THE UNIVERSITY OF CALGARY

DIFFERENTIABLE MANIFOLDS AND SOME BASIC NOTIONS OF CONTINUUM MECHANICS

by

© REUVEN SEGEV

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE

DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MECHANICAL ENGINEERING

CALGARY, ALBERTA

MAY 1981

© Reuven Segev 1981

THE UNIVERSITY OF CALGARY

FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled, "Differentiable Manifolds and Some Basic Notions of Continuum Mechanics" submitted by Mr. Reuven Segev in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

M. Epstein

M. Epstein, Supervisor
Department of Mechanical Engineering

Peter Glockner

P.G. Glockner
Department of Mechanical Engineering

Mansa Singh

M.C. Singh
Department of Mechanical Engineering

J. Śniatycki

J. Śniatycki
Department of Mathematics

Jerrold E. Marsden

J.E. Marsden, External Examiner
Department of Mathematics
University of California - Berkeley

May 1981

ABSTRACT

In an attempt to give an invariant formulation of continuum mechanics, the placements of a body in space are described in terms of two models. In the first, the global model, one considers the set of C^k, $1 \leq k < \infty$, embeddings of a compact manifold B, the body, in a manifold without a boundary S, the physical space. This set is known to be an infinite dimensional differentiable manifold and it is called the global configuration space. Then, a global virtual displacement and a global force are defined as elements of the tangent bundle and cotangent bundle of the global configuration space, respectively. In the second model, the local model, one considers the set of C^r, $1 \leq r \leq k$, vector bundle morphisms $\pi \to \rho$, where π is a vector bundle with base B, ρ is a vector bundle with base S and the vector bundle morphisms considered have elements of the global configuration space as base maps. This set, called the local configuration space, is an infinite dimensional vector bundle with the global configuration space as a base manifold. Again, a local virtual displacement and a local force are defined as elements of the tangent bundle and cotangent bundle of the local configuration space, respectively. In order to express the idea that both models represent the same physical phenomenon, compatibility conditions relating the local and global variables are given in terms of a section of the local configuration space. Since such a section relates global and local configurations, its tangent map relates local and global virtual displacements and the adjoint to the tangent map relates global forces to local forces. This last compatibility condition of forces is a generalization of the principle of virtual work and of the equilibrium

equation of continuum mechanics. The consistency of the suggested formulation with continuum mechanics is indicated by the following:

a. It is shown that local forces can be represented by sections of some vector bundle and if a connection is given on the manifold S a local force can be given by a tensor field which is a generalization of the stress field.

b. If the tangent functor, assigning to each configuration its tangent map, is used as a compatibility section, compatibility equations can be written and a general solution for the case where the global force is zero, is given in terms of a stress function.

c. If, in addition to (b), a connection is specified on S the equilibrium equation and boundary conditions of continuum mechanics are recovered.

This work is dedicated to my parents and grandparents to whom I owe my education.

ACKNOWLEDGEMENTS

I wish to express my gratitude to Professor Marcelo Epstein for the inspiration, guidance and help he provided during the thesis research.

Acknowledgements are also due to the Killam Foundation for the generous financial support during the last two years, to Professor P.G. Glockner and the Department of Mechanical Engineering for the support and encouragement, and to Mrs. Karen Undseth for her skillful typing.

LIST OF CONTENTS

CHAPTER 1 - INTRODUCTION

The purpose of this work is to present a covariant formulation of some basic notions of continuum mechanics, the most important of which are stress, compatibility and equilibrium.

1.1 Covariance in Physical Theories

The requirement that a physical theory have a covariant formulation originated with Einstein's theory of relativity and as more and more predictions of the theory were verified experimentally, the notion of covariance became generally accepted. Nevertheless, the importance of covariance reaches beyond the ability of general relativity to explain some physical observations. The main idea is that in classical physics one applies the mental construction of classical mathematics to the physical world and while using all its structure some of the naturality of the laws postulated is lost. Since no mathematical structure is inherent in the physical world, the less mathematical structure a physical theory exploits the more natural and hence preferable it is. In accordance with this philosophy we would rather use the term "mathematical model" than the more classical term "physical law".

A practical added benefit for such a covariant formulation is that it has a larger range of applications.

In the following text "covariant formulation" of a physical theory means a formulation on a differentiable manifold which is independent of coordinate representations.

1.2 Existing Formulations

In order that a comparison with the suggested formulation can be made, some assumptions, definitions and results concerning forces, stresses and equilibrium as given by two existing formulations will be reviewed concisely. We will consider classical continuum mechanics as given by C. Truesdell [1], [2] and a more recent covariant formulation given by J. Marsden and T. Hughes [3].

1.2.1 Classical Continuum Mechanics

a. In the framework of this theory it is assumed that physical phenomena occur in the 3-dimensional Euclidean space and this structure is being used intensively throughout the definitions.

b. Forces are not defined but are primitive quantities subjected to the requirement that mathematically they are vector measures.

c. Forces are assumed to be given as the sum of two different types of forces: body forces and surface forces given by integrals of vector fields over the interior of the body and its boundary, respectively.

d. The axiom of equilibrium states that the total force acting on the body is zero. (We restrict here the authors' dynamical treatment to the statical case.)

e. In classical continuum mechanics it is assumed that the traction on all like-oriented surfaces with a common tangent plane at some point of the body is the same. This assumption is called Cauchy's postulate and can be proven using the axiom of equilibrium and assumptions of smoothness [2].

f. Using Cauchy's postulate and the axiom of equilibrium, it is possible to prove Cauchy's theorem stating the existence of a stress tensor from which the traction on any plane can be obtained.

g. Using the axiom of equilibrium, the equation of equilibrium is derived.

Clearly, the mathematical definition of a force as a vector measure does not make any sense unless one assumes the space to be the Euclidean space. Hence, as the formulation of the axiom of equilibrium depends on the definition of force, it is not a covariant statement. Without being able to use the axiom of equilibrium, Cauchy's postulate and Cauchy's theorem cannot be proven and the equation of equilibrium cannot be derived.

1.2.2 The Marsden-Hughes Formulation

We consider here only the Marsden-Hughes generalization of the Green Rivlin theorem [4] to manifolds.

a. In this formulation the body is a differentiable manifold and the space is a Riemannian manifold.

b. The Cauchy theorem is assumed to hold so that assumption (c) of classical continuum mechanics is adopted too.

c. An energy balance equation of a given form which includes an internal energy term is assumed to be invariant under any superposed motion.

d. The form of the transformation laws of the variables is assumed.

e. The equations of motion, conservation of mass and energy are deduced.

1.3 The Suggested Formulation (Intuitive Description)

1.3.1 The Local Model

The main idea behind the suggested formulation is that in order to give some structure to the body one must describe explicitly the kinematics and statics of the neighborhoods of the material points. A mathematical model which reflects this point of view is called a local model. Specifically, this is done by attaching a mathematical object to each point in the body and each point in space and considering mappings between the corresponding objects. Thus, the body is conceived as a collection of the neighborhoods of the material points, each of which is represented by the affixed mathematical object. Similar construction will represent the space in the local model.

In this work these mathematical objects are restricted to be vector spaces, and we assume that the vector spaces attached to the various material points are isomorphic. Similar assumption is made concerning the vector spaces attached to the points in space.

In order to be more specific in describing the kinematics and statics of the local model we will discuss the notions of configuration spaces, virtual displacements and generalized forces.

1.3.2 Configuration Space, Virtual Displacements and Generalized Forces

By the "configuration space" of a certain physical system one means the set of all possible states of the system. If one considers for example, the temperature in a room, the configuration space will be the set of all temperature fields in the considered region. Similarly, considering the result of rolling dice the configuration space will be the set $\{(i,j);\ i,j = 1\ldots6\}$. In addition, we will assume here that the configuration space is a differentiable manifold. This means that any configuration (state of the system) has a neighborhood of close configurations, each of which can be identified with an element of a vector space and the state under consideration can be identified with the zero vector. This vector space is called the tangent space to the considered configuration. The collection of all tangent spaces is known as the tangent bundle. Thus, an element of the tangent bundle consists of a configuration, indicating which tangent space is under consideration, and a vector in this tangent space which can be conceived as a possible neighboring configuration. Such an element is known as a virtual displacement (away from the configuration under consideration). The assumption that the configuration space is a differentiable manifold should be justified in any particular case. (It does not hold, for example, in the case of the dice.)

In general, any vector space has another vector space - its dual vector space - corresponding to it, naturally. It is the vector space of all continuous linear mappings of the original vector space to real numbers. This allows one to define the cotangent bundle of the configuration space as the collection of the dual spaces to the tangent spaces. By definition, any element of the cotangent bundle consists of a point indicating which tangent space is under consideration, and a linear function mapping elements of this tangent space - virtual displacements - to real numbers. Customarily, linear functions which map virtual displacements to real numbers are known as generalized forces and the evaluation of a force on a virtual displacement is known as the virtual work.

For example, consider the placement of a body in space as a configuration so that the set of all such placements is the configuration space, and assume it is a manifold. Considering a certain placement, a virtual displacement away from this placement is a neighboring configuration, and the set of all such neighboring placements can be identified with elements of a vector space by the manifold assumption. (It is possible to prove the intuitive identification of this vector space with the collection of all vector fields, virtual displacement vector fields, over the placement under consideration.) A force in this case will be an operator which assigns real numbers to the neighboring possible placements - the virtual work required to arrive at them.

This description of continuum mechanics will be referred to as the global model in order to emphasize the difference between this point of view and the local model.

1.3.3 The Local Configuration Space

In accordance with the objective of exhibiting the mechanics of the neighborhoods explicitly, the configuration space, virtual displacement, force terminology is applied to the local model.

A local configuration is defined as a mapping which carries a point in the body to a point in the space, and maps linearly the vector space representing the neighborhood of the material point into the space representing the neighborhood of the corresponding point in the physical space. (In a more general setting where mathematical objects other than vector spaces represent the structure of the body, appropriate mapping should be required.)

Assuming that the local configuration space is a manifold, we immediately get the local virtual displacement and local force. The local virtual displacement away from a local configuration will have the meaning of a superposed small deformation of the collection of neighborhoods and the local force will have the meaning of stress since it performs work on the deformation of neighborhoods. Thus, using the configuration space terminology, the introduction of the local model leads in a natural way to generalizations of the concepts of stress and strain.

1.3.4 Compatibility

What is needed in order to complete the formulation is to express the fact that both local and global models represent the same physical phenomenon. More specifically, one has to relate the linear mappings of the vector spaces representing the neighborhoods of the material points with the placements of the actual neighborhoods they are supposed to represent. This is done by assigning to each global configuration the compatible local configuration or in other words, by mapping the global configuration manifold into the local configuration manifold so that a certain global configuration is compatible with its image. But then, the derivative of this compatibility mapping, which is a map from the set of global virtual displacements into the set of local virtual displacements, can serve to relate a global virtual displacement with its compatible local virtual displacement. Moreover, the adjoint of the derivative map, which by definition carries the cotangent bundle of the local configuration space into the cotangent bundle of the global configuration space, relates local forces and global forces and can be used as a compatibility condition for the forces. By the definition of the adjoint mapping, the compatibility condition for forces is actually a generalization of the principle of virtual work, i.e., a global force is compatible with a given local force if the virtual work as calculated using the local model is equal to the virtual work as calculated using the global model for all compatible virtual displacements.

The customary continuum mechanics can be obtained using this procedure if one uses the derivative map, assigning to each configuration its derivative, as a compatibility map.

1.4 The Suggested Formulation - Discussion

The applicability of such a formulation is restricted by the requirement that both global and local configuration spaces are manifolds and by the need for the specification of the compatibility map between these manifolds. The first condition is satisfied if this body is a compact manifold, the space is a manifold without a boundary and the configurations are differentiable up to a certain degree (this is not the most general case). In this case the tangent functor can provide the compatibility conditions.

In order to show that this formulation is a generalization of the classical continuum mechanics one has to use the structure that classical continuum mechanics allows, and retrieve the corresponding classical results. In this work it is shown that with the addition of a connection on the space manifold, local forces can be represented by tensor fields corresponding to the classical stress fields and the equation of equilibrium is recovered.

Summarizing the main assumptions and definitions we have:

a. The body and space are differentiable manifolds.

b. The force is defined as a functional allowing concentrated forces. No assumption concerning the form of the force (body or surface forces) is needed in the general case.

c. The existence of the local model (or the need for it) is postulated. This is clearly a very strong assumption which might be thought of as a generalization of Cauchy's postulate.

d. The compatibility conditions are defined.

No physical law is given or postulated here. This formulation is a mathematical model of what is done in analyzing the mechanics of deformable bodies.

Obviously, this is not a generalization of the whole rich theory of continuum mechanics. Nevertheless, any generalization of a certain theory applies only to a limited number of aspects of the theory. It is our hope that the suggested formulation can serve as a generalization of the concepts of stress, compatibility and equilibrium in continuum mechanics.

CHAPTER 2 - MATHEMATICAL PRELIMINARIES

The purpose of this chapter is to summarize the basic definition, notations and results of the mathematical theory of differentiable manifolds, that will be used in later chapters. Complete and systematic treatment of the subjects discussed here can be found in the references cited.

Sections 2.1-2.7 are devoted to manifolds, submanifolds and vector bundles and the exposition which is applicable to both finite and infinite dimensional manifolds is based on [5], [6] and [7].

In sections 2.8-2.10 the treatment is limited to the finite dimensional case and coordinate representation, integration and fibre bundles are introduced. Additional information concerning these topics can be found in [5], [6] and [8]. In section 2.11 we define linear connections and section 2.12 deals with the infinite dimensional manifold structure on the set of maps between two finite dimensional manifolds ([7], [9], [10], [11], [12], [13]). In general, [6] and [14] are suggested as references for the topological and analytical terminology.

2.1 Categories and Functors

Categories were defined in order to give precise meaning to the idea of a collection of spaces having the same mathematical structure and maps between these spaces that preserve the characteristic structure. A collection of such spaces will be called a category and the maps will be called morphisms.

2.1.1 Definition [5]

A category is a collection of objects $\{X,Y,\ldots\}$, such that for two objects X,Y, there exists a set $\mathrm{Mor}(X,Y)$, whose elements are called morphisms, satisfying:

(i) Two sets $\mathrm{Mor}(X,Y)$ and $\mathrm{Mor}(X',Y')$ are disjoint unless $X = X'$ and $Y = Y'$, in which case they are equal.

(ii) For the objects X,Y,Z, there exists a mapping, $\mathrm{Mor}(X,Y) \times \mathrm{Mor}(Y,Z) \to \mathrm{Mor}(X,Z)$, called a composition law, which is associative.

(iii) Each $\mathrm{Mor}(X,X)$ has an element id_X which acts as left and right identities under the composition law.

Once categories have been introduced it is possible to define a transformation from one category to another, such that the category properties are preserved.

2.1.2 Definition

A functor

$$\lambda: \quad A \to A'$$

from a category A into a category A' is a map which associates with each object X in A an object $\lambda(X)$ in A', and with each morphism $f: \; X \to Y$ in A a morphism $\lambda(f)$ in A' satisfying the condition (ii) and either condition (i) or (i') below:

(i) $\qquad \lambda(f): \; \lambda(X) \to \lambda(Y)$,

and whenever f an g can be composed in A

$$\lambda(f \circ g) = \lambda(f) \circ \lambda(g) \; .$$

(i') $\qquad \lambda(f): \; \lambda(Y) \to \lambda(X)$

and whenever f and g can be composed in A

$$\lambda(f \circ g) = \lambda(g) \circ \lambda(f) \; .$$

(ii) $$\lambda(id_X) = id_{\lambda(X)} .$$

A functor that satisfies (i) is said to be covariant, and a functor that satisfies (i') is said to be contravariant.

2.2 Manifolds. Maps.

The idea behind the construction of a finite dimensional manifold is to create a geometrical object which "looks like" $\tilde{R}^n$, locally. It is more general than $\tilde{R}^n$ because the identification is not unique and it is not global. Thus, the affine and metric properties of $\tilde{R}^n$ cannot be transferred into the manifold. In the case of infinite dimensional manifolds, the construction becomes more general as the local identification is allowed to be with a Banach space.

2.2.1 Definition

Let X be a set. An atlas of class C^p $(p \geq o)$ on X is a collection of pairs $\{(U_i,\phi_i)\}_{i\varepsilon I}$, where I is some indexing set, satisfying the following conditions:

(i) $\{U_i\}_{i\varepsilon I}$ is a covering of X.

(ii) Each ϕ_i in a bijection of U_i onto an open subset $\phi_i(U_i)$ of some Banach space $\tilde{E}$.

(iii) The map

$$\phi_j \circ \phi_i^{-1}: \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

is a C^p morphism for each pair of indices $i,j\varepsilon I$. As a rule it will be assumed that the topology induced on X by the atlas is Housdorff.

Each pair (U_i,ϕ_i) will be called a chart, and (U_i,ϕ_i) is said to be a chart at $x\varepsilon X$ if $x\varepsilon U_i$.

2.2.2 Definition

Let U be an open subset of X and let ϕ: $U \to U'$ be a homeomorphism onto an open subset of $\underset{\sim}{E}$. (U,ϕ) is said to be compatible with the atlas $\{(U_i,\phi_i)\}_{i\varepsilon I}$ iff

$$\phi_i \circ \phi^{-1}: \quad \phi(U\cap U_i) \to \phi_i(U\cap U_i)$$

is a C^p morphism, for all $i\varepsilon I$ such that $U\cap U_i \neq \phi$. Two atlases are said to be compatible iff each chart of one atlas is compatible with the other atlas. It is obvious that the compatibility relation is an equivalence relation between atlases.

2.2.3 Definition

A set X with an equivalence class of atlases of class C^p is a <u>C^p manifold</u> modelled on the Banach space $\underset{\sim}{E}$.

2.2.4 Definition

Let X,Y be two manifolds and f: $X \to Y$ a map. We say that f is a <u>C^p morphism</u> iff given $x\varepsilon X$, there exists a chart (U,ϕ) at x and (V,ψ) at f(x) such that $f(U)\subset V$, and

$$\psi \circ f \circ \phi^{-1}: \quad \phi(U) \to \psi(V) \ ,$$

called the local representative of f, is a C^p morphism of Banach spaces. A C^p morphism is a <u>C^p diffeomorphism</u> iff it is a bijection, and its inverse is a C^p morphism.

