \\ y = Y(t,\varepsilon) + \overline{Y}(\tau,\varepsilon) \end{cases}$$

into (3.1) and (3.2) to obtain

$$\frac{dX}{dt} + \varepsilon\frac{d\overline{X}}{dt} = U(t, X(t,\varepsilon) + \varepsilon\overline{X}(\tau,\varepsilon), Y(t,\varepsilon) + \overline{Y}(\tau,\varepsilon), \varepsilon),$$

and

$$\varepsilon^2(\frac{dY}{dt} + \frac{d\overline{Y}}{dt}) = V(t, X(t,\varepsilon) + \varepsilon\overline{X}(\tau,\varepsilon), \varepsilon Y(t,\varepsilon) + \varepsilon\overline{Y}(\tau,\varepsilon), \varepsilon).$$

Since

$$\begin{cases} \dfrac{dX}{dt} = U(t, X(t,\varepsilon), Y(t,\varepsilon), \varepsilon) \\ \varepsilon^2\dfrac{dY}{dt} = V(t, X(t,\varepsilon), \varepsilon Y(t,\varepsilon), \varepsilon) \end{cases}$$

and

$$\begin{cases} \dfrac{d\overline{X}}{dt} = \dfrac{d\overline{X}}{d\tau} \times \dfrac{d\tau}{dt} = \dfrac{1}{\varepsilon}\dfrac{d\overline{X}}{d\tau} \\ \dfrac{d\overline{Y}}{dt} = \dfrac{d\overline{Y}}{d\tau} \times \dfrac{d\tau}{dt} = \dfrac{1}{\varepsilon}\dfrac{d\overline{Y}}{d\tau}. \end{cases}$$

Then

$$\begin{aligned} \frac{d\overline{X}}{d\tau} &= U(t, X(\varepsilon\tau,\varepsilon) + \varepsilon\overline{X}(\tau,\varepsilon), Y(\varepsilon\tau,\varepsilon) + \overline{Y}(\tau,\varepsilon), \varepsilon) \\ &\quad - U(t, X(\varepsilon\tau,\varepsilon), Y(\varepsilon\tau,\varepsilon), \varepsilon) \end{aligned} \tag{3.36}$$

and

$$\begin{aligned} \varepsilon\frac{d\overline{Y}}{d\tau} &= V(\varepsilon\tau, X(\varepsilon\tau,\varepsilon) + \varepsilon\overline{X}(\tau,\varepsilon), \varepsilon Y(\varepsilon\tau,\varepsilon) + \varepsilon\overline{Y}(\tau,\varepsilon), \varepsilon) \\ &\quad - V(\varepsilon\tau, X(\varepsilon\tau,\varepsilon), \varepsilon Y(\varepsilon\tau,\varepsilon), \varepsilon) \end{aligned} \tag{3.37}$$

We first handle (3.37). We try to obtain Taylor series expansions for the left part of (3.37) about $\varepsilon = 0$, that is

$$V(\varepsilon\tau, X(\varepsilon\tau,\varepsilon) + \varepsilon\overline{X}(\tau,\varepsilon), \varepsilon Y(\varepsilon\tau,\varepsilon) + \varepsilon\overline{Y}(\tau,\varepsilon), \varepsilon) - V(\varepsilon\tau, X(\varepsilon\tau,\varepsilon), \varepsilon Y(\varepsilon\tau,\varepsilon), \varepsilon)$$
$$= \frac{1}{i!}\sum_{i=1}^{\infty}\{[\varepsilon\tau\frac{\partial}{\partial(\varepsilon\tau)} + (X(\varepsilon\tau,\varepsilon) + \varepsilon\overline{X}(\tau,\varepsilon) + \cdots - x_0(0))\frac{\partial}{\partial x} +$$

$$(\varepsilon Y(\varepsilon\tau,\varepsilon)+\varepsilon\overline{Y}(\varepsilon\tau,\varepsilon))\frac{\partial}{\partial(\varepsilon y)}+\varepsilon\frac{\partial}{\partial\varepsilon}]^i V(0,x_0(0),0,0)-$$
$$[\varepsilon\tau\frac{\partial}{\partial(\varepsilon\tau)}+(X(\varepsilon\tau,\varepsilon)-x_0(0))\frac{\partial}{\partial x}+\varepsilon Y(\varepsilon\tau,\varepsilon)\frac{\partial}{\partial(\varepsilon y)}+\varepsilon\frac{\partial}{\partial\varepsilon}]^i$$
$$V(0,x_0(0),0,0)\}$$

Since

$$X(\varepsilon\tau,\varepsilon)=x_0(\varepsilon\tau)+x_1(\varepsilon\tau)\varepsilon+x_2(\varepsilon\tau)\varepsilon^2+\cdots,$$

and

$$x_i(\varepsilon\tau)=x_i(0)+x_i'(0)\varepsilon\tau+\frac{1}{2!}x_i''(0)(\varepsilon\tau)^2+\cdots.$$

We equate the like powers of ε between the two sides of (3.37) to obtain

$$\frac{d\overline{y}_0}{d\tau}=[\overline{x}_0\frac{\partial}{\partial x}+\overline{y}_0\frac{\partial}{\partial(\varepsilon y)}]V(0,x_0(0),0,0) \tag{3.38}$$

and generally

$$\frac{d\overline{y}_i}{d\tau}=[\overline{x}_i\frac{\partial}{\partial x}+\overline{y}_i\frac{\partial}{\partial(\varepsilon y)}]V(0,x_0(0),0,0)+\overline{Q}_i(\tau)\quad(i\geq 1) \tag{3.39}$$

where

$$\overline{Q}_i(\tau)=\overline{Q}_i(\tau,x_0,x_1,\cdots x_i;\ y_0,y_1,\cdots y_i;\ \overline{x}_0,\overline{x}_1,\cdots\overline{x}_{i-1};\ \overline{y}_0,\overline{y}_1,\cdots\overline{y}_{i-1})$$

is infinitely differentiable with respect to its variables. For example

$$\begin{aligned}\overline{Q}_1=&\frac{1}{2}[\overline{x}_0(\tau)\frac{\partial}{\partial x}+\overline{y}_0(\tau)\frac{\partial}{\partial(\varepsilon y)}]^2V(0,x_0(0),0,0)+[\tau\frac{\partial}{\partial t}+\tau x_0'(0)\frac{\partial}{\partial x}\\&+y_0(0)\frac{\partial}{\partial(\varepsilon y)}+\frac{\partial}{\partial\varepsilon}][\overline{x}_0(\tau)\frac{\partial}{\partial x}+\overline{y}_0(\tau)\frac{\partial}{\partial(\varepsilon y)}]V(0,x_0(0),0,0).\end{aligned} \tag{3.40}$$

Applying the same method for (3.36) to obtain

$$\frac{d\overline{x}_0}{d\tau}=U(0,x_0(0),y_0(0)+\overline{y}_0(\tau),0)-U(0,x_0(0),y_0(0),0) \tag{3.41}$$

and in particular

$$\frac{d\overline{x}_i}{d\tau} = U_y(0, x_0(0), y_0(0) + \overline{y}_0(\tau), 0)\overline{y}_i(\tau) + \overline{P}_i(\tau) \quad (i \geq 1) \tag{3.42}$$

where

$$\overline{P}_i(\tau) = \overline{P}_i(\tau, x_0, x_1, \cdots x_i;\ y_0, y_1, \cdots y_i;\ \overline{x}_0, \overline{x}_1, \cdots \overline{x}_{i-1};\ \overline{y}_0, \overline{y}_1, \cdots \overline{y}_{i-1})$$

is infinitely differentiable with respect to its variables.

Now, it is time for us to consider one boundary condition

$$H_1(x(0,\varepsilon), y(0,\varepsilon), \varepsilon) = 0$$

Let

$$x(0,\varepsilon) = X(0,\varepsilon) + \varepsilon\overline{X}(0,\varepsilon); \qquad y(0,\varepsilon) = Y(0,\varepsilon) + \overline{Y}(0,\varepsilon)$$

We have

$$H_1(X(0,\varepsilon) + \varepsilon\overline{X}(0,\varepsilon), Y(0,\varepsilon) + \overline{Y}(0,\varepsilon)) = 0 \tag{3.43}$$

Next, we try to obtain Taylor series expansion for the right side of (3.43) about $\varepsilon = 0$, that is

$$\begin{aligned} 0 &= H_1(X(0,\varepsilon) + \varepsilon\overline{X}(0,\varepsilon), Y(0,\varepsilon) + \overline{Y}(0,\varepsilon), \varepsilon) \\ &= H_1(x_0(0), y_0(0) + \overline{y}_0(0), 0) + \frac{1}{i!}\sum_{i=1}^{\infty}[(X(0,\varepsilon) + \varepsilon\overline{X}(0,\varepsilon) - x_0(0))\frac{\partial}{\partial p} + \\ &\quad (Y(0,\varepsilon) + \overline{Y}(0,\varepsilon) - y_0(0) - \overline{y}_0(0))\frac{\partial}{\partial r} + \frac{\partial}{\partial \varepsilon}]^i H_1((x_0(0), y_0(0) + \overline{y}_0(0), 0). \end{aligned} \tag{3.44}$$

We have

$$\begin{aligned} &X(0,\varepsilon) + \varepsilon\overline{X}(0,\varepsilon) \\ &= x_0(0) + x_1(0)\varepsilon + x_2(0)\varepsilon^2 + \cdots + \overline{x}_0(0)\varepsilon + \overline{x}_1(0)\varepsilon^2 + \overline{x}_2(0)\varepsilon^3 + \cdots \end{aligned} \tag{3.45}$$

and

$$
\begin{aligned}
&Y(0,\varepsilon)+\overline{Y}(0,\varepsilon) \\
&= y_0(0)+y_1(0)\varepsilon+y_2(0)\varepsilon^2+\cdots+\overline{y}_0(0)+\overline{y}_1(0)\varepsilon+\overline{y}_2(0)\varepsilon^2+\cdots.
\end{aligned}
\tag{3.46}
$$

Substitute (3.45) and (3.46) into (3.44) and equate the coefficients of like powers of ε to obtain

$$
\begin{aligned}
&H_1(x_0(0),y_0(0)+\overline{y}_0(0),0)=0 \\
&H_{1,\,r}(x_0(0),y_0(0)+\overline{y}_0(0),0)\overline{y}_i(0)+M_i=0 \qquad (i\geq 1)
\end{aligned}
\tag{3.47}
$$

where

$$
M_i = M_i(x_0,x_1,\cdots x_i;y_0,y_1,\cdots y_i;\overline{x}_0,\overline{x}_1,\cdots\overline{x}_{i-1};\overline{y}_0,\overline{y}_1,\cdots\overline{y}_{i-1})|_{t=0,\tau=0}
$$

has a known value.

