THE UNIVERSITY OF CALGARY

Some Results Concerning Periodic Continued Fractions

by

Kell Hiu Fai Cheng

A DISSERTATION

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE

DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS AND STATISTICS

CALGARY, ALBERTA

September, 2003

THE UNIVERSITY OF CALGARY

FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a dissertation entitled "Some Results Concerning Periodic Continued Fractions" submitted by Kell Hiu-Fai Cheng in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Hugh Williams
Supervisor, Dr. H. C. Williams, Department of Mathematics and Statistics

Richard K. Guy
Co-supervisor, Dr. R. K. Guy, Department of Mathematics and Statistics

Renate S.
Dr. R. Scheidler, Department of Mathematics and Statistics

Peter D. Zvengrowski
Dr. P. Zvengrowski, Department of Mathematics and Statistics

Dr. K. Barker, Department of Computer Science

Kenneth S. Williams
Dr. K. S. Williams, Carleton University

Oct 15/2003
Date

Abstract

This dissertation discusses the regular continued fraction expansion of $\sqrt{D(X)}$ where $D(X)$ is a quadratic polynomial whose coefficients satisfy a certain divisibility condition.

There are two main results in this dissertation. We show that certain families of non-square D may be represented by some quadratic $D(X)$ satisfying the divisibility condition and give a surprising period length property of the product of some members of a family. The second result generalizes the work of van der Poorten and Williams [206] concerning the continued fraction expansion of $\sqrt{D(X)}$ for sufficiently large X, where $D(X)$ obeys the divisibility condition. Also, we establish an upper bound for the period length of the continued fraction expansion of $\sqrt{D(X)}$ using the Lucas-Lehmer theory, and construct the fundamental unit of the real quadratic order $[1, \sqrt{D(X)}]$.

Acknowledgments

I would like to thank my thesis advisor, Dr. Hugh Williams, for his patience, guidance, enlightenments and generous financial support. There will not be a thesis without him. I am indebted to Dr. Richard Guy for the inspiration he gave me over the years. I thank Dr. Renate Scheidler for her tireless and careful reading and insightful suggestions that improve the overall quality of this thesis. Last but not least, many thanks go to the Department of Mathematics and Statistics of the University of Calgary for their support.

Dedication

To Mom, Dad and Lea, this thesis would not be done without your support.

Table of Contents

List of Tables

Glossary

A	natural number whose square is the first coefficient of $D(X)$ 2
A^*	$A/(\Delta_2\Delta_4/\tau)$ 98
A'	A^*/Γ. In particular, $A' = 1$ when $A \mid B$ 99
A_i	numerator of the i-th convergent of a continued fraction 4
a	rational integer; also the real part of s in Section 1.6 36
a_i	i-th partial quotient of a continued fraction 4
$\langle a_0, a_1, a_2, \ldots, a_n\rangle$	formal continued fraction 4
$(a_0, a_1, a_2, \ldots, a_n)$	regular continued fraction; 6
$(\theta_1, \theta_2, \ldots, \theta_k)$	Ideal generated by $\theta_1, \theta_2, \ldots, \theta_k \in \mathcal{O}_\Delta$. 22
$\mathfrak{a}$	ideal of $\mathcal{O}_\Delta$ 21
$\overline{\mathfrak{a}}$	conjugate ideal of $\mathcal{O}_\Delta$ 25
B	rational integer; also, $2B$ is the second coefficient of $D(X)$ 2
B^*	$B/(\Delta_2\Delta_4/\tau)$ 98
B'	B^*/Γ 212
B_i	denominator of the i-th convergent of a continued fraction 4
b	rational integer; also the imaginary part of s in Section 1.6 36
C	rational integer; also, the constant coefficient of $D(X)$ 2
$\mathbb{C}$	the set of complex numbers 36
C_Δ	class group of $\mathcal{O}_\Delta$ 26
D	non-square natural number 9
D_0	radicand of a real quadratic field, square-free 16
$D(X)$	$A^2X^2 + 2BX + C$ 2

$\mathrm{lp}(\theta)$	period length of the c. f. expansion of a quadratic irrational θ	55
M	part of $L^2D = M^2 - N$ in Lemma 4.1.1	93
m	subscript or rational integer	7, 34
N	part of $L^2D = M^2 - N$ in Lemma 4.1.1	93
$\mathcal{N}(\mathfrak{a})$	norm of an ideal $\mathfrak{a}$	25
$\mathcal{N}(\theta)$	norm of a quadratic irrational θ	19
n	subscript or rational integer	2, 9
$\mathcal{O}_0$	maximal order of $\mathbb{Q}(\sqrt{D_0})$ with discriminant Δ_0	17
$\mathcal{O}_\Delta$	order of $\mathbb{Q}(\sqrt{D_0})$ with discriminant Δ	17
$O(g(x))$	$f(x) = O(g(x))$ means $f(x)/g(x) \to 1$ as $x \to \infty$	147
$o(g(x))$	$f(x) = o(g(x))$ means $f(x)/g(x) \to 0$ as $x \to \infty$	39
P	rational integer; also coefficient of $x^2 - Px + Q = 0$	9, 40
P_i	rational part of the numerator of the i-th complete quotient of a c.f.	11
p	rational prime	42
P^*	$(-1)^m (B_{m-1}B_{m-2}Q' - A_{m-1}A_{m-2}Q + (A_{m-1}B_{m-2} + A_{m-2}B_{m-1})P)$	
	In particular, $P^* = M/L - H\|N\|/(dL)$	93, 93
$\mathbf{P}_i$	$AX + B/A - \Delta_1^{\varepsilon_i}(g_i/A')(\Delta'/d_i)$	104
Q	non-zero rational integer	9
Q_i	denominator of the i-th complete quotient of a continued fraction	11
Q^*	$(-1)^m \left(A_{m-1}^2 Q - B_{m-1}^2 Q' - 2A_{m-1}B_{m-1}P\right)$ or $Q^* = \|N\|Q_0/d^2$	93
$\mathbf{Q}_i$	$\Delta_1^{\varepsilon_i}(\Delta'/d_i)^2$	104
$\mathbb{Q}$	the field of rational numbers	16
$\mathbb{Q}(\sqrt{D_0})$	real quadratic field	16
q	$\lfloor B/A \rfloor$ in Chapter 4	95

Chapter 1

Introduction

Continued fractions have been studied by mathematicians for over 2000 years. They have found application in a wide variety of areas in mathematics, particularly when accurate approximations of certain objects are required. One important place where continued fractions have found application is in number theory. For instance, we can use the properties of periodic continued fractions to determine the fundamental unit of a real quadratic order and subsequently to determine the class number and class group structure of that order. These invariants of the order are of enormous importance in understanding how to solve number theoretic and Diophantine problems within the order. Thus, any progress in understanding the structure of a continued fraction expansion of real quadratic irrational is of great value in any subsequent work to be done in the associated order. One of the objectives of this thesis is to produce the complete continued fraction period for irrationals in a particular class first studied by Schinzel in 1961.

Finding the fundamental unit of a real quadratic order, as we will show in Section 1.3, is essentially equivalent to finding the fundamental solution of the quadratic Diophantine equation

$$|x^2 - Dy^2| = 1, \tag{1.1}$$

where $x, y \in \mathbb{Z}$ and D is a non-square natural number. The well-known method for solving such an equation is the continued fraction method, in which we convert $\sqrt{D}$ into a continued fraction and subsequently obtain the fundamental solution.

Although the continued fraction method provides the fundamental solution of (1.1), getting the fundamental solution is by no means trivial in general. In many instances, the computation is extremely time-consuming when D is large. However, it is sometimes possible to write D as an integer-valued polynomial evaluated at some X, i.e., $D = A^2X^2 + 2BX + C$, and express the regular continued fraction expansion of $\sqrt{D}$ in terms of the coefficients of that polynomial.

In his 1961 paper [213], Schinzel showed that if $D(X)$ is an integer-valued polynomial of odd degree, or of even degree with non-square leading coefficient, then the regular continued fraction expansion of $\sqrt{D(X)}$ has unbounded period length as X varies. However, when $D(X)$ is a quadratic polynomial,

$$D(X) = A^2X^2 + 2BX + C, \tag{1.2}$$

with integer coefficients, where we may take $A > 0$, Schinzel showed that the period length of the continued fraction expansion of $\sqrt{D(X)}$ is bounded for all integers X if and only if it satisfies what is called the **Schinzel condition**

$$\Delta \mid 4\gcd(A^2, B)^2, \tag{1.3}$$

where the **discriminant**, $\Delta = B^2 - A^2C$, is non-zero.

van der Poorten and Williams [206] gave the regular continued fraction expansion of $\sqrt{D(X)}$ for sufficiently large X when $D(X)$ is a quadratic polynomial satisfying the Schinzel condition together with the additional condition that $\gcd(A^2, 2B, C)$ is squarefree. For example, they showed that $\sqrt{D(X)}$ has regular continued fraction expansion

$$\left(AX + a_0, \overline{a_1, a_2, \ldots, a_n, 2(AX + a_0)}\right),$$

where $B, C > 0$, $|\Delta| = |B^2 - A^2C| = 1$ and $B/A = (a_0, a_1, a_2, \ldots, a_n)$ for all non-negative integers X. It is clear that the above expansion of $\sqrt{D(X)}$ has fixed period length for all non-negative integers X. We will give a more detailed account of van der Poorten's and Williams's work.

There are two main results in this thesis. We show that certain families of non-square D may be represented by some quadratic $D(X)$ satisfying the Schinzel condition. By this result, we find a surprising period length property associated with the product of some members of a family.

The second result concerns the Schinzel condition. Schinzel's result regarding the period length of $\sqrt{D(X)}$ has two parts. Louboutin [113] studied the necessity part and essentially reproved this part by establishing a lower bound for the period length of $\sqrt{D(X)}$, where $D(X)$ does not satisfy the Schinzel condition. In Chapter 4, we reprove the sufficiency part of Schinzel's result, i.e, if $D(X)$ satisfies the Schinzel condition, then the continued fraction expansion of $\sqrt{D(X)}$ has a bounded period as X varies. We use a constructive approach to investigate the continued fraction expansion of $\sqrt{D(X)}$. Assuming the Schinzel condition, we show that as X varies, the continued fraction expansions of $\sqrt{D(X)}$ can be separated into a finite number of classes of constant period length. More importantly, we obtain the continued fraction expansion of $\sqrt{D(X)}$. By this result, we obtain an upper bound for the period length of $\sqrt{D(X)}$ and establish a formula for the fundamental unit of the real quadratic order $[1, \sqrt{D(X)}]$ for fixed X in Chapter 5. Our result generalizes the work of van der Poorten and Williams [206] by dropping the condition that $\gcd(A^2, 2B, C)$ is squarefree.

This thesis comprises six chapters of material along with a bibliography and an appendix. The first chapter is an introduction to continued fractions and real quadratic orders. The second chapter is a discussion of known results on the continued fraction expansion of $\sqrt{D(X)}$. The third chapter contains our first original result regarding some families of D. The fourth chapter contains the major result of the thesis, namely the continued fraction expansion of $\sqrt{D(X)}$ for sufficiently large X, where $D(X)$ satisfies the Schinzel condition. Based upon the work in Chapter 4, in Chapter 5 we present the upper bound for the period length of the expansion of $\sqrt{D(X)}$ as well as the fundamental unit of the real quadratic order $[1, \sqrt{D(X)}]$. Chapter 6 contains the concluding remarks and future research topics. The bibliography is intended to be as complete as possible, with the referenced that are explicitly referred to in this thesis being marked with an asterisk. The appendix consists of examples and relevant information pertaining to our work here.

1.1 Basic Definitions

The material in this section is available in the literature, such as Perron [190] and Rosen [211].

Definition 1.1.1 *By a* **formal continued fraction** *we mean an expression of the form:*

$$a_0 + \cfrac{1}{a_1 + \cfrac{\ddots}{\cfrac{1}{a_n + \cfrac{1}{\ddots}}}} \tag{1.4}$$

where a_i is a real number. We call a_i the i-th **partial quotient** *for $i = 0, 1, 2, \ldots$.*

Remark 1.1.1 The above partial quotients a_i can be complex numbers or even polynomials.

We rewrite (1.4) in the typographically simpler form

$$\langle a_0, a_1, a_2, \ldots, \rangle. \tag{1.5}$$

Let

$$\theta_i = \langle a_i, a_{i+1}, \ldots, \rangle \quad \text{for} \quad i = 0, 1, 2, \ldots, \tag{1.6}$$

then it is apparent that for any fixed $i \geq 0$,

$$\theta_i = a_i + \frac{1}{\theta_{i+1}}, \;\; \theta_{i-1} = a_{i-1} + \frac{1}{\theta_i}, \;\; \ldots, \;\; \theta_1 = a_1 + \frac{1}{\theta_2}, \;\; \theta_0 = a_0 + \frac{1}{\theta_1}, \tag{1.7}$$

and θ_0 is the continued fraction in (1.5). We refer to θ_i as the i-th **complete quotient**.

Put

$$A_{-2} = 0, \;\; B_{-2} = 1, \;\; A_{-1} = 1, \;\; \text{and} \;\; B_{-1} = 0,$$

and for $i \geq 0$, define recurrence relations

$$A_i = a_i A_{i-1} + A_{i-2} \quad \text{and} \quad B_i = a_i B_{i-1} + B_{i-2}. \tag{1.8}$$

Then a truncation, $\langle a_0, a_1, a_2, \ldots, a_i \rangle$ for some non-negative integer i, of the continued fraction in (1.5) may be written as ratio A_i/B_i. We call such a ratio the i-th **convergent** of the continued fraction. The equations in (1.8) were first introduced by John Wallis in his 1655 publication, *Arithmetica Infinitorum*.

It is well-known that the equations in (1.8) can be written as a matrix equation,

$$\begin{pmatrix} A_i & A_{i-1} \\ B_i & B_{i-1} \end{pmatrix} = \begin{pmatrix} A_{i-1} & A_{i-2} \\ B_{i-1} & B_{i-2} \end{pmatrix} \begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix} = \prod_{j=0}^{i} \begin{pmatrix} a_j & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.9}$$

This implies that

$$A_i B_{i-1} - A_{i-1} B_i = \det\left(\prod_{j=0}^{i} \begin{pmatrix} a_j & 1 \\ 1 & 0 \end{pmatrix} \right) = (-1)^{i+1} \tag{1.10}$$

and $\gcd(A_i, B_i) = \gcd(A_i, A_{i-1}) = \gcd(A_{i-1}, B_{i-1}) = \gcd(B_i, B_{i-1}) = 1$.

If we write

$$\prod_{j=1}^{i} \begin{pmatrix} a_j & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} u & z \\ v & w \end{pmatrix},$$

then

$$\begin{pmatrix} A_i & A_{i-1} \\ B_i & B_{i-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u & z \\ v & w \end{pmatrix}.$$

This implies that $A_i = a_0 u + v$, $B_i = u$, $A_{i-1} = a_0 z + w$ and $B_{i-1} = z$. When the rightmost matrix of the above equation is symmetric, that is $v = z$, we get

$$A_i = a_0 u + v, \quad B_i = u, \quad A_{i-1} = a_0 v + w, \quad \text{and} \quad B_{i-1} = v. \tag{1.11}$$

By (1.6) and (1.7), we may write

$$\theta_0 = \langle a_0, a_1, \ldots, a_{i-1}, \theta_i \rangle. \tag{1.12}$$

If $i = 1$, then

$$\theta_0 = \langle a_0, \theta_1 \rangle = \frac{a_0\,\theta_1 + 1}{\theta_1} = \frac{A_0\,\theta_1 + A_{-1}}{B_0\,\theta_1 + B_{-1}}.$$

Similarly, if $i = 2$, then

$$\theta_0 = \langle a_0, a_1, \theta_2 \rangle = \frac{A_1\,\theta_2 + A_0}{B_1\,\theta_2 + B_0}.$$

Repeating this process inductively on i, we get

$$\theta_0 = \langle a_0, a_1, a_2, \ldots, a_{i-1}, \theta_i \rangle = \frac{A_{i-1}\,\theta_i + A_{i-2}}{B_{i-1}\,\theta_i + B_{i-2}}. \tag{1.13}$$

Rearranging terms to express θ_i in terms of θ_0, we get a useful formula,

$$\theta_i = -\frac{B_{i-2}\,\theta_0 - A_{i-2}}{B_{i-1}\,\theta_0 - A_{i-1}}. \tag{1.14}$$

It follows that θ_i is uniquely determined by θ_0 and $\langle a_0, a_1, \ldots, a_{i-1} \rangle$.

When the initial partial quotient, a_0, is an integer and the other partial quotients, a_i, are natural numbers for $i \geq 1$, we call the continued fraction in (1.4) **regular** and write it as

$$(a_0, a_1, a_2, \ldots).$$

Here, the use of ( ) is to distinguish the regular continued fraction expansion from the formal continued fraction expansion denoted by $\langle\ \rangle$.

We use the Euclidean algorithm to compute the regular continued fraction expansion of a real number θ. Set $\theta_0 = \theta$ and $a_0 = \lfloor \theta_0 \rfloor$. If θ_0 is an integer, then the continued fraction expansion of θ is (a_0). If θ_0 is not an integer, then it can be written as $\lfloor \theta_0 \rfloor + 1/\theta_1$ for some positive real number $\theta_1 > 1$. Let $a_1 = \lfloor \theta_1 \rfloor$. If θ_1 is an integer, then we get $\theta = (a_0, a_1)$. Otherwise, θ_1 is not an integer and it can be written as $a_1 + 1/\theta_2$ for some positive real number $\theta_2 > 1$. Continuing with this process, it either terminates, or it doesn't, in which case we get

$$\theta_0 = \langle a_0, a_1, a_2, \ldots, a_{i+1}, \ldots \rangle, \tag{1.15}$$

where $a_{i+1} = \lfloor \theta_{i+1} \rfloor$ and $\theta_{i+1} = 1/(\theta_i - a_i)$ for $i \geq 0$. It is clear that $\theta_{i+1} > 1$ for all $i \geq 0$. Thus, $a_{i+1} = \lfloor \theta_{i+1} \rfloor \geq 1$. Hence, we may write

$$\theta_0 = (a_0, a_1, a_2, \ldots).$$

The expansion in (1.15) terminates exactly when $\theta_i = \lfloor \theta_i \rfloor$ for some i. By the Euclidean algorithm, this happens if and only if θ_0 is rational. Hence, we have a **finite** regular continued fraction expansion if and only if θ_0 is rational. In other words, as a contrapositive statement, we get an **infinite** regular continued fraction expansion if and only if θ_0 is irrational. Moreover, when θ_0 is irrational, the infinite regular continued fraction expansion of θ is unique by (1.15).