It is clear that the composite of two C^p morphisms is a C^p morphism and thus, the C^p manifolds and the C^p morphisms form a category.

2.2.5 Definition

Let $\underset{\sim}{E}$ be a Banach space and λ: $\underset{\sim}{E} \to \underset{\sim}{R}$ a continuous linear functional on $\underset{\sim}{E}$. The kernel of λ will be called a <u>hyperplane</u>, and the set of points $x \in \underset{\sim}{E}$ such that $\lambda(x) \geq 0$ (or $\lambda(x) \leq 0$) will be called a <u>half space</u>.

2.2.6 <u>Definition</u>

Let $\underset{\sim}{E}, \underset{\sim}{F}$ be Banach spaces, $\underset{\sim}{E}^+$ and $\underset{\sim}{F}^+$ two half spaces in $\underset{\sim}{E}$ and $\underset{\sim}{F}$ and U,V two open subsets of these half spaces, respectively. We say that

$$f: \; U \to V$$

is a morphism of class C^p between the open sets of the half spaces iff given $x \in U$, there exists an open neighborhood U_1 of x in $\underset{\sim}{E}$, an open neighborhood V_1 of f(x) in $\underset{\sim}{F}$ and a morphism f_1: $U_1 \to V_1$ (morphism of Banach spaces), such that the restriction of f_1 to $U_1 \cap U$ is equal to f.

With the definitions 2.2.5, 2.2.6, a category whose objects are open sets of half spaces and whose morphisms are morphisms of open sets of half spaces is obtained. The composition condition of definition 2.1.1 is satisfied as a consequence of definition 2.2.6.

If the procedure of defining a manifold is repeated with the image of the charts being open sets of a half space, <u>a manifold with boundary</u> is obtained. This is a well defined object since it can be shown ([5], p. 39) that the boundary of such a manifold is invariant. The boundary of a manifold X will be denoted by ∂X.

2.3 <u>Vector Bundles. Vector Bundle Morphisms.</u>

In this section vector bundles will be defined for a manifold modelled on a Banach space. A vector bundle can be thought of as a construction in which a Banach space is attached to each point on a manifold resulting in another manifold. In a later section a more general construction, a fibre bundle, will be defined. However, that discussion will be confined to finite-dimensional manifolds.

2.3.1 Definition

Let E be a set, X a C^r manifold and π: $E \to X$ a surjective map. A C^p vector vector bundle atlas for π ($p \leq r$), is a collection of triplets $\{(U_i,\phi_i,\Phi_i)\}_{i\varepsilon I}$, where $\{(U_i,\phi_i)\}_{i\varepsilon I}$ is an atlas of X, and in addition:

(i) For each $x\varepsilon X$, $\pi^{-1}(x) = E_x$ has a structure of a Banach space.

(ii) Φ_i: $\pi^{-1}(U_i) \to \phi_i(U_i)\times \underset{\sim}{F}$

where $\underset{\sim}{F}$ is a Banach space, is a bijection. The diagram

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\Phi_i} & \phi_i(U_i)\times \underset{\sim}{F} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle pr_1} \\ U_i & \xrightarrow{\phi_i} & \phi_i(U_i) \end{array}$$

is commutative, and for every $x\varepsilon U_i$,

$$\Phi_i(x) = \Phi_{ix}: \ \pi^{-1}(x) \to \underset{\sim}{F}$$

is an isomorphism.

(iii) If $U_i \cap U_j \neq \phi$ then the map $U_i \cap U_j \to L(\underset{\sim}{F},\underset{\sim}{F})$ given by $x \rightsquigarrow \Phi_j \circ \Phi_i^{-1}(x)$ is a C^p morphism.

A triplet (U_i,ϕ_i,Φ_i) is called a vector bundle chart, Φ_i is called a trivialising map, X is called the base manifold and $\underset{\sim}{F}$ is called the typical fibre. Remark: In the finite dimensional case (iii) is implied by (ii).

2.3.2 Definition

Two C^p vector bundle atlases are said to be vector bundle equivalent iff taken together they are a C^p vector bundle atlas. An equivalence class of vector bundle atlases is said to determine the structure of a vector bundle on π.

We will denote this vector bundle either by (E,X,π) or by π.

Remark: Note that every vector bundle is a manifold.

2.3.3 Definition

Let (E,X,π) and (E',X',π') be two vector bundles. A C^r vector bundle morphism $(E,X,\pi) \to (E',X',\pi')$ (it will be denoted also by $f\colon \pi \to \pi'$ or $\underset{=}{f}\colon E \to E'$ with abuse of notation) consists of two morphisms

$$f_o\colon X \to X' \quad \text{and} \quad f\colon E \to E'$$

satisfying:

(i) The diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ X & \xrightarrow[f_o]{} & X' \end{array}$$

is commutative, and the induced map on the fibre over $x \in X$,

$$f_x\colon E_x \to E'_{f_o(x)}$$

is a continuous linear map.

(ii) For each $x_o \in X$ there exist vector bundle charts (U,ϕ,Φ), (U',ϕ',Φ') at x_o and $f_o(x_o)$, respectively, such that $f_o(U)$ is contained in U', and the map of U into $L(\underset{\sim}{F},\underset{\sim}{F}')$ given by

$$x \leadsto \Phi'_{f_o(x)} \circ f_x \circ \Phi^{-1} ,$$

is a C^r morphism. ($\underset{\sim}{F}$ and $\underset{\sim}{F}'$ are the typical fibres of π and π', respectively.) Remark: As in the definition of vector bundles

(ii) is implied by (i) for the finite dimensional case.

By the definition 2.3.2 and 2.3.3, it can be shown that the vector bundles with the vector bundle morphisms form a category.

Let VB(X) be the category of vector bundles over the base manifold X. (The morphisms in this category being vector bundle morphisms with f_o: $X \to X$ being the identity.) We want to construct a functor from the category VB(Y) to the category VB(X).

2.3.4 <u>Definition</u>

Let ϕ: $X \to Y$ be a morphism. Then, the functor ϕ^* from VB(Y) into VB(X) is defined as follows:

(i) The pullback $(\phi^*E, X, \phi^*(\pi))$ of the object (E, Y, π) is given by:

(a) $$(\phi^*E)_x = E_{\phi(x)}$$

(b) The diagram

$$\begin{tikzcd} \phi^*E \arrow[r] \arrow[d, "\phi^*\pi"'] & E \arrow[d, "\pi"] \\ X \arrow[r, "\phi"'] & Y \end{tikzcd}$$

commutes, where the top horizontal map is the identity on each fibre.

(c) If $E = Y \times \underset{\sim}{E}$, then $\phi^*E = X \times \underset{\sim}{E}$ and $\phi^*\pi$ is the projection on the product.

(d) If V is open in Y and $U = \phi^{-1}(V)$, then

$$\phi^*(\pi^{-1}(V)) = (\phi^*\pi)^{-1}(U) ,$$

so that the following diagram is commutative.

$$\begin{tikzcd}
& \phi^*(\pi^{-1}(V)) \arrow[rr] \arrow[dl] \arrow[dd] & & \pi^{-1}(V) \arrow[dl] \arrow[dd] \\
\phi^*E \arrow[rr] \arrow[dd] & & E \arrow[dd] & \\
& U \arrow[rr] \arrow[dl] & & V \arrow[dl] \\
X \arrow[rr] & & Y &
\end{tikzcd}$$

(ii) The pullback of a morphism $f\colon (E,Y,\pi) \to (E',Y,\pi')$ in $VB(Y)$, under the functor ϕ^* is a morphism

$$\phi^* f\colon \ (\phi^* E, X, \phi^* \pi) \to (\phi^* E', X, \phi^* \pi')$$

given by

$$(\phi^* f)(e) = f(e)$$

for all $e \varepsilon (\phi^* E)$

$$\begin{tikzcd}
\phi^* E \arrow[r, "\phi^* f"] \arrow[d] & \phi^*(E') \arrow[d] \\
E \arrow[r, "f"'] & E'
\end{tikzcd}$$

Remark: Although ϕ^* is a covariant functor between $VB(Y)$ and $VB(X)$, by the above definition, we have a functor VB from the category of manifolds to the category of vector bundles which assigns $VB(X)$ to every manifold X. It is contravariant since it assigns a morphism (functor)

$$\phi^*\colon \ VB(Y) \to VB(X)$$

to a morphism

$$\phi\colon \ X \to Y \ .$$

2.3.5 <u>Definition</u>

Let (E,X,π) be a vector bundle. A <u>section</u> s of π is a map

$$s\colon \ X \to E \ ,$$

satisfying

$$\pi \circ s = id.$$

A section is therefore a generalization of the familiar notion of a vector field.

Remark: The set of all sections of a given vector bundle can be given a structure of a vector space. Let s_1 and s_2 be sections of (E,X,π), define

$$\alpha s_1: \quad X \to E$$

by

$$(\alpha s_1)\ (x) = \alpha(s_1(x)) \ , \qquad x\varepsilon X \ , \qquad \alpha\varepsilon \underset{\sim}{R}$$

and

$$s_1 + s_2: \quad X \to E$$

by

$$(s_1 + s_2)\ (x) = s_1(x) + s_2(x)$$

2.3.6 Definition [9]

Let VB denote the category of C^∞ vector bundles over C^∞ manifolds. Denote by C^k and S the functors from VB to the category of (infinite dimensional) vector spaces defined as follows:

(i) Given an object π of VB, $C^k(\pi)$ will be the vector space of C^k sections of π and $S(\pi)$ will be the vector space of all sections of π.

(ii) Given a VB-morphism f: $\pi \to \pi'$ in VB(X), the map

$$S(f) \equiv f_*: \quad S(\pi) \to S(\pi')$$

is defined by

$$(f_* s)(x) = f(s(x))$$

and f_* as defined will map $C^k(E)$ into $C^k(E')$.

(iii) Given a map ϕ: $X \to Y$ and a vector bundle (E,Y,π) we have the induced functor ϕ_π^*: $(E,Y,\pi) \rightsquigarrow (\phi^*(E),Y,\phi^*(\pi))$ as defined in 2.3.4(i). The map

$$S(\phi_\pi^*): \; S(\pi) \to S(\phi^*(\pi))$$

is defined by

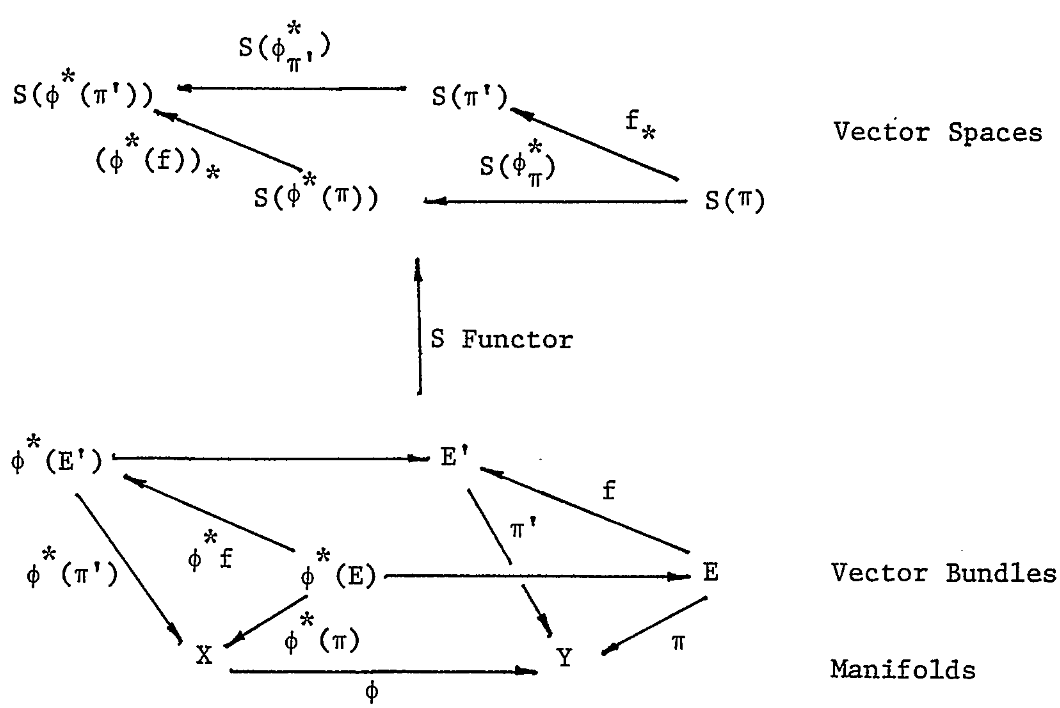
$$s \rightsquigarrow s \circ \phi , \quad s \in S(\pi) .$$

Here again $C^k(\phi_\pi^*)$ maps $C^k(\pi)$ into $C^k(\phi^*(\pi))$.

(iv) Given a morphism f: $\pi \to \pi'$ in $VB(Y)$ and a map ϕ: $X \to Y$ as in (iii), we have the induced map f_* as in (ii) and the map $\phi^*(f)$ as in 2.3.4 (ii). The map

$$S(\phi^*(f)) \equiv (\phi^*(f))_*: \; S(\phi^*(\pi)) \to S'(\phi^*(\pi'))$$

is defined as to make the upper part of the following diagram commutative (the rest of the diagram is shown for illustration).



2.3.7 Definition

Let (E,X,π) and (E',X,π') be vector bundles over X. Define

$$L(E,E')_x = L(E_x,E'_x)$$

It can be shown that $L(E,E') = \bigcup_{x \in X} L(E,E')_x$ has a fector bundle structure and will be denoted by $(L(E,E'),X,L(\pi,\pi'))$.

The space of C^k sections of this vector bundle is isomorphic with the space of C^k vector bundle morphisms $\pi \to \pi'$ over the identity.

2.4 Tangent Space. Tangent Bundle. Tangent Map.

Two equivalent ([5] p. 83) definitions of tangent vectors and tangent bundles will be given.

2.4.1 Definition

Let x be a point in a C^p, $p \geq 1$, manifold X. Consider all curves $c: \underset{\sim}{R} \to X$ with $c(a) = x$. Define an equivalence relation by saying $c_1 \sim c_2$ iff $\frac{dc_1}{dt}(a) = \frac{dc_2}{dt}(a)$ in some chart at x. An equivalence class of curves is a tangent vector. The tangent space T_xX is the set of tangent vectors at x and the tangent bundle TM is the disjoint union of the tangent spaces.

2.4.2. Definition

Consider triplets (U,ϕ,v) where (U,ϕ) is a chart at x and v is an element of the space $\underset{\sim}{E}$ on which X is modelled. Define an equivalence relation on such triplets by saying that $(U,\phi,v) \sim_x (U',\phi',v')$ iff

$$[D(\phi' \circ \phi^{-1})(\phi(x))]v = v'$$

Again, a tangent vector is an equivalence class, the tangent space is the set of tangent vectors and the tangent bundle is the disjoint union of the tangent spaces over X.

2.4.3 Definition

The tangent bundle projection

$$\tau : \quad TX \to X$$

assigns the point x to every element in the tangent space $T_x X$.

2.4.4 Proposition

The tangent bundle TX can be given a structure of a vector bundle (TX, X, τ) with fibres isomorphic to the vector space $\underset{\sim}{E}$ on which X is modelled.

The proof of this proposition is based on the definition 2.4.2 and can be found in [5] pp. 47–48.

2.4.5 Definition

Let f: $X \to Y$ be a morphism of class C^p, $p \geq 1$, and let $\psi \circ f \circ \phi^{-1}$ be its local representative with respect to the charts (U,ϕ) and (V,ψ) in X and Y respectively, where X and Y are modelled on $\underset{\sim}{E}$ and $\underset{\sim}{F}$.

The tangent map

$$Tf: \quad TX \to TY$$

is defined by its local representative

$$(Tf)_{\phi,\psi}: \quad U \times \underset{\sim}{E} \to V \times \underset{\sim}{F}$$

with respect to the natural charts in TX and TY (as in 2.4.2) by

$$(x,e) \rightsquigarrow (f(x),\ (D(\psi \circ f \circ \psi^{-1})\ (x))(e))\ , \qquad e \varepsilon \underset{\sim}{E}$$

The map Tf is a vector bundle morphism of class C^{p-1}. It can be also shown that the tangent map satisfies the rules:

$$T(f \circ g) = Tf \circ Tg$$

and

$$T(id_X) = id_{TX}\ ,$$

making T into a functor.

2.4.6 Definition

Let α: $J \to X$, where J is an open interval of $\underset{\sim}{R}$, be a curve. The canonical lifting of the curve α is the curve

$$\alpha': \quad J \to TX$$

in TX defined by

$$\alpha' = T\alpha o i$$

where i is the canonical cross-section of $TJ = J\times\underset{\sim}{R}$, i.e. $i(t) = 1$.

2.5 Submanifolds

2.5.1 Definition

A subset $Y\subset X$, where X is a manifold modelled on $\underset{\sim}{E}$, is a *sub-manifold* of X iff $\underset{\sim}{E} = \underset{\sim}{E}_1 \times \underset{\sim}{E}_2$, and for every $y\varepsilon Y$ there exists a chart (U,ψ) at y, whose image is $V\subset\underset{\sim}{E}$, such that

$$\psi(U\cap Y) = V\cap(\underset{\sim}{E}_1\times \{e_2\})$$

where $e_2\varepsilon\underset{\sim}{E}_2$.

Clearly, the set $\{(Y\cap U_i,\ \psi_i'\colon Y\cap U_i \to \underset{\sim}{E}_1)\}_{i\varepsilon I}$ defines a manifold structure for Y.

2.5.2 Definition

A morphism f: $Z \to X$ is an *immersion* at $z\varepsilon Z$ iff there exists an open neighborhood U of z such that the restriction of f to U induces a diffeomorphism of U onto a submanifold of X. The morphism f is an immersion if it is an immersion at every point. If in addition f(Z) is a submanifold of X, f is an *embedding*.