We combine the above results to obtain

$$
\begin{cases}
\dfrac{d\overline{x}_0}{d\tau}=U(0,x_0(0),y_0(0)+\overline{y}_0(\tau),0)-U(0,x_0(0),y_0(0),0) \\
\dfrac{d\overline{y}_0}{d\tau}=V_x(0,x_0(0),0,0)\overline{x}_0(\tau)+V_{\varepsilon y}(0,x_0(0),0,0)\overline{y}_0(\tau) \\
H_1(x_0(0),y_0(0)+\overline{y}_0(0),0)=0
\end{cases}
\tag{3.48}
$$

and

$$
\begin{cases}
\dfrac{d\overline{x}_i}{d\tau}=U_y(0,x_0(0),y_0(0)+\overline{y}_0(\tau),0)\overline{y}_i(\tau)+\overline{P}_i(\tau) \\
\dfrac{d\overline{y}_i}{d\tau}=V_x(0,x_0(0),0,0)\overline{x}_i(\tau)+V_{\varepsilon y}(0,x_0(0),0,0)\overline{y}_i(\tau)+\overline{Q}_i(\tau) \qquad (i\geq 1) \\
H_{1,\,r}(x_0(0),y_0(0)+\overline{y}_0(0),0)\overline{y}_i(0)+\overline{M}_i=0
\end{cases}
\tag{3.49}
$$

Substitute

$$
\begin{cases}
x=X(t,\varepsilon)+\varepsilon\widehat{X}(\sigma,\varepsilon) \\
y=Y(t,\varepsilon)+\widehat{Y}(\sigma,\varepsilon)
\end{cases}
$$

into (3.1) and (3.2) and use the same method to deal with the right boundary layer correction near $t = 1$ to obtain

$$\begin{cases} \dfrac{d\widehat{x}_0}{d\sigma} = -[U(1, x_0(1), y_0(1) + \widehat{y}_0(\sigma), 0) - U(1, x_0(1), y_0(1), 0)] \\ \dfrac{d\widehat{y}_0}{d\sigma} = -[V_x(1, x_0(1), 0, 0)\widehat{x}_0(\sigma) + V_{\epsilon y}(1, x_0(1), 0, 0)\widehat{y}_0(\sigma)] \\ H_2(x_0(1), y_0(1) + \widehat{y}_0(0), 0) = 0 \end{cases} \tag{3.50}$$

and

$$\begin{cases} \dfrac{d\widehat{x}_i}{d\sigma} = -U_y(1, x_0(1), y_0(1) + \widehat{y}_0(\sigma), 0)\widehat{y}_i(\sigma) + \widehat{P}_i(\sigma) \\ \dfrac{d\widehat{y}_i}{d\sigma} = -[V_x(1, x_0(1), 0, 0)\widehat{x}_i(\sigma) + V_{\epsilon y}(1, x_0(1), 0, 0)\widehat{y}_i(\sigma)] + \widehat{Q}_i(\sigma) \qquad (i \geq 1) \\ H_{2,\,s}(x_0(1), y_0(1) + \widehat{y}_0(0), 0)\widehat{y}_i(0) + \widehat{M}_i = 0 \end{cases} \tag{3.51}$$

where

$$\widehat{P}_i(\sigma) = \widehat{P}_i(\sigma, x_0, x_1, \cdots x_i; y_0, y_1, \cdots y_i; \widehat{x}_0, \widehat{x}_1, \cdots \widehat{x}_{i-1}; \widehat{y}_0, \widehat{y}_1, \cdots \widehat{y}_{i-1})$$

and

$$\widehat{Q}_i(\sigma) = \widehat{Q}_i(\sigma, x_0, x_1, \cdots x_i; y_0, y_1, \cdots y_i; \widehat{x}_0, \widehat{x}_1, \cdots \widehat{x}_{i-1}; \widehat{y}_0, \widehat{y}_1, \cdots \widehat{y}_{i-1})$$

are infinitely differentiable with respect to their variables respectively. Furthmore

$$\widehat{M}_i = \widehat{M}_i(x_0, x_1, \cdots x_i; y_0, y_1, \cdots y_i; \widehat{x}_0, \widehat{x}_1, \cdots \widehat{x}_{i-1}; \widehat{y}_0, \widehat{y}_1, \cdots \widehat{y}_{i-1})|_{t=1, \tau=0}$$

has a known value.

From the properties of the boundary layer corrections, it is natural to require

$$\overline{x}_i(+\infty) = \overline{y}_i(+\infty) = 0 \tag{3.52}$$

and

$$\widehat{x}_i(+\infty) = \widehat{y}_i(+\infty) = 0 \tag{3.53}$$

for all $i \geq 0$.

We have the following results for the problems (3.48) and (3.50)

Theorem 3.2.1: *Let Assumptions 1-4 hold, then the problems* (3.48) *and* (3.50) *have solutions*

$$\overline{x}_0(\tau),\ \overline{y}_0(\tau) \text{ and } \widehat{x}_0(\sigma),\ \widehat{y}_0(\sigma) \in C^\infty[0,+\infty)$$

such that

$$\frac{d^j}{d\tau^j}\begin{pmatrix}\overline{x}_0(\tau)\\ \overline{y}_0(\tau)\end{pmatrix} = O(e^{-\lambda_0\tau}),\ \ j\geq 0,\ \ \tau\in[0,+\infty)$$

and

$$\frac{d^j}{d\sigma^j}\begin{pmatrix}\widehat{x}_0(\sigma)\\ \widehat{y}_0(\sigma)\end{pmatrix} = O(e^{-\lambda_0\sigma}),\ j\geq 0,\ \sigma\in[0,+\infty).$$

Proof: We first consider the problem (3.48). After we rewrite the (3.48) and add another condition $\overline{y}_0(+\infty) = 0$, we have

$$\begin{cases} \dfrac{d^2\overline{y}_0}{d\tau^2} = V_{\epsilon y}(0,x_0(0),0,0)\dfrac{d\overline{y}_0}{d\tau} + V_x(0,x_0(0),0,0) \\ \qquad [(U(0,x_0(0),y_0(0)+\overline{y}_0(\tau),0) - U(0,x_0(0),y_0(0),0)], \\ H_1(x_0(0),y_0(0)+\overline{y}_0(0),0) = 0, \\ \overline{y}_0(+\infty) = 0. \end{cases} \tag{3.54}$$

In fact, (3.54) is (3.4) of Assumption 2 and has solution $\overline{y}_0(\tau)$. Substitute it back into (3.48) to obtain

$$\overline{x}_0(\tau) = V_x^{-1}(0,x_0(0),0,0)[\frac{d\overline{y}_0}{d\tau} - V_{\epsilon y}(0,x_0(0),0,0)\overline{y}_0(\tau)]$$

From (3.48), (3.52) and (3.54), we have

$$\overline{x}_0'(+\infty) = \overline{y}_0'(+\infty) = 0.$$

By Lemma (3.1.1)

$$\overline{y}_0(\tau) \in C^\infty[0, +\infty).$$

It is easy to show

$$\overline{x}_0(\tau) \in C^\infty[0, +\infty).$$

We differentiate both sides of ODE in (3.54) to obtain

$$\frac{d^3\overline{y}_0}{d\tau^3} = D_0(0)\frac{d^2\overline{y}_0}{d\tau^2} + C_0(0)\overline{B}_0(\tau)\frac{d\overline{y}_0}{d\tau},$$

where

$$\overline{B}_0(\tau) = \lim_{\sigma\to\infty} B_0(0, \tau, \sigma).$$

Define

$$z = \frac{d\overline{y}_0}{d\tau}.$$

We have

$$\begin{cases} \dfrac{d^2z}{d\tau^2} = D_0(0)\dfrac{dz}{d\tau} + C_0(0)\overline{B}_0(\tau)z, \\ z(0) = \dfrac{d\overline{y}_0}{d\tau}|_{\tau=0}, \\ z(+\infty) = 0. \end{cases} \tag{3.55}$$

Clearly, (3.55) has a solution

$$z = \frac{d\overline{y}_0}{d\tau}.$$

We will use Lemma 1.1 in Chapter 1 to show that this is the unique solution such that

$$z(\tau) = O(e^{-\lambda_0\tau}),$$

where λ_0 is defined in Assumption 3.

Let $r(\tau) = (|z(0)| + 1)e^{-\lambda_0\tau}$, we check that $r(\tau)$ satisfies the conditions (i)-(iii) in Lemma (1.1). Since here $n = 1$, so $\|r\| = |r|$.

It is clear that condition (i) holds.

For condition (ii), whenever $r(\tau) = |z(\tau)|$, we have

$$r'(\tau) = \frac{z \times z'}{|z(\tau)|}.$$

Since

$$r'(\tau) = (|z(0)| + 1)(-\lambda_0)e^{-\lambda_0 \tau} < 0.$$

Thus

$$z \times z' < 0.$$

Define

$$h(t, z, z') = D_0(0)z' + C_0(0)\overline{B}_0(\tau)z, \tag{3.56}$$

where

$$D_0(0) \leq 0, \qquad C_0(0)\overline{B}_0(\tau) \geq \lambda_0^2.$$

Thus

$$\begin{aligned}\frac{zh(t,z,z')}{|z(\tau)|} &= \frac{z(D_0(0)z' + C_0(0)\overline{B}_0(\tau)z)}{|z(\tau)|} = \frac{D_0(0)z \times z' + C_0(0)\overline{B}_0(\tau)z^2}{|z(\tau)|} \\ &\geq \frac{C_0(0)\overline{B}_0(\tau)z^2}{|z(\tau)|} \geq \lambda_0^2|z(\tau)| = \lambda_0^2 r(\tau) = r''(\tau)\end{aligned}$$

So, $r(\tau)$ satisfies the condition (ii).

As to the final condition (iii), by Lemma 3.1.2

$$|C_0(0)\overline{B}_0(\tau)| < k_0$$

for a certain constant k_0.

On the domain

$$D = \{(t, z) | \ |z(\tau)| \leq r(\tau)\},$$

we have

$$
\begin{aligned}
|h(t,z,z')| &= |D_0(0)z' + C_0(0)\overline{B}_0(\tau)z| \\
&\le |D_0(0)| \times |z'| + |C_0(0)\overline{B}_0(\tau)| \times |z| \\
&\le |D_0(0)| \times |z'| + k_0|z| \\
&\le |D_0(0)| \times |z'| + k_0 r(\tau) \\
&\le |D_0(0)| \times |z'| + k_0(|z(0)| + 1)
\end{aligned}
$$

Define

$$\varphi(s) = |D_0(0)| \times |s| + k_0(|z(0)| + 1).$$

It is easy to verify that this $\varphi(s)$ satisfies the Nagumo condition, thus condition (iii) still holds.

By Lemma 1.1 and 1.2, (3.55) has the unique solution

$$z(\tau) \in C^2[0, +\infty) \text{ and } |z(\tau)| \le r(\tau).$$

So

$$z(\tau) = O(e^{-\lambda_0 \tau}),$$

or

$$\frac{d\overline{y}_0}{d\tau} = O(e^{-\lambda_0 \tau}).$$

Since

$$\overline{y}_0(\tau) = \int_\infty^\tau \frac{d\overline{y}_0(s)}{ds} ds + \overline{y}_0(+\infty) = \int_\infty^\tau \frac{d\overline{y}_0(s)}{ds} ds.$$

Then

$$|\overline{y}_0(\tau)| \le \int_\infty^\tau \left|\frac{d\overline{y}_0(s)}{ds}\right| ds = O(e^{-\lambda_0 \tau}).$$

Based on the above results, from 2nd equation of (3.48), we obtain

$$\overline{x}_0(\tau) = O(e^{-\lambda_0 \tau}),$$

and

$$\frac{d\overline{x}_0}{d\tau} = O(e^{-\lambda_0\tau}).$$

Differentiating both sides of 2 ODE's in (3.48) for infinitely many times, we obtain, as required, that

$$\frac{d^j}{d\tau^j}\begin{pmatrix} \overline{x}_0(\tau) \\ \overline{y}_0(\tau) \end{pmatrix} = O(e^{-\lambda_0\tau}),\ j \geq 0,\ \tau \in [0, +\infty).$$

The problem (3.50) can be handled similarily. This completes the proof.

Based on the above results, with routine calculation, we have

$$\frac{d^j}{d\tau^j}\begin{pmatrix} \overline{P}_1(\tau) \\ \overline{Q}_1(\tau) \end{pmatrix} = O(e^{-(1-\mu_1)\lambda_0\tau}), \tag{3.57}$$

and

$$\frac{d^j}{d\tau^j}\begin{pmatrix} \widehat{P}_1(\sigma) \\ \widehat{Q}_1(\sigma) \end{pmatrix} = O(e^{-(1-\mu_1)\lambda_0\tau}), \tag{3.58}$$

for a certain constant $\mu_1 \in (0, 1)$ and any integer $j \geq 0$.

Theorem 3.2.2: *Let all assumptions 1-4 hold, then, for $i \geq 1$, the problems(3.49) and (3.51) have unique solutions*

$$\overline{x}_i(\tau),\ \overline{y}_i(\tau) \text{ and } \widehat{x}_i(\sigma),\ \widehat{y}_i(\sigma) \in C^\infty[0, +\infty\ )$$

such that

$$\frac{d^j}{d\tau^j}\begin{pmatrix} \overline{x}_i(\tau) \\ \overline{y}_i(\tau) \end{pmatrix} = O(e^{-(1-\mu_i)\lambda_0\tau}),\ \tau \in [0, +\infty),$$

and

$$\frac{d^j}{d\sigma^j}\begin{pmatrix} \widehat{x}_i(\sigma) \\ \widehat{y}_i(\sigma) \end{pmatrix} = O(e^{-(1-\mu_i)\lambda_0\sigma}),\ \sigma \in [0, +\infty)$$

for any integer $j \geq 0$, $\mu_i \in (0,1)$ *and*

$$0 < \mu_1 < \mu_2 < \cdots < 1.$$

Proof: We first consider the problem (3.49). Let $i = 1$, from (3.49), we have

$$\begin{cases} \dfrac{d^2\overline{y}_1}{d\tau^2} = D_0(0)\dfrac{d\overline{y}_1}{d\tau} + C_0(0)\overline{B}_0(\tau)\overline{y}_1 + f(\tau), \\ \overline{y}_1(0) = H_{1,\tau}^{-1}(x_0(0), y_0(0) + \overline{y}_0(0), 0)\overline{M}_1, \\ \overline{y}_1(+\infty) = 0, \end{cases} \tag{3.59}$$

where

$$f(\tau) = C_0(0)\overline{P}_1(\tau) + \frac{d\overline{Q}_1}{d\tau}.$$

From (3.57), we have

$$f(\tau) = O(e^{-(1-\mu_1)\lambda_0\tau}),$$

or

$$|f(\tau)| \leq k_1 e^{-(1-\mu_1)\lambda_0\tau}$$

for a certain constant k_1.