Lemma 1.1.1 *Let θ be an irrational number, $s_0, s_1, \ldots, s_{m-1} \in \mathbb{N}$ and θ^* defined by*

$$\theta = \langle s_0, s_1, \ldots, s_{m-1}, \theta^* \rangle.$$

If $s_0 \neq \lfloor \theta \rfloor$, then $\theta^* < 1$.

Proof: Let $\theta_1 = 1/(\theta - s_0)$. Then $\theta_1 = \langle s_1, s_2, \ldots, s_{m-1}, \theta^* \rangle$. If $s_0 \neq \lfloor \theta \rfloor$, then $0 < \theta_1 < 1$ if $s_0 < \lfloor \theta \rfloor$ and $\theta_1 < 0$ if $s_0 > \lfloor \theta \rfloor$. Write $\theta_2 = 1/(\theta_1 - s_1)$. Since $s_1 \in \mathbb{N}$, we get $\theta_2 = 1/(\theta_1 - s_1) < 0$. Similarly, $\theta_3 = 1/(\theta_2 - s_2) < 0$. Continuing with the process, we get $\theta^* = 1/(\theta_{m-1} - s_{m-1}) < 0$. Hence, $\theta^* < 1$. □

Theorem 1.1.1 *Let θ be an irrational number, $a_0 = \lfloor \theta \rfloor$, $s_0, s_1, \ldots, s_{m-1} \in \mathbb{N}$, $\theta_1 = 1/(\theta - a_0)$, $\theta_{i+1} = 1/(\theta_i - s_{i-1})$ for $1 \leq i \leq m$, and θ^* be a real number defined by the formal continued fraction*

$$\theta = \langle a_0, s_0, s_1, \ldots, s_{m-1}, \theta^* \rangle. \tag{1.16}$$

If $\theta^* > 1$, then $s_{i-1} = \lfloor \theta_i \rfloor$ for $1 \leq i \leq m$.

Proof: Suppose on the contrary that there exists some i such that $1 \leq i \leq m$ and $s_{i-1} \neq \lfloor \theta_i \rfloor$. Then by Lemma 1.1.1, we have $\theta^* < 1$. □

Remark 1.1.2 From Theorem 1.1.1, we see that θ^* is the $(m+1)$-th complete quotient θ_{m+1} of the regular continued fraction expansion of θ. Thus, if $(s_m, s_{m+1}, \ldots)$ is the regular continued fraction expansion of θ^*, then the regular continued fraction expansion of θ is given by

$$(a_0, s_0, s_1, \ldots, s_{m-1}, s_m, s_{m+1}, \ldots).$$

Hence, when conditions of the theorem are met, it is justified to write the continued fraction in (1.16) as $(a_0, s_0, s_1, \ldots, s_{m-1}, \theta^*)$ and refer to it as the regular continued fraction expansion of θ.

Unlike the infinite expansions, it is possible to express a finite regular continued fraction expansion in two ways. Suppose that we have a regular continued fraction expansion $(a_0, a_1, a_2, \ldots, a_i)$. If $a_i \geq 2$, then we may also write the expansion as $(a_0, a_1, a_2, \ldots, a_i - 1, 1)$. Similarly, if $a_i = 1$, then we may write $(a_0, a_1, a_2, \ldots, a_i)$ as $(a_0, a_1, a_2, \ldots, a_{i-1} + 1)$.

In the theorem below, we give a classical result on finite continued fractions and linear Diophantine equations.

Theorem 1.1.2 *Let a, b, and c be non-zero integers and x and y integer unknowns. If $d = \gcd(a, b)$ and $d \mid c$, then the solutions to the linear Diophantine equation*

$$ax - by = c$$

are given by

$$x = B_{n-1}(c/d) + m(b/d) \quad \text{and} \quad y = A_{n-1}(c/d) + m(a/d),$$

where A_{n-1}/B_{n-1} is $(n-1)$-th convergent of $a/b = (a_0, a_1, \ldots, a_n)$, n is odd and m is any integer.

Proof: See Rosen [211, Theorem 2.14, p. 113].

1.2 Periodicity and Quadratic Irrationals

Among the infinite regular continued fractions, the periodic ones are the most studied. Over the past few centuries, many interesting and important results have been established. References and proofs can be found in Davenport [31], Hurwitz [74] and [75], Perron [190] and Williams [251], for instance. We will distill this wealth of information into a few sections of results that are needed for the work in the sequel. Since we are primarily interested in regular continued fractions, we will omit the word *regular* from now on, unless ambiguity arises.

Definition 1.2.1 *An infinite continued fraction,* $(a_0, a_1, a_2, \ldots, \ldots)$*, is called* **periodic** *if there exists a natural number* ℓ *and an non-negative* k *such that for all non-negative integers* n*,*

$$a_{k+i} = a_{k+n\ell+i}$$

for $0 \leq i \leq \ell - 1$*.*

We write such a periodic continued fraction as

$$(a_0, a_1, \ldots, a_{k-1}, \overline{a_k, a_{k+2}, \ldots, a_{k+\ell-1}}) \tag{1.17}$$

and if ℓ is minimal, it is called the **period length.** When k is minimal, the integer sequence $a_0, a_1, \ldots, a_k$ is referred to as the **pre-period** and the integer sequence $a_k, a_{k+1}, \ldots, a_{k+\ell-1}$ under the overline notation as the **period.** Also, when the pre-period is empty, in this case $k = 0$, we call such a continued fraction expansion **purely periodic.**

The study of periodic continued fraction expansions focuses on a mathematical object called a quadratic irrational.

Definition 1.2.2 *A* **quadratic irrational** *is a real number of the form*

$$\frac{P + \sqrt{D}}{Q}$$

where D *is a non-square natural number,* P *is an integer,* Q *is a natural number, and*

$$P^2 \equiv D \bmod Q. \tag{1.18}$$

Note that since

$$\frac{P + \sqrt{D}}{Q} = \frac{PQ + \sqrt{DQ^2}}{Q^2} \quad \text{and} \quad (PQ)^2 \equiv DQ^2 \bmod Q^2,$$

we may always assume (1.18). For instance, the golden ratio, $(1 + \sqrt{5})/2$, satisfies (1.18), i.e.,

$$1^2 \equiv 5 \bmod 2,$$

and the other conditions required by Definition 1.2.2. Hence, it is a quadratic irrational.

In the eighteenth century, Euler and Lagrange proved that *a real number θ has a periodic continued fraction expansion if and only if it is a quadratic irrational.* Euler established the sufficiency condition and Lagrange proved the necessity condition. In light of this theorem, we understand the reason why the study of periodic continued fraction expansions is closely related to the topic of quadratic irrationals.

In the remark following Definition 1.2.1, we mentioned purely periodic continued fraction expansions. Certainly, any irrational number with a purely periodic continued fraction expansion must be a quadratic irrational by the theorem of Euler and Lagrange. But there is more: any quadratic irrational θ_0 must have a complete quotient θ_i that exhibits pure periodicity by (1.17) of Definition 1.2.1.

Definition 1.2.3 *A quadratic irrational $\theta = (P + \sqrt{D})/Q$ is said to be* **reduced** *if $\theta > 1$ and its conjugate $\overline{\theta} = (P - \sqrt{D})/Q$ lies strictly between -1 and 0.*

For example, since $(1 - \sqrt{5})/2$ lies strictly between -1 and 0, the golden ratio, $(1 + \sqrt{5})/2$ is reduced. From the above definition, we can deduce that P and Q must be positive.

In 1828, Galois showed that *a real number has a purely periodic regular continued fraction expansion if and only if it is a reduced quadratic irrational.* By Definition 1.2.1 and Galois' theorem, it is evident that a quadratic irrational θ will eventually have a reduced complete quotient. Moreover, once such a reduced complete quotient is determined, all ensuing complete quotients are reduced.

The results of Euler, Lagrange and Galois are central to the study of periodic continued fractions. Besides these three famous mathematicians, there are many more who contributed to the study of periodic continued fraction expansions. According to Dickson [36], Tenner [237] gave a convenient algorithm to calculate the continued fraction expansion of $\sqrt{D}$ for any non-square natural number D. Set $a_0 = \lfloor\sqrt{D}\rfloor$, $P_0 = 0$, $Q_{-1} = D$, $Q_0 = 1$ and $R_0 = 0$. For $i \geq 0$, compute

$$P_{i+1} = \lfloor\sqrt{D}\rfloor - R_i, \qquad Q_{i+1} = Q_{i-1} - a_i(P_{i+1} - P_i),$$

$$a_{i+1} = \left\lfloor \frac{P_{i+1} + \lfloor\sqrt{D}\rfloor}{Q_{i+1}} \right\rfloor, \qquad R_{i+1} = (P_{i+1} + \lfloor\sqrt{D}\rfloor) - a_{i+1}Q_{i+1}.$$

This algorithm is called **Tenner's algorithm.** We note that the computation for a_{i+1} and R_{i+1} can be done in one operation in a modern computer.

Now, to convert an arbitrary quadratic irrational, $\theta = (P + \sqrt{D})/Q$, into a continued fraction expansion, we set $P_0 = P$, $Q_0 = Q$, $a_0 = \lfloor\theta\rfloor$, and for $i \geq 0$, define

$$P_{i+1} = a_iQ_i - P_i, \quad Q_{i+1} = \frac{D - P_{i+1}^2}{Q_i} \quad \text{and} \quad a_{i+1} = \left\lfloor \frac{P_{i+1} + \sqrt{D}}{Q_{i+1}} \right\rfloor. \tag{1.19}$$

Then $(P_{i+1} + \sqrt{D})/Q_{i+1}$ is the $(i+1)$-th complete quotient of θ_0. Moreover, if we put $Q_{-1} = (D - P_0^2)/Q_0$, then Q_{i+1} in (1.19) can be written as

$$Q_{i+1} = \frac{D - P_{i+1}^2}{Q_i} = \frac{D - (a_iQ_i - P_i)^2}{Q_i} = \frac{D - P_i^2}{Q_i} - a_i^2Q_i + 2a_iP_i.$$

Hence,

$$Q_{i+1} = Q_{i-1} - a_i(a_iQ_i - 2P_i) = Q_{i-1} - a_i(P_{i+1} - P_i). \tag{1.20}$$

When $P_0 = 0$ and $Q_0 = 1$, we have $\theta = \sqrt{D}$ and when $P_0 = 1$ and $Q_0 = 2$, we get $\theta = (1 + \sqrt{D})/2$. These two cases play a significant role in the study of Pell equations and real quadratic fields, which are the topics of the next few sections.

Theorem 1.2.1 *If D is a non-square natural number, then $\sqrt{D} = (a_0, \overline{a_1, a_2, \ldots, a_{\ell-2}, a_{\ell-1}, a_\ell})$, where the period length is ℓ, the first $\ell - 1$ members of the period are* **palindromic**, *that is, read the same backwards as forwards, and $a_\ell = 2a_0$. In other words,*

$$\sqrt{D} = (a_0, \overline{a_1, a_2, \ldots, a_2, a_1, 2a_0}).$$

Moreover, if the period length of the continued fraction expansion of $\sqrt{D}$ *is* ℓ*, then the denominator* Q_i *of the* i*-th complete quotient is* 1 *if and only if* $i = n\ell$ *for some non-negative integer* n*. Furthermore, if* $\theta_{n\ell}$ *is the* $n\ell$*-th complete quotient of* $\sqrt{D}$*, then* $\theta_{n\ell} = a_0 + \sqrt{D}$*.*

Proof: See Perron [190, Satz 3.9, p. 79].

Theorem 1.2.2 *If* D *is a non-square natural number and* $\theta_i = (P + \sqrt{D})/Q$ *is the* i*-th complete quotient of* $\sqrt{D}$*, where the period length of* $\sqrt{D}$ *is* ℓ*, then*

$$0 < P_i < \sqrt{D}, \ \ 0 < Q_i < 2\sqrt{D} \ \ \textit{and} \ \ a_i Q_i \le 2\sqrt{D}$$

for all positive integers $i \le \ell$*.*

Proof: See Perron [190, p. 84].

Theorem 1.2.3 *Let* D *be a non-square natural number and* ℓ *the period length of the continued fraction expansion of* $\sqrt{D}$*. If* $(P_i + \sqrt{D})/Q_i$ *is the* i*-th complete quotient of* $\sqrt{D}$ *for some non-negative integer* i*, and* $Q_i \mid 2P_i$*, then* ℓ *is even and* $i \equiv \ell/2 \bmod \ell$*.*

Proof: See Perron [190, Satz 3.13, p. 85].

Theorem 1.2.4 *If* D *is a non-square natural number that is congruent to* 1 *modulo* 4*, then*

$$\frac{1+\sqrt{D}}{2} = (a_0, \overline{a_1, a_2, \ldots, a_2, a_1, 2a_0 - 1}).$$

In addition, if the period length of $(1 + \sqrt{D})/2$ *is* ℓ*, then the denominator of the* i*-th complete quotient,* Q_i*, is* 2 *if and only if* $i = n\ell$ *for some non-negative integer* n*. Moreover, if* $\theta_{n\ell}$ *is the* $n\ell$*-th complete quotient of* $(1+\sqrt{D})/2$ *for some natural number* n*, then* $\theta_{n\ell} = a_0 - 1 + (1+\sqrt{D})/2$*.*

Proof: See Perron [190, Satz 3.31, p. 105].

Recall from (1.13) that

$$\theta_0 = \frac{A_{i-1}\theta_i + A_{i-2}}{B_{i-1}\theta_i + B_{i-2}}.$$

Substituting $(P_0+\sqrt{D})/Q_0$ for θ_0 and $(P_i+\sqrt{D})/Q_i$ for θ_i, we get

$$\frac{P_0+\sqrt{D}}{Q_0}=\frac{A_{i-1}\left((P_i+\sqrt{D})/Q_i\right)+A_{i-2}}{B_{i-1}\left((P_i+\sqrt{D})/Q_i\right)+B_{i-2}}.$$

On cross-multiplying the equation and comparing rational and irrational parts, we get

$$A_{i-1}Q_0-B_{i-1}P_0=B_{i-1}P_i+B_{i-2}Q_i \tag{1.21}$$

and

$$B_{i-1}D=(A_{i-1}P_i+A_{i-2}Q_i)Q_0-(B_{i-1}P_i+B_{i-2}Q_i)P_0. \tag{1.22}$$

If we multiply (1.21) by $A_{i-1}Q_0-B_{i-1}P_0$ and (1.22) by B_{i-1} and take the difference of the two equations, then we get

$$(A_{i-1}Q_0-B_{i-1}P_0)^2-DB_{i-1}^2=(A_{i-1}B_{i-2}-A_{i-2}B_{i-1})Q_0Q_i.$$

By (1.10), we have

$$(A_{i-1}Q_0-B_{i-1}P_0)^2-DB_{i-1}^2=(-1)^iQ_0Q_i. \tag{1.23}$$

For any natural number i, if we put

$$G_{i-1}=A_{i-1}Q_0-B_{i-1}P_0, \tag{1.24}$$

then

$$G_{i-1}^2-B_{i-1}^2D=(-1)^iQ_0Q_i. \tag{1.25}$$

1.3 Pell Equations

A quadratic Diophantine equation of the form

$$x^2-y^2D=1 \tag{1.26}$$

with a non-square natural number D and integer unknowns x and y is known as the **Pell equation.** This simple equation was misattributed to John Pell by Euler in [48] for a method of solving it. This particular technique of solving the Pell equation was most likely due to Lord Brouncker as credited by Wallis. Pell equations have been studied by mathematicians all over the world, such as the Greeks, the Indians and the Europeans, for over 2000 years.

Although there is no definite evidence that the ancient Greeks had a solution to (1.26), they knew of such equations according to Fowler [53]. A strong case for such knowledge can be made by considering the *Cattle Problem of Archimedes.* There are two parts in this problem and we present them in contemporary notation: the first asks for the determination of 8 unknowns, numbers of bulls and cows in each of four herds of cattle, with seven linear relations among these numbers. The second part essentially asks for positive integer solutions of

$$x^2 - 410286423278424y^2 = 1.$$

The original wording of the cattle problem is poetic in nature. There were disputes on the mathematical formulation from the exact wording of the second part of the problem. However, most of the experts today agree that the second part stands as the above Pell equation.

In the twelfth century A.D., the Indian mathematician Bhaskara II gave a technique, which came to be known as the *cyclic method,* that solves Pell equations in general. However, neither he nor his countrymen were able to prove the method rigorously. Nevertheless, they were content that the technique always seemed to work empirically and they used it to solve (1.26) for $D = 61$, 67, 97 and 103.

According to Weil [245], Lagrange, completing earlier work of Euler, established in 1768 that a Pell equation always has a non-trivial solution, a solution with non-zero y. Moreover, there are infinitely many such solutions (x, y) given by

$$x + y\sqrt{D} = \pm(x_0 + y_0\sqrt{D})^n, \tag{1.27}$$

where n is any integer, $x_0+y_0\sqrt{D}>1$, (x_0, y_0) is the minimal positive solution, of the Pell equation (1.26), and no other solutions exist except those given in (1.27). The minimal positive solution is referred to as the **fundamental solution**. Thus, it is evident that solving (1.26) amounts to determining the values of the integers x_0 and y_0. In modern mathematical language, the key to solving (1.26) is the use of the convergents of the continued fraction expansion of $\sqrt{D}$ as we will see in the following theorem.

Theorem 1.3.1 *Suppose that D is a non-square natural number and n is an integer with $|n| < \sqrt{D}$. If x and y are co-prime positive integers and (x, y) is a solution of $x^2 - Dy^2 = n$, then x/y is a convergent in the continued fraction expansion of $\sqrt{D}$.*

Proof: See Mollin [134, Theorem 5.2.5, p. 232].

Recall from (1.25) that

$$G_{i-1}^2 - B_{i-1}^2 D = (-1)^i Q_0 Q_i.$$

Since we are looking at the continued fraction expansion of $\sqrt{D}$, we have $P_0 = 0$, $Q_0 = 1$, $G_{i-1} = A_{i-1}$ and $A_{i-1}^2 - B_{i-1}^2 D = (-1)^i Q_i$ for natural numbers i. If ℓ is the period length of $\sqrt{D}$, then ℓ is the first positive index that $Q_\ell = 1$ by Theorem 1.2.1 and

$$A_{\ell-1}^2 - B_{\ell-1}^2 D = (-1)^\ell.$$

Now, if ℓ is even, then the fundamental solution to (1.26) is $(x_0, y_0) = (A_{\ell-1}, B_{\ell-1})$. Yet, when ℓ is odd, the fundamental solution to (1.26) is $(x_0, y_0) = (A_{2\ell-1}, B_{2\ell-1})$ since $A_{\ell-1}^2 - B_{\ell-1}^2 D = -1$ when ℓ is odd.