2.5.3 Definition

A morphism $f: X \to Z$ is a submersion at a point $x \in X$ iff there exists a chart (U,ϕ) at x and a chart (V,ψ) at f(x), such that ϕ gives a diffeomorphism of U on a product $U_1 \times U_2$ (U_1 and U_2 open in some Banach spaces), and the map

$$\psi o f o \phi^{-1}: U_1 \times U_2 \to V$$

is a projection. We say that f is a subersion if it is a submersion at every point, making $f^{-1}(\{z\})$, $z \in Z$, a submanifold of X.

2.5.4 Proposition

Let X,Y be manifolds of class C^p $(p \geq 1)$. Let $f: X \to Y$ be a C^p-morphism and let $x \in X$. Then:

(i) f is an immersion at x iff $T_x f \equiv Tf|_x$ is injective and splits (splits as in [5] p. 4).

(ii) f is a submersion at x iff $T_x f$ is surjective and its kernel splits.

2.6 Differential Forms

2.6.1 Definition

The cotangent bundle T^*X of a manifold X is a bundle whose fibre T_x^*X at $x \in X$ is the topological dual of T_xX. It can be shown that the cotangent bundle is a vector bundle.

A section of the tangent bundle is called a vector field and a section of the cotangent bundle is called a one form.

2.6.2 Definition

Let $\Lambda^k X$ be the vector bundle over X whose fibre at $x \in X$ is the set of k-multilinear alternating continuous maps $T_xX\times\ldots\times T_xX \to \underset{\sim}{R}$. A section of Λ^kX is a k-form. Real valued functions are o-forms.

2.6.3 Definition

Let α be a k-form and β an ℓ-form. Then the exterior product $\alpha\wedge\beta$ is the k+ℓ-form defined by

$$(\alpha\wedge\beta)_x(v_1,\ldots,v_{k+\ell}) = \frac{1}{k!\,\ell!}\sum_{\sigma}(\mathrm{sgn}\,\sigma)\alpha_x(v_{\sigma(1)},\ldots,v_{\sigma(k)})\beta_x(v_{\sigma(k+1)},\ldots,v_{\sigma(k+\ell)})$$

where the sum is taken over all the permutations σ such that

$$\sigma(1) < \ldots < \sigma(k)\ ,\quad \sigma(k+1) < \ldots < \sigma(k+1)$$

2.6.4 Definition

Let $f\colon\ X \to Y$ be a C^p morphism and let α be a C^p k-form on Y. The pull back $f^*\alpha$ of α by f is the C^{p-1} k-form on X defined by

$$(f^*\alpha)_x(v_1,\ldots,v_k) = \alpha_{f(x)}(Tf(v_1),\ldots,Tf(v_k))\ .$$

2.6.5 Definition

The exterior derivative d is an operator which transforms a smooth k-form α to a k+1 form $d\alpha$ by

$$d\alpha_x(v_o,\ldots v_k) = \sum_{i=o}^{k}(-1)^i\,[D\alpha_x\ (v_i)]\ (v_o,\ldots,v_{i-1},v_{i+1},\ldots v_K)$$

where $D\alpha_x$ is the derivative of α in a chart (U,ϕ) at x. $D\alpha_x$ maps $\underset{\sim}{E}$ into $\Lambda^k(E)$ since $\alpha\colon\ \phi(U)\subset\underset{\sim}{E}\to\Lambda^k(\underset{\sim}{E})$ in the chart. It can be verified that the exterior derivative is chart independent.

2.6.6 The following identities hold:

(i) d is linear.

(ii) $d^2 = d\circ d = 0$

(iii) $d(\alpha\wedge\beta) = d\alpha\wedge\beta+(-1)^k\alpha\wedge d\beta$ for a k-form α.

2.6.7 Definition

Let α be a k-form and v a vector field. The contraction $v \lrcorner \alpha$ of α by v is the k-1 form defined by

$$(v \lrcorner \alpha)_x \, (v_2, \ldots, v_k) = \alpha_x \, (v(x), \, v_2, \, \ldots, \, v_k)$$

2.6.8 Poincare Lemna

If $d\alpha = 0$, there exists a neighbourhood U about each point on which $\alpha = d\beta$.

2.6.9 Definition

The Lié derivative $L_v \alpha$ of a k-form α by a vector field v is the k-form defined by

$$L_v \alpha = d(v \lrcorner \alpha) + v \lrcorner d\alpha \quad .$$

2.7 Flow

2.7.1 Definition

A flow or a one-parameter group of diffeomorphisms is a collection of smooth maps

$$F_t : \; X \to X \; , \quad t \varepsilon \underset{\sim}{R}$$

satisfying:

(i) $F_{t+s} = F_t \circ F_s$

(ii) $F_o = id_X$.

2.7.2 Definition

The flow F is the flow of a vector field v iff

$$\frac{d}{dt} F_t(x) = v(F_t(x)) \quad .$$

2.7.3 Local Existence and Uniqueness Theorem

If v is a C^p-vector field, $p \geq 1$, then v has a locally defined, unique C^p flow F_t.

2.8 Coordinate Representations

In this section the coordinate representations of objects already defined is considered. The discussion is therefore limited to finite dimensional manifolds.

2.8.1 Coordinates of a Point. Maps.

Let X be a manifold modelled on $\underset{\sim}{R}^n$. Then if $x \varepsilon X$ and (U,ϕ) is a chart at x, the coordinates $\{x^i\}$ of the image $\phi(x) \varepsilon \underset{\sim}{R}^n$ are called the coordinates of x in the chart (U,ϕ). The functions ϕ_i: $x \rightsquigarrow x^i$ are called coordinate functions. If (V,ψ) is another chart and $U \cap V \neq \phi$ the coordinates of x in this chart in terms of the first chart are given by n real valued (C^k) functions of n variables

$$x^{i'} = f^{i'}(x^i) \ .$$

Let g: $X \to Y$ be a morphism between the m-dimensional manifold X and the n-dimensional manifold Y. The coordinate (local) representation for $y = g(x)$, $y \varepsilon Y$, $x \varepsilon X$ is

$$y^i = g^i(x^j)$$

2.8.2 Tangent Vectors

Each coordinate function ϕ_i at x defines a unique curve

$$\phi_i^{-1}: \ J \subset \underset{\sim}{R} \to U \subset X$$

by requiring that $x'^j = \text{const}$, $j \neq i$, for all points x' on the curve. According to definition 2.4.1 the tangent vector associated with this curve is characterized by $\frac{\partial}{\partial x^i} \phi_i^{-1}$. In a given chart, the ϕ_i^{-1} is omitted, the set $\{\frac{\partial}{\partial x^j}\}$ can serve as a basis for T_xX and a vector $v \varepsilon T_xX$ can be represented by $v^i \frac{\partial}{\partial x^i}$. In case of a different coordinate system we have by the chain rule

$$\frac{\partial}{\partial x^{i'}} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial}{\partial x^i}$$

and by

$$v^{i'} \frac{\partial}{\partial x^{i'}} = v^{i'} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial}{\partial x^i} = v^i \frac{\partial}{\partial x^i} ,$$

we have

$$v^i = \frac{\partial x^i}{\partial x^{i'}} v^{i'} .$$

The coordinates of an element in the tangent bundle are therefore (x^i, v^i) with respect to the natural chart and

$$\tau: \quad (x^i, v^i) \rightsquigarrow (x^i) .$$

2.8.3 Tangent Maps

By the definition of the tangent map (2.4.5) we have the functions $(y^i, \frac{\partial y^i}{\partial x^j})$ as representatives of Tg: $TX \to TY$ and

$$(y^i, \frac{\partial y^i}{\partial x^j}): \quad (x^n, v^k) \rightsquigarrow (y^i(x^n), \ (\frac{\partial y^i}{\partial x^k} (x^n)) \ (v^k)) .$$

2.8.4 Submanifolds

Since every subspace of a finite dimensional vector space splits, the following holds: Y is a submanifold of X of dimension r iff each point of Y has an open neighborhood U in X with local coordinates $(x^1,\ldots,x^n)$ such that the points of Y in U are those having coordinates of the type:

$$(x^1,\ldots,x^r,o,\ldots,o) .$$

2.8.5 Differential Forms

It can be shown that the $\binom{n}{p}$ p-forms

$$\{\theta^{I_1} \wedge \theta^{I_2} \wedge \dots \wedge \theta^{I_p} , I_i < I_{i+1} , I_j = 1,\dots,n\} ,$$

where the θ^k are the basis of T_x^*X, form a basis for Λ_x^pX. Thus, let $\alpha\varepsilon\Lambda_x^pX$, then

$$\alpha = \alpha_{I_1\dots I_p} \theta^{I_1} \wedge\dots\wedge \theta^{I_p} .$$

Once we have a base in T_xX we have the dual base induced in T_x^*X. This dual base is denoted by $\{dx^i\}$. By the definition of the dual base we have

$$dx^i \left(\frac{\partial}{\partial x^j}\right) = \delta^i_j$$

and

$$w(v) = w_i dx^i \left(v^j \frac{\partial}{\partial x^j}\right) = w_i v^i , \quad w\varepsilon T_x^*X , \quad v\varepsilon T_xX .$$

Thus, in natural coordinates dx^{I_j} will replace the θ^{I_j} and

$$\alpha = \alpha_{I_1\dots I_p} dx^{I_1}\wedge\dots\wedge dx^{I_p}$$

A change of coordinates $x^i \rightsquigarrow x^{i'}$ will induce a change of the natural base according to

$$dx^{i'} = \frac{\partial x^{i'}}{\partial x^i} dx^i ,$$

and a change of components by

$$\alpha_{I_1\dots I_p} = \alpha_{I_1'\dots I_p'} \frac{D(x^{I_1'},\dots,x^{I_p'})}{D(x^{I_1},\dots,x^{I_p})} ,$$

where $\frac{D(x^{I'})}{D(x^I)}$ is the Jacobian determinant of the coordinate transformation. This last formula is similar to the rule for the change of variables of a p-multiple integral in an n-dimensional space and it suggests the role of differential forms in the theory of integration on manifolds.

2.8.6 Exterior Derivative. Lie Derivative. Contraction. Pull-Back.

(i) Let α be the differential p-form

$$\alpha = \alpha_{I_1 \ldots I_p} dx^{I_1} \wedge \ldots \wedge dx^{I_p} \ .$$

It can be shown that by definition

$$d\alpha = \frac{\partial \alpha_{I_1 \ldots I_p}}{\partial x^k} dx^k \wedge dx^{I_1} \wedge \ldots \wedge dx^{I_p}$$

(ii) The Lie derivative $L_v w$ of a contravariant vector field w by the vector field v is the contravariant vector field defined by

$$L_v w \Big|_x = \frac{\partial}{\partial t} [(TF_t)^{-1} (w(F_t(x))] \ ,$$

where F_t is the flow of v, and t is the parameter of this flow.

The Lie derivative $L_v \alpha$ of a covariant vector field is defined by

$$L_v \alpha \Big|_x = \frac{\partial}{\partial t} [F_t^* (\alpha(F_t(x))] \ .$$

In terms of components we have

$$L_v \frac{\partial}{\partial x^i} = - \frac{\partial v^j}{\partial x^i} \frac{\partial}{\partial x^j}$$

$$L_v \, dx^i = \frac{\partial v_i}{\partial x^j} dx^j$$

For arbitrary tensor the Lie derivative satisfies

$$L_v (u \otimes w) = L_v u \otimes w + u \otimes L_v w \ .$$

The Lie derivative of a tensor is useful as it calculates the rate of change of a tensor field with respect to its image under a flow. If the Lie derivative of a tensor field with respect to a certain vector field vanishes the tensor field is invariant under the group of transformations generated by the vector field.

(iii) In terms of components we have for the contraction of the p-form

$$w = w_{I_1 \ldots I_p} \, dx^{I_1} \wedge \ldots \wedge dx^{I_p}$$

with the vector v:

$$v \lrcorner w = v^j w_{jI_1 \ldots I_{p-1}} \, dx^{I_1} \wedge \ldots \wedge dx^{I_{p-1}}$$

(iv) The coordinate representation of the pullback of α by f is

$$f^*\alpha = (f^*\alpha)_{I_1 \ldots I_p} \, dx^{I_1} \wedge \ldots \wedge dx^{I_p} \quad ,$$

where the components are given by

$$(f^*\alpha)_{I_1 \ldots I_p}(x) = \frac{D(y^{J_1} \ldots y^{J_p})}{D(x^{I_1} \ldots x^{I_p})} \; \alpha_{J_1 \ldots J_p}(f(x)) \; .$$

2.9 Integration on Manifolds

The discussion in this section is limited to finite dimensional manifolds.

2.9.1 Orientation

Two coordinate systems (x^i) and (y^j) on an open set of $\underset{\sim}{R}^n$ are said to define the same orientation if the Jacobian determinant, $J = \frac{D(x^i)}{D(y^j)}$, is positive for all points on the set. A chart (U,ϕ) on a manifold X induces an orientation of U by means of the orientation of $\phi(U)$. The manifold X is said to be orientable if there exists an atlas such

that on any overlap $U \cap V \neq \phi$ of charts (U,ϕ) and (V,ψ), $\frac{D(\phi_i)}{D(\psi_j)} > 0$.
By the discussion in (2.8.5) concerning the dimension of $\Lambda^n_x(X)$, it follows that any n-form on an n dimensional manifold has only one component and can be represented as

$$w = w_{12\ldots n}(x)\, dx^1 \wedge \ldots \wedge dx^n$$

2.9.2 Definition

Let w be an n-form on an n-dimensional manifold X vanishing outside a compact set contained in the domain of a chart U. The form is said to be integrable if its representative is integrable on $\underset{\sim}{R}^n$. In this case its integral on U (and X) is

$$\int_X w = \int_U w = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} w_{12\ldots n}\, dx^1 dx^2 \ldots dx^n$$

By the rule for transformation of coordinates of differential forms and the properties of the integral the definition is independent on the coordinates on U.

2.9.3 Definition

A C^p partition of unity on X is a collection of C^p functions $\theta_k \geq 0$ on X satisfying:

(i) At each point $x \varepsilon X$ there is a finite number of functions $\theta_k(x) \neq 0$.

(ii) The support of each function is compact.

(iii) $\sum_k \theta_k(x) = 1$ for all $x \varepsilon X$.

A partition of unity $\{\theta_k\}$ is subordinate to a covering $\{U_i\}$ of X iff one can find $U_i \supset \text{supp}\, \theta_k$ for every k. It can be shown that if X is paracompact, it is always possible to find on it a partition of unity subordinate to any preassigned locally finite covering.

2.9.4 Definition

Let $\{\theta_k\}$ be a partition of unity subordinate to the atlas of X. The n-form w is integrable on X iff the series

$$\int_X w = \sum_k \int_X \theta_k w$$

converges. In such a case the sum is the integral of w. It can be shown that the integral depends neither on the choice of atlas nor the partition of unity.

2.9.5 Stokes' Theorem

Let X be an oriented manifold of class C^2, dimension n, and let w be an (n-1) form on X of class C^1, with compact support. Then $\int_X dw = \int_{\partial X} w$.

This theorem is a generalization of the classical Gauss and Stokes theorems and its proof can be found in Lang [5, p. 194].

2.10 Fibre Bundles

The discussion in this section is limited to finite dimensional manifolds.

2.10.1 Definition

A Lie group G is a group that is also a differentiable manifold such that the differentiable structure is compatible with the group structure, i.e. the operation $G \times G \to G$ by $(x,y) \to xy^{-1}$ is a differentiable mapping.

2.10.2 Definition

Let E, X be finite dimensional C^k manifolds (possibly with boundary), let F be a finite dimensional C^∞ manifold without a boundary. and let π: $E \to X$ be a C^k map. Then (E,X,F,π) is a C^k fibre bundle with base X, fibre F and projection π iff for each $x \in X$ there is an open neighborhood U of x in X and a C^k morphism

$$\phi: \quad \pi^{-1}(U) \to U \times F \;,$$

such that the following diagram commutes.

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & U\times F \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

Similarly to vector bundles $E_x = \pi^{-1}(x)$ will be called the fibre over x. A fibre bundle is trivial iff $E = X\times F$. By definition each fibre bundle is locally trivial. Let $\{U_i\}_{i\in I}$ be an open covering of X such that $\pi^{-1}(U_i)$ is a trivial bundle. The morphism

$$\phi_i: \quad \pi^{-1}(U_i) \to U_i \times F$$

is called a trivialization of E over U_i. Let $i,j \in I$ and $U_i \cap U_j \neq \phi$, there is a morphism

$$\phi_{ji} = \phi_j \left|\pi^{-1} (U_i \cap U_j) \circ (\phi_i \left|\pi^{-1} (U_i \cap U_j)\right.)^{-1}\right.: \quad (U_i \cap U_j) \times F \to$$

$$\to (U_i \cap U_j) \times F$$

which can be thought of as a family of morphisms

$$\phi_{ji}(x): \quad F \to F$$

called transition functions such that

$$\phi_{ji}(x,e) = (x,\phi_{ji}(x)(e)) \ , \quad x\varepsilon X \ , \quad e\varepsilon F$$

2.10.3 Definition

A fibre bundle morphism between the fibre bundles (E,X,F,π) and (E',X',F',π') is a pair of maps (f,f_o).

$$f: \ E \to E' \qquad f_o: \ X \to X'$$

such that the following diagram is commutative.

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ X \arrow[r, "f_o"] & X' \end{tikzcd}$$

With this definition we have the category of fibre bundles and fibre bundle morphisms.

2.10.4 Definition

A section s of the fibre bundle (E,X,F,π) is a map

$$s: \ X \to E$$

such that

$$\pi \circ s = id_X$$

2.10.5 Definition

A C^k coordinate fibre bundle (E,X,F,G,π) is a fibre bundle (E,X,F,π) such that

(i) G, called the structural group, is a Lie group acting affectively on the fibre F.

(ii) The transition function are elements of G and will be denoted by $g_{ij}(x)$.

(iii) The transition functions $g_{ij}(x)$ are functions of x of class C^k.

2.10.6 Definition

A fibre bundle, in which the typical fibre F and the structural group G are identical, and G acts on F by left translation, is called a principal fibre bundle.