Let

$$m = \max\{1, \frac{k_1}{2\lambda_0^2(|\overline{y}_0(0)| + 1)(1-\mu_1)(2-\mu_1)}\}.$$

Define

$$\begin{aligned} r(\tau) &= m(|\overline{y}_1(0)| + 1)e^{-(1-\mu_1)\lambda_0\tau} \\ &\quad + \frac{1}{2\lambda_0}[e^{\lambda_0\tau}\int_\tau^\infty |f(s)|e^{-\lambda_0 s}ds + e^{-\lambda_0\tau}\int_0^\tau |f(s)|e^{\lambda_0 s}]. \end{aligned}$$

Thus

$$\begin{aligned} r'(\tau) &= -(1-\mu_1)\lambda_0 m(|\overline{y}_1(0)| + 1)e^{-(1-\mu_1)\lambda_0\tau} \\ &\quad + \frac{1}{2}[e^{\lambda_0\tau}\int_\tau^\infty |f(s)|e^{-\lambda_0 s}ds - e^{-\lambda_0\tau}\int_0^\tau |f(s)|e^{\lambda_0 s}] \end{aligned}$$

with

$$r''(\tau) = (1-\mu_1)^2\lambda_0^2 m(|\overline{y}_1(0)|+1)e^{-(1-\mu_1)\lambda_0\tau}$$
$$+\frac{\lambda}{2}[e^{\lambda_0\tau}\int_\tau^\infty |f(s)|e^{-\lambda_0 s}ds + e^{-\lambda_0\tau}\int_0^\tau |f(s)|e^{\lambda_0 s}] - |f(\tau)|$$

Now we check that $r(\tau)$ satisfies the conditions (i)-(iii) in Lemma 1.1. Here $n = 1, so\ \|r\| = |r|$.

It is clear that condition (i) holds.

We consider condition (ii), here

$$h(t, \overline{y}_1, \overline{y}_1') = D_0(0)\overline{y}_1' + C_0(0)\overline{B}_0(\tau)\overline{y}_1 + f(\tau)$$

Whenever

$$r(\tau) = |\overline{y}_1(\tau)|.$$

We have

$$r'(\tau) = \frac{\overline{y}_1 \times \overline{y}_1'}{|\overline{y}_1|}.$$

Since

$$\begin{aligned}
&\frac{1}{2}e^{\lambda_0\tau}\int_\tau^\infty |f(s)|e^{-\lambda_0 s}ds \\
&\leq \frac{1}{2}e^{\lambda_0\tau}\int_\tau^\infty k_1 e^{-(1-\mu_1)\lambda_0 s}e^{-\lambda_0 s}ds \\
&= \frac{k_1}{2}e^{\lambda_0\tau}\int_\tau^\infty k_1 e^{-(2-\mu_1)\lambda_0 s}ds \\
&= \frac{k_1}{2}e^{\lambda_0\tau}\frac{e^{-(2-\mu_1)\lambda_0\tau}}{(2-\mu_1)\lambda_0} \\
&= \frac{k_1 e^{-(1-\mu_1)\lambda_0\tau}}{2(2-\mu_1)\lambda_0}
\end{aligned}$$

and

$$\begin{aligned}
&(1-\mu_1)\lambda_0 m(|\overline{y}_1(0)|+1)e^{-(1-\mu_1)\lambda_0\tau} \\
&\geq (1-\mu_1)\lambda_0\frac{k_1}{2\lambda_0^2(|\overline{y}_0(0)|+1)(1-\mu_1)(2-\mu_1)}(|\overline{y}_1(0)|+1)e^{-(1-\mu_1)\lambda_0\tau} \\
&= \frac{k_1 e^{-(1-\mu_1)\lambda_0\tau}}{2(2-\mu_1)\lambda_0}
\end{aligned}$$

From the above inequalities, we obtain

$$r'(\tau) < 0,$$

thus

$$\overline{y}_1 \times \overline{y}_1' < 0.$$

Since

$$\begin{aligned}
&\frac{\overline{y}_1 h(t, \overline{y}_1, \overline{y}_1')}{|\overline{y}_1|} \\
&= \frac{\overline{y}_1 (D_0(0)\overline{y}_1' + C_0(0)\overline{B}_0(\tau)\overline{y}_1 + f(\tau))}{|\overline{y}_1|} \\
&\geq \frac{(\lambda_0^2 |\overline{y}_1|^2 + \overline{y}_1 f(\tau))}{|\overline{y}_1|} \\
&\geq \lambda_0^2 |\overline{y}_1| - |f(\tau)| \\
&\geq r''(\tau)
\end{aligned}$$

Thus, the condition (ii) holds.

It is easy to verify that the condition (iii) still holds here.

By Lemma 1.1 and 1.2, the problem (3.59) has the unique solution $\overline{y}_1(\tau)$ such that

$$|\overline{y}_1(\tau)| \leq r(\tau),$$

or

$$\overline{y}_1(\tau) = O(e^{-(1-\mu_1)\lambda_0 \tau}).$$

Thus, we find the unique solution $\overline{x}_1(\tau)$ such that

$$\overline{x}_1'(\tau) = O(e^{-(1-\mu_1)\lambda_0 \tau}).$$

We conclude

$$\overline{x}_1(\tau) = O(e^{-(1-\mu_1)\lambda_0 \tau}), \qquad \overline{y}_1'(\tau) = O(e^{-(1-\mu_1)\lambda_0 \tau}).$$

It is easy to obtain

$$\frac{d^j}{d\tau^j}\begin{pmatrix}\overline{x}_1(\tau)\\ \overline{y}_1(\tau)\end{pmatrix} = O(e^{-(1-\mu_1)\lambda_0\tau}),\ \tau\in[0,+\infty).$$

for any integer $j \geq 0$.

By the same method, we obtain that, for $i = 2$, the problem (3.49) has the unique solution $\overline{x}_2(\tau)$, $\overline{y}_2(\tau)$ such that

$$\frac{d^j}{d\tau^j}\begin{pmatrix}\overline{x}_2(\tau)\\ \overline{y}_2(\tau)\end{pmatrix} = O(e^{-(1-\mu_2)\lambda_0\tau}),\ \tau\in[0,+\infty)$$

for any $j \geq 0$, in particular

$$0 < \mu_1 < \mu_2 < 1.$$

By induction, we know that for any $i \geq 1$, the problem (3.49) has the unique solution $\overline{x}_i(\tau), \overline{y}_i(\tau)$ such that

$$\frac{d^j}{d\tau^j}\begin{pmatrix}\overline{x}_i(\tau)\\ \overline{y}_i(\tau)\end{pmatrix} = O(e^{-(1-\mu_i)\lambda_0\tau}),\ \tau\in[0,+\infty)$$

for any $j \geq 0$, in particular

$$0 < \mu_1 < \mu_2 < \cdots < 1.$$

The above method can be applied to deal with problem (3.51) analogously.

This completes the proof.

Theorem 3.5: *Let*

$$\begin{cases} x^N(t,\varepsilon) = \sum\limits_{i=0}^{N}(x_i(t) + \overline{x}_{i-1}(\tau) + \widehat{x}_{i-1}(\sigma))\varepsilon^i \\ y^N(t,\varepsilon) = \sum\limits_{i=0}^{N}(y_i(t) + \overline{y}_i(\tau) + \widehat{y}_i(\sigma))\varepsilon^i \end{cases} \tag{3.60}$$

where $N \geq 0$ and $\overline{x}_{-1}(\tau) = \widehat{x}_{-1}(\sigma) = 0$. *Then, there exists a constant* $\delta(N) > 0$ *such that*

$$\begin{cases} |x^N(t,\varepsilon)' - U(t,\ x^N(t,\varepsilon),\ y^N(t,\varepsilon),\ \varepsilon)| \leq \delta(N)\varepsilon^N, \\ |\varepsilon^2 y^N(t,\varepsilon)' - V(t,\ x^N(t,\varepsilon),\ \varepsilon y^N(t,\varepsilon),\ \varepsilon)| \leq \delta(N)\varepsilon^{N+1}, \end{cases} \tag{3.61}$$

where

$$x^N(t,\varepsilon)' = \frac{d}{dt}x^N(t,\varepsilon), \qquad y^N(t,\varepsilon)' = \frac{d}{dt}y^N(t,\varepsilon).$$

Proof: First we consider the case when t is far away from 1, then

$$\sigma = \frac{1-t}{\varepsilon} \to +\infty, \text{ if } \varepsilon \text{ is sufficiently small.}$$

Since

$$\frac{d^j}{d\sigma^j}\begin{pmatrix} \widehat{x}_i(\sigma) \\ \widehat{y}_i(\sigma) \end{pmatrix} = O(e^{-(1-\mu_i)\lambda_0\sigma})$$

where $\sigma \in [0, +\infty)$, $N-1 \geq i \geq 0$, $j \geq 0$.

Thus

$$\frac{d^j}{d\sigma^j}\begin{pmatrix} \widehat{x}_i(\sigma) \\ \widehat{y}_i(\sigma) \end{pmatrix} = O(\varepsilon^{N+1}).$$

We consider

$$X^N(t,\varepsilon)' - U(t, X^N(t,\varepsilon), Y^N(t,\varepsilon), \varepsilon)$$

where

$$\begin{cases} X^N(t,\varepsilon) = \sum_{i=0}^{N} x_i(t)\varepsilon^i, \\ Y^N(t,\varepsilon) = \sum_{i=0}^{N} y_i(t)\varepsilon^i. \end{cases}$$

Since

$$X^N(t,\varepsilon)' - U(t, X^N, Y^N, \varepsilon)$$
$$= \sum_{i=0}^{N} x_i'\varepsilon^i - U(t, \sum_{i=0}^{N} x_i\varepsilon^i, \sum_{i=0}^{N} y_i\varepsilon^i, \varepsilon)$$

$$= (x_0' + x_1'\varepsilon + \cdots + x_N'\varepsilon^N) - \{U(t, x_0, y_0, 0) + \frac{1}{i!}\varepsilon^i \sum_{i=1}^{N}[(x_1 + \cdots + x_N\varepsilon^{N-1})\frac{\partial}{\partial x} +$$

$$(y_1 + \cdots + y_N\varepsilon^{N-1})\frac{\partial}{\partial y} + \frac{\partial}{\partial \varepsilon}]^i U(t, x_0(t), y_0(t), 0) + \frac{1}{(N+1)!}\varepsilon^{N+1}[(x_1$$

$$+ \cdots + x_N\varepsilon^{N-1})\frac{\partial}{\partial x} + (y_1 + \cdots + y_N\varepsilon^{N-1})\frac{\partial}{\partial y} + \frac{\partial}{\partial \varepsilon}]^i U(t, x_0(t) +$$

$$\theta(X^N - x_0(t)), y_0(t) + \theta(Y^N - y_0(t)), \theta\varepsilon)$$

$$= x_0' - U(t, x_0, y_0, 0) + \sum_{i=1}^{N}[x_i'(t) - (\frac{\partial}{\partial x}x_i + \frac{\partial}{\partial y}y_i)U(t, x_0, y_0, 0) - P_i(t)]\varepsilon^i$$

$$+ O(\varepsilon^{N+1})$$

$$= O(\varepsilon^{N+1}).$$

where $\theta \in [0, 1]$, $P_i's$ are defined in (3.25).