Theoretically, we have completely solved the Pell equation by means of finding the period length and the appropriate convergent of $\sqrt{D}$. However, in practice, we are far from a useful solution still. It is a relatively easy task to find the fundamental solution if D in the Pell equation at hand is small. Yet, it is another story when D is large; it could be a tremendous computational effort to find the period length and the appropriate convergent of $\sqrt{D}$. For this reason, research on Pell equations is still very active today.

1.4 A Brief Description of Real Quadratic Fields

In this section, we review the basics of the theory of real quadratic fields. Notions such as orders, discriminants, units and conductors will be covered here. We will also establish the link between Pell equations and quadratic fields.

Definition 1.4.1 *A* **real quadratic field** *is formed by joining $\sqrt{D_0}$ to the rational field $\mathbb{Q}$ for some squarefree natural number D_0, the* **radicand**. *By convention, we denote such a field by*

$$\mathbb{Q}(\sqrt{D_0}) = \{p + q\sqrt{D_0} \mid p, q \in \mathbb{Q}\}.$$

In Definition 1.4.1, we assume that D_0 is squarefree. We justify such an assumption by the fact that if the radicand D is s^2D_0 for some integer $s > 0$, then

$$\mathbb{Q}(\sqrt{D}) = \{p+q\sqrt{D} \mid p, q \in \mathbb{Q}\} = \{p+q\sqrt{s^2D_0} \mid p, q \in \mathbb{Q}\} = \{p+sq\sqrt{D_0} \mid p, q \in \mathbb{Q}\} = \mathbb{Q}(\sqrt{D_0}).$$

The basic invariant of $\mathbb{Q}(\sqrt{D_0})$ is its **fundamental discriminant** Δ_0, which is defined by

$$\Delta_0 = \begin{cases} D_0 & \text{if } D_0 \equiv 1 \bmod 4, \\ 4D_0 & \text{otherwise.} \end{cases} \tag{1.28}$$

So Δ_0 is congruent to 0 or 1 modulo 4, and $\mathbb{Q}(\sqrt{D_0})$ may also be written as $\mathbb{Q}(\sqrt{\Delta_0})$. This implies that a quadratic field is determined by its fundamental discriminant.

Similar to the rational field $\mathbb{Q}$, which contains the ring of rational integers, $\mathbb{Z}$, the quadratic field $\mathbb{Q}(\sqrt{D_0})$ contains a ring of **algebraic integers** that behaves in much the same manner as $\mathbb{Z}$ in $\mathbb{Q}$. Define

$$\omega_0 = \frac{\sigma_0 - 1 + \sqrt{D_0}}{\sigma_0} \quad \text{and} \quad \sigma_0 = \begin{cases} 2 & \text{if } D_0 \equiv 1 \bmod 4, \\ 1 & \text{otherwise.} \end{cases} \tag{1.29}$$

Thus, $\omega_0 = \left(1+\sqrt{D_0}\right)/2$ if $D_0 \equiv 1 \bmod 4$ and $\omega_0 = \sqrt{D_0}$ otherwise.

The ring of algebraic integers of $\mathbb{Q}(\sqrt{D_0})$ consists of integer linear combinations of 1 and ω_0 and is denoted by $\mathcal{O}_0$, namely,

$$\mathcal{O}_0 = \{a + b\omega_0 \mid a, b \in \mathbb{Z}\}, \tag{1.30}$$

the $\mathbb{Z}$-module generated by $\{1, \omega_0\}$. We often use $[\alpha, \beta]$ to denote the $\mathbb{Z}$-module generated by $\{\alpha, \beta\}$, thus we can write $\mathcal{O}_0 = [1, \omega_0]$. Moreover, it is clear that $\{1, \omega_0\}$ is a $\mathbb{Q}$-basis for $\mathbb{Q}(\sqrt{D_0})$.

Definition 1.4.2 *An* **order** *of a quadratic field $\mathbb{Q}(\sqrt{D})$ is a ring, $\mathcal{O}$, in $\mathbb{Q}(\sqrt{D})$ that satisfies the following conditions.*

(1) *$\mathcal{O}$ is a subring of $\mathbb{Q}(\sqrt{D})$ containing* 1.

(2) *$\mathcal{O}$ contains a $\mathbb{Q}$-basis of $\mathbb{Q}(\sqrt{D})$.*

(3) *The additive group of $\mathcal{O}$ is finitely generated.*

Let $\triangle = f^2\triangle_0$ for some natural number f and set

$$g = \gcd(f, \sigma_0), \quad \sigma = \frac{\sigma_0}{g}, \quad D = \left(\frac{f}{g}\right)^2 D_0, \quad \triangle = \frac{4D}{\sigma^2}, \quad \text{and } \omega_\triangle = \frac{\sigma - 1 + \sqrt{D}}{\sigma}. \tag{1.31}$$

Then $\omega_\triangle = f\omega_0 + h$ for some integer h. It can be checked that the $\mathbb{Z}$-module generated by 1 and $\omega_\triangle$, denoted by

$$[1, \omega_\triangle] = \{a + b\omega_\triangle \mid a, b \in \mathbb{Z}\},$$

is an order of $\mathbb{Q}(\sqrt{D_0})$. In fact, every order of $\mathbb{Q}(\sqrt{D_0})$ can be written as $[1, \omega_\triangle] = [1, f\,\omega_0]$, where $\omega_\triangle$ is given as above. So we write $[1, \omega_\triangle] = \mathcal{O}_\triangle$. It is not difficult to see that $\mathcal{O}_\triangle$ is a subring of $\mathcal{O}_0$. In fact, the **index** of $\mathcal{O}_\triangle$ in $\mathcal{O}_0$ is finite and is equal to

$$|\mathcal{O}_0 : \mathcal{O}_\triangle| = f.$$

We call f the **conductor** of $\mathcal{O}_\triangle$. Moreover,

$$(\omega_\triangle - \overline{\omega}_\triangle)^2 = \left(\frac{2\sqrt{D}}{\sigma}\right)^2 = \frac{4D}{\sigma^2} = \triangle.$$

and call $\triangle$ the **discriminant** of $\mathcal{O}_\triangle$. Similar to the fundamental discriminant of $\mathbb{Q}(\sqrt{D_0})$, $\triangle$ is an invariant of $\mathcal{O}_\triangle$, i.e., it is independent of the $\mathbb{Z}$-basis used. If $f = 1$, then $D = D_0$. It can be shown that any order of $\mathbb{Q}(\sqrt{D_0})$ is a suborder of $\mathcal{O}_0$, thus, we refer to $\mathcal{O}_0$ as the maximal order of $\mathbb{Q}(\sqrt{D_0})$.

Example 1.4.1 Let $D_0 = 5$, then $\sigma_0 = 2$, $\triangle_0 = 5$,

$$\omega_0 = \frac{1+\sqrt{5}}{2}, \quad \text{and} \quad \mathcal{O}_0 = \left[1, \frac{1+\sqrt{5}}{2}\right].$$

Consider $\triangle = 45$, then $\triangle = 3^2 \cdot 5$ with conductor $f = 3$. Hence,

$$g = \gcd(f, \sigma_0) = 1, \quad \sigma = \frac{\sigma_0}{g} = 2, \quad D = \left(\frac{f}{g}\right)^2 D_0 = 45,$$

$$\omega_\triangle = \frac{1+\sqrt{45}}{2}, \quad \text{and} \quad \mathcal{O}_\triangle = \left[1, \frac{1+\sqrt{45}}{2}\right].$$

However, if we take $\triangle' = 180$, then $\triangle' = 6^2 \cdot 5$ with conductor $f' = 6$. Hence,

$$g' = \gcd(f', \sigma_0) = 2, \quad \sigma' = \frac{\sigma_0}{g'} = 1, \quad D' = \left(\frac{f'}{g'}\right)^2 D_0 = 45,$$

$$\omega_{\triangle'} = \sqrt{45}, \quad \text{and} \quad \mathcal{O}_{\triangle'} = \left[1, \sqrt{45}\right].$$

Observe that

$$\left[1, \sqrt{45}\right] \subset \left[1, \frac{1+\sqrt{45}}{2}\right] \subset \left[1, \frac{1+\sqrt{5}}{2}\right]$$

and

$$|\mathcal{O}_0 : \mathcal{O}_\triangle| = 3, \quad |\mathcal{O}_0 : \mathcal{O}_{\triangle'}| = 6, \quad \text{and} \quad |\mathcal{O}_\triangle : \mathcal{O}_{\triangle'}| = 2.$$

$\square$

The **norm** of an element, $\theta = x + y\,\omega_{\triangle}$, in $\mathcal{O}_{\triangle}$ is the product of θ and its algebraic conjugate, $\overline{\theta} = x + y\overline{\omega}_{\triangle}$, and is denoted by

$$\mathcal{N}(\theta) = \left(x + y\,\omega_{\triangle}\right)\left(x + y\,\overline{\omega}_{\triangle}\right) = \frac{(\sigma x + (\sigma-1)y)^2 - y^2 D}{\sigma^2}. \tag{1.32}$$

Since σ is either 1 or 2, it is easy to verify that $\mathcal{N}(\theta)$ is always a rational integer. Also, the norm operation is *completely multiplicative*, i.e., $\mathcal{N}(\theta\phi) = \mathcal{N}(\theta)\mathcal{N}(\phi)$.

An element, μ, of an integral domain is called a **unit** if it has a multiplicative inverse in the integral domain. For instance, the identity of the ring is a unit since its multiplicative inverse is itself. Since $\mathcal{O}_{\triangle}$ is an integral domain, it has units. If μ is a unit in $\mathcal{O}_{\triangle}$, then there exists ν in $\mathcal{O}_{\triangle}$ such that $\mu\nu = 1$. It follows that *μ is a unit in $\mathcal{O}_{\triangle}$ if and only if* $|\mathcal{N}(\mu)| = 1$. Hence, to find all the units in $\mathcal{O}_{\triangle}$, we look at the equation,

$$\mathcal{N}(\mu) = (x + y\,\omega_{\triangle})(x + y\overline{\omega}_{\triangle}) = \pm 1.$$

By (1.32), we have

$$\frac{(\sigma x + (\sigma-1)y)^2 - y^2 D}{\sigma^2} = \pm 1.$$

This is equivalent to

$$(\sigma x + (\sigma-1)y)^2 - y^2 D = \pm\sigma^2. \tag{1.33}$$

Since σ is either 1 or 2, we are solving a quadratic Diophantine equation. All the solutions to the above two equations are implicitly given by Theorem 1.3.1. For instance, when the right hand side of (1.33) is 1, we get the Pell equation, $x^2 - y^2 D = 1$, and we have presented its fundamental solution on page 15. Moreover, if $(\sigma x_0 + (\sigma-1)y_0 + y_0\sqrt{D})/\sigma > 1$ is the smallest positive unit of $\mathcal{O}_{\triangle}$, then all other units can be found by

$$x + y\sqrt{D} = \pm\left(\frac{\sigma x_0 + (\sigma-1)y_0 + y_0\sqrt{D}}{\sigma}\right)^n \quad \text{for some integer } n,$$

and such a unit is called the **fundamental unit** of $\mathcal{O}_{\triangle}$ and denoted by $\varepsilon_{\triangle}$.

Theorem 1.4.1 *If $D > 0$ is a non-square integer and $\mathcal{O}_\Delta = [1, (\sigma - 1 + \sqrt{D})/\sigma]$ is a real quadratic order, then the fundamental unit of $\mathcal{O}_\Delta$ is*

$$\varepsilon_\Delta = \frac{G_{\ell-1} + B_{\ell-1}\sqrt{D}}{\sigma},$$

where ℓ is the period length of the continued fraction of $(\sigma - 1 + \sqrt{D})/\sigma$, $A_{\ell-1}/B_{\ell-1}$ is the $(\ell-1)$-th convergent of $(\sigma - 1 + \sqrt{D})/\sigma$ and $G_{\ell-1} = \sigma A_{\ell-1} - (\sigma - 1)B_{\ell-1}$.

Proof: This result follows from (1.25).

For example, the fundamental unit of the maximal order $[1, (1+\sqrt{5})/2]$ is $(1+\sqrt{5})/2$ since $((1+\sqrt{5})/2) = (1-5)/4 = -1$. It can be checked that the fundamental unit of the order $[1, (1+\sqrt{45})/2]$ is $(7+\sqrt{45})/2$ and the fundamental unit of $[1, \sqrt{45}]$ is $161 + 24\sqrt{45}$.

Since $\varepsilon_\Delta \in \mathcal{O}_\Delta$, we can always write

$$\varepsilon_\Delta = \frac{a + b\sqrt{\Delta}}{2} \tag{1.34}$$

for some integers a and b. For instance, if $\Delta = D$ and $\sigma = 1$ in Theorem 1.4.1, then

$$\varepsilon_\Delta = A_{\ell-1} + B_{\ell-1}\sqrt{D} = \frac{a + b\sqrt{\Delta}}{2},$$

where $a = 2A_{\ell-1}$ and $b = 2B_{\ell-1}$. Similarly, we can establish (1.34) for other cases.

1.5 Ideals of an Order

We introduce in this section an ideal of a quadratic order. Terminology such as proper ideal, fractional ideal, principal ideal, reduced ideal, and class group and class number of a quadratic order will be defined and discussed. We will pay special attention to the relation between quadratic irrationals and ideals of an order in a real quadratic field using continued fractions. The proofs for the results in this section can be found in the literature, such as Cohn [28], Marcus [126] and Williams and Wunderlich [260].

Definition 1.5.1 *An* **ideal** *of a commutative ring $\mathcal{R}$ is a non-empty subset $\mathfrak{a}$ of $\mathcal{R}$ satisfying:*

(1) *If $\alpha, \beta \in \mathfrak{a}$, then $\alpha - \beta \in \mathfrak{a}$.*

(2) *If $r \in \mathcal{R}$ and $\alpha \in \mathfrak{a}$, then $\alpha r \in \mathfrak{a}$.*

We only consider non-zero ideals, thus, the term *ideal* will always refer to a non-zero ideal in the sequel.

Since $\mathcal{O}_\Delta$ is an integral domain, it is a commutative ring and has ideals. Also, since $\mathcal{O}_\Delta$ is a $\mathbb{Z}$-module, ideals of $\mathcal{O}_\Delta$ are $\mathbb{Z}$-submodules of $\mathcal{O}_\Delta$. However, not all $\mathbb{Z}$-submodules of $\mathcal{O}_\Delta$ are ideals. To determine when a $\mathbb{Z}$-submodule of $\mathcal{O}_\Delta$ is an ideal, we use the following theorem.

Theorem 1.5.1 *Let $\mathfrak{a} = [a, b + c\omega_\Delta]$, where a is a natural number, $b, c \in \mathbb{Z}$ with c non-zero. Then $\mathfrak{a}$ is an ideal of $\mathcal{O}_\Delta$ if and only if $c \mid a$, $c \mid b$ and $ac \mid \mathcal{N}(b + c\omega_\Delta)$.*

The set $\{a, b + c\omega_\Delta\}$ in the above theorem is referred to as a $\mathbb{Z}$-basis of $\mathfrak{a}$. Since the ideal $[a, b + c\omega_\Delta]$ is a Z-module, we can write it as $[a, an + b + c\omega_\Delta]$ for any integer n. The quantity a is the least positive rational integer in $\mathfrak{a}$ and hence it is unique in $\mathfrak{a}$. If m is any rational integer in $\mathfrak{a}$, then $a \mid m$. The quantity c is also unique in $\mathfrak{a}$. If $c = 1$, then $\mathfrak{a}$ is called a **primitive ideal**; that is, $\mathfrak{a}$ does not have any non-trivial rational integer divisors. If $c \geq 2$, $a = cm$ and $b = cn$ for some integers m and n, then we may write the ideal $[a, b + c\omega_\Delta] = c[m, n + \omega_\Delta]$, where $[m, n + \omega_\Delta]$ is a primitive ideal.

Example 1.5.1 From Example 1.4.1, we see that $\mathcal{O}_\Delta = \left[1, \sqrt{45}\right]$ is an order in $\mathbb{Q}(\sqrt{5})$. $\mathcal{O}_\Delta$ has discriminant $\Delta = 180$ and conductor $f = 6$. Consider the $\mathbb{Z}$-submodule, $[9, 6 + \sqrt{45}]$, of $\mathcal{O}_\Delta$. It is easy to see that the difference of any two elements in $[9, 6 + \sqrt{45}]$ resides in $[9, 6 + \sqrt{45}]$. For any $\alpha \in [9, 6 + \sqrt{45}]$, we may write $\alpha = 9m + n(6 + \sqrt{45})$, where $m, n \in \mathbb{Z}$. For any $\gamma \in \mathcal{O}_\Delta$, we write $\gamma = m' + n'\sqrt{45}$, where $m', n' \in \mathbb{Z}$. Then $\gamma\alpha = (m' + n'\sqrt{45})(9m + n(6 + \sqrt{45}))$. It can be checked that

$$\gamma\alpha = 9(mm' - 6mn' - nn') + (9mn' + 6nn' + m'n)(6 + \sqrt{45}) \in [9, 6 + \sqrt{45}].$$

Thus, $[9, 6 + \sqrt{45}]$ is an ideal in $\mathcal{O}_\Delta$ by Definition 1.5.1.

On the other hand, if we apply Theorem 1.5.1, we have $a = 9$, $b = 6$ and $c = 1$. Clearly, $c \mid a$ and $c \mid b$. Also, since $\mathcal{N}(6 + \sqrt{45}) = -9$, $ac \mid \mathcal{N}(6 + \sqrt{45})$. Therefore, $[9, 6 + \sqrt{45}]$ is an ideal of $\mathcal{O}_\Delta$. In fact, $[9, 6 + \sqrt{45}]$ is primitive since $c = 1$.