2.11 Linear Connections [12]

2.11.1 Definition

A connection map (connection) C for a C^r vector bundle (E,X,π) is a map C: $TE \to E$ such that for any vector bundle chart (U,ϕ,Φ) of π, there is a map $\underset{\sim}{\Gamma}$: $\phi(U) \to L(\underset{\sim}{E}_o,\underset{\sim}{E};\underset{\sim}{E})$ of class C^{r-1}, which gives a local representative of C, $\underset{\sim}{C} = \Phi \circ C \circ T\Phi^{-1}$, by

$$\underset{\sim}{C}(\underset{\sim}{x},\underset{\sim}{\xi},\underset{\sim}{y},\underset{\sim}{\eta}) = (\underset{\sim}{x}, \underset{\sim}{\eta}+\underset{\sim}{\Gamma}(\underset{\sim}{x})(\underset{\sim}{y},\underset{\sim}{\xi}))$$

Here, $\underset{\sim}{E}_o$ is the space on which X is modelled and $\underset{\sim}{E}$ is the typical fibre.

Let $E \oplus TX \oplus E$ be the vector bundle over X with $(E \oplus TX \oplus E)_x = E_x \oplus T_xX \oplus E_x$, then the map

$$(\tau_E, T\pi, C): TE \to E \oplus TX \oplus E$$

is a C^{r-1} diffeomorphism.

2.11.2 Definition

An element $v \in TE$ is called vertical iff $T\pi(v) = 0$. The representative of a vertical vector in any vector bundle chart is $(\underset{\sim}{x},\underset{\sim}{\xi},\underset{\sim}{o},\underset{\sim}{\eta})$ and it can be thought of as an element of $T(E_x)$, $x = \pi(\tau_F(v))$. Since E_x is a vector space, it is isomorphic to its tangent space at any point and we have the induced mapping $\dot{i}$ on vertical vectors whose local representative in any vector bundle chart is given by

$$\underset{\sim}{i}: \quad (\underset{\sim}{x},\underset{\sim}{\xi},o,\underset{\sim}{\eta}) \rightsquigarrow (\underset{\sim}{x},\underset{\sim}{\eta})$$

An element $u\varepsilon TE$ is called horizontal iff $C(v) = 0$. In general, the second and third components of the map $(\tau_E,\ T\pi,\ C)$ are called the horizontal and vertical projections and $T\pi(w)$, $C(w)$ are the horizontal and vertical components of $w\varepsilon TE$.

2.11.3 Definition

Let s be a section of the vector bundle (E,X,π). Given a connection C on the vector bundle π, the covariant derivative of s is the section ∇s of $(L(TM,E),\ X,\ L(\tau,\pi))$ given by

$$\nabla s = C \circ Ts.$$

2.11.4 Theorem

Let s, s_2 be sections of π and $f\colon\ X \to \underset{\sim}{R}$ a differentiable map then

$$\nabla(s_1+s_2) = \nabla s_1 + \nabla s_2$$

and

$$\nabla(fs_1)(v) = f\nabla s_1(v) + df(v)s_1(\tau(v))$$

$v\varepsilon TX$. A connection on the vector bundle (TX,X,τ) is called a connection on the manifold X. Such a connection induces a decomposition of T^2X, the second tangent bundle.

2.11.5 Definition

Let $(\underset{\sim}{x},\underset{\sim}{a},\underset{\sim}{b},\underset{\sim}{c})$ be the representative of an element in T^2X. Define the canonical involution $\omega\colon\ T^2X \to T^2X$ through its local representative $\underset{\sim}{\omega}$ by

$$\underset{\sim}{\omega}(\underset{\sim}{x},\underset{\sim}{a},\underset{\sim}{b},\underset{\sim}{c}) = (\underset{\sim}{x},\underset{\sim}{b},\underset{\sim}{a},\underset{\sim}{c})$$

It can be shown that the definition is valid as it is independent of the chart representation.

The double tangent T^2X has two vector bundle structures, namely (T^2X, TX, τ_{TX}) and $(T^2X, TX, T\tau_X)$. The canonical involution defines a vector bundle isomorphism $T\tau_X \to \tau_{TX}$ by the diagram

$$\begin{tikzcd} T^2X \arrow[r, "\omega"] \arrow[d, "T\tau_X"'] & T^2X \arrow[d, "\tau_{TX}"] \\ TX \arrow[r, "id"] & TX \end{tikzcd}$$

and by definition we have $\omega = \omega^{-1}$.

2.12 Manifolds of Maps

In this section we summarize the definitions, theorems and corollaries of the theory of manifolds of maps that are needed in the following chapters. Since the constructions involved in the theory are complicated they are not presented here for the sake of brevity. In addition, different authors utilize different starting points and different methods in their constructions. For example, Marsden [7] uses 2.12.4 as a definition of $TC^k(X,Y)$ in order to prove 2.12.3. Similarly, 2.12.6.ii can be used as a definition in order to arrive at 2.12.6.i.

The results 2.12.1-2.12.4 can be found in [9], [12] and 2.12.5 can be found in [11]. Verona [13] obtained 2.12.6.i,ii is a corollary and iii,iv can be found in [12]. The results concerning sections of jet bundles are given in [9],[10] ([10] discusses only the

C^∞ case), and the relation $TT_* = \omega \circ T$ follows from the fact that T_* is linear in the representatives of maps (in the manifold of maps), [9, p. 64].

2.12.1 Definition

Let (E,X,π) be a vector bundle. A continuous map $\phi\colon\ E \to \underset{\sim}{R}$ is called a Finsler structure on π iff for each $x \in X$, $\phi|E_x$ is a norm for E_x.

2.12.2 Theorem

Let (E,X,π) be a vector bundle with finite dimensional fibre and with X compact. Then,

(i) There exists a Finsler structure on π.

(ii) The Finsler structure induce a Banach space topology on $C^k(\pi)$.

(iii) The topology is independent of the Finsler structure, so that $C^k(\pi)$ will denote the space of sections together with the Banachable topology.

2.12.3 Theorem

Let X be a compact C^∞ manifold and Y a C^∞ manifold without a boundary, then the set $C^k(X,Y)$ of C^k mappings $X \to Y$, $0 \leq k < \infty$, can be given a structure of a C^∞ Banach manifold modelled on a space of sections.

2.12.4 Theorem

Let $f \in C^k(X,Y)$ then $T_f C^k(X,Y) = \{g \in C^k(X,TY) \mid \tau_Y \circ g = f\}$, and for any $h \in TC^k(X,Y)$

$$\tau\colon\ TC^k(X,Y) \to C^k(X,Y)\colon\ h \leadsto \tau_Y \circ h\ .$$

2.12.5 Theorem

The set of all C^k embeddings is open in $C^k(X,Y)$, $k \geq 1$ and hence it is a submanifold of $C^k(X,Y)$.

2.12.6 Vector Bundles Over $C^k(X,Y)$

i. Let (E,X,π) and (F,Y,ρ) be C^∞ vector bundles where X and Y are as in 2.12.3. Then, the set $C_k^r(\pi,\rho)$ of C^r vector bundle morphisms over C^k base maps has a C^∞ vector bundle structure $C_k^r(\pi,\rho) \to C^k(X,Y)$ and the projection map of the vector bundle assigns to each vector bundle morphism its base map.

ii. Let $f \varepsilon C_k^r(\pi,\rho)$, then

$$T_f C_k^r(\pi,\rho) = \{g \varepsilon C_k^r(\pi, T\rho) | \tau_F \circ g = f\}$$

We use here the fact that $T\rho$: $TF \to TY$ defines a vector bundle structure on TF in addition to the τ_F: $TF \to F$ structure. The tangent bundle projection of $TC_k^r(\pi,\rho)$ is given by $h \leadsto \tau_F \circ h$ for $h \varepsilon TC_k^r(\pi,\rho)$.

iii. As a particular case of i, the set of all C^{k-1} vector bundle morphisms $\tau_X \to \tau_Y$ is a vector bundle $C_k^{k-1}(\tau_X,\tau_Y) \to C^k(X,Y)$. Remark: Let $J^1(X,Y)$ be the vector bundle over X×Y whose fibre $J^1(X,Y)_{x,y}$ consist of all possible values of derivatives at x of maps f: $X \to Y$ with $f(x) = y$. Using charts at x and y, $J^1(X,Y)_{x,y}$ is clearly isomorphic to $L(\underset{\sim}{E}_o,\underset{\sim}{F}_o)$ ($\underset{\sim}{E}_o,\underset{\sim}{F}_o$ are the spaces on which X and Y are modelled, respectively). Let $\pi_o(J^1)$: $J^1(X,Y) \to X\times Y$ be the projection of this vector bundle, then $\pi(J^1) = pr_1 \circ \pi_o(J^1)$: $J\ (X,Y) \to X$ is a fibre bundle over X. The set $C_k^{k-1}(\pi(J^1))$ of C^{k-1} sections of this fibre bundle over C^k sections of pr_1: $X\times Y \to X$ (clearly $C^k(X,Y)$) is identical to $C_k^{k-1}(\tau_x,\tau_y)$. The space $J^1(X,Y)$ is called the first jet. The tangent functor T maps C^k sections of X×Y into $C_k^{k-1}(\pi(J^1))$.

iv. Let T_*: $C^k(X,Y) \to C_k^{k-1}(\tau_X,\tau_Y)$ be the operation of the tangent functor on the manifold of maps, i.e. $T_*(f) = Tf$. Then, T_* is a smooth section of the vector bundle $C_k^{k-1}(\tau_x,\tau_y) \to C^k(X,Y)$ (alternatively, of $C_k^{k-1}(\pi(J^1)) \to C^k(X,Y))$ and its tangent map

$$TT_*:\ TC^k(X,Y) \to TC_k^{k-1}(\tau_X,\tau_Y)$$

$$TT_*:\ C^k(X,TY) \to C_k^{k-1}(\tau_X,T\tau_Y)$$

is given by

$$TT_*(v) = \omega \circ Tv \quad ,$$

ω being the canonical involution.

CHAPTER 3 - THE GLOBAL MODEL

The ideas of section 1.3.2 will be formulated in this chapter in precise terms using the theory of manifolds and the theory of manifolds of maps. In the global model the assumption that the body and the space are continua is made, but we are not concerned with the internal structure of the body and with interactions within the body.

3.1 Physical Space

In the global model the space is an m-dimensional differentiable manifold S without a boundary.

3.2 Body

A body is an n-dimensional compact differentiable manifold which may have a boundary, with $n \leq m$. A typical body will be denoted by B.

3.3 Configuration

A configuration (or a placement) is an embedding $\kappa: B \to S$ of a body in the space, of class C^k, $0 < k < \infty$. By the results of section 2.12, concerning the set of embeddings of a compact manifold in a manifold without a boundary, the set of all configurations - the configuration space - is a Banach manifold. The configuration space will be denoted by Q.

The global configuration space is a generalization of the concept of a generalized coordinates space, originated in classical mechanics, to the case of a continuum, and it is infinite dimensional since we have an infinite number of degrees of freedom.

3.4 Virtual Displacement

Let (TQ, Q, τ_Q) be the tangent bundle of the configuration space. A virtual displacement is an element $\delta\kappa \varepsilon TQ$. Following section 2.12, we identify the virtual displacement $\delta\kappa$ with a map

$$\delta\kappa: \ B \to TS$$

such that $\tau_Q(\delta\kappa)$, the configuration which is the base point for the virtual displacement, is given by $\tau_S o \delta\kappa$. Here, τ_S is the tangent bundle projection of TS.

3.5 Force

A global force is an element $f \varepsilon T^* Q$. Except for a few remarks in section 3.7.2 we do not discuss representations of forces in general within the framework of this thesis.

3.6 Virtual Work

Let $f \varepsilon T_k^* Q$, $\delta\kappa \varepsilon T_k Q$ for some configuration κ. Then, the action $f(\delta\kappa)$ is called the virtual work of the force f on the virtual displacement $\delta\kappa$.

3.7 Remarks

3.7.1 Material and Spatial Fields

Let (E, B, π) be a vector bundle over the body manifold. Then, we can use κ^{-1}: $\kappa(B) \to B$ to pull back the vector bundle π onto $\kappa(B)$. Since κ is an embedding, $\kappa(B)$ is diffeomorphic with B and therefore $(\kappa^{-1})^* \pi$ is vector-bundle-isomorphic with π.

A similar situation occurs when we have a vector bundle $(F, \kappa(B), \rho)$. In this case we can use κ in order to pull back ρ, and then, $\kappa^* \rho$ will be vector bundle isomorphic with ρ. Moreover, one can use definition 2.6.3 in order to relate sections of the corresponding vector bundles. Thus

$$S((\kappa^{-1})^*_\pi): \quad C^k(\pi) \to C^k((\kappa^{-1})^*\pi)$$

and

$$S(\kappa^*_\rho): \quad C^k(\rho) \to C^k(\kappa^*\rho)$$

are Banach spaces isomorphisms.

Customarily, a section of a vector bundle over the body is called a material or Lagrangian vector field, and a section of a vector bundle over $\kappa(B)$ is called a spatial or Eulerian vector field. We have shown that for any vector bundle (either over the body or over its image) there is an isomorphism between the space of spatial fields and the space of material fields.

In particular, since $\kappa(B)$ is diffeomorphic with B, $T_\kappa Q$ is isomorphic with $C^k(\kappa^*\tau_S)$ and with $C^k(TS|\kappa(B))$, so that every virtual displacement has an Eulerian version and a Lagrangian version.

3.7.2 Concerning the Definition of Global Forces

In the case of classical continuum mechanics one assumes that the space is the three-dimensional Euclidean space and that the body is a three dimensional submanifold of the Euclidean space. Assuming in addition that configurations are continuous embeddings, forces are linear functionals on the vector space of continuous vector fields (virtual displacements) over the body. But then, omitting technicalities, by some type of Riesz representation theorem, a force is composed of a triplet of measures over the body (a measure for each component of the virtual displacement field). On the other hand, a triplet of measures defines a vector measure over the body as required in the classical definition of force. Conversely, every vector measure can be decomposed into three scalar measures [2, p. 20] and can be used as a global force according to the definition given here. Hence, for continuous embeddings the definition given here becomes identical to that given by Truesdell [1], [2].

In the case of C^k embeddings the forces will consist of derivatives of measures as in distributions of finite order.

3.7.3 Body Forces and Surface Forces

As was mentioned in the introduction it is assumed in classical continuum mechanics that forces are composed of surface forces and body forces and can be represented in the form

$$\bar{f} = \int_{\kappa(B)} \bar{b}dv + \int_{\kappa(\partial B)} \bar{t}da$$

Here, $\bar{b}$ and $\bar{t}$ are vector fields called the body force field and surface force field, respectively.

Since we are going to use this type of force as a standard example we will formulate it invariantly. The virtual work of $\bar{f}$ on a virtual displacement field $\overline{\delta\kappa}$ (in the 3-Euclidean space) is

$$\bar{f}(\overline{\delta\kappa}) = \int_{\kappa(B)} \bar{b}(x)\cdot\overline{\delta\kappa}(x)\,dv + \int_{\kappa(\partial B)} \bar{t}(x)\cdot\overline{\delta\kappa}(x)\,da$$

A similar expression can be written on a manifold. Assuming that a volume element θ is given on $\kappa(B)$ and that a volume element θ' is given on $\kappa(\partial B)$, given the 1-forms $\tilde{b}$ and $\tilde{t}$ we can write

$$f(\delta\kappa) = \int_{\kappa(B)} \tilde{b}_x(\delta\kappa_x)\theta + \int_{\kappa(\partial B)} \tilde{t}_x(\delta\kappa_x)\theta'$$

Here, the forms $\tilde{b}$ and $\tilde{t}$ will represent the body force and surface force fields, respectively.

We can even drop the assumption that θ and θ' are given and give the body force field and the surface force field in terms of the vector bundle morphisms

$$\hat{b}: \ TS \to \Lambda^n S$$

and

$$\hat{t}: \ TS \to \Lambda^{n-1} S$$

satisfying $\pi^n \circ \hat{b} = \mathrm{id}$, $\pi^{n-1} \circ \hat{t} = \mathrm{id}$ (π^n is the projection of $\Lambda^n S$ and π^{n-1} is the projection of $\Lambda^{n-1} S$).

Equivalently, we can regard $\hat{b}$ and $\hat{t}$ as sections of $L(TS,\Lambda^n S)$ and $L(TS,\Lambda^{n-1} S)$, respectively.

The vector bundle morphisms $\hat{b}$ and $\hat{t}$ contain the volume element in them implicitly, and in case $\theta,\theta',\tilde{b}$ and $\tilde{t}$ are given, we can construct $\hat{b}$ and $\hat{t}$ by

$$\hat{b}_x(\delta\kappa_x) = \tilde{b}_x(\delta\kappa_x)\theta_x \ , \qquad \hat{t}_x(\delta\kappa_x) = \tilde{t}_x(\delta\kappa_x)\theta'_x \ .$$

Thus, in the most general case we have

$$f(\delta\kappa) = \int_{\kappa(B)} \hat{b}\circ\delta\kappa + \int_{\kappa(\partial B)} \hat{t}\circ\delta\kappa$$

Clearly, one can pullback all the sections defined here onto B to obtain the corresponding material fields.

CHAPTER 4 - THE LOCAL MODEL

In order to consider local deformations and internal forces we introduce in this chapter the local model. As was mentioned in the introduction, the idea here is to treat the neighborhood of the material points as entities which are independent (until compatibility is introduced) of the global model and for which configurations, virtual displacements and forces are defined explicitly. These neighborhoods will be represented mathematically by the fibres of the vector bundles introduced in the local model.

4.1 Physical Space

In the local model the space is conceived as a vector bundle (F,S,ρ).

4.2 Body

The body is modelled mathematically by a vector bundle (E,B,π).

4.3 Configuration

A local configuration is a C^r vector bundle morphism

$$\chi: \quad \pi \to \rho$$

such that the induced base map, χ_o: $B \to S$, is a C^k embedding with $k \geq r$. We say that such a vector bundle morphism is of class C^r_k.

The local configuration space is the set of all local configurations and we will denote it by R. Thus, using the properties of manifolds of maps, the local configuration space has a structure of a vector bundle (R,Q,π_Q) over the global configuration space.

4.4 Virtual Displacement

A local virtual displacement is an element $\delta\chi$ in TR, the tangent bundle of the local configuration manifold.

Since R has a structure of a vector bundle TR has two different vector bundle structures. The first, (TR,R,τ_R), is the tangent bundle structure in which the projection map, τ_R: $TR \to R$, will assign to each virtual displacement its base local configuration.