Thus

$$X^N(t, \varepsilon)' - U(t, X^N, Y^N, \varepsilon) = O(\varepsilon^{N+1}). \tag{3.62}$$

On the other hand, we can prove

$$\varepsilon\frac{d}{dt}\overline{X}^{N-1} - [U(t, X^N + \varepsilon\overline{X}^{N-1}, Y^N + \overline{Y}^N, \varepsilon) - U(t, X^N, Y^N, \varepsilon)] = O(\varepsilon^N) \tag{3.63}$$

where

$$\begin{cases} \overline{X}^{N-1}(\tau, \varepsilon) = \sum_{i=0}^{N-1} \overline{x}_i\varepsilon^i, \\ \overline{Y}^N(\tau, \varepsilon) = \sum_{i=0}^{N} \overline{y}_i\varepsilon^i. \end{cases}$$

Add (3.62) to (3.63) to obtain

$$x^N(t, \varepsilon)' - U(t,\ x^N,\ y^N,\ \varepsilon) = O(\varepsilon^N) \tag{3.64}$$

If t is near 1, we can use the same method to obtain (3.64).

Thus

$$|x^N(t, \varepsilon)' - U(t,\ x^N,\ y^N,\ \varepsilon)| \le \delta_1(N)\varepsilon^N$$

for all $t \in [0, 1]$, $\delta_1(N)$ is a positive constant.

The above method can also be applied to obtain the second inequality

$$|\varepsilon^2 y^N(t,\varepsilon)' - V(t,\ x^N,\ \varepsilon y^N,\ \varepsilon)| \leq \delta_2(N)\varepsilon^{N+1}$$

for all $t \in [0,1]$, $\delta_2(N)$ is a positive constant.

Let

$$\delta(N) = \max(\delta_1(N), \delta_2(N)).$$

We have

$$\begin{cases} |x_N' - U(t,\ x^N,\ y^N,\ \varepsilon)| \leq \delta(N)\varepsilon^N, \\ |\varepsilon^2 y_N' - V(t,\ x^N,\ \varepsilon y^N,\ \varepsilon)| \leq \delta(N)\varepsilon^{N+1}. \end{cases}$$

This completes the proof.

We can use the same method to obtain the following results

$$\begin{cases} |H_1(x^N(0,\varepsilon),\ y^N(0,\varepsilon),\ \varepsilon)| \leq \rho(N)\varepsilon^{N+1} \\ |H_2(x^N(1,\varepsilon),\ y^N(1,\varepsilon),\ \varepsilon)| \leq \rho(N)\varepsilon^{N+1} \end{cases}$$

where $x^N(t,\varepsilon)$, $y^N(t,\varepsilon)$ are defined in (3.60), $\rho(N)$ is a constant.

3.3 The Main Theorem

Theorem 3.3.1: *Let Assumptions 1-4 hold, then, when* $N \geq 0$ is an integer and ε *is sufficiently small, the problem (3.1)-(3.2) has a unique solution* $x(t,\varepsilon)$ *and* $y(t,\varepsilon)$ $\in C^\infty[0,1]$ *satisfying*

$$\begin{cases} x(t,\varepsilon) = x^N(t,\varepsilon) + O(\varepsilon^{N+1}) \\ y(t,\varepsilon) = y^N(t,\varepsilon) + O(\varepsilon^{N+1}) \end{cases} \tag{3.65}$$

where $(x^N(t,\varepsilon), y^N(t,\varepsilon))$ *is defined in (3.60).*

The proof of this theorem involves steps (A)-(E).

Step A:

Let

$$z = x(t,\varepsilon) - x^N(t,\varepsilon), \ w = y(t,\varepsilon) - y^N(t,\varepsilon). \tag{3.66}$$

Substitute (3.66) into (3.1) and (3.2), we have

$$\int_0^1 (1-s)\frac{d^2}{ds^2}U(t,\, x^N(t,\varepsilon)+sz,\, y^N(t,\varepsilon)+sw,\, \varepsilon)ds$$
$$= (1-s)\frac{d}{ds}U(t, x^N(t,\varepsilon)+sz, y^N(t,\varepsilon)+sw, \varepsilon)|_{s=0}^{s=1} +$$
$$\int_0^1 \frac{d}{ds}U(t, x^N(t,\varepsilon)+sz, y^N(t,\varepsilon)+sw, \varepsilon)ds$$
$$= -\frac{d}{ds}U(t, x^N(t,\varepsilon)+sz, y^N(t,\varepsilon)+sw, \varepsilon)|_{s=0} +$$
$$[U(t, x^N(t,\varepsilon)+z, y^N(t,\varepsilon)+w, \varepsilon) -$$
$$U(t, x^N(t,\varepsilon), y^N(t,\varepsilon), \varepsilon)]$$
$$= -[U_x(t, x^N(t,\varepsilon), y^N(t,\varepsilon), \varepsilon)z + U_y(t, x^N(t,\varepsilon), y^N(t,\varepsilon), \varepsilon)w] +$$
$$[x^N(t,\varepsilon)' + z' - U(t, x^N(t,\varepsilon), y^N(t,\varepsilon), \varepsilon)]$$

So

$$z' = [U_x(t, x^N(t,\varepsilon), y^N(t,\varepsilon), \varepsilon)z + U_y(t, x^N(t,\varepsilon), y^N(t,\varepsilon), \varepsilon)w]$$
$$+ [U(t, x^N(t,\varepsilon), y^N(t,\varepsilon), \varepsilon) - x^N(t,\varepsilon)'] +$$
$$\int_0^1 (1-s)\frac{d^2}{ds^2}[U(\,t,\, x^N(t,\varepsilon)+sz,\, y^N(t,\varepsilon)+sw, \varepsilon)]ds$$

On the other hand

$$\varepsilon^2 w' = [V_x(t, x^N(t,\varepsilon), \varepsilon y^N(t,\varepsilon), \varepsilon)z + V_{\varepsilon y}(t, x^N(t,\varepsilon), \varepsilon y^N(t,\varepsilon), \varepsilon)\varepsilon w]$$
$$+ [V(t, x^N(t,\varepsilon), \varepsilon y^N(t,\varepsilon), \varepsilon) - \varepsilon^2 y^N(t,\varepsilon)'] +$$
$$+ \int_0^1 (1-s)\frac{d^2}{ds^2}[V(\,t,\, x^N(t,\varepsilon)+sz, \varepsilon(y^N(t,\varepsilon)+sw), \varepsilon)]ds$$

Combine the above results to yield

$$\begin{pmatrix} z' \\ \varepsilon^2 w' \end{pmatrix} = \begin{pmatrix} A & B \\ C & \varepsilon D \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} + \begin{pmatrix} E_1(t,z,w,\varepsilon) + \rho_1(t,\varepsilon) \\ E_2(t,z,w,\varepsilon) + \rho_2(t,\varepsilon) \end{pmatrix} \tag{3.67}$$

where

$$\begin{cases} A = U_x(t, x^N(t,\varepsilon), y^N(t,\varepsilon), \varepsilon) \\ B = U_y(t, x^N(t,\varepsilon), y^N(t,\varepsilon), \varepsilon) \\ C = V_x(t, x^N(t,\varepsilon), \varepsilon y^N(t,\varepsilon), \varepsilon) \\ D = V_{\varepsilon y}(t, x^N(t,\varepsilon), \varepsilon y^N(t,\varepsilon), \varepsilon) \end{cases}, \tag{3.68}$$

and

$$\begin{pmatrix} E_1(t,z,w,\varepsilon) \\ E_2(t,z,w,\varepsilon) \end{pmatrix} = \int_0^1 (1-s) \frac{d^2}{ds^2} \begin{pmatrix} U(t, x^N(t,\varepsilon) + sz, y^N(t,\varepsilon) + sw, \varepsilon) \\ V(t, x^N(t,\varepsilon) + sz, \varepsilon(y^N(t,\varepsilon) + sw), \varepsilon) \end{pmatrix} ds, \tag{3.69}$$

and

$$\begin{pmatrix} \rho_1(t,\varepsilon) \\ \rho_2(t,\varepsilon) \end{pmatrix} = \begin{pmatrix} U(t, x^N(t,\varepsilon), y^N(t,\varepsilon), \varepsilon) \\ V(t, x^N(t,\varepsilon), \varepsilon y^N(t,\varepsilon), \varepsilon) \end{pmatrix} - \begin{pmatrix} x^N(t,\varepsilon)' \\ \varepsilon^2 y^N(t,\varepsilon)' \end{pmatrix} = \begin{pmatrix} O(\varepsilon^N) \\ O(\varepsilon^{N+1}) \end{pmatrix}. \tag{3.70}$$

The same linearization method can be applied to handle the boundary conditions to obtain

$$L(\varepsilon) \begin{pmatrix} z(0,\varepsilon) \\ w(0,\varepsilon) \end{pmatrix} + R(\varepsilon) \begin{pmatrix} z(1,\varepsilon) \\ w(1,\varepsilon) \end{pmatrix} = - \begin{pmatrix} H_1(x^N(0,\varepsilon), y^N(0,\varepsilon), \varepsilon) + F_1(z,w,\varepsilon) \\ H_2(x^N(1,\varepsilon), y^N(1,\varepsilon), \varepsilon) + F_2(z,w,\varepsilon) \end{pmatrix} \tag{3.71}$$

where

$$L(\varepsilon) = \begin{pmatrix} H_{1,p}(x^N(0,\varepsilon), y^N(0,\varepsilon), \varepsilon) & H_{1,r}(x^N(0,\varepsilon), y^N(0,\varepsilon), \varepsilon) \\ 0 & 0 \end{pmatrix}, \tag{3.72}$$

and

$$R(\varepsilon) = \begin{pmatrix} 0 & 0 \\ H_{2,q}(x^N(1,\varepsilon), y^N(1,\varepsilon), \varepsilon) & H_{2,s}(x^N(1,\varepsilon), y^N(1,\varepsilon), \varepsilon) \end{pmatrix}. \tag{3.73}$$

and

$$\begin{pmatrix} F_1(z,w,\varepsilon) \\ F_2(z,w,\varepsilon) \end{pmatrix} = \int_0^1 (1-s)\frac{d^2}{ds^2} \begin{pmatrix} H_1(x^N(0,\varepsilon)+sz(0,\varepsilon), y^N(0,\varepsilon)+sw(0,\varepsilon), \varepsilon) \\ H_2(x^N(1,\varepsilon)+sz(1,\varepsilon), y^N(1,\varepsilon)+sw(1,\varepsilon), \varepsilon) \end{pmatrix} ds, \tag{3.74}$$

From (3.69)

$$\begin{aligned}
&E_1(t,z,w,\varepsilon) \\
&= \int_0^1 (1-s)\frac{d^2}{ds^2} U(t, x^N(t,\varepsilon)+sz, y^N(t,\varepsilon)+sw, \varepsilon) ds \\
&= \int_0^1 (1-s)\{\frac{d}{ds}[z\frac{\partial}{\partial x} + w\frac{\partial}{\partial y}]U(t, x^N(t,\varepsilon)+sz, y^N(t,\varepsilon)+sw, \varepsilon)\} ds \\
&= \int_0^1 (1-s)[z^2\frac{\partial^2}{\partial x^2} + w^2\frac{\partial^2}{\partial y^2} + 2zw\frac{\partial^2}{\partial x \partial y}]U(t, x^N(t,\varepsilon)+sz, y^N(t,\varepsilon)+sw, \varepsilon) ds
\end{aligned}$$

Thus, it is easy to conclude

$$|E_1(t,z,w,\varepsilon)| \leq |E_1|(\|z\|_1 + \|w\|_1)^2$$

where $|E_1|$ is a positive constant, and norm $\|\cdot\|_1$is defined by

$$\|x(t)\|_1 = \sup_{t\in[0,1]} |x(t)|.$$

Furthmore

$$|E_1(t,z_1,w_1,\varepsilon) - E_1(t,z_2,w_2,\varepsilon)| \leq \alpha_1 |E_1|(\|z_1 - z_2\|_1 + \|w_1 - w_2\|_1)$$

where

$$\alpha_1 = \max_{i=1,2}\{\|z_i\|_1, \|w_i\|_1\}.$$

The same method can be used to obtain

$$|E_2(t,z,w,\varepsilon)| \le |E_2|(\|z\|_1 + \varepsilon\|w\|_1)^2$$

and

$$|E_2(t,z_1,w_1,\varepsilon) - E_2(t,z_2,w_2,\varepsilon)| \le \alpha_2|E_2|(\|z_1 - z_2\|_1 + \|w_1 - w_2\|_1)$$

where $|E_2|$ is a positive constant, and

$$\alpha_2 = \max_{i=1,2}\{\|z_i\|_1, \varepsilon\|w_i\|_1\}.$$

Let

$$|E| = \max\{|E_1|, |E_2|\}.$$

Combine the above results to obtain

$$\begin{cases} |E_1(t,z,w,\varepsilon)| \le |E|(\|z\|_1 + \|w\|_1)^2 \\ |E_2(t,z,w,\varepsilon)| \le |E|(\|z\|_1 + \varepsilon\|w\|_1)^2 \end{cases} \tag{3.75}$$

and

$$\begin{cases} |E_1(t,z_1,w_1,\varepsilon) - E_1(t,z_2,w_2,\varepsilon)| \le \alpha_1|E|(\|z_1 - z_2\|_1 + \|w_1 - w_2\|_1) \\ |E_2(t,z_1,w_1,\varepsilon) - E_2(t,z_2,w_2,\varepsilon)| \le \alpha_2|E|(\|z_1 - z_2\|_1 + \|w_1 - w_2\|_1) \end{cases}. \tag{3.76}$$