Consider the order $\mathcal{O}_{\Delta'} = [1, \sqrt{5}]$ in $\mathbb{Q}(\sqrt{5})$. $\mathcal{O}_{\Delta'}$ has discriminant $\Delta' = 20$ and conductor $f = 2$. We see that $[9, 6 + \sqrt{45}] = [9, 6 + 3\sqrt{5}]$ is an ideal of $\mathcal{O}_{\Delta'}$. However, $[9, 6 + 3\sqrt{5}]$ is not primitive in $\mathcal{O}_{\Delta'}$ since every element in $[9, 6 + 3\sqrt{5}]$ is a multiple of 3 in $\mathcal{O}_{\Delta'}$. Hence, when an ideal is primitive in a given order, it is not necessarily primitive in other orders in the same real quadratic field. $\square$

Let $\theta_1, \theta_2, \ldots, \theta_k \in \mathcal{O}_\Delta$ and define

$$(\theta_1, \theta_2, \ldots, \theta_k) = \{\gamma_1\theta_1 + \gamma_2\theta_2 + \cdots + \gamma_k\theta_k \mid \gamma_1, \gamma_2, \ldots, \gamma_k \in \mathcal{O}_\Delta\}.$$

It can be shown that $(\theta_1, \theta_2, \ldots, \theta_k)$ is an ideal of $\mathcal{O}_\Delta$ and we call it the ideal **generated** by $\theta_1, \theta_2, \ldots, \theta_k$. Note that the notation $(\theta_1, \theta_2, \ldots, \theta_k)$ should not be confused with that of the regular continued fraction expansion.

If we have two ideals, $\mathfrak{a} = (\theta_1, \theta_2, \ldots, \theta_k)$ and $\mathfrak{b} = (\phi_1, \phi_2, \ldots, \phi_l)$, of $\mathcal{O}_\Delta$, then the **ideal product**, $\mathfrak{ab}$, is the ideal generated by the products $\theta_i\phi_j$ for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, l$; that is,

$$\mathfrak{ab} = \left\{ \sum_{1 \le i \le k, 1 \le j \le l} \gamma_{i,j}\theta_i\phi_j \;\middle|\; \gamma_{i,j} \in \mathcal{O}_\Delta \right\}.$$

If an ideal $\mathfrak{a}$ has a single generator, then we write $\mathfrak{a} = (\theta)$ for some $\theta \in \mathcal{O}_\Delta$ and call it a **principal ideal**. For instance, if we take $\theta = 1 + \sqrt{45}$, which is in $\mathcal{O}_\Delta = [1, \sqrt{45}]$, and let

$\gamma = a + b\sqrt{45} \in \mathcal{O}_\triangle$, where $a, b \in \mathbb{Z}$, then $\gamma\theta = (a + 45b) + (a + b)\sqrt{45}$. Hence, the principal ideal generated by θ is $\{(a + 45b) + (a + b)\sqrt{45} \mid a, b \in \mathbb{Z}\}$.

It can be shown that an ideal generated by two elements, θ_1 and θ_2, in $\mathcal{O}_\triangle$ is the same as the $\mathbb{Z}$-module generated by the two elements, i.e., $(\theta_1, \theta_2) = [\theta_1, \theta_2]$. Thus, any ideal of $\mathcal{O}_\triangle$ need have at most two generators.

Theorem 1.5.2 *Suppose that we have two primitive ideals, $\mathfrak{a}_1 = [a_1, b_1 + \omega_\triangle]$ and $\mathfrak{a}_2 = [a_2, b_2 + \omega_\triangle]$, of $\mathcal{O}_\triangle$ with $\gcd(a_1, a_2) = 1$. If we put $a_3 = a_1 a_2$ and solve for the least positive integer b_3 in the congruences,*

$$b_3 \equiv b_1 \bmod a_1 \quad \textit{and} \quad b_3 \equiv b_2 \bmod a_2,$$

then the product, $\mathfrak{a}_1\mathfrak{a}_2$ is given by $\mathfrak{a}_3 = [a_3, b_3 + \omega_\triangle]$.

There is a more general result than Theorem 1.5.2 for the case where a_1 and a_2 are not relatively prime, for example see [106] and [218]. However, Theorem 1.5.2 is adequate for our work here.

When we have ideals $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ of $\mathcal{O}_\triangle$ and $\mathfrak{a}\mathfrak{b} = \mathfrak{c}$, we say $\mathfrak{a}$ **divides** $\mathfrak{c}$ and write $\mathfrak{a}|\mathfrak{c}$. This is equivalent to $\mathfrak{a} \supseteq \mathfrak{c}$. We call $\mathfrak{a} \neq \mathcal{O}_\triangle$ a **prime ideal** $\mathcal{O}_\triangle$ whenever $\mathfrak{a} \mid \mathfrak{b}\mathfrak{c}$ implies that $\mathfrak{a} \mid \mathfrak{b}$ or $\mathfrak{a} \mid \mathfrak{c}$. We may reformulate the definition of prime ideals as follows. A ideal $\mathfrak{a}$ of $\mathcal{O}_\triangle$ is a prime ideal if and only if whenever there is an ideal $\mathfrak{b}$ such that $\mathfrak{b} \mid \mathfrak{a}$, then $\mathfrak{b} = (1) = \mathcal{O}_\triangle$ or $\mathfrak{b} = \mathfrak{a}$.

Example 1.5.2 Consider the ideal $[2, 1 + \sqrt{45}]$ of $\mathcal{O}_\triangle = [1, \sqrt{45}]$. If there is an ideal $\mathfrak{a} = [a, b + \sqrt{45}]$ that divides $[2, 1 + \sqrt{45}]$, then a must divide 2. This means that $a = 1$ or 2. If $a = 1$, we have $\mathfrak{a} = \mathcal{O}_\triangle$. If $a = 2$, then $\mathfrak{a} = [2, b + \sqrt{45}]$ and b must be odd. Hence, $\mathfrak{a} = [2, 2n + 1 + \sqrt{45}] = [2, 1 + \sqrt{45}]$, by the comment following Theorem 1.5.1. Therefore, $[2, 1 + \sqrt{45}]$ is a prime ideal of $\mathcal{O}_\triangle = [1, \sqrt{45}]$. $\square$

Note that a prime ideal is always a primitive ideal but the converse is not necessarily true by Theorem 1.5.2.

A **fractional ideal** $\mathfrak{f}$ of $\mathcal{O}_\Delta$ is a generalization of an ideal of $\mathcal{O}_\Delta$. Instead of residing in $\mathcal{O}_\Delta$, $\mathfrak{f}$ resides in the number field $\mathbb{Q}(\sqrt{D_0})$, but has the property that there is an element $\alpha \in \mathbb{Q}(\sqrt{D_0})$ such that $\mathfrak{a} = \alpha\mathfrak{f}$ is an ideal in $\mathcal{O}_\Delta$. It follows that a subset of $\mathbb{Q}(\sqrt{D_0})$ is a fractional ideal of $\mathcal{O}_\Delta$ if and only if it is of the form $\gamma\mathfrak{a}$ for some non-zero element $\gamma \in \mathbb{Q}(\sqrt{D_0})$ and an ideal $\mathfrak{a}$ of $\mathcal{O}_\Delta$.

For example, consider $\gamma = 1/\sqrt{45}$ and the ideal $\mathfrak{a} = [9, 6+\sqrt{45}]$ of the order $\mathcal{O}_\Delta = \left[1, \sqrt{45}\right]$. We compute

$$\gamma\mathfrak{a} = \frac{1}{\sqrt{45}}[9, 6+\sqrt{45}] = \left[\frac{9}{\sqrt{45}}, \frac{6+\sqrt{45}}{\sqrt{45}}\right] = \left[\frac{9}{\sqrt{45}}, 1+\frac{6}{\sqrt{45}}\right] = \left[\frac{9}{\sqrt{45}}, 1-\frac{3}{\sqrt{45}}\right].$$

Since $3 = 9/\sqrt{45}+3(1-3/\sqrt{45})$, we write the fractional ideal $\gamma\mathfrak{a} = [3, 1-3/\sqrt{45}] = [3, 1-\sqrt{45}/15]$.

A fractional ideal of $\mathcal{O}_\Delta$ is not necessarily an ideal of $\mathcal{O}_\Delta$. It is clear that the above fractional ideal is not an ideal of $\mathcal{O}_\Delta$ since $1-\sqrt{45}/15 \notin \mathcal{O}_\Delta = \left[1, \sqrt{45}\right]$. However, in some cases, we may have a fractional ideal of $\mathcal{O}_\Delta$ being an ideal of $\mathcal{O}_\Delta$. If we have an ideal $\mathfrak{a}$ of $\mathcal{O}_\Delta$, then $\mathfrak{a}$ is a fractional ideal since $\mathfrak{a} = 1 \cdot \mathfrak{a}$. Further, consider $(\sqrt{5}/5)[45, 15-\sqrt{45}]$. Clearly, $\sqrt{5}/5 \in \mathbb{Q}(\sqrt{5})$ and it can be checked that $[45, 15-\sqrt{45}]$ is an ideal in $[1, \sqrt{45}]$. We find

$$\frac{\sqrt{5}}{5}[45, 15-\sqrt{45}] = [9, 6+\sqrt{45}],$$

which is an ideal in $[1, \sqrt{45}]$ even though $(\sqrt{5}/5)[45, 15-\sqrt{45}]$ appears not to be an ideal.

If $\mathfrak{a}$ is an ideal of $\mathcal{O}_\Delta$, then it is not difficult to see that

$$\mathcal{O}_\Delta \subseteq \left\{\gamma \in \mathbb{Q}\left(\sqrt{D_0}\right) \middle|\ \gamma\mathfrak{a} \subseteq \mathfrak{a}\right\};$$

and we call $\mathfrak{a}$ a **proper** ideal of $\mathcal{O}_\Delta$ whenever equality holds, i.e.

$$\mathcal{O}_\Delta = \left\{\gamma \in \mathbb{Q}\left(\sqrt{D_0}\right) \middle|\ \gamma\mathfrak{a} \subseteq \mathfrak{a}\right\}.$$

For example, principal ideals of $\mathcal{O}_\Delta$ and ideals of the maximal order $\mathcal{O}_0$ are proper.

A fractional ideal $\mathfrak{f}$ of $\mathcal{O}_\Delta$ is **invertible** if there is another fractional ideal $\mathfrak{g}$ of $\mathcal{O}_\Delta$ such that $\mathfrak{f}\mathfrak{g} = \mathcal{O}_\Delta$. In fact, in a quadratic field, $\mathfrak{f}$ *is invertible if and only if* $\mathfrak{f}$ *is proper.* Since ideals of the maximal order are proper, they are invertible.

If $\mathfrak{a}$ is an ideal of $\mathcal{O}_\Delta$, it can be shown that the quotient ring $\mathcal{O}_\Delta/\mathfrak{a}$ is finite. We define the **norm** of $\mathfrak{a}$ to be

$$\mathcal{N}(\mathfrak{a}) = |\,\mathcal{O}_\Delta/\mathfrak{a}\,|.$$

It is not difficult to see that an ideal $\mathfrak{a} = [a, b+c\omega_\Delta]$ with $\mathbb{Z}$-basis $\{a, b+c\omega_\Delta\}$ has norm $\mathcal{N}(\mathfrak{a}) = ac$. In particular, if $\mathfrak{a}$ is primitive, then $\mathcal{N}(\mathfrak{a}) = a$. For example, the norm of the ideal $[2, 1+\sqrt{45}]$ in $\mathcal{O}_\Delta = [1, \sqrt{45}]$ is 2. It is known that if $\mathcal{N}(\mathfrak{a}) = p$, where p is a rational prime, then $\mathfrak{a}$ is a prime ideal. Thus, the ideal $[2, 1+\sqrt{45}]$ is a prime ideal. However, the converse is not necessarily true. If $\mathfrak{a}$ is a prime ideal, then $\mathcal{N}(\mathfrak{a})$ is a prime or the square of a prime.

One of the important properties of the norm of an ideal is its *multiplicativity* for proper ideals, i.e., when $\mathfrak{a}$ and $\mathfrak{b}$ are proper ideals of $\mathcal{O}_\Delta$,

$$\mathcal{N}(\mathfrak{a}\mathfrak{b}) = \mathcal{N}(\mathfrak{a})\mathcal{N}(\mathfrak{b}).$$

If $\mathfrak{a} = (\theta)$ is principal, then $\mathfrak{a}$ is proper and $\mathcal{N}(\mathfrak{a}) = |\mathcal{N}(\theta)|$.

Let $\mathfrak{a} = [a, \beta]$ be an ideal of $\mathcal{O}_\Delta$. The **conjugate ideal** of $\mathfrak{a}$ is defined as $\bar{\mathfrak{a}} = [a, \overline{\beta}]$. For example, the ideal $[2, 1-\sqrt{45}]$ is the conjugate ideal of $[2, 1+\sqrt{45}]$. The ideal product $\mathfrak{a}\bar{\mathfrak{a}}$ is the principal ideal generated by $\mathcal{N}(\mathfrak{a})$, that is $\mathfrak{a}\bar{\mathfrak{a}} = (\mathcal{N}(\mathfrak{a}))$.

If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of $\mathcal{O}_\Delta$ and there exist non-zero $\alpha, \beta \in \mathcal{O}_\Delta$ such that

$$(\alpha)\mathfrak{a} = (\beta)\mathfrak{b},$$

then $\mathfrak{a}$ and $\mathfrak{b}$ are said to be **equivalent** and this relation is denoted by $\mathfrak{a} \sim \mathfrak{b}$.

Example 1.5.3 Consider the ideals $[2, 5+\sqrt{45}]$ and $[10, 5+\sqrt{45}]$ of $\mathcal{O}_\Delta = [1, \sqrt{45}]$. Note that

$$(10)[2, 5+\sqrt{45}] = [20, 50+10\sqrt{45}]$$

and

$$(5+\sqrt{45})[10,5+\sqrt{45}] = [50+10\sqrt{45}, 70+10\sqrt{45}] = [20, 50+10\sqrt{45}].$$

Hence,

$$(5+\sqrt{45})[10,5+\sqrt{45}] = (10)[2,5+\sqrt{45}]$$

and $[2,5+\sqrt{45}] \sim [10,5+\sqrt{45}]$. □

It can be shown that if $\mathfrak{a}_1 = [a_1, b_1 + c_1\omega_\triangle]$ and $\mathfrak{a}_2 = [a_2, b_2 + c_2\omega_\triangle]$ are equivalent ideals in $\mathcal{O}_\triangle$, then there exists some $\gamma \in \mathfrak{a}_1$ such that

$$(\gamma)\mathfrak{a}_2 = (a_2)\mathfrak{a}_1 \tag{1.35}$$

and $0 < \gamma < a_1$. In fact, γ can be found by the *continued fraction algorithm*, which is the main topic later in this section.

The relation $\sim$ is an equivalence relation and it therefore partitions all the proper ideals of $\mathcal{O}_\triangle$ into disjoint equivalence classes. It can be shown that the number of these classes is finite, denoted by $h_\triangle$, and we call $h_\triangle$ the **class number** of the order $\mathcal{O}_\triangle$. In case $\mathcal{O}_\triangle = \mathcal{O}_0$ is maximal, we call $h_\triangle$ the class number of the quadratic field $\mathbb{Q}(\sqrt{D_0})$ and may sometimes write it as h_0. If we write $[\mathfrak{a}]$ as the class of all ideals of $\mathcal{O}_\triangle$ equivalent to $\mathfrak{a}$, then we can define a multiplicative operation of these classes by $[\mathfrak{a}][\mathfrak{b}] = [\mathfrak{a}\mathfrak{b}]$. It is easy to see that this operation is well-defined. Under this operation, the set of all ideal classes forms a finite abelian group, $\mathcal{C}_\triangle$, with identity $[(1)] = [\mathcal{O}_0]$ and order $h_\triangle$. We note that $[(1)]$ is the class of all principal ideals. This group is referred to as the **ideal class group** of $\mathcal{O}_\triangle$. If all ideals of $\mathcal{O}_\triangle$ are principal, then the group $\mathcal{C}_\triangle$ consists of only one element, the class of all principal ideals of $\mathcal{O}_\triangle$, and $\mathcal{O}_\triangle$ has class number 1.

Example 1.5.4 Consider $\mathcal{O}_\triangle = [1, \sqrt{45}]$. We find that $\mathcal{O}_\triangle$ has class number 4, i.e, it has 4 equivalence classes. The first class contains all the principal ideals and is denoted by $[(1)]$. Put

$\mathfrak{a} = [6, 3+\sqrt{45}]$, $\mathfrak{b} = [3, \sqrt{45}]$ and $\mathfrak{c} = [2, 1+\sqrt{45}]$. The ideal class group, $C_\triangle$, is $\{[(1)], [\mathfrak{a}], [\mathfrak{b}], [\mathfrak{c}]\}$ and we expect to have the product of two any classes reside in $C_\triangle$.

If we put $a_1 = 3$, $b_1 = 0$, $a_2 = 2$, $b_2 = 1$, then by Theorem 1.5.2, we get $a_3 = 6$ and $b_3 = 3$ and the product $\mathfrak{bc} = [a_3, b_3 + \sqrt{45}] = [6, 3+\sqrt{45}]$, which is $\mathfrak{a}$. Since $C_\triangle$ is a group, we have $[\mathfrak{b}][\mathfrak{c}] = [\mathfrak{a}]$. □

A primitive ideal $\mathfrak{a}$ of $\mathcal{O}_\triangle$ is **reduced** if there does not exist any non-zero $\alpha \in \mathfrak{a}$ such that

$$|\alpha| < \mathcal{N}(\mathfrak{a}) \quad \text{and} \quad |\overline{\alpha}| < \mathcal{N}(\mathfrak{a}).$$

We may write $\mathcal{O}_\triangle = [1, \omega_\triangle]$ with normal $\mathbb{Z}$-basis $\{1, \omega_\triangle\}$ and treat it as an ideal of itself. Then its norm is 1 and certainly, there is no non-zero element $\alpha \in \mathcal{O}_\triangle$ statisfying the above reduction conditions. Thus, $\mathcal{O}_\triangle$ is a reduced ideal.