In order to consider the second vector bundle structure on TR, we first note that if (A,M,ξ) is a vector bundle, we have for the tangent map of the projection ξ,

$$T\xi\colon\ TA \to TM$$

and it can be shown that $(TA,TM,T\xi)$ is a vector bundle. Thus, in our case we have the vector bundle structure $(TR,TQ,T\pi_Q)$, in which the projection map, $T\pi_Q$: $TR \to TQ$, will assign to any local virtual displacement a global virtual displacement.

The relation between the various vector bundles is represented in the following commutative diagram.

$$\begin{tikzcd} & TR \arrow[dl, "\tau_R"'] \arrow[dr, "T\pi_Q"] & \\ R \arrow[dr, "\pi_Q"'] & & T_Q \arrow[dl, "\tau_Q"] \\ & Q & \end{tikzcd}$$

We can make now the identification of TR with the set of C_k^r vector bundle morphisms

$$(E,B,\pi) \to (TF,TS,T\rho)$$

and we have

$$\tau_R(\delta\chi) = \chi$$

$$T\pi_Q(\delta\chi) = \delta\kappa$$

$$\pi_Q(\chi) = \kappa$$

$$\tau_Q(\delta\kappa) = \kappa$$

such that χ, $\delta\kappa$ and κ make the following diagram commutative.

$$\begin{tikzcd}
& & & TF \arrow[dl, "\tau_F"] \arrow[dd, "\tau\rho"] \\
E \arrow[urrr, "\delta\chi"] \arrow[rr, "\chi"'] \arrow[dd, "\pi"'] & & F \arrow[dd, "\rho"] & \\
& & & TS \arrow[dl, "\tau_S"] \\
B \arrow[urrr, "\delta\kappa"] \arrow[rr, "\kappa"'] & & S &
\end{tikzcd}$$

4.5 Force

A local force is an element $\sigma \varepsilon T^*R$. We will not discuss general representations of local forces in this work.

4.6 Virtual Work

Let $\sigma \varepsilon T^*_\chi R$ and $\delta\chi \varepsilon T_\chi R$. The evaluation $\sigma(\delta\chi)$ is called the virtual work of the local force σ on the local virtual displacement $\delta\chi$.

CHAPTER 5 - FIELD REPRESENTATIONS OF THE LOCAL VARIABLES

In the previous chapter it was shown that local configurations are elements of a vector bundle (R,Q,π_Q) and local virtual displacements were defined as elements of the vector bundle (TR,R,τ_R). It follows that a set of local configurations that constitutes a fibre can be given the structure of a Banach space. Similarly, a set of local virtual displacements that constitutes a fibre of τ_R can be given the structure of a Banach space.

In this chapter we construct two vector bundles with the property that the fibres mentioned can be identified with spaces of sections of these bundles. Although this property seems technical, it is of physical importance since it allows us to relate the local configuration to the classical concept of a deformation gradient tensor field and similarly, we can represent a local virtual displacement by a field corresponding to the field of variation of the deformation gradient. Moreover, having the field representation for the local virtual displacements, another vector bundle can be constructed whose sections represent local forces so that we can relate the concept of a local force to the notion of a stress field.

In addition, it is shown that a connection on the vector bundle (F,S,ρ) is sufficient in order that local virtual displacements and local forces can be represented by 2-tensor fields, carrying us one step further towards the classical continuum mechanics.

5.1 Local Representations

Let χ be a local configuration and $\kappa = \pi_Q(\chi)$ its base map.

Given a point $x \varepsilon B$ there is a vector bundle chart (U,ϕ,Φ) at x and a vector bundle chart (V,ψ,Ψ) at $\kappa(x)$ with $\kappa(U) \subset V$. We will denote by $\underset{\sim}{x}$, $(\underset{\sim}{x},\underset{\sim}{e})$, $\underset{\sim}{\kappa}$, $\underset{\sim}{\chi}$ the local representatives of x, $e \varepsilon E_x$, κ, χ, in these charts, respectively. If $\underset{\sim}{E}$ and $\underset{\sim}{F}$ are the typical fibres of π and ρ respectively, we have the following commutative diagram.

$$\begin{array}{ccc} \phi(U)\times\underset{\sim}{E} & \xrightarrow{\underset{\sim}{\chi}} & \psi(V)\times\underset{\sim}{F} \\ \downarrow & \underset{\sim}{\kappa} & \downarrow \\ \phi(U) & \longrightarrow & \psi(V) \end{array}$$

Let (f_1,f_2) be the two components of $\underset{\sim}{\chi}$. Then, it is clear that

$$f_1: \quad \phi(U)\times\underset{\sim}{E} \to \psi(V)$$

is independent of its second argument and is identical to $\underset{\sim}{\kappa}$. The second component

$$f_2: \quad \phi(U)\times\underset{\sim}{E} \to \underset{\sim}{F}$$

is by definition linear in its second argument and the induced map

$$f_2': \quad \phi(U) \to L(\underset{\sim}{E},\underset{\sim}{F})$$

defined by

$$f_2'(\underset{\sim}{x})(\underset{\sim}{e}) = f_2(\underset{\sim}{x},\underset{\sim}{e})$$

is of class C^r.

Let $\delta\chi \varepsilon T_\chi R$ be a local virtual displacement with base map $\delta\kappa$ (i.e. $\delta\kappa = T\pi_Q(\delta\chi)$) and denote by $\underset{\sim}{\delta\chi}$ and $\underset{\sim}{\delta\kappa}$ their respective local representatives in the vector bundle charts induced on τ_F and $T\rho$ by (V,ψ,Ψ). We can now rewrite the commutative diagram of section 4.4 in terms of local representatives as

$$
\begin{tikzcd}
 & & & \psi(V)\times\underset{\sim}{F}\times\underset{\sim}{R}^m\times\underset{\sim}{F} \arrow[dl, "pr_{1,2}"'] \arrow[dd, "pr_{1,3}"] \\
\phi(U)\times\underset{\sim}{E} \arrow[urrr, "\delta\underset{\sim}{\chi}"] \arrow[rr, "\underset{\sim}{\chi}"] \arrow[dd] & & \psi(V)\times\underset{\sim}{F} \arrow[dd] & \\
 & & & \psi(V)\times\underset{\sim}{R}^m \arrow[dl] \\
\phi(U) \arrow[urrr, "\underset{\sim}{\delta\kappa}"] \arrow[rr, "\underset{\sim}{\kappa}"'] & & \psi(V) &
\end{tikzcd}
$$

Denoting by f_1, f_2, f_3, f_4 the four components of $\delta\underset{\sim}{\chi}$ we note that by the commutativity of the top triangle, f_1 and f_2 are indeed those f_1 and f_2 defined above. In addition,

$$f_3\colon \quad \phi(U)\times\underset{\sim}{E} \to \underset{\sim}{R}^m$$

is independent of its second argument, and

$$f_4\colon \quad \phi(U)\times\underset{\sim}{E} \to \underset{\sim}{F},$$

is linear in the second argument and induces a C^r map

$$f_4'\colon \quad \phi(U) \to L(\underset{\sim}{E},\underset{\sim}{F}),$$

defined by

$$f_4'(\underset{\sim}{x})(\underset{\sim}{e}) = f_4(\underset{\sim}{x},\underset{\sim}{e}) \ .$$

5.2 The Local Configuration Field

Consider the vector bundle $(L(E,\kappa^*F),\ B,\ L(\pi,\kappa^*\rho))$ where κ is some configuration. By the definition of the linear map bundle and the pullback of a vector bundle, the fibre of this vector bundle over the point $x\varepsilon B$ is

$$L(E,\overset{*}{\kappa}F)_x = L(E_x,(\overset{*}{\kappa}F)_x) = L(E_x,F_{\kappa(x)}) \quad .$$

We claim that the space $C^r(L(\pi,\overset{*}{\kappa}\rho))$ of C^r sections of this vector bundle is identical to the set of all local configurations over κ and therefore it is isomorphic to $R_\kappa = \pi_Q^{-1}(\kappa)$ (this is in fact the way R_κ is defined in [13]).

Firstly, we use the fact that by the definition of the pull-back for every vector bundle morphism χ there exists a unique vector bundle morphism A: $\pi \to \overset{*}{\kappa}(\rho)$ such that the following diagram is commutative

$$\begin{tikzcd}
E \arrow[dr, "A"] \arrow[drr, bend left, "\chi"] \arrow[ddr, bend right, "\pi"'] & & \\
& \overset{*}{\kappa}F \arrow[r, "\overset{*}{\rho}\kappa"] \arrow[d, "\overset{*}{\kappa}\rho"] & F \arrow[d] \\
& B \arrow[r, "\kappa"] & S
\end{tikzcd}$$

Secondly, we can identify A with the C^r section of $L(\pi,\overset{*}{\kappa}\rho)$ it induces to complete the identification.

One should observe the fact that the elements of $L(\pi,\overset{*}{\kappa}\rho)$ have the properties of a two point tensor. This means that representatives of elements transform under coordinate changes both in ρ and in π, with a different transformation rule in each case. Hence, a section of $L(\pi,\overset{*}{\kappa}\rho)$ is a two point tensor field. This feature, very well known in classical continuum mechanics, will also characterize the rest of the vector bundles that we introduce in this chapter.

5.3 The Local Virtual Displacement Field

We now construct a vector bundle $(\delta_\chi,B,\varepsilon_\chi)$ such that a certain space of its sections is isomorphic to $T_\chi R$ for a given $\chi \varepsilon R$.

Recalling that $T_\chi R$ was the set of C_k^r vector bundle morphisms $\delta\chi$: $\pi \to T\rho$ with $\tau_F \circ \delta\chi = \chi$, we define an equivalence relation $\sim_x$ as follows: Let $\delta\chi_1$, $\delta\chi_2 \varepsilon T_\chi R$, we say that

$$\delta\chi_1 \sim_x \delta\chi_2 \quad \text{iff} \quad \delta\chi_{1x} = \delta\chi_{2x}$$

($\delta\chi_{1x} = \delta\chi_1|\pi^{-1}(x)$ and similarly for $\delta\chi_{2x}$). Thus the quotient set $T_\chi R/\sim_x$ is the set of all possible values that a local virtual displacement can assume over the point x. Clearly, $T_\chi R/\sim_x$ is a vector space. For let $[\delta\chi_1]_x$ and $[\delta\chi_2]_x$ be the equivalence classes of $\delta\chi_1$ and $\delta\chi_2$, respectively, then, since $\tau_F \circ \delta\chi_{1x} = \tau_F \circ \delta\chi_{2x} = \chi_x$, we can use the τ_F vector bundle structure to define

$$a[\delta\chi_1]_x + b[\delta\chi_2]_x = [a\delta\chi_{1x} + b\delta\chi_{2x}]_x \ .$$

In addition, we claim that the vector space $T_\chi R$ is isomorphic to $\underset{\sim}{R}^m \times L(\underset{\sim}{E},\underset{\sim}{F})$. In order to prove this claim, consider a chart (V,ψ,Ψ) at $\kappa(x)$, $\kappa = \pi_Q(\chi)$ and a chart (U,ϕ,Φ) at x with $\kappa(U) \subset V$. Using the notation of section 5.1 it is clear that $\delta\chi_1 \sim_x \delta\chi_2$ iff

$$(f_1(\underset{\sim}{x}),\ f_2'(\underset{\sim}{x}),\ f_3(\underset{\sim}{x}),\ f_4'(\underset{\sim}{x}))_1 = (f_1(\underset{\sim}{x}),\ f_2'(\underset{\sim}{x}),\ f_3(\underset{\sim}{x}),\ f_4'(\underset{\sim}{x}))_2$$

where the 1,2 indices outside the brackets indicate local representations of $\delta\chi_1$ and $\delta\chi_2$, respectively. Hence, given the charts at x and $\kappa(x)$, $T_\chi R/\sim_x$ is identical to the values that (f_1,f_2',f_3,f_4') can assume. However, for all $\delta\chi \varepsilon T_\chi R$ the functions f_1 and f_2' remain constant and since $(f_3,f_4') \varepsilon \underset{\sim}{R}^m \times L(\underset{\sim}{E},\underset{\sim}{F})$ we have the identification of $T_\chi R/\sim_x$ with $\underset{\sim}{R}^m \times L(\underset{\sim}{E},\underset{\sim}{F})$. In order to show that the linear structure is preserved under the identification, let $(f_3,f_4')_1$ and $(f_3,f_4')_2$ be the elements of $\underset{\sim}{R}^m \times L(\underset{\sim}{E},\underset{\sim}{F})$ corresponding to $[\delta\chi_1]_x$ and $[\delta\chi_2]_x$, respectively. Then, if I_x: $T_\chi R/\sim_x \to \underset{\sim}{R}^m \times L(\underset{\sim}{E},\underset{\sim}{F})$ denotes the identification we have

$$I_x(a[\delta\chi_1]_x + b[\delta\chi_2]_x) = I_x([a\delta\chi_{1x} + b\delta\chi_{2x}]_x)$$

$$= (a(f_3)_1 + b(f_3)_2,\ a(f_4')_1 + b(f_4')_2)$$

$$= a(f_3,f_4')_1 + b(f_3,f_4')_2$$

$$= aI_x([\delta\chi_1]_x) + bI_x([\delta\chi_2]_x)\ ,$$

where we used

$$(a\delta\chi_{1x} + b\delta\chi_{2x})(\underset{\sim}{e}) = a(f_3,f_4'(x)(\underset{\sim}{e}))_1 + b(f_3,\ f_4'\ (x)(\underset{\sim}{e}))_2$$

$$= (a(f_3)_1 + b(f_3)_2,\ a(f_4')_1 + b(f_4')_2)(\underset{\sim}{e})\ .$$

We define now

$$\delta_\chi = \bigcup_{x\varepsilon B} T_\chi R/\sim_x$$

and

$$\varepsilon_\chi:\quad \delta\chi \to B$$

by

$$\varepsilon_\chi(a) = x \quad \text{for} \quad a\varepsilon T_\chi R/\sim_x\ .$$

Then, using I_x as trivializing maps, $(\delta_\chi,B,\varepsilon_\chi)$ is a vector bundle with fibre $\underset{\sim}{R}^m\times L(\underset{\sim}{E},\underset{\sim}{F})$, provided we can establish the differentiability of the transition functions.

Let (U^*,ϕ^*,Φ^*) be another chart at x and (V^*,ψ^*,Ψ^*) another chart at $\kappa(x)$. We will denote by $\underset{\sim}{\kappa}^*$, $\delta\underset{\sim}{\kappa}^*$, $\delta\underset{\sim}{\chi}^*$, etc. the local representatives of $\kappa,\delta\kappa$, $\delta\chi$ etc., with respect to these charts. Following this convention, an element $e\varepsilon E$ with $\Phi^*(e) = (\underset{\sim}{x}^*,\underset{\sim}{e}^*)$ will be mapped under $\delta\chi$ into the element in TF whose representation using the new charts induced in TF is $(f_1^*,\ f_2'^*(\underset{\sim}{e}^*),\ f_3^*,\ f_4'^*(\underset{\sim}{e}^*))$. We want to construct the transition functions

$$I \circ I^{*-1}: \ U \cap U^* \to L(\underset{\sim}{R}^m \times L(\underset{\sim}{E},\underset{\sim}{F}),\ \underset{\sim}{R}^m \times L(\underset{\sim}{E},\underset{\sim}{F}))$$

where I^*_x is the identification $T_\chi R/\sim_x \to \underset{\sim}{R}^m \times L(\underset{\sim}{E},\underset{\sim}{F})$ induced by the *-charts.

Since $f_3(x)$ is the third component of an element in τ_F we have

$$f_3 = D(\psi \circ \psi^{*-1})\ (f_3^*)$$

for the transformation rule of f_3.

Let $(\underset{\sim}{x}^*,\ \underset{\sim}{e}_2^*,\ \underset{\sim}{y}^*,\ \underset{\sim}{e}_4^*)$ be the representation of an element in τ_F, then by the transformation rule induced in τ_F by the one in F we have

$$e_4 = D(\Psi \circ \Psi^{*-1})\ (\underset{\sim}{y}^*, \underset{\sim}{e}_2^*) + \Psi \circ \Psi^{*-1}\ (\underset{\sim}{e}_4^*)$$

Hence we can write

$$f_4'(\underset{\sim}{e}) = D(\Psi \circ \Psi^{*-1})(f_3^*, f_2'^*(\underset{\sim}{e}^*)) + \Psi \circ \Psi^{*-1}(f_4'^*(\underset{\sim}{e}^*)) ,$$

and using $\underset{\sim}{e}^* = \Phi^* \circ \Phi^{-1}(\underset{\sim}{e})$ we have

$$f_4'(\underset{\sim}{e}) = D(\Psi \circ \Psi^{*-1})(f_3^*, f_2'^* \circ \Phi^* \circ \Phi^{-1}(\underset{\sim}{e})) + \Psi \circ \Psi^{*-1}(f_4'^* \circ \Phi^* \circ \Phi^{-1}(\underset{\sim}{e}))$$

or

$$f_4' = D(\Psi \circ \Psi^{*-1})(f_3^*, f_2'^* \circ \Phi^* \circ \Phi^{-1}) + \Psi \circ \Psi^{*-1} \circ f_4'^* \circ \Phi^* \circ \Phi^{-1} .$$

Therefore

$$I \circ I^{*-1}: \ U \cap U^* \to L(\underset{\sim}{R}^m \times L(\underset{\sim}{E},\underset{\sim}{F}),\ \underset{\sim}{R}^m \times L(\underset{\sim}{E},\underset{\sim}{F}))$$

is given by

$$I \circ I_x^{*-1}(f_3^*, f_4'^*) = (D(\psi \circ \psi^{*-1})(f_3^*),\ D(\Psi \circ \Psi^{*-1})(f_3^*, f_2'^* \circ \Phi^* \circ \Phi^{-1}) +$$

$$+ \Psi \circ \Psi^{*-1} \circ f_4'^* \circ \Phi^* \circ \Phi^{-1}) ,$$

where the x dependence is understood on the right hand side of the equation.

Note that all compositions are just matrix multiplications and the fact that $f_2'^*$ is present in the right hand side makes ε_χ a vector bundle of class C^r.