The same analysis can be applied to handle functions F_1 and F_2 to yield

$$\begin{cases} |F_1(t,z,w,\varepsilon)| \le |F|(|z(0,\varepsilon)| + |w(0,\varepsilon)|)^2 \\ |F_2(t,z,w,\varepsilon)| \le |F|(|z(1,\varepsilon)| + |w(1,\varepsilon)|)^2 \end{cases} \tag{3.77}$$

and

$$\begin{cases} |F_1(t,z_1,w_1,\varepsilon) - F_1(t,z_2,w_2,\varepsilon)| \le \beta_1|F|(|z_1(0,\varepsilon) - z_2(0,\varepsilon)| + |w_1(0,\varepsilon) - w_2(0,\varepsilon)|) \\ |F_2(t,z_1,w_1,\varepsilon) - F_2(t,z_2,w_2,\varepsilon)| \le \beta_2|F|(|z_1(1,\varepsilon) - z_2(1,\varepsilon)| + |w_1(1,\varepsilon) - w_2(1,\varepsilon)|) \end{cases} \tag{3.78}$$

where $|F|$ is a positive constant, and

$$\beta_1 = \max_{i=1,2}\{\|z_i(0,\varepsilon)\|_1, \|w_i(0,\varepsilon)\|_1\},\ \beta_2 = \max_{i=1,2}\{\|z_i(1,\varepsilon)\|_1, \|w_i(1,\varepsilon)\|_1\}.$$

Let

$$u = z,\ v = \varepsilon w.$$

We can transform (3.67) and (3.71) into the following

$$\varepsilon \begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} \varepsilon A & B \\ C & D \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \varepsilon \widehat{E}_1(t,u,v,\varepsilon) + \varepsilon\rho_1(t,\varepsilon) \\ \widehat{E}_2(t,u,v,\varepsilon) + \rho_2(t,\varepsilon) \end{pmatrix} \tag{3.79}$$

and

$$\widehat{L}(\varepsilon) \begin{pmatrix} u(0,\varepsilon) \\ v(0,\varepsilon) \end{pmatrix} + \widehat{R}(\varepsilon) \begin{pmatrix} u(1,\varepsilon) \\ v(1,\varepsilon) \end{pmatrix} = -\begin{pmatrix} H_1(x^N(0,\varepsilon), y^N(0,\varepsilon), \varepsilon) + \widehat{F}_1 \\ H_2(x^N(1,\varepsilon), y^N(1,\varepsilon), \varepsilon) + \widehat{F}_2 \end{pmatrix} \tag{3.80}$$

where

$$\begin{pmatrix} \widehat{E}_i(t,u,v,\varepsilon) \\ \widehat{F}_i(t,u,v,\varepsilon) \end{pmatrix} = \begin{pmatrix} E_i(t,u,\varepsilon^{-1}v,\varepsilon) \\ F_i(t,u,\varepsilon^{-1}v,\varepsilon) \end{pmatrix} \quad (i = 1,2), \tag{3.81}$$

and

$$\widehat{L}(\varepsilon) = \begin{pmatrix} H_{1,p}(x^N(0,\varepsilon), y^N(0,\varepsilon), \varepsilon) & \varepsilon^{-1}H_{1,r}(x^N(0,\varepsilon), y^N(0,\varepsilon), \varepsilon) \\ 0 & 0 \end{pmatrix}, \tag{3.82}$$

and

$$\widehat{R}(\varepsilon) = \begin{pmatrix} 0 & 0 \\ H_{2,q}(x^N(1,\varepsilon), y^N(1,\varepsilon), \varepsilon) & \varepsilon^{-1}H_{2,s}(x^N(1,\varepsilon), y^N(1,\varepsilon), \varepsilon) \end{pmatrix}. \tag{3.83}$$

Rewrite (3.75)-(3.78) to obtain the following

$$\begin{cases} |\widehat{E}_1(t,u,v,\varepsilon)| \le |E|(\|u\|_1 + \varepsilon^{-1}\|v\|_1)^2 \\ |\widehat{E}_2(t,u,v,\varepsilon)| \le |E|(\|u\|_1 + \|v\|_1)^2 \\ |\widehat{F}_1(t,u,v,\varepsilon)| \le |F|(\|u(0,\varepsilon)\|_1 + \varepsilon^{-1}\|v(0,\varepsilon)\|_1)^2 \\ |\widehat{F}_2(t,u,v,\varepsilon)| \le |F|(\|u(1,\varepsilon)\|_1 + \varepsilon^{-1}\|v(1,\varepsilon)\|_1)^2 \end{cases}$$

and

$$\begin{cases} |\widehat{E}_1(t,u_1,v_1,\varepsilon) - \widehat{E}_1(t,u_2,v_2,\varepsilon)| \le \widehat{\alpha}_1|E|(\|u_1-u_2\|_1 + \varepsilon^{-1}\|v_1-v_2\|_1) \\ |\widehat{E}_2(t,u_1,v_1,\varepsilon) - \widehat{E}_2(t,u_2,v_2,\varepsilon)| \le \widehat{\alpha}_2|E|(\|u_1-u_2\|_1 + \varepsilon^{-1}\|v_1-v_2\|_1) \\ |\widehat{F}_1(t,u_1,v_1,\varepsilon) - \widehat{F}_1(t,u_2,v_2,\varepsilon)| \le \widehat{\beta}_1|F|(|u_1(0,\varepsilon)-u_2(0,\varepsilon)| + \varepsilon^{-1}|v_1(0,\varepsilon)-v_2(0,\varepsilon)|) \\ |\widehat{F}_1(t,u_1,v_1,\varepsilon) - \widehat{F}_1(t,u_2,v_2,\varepsilon)| \le \widehat{\beta}_2|F|(|u_1(1,\varepsilon)-u_2(1,\varepsilon)| + \varepsilon^{-1}|v_1(1,\varepsilon)-v_2(1,\varepsilon)|) \end{cases} \tag{3.84}$$

where

$$\widehat{\alpha}_1 = \max_{i=1,2}\{\|u_i\|_1, \varepsilon^{-1}\|v_i\|_1\}, \widehat{\alpha}_2 = \max_{i=1,2}\{\|u_i\|_1, \|v_i\|_1\},$$

and

$$\widehat{\beta}_1 = \max_{i=1,2}\{\|u_i(0,\varepsilon)\|_1, \varepsilon^{-1}\|v_i(0,\varepsilon)\|_1\}, \widehat{\beta}_2 = \max_{i=1,2}\{\|u_i(1,\varepsilon)\|_1, \varepsilon^{-1}\|v_i(1,\varepsilon)\|_1\}.$$

Step B:

Next, we consider the homogeneous part of (3.79)

$$\varepsilon \begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} \varepsilon A & B \\ C & D \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{3.85}$$

Rewrite (3.85) into the following

$$\varepsilon \begin{pmatrix} u \\ v \end{pmatrix}' = \left[\begin{pmatrix} 0 & B_0 \\ C_0 & D_0 \end{pmatrix} + \varepsilon Z \right] \begin{pmatrix} u \\ v \end{pmatrix} \tag{3.86}$$

where $B_0 = B_0(t,\tau,\sigma)$, $C_0 = C_0(t)$, $D_0 = D_0(t)$ and

$$Z = \begin{pmatrix} A & (B-B_0)/\varepsilon \\ (C-C_0)/\varepsilon & (D-D_0)/\varepsilon \end{pmatrix}$$

is bounded.

We will transform (3.86) into the diagonalized form.

Now, we try to find a nonsingular matrix $\widehat{P}$ such that

$$\widehat{P}^{-1} \begin{pmatrix} 0 & B_0 \\ C_0 & D_0 \end{pmatrix} \widehat{P} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

where λ_1 and λ_2 are the eigenvaules of matrix

$$\begin{pmatrix} 0 & B_0 \\ C_0 & D_0 \end{pmatrix}.$$

In fact

$$\lambda_1(t,\varepsilon) = \frac{D_0 - \sqrt{D_0^2 + 4C_0B_0}}{2}, \tag{3.87}$$

and

$$\lambda_2(t,\varepsilon) = \frac{D_0 + \sqrt{D_0^2 + 4C_0B_0}}{2}. \tag{3.88}$$

By Assumption 3

$$C_0B_0 \geq \lambda_0^2$$

Thus, λ_1 and λ_2 are nonzero for $t \in [0,1], 0 < \varepsilon \leq \varepsilon_0$. Since λ_1 and λ_2 are the continuous functions with respect to $t \in [0,1], 0 < \varepsilon \leq \varepsilon_0$, they are bounded, say,

$$0 > m_2 \geq \lambda_1 \geq m_1 \tag{3.89}$$

and

$$m_4 \geq \lambda_2 \geq m_3 > 0 \tag{3.90}$$

where m_1, m_2, m_3 and m_4 are constants.

The eigenvector for the eigenvalue λ_1 is

$$\begin{pmatrix} 1 \\ \lambda_1/B_0 \end{pmatrix}.$$

The eigenvector for the eigenvalue λ_2 is

$$\begin{pmatrix} 1 \\ \lambda_2/B_0 \end{pmatrix}.$$

Define

$$\widehat{P}(t,\varepsilon) = \begin{pmatrix} 1 & 1 \\ \lambda_1/B_0 & \lambda_2/B_0 \end{pmatrix}.$$

Thus

$$\widehat{P}^{-1} = \frac{B_0}{\lambda_1 - \lambda_2} \begin{pmatrix} -\lambda_2/B_0 & 1 \\ \lambda_1/B_0 & -1 \end{pmatrix},$$

with

$$\widehat{P}^{-1} \begin{pmatrix} 0 & B_0 \\ C_0 & D_0 \end{pmatrix} \widehat{P} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Clearly, $\widehat{P}$, $\widehat{P}^{-1}$ and $\widehat{P}'$ are bounded.

Let

$$\begin{pmatrix} u \\ v \end{pmatrix} = \widehat{P} \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}. \tag{3.91}$$

From (3.86), we have

$$\varepsilon \left[\widehat{P} \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}' + \widehat{P}' \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} \right] = \left[\begin{pmatrix} 0 & B_0 \\ C_0 & D_0 \end{pmatrix} + \varepsilon Z \right] \widehat{P} \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix},$$

thus

$$\varepsilon\widehat{P}\begin{pmatrix}\widehat{u}\\ \widehat{v}\end{pmatrix}' = \left[\begin{pmatrix}0 & B_0\\ C_0 & D_0\end{pmatrix}\widehat{P}+\varepsilon(Z\widehat{P}-\widehat{P}')\right]\begin{pmatrix}\widehat{u}\\ \widehat{v}\end{pmatrix},$$

so

$$\varepsilon\begin{pmatrix}\widehat{u}\\ \widehat{v}\end{pmatrix}' = \left[\widehat{P}^{-1}\begin{pmatrix}0 & B_0\\ C_0 & D_0\end{pmatrix}\widehat{P}+\varepsilon\widehat{P}^{-1}(Z\widehat{P}-\widehat{P}')\right]\begin{pmatrix}\widehat{u}\\ \widehat{v}\end{pmatrix},$$

finally

$$\varepsilon\begin{pmatrix}\widehat{u}\\ \widehat{v}\end{pmatrix}' = \left[\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}+\varepsilon\widehat{P}^{-1}(Z\widehat{P}-\widehat{P}')\right]\begin{pmatrix}\widehat{u}\\ \widehat{v}\end{pmatrix}. \tag{3.92}$$