Example 1.5.5 Consider the ideal $\mathfrak{a} = [2, 1+\sqrt{45}]$ of $\mathcal{O}_\triangle = [1, \sqrt{45}]$. Let $\alpha = 2n+m+m\sqrt{45} \in \mathfrak{a}$ for integers m and n. If $|\alpha| < \mathcal{N}(\mathfrak{a}) = 2$ and $|\overline{\alpha}| < \mathcal{N}(\mathfrak{a}) = 2$, then $-2 < 2n+m+m\sqrt{45} < 2$ and $-2 < 2n+m-m\sqrt{45} < 2$. On adding the two inequalities, we get $-4 < 4n+2m < 4$. Thus, $2n+m = -1$, 0, or 1. If $2n+m = -1$, then $\alpha = -1+m\sqrt{45}$. Notice that m cannot be 0; otherwise, $2n = -1$, an absurdity. Since $|-1+m\sqrt{45}| \not< 2$ and $|-1-m\sqrt{45}| \not< 2$ for $m \neq 0$, we have $2n+m \neq -1$. Similarly, it can be shown that $2n+m \neq 0$, 1. So, there does not exist any non-zero $\alpha \in \mathfrak{a}$ such that

$$|\alpha| < \mathcal{N}(\mathfrak{a}) \quad \text{and} \quad |\overline{\alpha}| < \mathcal{N}(\mathfrak{a}).$$

Hence, $\mathfrak{a}$ is reduced. □

Using the definition of reduction, it is possible to prove

Theorem 1.5.3 *$\mathfrak{a}$ is a reduced ideal of $\mathcal{O}_\triangle$ if and only if there exists some $\beta \in \mathfrak{a}$ such that $\mathfrak{a} = [\mathcal{N}(\mathfrak{a}), \beta]$ with $\beta > \mathcal{N}(\mathfrak{a})$ and $-\mathcal{N}(\mathfrak{a}) < \overline{\beta} < 0$.*

The validity of Theorem 1.5.3 may seem suspect if we consider $\mathfrak{a} = [2, 1 + \sqrt{45}]$ with $\beta = 1 + \sqrt{45}$. When $\beta = 1 + \sqrt{45}$, we have $\overline{\beta} < -2 = -\mathcal{N}(\mathfrak{a})$, which violates the second condition in Theorem 1.5.3. However, by the remark following Theorem 1.5.1, we may rewrite the ideal $\mathfrak{a} = [2, 1 + \sqrt{45}]$ as $[2, 5 + \sqrt{45}]$. In this case, $\beta = 5 + \sqrt{45} > 2$ and $-2 < \overline{\beta} < 0$. Hence, $\mathfrak{a}$ is reduced by Theorem 1.5.3.

As a consequence of Theorem 1.5.3, if $\mathfrak{a}$ is reduced, then $\mathcal{N}(\mathfrak{a}) < \sqrt{\Delta}$. This implies that there can only be a finite number of reduced ideals in $\mathcal{O}_\Delta$.

In what follows, we examine the relation between ideals of $\mathcal{O}_\Delta$ and quadratic irrationals in $\mathcal{O}_\Delta$. Suppose that $\{a, b + \omega_\Delta\}$ is a $\mathbb{Z}$-basis of a primitive ideal $\mathfrak{a}$. If we set $\theta = (b + \omega_\Delta)/a$ and put

$$P = \frac{b\sigma_0 + f(\sigma_0 - 1) + h\sigma_0}{g} \in \mathbb{Z} \quad \text{and} \quad Q = \frac{a\sigma_0}{g} \in \mathbb{Z}, \tag{1.36}$$

then $\theta = (P + \sqrt{D})/Q$, where σ_0 was defined in (1.29) while f, g and h were defined in (1.31).

If we have $P, Q \in \mathbb{Z}$ such that $P \equiv 1 \bmod \sigma$, $\sigma \mid Q$ and $\sigma Q \mid D - P^2$, where σ was defined in (1.31), then the $\mathbb{Z}$-module $[Q/\sigma, (P + \sqrt{D}/\sigma]$ is an ideal of $\mathcal{O}_\Delta$. For, if we put

$$a = \frac{|Q|}{\sigma} \in \mathbb{Z} \quad \text{and} \quad b = \frac{P-1}{\sigma} - f - h + \frac{f+g}{\sigma_0} \in \mathbb{Z}, \tag{1.37}$$

then $\mathfrak{a} = [a, b + \omega_\Delta]$ is an ideal of $\mathcal{O}_\Delta$.

We are now in a position to bring continued fractions to the discussion. Let $\{a, b + \omega_\Delta\}$ be a $\mathbb{Z}$-basis of a primitive ideal $\mathfrak{a}$. Put $P_0 = P$, $Q_0 = Q$ and $\theta_0 = (P_0 + \sqrt{D})/Q_0$, where P and Q were defined in (1.36). By the fact that $a \mid \mathcal{N}(b + \omega_\Delta)$, it can be shown that $\sigma Q_0 \mid D - P_0^2$, where σ was defined in (1.31). Hence, θ_0 is a quadratic irrational. Moreover, we may write

$$\mathfrak{a} = [a, b + \omega_\Delta] = \left[\frac{Q_0}{\sigma}, \frac{P_0 + \sqrt{D}}{\sigma}\right]. \tag{1.38}$$

We put $\mathfrak{a}_1 = \mathfrak{a}$ and convert θ_0 into a continued fraction and produce a sequence of sets, $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \ldots$, such that

$$\mathfrak{a}_i = \frac{Q_{i-1}}{\sigma}\mathbb{Z} + \frac{P_{i-1} + \sqrt{D}}{\sigma}\mathbb{Z}, \tag{1.39}$$

where P_{i-1} and Q_{i-1} were defined in (1.19). It can be shown by induction that $P_{i-1} \equiv 1 \bmod \sigma$, $\sigma \mid Q_{i-1}$ and $\sigma Q_{i-1} \mid D - P_{i-1}^2$. Hence, if we put

$$a_i = \frac{Q_{i-1}}{\sigma} \quad \text{and} \quad b_i = \frac{P_i - 1}{\sigma} - f - h + \frac{f+g}{\sigma_0},$$

then $\mathfrak{a}_i = [a_i, b_i + \omega_\Delta]$ are ideals for all $i \geq 0$. In addition, they are primitive. It can also be checked that

$$(Q_0 \varphi_i)\mathfrak{a}_i = (Q_{i-1})\mathfrak{a},$$

where $\varphi_i = (-1)^{i-1}(A_{i-2} - \theta_0 B_{i-2})$ and A_{i-2}/B_{i-2} is the $(i-2)$-th convergent of θ_0. Hence, for all natural numbers i, $\mathfrak{a}_i \sim \mathfrak{a}$.

By the remark following Galois's theorem on page 10, since θ_0 is a quadratic irrational, eventually some k-th complete quotient of θ_0 must be a reduced quadratic irrational. In other words, the k-th complete quotient, $(P_k + \sqrt{D})/Q_k$, satisfies

$$\frac{P_k + \sqrt{D}}{Q_k} > 1 \quad \text{and} \quad -1 < \frac{P_k - \sqrt{D}}{Q_k} < 0,$$

which is equivalent to

$$\frac{P_k + \sqrt{D}}{\sigma} > \frac{Q_k}{\sigma} \quad \text{and} \quad -\frac{Q_k}{\sigma} < \frac{P_k - \sqrt{D}}{\sigma} < 0.$$

Thus, the ideal

$$\mathfrak{a}_{k+1} = \left[\frac{Q_k}{\sigma}, \frac{P_k + \sqrt{D}}{\sigma}\right]$$

is reduced by Theorem 1.5.3. Therefore, the above **continued fraction algorithm** gives us a method of computing a reduced ideal $\mathfrak{a}_{k+1}$ equivalent to $\mathfrak{a}$. Moreover, we see that $\mathfrak{a}_{k+1}$ corresponds to the quadratic irrational $(P_k + \sqrt{D})/Q_k$ in such a way that $(P_k + \sqrt{D})/Q_k$ is a reduced quadratic irrational just in case $\mathfrak{a}_{k+1}$ is a reduced ideal. So, if k is the first index such that $\mathfrak{a}_{k+1}$ is a reduced ideal, then $\mathfrak{a}_{k+i}$ is reduced because $(P_{k+i-1} + \sqrt{D})/Q_{k+i-1}$ is a reduced quadratic irrational for all natural numbers i.

If we apply the continued fraction algorithm to ω_Δ, then we expect to obtain a sequence of primitive ideals, $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \ldots$. The quadratic irrational $\omega_\Delta = ((\sigma - 1) + \sqrt{D})/\sigma$ corresponds to the ideal

$$\left[\frac{\sigma}{\sigma}, \frac{\sigma - 1 + \sqrt{D}}{\sigma}\right] = \mathcal{O}_\Delta = (1),$$

the principal ideal generated by 1, which we call $\mathfrak{a}_1$. By Theorems 1.2.1 and 1.2.4, the i-th complete quotient $(P_i + \sqrt{D})/Q_i$ of the continued fraction expansion of ω_Δ is reduced for all $i \geq 1$; thus, its corresponding ideal, $\mathfrak{a}_{i+1}$, is reduced. Moreover, since $\mathfrak{a}_1$ is principal, $\mathfrak{a}_i$ is principal for all $i \geq 1$. In fact, $\mathfrak{a}_i$ can be written as $\mathfrak{a}_i = (\varrho_i)$, where

$$\varrho_i = \frac{G_{i-2} + B_{i-2}\sqrt{D}}{\sigma} \tag{1.40}$$

and G_{i-2} was defined in (1.24). Furthermore, since the number of reduced ideals is finite, we have $\mathfrak{a}_{k+1} = \mathfrak{a}_1$ for some minimal k. If we put $\mathcal{P} = \{\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_k\}$, then *$\mathcal{P}$ is the set of all reduced principal ideals of $\mathcal{O}_\Delta$.*

Example 1.5.6 Recall from Example 1.5.4 that in the ideal class group C_Δ of $\mathcal{O}_\Delta = [1, \sqrt{45}]$ consists of four equivalence classes with representatives: (1), $\mathfrak{a} = [6, 3 + \sqrt{45}]$, $\mathfrak{b} = [3, \sqrt{45}]$ and $\mathfrak{c} = [2, 1 + \sqrt{45}]$.

In $[(1)]$, we write $\theta = \sqrt{45}$ and apply the continued fraction algorithm to get

$$(1) = [1, \sqrt{45}] \sim [9, 6 + \sqrt{45}] \sim [4, 3 + \sqrt{45}] \sim [5, 5 + \sqrt{45}] \sim [4, 5 + \sqrt{45}]$$

$$\sim [9, 3 + \sqrt{45}] \sim [1, 6 + \sqrt{45}] \sim [9, 6 + \sqrt{45}].$$

By (1.40), we compute $[9, 6 + \sqrt{45}] = (6 + \sqrt{45})$, $[4, 3 + \sqrt{45}] = (7 + \sqrt{45})$, $[5, 5 + \sqrt{45}] = (20 + 3\sqrt{45})$, $[4, 5 + \sqrt{45}] = (47 + 7\sqrt{45})$, $[9, 3 + \sqrt{45}] = (114 + 17\sqrt{45})$ and $[1, 6 + \sqrt{45}] = (161 + 24\sqrt{45})$ are all distinct reduced principal ideals.

In $[\mathfrak{a}]$, we use the representative $[6, 3+\sqrt{45}]$ and convert it to the quadratic irrational, $(3+\sqrt{45})/6$. We compute the continued fraction expansion of $(3+\sqrt{45})/6$ and find that there is one reduced ideal, namely,

$$[6, 3+\sqrt{45}] \sim [6, 3+\sqrt{45}].$$

Similarly, in $[\mathfrak{b}]$, we have

$$[3, \sqrt{45}] \sim [3, 6+\sqrt{45}] \sim [3, 6+\sqrt{45}]$$

and $[3, 6+\sqrt{45}]$ is the only reduced ideal in $[\mathfrak{b}]$.

In $[\mathfrak{c}]$, we have

$$[2, 1+\sqrt{45}] \sim [10, 5+\sqrt{45}] \sim [2, 5+\sqrt{45}] \sim [10, 5+\sqrt{45}].$$

So, there are two reduced ideals in $[\mathfrak{c}]$. $\square$

1.6 The Class Number Formula

Recall from the previous section that the class number h_Δ of $\mathcal{O}_\Delta$ is the number of elements of the class group $\mathcal{C}_\Delta$. The class number formula provides us with a tool for computing the class number of a quadratic order. The results here can be found in the literature, such as Davenport [30] and Hecke [68] .

According to Davenport [30], the class number formula, *in its simplest and most striking form*, was originally conjectured by Jacobi [78] in 1832 and later proved in complete detail by Dirichlet [39] in 1840. Since we are only interested in real quadratic fields, we will present the real quadratic field version of Dirichlet's class number formula. Also, other results concerning class numbers will be discussed within the framework of real quadratic fields.

When the study of class numbers was initiated by Gauss, the term *class number* referred to the number of classes of equivalent binary quadratic forms of a given discriminant. All the results on class numbers at that time were discussed in terms of quadratic forms. In the mid-nineteenth century when the theory of ideals was developed, it became clear that the theory of binary quadratic forms was essentially identical to the theory of class groups of quadratic fields, and the term class number was then used to stand for the number of elements of the class group.

According to Dickson [37], Gauss was interested in the relation between the number, $h_\triangle$, of properly primitive classes of quadratic forms of negative discriminant, $\triangle$, and the number of proper representations of $|\triangle|$ as the sum of three squares. One of the conjectures that Gauss [56] made regarding class numbers is the so-called *class number one problem*. In the language of real quadratic fields, it conjectures that there are infinitely many real quadratic fields with class number one. Advances concerning the class number one problem have been made, for example, in Byeon and Kim [15], Louboutin [112] and Lu [121]. In particular, Lu [121] gave a nice criterion to determine whether a real quadratic field has class number one. Lu's result can be written as follows. If $\triangle_0 > 0$ is a fundamental discriminant, then the class number h_0 is 1 if and only if

$$\sum_{i=1}^{\ell} a_i = n_1(\triangle_0) + n_2(\triangle_0) - c,$$

where ℓ is the period length of ω_0 with $\omega_0 = (a_0, \overline{a_1, \ldots, a_\ell})$,
$n_1(\triangle_0)$ is the number of solutions of $x^2 + 4yz = \triangle_0$ for non-negative $x, y, z \in \mathbb{Z}$,
$n_2(\triangle_0)$ is the number of solutions of $x^2 + 4y^2 = \triangle_0$ for non-negative $x, y \in \mathbb{Z}$,
and

$$c = \begin{cases} 0 & \text{if } \ell \text{ is even, } a_{\ell/2} \text{ odd if } \triangle_0 \equiv 1 \bmod 4, \\ 1 & \text{otherwise if } \triangle_0 \equiv 1 \bmod 4, \\ 1 & \text{if } \ell \text{ is even, } a_{\ell/2} \text{ odd if } \triangle_0 \equiv 0 \bmod 4, \\ 2 & \text{otherwise if } \triangle_0 \equiv 0 \bmod 4. \end{cases}$$

Dirichlet's original result was in the language of quadratic forms. We will present it in the context of quadratic fields and quadratic orders. Moreover, we will first present the formula for the class number h_0 of the maximal order $\mathcal{O}_0$ and then give another formula of Dirichlet to find the class number h_Δ of any order $\mathcal{O}_\Delta$ using h_0. The class number formula relates the class number h_0 to the **regulator** $R_0 = \ln(\varepsilon_0)$, to the fundamental discriminant Δ_0, and to a particular value of a function called the Dirichlet L-function.

The Dirichlet L-function is an example of a **Dirichlet series**,

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where $a_1, a_2, \ldots$ is a given sequence of complex numbers and s is a complex variable.

A **character** of a group G is a complex valued function f defined on G such that for all $m_1, m_2 \in G$, $f(m_1 m_2) = f(m_1)f(m_2)$ and $f(m) \neq 0$ for some non-identity element m of G. It is well-known that if f is a character of a finite group G with identity element e, then $f(e) = 1$ and e is the only element of G such that $f(e) = 1$ for all characters of G. Moreover, each functional value $f(m)$ is a root of unity. Indeed, if $m^n = e$, then $f(m)^n = 1$.

Example 1.6.1 Consider the multiplicative group modulo 7, $\mathbb{Z}_7^*$, and the Legendre symbol $(\frac{m}{p}) \equiv m^{(p-1)/2} \bmod p$, where $m \in \mathbb{Z}_7^*$ and p is an odd prime. Define $f : \mathbb{Z}_7^* \to \mathbb{C}$ by $f(m) = (\frac{m}{7})$. Then, if $m_1, m_2 \in \mathbb{Z}_7^*$, then $(\frac{m_1 m_2}{7}) = (\frac{m_1}{7})(\frac{m_1}{7})$, and hence, $f(m_1 m_2) = f(m_1)f(m_2)$. $\square$

Let m be a natural number and $\mathbb{Z}_m$ the group of reduced residue classes modulo m. For each character f of $\mathbb{Z}_m^*$, we define a corresponding arithmetic function $\chi = \chi_f$ by

(1) $\chi(n) = f([n])$ if $\gcd(m, n) = 1$,

(2) $\chi(n) = 0$ if $\gcd(m, n) > 1$.

This function χ is called a **Dirichlet character** modulo m. It can be checked that if χ is a Dirichlet character modulo m, then for $r, s \in \mathbb{Z}$,

$$\chi(rs) = \chi(r)\chi(s) \quad \text{and} \quad \chi(r+m) = \chi(r). \tag{1.41}$$

On the other hand, if a character χ satisfies (1.41) and $\chi(n) = 0$ for $\gcd(n,m) > 1$, then χ is a Dirichlet character modulo m.

Example 1.6.2 Consider the character on $\mathbb{Z}_7^*$ defined in in Example 1.6.1, and define $\chi : \mathbb{Z} \to \mathbb{C}$ by

$$\chi(n) = \left(\frac{n}{7}\right) \quad \text{if} \quad 7 \nmid n \quad \text{and} \quad \chi(n) = 0 \quad \text{if} \quad 7 \mid n.$$

Then χ is a Dirichlet Character modulo 7. □

The **principal Dirichlet character modulo** m, χ_m, is the Dirichlet character modulo m that satisfies

(1) $\chi_m(n) = 1$ if $\gcd(m,n) = 1$,

(2) $\chi_m(n) = 0$ if $\gcd(m,n) > 1$.

Also, we call a Dirichlet character modulo m that does not satisfy the above condition **non-principal**. An example of such a character is the Kronecker symbol $(\frac{m}{*})$, where $m \equiv 0$ or $1 \bmod 4$, defined as follows.

Suppose that m is congruent to 0 or 1 modulo 4. If n is odd, then the Kronecker symbol is the Jacobi symbol, which extends the Legendre symbol to odd composite rational integers. That is, if n be an odd integer with prime factorization $n = \prod_{i=1}^{a} p_i^{k_i}$, then

$$\left(\frac{m}{n}\right) = \left(\frac{m}{p_1}\right)^{k_1} \left(\frac{m}{p_2}\right)^{k_2} \cdots \left(\frac{m}{p_a}\right)^{k_a},$$

where $(\frac{m}{p_i})$ is the Legendre symbol defined by $m^{(p_i-1)/2} \bmod p_i$ for odd prime p_i.