By the construction given, we have a vector bundle morphism

$$pr: \quad \varepsilon_\chi \to \kappa^* \ (\tau_S)$$

over the identity morphism on B, which is given locally by $(f_3, f_4') \to f_3$.

The morphism pr induces a morphism pr_* assigning sections of $\kappa^*(\tau_S)$ to sections of ε_χ. Thus, identifying global virtual displacements with the sections of $\kappa^*(\tau_S)$ they induce, we identify $T_\chi R$ with the set $C_k^r(\varepsilon_\chi)$ of C^r sections of ε_χ whose image under pr_* is of class C^k.

5.4 The Case of a Connection

We now assume that a connection is given on the vector bundle (F,S,ρ) by means of a connection map C: $TF \to F$.

By the diffeomorphism

$$(\tau_F, T_\rho, C): \quad TF \to F \oplus TS \oplus F \ ,$$

each element of $T_\chi R$ can be identified with the pair of maps $(T\rho\circ\delta\chi,\ C\circ\delta\chi)$. The map $T\rho\circ\delta\chi$: $E \to TS$ is in fact given by

$$T\rho\circ\delta\chi = T\pi_Q(\delta\chi) = \delta\kappa: \quad B \to TS$$

and does not depend on the fibres of E. The map $C\circ\delta\chi$ is given in local representation by

$$\underset{\sim}{C}o\delta\chi(\underset{\sim}{x},\underset{\sim}{e}) = (\underset{\sim}{\kappa}(\underset{\sim}{x}),\ (f_4'(\underset{\sim}{x})(\cdot) + \Gamma(\underset{\sim}{\kappa}(\underset{\sim}{x}),\ f_3(\underset{\sim}{x}),\ f_2'(\underset{\sim}{x})(\cdot)))(\underset{\sim}{e})) ,$$

where Γ is the Christoffel symbol of the connection, and it follows that $Co\delta\chi$ is a C^r_k vector bundle morphism.

Conversely, a C^k morphism $B \to TS$ over κ and a C^r_k vector bundle morphism over κ will determine a unique $\delta\chi \varepsilon T_\chi R$. Hence, identifying a C^r_k vector bundle morphism $\pi \to \rho$ with a C^r section of $(L(E,\kappa^*F),\ B,\ L(\pi,\kappa^*\rho))$ and a C^k map $B \to TS$ over κ with a C^k section of $\kappa^*(\tau_S)$. We have

$$T_\chi R \simeq C^k(\kappa^*(\tau_S)) \times C^r(L(\pi,\kappa^*\rho)) .$$

The decomposition of $T_\chi R$ that was introduced here induces a connection on the vector bundle (R,Q,π_Q) whose connection map

$$C_*:\ \ TR \to R$$

is given by

$$\delta\chi \rightsquigarrow Co\delta\chi .$$

5.5 <u>The Stress Field</u>

It was shown in section 5.3 that $T_\chi R$ was isomorphic to $C^r_k(\varepsilon_\chi)$. Hence, local forces which are elements of $T^*_\chi R$, can be identified with elements of $C^r_k(\varepsilon_\chi)^*$.

Consider ε^*_χ, the dual bundle of ε_χ, i.e. the vector bundle $(\delta^*_\chi, B, \varepsilon^*_\chi)$, with $(\delta^*_\chi)_x = (\delta_{\chi x})^*$, so that its typical fibre is $(\underset{\sim}{R}^m \times L(\underset{\sim}{E},\underset{\sim}{F}))^* \simeq \underset{\sim}{R}^m \times L(\underset{\sim}{F},\underset{\sim}{E})$. Then, if a volume element θ is given on the body manifold, a section p of ε^*_χ can operate on any section $\mathcal{s}$ of ε_χ by

$$p:\ \ \mathcal{s} \rightsquigarrow \int_B p_x(\mathcal{s}_x)\theta .$$

It follows that p represents a local force σ by

$$\sigma(\delta\chi) = \int_B p_x(\delta_x)\theta$$

where δ is the section of ε_χ representing $\delta\chi$. The section p is then called the stress field representing the local force σ.

In the case where a connection is specified on the vector bundle ρ, the decomposition of $T_\chi R$ into $C^k(\kappa^*(\tau_S)) \times \dot{C}^r(L(\pi,\kappa^*\rho))$ allows us to consider local forces represented by sections of $L(\pi,\kappa^*\rho)^*$ only. As the typical fibre of $L(\pi,\kappa^*\rho)^*$ is $L(\underset{\sim}{F},\underset{\sim}{E})$ local forces can be represented by tensor fields over the body (or equivalently over $\kappa(B)$), as in the classical case. This type of force is of importance because the component of the force that operates on sections of $\kappa^*(\tau_S)$ can be represented by a global force and is of no importance in the local model, while the section of $L(\pi,\kappa^*\rho)^*$ represents the work performed on the deformation of neighborhoods.

CHAPTER 6 - THE COMPATIBILITY OF THE MODELS

Now that the two mathematical models for the placements of a body in the space have been introduced, one has to express the idea that both global and local models represent the same physical phenomenon. In other words, as the transformations of the fibres of the body vector bundle represent configurations of the neighborhoods of the material points, a rule should be given, specifying what local configuration represents the configuration of the neighborhoods corresponding to a known global configuration. Similarly, local virtual displacements should be related to global virtual displacements and local forces to global forces. The rules relating the respective global and local variables are called the compatibility conditions.

6.1 The Compatibility Functor

Let λ be a covariant functor from the category of manifolds and C^k manifold morphisms to the category of vector bundles and C^r_k vector bundle morphisms, assigning to the manifold X a vector bundle over X and to f: $X \to Y$ a vector bundle morphism over f, with

$$\lambda(B) = E , \qquad \lambda(S) = F .$$

It follows that the local models of the body and space are the image of the global models of the body and space, respectively and in addition, the operation of the functor on manifold morphisms induces a section

$$\lambda: \ Q \to R$$

of the local configuration space (R,Q,π_Q). We say that λ is a compatibility functor iff this section of π_Q is differentiable. In the following λ will denote a compatibility functor.

6.2 Configuration

We say that a local configuration χ is compatible with the global configuration κ iff

$$\chi = \lambda(\kappa) \ .$$

6.3 Virtual Displacements

As the section of π_Q induced by λ is differentiable we have the tangent map

$$T\lambda\colon \ TQ \to TR$$

which is a section of $(TR,TQ,T\pi_Q)$. We say that a local virtual displacement $\delta\chi$ is compatible with the global virtual displacement $\delta\kappa$ iff

$$\delta\chi = T\lambda(\delta\kappa) \ .$$

6.4 Forces

Consider the adjoint map of $T\lambda$

$$T^*\lambda\colon \ T^*R|\lambda(Q) \to T^*Q \ .$$

We say that a global force f is compatible with the local force σ iff

$$f = T^*\lambda(\sigma) \ .$$

In the next chapter it will be shown that the compatibility of forces is a generalization of the classical equation of equilibrium.

6.5 Remarks

a. Since λ is a differentiable section and therefore an embedding $T_\kappa\lambda\colon \ T_\kappa Q \to T_{\lambda(\kappa)}R$ splits. Hence we can say that the set of compatible local virtual displacements splits the Banach space of all local virtual displacements.

b. The compatibility of the global force f with the local force σ implies that

$$f(\delta\kappa) = (T^*\lambda(\sigma))(\delta\kappa) \quad \text{for all} \quad \delta\kappa\varepsilon TQ .$$

However, by the definition of the adjoint map

$$(T^*\lambda(\sigma))(\delta\kappa) = \sigma(T\lambda(\delta\kappa))$$

and using the compatibility condition for the virtual displacements we have

$$f(\delta\kappa) = \sigma(\delta\chi)$$

for all compatible pairs $\delta\kappa$, $\delta\chi$.

Hence, the compatibility condition for the forces can be interpreted as saying that a global force f is compatible with a local force σ if the virtual work as calculated using the global model is equal to the virtual work as calculated using the local model. It follows that the given compatibility condition is a general form of the principle of virtual work in continuum mechanics.

c. Let $T^*\pi_Q$: $T^*Q \to T^*R$ be the adjoint map of $T\pi_Q$. Consider elements of T^*R of the form $\sigma = T^*\pi_Q(f)$, where f is any element of T^*Q. In this case we have

$$\begin{aligned} \sigma(\delta\chi) &= (T^*\pi_Q(f))(\delta\chi) \\ &= f(T\pi_Q(\delta\chi)) \\ &= f(\delta\kappa) \end{aligned}$$

It follows that if σ is given by the above relation, compatibility is satisfied identically. However, $\delta\chi$ above is any local virtual displacement with base map $\dot{\delta}\kappa$ and in general it is not compatible with $\delta\kappa$. Hence, since compatibility of forces holds even for virtual displacements which are not compatible, this case is of no importance and the local description becomes trivial. In the case where a connection is specified on ρ and $T^*_\chi R \simeq C^k\ (k^*(\tau_S))^* \times C^r\ (L(\pi,k^*\rho))^*$, local forces of the form $T^*\pi_Q(f)$, $f\varepsilon T^*_k Q$ are those for which the second component vanishes.

CHAPTER 7 - THE LOCAL MODEL OF THE TANGENT BUNDLES

The most natural vector bundle over a manifold is its tangent bundle. Therefore, the vector bundles (TB,B,τ_B) and (TS,S,τ_S) can serve as natural examples of local models for the body and space, respectively. In the local model based on τ_B and τ_S, the local model of tangent bundles, it is the tangent space to a point that represents the neighborhood of this point. Using the obvious choice, the tangent functor, as a compatibility functor, a compatible local configuration assumes the meaning of a deformation gradient and with some assumptions the equations of equilibrium of continuum mechanics can be obtained. In addition, the tangent functor allows us to write equations of compatibility as conditions for kinematical compatibility and it is possible to write a general solution for the case f=o in terms of a stress function.

One could generalize the local model of the tangent bundles to higher tangents, taking powers of the tangent functor as compatibility functors. The local model corresponding to T^n will give continuum mechanics of order n, and as n will approach infinity the local model will approach the nonlocal continuum mechanics.

7.1 The Compatibility of Configurations and Virtual Displacements

In the local model of tangent bundles the body is modelled by the vector bundle (TB,B,τ_B), the space is modelled by (TS,S,τ_S) and the configuration space R consists of C_k^{k-1} vector bundle moprhisms $\tau_B \to \tau_S$. Since we are going to use the compatibility functor as a map and take its tangent map, we will denote the tangent functor by λ_T. Note that by 2.12.6.iv, λ_T: $Q \to R$ is a C^∞ section so that λ_T can serve as a compatibility functor.

The compatibility condition for the configurations implies that χ is compatible with κ iff $\chi = \lambda_T(\kappa)$ or alternatively, $\chi = T\kappa$. Hence, compatibility means that the configuration of a neighborhood of a point is represented by the derivative of the placement at that point.

As an element of TR, a local virtual displacement will be identified in the case of the tangent bundles with a C_k^{k-1} vector bundle morphism $\delta\chi$: $\tau_B \to T\tau_S$. Such a local virtual displacement will be compatible with a global virtual displacement $\delta\kappa\varepsilon TQ$ iff $\delta\chi = T\lambda_T(\delta\kappa)$.

7.2 Alternative Definition for the Kinematical Compatibility

The expression given in the last section for the compatibility of virtual displacement is of abstract nature since it does not specify the local representatives of $\delta\chi$. Another expression for the compatibility condition can be obtained if we use the relation $T\lambda_T(\delta\kappa) = \omega \circ T\delta\kappa$, given in 2.12.6.iv. To see how this relation originates consider a close submanifold $D\subset B$ which is contained in the coordinate neighborhood U with chart ϕ and let (V,ψ) be a chart in S. Then, the manifold of maps $C^k(D,V)$ will be identified with $C^k(\phi(D),\psi(V))$ which is an open subset of the Banach space $C^k(\phi(D),\underset{\sim}{R}^m)$.

Let $\underset{\sim}{\kappa}\varepsilon C^k(\phi(D),\psi(V))$ be a local representative of a configuration $\kappa\varepsilon Q$, then

$$\lambda_T\colon\ C^k(\phi(D),\psi(V)) \to C_k^{k-1}(T(\phi(D)),\ T(\psi(V)) \simeq$$

$$\simeq C^k(\phi(D),\psi(V)) \times C^{k-1}\ (\phi(D),\ L(\underset{\sim}{R}^n,\underset{\sim}{R}^m))$$

will operate on $\underset{\sim}{\kappa}$ by

$$\lambda_T(\underset{\sim}{\kappa}) = (\underset{\sim}{\kappa},\ D\underset{\sim}{\kappa})\ .$$

The tangent map

$$T\lambda_T\colon\ TC^k(\phi(D),\psi(V)) \simeq C^k(\phi(D),T(\psi(V))) \simeq C^k(\phi(D),\psi(V)\times R^m) \to$$
$$TC_k^{k-1}(T(\phi(D)),T(\psi(V))) \simeq C_k^{k-1}(T(\phi(D)),T^2(\psi(V)))$$

will operate on $(\underset{\sim}{\kappa},\underset{\sim}{w})\varepsilon TC^k(\phi(D),\psi(V))$, which can be regarded as a local representative of $\delta\kappa\varepsilon TQ$, by

$$T\lambda_T(\underset{\sim}{\kappa},\underset{\sim}{w}) = (\lambda_T(\underset{\sim}{\kappa}),(D\lambda_T(\underset{\sim}{\kappa}))(\underset{\sim}{w}))$$
$$= (\underset{\sim}{\kappa},D\underset{\sim}{\kappa},\underset{\sim}{w},(D_{\tilde{\kappa}}(D\underset{\sim}{\kappa}))(\underset{\sim}{w}))\ .$$

Now, since D: $C^k(\phi(D),\psi(V)) \to C^{k-1}(\phi(D),L(\underset{\sim}{R}^n,\underset{\sim}{R}^m)$ is linear, we have

$$T\lambda_T(\underset{\sim}{\kappa},\underset{\sim}{w}) = (\underset{\sim}{\kappa},D\underset{\sim}{\kappa},\underset{\sim}{w},D\underset{\sim}{w})\ .$$

It follows that $T\lambda_T(\underset{\sim}{\kappa},\underset{\sim}{w}) = \omega o(\underset{\sim}{\kappa},\underset{\sim}{w},D\underset{\sim}{\kappa},D\underset{\sim}{w}) = \omega oT(\underset{\sim}{\kappa},\underset{\sim}{w})$. This is not the proof of the relation because we did not use charts on the manifolds of maps, however, it is analogous to the proof of the relation in using the linearity of the derivative map.

Thus, using $T\lambda_T(\delta\kappa) = \omega oT\delta\kappa$, the compatibility condition for the local virtual displacements can be written in the form $\delta\chi = \omega oT\delta\kappa$.

This last expression suggests a different point of view regarding the compatibility conditions. Instead of considering $\delta\kappa$ as an element of TQ and using $T\lambda_T$ in order to arrive at the corresponding $\delta\chi$ it is possible to treat $\delta\kappa$ as a map from the manifold B to the manifold TS and apply λ_T to it. However, as $T\delta\kappa$ is a vector bundle morphism $\tau_B \to \tau_{TS}$ and $\delta\chi$ is a vector bundle morphism $\tau_B \to T\tau_S$ we have to use the canonical involution ω satisfying $T\tau_S = \tau_{TS}o\omega$ in order to complete the compatibility picture (see diagram).

$$
\begin{tikzcd}
 & & TTS \arrow[dl, "\tau_{TS}"] & & \text{Local Model} \\
TB \arrow[urr, "\delta\chi"] \arrow[r, "\chi"'] & TS & & {} & \\
 & & TTS \arrow[dl, "T\tau_S"] & {} \arrow[u, "\omega"'] & \\
TB \arrow[urr, "T\delta\kappa"] \arrow[r, "T\kappa"'] & TS & & {} & \text{Functor} \\
 & & TS \arrow[dl, "\tau_S"] & {} \arrow[u, "\lambda_T"'] & \\
B \arrow[urr, "\delta\kappa"] \arrow[r, "\kappa"'] & S & & & \text{Global Model}
\end{tikzcd}
$$

7.3 The Compatibility Equations

Given a local configuration χ (or a local virtual displacement $\delta\chi$), the compatibility equation provides a necessary and sufficient condition for the compatibility of χ ($\delta\chi$, respectively) with its base map $\kappa = \pi_Q(\chi)$ ($\delta\kappa = T\pi_Q(\delta\chi)$, respectively).

Let f be a vector bundle morphism $(TX,X,\tau_X) \to (TY,Y,\tau_Y)$ and denote by (f_1,f_2') its local representative with respect to some charts in X and Y, (we use f_2' as in section 5.1). Let $e \varepsilon TTX$ be in the domain of the chart induced in TTX and denote its representative by $(\underset{\sim}{x},\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3)$, then the local representative of $Tf(e)$ is

$$(f_1(\underset{\sim}{x}),f_2'(\underset{\sim}{x})(\underset{\sim}{e}_1),Df_1(\underset{\sim}{x})(\underset{\sim}{e}_2),Df_2'(\underset{\sim}{x})(\underset{\sim}{e}_2,\underset{\sim}{e}_1) + f_2'(\underset{\sim}{x})(\underset{\sim}{e}_3)) \ .$$

By definition, Tf is a vector bundle morphism $\tau_{TX} \to \tau_{TY}$ but in addition, observing its local representatives, it is clear that $\omega_Y \circ Tf \circ \omega_X$ whose local representative is given by

$$(f_1(\underset{\sim}{x}),Df_1(\underset{\sim}{x})(\underset{\sim}{e}_1),f_2'(\underset{\sim}{x})(\underset{\sim}{e}_2),Df_2'(\underset{\sim}{x})(\underset{\sim}{e}_1,\underset{\sim}{e}_2) + f_2'(\underset{\sim}{x})(\underset{\sim}{e}_3)) \ ,$$

is also a vector bundle morphism $\tau_{TX} \to \tau_{TY}$. Alternatively, we can say that if f is a vector bundle morphism $\tau_X \to \tau_Y$, Tf is both a vector bundle morphism $\tau_{TX} \to \tau_{TY}$ and a vector bundle morphism $T\tau_X \to T\tau_Y$. Note that the base maps for all the vector bundle morphisms $T^2X \to T^2Y$ considered, are vector bundle morphisms $\tau_X \to \tau_Y$.