Define

$$\begin{pmatrix}\Delta_{11} & \varepsilon\Delta_{12}\\ \varepsilon\Delta_{21} & \Delta_{22}\end{pmatrix} = \left[\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}+\varepsilon\widehat{P}^{-1}(Z\widehat{P}-\widehat{P}')\right].$$

where $\Delta_{11}, \Delta_{12}, \Delta_{21}$ and Δ_{22} are bounded. Since $\widehat{P}^{-1}(Z\widehat{P}-\widehat{P}')$ is bounded, when ε is sufficiently small, from (3.89) and (3.90), we conclude

$$0 > \frac{m_2}{2} \geq \Delta_{11} \geq 2m_1, \tag{3.93}$$

and

$$2m_4 \geq \Delta_{22} \geq \frac{m_3}{2} > 0. \tag{3.94}$$

Thus

$$\varepsilon\begin{pmatrix}\widehat{u}\\ \widehat{v}\end{pmatrix}' = \begin{pmatrix}\Delta_{11} & \varepsilon\Delta_{12}\\ \varepsilon\Delta_{21} & \Delta_{22}\end{pmatrix}\begin{pmatrix}\widehat{u}\\ \widehat{v}\end{pmatrix}. \tag{3.95}$$

Now, we adopt a Riccati transformation

$$\begin{pmatrix}\widehat{u}\\ \widehat{v}\end{pmatrix} = R(t,\varepsilon)\begin{pmatrix}\xi\\ \eta\end{pmatrix} \tag{3.96}$$

where

$$R(t,\varepsilon) = \begin{pmatrix} 1 & -\varepsilon S \\ -T & 1+\varepsilon TS \end{pmatrix}, \tag{3.97}$$

with

$$R^{-1}(t,\varepsilon) = \begin{pmatrix} 1+\varepsilon ST & \varepsilon S \\ T & 1 \end{pmatrix}. \tag{3.98}$$

Thus, (3.96) takes (3.95) into the following

$$\varepsilon \begin{pmatrix} \xi \\ \eta \end{pmatrix}' = \begin{pmatrix} \Delta_{11} - \varepsilon \Delta_{12} T & 0 \\ 0 & \Delta_{22} + \varepsilon T \Delta_{12} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{3.99}$$

where S, T are the solutions of the problems

$$\begin{cases} \varepsilon \dfrac{dT}{dt} = (\Delta_{22} - \Delta_{11})T + \varepsilon \Delta_{12} T^2 - \varepsilon \Delta_{21} \\ T(1) = 0 \end{cases} \tag{3.100}$$

and

$$\begin{cases} \varepsilon \dfrac{dS}{dt} = (\Delta_{11} - \Delta_{22} - \varepsilon \Delta_{12} T - \varepsilon T \Delta_{12})S - \Delta_{12} \\ S(0) = 0 \end{cases} \tag{3.101}$$

Lemma 3.3.1 *Let Assumptions 1-3 hold, then for all sufficiently small $\varepsilon > 0$, the problem (3.100) and (3.101) have unique solutions $T(t,\varepsilon)$, $S(t,\varepsilon)$ such that*

$$T(t,\varepsilon) = O(\varepsilon),\ S(t,\varepsilon) = O(1),\ t \in [0,1].$$

Proof: First we consider (3.100), we can rewrite (3.100) as

$$T(t,\varepsilon) = \int_1^t e^{-\int_t^s \frac{1}{\varepsilon}(\Delta_{22}-\Delta_{11})du}(\Delta_{12}T^2 - \Delta_{21})ds. \tag{3.102}$$

Define the integral operator $I(T)$ equal to the right side of (3.102), from (3.93) and (3.94), we have

$$\begin{cases} \|I(T)\|_1 \leq \varepsilon k(\|T\|_1^2 + 1) \\ \|I(T_1) - I(T_2)\|_1 \leq \varepsilon k(\|T_1\|_1 + \|T_2\|_1)(\|T_1 - T_2\|_1) \end{cases}$$

for any $T(t,\varepsilon)$, $T_1(t,\varepsilon)$, $T_2(t,\varepsilon) \in C[0,1]$, where k is a constant and norm $\|\cdot\|_1$ is defined by $\|x(t)\|_1 = \sup\limits_{0\leq t\leq 1} |x(t)|$

Define

$$\widehat{B} = \{T(t,\varepsilon) | T(t,\varepsilon) \in C[0,1], \text{ and } \|T\|_1 \leq 2\varepsilon k\}.$$

Thus, $\widehat{B}$ is a closed ball in the Banach space $B = \{T(t,\varepsilon)|T(t,\varepsilon) \in C[0,1]\}$.

When

$$0 < \varepsilon \leq \min\{1, \frac{1}{4k}, \varepsilon_0\}.$$

We have

$$\|I(T)\|_1 \leq \varepsilon k(\|T\|_1^2 + 1) < \varepsilon k(1+1) = 2\varepsilon k$$

and

$$\begin{aligned} &\|I(T_1) - I(T_2)\|_1 \\ &\leq \varepsilon k(\|T_1\|_1 + \|T_2\|_1)(\|T_1 - T_2\|_1) \\ &\leq \varepsilon k(2\varepsilon k + 2\varepsilon k)(\|T_1 - T_2\|_1) \\ &= 4\varepsilon^2 k^2(\|T_1 - T_2\|_1) \\ &\leq 4k^2 \frac{1}{16k^2}\|T_1 - T_2\|_1 \\ &= \frac{1}{4}\|T_1 - T_2\|_1 \end{aligned}$$

where T, T_1, $T_2 \in \widehat{B}$

From the above results, we use Theorem (1.1) to show (3.102) has the unique solution $T(t,\varepsilon) \in \widehat{B}$, obviously this solution $T(t,\varepsilon) = O(\varepsilon)$. Put this solution $T(t,\varepsilon)$

into (3.101) to find the unique solution

$$S(t,\varepsilon)=\int_0^t e^{\frac{1}{\varepsilon}\int_s^t(\Delta_{11}-\Delta_{22}-\varepsilon\Delta_{12}T-\varepsilon T\Delta_{12})}\frac{1}{\varepsilon}(-\Delta_{12})ds.$$

Clearly

$$S(t,\varepsilon)=O(1).$$

This completes the proof.

Step C:

We can find the fundamental solution matrix $\phi(t,\varepsilon)$ for (3.99)

$$\phi(t,\varepsilon)=\begin{pmatrix} e^{\frac{1}{\varepsilon}\int_0^t(\Delta_{11}-\varepsilon\Delta_{12}T)ds} & 0 \\ 0 & e^{\frac{1}{\varepsilon}\int_1^t(\Delta_{22}+\varepsilon\Delta_{21}T)ds}\end{pmatrix}. \tag{3.103}$$

Thus, (3.85) has the fundamental solution matrix $\psi(t,\varepsilon)$

$$\psi(t,\varepsilon)=\widehat{P}R\begin{pmatrix} e^{\frac{1}{\varepsilon}\int_0^t(\Delta_{11}-\varepsilon\Delta_{12}T)ds} & 0 \\ 0 & e^{\frac{1}{\varepsilon}\int_1^t(\Delta_{22}+\varepsilon\Delta_{21}T)ds}\end{pmatrix}. \tag{3.104}$$

Clearly, when ε is sufficiently small, we have

$$\begin{pmatrix} \psi(t,\varepsilon)P\psi^{-1}(s,\varepsilon) \\ \psi(s,\varepsilon)(I_2-P)\psi^{-1}(t,\varepsilon)\end{pmatrix}=O(e^{-\frac{k_2}{\varepsilon}(t-s)}) \tag{3.105}$$

where

$$0\le s\le t\le 1,\ k_2=\min(-\frac{m_2}{4},\frac{m_3}{4}),\ P=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}.$$

Let

$$M(\varepsilon)=\widehat{L}\psi(0,\varepsilon)+\widehat{R}\psi(1,\varepsilon).$$

Lemma 3.3.2: *For all sufficiently small $\varepsilon > 0$, $M(\varepsilon)$ is invertible and $M^{-1}(\varepsilon)$ satisfies*

$$M^{-1}(\varepsilon) = O(\varepsilon)$$

Proof: From (3.104), we have

$$\begin{aligned}
&\psi(0,\varepsilon)\\
&= P(0,\varepsilon)R(0,\varepsilon)\begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{1}{\varepsilon}\int_1^0(\Delta_{22}+\varepsilon\Delta_{21}T)ds} \end{pmatrix}\\
&= \begin{pmatrix} 1 & 1 \\ \dfrac{\lambda_1(0,\varepsilon)}{B_0} & \dfrac{\lambda_2(0,\varepsilon)}{B_0} \end{pmatrix}\begin{pmatrix} 1 & -\varepsilon S(0,\varepsilon) \\ -T(0,\varepsilon) & 1+\varepsilon T(0,\varepsilon)S(0,\varepsilon) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & O(e^{-\frac{k_2}{\varepsilon}}) \end{pmatrix}\\
&= \begin{pmatrix} 1 & 1 \\ \dfrac{\lambda_1(0,\varepsilon)}{B_0} & \dfrac{\lambda_2(0,\varepsilon)}{B_0} \end{pmatrix}\begin{pmatrix} 1 & O(e^{-\frac{k_2}{\varepsilon}}) \\ -T(0,\varepsilon) & O(e^{-\frac{k_2}{\varepsilon}}) \end{pmatrix}\\
&= \begin{pmatrix} 1-T(0,\varepsilon) & O(e^{-\frac{k_2}{\varepsilon}}) \\ \dfrac{\lambda_1(0,\varepsilon)-\lambda_2(0,\varepsilon)T(0,\varepsilon)}{B_0} & O(e^{-\frac{k_2}{\varepsilon}}) \end{pmatrix}
\end{aligned}$$

From (3.82), we have

$$\widehat{L}(\varepsilon) = \begin{pmatrix} H_{1,p}(x^N(0,\varepsilon), y^N(0,\varepsilon),\varepsilon) & \varepsilon^{-1}H_{1,r}(x^N(0,\varepsilon), y^N(0,\varepsilon),\varepsilon) \\ 0 & 0 \end{pmatrix}.$$

Since

$$\begin{aligned}
&H_{1,r}(x^N(0,\varepsilon), y^N(0,\varepsilon),\varepsilon)\\
&= H_{1,r}(x_0(0), y_0(0)+\overline{y}_0(0)+\widehat{y}_0(\tfrac{1}{\varepsilon}),0) + \varepsilon\frac{d}{d\varepsilon}H_{1,r}(x^N(0,\theta_1\varepsilon), y^N(0,\theta_1\varepsilon),\theta_1\varepsilon)\\
&= H_{1,r}(x_0(0), y_0(0)+\overline{y}_0(0),0) + [\frac{\partial}{\partial r}H_{1,r}(x_0(0), y_0(0)+\overline{y}_0(0)+\theta_2\widehat{y}_0(\frac{1}{\varepsilon}),0)]\times\\
&\qquad \widehat{y}_0(\frac{1}{\varepsilon}) + \varepsilon\frac{d}{d\varepsilon}H_{1,r}(x^N(0,\theta_1\varepsilon), y^N(0,\theta_1\varepsilon),\theta_1\varepsilon)
\end{aligned}$$

where $\theta_1, \theta_2 \in [0,1]$, $\hat{y}_0(\frac{1}{\varepsilon}) = O(e^{-\lambda_0/\varepsilon})$. By Assumption 4, when ε is enough small

$$H_{1,r}(x^N(0,\varepsilon), y^N(0,\varepsilon), \varepsilon) \neq 0.$$

Then

$$\begin{aligned}
&\hat{L}\phi(0,\varepsilon)\\
&= \begin{pmatrix} H_{1,p}(x^N(0,\varepsilon), y^N(0,\varepsilon), \varepsilon) & \varepsilon^{-1} H_{1,r}(x^N(0,\varepsilon), y^N(0,\varepsilon), \varepsilon) \\ 0 & 0 \end{pmatrix}\\
&\quad \times \begin{pmatrix} 1 - T(0,\varepsilon) & O(e^{-\frac{k_2}{\varepsilon}}) \\ \dfrac{\lambda_1(0,\varepsilon) - \lambda_2(0,\varepsilon)T(0,\varepsilon)}{B_0} & O(e^{-\frac{k_2}{\varepsilon}}) \end{pmatrix}\\
&= \begin{pmatrix} C_1\varepsilon^{-1} + O(1) & O(e^{-\frac{k_2}{\varepsilon}}) \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

where $C_1 \neq 0$.

Similarly

$$\hat{R}\phi(1,\varepsilon) = \begin{pmatrix} 0 & 0 \\ O(e^{-\frac{k_2}{\varepsilon}}) & C_2\varepsilon^{-1} + O(1)) \end{pmatrix}.$$

where $C_2 \neq 0$.