If $n = 2$, then

$$\left(\frac{m}{2}\right) = \begin{cases} 0 & \text{if } 2 \mid m, \\ 1 & \text{if } m \equiv 1 \bmod 8, \\ -1 & \text{if } m \equiv 5 \bmod 8. \end{cases}$$

If $n = 2^a n_0$ for some $a \geq 1$ and odd n_0, then

$$\left(\frac{m}{n}\right) = \left(\frac{m}{2}\right)^a \left(\frac{m}{n_0}\right).$$

It follows that for all $m, n, k \in \mathbb{Z}$,

$$\left(\frac{m}{nk}\right) = \left(\frac{m}{n}\right)\left(\frac{m}{k}\right).$$

The Kronecker symbol extends the Jacobi symbol to even n but restricts m to be congruent to 0 or 1 modulo 4.

Assuming that χ is a Dirichlet character modulo m and s is a complex variable, the **Dirichlet L-function** is

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}. \tag{1.42}$$

When the real part of s is strictly greater than 1, $L(s,\chi)$ is absolutely convergent and we may write it as a product,

$$L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},$$

where the product is taken over all rational prime numbers p. This product is known as the **Euler product**. In fact, if the character χ is non-principal, then $L(s,\chi)$ converges even for $Re(s) > 0$.

Let $\triangle_0$ be the fundamental discriminant of a real quadratic field. By (1.28), $\triangle_0$ is congruent to 0 or 1 modulo 4. So, if we write $\chi_{\triangle_0}(n) = (\frac{\triangle_0}{n})$, the Kronecker symbol, then we get a special instance of (1.42),

$$L(s,\chi_{\triangle_0}) = \sum_{n=1}^{\infty} \left(\frac{\triangle_0}{n}\right) \frac{1}{n^s}. \tag{1.43}$$

If χ_1 is the principal Dirichlet character modulo 1, then $\chi_1(n) = 1$ for all natural numbers n. The L-function with character χ_1,

$$L(s,\chi_1) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

is the celebrated **Riemann zeta function** $\zeta(s)$. It converges absolutely and is analytic on the half-plane $Re(s) > 1$. It has a simple pole with residue 1 at $s = 1$, i.e.

$$\lim_{s\to 1}(s-1)\zeta(s) = 1. \tag{1.44}$$

When $Re(s) > 1$, the Riemann zeta function can be written as an infinite product,

$$\zeta(s) = \prod_{p \text{ rational prime}} (1 - p^{-s})^{-1}.$$

The zeros of $\zeta(s)$ come in two different types. The so-called trivial zeros occur at $s = -2, -4, -6, \ldots$, and the non-trivial ones occur at certain $s \in \mathbb{C}$. The **Riemann Hypothesis** asserts that if s is any non-trivial zero of $\zeta(s)$, then the real part of s must be $1/2$.

It is well-known that there is an interesting relationship between the Riemann zeta function and the Prime Number theorem which speaks of the density of primes. Dirichlet discovered that the density of ideals in a fixed ideal class of a quadratic field $K = \mathbb{Q}(\sqrt{D_0})$ with fundamental discriminant Δ_0 is the same for all classes of ideals of K. More precisely, for the case of real quadratic fields $K = \mathbb{Q}(\sqrt{D_0})$ with discriminant Δ_0, if A is any class of ideals of K and we denote the number of ideals in the class A whose norm is less than or equal to t by $Z(t, A)$, then the limit

$$\lim_{t\to\infty} \frac{Z(t,A)}{t} = \kappa_0$$

exists and is given by the formula

$$\kappa_0 = \frac{2R_0}{\sqrt{\Delta_0}},$$

where $R_0 = \ln(\varepsilon_0)$ is the regulator of $\mathcal{O}_0$. The number κ_0 is called the **Dirichlet structure constant.** Note that κ_0 is independent of A and is determined by the field only. On putting $Z(t)$ to be the number of ideals of the field whose norm is less than or equal to t, we get

$$\lim_{t\to\infty} \frac{Z(t)}{t} = \kappa_0 h_0. \tag{1.45}$$

If we let $F(n)$ be the number of ideals of the field K whose norm is n, then

$$Z(t) = \sum_{n=1}^{t} F(n).$$

Recall that a Dirichlet series is of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where a_n are complex numbers for $n \in \mathbb{N}$ and s is a complex variable. At this juncture, we restrict our attention to the case where the a_n are rational integers. Let $S(n) = a_1 + a_2 + \cdots + a_n$. If the limit

$$\lim_{n\to\infty} \frac{S(n)}{n} \tag{1.46}$$

exists and is equal to c, it can shown that

$$\lim_{s\to 1}(s-1)\sum_{n=1}^{\infty} \frac{a_n}{n^s} = c. \tag{1.47}$$

The function

$$\zeta_K(s) = \sum \frac{1}{\mathcal{N}(\mathfrak{a})^s} \tag{1.48}$$

where the sum ranges over all non-zero ideals of the maximal order of $K = \mathbb{Q}(\sqrt{D_0})$ was first introduced by Dirichlet for quadratic fields and extended to arbitrary number fields by R. Dedekind and is known as the **Dedekind zeta function**. If $Re(s) > 1$, then $\zeta_K(s)$ converges.

We may rewrite (1.48) in

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{F(n)}{n^s}. \tag{1.49}$$

In view of (1.45), (1.46), (1.47) and (1.49), it follows that

$$\lim_{s\to 1}(s-1)\zeta_K(s) = \kappa_0 h_0. \tag{1.50}$$

When $Re(s) > 1$, it is well-known that

$$\frac{\zeta_K(s)}{\zeta(s)} = L(s, \chi_{\Delta_0}) = \sum_{n=1}^{\infty} \left(\frac{\Delta_0}{n}\right) \frac{1}{n^s},$$

and at $s = 1$,

$$L(1, \chi_{\Delta_0}) = \lim_{s \to 1}(s-1)\zeta_K(s).$$

This means that

$$\kappa_0 h_0 = L(1, \chi_{\Delta_0}).$$

In other words,

$$2R_0\, h_0 = \sqrt{\Delta_0}\, L(1, \chi_{\Delta_0}), \tag{1.51}$$

and this is the **Dirichlet class number formula** for the real quadratic field with fundamental discriminant Δ_0.

In 1935, Siegel [226] showed that in a real quadratic field with discriminant Δ_0,

$$R_0 h_0 \sim \sqrt{\Delta_0} \quad \text{as } \Delta_0 \to \infty.$$

This is a remarkable asymptotic result relating h_0 to R_0 and $\sqrt{\Delta_0}$, but it has little practical value if we want to compute h_0. By (1.51), it is clear that to find h_0, we need to know the values of R_0 and $L(1, \chi_{\Delta_0})$. To find R_0, it suffices to find ε_0. Since ε_0 can be found by using the continued fraction expansion of ω_0, we can determine R_0. In fact, for certain values of Δ_0, finding ε_0 is easy; for example, if D_0 is of R-D type which will be introduced in Section 2.1. However, when Δ_0 is large, finding $L(1, \chi_{\Delta_0})$ could be difficult and remains a deep and open problem.

There are known lower and upper bounds for $L(1, \chi_{\Delta_0})$. The best known lower bound on $L(1, \chi_{\Delta_0})$ at present was given by Hoffstein [73], which is an improvement of Tatuzawa's lower bound in [238]. Let $\Delta_0 > 0$ be a fundamental discriminant and $0 < \eta < 1/(6 \ln 10)$. If $\Delta_0 > e^{1/\eta}$, then with at most one exceptional value of Δ_0,

$$L(1, \chi_{\Delta_0}) > \min\left\{\frac{1}{7.735 \ln \Delta_0}, \frac{\eta}{0.349\Delta_0^{\eta}}\right\}$$

and

$$L(1, \chi_{\Delta_0}) > \min\left\{\frac{1}{7.735 \ln \Delta_0}, \frac{\eta}{0.596(1+\eta \ln \Delta_0)^2 \Delta_0^{0.138\eta}}\right\}.$$

The current best upper bound was given by S. Louboutin in [117], namely

$$|L(1,\chi_{\triangle_0})| \leq \frac{1}{2}(\ln \triangle_0 + k_0),$$

where $k_0 = 2 + \gamma - \ln(4\pi) = 0.046\ldots$ and $\gamma = 0.577\ldots$ is Euler's constant.

The above lower and upper bounds can be greatly improved if we assume the truth of the extended Riemann Hypothesis. The **extended Riemann Hypothesis (ERH)** asserts that $L(s,\chi_{\triangle_0}) \neq 0$ for any value of s such that the real part of s is strictly greater than $1/2$.

Assuming the truth of ERH, Littlewood [111] proved that

$$\frac{\pi^2(1+o(1))}{12e^{\gamma}\ln\ln\triangle_0} < L(1,\chi_{\triangle_0}) \leq (1+o(1))2e^{\gamma}\ln\ln\triangle_0,$$

where γ is the Euler constant and the error term $o(1)$ tends to zero as $\triangle_0$ approaches infinity. Although the error term $o(1)$ makes Littlewood's bounds impractical, explict practical bounds can be achieved by applying Bach's averaging method in [6], which also assumes the ERH.

We conclude this section by presenting Dirichlet's result [40] on the relation between the class number, $h_\triangle$, of an order $\mathcal{O}_\triangle$ and the class number h_0 of the maximal order $\mathcal{O}_0$ (or of $\mathbb{Q}(\sqrt{D_0})$).

Note that since $\mathcal{O}_\triangle$ is a subring of $\mathcal{O}_0$, any unit in $\mathcal{O}_\triangle$ must be a unit in $\mathcal{O}_0$. In particular, the fundamental unit, $\varepsilon_\triangle$, of $\mathcal{O}_\triangle$ must be a unit in $\mathcal{O}_0$. Therefore, $\varepsilon_\triangle = \varepsilon_0^m$ for some natural number m. The minimal such m is called the **unit index** of $\mathcal{O}_\triangle$ and denoted by $u_\triangle$. There will be a discussion in the next section on the determination of $u_\triangle$.

If $f > 1$ is the conductor of an order $\mathcal{O}_\triangle$ with unit index $u_\triangle$ and fundamental discriminant $\triangle_0$, then

$$h_\triangle = \frac{h_0\,\psi(f)}{u_\triangle}, \tag{1.52}$$

where $\psi(f) = f\prod\left(1-\left(\frac{\triangle_0}{p}\right)\right)$ and the product ranges over all the distinct primes p dividing f and $\left(\frac{*}{*}\right)$ denotes the Kronecker symbol.

1.7 Lucas Functions

Consider a monic quadratic polynomial equation

$$x^2 - Px + Q = 0 \tag{1.53}$$

with integers P and Q and roots α and β. The **Lucas functions** are given by

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n. \tag{1.54}$$

Lucas functions play a significant role in the theory of continued fractions. In this section, we will present a number of identities and results concerning the Lucas functions and use the results to determine the unit index introduced in the previous section. Also, some of the results here are crucial to our work in Chapter 5.

Material presented here is available in the literature. For a thorough discussion on the Lucas functions, see Lehmer's articles [100], [101] and [102] as well as Lehmer's selected works [129, V. I, pp. 1-49] and the books of Ribenboim [208] and Williams [246].

Although the functions U_n and V_n are called the Lucas functions, Lucas was not the first person to study them. They were attributed to him because of the great variety of results he discovered pertaining to them. Another major contributor to the study of the Lucas functions is Lehmer. Some of Lucas's results were not proved rigorously; it was Lehmer who refined the details and put the study on a solid mathematical foundation. Also, in his Ph.D dissertation [101], Lehmer generalized (1.53) to the case in which P is replaced by $\sqrt{R}$ where R is any integer prime to Q. Because of Lehmer's work in the area, the study of the functions U_n and V_n is referred to as the Lucas-Lehmer theory.

Since α and β are roots of (1.53), their sum $\alpha + \beta = P$ and their product $\alpha\beta = Q$. If $\alpha = \beta$, then $P^2 - 4Q = 0$ and the Lucas function U_n is said to be **degenerate**. In this case, U_n is defined by

$$U_n = \lim_{\beta \to \alpha} \frac{\alpha^n - \beta^n}{\alpha - \beta} = n\alpha^{n-1}.$$

Henceforth, we assume that $\alpha \neq \beta$. Note that

$$\alpha^{n+1} - \beta^{n+1} = (\alpha + \beta)(\alpha^n - \beta^n) - \alpha\beta(\alpha^{n-1} - \beta^{n-1})$$

and

$$\alpha^{n+1} + \beta^{n+1} = (\alpha + \beta)(\alpha^n + \beta^n) - \alpha\beta(\alpha^{n-1} + \beta^{n-1}).$$

Thus,

$$U_{n+1} = PU_n - QU_{n-1} \quad \text{and} \quad V_{n+1} = PV_n - QV_{n-1}.$$

Since $U_0 = 0$, $U_1 = 1$, $V_0 = 2$ and $V_1 = P$, U_n and V_n are integers for $n \geq 0$. Also,

$$Q^n U_{-n} = (\alpha\beta)^n \left(\frac{\alpha^{-n} - \beta^{-n}}{\alpha - \beta} \right) = \frac{\beta^n - \alpha^n}{\alpha - \beta} = -U_n$$

and

$$Q^n V_{-n} = (\alpha\beta)^n (\alpha^{-n} + \beta^{-n}) = \beta^n + \alpha^n = V_n.$$

If we set $\delta = \alpha - \beta$, then

$$2\alpha^n = V_n + \delta U_n \quad \text{and} \quad 2\beta^n = V_n - \delta U_n. \tag{1.55}$$

This implies that

$$\frac{V_{mn} \pm \delta U_{mn}}{2} = \left(\frac{V_n \pm \delta U_n}{2} \right)^m. \tag{1.56}$$

It follows that

$$2^{m-1} U_{mn} = \sum_{i=0}^{\lfloor m/2 \rfloor} \binom{m}{2i+1} \Delta^i U_n^{2i+1} V_n^{m-2i-1} \tag{1.57}$$

and

$$2^{m-1} V_{mn} = \sum_{i=0}^{\lfloor m/2 \rfloor} \binom{m}{2i} \Delta^i U_n^{2i} V_n^{m-2i}, \tag{1.58}$$

where $\triangle = \delta^2 = P^2 - 4Q \neq 0$.

Note that p divides the binomial coefficient $\binom{p}{j}$ if j is positive and strictly less than p. Hence, by (1.57) with $n = 1$ and $m = p$, we get

$$2^{p-1}U_p \equiv \triangle^{(p-1)/2} \bmod p.$$

If p is an odd prime, then by Fermat's Little Theorem, we have

$$U_p \equiv \triangle^{(p-1)/2} \bmod p. \tag{1.59}$$

Thus, if $p \mid \triangle$, then $p \mid U_p$.

If a prime p divides Q, it can be shown by induction that $U_n \equiv P^{n-1} \bmod p$ for all $n \geq 0$. This implies that if $p \nmid P$, then $p \nmid U_n$ except for U_0. If $p \mid P$, then $p \mid U_n$ for $n = 0$ and $n \geq 2$.

Henceforth, we assume that $p \nmid Q$. Define $\epsilon(p) = \left(\frac{\triangle}{p}\right)$ where $\left(\frac{*}{*}\right)$ is the Kronecker symbol. We will write $\epsilon = \epsilon(p)$ when the context is clear. Also, when p is an odd prime, the notation $\epsilon(p)$ simply stands for the Legendre symbol. It can be shown by induction that if $2 \mid P$, then $2 \mid U_n$ if and only if $2 \mid n$; and if $2 \nmid P$, then $2 \mid U_n$ if and only if $3 \mid n$. Hence,

$$\epsilon(2) = \begin{cases} 0 & \text{if } 2 \mid P, \\ -1 & \text{if } 2 \nmid P, \end{cases} \tag{1.60}$$

implies that

$$2 \mid U_{2-\epsilon(2)}. \tag{1.61}$$

Also, it can be shown that for any odd prime p,

$$p \mid U_{p-\epsilon(p)}. \tag{1.62}$$

Theorem 1.7.1 *If p is an odd prime, $\epsilon = \epsilon(p)$ and $p \nmid Q\triangle$, then $p \mid V_{(p-\epsilon)/2}$ when the Legendre symbol $\left(\frac{Q}{p}\right) = -1$ and $p \mid U_{(p-\epsilon)/2}$ when $\left(\frac{Q}{p}\right) = 1$.*

Theorem 1.7.2 *If $p \nmid Q$ and $p^k \parallel U_n$, then $p^{k+1} \parallel U_{pn}$ when $p^k \neq 2$. If $p^k = 2$, i.e. $p = 2$, then $4 \mid U_{2n}$.*

By (1.62), if p is a prime and does not divide Q, then there exists a natural number k such that $p \mid U_k$. By Theorem 1.7.2, we see that for any integer a, there exists some $n > 0$ such that $p^a \mid U_n$. If $r \mid U_n$ and $r' \mid U_{n'}$ where $\gcd(r, r') = 1$, then $U_n \mid U_{nn'}$ and $U_{n'} \mid U_{nn'}$ implies that $rr' \mid U_{nn'}$. Thus, if m is any integer such that $\gcd(m, Q) = 1$, then there is a natural number n such that $m \mid U_n$. The least such positive n, denoted by $\omega(m)$, is called the **rank of apparition of** m in U_n.

Theorem 1.7.3 *Let* $\gcd(m, Q) = 1$. *If $m \mid U_n$ for some natural number n, then $\omega(m) \mid n$.*

Theorem 1.7.4 (Law of Repetition for a prime p)

If p is a prime and a is a natural number such that $p^a \neq 2$ and $p^a \parallel U_m$ for some m, then $p^{a+b} \parallel U_{p^b mn}$ when $p \nmid n$ and $b > 0$. If $p^a = 2$, then $p^{a+b} = 2^{b+1}$ and $2^{b+1} \mid U_{2^b mn}$ and $U_{mn/2}$ is odd when n is odd.

Definition 1.7.1 *Suppose that* $\gcd(m, Q) = 1$ *and the prime factorization of m is $\prod_{i=1}^k p_i^{a_i}$ where p_i are distinct primes. We define the functions $\Phi(m)$ and $\Lambda(m)$ as follows:*

$$\Phi(m) = \prod_{i=1}^{k} p_i^{a_i - 1}(p_i - \epsilon(p_i))$$

and

$$\Lambda(m) = \text{ the least common multiple of } p_i^{a_i-1}(p_i - \epsilon(p_i)) \text{ for } i = 1, 2, \ldots, k.$$

We note that $\Lambda(m) \mid \Phi(m)$. The function $\Lambda(m)$ is a generalization of the Carmichael λ-function [21] and the function $\Phi(m)$ is a generalization of the Euler ϕ-function to U_n. To see the latter, let $m = \prod_{i=1}^k p_i^{a_i}$ be an odd natural number and $b \in \mathbb{N}$ such that $\gcd(m, b(b-1)) = 1$. Take $P = b+1$

and $Q = b$. Then $\gcd(P, Q) = 1$ and $U_n = (b^n - 1)/(b - 1)$. Since $\Delta = (\alpha - \beta)^2 = (b - 1)^2$ is a square, we have $\epsilon(p) = (\frac{\Delta}{p}) = 1$ for all prime divisor p of m. Thus,

$$\Phi(m) = \prod_{i=1}^{k} p_i^{a_i - 1}(p_i - 1) = \phi(m).$$

Theorem 1.7.5 (Law of Apparition of m)

If $\gcd(m, Q) = 1$, *then* $\omega(m)$ *exists and* $\omega(m) \mid \Lambda(m)$.