Let $g = Tg_o$, where g_o: $X \to Y$, then, the local representative of g is $(\underset{\sim}{g}_o, D\underset{\sim}{g}_o)$, where $\underset{\sim}{g}_o$ is the local representative of g_o. The local representative of $Tg(e) = T^2 g_o(e)$ is given by

$$(\underset{\sim}{g}_o(\underset{\sim}{x}), D\underset{\sim}{g}_o(\underset{\sim}{x})(\underset{\sim}{e}_1), D\underset{\sim}{g}_o(\underset{\sim}{x})(\underset{\sim}{e}_2), D^2\underset{\sim}{g}_o(\underset{\sim}{x})(\underset{\sim}{e}_1, \underset{\sim}{e}_2) + D\underset{\sim}{g}_o(\underset{\sim}{x})(\underset{\sim}{e}_3))$$

It follows that a necessary and sufficient condition for f: $\tau_X \to \tau_Y$ to be given by $f = Tf_o$, where f_o is the base map of f, is that

$$\omega_Y \circ Tf \circ \omega_X = Tf$$

or alternatively, that the two vector bundle morphism induced by Tf are identical.

Applying this result to local configurations, we can say immediately that χ is a compatible local configuration iff

$$\omega_S \circ T\chi \circ \omega_B = T\chi \ .$$

In order to arrive at the compatibility equation for the virtual displacements we use the fact that $\omega = \omega^{-1}$ so that $\delta\chi$ is compatible with $\delta\kappa$ iff $\omega_S \circ \delta\chi = T\delta\kappa$. Next, we identify B, TS and $\omega_S \circ \delta\chi$ with X, Y and f above, respectively, to obtain the compatibility equation

$$\omega_{TS} \circ T(\omega_S \circ \delta\chi) \circ \omega_B = T(\omega_S \circ \delta\chi)$$

or

$$\omega_{TS} \circ T\omega_S \circ T\delta\chi \circ \omega_B = T\omega_S \circ T\delta\chi$$

Let $(\underset{\sim}{x}, \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3, \underset{\sim}{e}_4, \underset{\sim}{e}_5, \underset{\sim}{e}_6, \underset{\sim}{e}_7)$ be the coordinates of an element in T^3S. Then, denoting the local representative of ω_{TS} by $\underset{\sim}{\omega}_{TS}$, we have

$$\underset{\sim}{\omega}_{TS}(\underset{\sim}{x},\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3,\underset{\sim}{e}_4,\underset{\sim}{e}_5,\underset{\sim}{e}_6,\underset{\sim}{e}_7) = (\underset{\sim}{x},\underset{\sim}{e}_1,\underset{\sim}{e}_4,\underset{\sim}{e}_5,\underset{\sim}{e}_2,\underset{\sim}{e}_3,\underset{\sim}{e}_6,\underset{\sim}{e}_7)$$

and similarly for $T\omega_S$ we have

$$T\underset{\sim}{\omega}_S(\underset{\sim}{x},\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3,\underset{\sim}{e}_4,\underset{\sim}{e}_5,\underset{\sim}{e}_6,\underset{\sim}{e}_7) = (\underset{\sim}{x},\underset{\sim}{e}_2,\underset{\sim}{e}_1,\underset{\sim}{e}_3,\underset{\sim}{e}_4,\underset{\sim}{e}_6,\underset{\sim}{e}_5,\underset{\sim}{e}_7) .$$

Using these expressions one obtains immediately

$$\omega_{TS} \circ \omega_{TS} = \mathrm{id}$$

$$T\omega_S \circ T\omega_S = \mathrm{id}$$

and

$$\omega_{TS} \circ T\omega_S \circ \omega_{TS} \circ T\omega_S = T\omega_S \circ \omega_{TS} .$$

Thus, operating $\omega_{TS} \circ T\omega_S$ on both sides of the compatibility equation for virtual displacements and using the last two relations, we obtain the equivalent equation

$$T\omega_S \circ \omega_{TS} \circ T\delta\chi \circ \omega_B = \omega_{TS} \circ T\delta\chi .$$

If the local representative of $\delta\chi$ is given in terms of (f_1, f_2', f_3, f_4') as in section 5.1 and $(\underset{\sim}{x},\underset{\sim}{a},\underset{\sim}{b},\underset{\sim}{c})$ are the coordinates of an element in T^2B, then the last equation can be written as

$$(f_1(\underset{\sim}{x}),\ Df_1(\underset{\sim}{a}),\ f_2'(\underset{\sim}{b}),\ Df_2'(\underset{\sim}{a},\underset{\sim}{b}) + f_2'(\underset{\sim}{c}),\ f_3(\underset{\sim}{x}),\ Df_3(a),\ f_4'(\underset{\sim}{b}),\ Df_4'(\underset{\sim}{a},\underset{\sim}{b}) +$$
$$+ f_4'(\underset{\sim}{c})) = (f_1(\underset{\sim}{x}),\ f_2'(\underset{\sim}{a}),\ Df_1(\underset{\sim}{b}),\ Df_2'(\underset{\sim}{b},\underset{\sim}{a}) +$$
$$+ f_2'(\underset{\sim}{c}),\ f_3(\underset{\sim}{x}),\ f_4'(\underset{\sim}{a}),\ Df_3(\underset{\sim}{b}),\ Df_4'(\underset{\sim}{b},\underset{\sim}{a}) + f_4'(\underset{\sim}{c})) .$$

It follows that

$$\tau_{TTS} \circ T\omega_S \circ \omega_{TS} \circ T\delta\chi \circ \omega_B = \tau_{TTS} \circ \omega_{TS} \circ T\tau\chi$$

is the compatibility equation of $\chi = \tau_R(\delta\chi)$, and it will be satisfied for all $\delta\chi\varepsilon T_\chi R$ if χ is compatible.

7.4 The Space P

Our objective in this section is to develop the appropriate structure so that in the next section we can construct a linear operator on $T_{\dot{X}}R$, representing the compatibility equation, whose kernel consists of the compatible virtual displacements.

Let P be the space of C^{k-2} maps $T^2B \to T^2S$ such that for each $g\varepsilon P$, g is both a C^{k-2} vector bundle morphism $\tau_{TB} \to \tau_{TS}$ over a C_k^{k-1} vector bundle morphism $\tau_B \to \tau_S$ and a C^{k-2} vector bundle morphism $T\tau_B \to T\tau_S$ over a C_k^{k-1} vector bundle morphism $\tau_B \to \tau_S$. From this definition it follows that if $g\varepsilon P$, $\omega_S o g o \omega_B \varepsilon P$, and in particular, if f is a vector bundle morphism $\tau_B \to \tau_S$, $Tf\varepsilon P$. Let $g\varepsilon P$, then it follows from the definition of P that the local representative of g with respect to the natural charts in T^2B and T^2S induced by the charts (U,ϕ) and (V,ψ) in B and S, is of the form

$$\underset{\sim}{g} = (g_1,g_2,g_3,g_4',g_4'') ,$$

where

$$g_1\colon\ \phi(U) \to V \quad \text{is a } C^k \text{ map} ,$$

$$g_2\colon\ \phi(U) \to L(\underset{\sim}{R}^n,\underset{\sim}{R}^m) \quad \text{is a } C^{k-1} \text{ map} ,$$

$$g_3\colon\ \phi(U) \to L(\underset{\sim}{R}^n,\underset{\sim}{R}^m) \quad \text{is a } C^{k-1} \text{ map} ,$$

$$g_4'\colon\ \phi(U) \to L(R^n,R^n,R^m) \quad \text{is a } C^{k-2} \text{ map} ,$$

$$g_4''\colon\ \phi(U) \to L(\underset{\sim}{R}^n,\underset{\sim}{R}^m) \quad \text{is a } C^{k-2} \text{ map} ,$$

such that if $(\underset{\sim}{x},\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3)$ is the local representative of $e\varepsilon T^2B$.

$$g(\underset{\sim}{x},\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3) = (g_1(\underset{\sim}{x}),g_2(\underset{\sim}{x})(\underset{\sim}{e}_1),g_3(\underset{\sim}{x})(\underset{\sim}{e}_2),g_4'(\underset{\sim}{x})(\underset{\sim}{e}_1,\underset{\sim}{e}_2) + g_4''(\underset{\sim}{x})(\underset{\sim}{e}_3)) .$$

Define an equivalence relation $\sim_{\dot{x}}$ on P as follows: Let $h_1,h_2\varepsilon P$ and $\dot{x}\varepsilon B$, we say that $h_1 \sim_{\dot{x}} h_2$ iff $h_{1_x} = h_{2_x}$, where $h_{i_x} = h_i|(\tau_{TB} o \tau)^{-1}(x)$.

Clearly, $h_1 \sim_x h_2$ iff $(g_1(\underset{\sim}{x}), g_2(\underset{\sim}{x}), g_3(\underset{\sim}{x}), g_4'(\underset{\sim}{x}), g_4''(\underset{\sim}{x}))_1$, the local representative of h_1 is equal to $(g_1(\underset{\sim}{x}), g_2(\underset{\sim}{x}), g_3(\underset{\sim}{x}), g_4'(\underset{\sim}{x}), g_4''(\underset{\sim}{x}))$, the local representative of h_2. Let $P/\sim_x$ be the quotient space, then

$$W = \bigcup_{x \varepsilon B} P/\sim_x$$

is a fibre bundle over B whose typical fibre is $S \times L(\underset{\sim}{R}^n, \underset{\sim}{R}^m) \times L(\underset{\sim}{R}^n, \underset{\sim}{R}^m) \times L(\underset{\sim}{R}^n, \underset{\sim}{R}^n, \underset{\sim}{R}^m) \times L(\underset{\sim}{R}^n, \underset{\sim}{R}^m)$. If $[g]_x$ an element in W, the projection map

$$\pi_W\colon\ W \to B$$

of this fibre bundle will be given by

$$\pi_W([g]_x) = x \ .$$

The trivializing maps and transition functions will be induced by those in B and S, similarly to the construction of the jet bundles and the element $[g]_x \varepsilon W$ will have the typical coordinates $(g_1, g_2, g_3, g_4', g_4'')$ in the typical fibre. The fibre bundle $\pi_W\colon\ W \to B$ will have the following properties:

(i) By its definition, W has two projections π_1 and π_2 on $J^1(B,S)$ defined as follows: Let $g \varepsilon P$ be a representative of $[g]_x \varepsilon P/\sim_x$ and let χ_1, χ_2 be its base maps with respect to the $\tau_{TB} \to \tau_{TS}$ and $T\tau_B \to T\tau_S$ vector bundle morphisms, respectively. Then,

$$\pi_1([g]_x) = [\chi_1]_x \ \varepsilon \ J^1(B,S)$$

$$\pi_2([g]_x) = [\chi_2]_x \ \varepsilon \ J^1(B,S)$$

where the equivalence classes of χ_1 and χ_2 are those with respect to the relation

$$\chi_i' \sim_x \chi_i'' \quad \text{iff} \quad \chi_i' | \tau_B^{-1}(x) = \chi_i'' | \tau_B^{-1}(x)$$

which is used to construct $J^1(B,S)$.

These projections define two vector bundle structures $(W, J^1(B,S),$

π_1), $(W, J^1(B,S), \pi_2)$, where the τ_{TS} and the $T\tau_S$ bundles are used to define the linear structure on the fibres. Locally the projections π_1 and π_2 are given by

$$(x,y,g_2,g_3,g_4',g_4'') \rightsquigarrow (x,y,g_2)$$

and

$$(x,y,g_2,g_3,g_4',g_4'') \rightsquigarrow (x,y,g_3) \quad ,$$

respectively. The typical fibre for both vector bundles is $L(\underset{\sim}{R}^n,\underset{\sim}{R}^m) \times L(\underset{\sim}{R}^n,\underset{\sim}{R}^n;\underset{\sim}{R}^m) \times L(\underset{\sim}{R}^n,\underset{\sim}{R}^m)$, and if $\phi o \phi^{*-1}$ and $\psi o \psi^{*-1}$ are transition functions in B and S respectively, the induced transformation rules for g_2, g_3, g_4', g_4'' are given by

$$g_2 \rightsquigarrow D(\psi^* o \psi^{-1}) \circ g_2 \circ D(\phi o \phi^{*-1}) \ ,$$

$$g_3 \rightsquigarrow D(\psi^* o \psi^{-1}) \circ g_3 \circ D(\phi o \phi^{*-1}) \ ,$$

$$g_4'(\cdot,\cdot) \rightsquigarrow D(\psi^* o \psi^{-1}) \circ [g_4'(D(\phi o \phi^{*-1})(\cdot),\ D(\phi o \phi^{*-1})(\cdot))$$

$$+ g_4'' \circ D^2\ (\phi o \phi^{*-1})(\cdot,\cdot)] + D^2(\psi^* o \psi^{-1})(g_2 o D(\phi o \phi^{*-1})(\cdot),\ g_3 o D(\phi o \phi^{*-1})(\cdot)) \ ,$$

$$g_4'' \rightsquigarrow D(\psi^* o \psi^{-1}) \circ g_4'' \circ D(\phi o \phi^{*-1}) \ ,$$

which reflect the vector bundle structure of π_1 and π_2.

(ii) Note that if $\xi \varepsilon T^2B$ is vertical, i.e. $T\tau(\xi) = 0$, then $g(\xi)$ is vertical for all $g \varepsilon P$. Thus, we have a map π_{3*}: $P \to R$ such that for $g \varepsilon P$, the corresponding χ in R makes the following diagram commutative.

$$\begin{tikzcd}
V(T^2B) \arrow[r, "g"] \arrow[d, "i_B"'] & V(T^2S) \arrow[d, "i_S"] \\
TS \arrow[r, "\chi"] & TS'
\end{tikzcd}$$

Here, $V(T^2B)$ and $V(T^2S)$ are the vertical subbundles of the respective bundles and i_B, i_S are the isomorphisms $V(T^2B) \to TB$, $V(T^2S) \to TS$ defined in 2.12.2.

It follows that we can define a third vector bundle structure $(W, J^1(B,S), \pi_3)$ by

$$\pi_3([g]_x) = [\chi]_x$$

where $\chi = \pi_{3*}g$ for any representative g of $[g]_x$.

Consider the fibre bundle $J^1(B,J^1(B,S))$ which is the bundle of first jets of maps $B \to J^1(B,S)$ or equivalently jets of vector bundle morphisms $\tau_B \to \tau_S$. Observing the local expression for the tangent map of a vector bundle morphisms χ, as was given in section 7.3, it is clear that $(\underset{\sim}{x}, g_1, g_2, g_3, g_4) \varepsilon \{x\} \times S \times L(\underset{\sim}{R}^n, \underset{\sim}{R}^m) \times L(\underset{\sim}{R}^n, \underset{\sim}{R}^m) \times L(\underset{\sim}{R}^n, \underset{\sim}{R}^n; \underset{\sim}{R}^m)$ are the coordinates of a typical element in $J^1(B,J^1(B,S))$, and we have a C^∞ inclusion in

$$J^1(B,J^1(B,S)) \to W$$

given locally by

$$(x, g_1, g_2, g_3, g_4) \rightsquigarrow (x, g_1, g_2, g_3, g_4, g_2)$$

reflecting the fact that for $\chi \varepsilon R$, $T\chi \varepsilon P$ and $\pi_{3*}(T\chi) = \chi$. From the construction of W and its properties we can draw the following conclusions:

(i) The space P can be identified with the space of C^{k-2} sections of W such that the two sections of $J^1(B,S)$ induced by π_1 and π_2 are C^{k-1}. It follows from [9], where the general case of sections of fibre bundles is treated, that P is a differentiable manifold. In addition, since W has a structure of two vector bundles π_1 and π_2 over $J^1(B,S)$, P will have two vector bundle structures (P, R, π_R), (P, R, π_R') induced by the association of base maps to the $\tau_{TB} \to \tau_{TS}$ and $T\tau_B \to T\tau_S$ vector bundle morphisms respectively (see diagrams).

$$\begin{tikzcd}
& W \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\
J^1(B,S) \arrow[dr, "\pi'_R(f)"'] & & J^1(B,S) \arrow[dl, "\pi_R(f)"] \\
& B \arrow[uu, "f"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& P \arrow[dl, "\pi'_R"'] \arrow[dr, "\pi_R"] & \\
R \arrow[dr, "\pi_Q"'] & & R \arrow[dl, "\pi_Q"] \\
& Q &
\end{tikzcd}$$

$$\begin{tikzcd}
T^2B \arrow[r, "f"] \arrow[d, "\tau_{TB}"'] & T^2S \arrow[d, "\tau_{TS}"] \\
TB \arrow[r, "\pi_R(f)"'] & TS
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2B \arrow[r, "f"] \arrow[d, "T\tau_B"'] & T^2S \arrow[d, "T\tau_S"] \\
TB \arrow[r, "\pi'_R(f)"'] & TS
\end{tikzcd}$$

Thus, we can define a canonical involution ω_*: $P \to P$ by

$$\omega_*(f) = \omega_S \circ f \circ \omega_B$$

and ω_* is clearly an isomorphism satisfying

$$\omega_* = \omega_*^{-1}$$

and

$$\omega_* \circ \pi'_R = \pi_R \ .$$

(ii) The tangent functor is a nonlinear differential operator

$$C^\infty(J^1(B,S)) \overset{j_1}{\to} C^\infty(J^1(B,J^1(B,S)) \overset{in_*}{\to} C^\infty(W)$$

where the first map is the association of the appropriate section of $C^\infty(J^1(B,J^1(B,S))$ with the tangent map of a vector bundle morphism, and the second map is induced by the inclusion $J^1(B,J^1(B,S)) \to W$. It follows [9, p. 67] that

$$\lambda_T\colon \quad R \to P$$

is a C^∞ section of π_R.

(iii) The tangent bundle (TP,P,τ_P) consists of C^{k-2} maps h: $T^2B \to T^3S$, such that for each h:

(a) $\tau_P(h) = \tau_{TTS} \circ h$;

(b) h is a C^{k-2} vector bundle morphism $\tau_{TB} \to T\tau_{TS}$ over a map $h_o \varepsilon TR$;

(c) h is a C^{k-2} vector bundle morphism $T\tau_B \to T^2\tau_S$ over a C^{k-1}_k vector bundle morphism $\tau_B \to T\tau_S$ in TR.