Thus

$$M(\varepsilon) = \begin{pmatrix} C_1\varepsilon^{-1} + O(1) & O(e^{-\frac{k_2}{\varepsilon}}) \\ O(e^{-\frac{k_2}{\varepsilon}}) & C_2\varepsilon^{-1} + O(1)) \end{pmatrix}.$$

So, $M(\varepsilon)$ is invertible such that

$$M^{-1}(\varepsilon) = O(\varepsilon). \tag{3.106}$$

This completes the proof.

Step D:

By Theorem 1.2, (3.79) can be rewriten as

$$\begin{pmatrix} u \\ v \end{pmatrix} = \psi(t,\varepsilon)M^{-1}H + \int_0^1 G(t,s)Eds \tag{3.107}$$

where

$$G(t,s) = \begin{cases} \psi(t,\varepsilon)M^{-1}\widehat{L}\psi(0,\varepsilon)\psi(s,\varepsilon)^{-1} & \text{for } t > s \\ -\psi(t,\varepsilon)M^{-1}\widehat{R}\psi(1,\varepsilon)\psi(s,\varepsilon)^{-1} & \text{for } t \le s \end{cases}$$

and

$$H = -\begin{pmatrix} H_1(x^N(0,\varepsilon), y^N(0,\varepsilon),\varepsilon) + \widehat{F}_1 \\ H_2(x^N(1,\varepsilon), y^N(1,\varepsilon),\varepsilon) + \widehat{F}_2 \end{pmatrix},$$

and

$$E = \begin{pmatrix} \widehat{E}_1(s,u,v,\varepsilon) + \rho_1(s,\varepsilon) \\ \varepsilon^{-1}(\widehat{E}_2(s,u,v,\varepsilon) + \varepsilon^{-1}\rho_2(s,\varepsilon) \end{pmatrix}.$$

Since

$$\begin{aligned}
&\int_0^t \psi(t,\varepsilon)M^{-1}\widehat{L}\psi(0,\varepsilon)\psi(s,\varepsilon)^{-1}Eds + \int_t^1 \psi(t,\varepsilon)M^{-1}\widehat{L}\psi(0,\varepsilon)[I_2 - P]\psi(s,\varepsilon)^{-1}Eds \\
&\quad + \int_0^t \psi(t,\varepsilon)M^{-1}\widehat{R}\psi(1,\varepsilon)P\psi(s,\varepsilon)^{-1}Eds \\
&= \int_0^t \psi(t,\varepsilon)M^{-1}\widehat{L}\psi(0,\varepsilon)(I_2 - P + P)\psi(s,\varepsilon)^{-1}Eds + \int_t^1 \psi(t,\varepsilon)M^{-1}\widehat{L}\psi(0,\varepsilon) \\
&\quad [I_2 - P]\psi(s,\varepsilon)^{-1}Eds + \int_0^t \psi(t,\varepsilon)M^{-1}\widehat{R}\psi(1,\varepsilon)P\psi(s,\varepsilon)^{-1}Eds \\
&= \int_0^1 \psi(t,\varepsilon)M^{-1}\widehat{L}\psi(0,\varepsilon)(I_2 - P)\psi(s,\varepsilon)^{-1}Eds + \int_0^t \psi(t,\varepsilon)M^{-1}\widehat{L}\psi(0,\varepsilon)P \\
&\quad \psi(s,\varepsilon)^{-1}Eds + \int_0^t \psi(t,\varepsilon)M^{-1}\widehat{R}\psi(1,\varepsilon)P\psi(s,\varepsilon)^{-1}Eds \\
&= \psi(t,\varepsilon)M^{-1}\widehat{L}\int_0^1 \psi(0,\varepsilon)(I_2 - P)\psi(s,\varepsilon)^{-1}Eds + \int_0^t \psi(t,\varepsilon)P\psi(s,\varepsilon)^{-1}Eds
\end{aligned}$$

Similarly

$$-\int_t^1 \psi(t,\varepsilon)M^{-1}\widehat{R}\psi(1,\varepsilon)\psi(s,\varepsilon)^{-1}Eds - \int_t^1 \psi(t,\varepsilon)M^{-1}\widehat{L}\psi(0,\varepsilon)[I_2 - P]\psi(s,\varepsilon)^{-1}Eds$$

$$-\int_0^t \psi(t,\varepsilon)M^{-1}\widehat{R}\psi(1,\varepsilon)P\psi(s,\varepsilon)^{-1}Eds$$
$$= -\psi(t,\varepsilon)M^{-1}\widehat{R}\int_0^1 \psi(1,\varepsilon)P\psi(s,\varepsilon)^{-1}Eds - \int_t^1 \psi(t,\varepsilon)(I_2 - P)\psi(s,\varepsilon)^{-1}Eds$$

Add the above results to obtain

$$\int_0^1 G(t,s)Eds$$
$$= \int_0^t \psi(t,\varepsilon)M^{-1}\widehat{L}\psi(0,\varepsilon)\psi(s,\varepsilon)^{-1}Eds - \int_t^1 \psi(t,\varepsilon)M^{-1}\widehat{R}\psi(1,\varepsilon)\psi(s,\varepsilon)^{-1}Eds$$
$$= \psi(t,\varepsilon)M^{-1}\{\widehat{L}\int_0^1 \psi(0,\varepsilon)(I_2 - P)\psi(s,\varepsilon)^{-1}Eds - \widehat{R}\int_0^1 \psi(1,\varepsilon)P\psi(s,\varepsilon)^{-1}Eds\}$$
$$+ \int_0^t \psi(t,\varepsilon)P\psi(s,\varepsilon)^{-1}Eds - \int_t^1 \psi(t,\varepsilon)(I_2 - P)\psi(s,\varepsilon)^{-1}Eds$$

Then, (3.107) can be rewritten as

$$\begin{pmatrix} u \\ v \end{pmatrix} = \psi(t,\varepsilon)M^{-1}(\varepsilon)\widehat{H}(u,v,\varepsilon) + \ell_1(t,u,v,\varepsilon) + \ell_2(t,u,v,\varepsilon) \tag{3.108}$$

where

$$\widehat{H}(u,v,\varepsilon) = H(u,v,\varepsilon) - \widehat{L}\ell_2(0,u,v,\varepsilon) - \widehat{R}\ell_1(1,u,v,\varepsilon), \tag{3.109}$$

with

$$\ell_1(t,u,v,\varepsilon) = \int_0^t \psi(t,\varepsilon)P\psi^{-1}(s,\varepsilon)Eds, \tag{3.110}$$

and

$$\ell_2(t,u,v,\varepsilon) = -\int_t^1 \psi(t,\varepsilon)(I_2 - P)\psi^{-1}(s,\varepsilon)Eds. \tag{3.111}$$

Step E:

We provide the final proof for Theorem 3.3.1.

Define the norm

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_\varepsilon = \|u\|_1 + \varepsilon^{-1}\|v\|_1, \text{ for } \varepsilon > 0, \text{ and } u,v \in C[0,1]. \tag{3.112}$$

From (3.105), we have

$$\left\|\psi(t,\varepsilon)P\psi^{-1}(s,\varepsilon)\right\|_{\varepsilon} \le c_1 e^{-\frac{k_2}{\varepsilon}(t-s)}$$

and

$$\left\|\psi(s,\varepsilon)(I_2-P)\psi^{-1}(t,\varepsilon)\right\|_{\varepsilon} \le c_1 e^{-\frac{k_2}{\varepsilon}(t-s)}$$

where $0 \le s \le t \le 1$, c_1 is a constant.

Since

$$\begin{aligned}
&\|\ell_1(t,u,v,\varepsilon)\|_{\varepsilon} \\
&= \left\|\int_0^t \psi(t,\varepsilon)P\psi^{-1}(s,\varepsilon)E ds\right\|_{\varepsilon} \\
&\le \int_0^t \left\|\psi(t,\varepsilon)P\psi^{-1}(s,\varepsilon)\right\|_{\varepsilon} \times \left\|\begin{pmatrix} \widehat{E}_1(t,u,v,\varepsilon)+\rho_1(t,\varepsilon) \\ \varepsilon^{-1}(\widehat{E}_2(t,u,v,\varepsilon)+\varepsilon^{-1}\rho_2(t,\varepsilon)\end{pmatrix}\right\|_{\varepsilon} ds \\
&\le \int_0^t c_1 e^{-\frac{k_2}{\varepsilon}(t-s)}[\left\|\widehat{E}_1(t,u,v,\varepsilon)\right\|_1 + \|\rho_1(t,\varepsilon)\|_1 + \\
&\qquad \varepsilon^{-2}(\left\|\widehat{E}_2(t,u,v,\varepsilon)\right\|_1 + \|\rho_2(t,\varepsilon)\|_1)]ds \\
&\le \int_0^t c_1 e^{-\frac{k_2}{\varepsilon}(t-s)}[|E|(\|u\|_1+\varepsilon^{-1}\|v\|_1)^2+\delta(N)\varepsilon^N + \\
&\qquad \varepsilon^{-2}(|E|(\|u\|_1+\|v\|_1)^2+\delta(N)\varepsilon^{N+1})]ds \\
&\le [(|E|+\varepsilon^{-2})(\|u\|_1+\varepsilon^{-1}\|v\|_1)^2+\delta(N)(\varepsilon^N+\varepsilon^{N-1})]\int_0^t c_1 e^{-\frac{k_2}{\varepsilon}(t-s)}ds \\
&\le [(|E|+\varepsilon^{-2})(\|u\|_1+\varepsilon^{-1}\|v\|_1)^2+\delta(N)(\varepsilon^N+\varepsilon^{N-1})] \times (c_1\frac{\varepsilon}{k_2}) \\
&\le \frac{c_1}{k_2}[(|E|\varepsilon+\varepsilon^{-1})\left\|\begin{pmatrix} u \\ v \end{pmatrix}\right\|_{\varepsilon}^2 + \delta(N)(\varepsilon^{N+1}+\varepsilon^N)] \\
&\le \frac{c_1}{k_2}[2\varepsilon^{-1}\left\|\begin{pmatrix} u \\ v \end{pmatrix}\right\|_{\varepsilon}^2 + 2\delta(N)\varepsilon^N] \qquad \text{(whenever } \varepsilon^2 < 1/|E|\text{, and } \varepsilon < 1)
\end{aligned}$$

$$\leq \frac{c_1}{k_2} c_2 [\varepsilon^{-1} \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_\varepsilon^2 + \varepsilon^N] \qquad (\text{ define } c_2 = \max\{2, 2\delta(N)\})$$

$$\leq \|\ell_1\| \, [\varepsilon^{-1} \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_\varepsilon^2 + \varepsilon^N] \qquad (\text{ define } \|\ell_1\| = \frac{c_1 c_2}{k_2})$$

Thus

$$\|\ell_1(t, u, v, \varepsilon)\|_\varepsilon \leq \|\ell_1\| \, (\varepsilon^{-1} \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_\varepsilon^2 + \varepsilon^N) \tag{3.113}$$

On the other hand

$$\|\ell_1(t, u_1, v_1, \varepsilon) - \ell_1(t, u_2, v_2, \varepsilon)\|_\varepsilon$$

$$= \left\| \int_0^t \psi(t, \varepsilon) P \psi^{-1}(s, \varepsilon) \begin{pmatrix} \widehat{E}_1(t, u_1, v_1, \varepsilon) - \widehat{E}_1(t, u_1, v_1, \varepsilon) \\ \varepsilon^{-1}(\widehat{E}_2(t, u_1, v_1, \varepsilon) - \widehat{E}_2(t, u_2, v_2, \varepsilon)) \end{pmatrix} ds \right\|_\varepsilon$$

$$\leq \int_0^t \left\| \psi(t, \varepsilon) P \psi^{-1}(s, \varepsilon) \right\|_\varepsilon [\widehat{\alpha}_1 |E| (\|u_1 - u_2\|_1 + \varepsilon^{-1} \|v_1 - v_2\|_1) +$$

$$\widehat{\alpha}_2 \varepsilon^{-2} |E| (\|u_1 - u_2\|_1 + \varepsilon^{-1} \|v_1 - v_2\|_1)] ds$$

$$\leq (\widehat{\alpha}_1 + \widehat{\alpha}_2 \varepsilon^{-2}) |E| (\|u_1 - u_2\|_1 + \varepsilon^{-1} \|v_1 - v_2\|_1)(c_1 \frac{\varepsilon}{k_2})$$

$$\leq (c_1 \frac{\varepsilon}{k_2})(\widehat{\alpha}_1 + \widehat{\alpha}_2 \varepsilon^{-2}) |E| \left\| \begin{pmatrix} u_1 - u_2 \\ v_1 - v_2 \end{pmatrix} \right\|_\varepsilon \qquad (\text{define } c_3 = \max(\widehat{\alpha}_1, \widehat{\alpha}_2))$$

$$\leq 2 c_1 \frac{1}{k_2} c_3 \varepsilon^{-1} |E| \left\| \begin{pmatrix} u_1 - u_2 \\ v_1 - v_2 \end{pmatrix} \right\|_\varepsilon \qquad (\text{whenever } 0 < \varepsilon < 1)$$

$$\leq \|\ell_1\| \, d \varepsilon^{-1} |E| \left\| \begin{pmatrix} u_1 - u_2 \\ v_1 - v_2 \end{pmatrix} \right\|_\varepsilon \qquad (\text{define } d = \frac{c_3}{c_2})$$