We now confine our attention to the relation between the fundamental unit of a quadratic order and the fundamental unit of the maximal order. By Theorem 1.4.1, the fundamental unit of the maximal order $\mathcal{O}_0$ can always be found by using the continued fraction of ω_0. We may write the fundamental unit as

$$\varepsilon_0 = \frac{a + b\sqrt{\Delta_0}}{2}$$

for some integers a and b.

Recall that all units in $\mathcal{O}_0$ are of the form $\pm\varepsilon_0^n$ or $\pm\bar{\varepsilon}_0^n$ for non-negative integers n. We may consider only the ones of the form ε_0^n and write them as $(a_n + b_n\sqrt{\Delta_0})/2$ with $a_1 = a$ and $b_1 = b$. Suppose that $\mathcal{O}_\Delta$ is an order of $\mathbb{Q}(\sqrt{\Delta_0})$ with conductor f. Then the fundamental unit ε_Δ of $\mathcal{O}_\Delta$ is a unit in $\mathcal{O}_0$. Thus, $\varepsilon_\Delta = (a_n + b_n\sqrt{\Delta_0})/2$ for some n. Since ε_Δ can be written as

$$\frac{a' + b'\sqrt{f^2\Delta_0}}{2} = \frac{a' + b'f\sqrt{\Delta_0}}{2}$$

for some integers a' and b', it is clear that

$$\frac{a' + b'f\sqrt{\Delta_0}}{2} = \frac{a_n + b_n\sqrt{\Delta_0}}{2}$$

and hence,

$$f \mid b_n.$$

The minimal such n is the unit index u_Δ of $\mathcal{O}_\Delta$.

To facilitate the calculations of the sequences $\{a_i\}$ and $\{b_i\}$, we consider the monic polynomial equation $x^2 - Px + Q = 0$ with roots $\alpha = (a_1 + b_1\sqrt{\triangle_0})/2 = \varepsilon_0$ and $\beta = (a_1 - b_1\sqrt{\triangle_0})/2 = \overline{\varepsilon}_0$. Then $P = \alpha + \beta = a_1$ and $Q = \alpha\beta = \mathcal{N}(\alpha)$, which is either 1 or -1. If we put

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n,$$

then $U_1 = 1$, $V_1 = P = a_1$, and since $\delta = b_1\sqrt{\triangle_0}$,

$$\frac{V_1 + \delta U_1}{2} = \frac{a_1 + b_1\sqrt{\triangle_0}}{2}.$$

By (1.56), we see that

$$\frac{a_n + b_n\sqrt{\triangle_0}}{2} = \left(\frac{a_1 + b_1\sqrt{\triangle_0}}{2}\right)^n = \left(\frac{V_1 + \delta U_1}{2}\right)^n = \frac{V_n + \delta U_n}{2} = \frac{V_n + U_n b_1\sqrt{\triangle_0}}{2}.$$

Therefore, $a_n = V_n$ and $b_n = b_1 U_n$.

We want to find the least positive n such that $f \mid b_1 U_n$. It is easy to see that if $f \mid b_1$, then $n = 1$ and $u_\triangle = 1$. For the case where $f \nmid b_1$, let $\gcd(f, b_1) = d$ for some natural number d. Then $f \mid b_1 U_n$ is equivalent to $(f/d) \mid (b_1/d)\, U_n$. Since $\gcd(f/d, b_1/d) = 1$, we have $(f/d) \mid (b_1/d)\, U_n$ if and only if $(f/d) \mid U_n$. This means that the least positive such n is the rank of apparition of f/d, i.e., $\omega(f/d)$. The existence of $\omega(f/d)$ is guaranteed by the law of apparition since Q in this case is ± 1 and $\gcd(f/d, Q) = 1$. Therefore, $u_\triangle = \omega(f/d)$. Since $\omega(f/d) \mid \Lambda(f/d)$ by the law of apparition, we have $u_\triangle \leq \Lambda(f/d)$.

Chapter 2

Known Results on the Continued Fraction Expansion of $\sqrt{D(X)}$

The purpose of this chapter is to motivate the new results in Chapters 3 and 4. As we indicated at the end of Section 1.3, finding solutions to a Pell equation could be a difficult computational problem when D is large. We will solve this problem in cases where D is given by an integer-valued quadratic polynomial and express solutions of the corresponding Pell equation in terms of the coefficients of the polynomial.

The first section is a review of relevant results pertaining to the continued fraction expansion of $\sqrt{D(X)}$ produced from the time of Stern (1834) to the present. The second section discusses Kraitchik's work in the area. The third section focuses on Schinzel's work on the period length of the continued fraction expansion of $\sqrt{D(X)}$. The last section presents the joint work of van der Poorten and Williams on the subject.

2.1 A Short History of Certain Parametric Families of D

Let $D(X) = AX^2 + BX + C$, where A, B, C and X are integers. We want to express solutions (x, y) of the Pell equation

$$x^2 - D(X)y^2 = 1 \tag{2.1}$$

in terms A, B, C and X. This problem has been studied since the time of Stern [236] in 1834. He looked at 42 different quadratic polynomials. Numerous subsequent results were achieved for specific quadratics by other researchers. However, there is no definite indication of a systematic approach to solving the problem in general since Stern's time. To solve (2.1) for general $D(X)$, we need to know the continued fraction expansion of $\sqrt{D(X)}$ for all X. It is possible that $\sqrt{D(X)}$ could have unbounded period length as X varies. So, to solve the general problem, we first need to have a criterion to determine if the continued fraction expansion of $\sqrt{D(X)}$ has bounded period length. Next, we proceed to find the continued fraction expansion of $\sqrt{D(X)}$ where $D(X)$ obeys the criterion and obtain solutions of (2.1) therein.

The 42 different quadratic polynomials that Stern examined can be found in [236, p. 332], for instance, $D(X) = (mX)^2+m$, $(mX)^2+2m$ and $(6X\pm1)^2+(8X\pm2)^2$. He computed the continued fraction expansions of these $\sqrt{D(X)}$ in terms of m and X. Then he used the continued fraction expansions to calculate the solutions of the corresponding Pell equations.

Richaud [209] studied quadratics of the form $D(X) = X^2 \pm r$, where $r \mid 2X$, and gave fundamental solutions to the corresponding Pell equations. He also supplied examples such as $D(X) = (9X+3)^2 \pm 9$, $(9X+6)^2 \pm 9$ and $(25X+5)^2 - 25$. We note here that the case $X^2 \pm r$ with $r \mid 2X$ is similar to the case $D(X) = (mX)^2 + m$ of Stern. One may argue that Richaud's result is the first indication of a systematic approach to solving (2.1). However, this approach never appeared in the literature again until a generalization of it appeared in Degert's 1958 article [32].

In Dickson's **History of the Theory of Numbers** [36, Chapter XII], there is a list of contributions pertaining to solving (2.1). Most of the articles cited in [36] share a common underlying theme. They investigate a quadratic polynomial with fixed coefficients and find solutions of the corresponding Pell equation in terms of the coefficients and the variable X. For example, Speckmann showed that the fundamental solution of $x^2 - D(X)y^2 = 1$ is $x = X+2$, $y = 1$ if $D(X) = X^2+4X+3$, and $x = 2X+3$, $y = 2$ if $D(X) = X^2+3X+2$. Didon and Moreau considered $D(X) = (4X+2)^2+1$ for natural numbers X and proved that $x^2 - D(X)y^2 = 4$ has no solution in odd integers and the fundamental solution is $x = 16(2X+1)^2+2$ and $y = 8(2X+1)$. Ricalde considered D of the form $n(k^2n \pm 2)$ and gave the fundamental solution $x = k^2n \pm 1$ and $y = k$ to the corresponding Pell equation.

In 1926, Kraitchik [92, V. II, pp. 30-71] considered non-square natural numbers D that are less than 1000 and classified them according to the period length of $\sqrt{D}$. He gave parametrizations of D for the cases where the continued fraction expansions of $\sqrt{D}$ have period length less than 7. For instance, he gave the continued fraction expansions of $\sqrt{(9X+6)^2+(10X+7)}$ and $\sqrt{(9X^2+2)^2+(8X+2)}$. There will be a more detailed account of his work in the next section.

In 1957, Degert investigated squarefree $D_0 = X^2 + r$ with the condition that

$$-X < r \leq X \quad \text{and} \quad r \mid 4X,$$

and found the fundamental units of the real quadratic fields $\mathbb{Q}(\sqrt{D_0})$. Since Richaud's result was not mentioned in Degert's paper, it is likely that Degert was not aware of Richaud's work. Hasse [67] pointed out the similarities of Richaud's and Degert's results and coined the term **Richaud-Degert type** (R-D type) for the families of D_0 that Degert studied. Hasse was interested in the class numbers of R-D type quadratic fields $\mathbb{Q}(\sqrt{D_0})$ and was able to get a lower bound for these class numbers. Numerous results concerning the R-D types can be found in the literature, such as [15], [83], [140] and [177]. Although Degert did not use continued fractions in his work, it is well-known today that for any non-square natural number D that satisfies Degert's condition, the continued fraction expansion of $\sqrt{D}$ has a period length at most 12.

Theorem 2.1.1 ([133, Theorem 3.2.1, p. 78]) *Let $D = X^2 + r$ be a non-square natural number such that $r \mid 4X$ and $-2X+1 < r \leq 2X$. Then $\sqrt{D}$ has period length at most 12. More specifically:*

(1) *If $X = \lfloor \sqrt{D} \rfloor$ and $r \mid 2X$, then $\sqrt{D} = (X, \overline{2X/r,\, 2X})$, unless $r = 1$ when $\sqrt{D} = (X, \overline{2X})$.*

(2) *If* $X = \lfloor\sqrt{D}\rfloor$, *X is odd*, $r \nmid 2X$, *then*

$$\sqrt{D} = \left(X, \overline{\frac{4X-r}{2r}, 1, 1, \frac{X-1}{2}, \frac{8X}{r}, \frac{X-1}{2}, 1, 1, \frac{4X-r}{2r}, 2X}\right),$$

unless $r = 4$ *when* $\sqrt{D} = (X, \overline{(X-1)/2, 1, 1, (X-1)/2, 2X})$.

(3) *If* $X = \lfloor\sqrt{D}\rfloor$, *X is even*, $r \nmid 2X$, *then*

$$\sqrt{D} = \left(X, \overline{\frac{4X-r}{2r}, 1, 1, \frac{X}{2} - 1, 1, 1, \frac{4X-r}{2r}, 2X}\right).$$

(4) *If* $X = \lfloor\sqrt{D}\rfloor + 1$, $r \mid 2X$, *then* $\sqrt{D} = \left(X-1, \overline{1, (-2X/r) - 2, 1, 2X-2}\right)$, *unless* $r = -1$ *when* $\sqrt{D} = (X-1, \overline{1, 2X-2})$ *or unless* $r = -X$ *when* $\sqrt{D} = (X-1, \overline{2, 2X-2})$.

(5) *If* $X = \lfloor\sqrt{D}\rfloor + 1$, $r \nmid 2X$, *X is even and* $X \geq r$, *then*

$$\sqrt{D} = \left(X-1, \overline{1, -\left(\frac{4X+3r}{2r}\right), 2, \frac{X}{2} - 1, 2, -\left(\frac{4X+3r}{2r}\right), 1, 2X-2}\right).$$

(6) *If* $X = \lfloor\sqrt{D}\rfloor + 1$, $r \nmid 2X$, *X is odd, then*

$$\sqrt{D} = \left(X-1, \overline{1, -\left(\frac{4X+3r}{2r}\right), 2, \frac{X-3}{2}, 1, -\frac{8X}{r} - 2, 1, \frac{X-3}{2}, 2, -\left(\frac{4X+3r}{2r}\right), 1, 2X-2}\right),$$

unless $r = -4$ *when* $\sqrt{D} = (X-1, \overline{1, (X-3)/2, 2, (X-3)/2, 1, 2X-2})$.

(7) *If* $X = \lfloor\sqrt{D}\rfloor + 1$, $r = -X$, *then* $\sqrt{D} = (X-1, \overline{2, 2X-2})$.

(8) *If* $X = \lfloor\sqrt{D}\rfloor + 1$, *X is a multiple of* 6 *and* $r = -4X/3$, *then* $\sqrt{D} = (X-1, \overline{3, X/2 - 1, 3, 2X-2})$.

(9) *If* $X = \lfloor\sqrt{D}\rfloor + 1$, *X is an odd multiple of* 3 *and* $r = -4X/3$, *then*

$$\sqrt{D} = \left(X-1, \overline{3, \left(\frac{X-3}{2}\right), 1, 4, 1, \left(\frac{X-3}{2}\right), 3, 2X-2}\right).$$

A systematic approach to the study of (2.1) was realized in 1961 when A. Schinzel [213], [214] studied whether the continued fraction expansion of $\sqrt{D(X)}$ has bounded period length as X varies. He was inspired by a theorem of H. Schmidt [217, Satz 10] which can be written as follows. If $D(X) = X^2 + r$ with $X \in \mathbb{Z}$ and $r \neq 0, \pm 1, \pm 2, \pm 4$, then the period length of the continued fraction expansion of $\sqrt{D(X)}$ is unbounded as X varies. Schinzel's results in [213] and [214] generalize Schmidt's theorem to arbitrary integer-valued polynomials $f(X)$ of degree n and provide simple criteria to determine whether the continued fraction expansion of $\sqrt{f(X)}$ has bounded period length as X varies.

There are three remarkable results in [213] and [214]. The first result in [213] states that if n is odd or n is even with a non-square leading coefficient of $f(X)$, then the period length of the continued fraction expansion of $\sqrt{f(X)}$ is unbounded as X gets large.

The second result in [213] deals with the quadratic case, $D(X) = A^2X^2 + BX + C$. We will see later in (2.5) and (2.6) that by considering even X and odd X separately, we may without loss of generality restrict ourselves to

$$D(X) = A^2X^2 + 2BX + C.$$

Schinzel's second result asserts that if $D(X) = A^2X^2 + 2BX + C$, with A, B, C integers, $A > 0$ and $\Delta = B^2 - A^2C \neq 0$, then the period length of the continued fraction expansion of $\sqrt{D(X)}$ is bounded if and only if $\Delta \mid 4\gcd(A^2, B)^2$. Recall from the beginning of Chapter 1 that the condition

$$\Delta \mid 4\gcd(A^2, B)^2$$

is called the **Schinzel condition**. This condition plays a significant role in our work later in Chapter 4 and will be discussed in detail in Section 2.3.

The third result appeared a year later in [214]. This result completely solved the initial problem of deciding whether $\sqrt{f(X)}$ has bounded period length for a general integer-valued polynomial $f(X)$ of degree n. Since we will not require the results in [214] for our work, we will not discuss the details here. Several analytic results concerning the period length of the continued fraction expansion of $\sqrt{D(X)}$ were established following Schinzel's work. Louboutin [113] studied quadratic polynomials, $D(X) = A^2X^2+2BX+C$, that violate the Schinzel condition, i.e. $\Delta \nmid 4\gcd(A^2,B)^2$, and gave the lower bound,

$$1+\frac{2\ln\sqrt{D(X)}}{\ln|\delta|} \quad \text{where} \quad \delta = \frac{\Delta}{\gcd(A,B)^2},$$

on the period length of the continued fraction expansion of $\sqrt{D(X)}$. Louboutin's result was later improved by Farhane [52].

In [206], A. J. van der Poorten and H. C. Williams used Schinzel's result on quadratic polynomials to investigate the exact expansion of $\sqrt{D(X)}$ where $D(X) = A^2X^2 + 2BX + C$ obeys the Schinzel condition. They considered the case where $\gcd(A^2, 2B, C)$ is squarefree. Among other results, they demonstrated that the period length of the continued fraction expansion of $\sqrt{D(X)}$ is not only bounded according to Schinzel, but is in fact constant for a fixed triple (A, B, C) with sufficiently large X. A comprehensive discussion of the results in [206] will be given in Section 2.3.

R. A. Mollin [148] studied the continued fraction expansion of certain $\sqrt{D(X)}$ where $D(X) = A^2X^2+2BX+C$. He let (x_0,y_0) be a solution to the Pell equation $x^2-Cy^2 = 1$ with non-square integer C. Then he set $A = (x_0-1)y_0$, $B = (x_0-1)^2$ and found the continued fraction expansion expansion of $\sqrt{D(X)}$ in terms of A, B and X.

In Chapter 4, we will generalize the work in [206] by dropping the condition that $\gcd(A^2, 2B, C)$ is squarefree and give the continued fraction expansion of $\sqrt{D(X)}$ for all quadratics $D(X)$ that satisfy the Schinzel condition. In this fashion, we completely solve

$$x^2 - D(X)y^2 = 1$$

for a general quadratic $D(X)$ that fulfills the Schinzel condition. Moreover, we will show the relationship between the continued fraction expansion of $\sqrt{D(X)}$ and the Lucas function, U_n, introduced in Section 1.7.

2.2 Kraitchik's Work

In his 1926 book [92], **Théorie des Nombres**, Kraitchik introduced a different perspective to the study of parametric families of D. Instead of looking for particular quadratics $D(X)$ where $\sqrt{D(X)}$ has a predictable continued fraction expansion, Kraitchik studied collections of non-square natural numbers D in which the continued fraction expansions of $\sqrt{D}$ have the same period length. For the case where the period length is less than or equal to 7, he gave explicit parametrizations of D using the partial quotients of the continued fraction expansion of $\sqrt{D}$. Also, he numerically classified all non-square natural numbers D that are less than 1000 into collections of numbers according to the period length of the continued fraction expansion of $\sqrt{D}$. We will now describe some of Kraitchik's work.