In addition, $T\omega_*$ is a canonical involution on TP with $T\omega_*(h) = T\omega_S o h o \omega_B$.

(iv) The tangent map of the tangent functor (its linearization as a nonlinear differential operator [9, Chapter 17])

$$T\lambda_T\colon \quad TR \to TP$$

which is a section of $T\pi_R$ is given by

$$T\lambda_T(h) = \omega_{TS} \circ Th$$

Again, this relation is based on the linearity of the tangent functor on representatives of elements in R.

7.5 The Compatibility Equations in Terms of Manifolds of Maps

In terms of the language developed in the previous section the compatibility equation for the local configuration assumes the form

$$\omega_* \circ \lambda_T(\chi) = \lambda_T(\chi)$$

and the compatibility equation for the local virtual displacements can rewritten as

$$T\omega_* \circ T\lambda_T(\delta\chi) = T\lambda_T(\delta\chi)$$

or

$$T(\omega_* \circ \lambda_T)(\delta\chi) = T\lambda_T(\delta\chi) \ .$$

Note that the left and right hand sides of the compatibility equation for the local virtual displacements can be obtained by taking the tangent of the maps on the left and right hand sides of the compatibility equation for the configurations. It follows that if χ is a compatible local configuration

$$\tau_P \circ T(\omega_* \circ \lambda_T)(\delta\chi) = \tau_P \circ T\lambda_T(\delta\chi)$$

for all $\delta\chi \varepsilon T_\chi R$. This is just the last expression in section 7.3 written in terms of manifolds of maps.

Let i denote the inclusion $\lambda_T(Q) \to R$. Since λ_T is a section of π_Q, $\lambda_T(Q)$ is a submanifold and we have the vector bundle $(i^*TR, \lambda_T(Q), i^*\tau_R)$ which is just the restriction of TR to virtual displacements over compatible local configurations. Let $\tau_R^* i$ denote the induced map $i^*TR \to TR$, then we can define the vector bundle $((\tau_R^* i)^* TP, i^*TR, (\tau_R^* i)^* T\pi_R)$, which is again just the restriction of $T\pi_R$ to i^*TR.

$$\begin{tikzcd}
(\tau_R^* i)^* TP \arrow[r] \arrow[d, "(\tau_R^* i)^* T\pi_R"'] & TP \arrow[d, "T\pi_R"] \\
i^* TR \arrow[r, "\tau_R^* i"] \arrow[d, "i^*\tau_R"'] & TR \arrow[d, "\tau_R"] \\
\lambda_T(Q) \arrow[r, "i"] & R
\end{tikzcd}$$

Since the base maps $\omega_* \circ \lambda_T$ and λ_T of $T(\omega_* \circ \lambda_T)$ and $T\lambda_T$, respectively, are identical on $i^* TR$, we can define a map

$$L: \quad i^* TR \to (\tau_R^* i)^* TP$$

by

$$L = T(\omega_* \circ \lambda_T) - T\lambda_T$$

where the linear structure of τ_P is used. The restriction of L to $T_\chi R$, where χ is a compatible configuration, induces a map

$$L_\chi: \quad T_\chi R \to T_{\lambda_T(\chi)} P$$

which is clearly linear. The kernel of the map L_χ consists of the compatible virtual displacements away from χ.

7.6 The Compatibility of Forces

In this sections we deal with the form that the compatibility condition for forces assumes in two special important cases. Firstly, we assume that the space manifold S is given a connection and we arrive at the equilibrium equations of continuum mechanics. In the second part we return to the general geometry but we assume that f = 0 and obtain a general solution for this case in terms of a stress function.

7.6.1 The Equilibrium Field Equation

We now assume that a symmetric connection C: $T^2 S \to TS$ is specified on the vector bundle (TS, S, τ_S). There are no further restrictions on the existence of such a connection, and we have $C = C \circ \omega$.

Having a connection on S, the compatibility condition for the virtual displacements $\delta\chi = \omega \circ T\delta\kappa$ can be decomposed into

$$T\tau_S \circ \delta\chi = T\tau_S \circ \omega \circ T\delta\kappa = \delta\kappa$$

and

$$Co\delta\chi = Co\omega oT\delta\kappa = CoT\delta\kappa\ .$$

Let $\delta\kappa'$ denote the spatial vector field corresponding to $\delta\kappa$, i.e., $\delta\kappa'$ is a vector field over $\kappa(B)$ such that $\delta\kappa = \delta\kappa' o\kappa$. It follows that $T\delta\kappa = T(\delta\kappa' o\kappa) = T\delta\kappa' oT\kappa$, and the vertical part of the compatibility condition becomes

$$Co\delta\chi = CoT\delta\kappa = CoT\delta\kappa' oT\kappa = \nabla\delta\kappa' oT\kappa$$

Since $Co\delta\chi$ is a vector bundle morphism $TB \rightarrow TS|\kappa(B)$, any vector bundle morphism $\bar{\sigma}$: $TS|\kappa(B) \rightarrow TB$ with base map κ^{-1}, represents a local force σ by

$$\sigma(\delta\chi) = \int_B \mathrm{tr}\ (\bar{\sigma} o\nabla\delta\kappa' oT\kappa)\theta_B$$

where θ_B is a volume element on B. The vector bundle morphism $\bar{\sigma}$ is customarily called the first Piola-Kirchoff stress. Alternatively, since $\mathrm{tr}(\bar{\sigma} o\nabla\delta\kappa' oT\kappa) = \mathrm{tr}(T\kappa o\bar{\sigma} o\nabla\delta\kappa')$, we can define the Cauchy stress s: $TS|\kappa(B) \rightarrow TS|\kappa(B)$ which is a vector bundle morphism over the identity by $s = T\kappa o\bar{\sigma}$ and we have

$$\sigma(\delta\chi) = \int_{(B)} \mathrm{tr}(s o\nabla\delta\kappa')\theta$$

where θ is a volume element on $\kappa(B)$, which is the pullback $\kappa^{-1*}\theta_B$ of θ_B by κ^{-1}. In addition, let $\hat{s}$ be the vector bundle morphism

$$\hat{s}:\ TS|\kappa(B) \rightarrow \Lambda^{n-1}\ (TS|\kappa(B))$$

over the identity on $\kappa(B)$, defined by

$$\hat{s} o\delta\kappa' = (s o\delta\kappa')\lrcorner\theta\ .$$

We now define the divergence operator

$$\mathrm{div}\hat{s}:\ TS|\kappa(B) \rightarrow \Lambda^{n}\ (TS|\kappa(B))$$

as the vector bundle morphism over the identity of $\kappa(B)$ given by

$$\operatorname{div}\hat{s}\circ\delta\kappa' = d\circ\hat{s}\circ\delta\kappa' - \operatorname{tr}(s\circ\nabla\delta\kappa')\theta$$

where d denotes exterior differentiation. In order to motivate the last two definitions we will write them in local coordinates. Let x^i, $i = 1,\ldots,n$ be local coordinates on an open set in $\kappa(B)$, so that $\theta = \theta_o dx^1\wedge\ldots\wedge dx^n$, s is given in terms of a tensor field $s^i{}_j \frac{\partial}{\partial x^i}\otimes dx^j$ and $\delta\kappa'$ is given in terms of the vector field $\delta\kappa^i \frac{\partial}{\partial x^i}$. Then, by definition

$$\hat{s}\circ\delta\kappa' = (s\circ\delta\kappa')\lrcorner\theta = \sum_{i=1}^{n} (-1)^{i+1}\, \theta_o s^i{}_j \delta\kappa^j dx^1\wedge\ldots\wedge\overline{dx^i}\wedge\ldots\wedge dx^n$$

where the bar denotes omission of the dx^i factor, and by the definition of the exterior derivative

$$\begin{aligned} d\circ\hat{s}\circ\delta\kappa' &= \frac{\partial}{\partial x^i}(\theta_o s^i{}_j \delta\kappa^j) dx^1\wedge\ldots\wedge dx^n \\ &= \frac{\partial}{\partial x^i}(\theta_o s^i{}_j)\delta\kappa^j dx^1\wedge\ldots\wedge dx^m + \\ &\quad + \theta_o s^i{}_j \frac{\partial\delta\kappa^j}{\partial x^i} dx^1\wedge\ldots\wedge dx^n \ . \end{aligned}$$

If Γ^i_{jk} are the components of the Christoffel symbol in this chart, then

$$\nabla\delta\kappa' = \left(\frac{\partial\delta\kappa^j}{\partial x^i} + \Gamma^j_{i\ell}\delta\kappa^\ell\right)\frac{\partial}{\partial x^j}\otimes dx^i$$

and

$$\operatorname{tr}(s\circ\nabla\delta\kappa')\theta = s^i{}_j \left(\frac{\partial\delta\kappa^j}{\partial x^i} + \Gamma^j_{i}\ \delta\kappa^\ell\right)\theta_o dx^1\wedge\ldots\wedge dx^n$$

so that

$$\operatorname{div}\hat{s}\circ\delta\kappa' = \left(\frac{1}{\theta_o}\frac{\partial}{\partial x^i}(\theta_o s^i{}_j) - \Gamma^k_{ij} s^i{}_k\right)\delta\kappa^j\theta_o dx^1\wedge\ldots\wedge dx^n \ .$$

Consider the special case in which $\frac{\partial\theta_o}{\partial x^i} = \theta_o\Gamma^r_{ri}$ (this will be the case if both θ and Γ are based on a Riemannian metric), then

$$\operatorname{div}\hat{s}\circ\delta\kappa' = \left(\frac{\partial}{\partial x^i}(s^i{}_j) + s^i{}_j\Gamma^r_{ri} - s^i{}_k\Gamma^k_{ij}\right)\delta\kappa^j\theta_0 dx^1\wedge\ldots\wedge dx^n$$

and the expression inside the parentheses is the usual definition of the divergence of $s^i{}_j$.

With these definitions the expression for the local virtual work assumes the form

$$\sigma(\delta\chi) = \int_{\kappa(B)} d\hat{s}\circ\delta\kappa' - \operatorname{div}\hat{s}\circ\delta\kappa' \ .$$

We now assume that the local force is given in terms of a body force field and a surface force field as in section 3.7.3, i.e.

$$f(\delta\kappa) = \int_{\kappa(B)} \hat{b}\circ\delta\kappa' + \int_{\kappa(\partial B)} \hat{t}\circ\delta\kappa'$$

where $\hat{b}$ and $\hat{t}$ are spatial fields.

Since compatibility of forces requires that $\sigma(\delta\chi) = f(\delta\kappa)$ for compatible virtual displacements, we have

$$\int_{\kappa(B)} (\operatorname{div}\hat{s}+\hat{b})\circ\delta\kappa' + \int_{\kappa(\partial B)} (\hat{t}-\hat{s})\circ\delta\kappa' = 0$$

where we used the Stokes' theorem for the integral $\int_{\kappa(B)} d\hat{s}\circ\delta\kappa'$. Finally, as this last equation holds for every virtual displacement $\delta\kappa'$ we have

$$\operatorname{div}\hat{s} + \hat{b} = 0 \quad \text{on} \quad \kappa(B)$$

and

$$\hat{t} = \hat{s} \quad \text{on} \quad \kappa(\partial B) \ .$$

Hence, for the case where a connection is specified on the space manifold and the global force is given in terms of a body force and a surface force, we obtain the generalization of the equilibrium equation and boundary conditions to manifolds.

7.6.2 Stress Function. A General Solution for the Case $f = 0$.

This short section was motivated by a paper [15] by C. Truesdell and the treatment here is simply an application of the method given by Truesdell to the general case under consideration here. In addition, it seems that the standard lemma stated below together with the general construction, can serve as a Lagrange multiplier theorem which was not available to Truesdell [15, p. 15]. Moreover, the ease in which the method is adapted to the suggested formulation can indicate the consistency of the suggested formulation with the classical continuum mechanics.

In section 7.5 it was shown that a necessary and sufficient condition that $\delta\chi \varepsilon T_\chi R$ is compatible, is that $\delta\chi \varepsilon \ker L_\chi$, where L_χ: $T_\chi R \to T_{\lambda_T(\chi)}P$ is a linear map and χ is a compatible configuration. We also claim that $L_\chi(T_\chi R)$ is closed in $T_{\lambda_T(\chi)}P$. This will follow if $\ker L_\chi$ splits $T_\chi R$ because in that case, $L_\chi(T_\chi R) = L'_\chi(T_\chi R/\ker L_\chi)$, where L'_χ is the canonical map

$$T_\chi R/\ker L_\chi \to T_{\lambda_T(\chi)}P$$

induced on the $T_\chi R/\ker L_\chi$ by L_χ. Since $T_\chi R/\ker L_\chi$ is isomorphic to the closed complement of $\ker L_\chi$ and since L'_χ is an isomorphism, $L_\chi(T_\chi R)$ is closed. The space $\ker L_\chi$ splits $T_\chi R$ because it consists of compatible virtual displacements, i.e.

$$\ker L_\chi = T_\kappa \lambda_T(T_\kappa Q)$$

and since λ_T is a section and hence an embedding $T_\kappa \lambda_T$ splits.

Lemma [16]: Let X,Y be complete normed linear spaces and

$A: X \to Y$ a continuous linear map whose image is closed in Y. Denote by A^* the adjoint map of A, then the annihilator of $\ker A$ is the image of A^*.

Let $f = 0$, then the compatibility of forces requires that $\sigma(\delta\chi) = 0$ for all compatible $\delta\chi$ or in other words, $\sigma(\delta\chi) = 0$ for all $\delta\chi \varepsilon \ker L_\chi$, so that σ is an element of the annihilator of $\ker L_\chi$. Hence, since L_χ satisfies the conditions of the lemma, every compatible σ is given by $\sigma = L_\chi^*(p)$ for some $p \varepsilon T^*_{\lambda_T(\chi)}P$. The element p is traditionally known as a stress function.

CHAPTER 8 - CONCLUSIONS

The foregoing formulation generalizes the following aspects of classical continuum mechanics:

a. Body and space. The body and space are assumed to be differentiable manifolds.

b. Forces. The formulation is not restricted to forces given in terms of a body force field and a surface force field.

c. Stress. A stress field is just an example of a local force and in general, local force is a distribution of some type.

d. Equilibrium. Rather than a physical law, equilibrium is presented, in the case of statical continuum mechanics, as a requirement of compatibility of two mathematical models of the same physical phenomenon, arising naturally from the kinematical compatibility.

In addition, although no example is given, the formulation makes sense even if a general vector bundle rather than the tangent bundle is used to construct the local model.

Within the framework of the suggested formulation further research can be carried out in order to establish representation of global and local forces. In particular, the possibility that distributional derivatives will appear in the equilibrium equation so that it holds for "generalized" stresses, should be examined. In addition, one can investigate into the relation between the local model and the Cauchy theorem.

This work does not deal with three important subjects included in classical continuum mechanics which are dynamics, thermodynamics and constitutive theory.

In trying to formulate a Gálilean type dynamics on manifolds, one encounters difficulties originating from the fact that no inertial frames are available. Hence, Newton's second law cannot be postulated, as it holds only in inertial frames. One way to avoid this problem is to use Greek-like space time, i.e. space time which is a product of the space manifold with the time dimension so that position in space is absolute. This way one can formulate Newton's law but this law will hold only in one preferred frame. Alternatively, it is possible to assume that space time is locally diffeomorphic with the Greek space time, specifically, that space time has a structure of a fibre bundle over the time dimension. In this case Newton's law can be stated in any trivialization but it will be trivialization or frame dependent [17]. It seems that both descriptions are not satisfactory and one might say that just as ruling out inertial frames led to a structure of space-time which is not affine, inertial forces are not frame invariant if the space is assumed to be a general manifold.

REFERENCES

[1] Truesdell, C. and Toupin, R., "The Classical Field Theories", in Handbuch der Physik, Vol. III/1, edited by S. Flugge, Springer-Verlag, Berlin (1960).

[2] Truesdell, C., "A First Course in Rational Continuum Mechanics", Vol. 1, Academic Press, New York (1977).

[3] Marsden, J. and Hughes, T., "Topics in the Mathematical Foundations of Elasticity", in "Nonlinear Analysis and Mechanics", Heriot-Watt Symposium, Vol. II, edited by R. Knops, Pitman, London (1978).

[4] Green, A.E. and Rivlin, R.S., "On Cauchy's Equations of Motion", Z. Angew Math. Phys., 15 (1964), 290-292.

[5] Lang, S., "Differential Manifolds", Addison-Wesley, Reading, Massachusetts (1972).

[6] Choquet-Bruhat, Y., Dewitt-Morette, C. and Dillard-Bleick, M., "Analysis, Manifolds and Physics", North-Holland, Amsterdam (1977).

[7] Marsden, J., "Application of Global Analysis in Mathematical Physics", Publish or Perish, Boston (1974).

[8] Grueb, W., Halperin, S. and Vanstone, R., "Connections, Curvature and Cohomology", Academic Press, New York (1972).

[9] Palais, R., "Foundations of Global Non-Linear Analysis", Benjamin, New York (1968).

[10] Michor, P.W., "Manifolds of Differentiable Mappings", Shiva, London (1980).

[11] Golubitsky, M. and Guillemin, V., "Stable Mappings and Their Singularities", Springer-Verlag, New York (1973).

[12] Eliasson, I.E., "Geometry of Manifolds of Maps", J. Diff. Geom. 1 (1967), 169-194.

[13] Verona, A., "Bundles Over Manifolds of Maps and Connections", Rev. Roum. Math. Pures Appl., 15 (1970), 1097-1112.

[14] Lang, S., "Analysis II", Addison-Wesley, Reading, Massachusetts (1969).

[15] Truesdell, C., "Invariant and Complete Stress Functions for General Continua", Arch. Rat. Mech. Anal., 4 (1960), 1-29.

[16] Taylor, A.E., "Introduction to Functional Analysis", John Wiley, New York (1958).

[17] Segev, R. and Epstein, M., "Some Geometrical Aspects of Continuum Mechanics", Report No. 158, Dept. of Mech. Engg., Univ. of Calgary, March 1980.