Thus

$$\|\ell_1(t, u_1, v_1, \varepsilon) - \ell_1(t, u_2, v_2, \varepsilon)\|_\varepsilon \leq \|\ell_1\| \, d \varepsilon^{-1} \left\| \begin{pmatrix} u_1 - u_2 \\ v_1 - v_2 \end{pmatrix} \right\|_\varepsilon . \tag{3.114}$$

Using the same method, we conclude for all sufficiently small $\varepsilon > 0$, there exists positive constants $\|\ell_2\|$ and $\|H\|$ such that

$$\begin{cases} \|\ell_2(t,u,v,\varepsilon)\|_\varepsilon \leq \|\ell_2\| \, [\varepsilon^{-1} \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_\varepsilon^2 + \varepsilon^N] \\ \left\| \widehat{H}(u,v,\varepsilon) \right\|_\varepsilon \leq \|H\| \, [\varepsilon^{-1} \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_\varepsilon^2 + \varepsilon^N] \end{cases} \tag{3.115}$$

and

$$\begin{cases} \|\ell_2(t,u_1,v_1,\varepsilon) - \ell_2(t,u_2,v_2,\varepsilon)\|_\varepsilon \leq \|\ell_2\| \, d\varepsilon^{-1} \left\| \begin{pmatrix} u_1 - u_2 \\ v_1 - v_2 \end{pmatrix} \right\|_\varepsilon \\ \left\| \widehat{H}(u_1,v_1,\varepsilon) - \widehat{H}(u_2,v_2,\varepsilon) \right\|_\varepsilon \leq \|H\| \, d\varepsilon^{-1} \left\| \begin{pmatrix} u_1 - u_2 \\ v_1 - v_2 \end{pmatrix} \right\|_\varepsilon \end{cases} \tag{3.116}$$

Let

$$B = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \mid u(t), \; v(t) \in C[0,1] \right\}$$

with norm (3.112).

Thus, B is a Banach space.

Let

$$\tilde{B} = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \mid u(t), \; v(t) \in C[0,1] \text{ and } \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_\varepsilon \leq k_3 \varepsilon^N \right\} \tag{3.117}$$

where

$$k_3 = 2(k_4 k_5 \|H\| + \|\ell_1\| + \|\ell_2\|),$$

k_4 and k_5 are constants satisfying

$$\|\psi(t,\varepsilon)\|_\varepsilon \leq k_4, \qquad \left\| M^{-1}(\varepsilon) \right\|_\varepsilon \leq k_5 \varepsilon. \tag{3.118}$$

Thus, $\widetilde{B}$ is a closed ball inside B.

Define an integral operator

$$\widehat{I}: \begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} u \\ v \end{pmatrix} \text{ for any } \begin{pmatrix} u \\ v \end{pmatrix} \in \widetilde{B} \tag{3.119}$$

where $\widehat{I}\left[\begin{pmatrix} u \\ v \end{pmatrix}\right]$ is the right side of the (3.108).

From the above inequalities for ℓ_1, ℓ_2 and $\widehat{H}$, we conclude that when

$$N \geq 2,\ 0 < \varepsilon < \min\{1, 1/(4k_3^2+1)\},$$

we have

$$\begin{aligned}
&\left\| \widehat{I}\left[\begin{pmatrix} u \\ v \end{pmatrix}\right] \right\|_\varepsilon \\
&\leq \|\psi(t,\varepsilon)M^{-1}(\varepsilon)H(u,v,\varepsilon)\|_\varepsilon + \|\ell_1(t,u,v,\varepsilon)\|_\varepsilon + \|\ell_2(t,u,v,\varepsilon)\|_\varepsilon \\
&\leq (k_4k_5\varepsilon\,\|H\| + \|\ell_1\| + \|\ell_2\|)[\varepsilon^{-1}\left\|\begin{pmatrix} u \\ v \end{pmatrix}\right\|_\varepsilon^2 + \varepsilon^N] \\
&\leq \frac{1}{2}k_3(k_3^2\varepsilon^{2N-1} + \varepsilon^N) \\
&< \frac{1}{2}k_3(\varepsilon^N + \varepsilon^N) \\
&= k_3\varepsilon^N
\end{aligned}$$

so, $\widehat{I}$ maps $\widetilde{B}$ into $\widetilde{B}$.

Since

$$\begin{aligned}
&\left\| \widehat{I}\left[\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}\right] - \widehat{I}\left[\begin{pmatrix} u \\ v_2 \end{pmatrix}\right] \right\|_\varepsilon \\
&\leq (k_4k_5\varepsilon\,\|H\| + \|\ell_1\| + \|\ell_2\|)d\varepsilon^{-1}\left\|\begin{pmatrix} u_1 - u_2 \\ v_1 - v_2 \end{pmatrix}\right\|_\varepsilon
\end{aligned}$$

$$\leq \frac{1}{2}k_3 \times k_3\varepsilon^{-1} \left\| \begin{pmatrix} u_1 - u_2 \\ v_1 - v_2 \end{pmatrix} \right\|_\varepsilon$$

$$\leq \frac{1}{2} \left\| \begin{pmatrix} u_1 - u_2 \\ v_1 - v_2 \end{pmatrix} \right\|_\varepsilon$$

thus, $\widehat{I}$ is contrative on $\widetilde{B}$ for all $\begin{pmatrix} u_i \\ v_i \end{pmatrix} \in \widetilde{B}$, $i = 1, 2$

By Banach/Picard fixed-point theorem (Theorem 1.1), there exists a unique solution $\begin{pmatrix} u \\ v \end{pmatrix} \in \widetilde{B}$ to the equation

$$\widehat{I}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

for all sufficiently small $\varepsilon > 0$, thus the problems (3.108) has the unique solution $\begin{pmatrix} u \\ v \end{pmatrix}$ such that

$$\|u\|_1 \leq k_3\varepsilon^N, \ \ \|v\|_1 \leq k_3\varepsilon^{N+1}$$

or

$$\|z\|_1 \leq k_3\varepsilon^N, \ \ \|w\|_1 \leq k_3\varepsilon^N.$$

On the other hand

$$\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} x - x^N \\ y - y^N \end{pmatrix} = \begin{pmatrix} x - x^{N+1} \\ y - y^{N+1} \end{pmatrix} + \begin{pmatrix} x^{N+1} - x^N \\ y^{N+1} - y^N \end{pmatrix}$$

$$= O(\varepsilon^{N+1}) + O(\varepsilon^{N+1})$$

$$= O(\varepsilon^{N+1})$$

The result for $N = 0$ or 1 follows from

$$x = x^{[2]} + O(\varepsilon^3), \qquad y = y^{[2]} + O(\varepsilon^3)$$

Since

$$x^{[2]} = x^{[1]} + O(\varepsilon^2), \qquad y^{[2]} = y^{[1]} + O(\varepsilon^2)$$

so

$$x = x^{[1]} + O(\varepsilon^2), \qquad y = y^{[1]} + O(\varepsilon^2)$$

clearly

$$x = x^{[0]} + O(\varepsilon), \qquad y = y^{[0]} + O(\varepsilon)$$

We remark that in order to avoid confusion, we denote $(x^N,\ y^N)$ for $N = 0$, 1 and 2 by

$$(x^{[0]},\ y^{[0]}),\ (x^{[1]},\ y^{[1]}) \text{ and } (x^{[2]},\ y^{[2]})$$

respectively.

This completes the proof.

3.4 Application

Consider the singular perturbation problem P_ε consisting of

$$\begin{cases} \dfrac{dx}{dt} = \varepsilon x + y \\ \varepsilon^2 \dfrac{dy}{dt} = x^2 + t\varepsilon y - (t+1)^4 \end{cases} \qquad t \in [0, 1] \tag{3.120}$$

and the boundary conditions

$$\begin{cases} y(0, \varepsilon) = 0 \\ y(1, \varepsilon) = 1 \end{cases} \tag{3.121}$$

This problem is of the form (3.1)-(3.2) with

$$\begin{cases} U(t,x,y,\varepsilon) = \varepsilon x + y \\ V(t,x,\varepsilon y,\varepsilon) = x^2 + t\varepsilon y - (t+1)^4 \end{cases}$$

and

$$\begin{cases} H_1(x(0,\varepsilon),y(0,\varepsilon),\varepsilon) = y(0,\varepsilon) \\ H_1(x(1,\varepsilon),y(1,\varepsilon),\varepsilon) = y(1,\varepsilon) - 1 \end{cases}$$

The reduced problem is given by

$$\begin{cases} \dfrac{dx_0}{dt} = y_0 \\ 0 = x_0^2 - (t+1)^4 \end{cases} \tag{3.122}$$

It has two solutions

$$\begin{cases} x_0(t) = (t+1)^2 \\ y_0(t) = 2(t+1) \end{cases} \tag{3.123}$$

and

$$\begin{cases} x_0(t) = -(t+1)^2 \\ y_0(t) = -2(t+1) \end{cases} \tag{3.124}$$

Since (3.124) does not satisfy Assumption 3, we choose (3.123).

For Assumption 2, we have

$$\begin{cases} \dfrac{d^2\overline{y}_0}{d\tau^2} = 2\overline{y}_0 \\ \overline{y}_0(0) = -2 \\ \overline{y}_0(+\infty) = 0 \end{cases} \tag{3.125}$$

and

$$\begin{cases} \dfrac{d^2\widehat{y}_0}{d\tau^2} = -\dfrac{d\widehat{y}_0}{d\tau} + 8\widehat{y}_0 \\ \widehat{y}_0(0) = -3 \\ \widehat{y}_0(+\infty) = 0 \end{cases} \tag{3.126}$$

The solutions for (3.125) and (3.126) are

$$\overline{y}_0(\tau) = -2e^{-\sqrt{2}\tau} \tag{3.127}$$

and

$$\widehat{y}_0(\sigma) = -3e^{-\frac{1+\sqrt{33}}{2}\sigma} \tag{3.128}$$

respectively.

For Assumption 3, we have

$$B_0(t,\tau,\sigma) = 1, C_0(t) = 2(t+1), D_0(t) = t.$$

Thus

$$C_0(t)B_0(t,\tau,\sigma) \geq 2,\ \ D_0(0) = 0,\ \ D_0(1) = 1 \geq 0.$$

For Assumption 4, we have

$$H_{1,\tau}(x_0(0), y_0(0) + \overline{y}_0(0), 0) = 1,$$

and

$$H_{2,s}(x_0(1), y_0(1) + \widehat{y}_0(0), 0) = 1.$$

Since four Assumptions hold in this example, the problem has the solution such that

$$\begin{cases} x = (t+1)^2 + O(\varepsilon) \\ y = 2(t+1) - 2e^{-\sqrt{2}\frac{t}{\varepsilon}} - 3e^{-\frac{1+\sqrt{33}}{2}\frac{(1-t)}{\varepsilon}} + O(\varepsilon) \end{cases} \tag{3.129}$$

uniformly for $t \in [0,1]$ and all sufficiently small $\varepsilon > 0$.

The above result for (3.120) and (3.121) cannot be deduced from [1]-[21].

3.5 Conclusion

In this thesis, we study a new class of scalar singular singularly perturbed problems under appropriate assumptions and obtain some new results. Compared to Shi[16], our result has the improvement and also the restriction. The improvement is that we can change $V(t, x, \varepsilon^2 y, \varepsilon)$ in Shi's case to $V(t, x, \varepsilon y, \varepsilon)$ in our case, which extends Shi's problem for scalar case. The restriction is that we can only handle the problem with scalar functions and the problem with vector functions still remains open, which we would continue to investigate.

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