For any non-square natural number D, it is convenient to write $D = a^2+b$ where $1 \le b \le 2a$. If the continued fraction expansion of $\sqrt{D}$ has period length 1, then it is well-known that $D = a^2+1$ and $\sqrt{D} = (a, \overline{2a})$ where $a \in \mathbb{N}$. Hence, we obtain a parametrization for D in which the continued fraction expansion of $\sqrt{D}$ has period length 1. For example, we have $D = 2, 5, 10, 17, \ldots$ etc. In the cases where the period length is greater than one, recall from Theorem 1.2.1 that if the continued fraction expansion of $\sqrt{D}$ has period length $\ell \in \mathbb{N}$, then

$$\sqrt{D} = (a, a_1, a_2, \ldots, a_{\ell-1}, a+\sqrt{D}) \quad \text{where} \quad a_i = a_{\ell-i} \text{ for } 1 \le i \le \ell-1.$$

We may write

$$\sqrt{D} - a = (0, a_1, a_2, \ldots, a_{\ell-1}, a+\sqrt{D}).$$

If we put

$$\frac{P}{Q} = (0, a_1, a_2, \ldots, a_{\ell-1}), \quad \text{and} \quad \frac{R}{P} = (0, a_1, a_2, \ldots, a_{\ell-2}),$$

where $P, Q, R \ge 0$ and $\gcd(P,Q) = 1 = \gcd(P,R)$, then by (1.13), we get

$$\sqrt{D} - a = \frac{P(\sqrt{D}+a)+R}{Q(\sqrt{D}+a)+P}.$$

We multiply the above equation by $Q(\sqrt{D}+a)+P$ to get $Q(D-a^2)=2aP+R$, which can be written as

$$D = a^2 + \frac{2aP+R}{Q}. \tag{2.2}$$

This means that D is an integer if and only if

$$Q \mid 2aP + R. \tag{2.3}$$

If the period length is 2, i.e. $\sqrt{D} = (a, \overline{a_1, 2a})$, then $P = 1$, $Q = a_1$, and $R = 0$. So, $D = a^2 + 2a/a_1$. Hence, D is an integer if and only if $a_1 \mid 2a$. Let $a_1 m = 2a$ for some integer m. Then $D = (a_1 m/2)^2 + m$ is a parametrization of D. For instance, if we take $m = 3$, $a_1 = 4$, then $D = 38$ and $\sqrt{38} = (6, \overline{4, 12})$.

When the period length is 3, i.e. $\sqrt{D} = (a, \overline{a_1, a_1, 2a})$, we compute $P = a_1$, $Q = a_1^2 + 1$, and $R = 1$. By (2.3), we have $(a_1^2+1) \mid 2aa_1 + 1$. This implies that a_1 must be even. By (2.2), we get

$$(D-a^2)(a_1^2+1) - 2aa_1 = 1.$$

If we treat the above equation as a linear Diophantine equation with unknowns $D - a^2$ and $2a$, then by Theorem 1.1.2, we find that

$$2a = a_1 + 2(a_1^2+1)m \quad \text{and} \quad D = a^2 + 1 + 2a_1 m$$

for some integer m. If we take $a_1 = 4$ and $m = 3$, then $a = 53$, $D = 2834$, and $\sqrt{2834} = (53, \overline{4, 4, 106})$.

If the period length is 4, then the continued fraction expansion of $\sqrt{D}$ is of the form $(a, \overline{a_1, a_2, a_1, 2a})$. We compute $P = a_1 a_2 + 1$, $Q = a_1^2 a_2 + 2a_1$, $R = a_2$. By (2.2), we get

$$(D-a^2)(a_1^2 a_2 + 2a_1) = 2a(a_1 a_2 + 1) + a_2.$$

Note that if a_2 is odd, then a_1 cannot be even. The above equation can be thought of as a linear Diophantine equation with unknowns $(D - a^2)$ and $2a$, namely,

$$(D - a^2)(a_1^2 a_2 + 2a_1) - 2a(a_1 a_2 + 1) = a_2.$$

By Theorem 1.1.2, we get

$$2a = -(a_1 a_2 + 1)a_2 + (a_1^2 a_2 + 2a_1)m \quad \text{and} \quad D = a^2 - a_2^2 + (a_1 a_2 + 1)m,$$

where the integer m is chosen to ensure that $2a$ is positive. Consider $a_1 = 1$, $a_2 = 4$, and $m = 6$. Then $a = 8$, $D = 78$, and $\sqrt{78} = (8, \overline{1, 4, 1, 16})$.

Kraitchik used the above method to parametrize D for period length from 1 to 7. A discussion of these seven parametrizaions is available in Appendix A. Besides the parametrizations of D for short periods, Kraitchik computed the continued fraction expansion of $\sqrt{D}$ where $D < 1000$ and categorized D into 45 collections of numbers according to the period length of the continued fraction expansion of $\sqrt{D}$. Also, he gave a number of specific quadratics $D(X)$ where the period length of the continued fraction expansion of $\sqrt{D(X)}$ remains constant as X varies, for example,

$$\sqrt{(569X + 9)^2 + (966X + 16)} = (569X + 9, \overline{1, 5, 1, 1, 1, 1, 1, 1, 5, 1, 2(569X + 9)}).$$

The above equation is of particular interest to us. In Chapter 3, we study quadratics $D(X)$ where the continued fraction expansion of $\sqrt{D(X)}$ has a fixed symmetric part as X varies.

2.3 Schinzel Sleepers

In his 1961 article [213] entitled *On some problems of the arithmetical theory of continued fractions*, Schinzel studied the period length of the continued fraction expansion of $\sqrt{f(X)}$ where $f(X)$ is an integer-valued polynomial of degree n. His aim was to determine whether the continued fraction expansion of $\sqrt{f(X)}$ has bounded period length as X varies. As we mentioned in Section 2.1, he obtained two remarkable results. It is the result on the quadratic case, $A^2X^2 + 2BX + C$, that we are interested in at this time. This result states that the continued fraction expansion of $\sqrt{A^2X^2 + 2BX + C}$ has bounded period length if and only if $B^2 - A^2C \mid 4\gcd(A^2, B)^2$, the Schinzel condition.

For the remainder of this section, we assume θ to be a quadratic irrational. Also, we follow Schinzel's notation in denoting the period length of the continued fraction expansion of θ by $\mathrm{lp}\,(\theta)$ and the pre-period length of the continued fraction expansion of θ by $\mathrm{lap}\,(\theta)$.

We will outline the proof of Schinzel's result with the aid of five preliminary results. These preliminary results establish the relationship between the upper bounds of $\mathrm{lap}\,((r\theta + t)/(u\theta + s))$ and $\mathrm{lap}\,(\theta)$ as well as the upper bounds of $\mathrm{lp}\,(\theta)$ and $\mathrm{lp}\,((r\theta + t)/(u\theta + s))$ for some $r, s, t, u \in \mathbb{Z}$.

Lemma 2.3.1 [213, Lemma 1] *Let the continued fraction expansion of θ be given by*

$$(a_0, a_1, \dots . a_{k-1}, \overline{a_k, a_{k+1}, \dots, a_{k+\ell-1}}). \tag{2.4}$$

For any non-negative integer i, we denote the i-th convergent of θ by x_i/y_i . If $N \geq 2$ and $N \geq a_i$ for $i = 1, \dots, k-1$, then $i \leq y_i \leq N^i$. Moreover, for integers r, $s \neq 0$ and t, $\mathrm{lap}((r\theta+t)/s) < 2sN^k$.

Theorem 2.3.1 [213, Theorem 1] *Let r and t be integers and m and s be natural numbers. If $\mathrm{lap}\,(\theta) \leq m$, then there exists a natural number M depending on m and the product rs such that*

$$\mathrm{lap}\left(\frac{r\theta + t}{s}\right) \leq M.$$

Corollary 2.3.1 [213, Corollary, p. 399] *Let r, s, t, u be integers and write $d = rs - tu$. Let m be a natural number. If $\mathrm{lap}\,(\theta) \leq m$, then there exists a natural number M depending on m, d and u such that*

$$\mathrm{lap}\left(\frac{r\theta + t}{u\theta + s}\right) \leq M,$$

where $u\theta + s \neq 0$.

Since θ is a quadratic irrational, we may write $\theta = (a_0, a_1, \ldots, a_{k-1}, \overline{a_k, a_{k+1}, \ldots, a_{k+\ell-1}})$. So, the k-th complete quotient, θ_k, has a purely periodic expansion with period length ℓ, i.e.,

$$\theta_k = (\overline{a_k, a_{k+1}, \ldots, a_{k+\ell-1}}).$$

We may rewrite the above expression as

$$\theta_k = (a_k, a_{k+1}, \ldots, a_{k+\ell-1}, \overline{a_k, a_{k+1}, \ldots, a_{k+\ell-1}}).$$

That is, the period of θ can be viewed as the pre-period of the above continued fraction expansion. This leads to $\operatorname{lp}(\theta) = \operatorname{lap}(\theta_k)$. By Corollary 2.3.1, it follows that

Theorem 2.3.2 [213, Theorem 2] *Let r, s, t, u be integers and write $d = rs - tu$. Let m be a natural number. If $\operatorname{lp}(\theta) \leq m$, then there exists a natural number M depending on m and d such that*

$$\operatorname{lp}\left(\frac{r\theta + t}{u\theta + s}\right) \leq M,$$

where $u\theta + s \neq 0$.

Theorem 2.3.3 [213, Theorem 4] *Let g be a non-zero integer. Denote the set of all integers X such that $g \mid 4X^2$ by $\mathcal{E}$. Then*

$$\lim_{X \notin \mathcal{E}} \operatorname{lp}\left(\sqrt{X^2 + g}\right) = \infty$$

and

$$\limsup_{X \in \mathcal{E}} \operatorname{lp}\left(\sqrt{X^2 + g}\right) < \infty.$$

Schinzel proved Theorem 2.3.3 by establishing a lower bound on the period length of $\sqrt{X^2 + g}$ for the case $X \notin \mathcal{E}$ and an upper bound for the case $X \in \mathcal{E}$. For the latter case, he showed that there exists a natural number M such that

$$\operatorname{lp}\left(\sqrt{X^2 + g}\right) \leq M.$$

We present Schinzel's approach in the following. Since $X \in \mathcal{E}$, we have $g \mid 4X^2$. Let a be a non-zero squarefree integer and b and x be natural numbers such that $g = ab^2$ and $2X = abx$. Then

$$\sqrt{X^2+g} = \sqrt{\left(\frac{abx}{2}\right)^2 + ab^2} = \frac{b}{2}\sqrt{(ax)^2+4a}.$$

Since $4a \mid 4(ax)$, we see that $(ax)^2+4a$ is of R-D type, which was introduced in Section 2.1. Thus, by Theorem 2.1.1, $\mathrm{lp}\,(\sqrt{(ax)^2+4a}) \leq 12$ for all integers x. Now, by Theorem 2.3.2 with $r = b$, $t = 0 = u$ and $s = 2$, there exists a natural number M such that $\mathrm{lp}\,(\sqrt{X^2+g}) \leq M$.

For an arbitrary quadratic polynomial, $D(X) = AX^2 + BX + C$, if the first coefficient, A, is not a square, then by Schinzel's first main result [213, Theorem 5], $\mathrm{lp}(\sqrt{D(X)})$ is unbounded as X varies. Hence, we assume the first coefficient to be a square and write $D(X) = A^2X^2 + BX + C$.

Although Schinzel's result on the quadratic case [213, Theorem 5] does not restrict the second coefficient B to be even, we will show in the argument below that the second coefficient may be assumed to be even without any loss of generality. The benefit of such an assumption is that we get a simpler expression to work with in the sequel.

If B is odd, then we divide the possible values of X into even integers $X = 2x$ or odd integers $X = 2x+1$. Thus,

$$D(X) = A^2X^2 + BX + C = a^2x^2 + bx + c,$$

where

$$a = 2A, \quad b = 2B \quad \text{and} \quad c = C \tag{2.5}$$

when $X = 2x$ and

$$a = 2A, \quad b = 2(2A^2+B) \quad \text{and} \quad c = A^2 + B + C \tag{2.6}$$

when $X = 2x+1$. Therefore, we may assume that the second coefficient is even. Note that we may further assume the coefficient A to be even. But as there is no advantage of such an assumption in this section, we will not make this assumption.

Henceforth, we write

$$D(X) = A^2X^2 + 2BX + C \tag{2.7}$$

with discriminant $\Delta = B^2 - A^2C$.

We may rewrite (2.7) as

$$D(X) = \frac{(A^2X+B)^2 - (B^2 - A^2C)}{A^2} = \frac{(A^2X+B)^2 - \Delta}{A^2}.$$

Since $\sqrt{D(X)} = \left(\sqrt{(A^2X+B)^2 - \Delta}\right) \big/ A$, by Theorem 2.3.2 with $\theta = \sqrt{(A^2X+B)^2 - \Delta}$, $r = 1$, $s = A$, $t = 0$ and $u = 0$, we get

$$\limsup_{X\to\infty} \operatorname{lp}\left(\sqrt{D(X)}\right) < \infty$$

if and only if for some X_0,

$$\Delta \mid 4(A^2X+B)^2 \quad \text{for all} \quad X \geq X_0.$$

The condition

$$\Delta \mid 4(A^2X+B)^2 \tag{2.8}$$

can sometimes be difficult to apply because of its dependence on X. Note that

$$4(A^2X+B)^2 = 4\gcd(A^2,B)^2\left(\frac{A^2}{\gcd(A^2,B)}X + \frac{B}{\gcd(A^2,B)}\right)^2.$$

The expression

$$\frac{A^2}{\gcd(A^2,B)}X + \frac{B}{\gcd(A^2,B)} \tag{2.9}$$

is an arithmetic progression in non-negative integers X whose first term $B/\gcd(A^2,B)$ and difference $A^2/\gcd(A^2,B)$ are relatively prime. Dirichlet's theorem on primes in arithmetic progression [38, p. 108] states that if a and b are relatively prime natural numbers, then the arithmetic progression $an+b$ in natural numbers n contains infinitely many primes. Thus, the arithmetic progression

(2.9) contains infinitely many primes; hence, it contains infinitely many numbers relatively prime to Δ. This means that the divisibility condition (2.8) for all sufficiently large X is equivalent to

$$\Delta \mid 4\gcd(A^2, B)^2, \tag{2.10}$$

the Schinzel condition.

Theorem 2.3.4 [213, Theorem 5] *Let $D(X) = A^2X^2 + 2BX + C$ where $A > 0$, B and C are integers such that $\Delta = B^2 - A^2C \neq 0$. Then*

$$\limsup_{X\to\infty} \operatorname{lp}\left(\sqrt{D(X)}\right) < \infty \quad \textit{if and only if} \quad \Delta \mid 4\gcd(A^2, B)^2.$$

Recall from Section 2.1 that the study of $D(X)$ is motivated by the difficulty of solving a Pell equation with large non-square D. As we noted in the beginning of Section 2.1, in order to study $D(X)$, we need a criterion to determine when $\sqrt{D(X)}$ has a bounded periodic continued fraction expansion as X gets large. As soon as such a criterion is found, we may proceed to find the solutions of the Pell equation,

$$x^2 - D(X)y^2 = 1,$$

where $D(X)$ satisfies the criterion. In light of Theorem 2.3.4, the criterion is (2.10). So we may now seek solutions to the above Pell equation.

Since the fundamental solution of a Pell equation is the fundamental unit of a real quadratic order by the discussion in Section 1.4, we may look for fundamental units instead. Stender [233] made use of Theorem 2.3.4 and studied the fundamental units of the maximal order of the real quadratic field $\mathbb{Q}(\sqrt{D(X)})$ where $D(X)$ is positive and squarefree. In addition to the assumption that the coefficients of $D(X)$ satisfy (2.10), Stender assumed that $\gcd(A^2, 2B, C)$ is squarefree. He showed that for sufficiently large X, the fundamental unit of $\mathbb{Q}(\sqrt{D(X)})$ is given by

$$\varepsilon = \begin{cases} \dfrac{A^2X + B + A\sqrt{D(X)}}{\sqrt{|\Delta|}} & \text{if } |\Delta| \text{ is a square,} \\ \\ \dfrac{\left(A^2X + B + A\sqrt{D(X)}\right)^2}{|\Delta|} & \text{otherwise.} \end{cases} \tag{2.11}$$

We note that the above two quantities are in $\mathcal{O}_{D(X)}$ because of the Schinzel condition.

As in Degert's approach in [32], Stender computed the fundamental units algebraically without using continued fractions.

Irving Kaplansky in a letter written in 1998 to Richard Mollin, Hugh Williams and Kenneth Williams, see Appendix B, suggested the term *sleepers* for families of continued fractions with bounded period length. We use Kaplansky's terminology and make the following definition.

Definition 2.3.1 *Let $D(X) = A^2X^2 + 2BX + C$ where A is a natural number, B, C and X are integers, the discriminant is $\Delta = B^2 - A^2C \neq 0$, and $\Delta \mid 4\gcd(A^2, B)^2$. Then $D(X)$ is called a* **Schinzel sleeper.**

2.4 Work of van der Poorten and Williams

In their article [206] on Schinzel sleepers, $D(X) = A^2X^2 + 2BX + C$, where $B^2 - A^2C$ divides $4\gcd(A^2, B)^2$, van der Poorten and Williams sought the exact continued fraction expansion for $\sqrt{D(X)}$ with the assumption that $\gcd(A^2, 2B, C)$ is squarefree. As we have seen in the remark following (2.5) and (2.6), they could assume that the coefficient A is even.

Lemma 2.4.1 [206, Lemma 2.1] *Put $S = \gcd(A, B)$ and*

$$(B/S)^2 - (A/S)^2C = G^2H, \tag{2.12}$$

where H is squarefree. Let $B^2 - A^2C \mid 4\gcd(A^2,B)^2$. Then GH divides $2A$, $2B/S$ and $2S$. Also, $G^2H \mid 4\gcd(A^2,2B,C)$. Therefore, if $\gcd(A^2,2B,C)$ is squarefree, then $G=1$ or 2.

For the remainder of this section, when we speak of G, H and S, we mean the quantities defined in Lemma 2.4.1.

Theorem 2.4.1 [206, Theorem 2.2] *Suppose that $B^2 - A^2C \mid 4\gcd(A^2,B)^2$ and $\gcd(A^2,2B,C)$ is squarefree. Then $D(X) = A^2X^2 + 2BX + C$ is of R-D type when $C \le 0$ or C is a perfect square. In other words, $D(X)$ is of the form $n^2 + r$ where $r \mid 4n$ and $-2n+1 < r \le 2n$.*

Since we want to exclude R-D types from our discussion, we henceforth assume that C is positive and not a perfect square. Note that when $X < 0$, $D(X)$ may be written as $A^2X^2 - 2B|X| + C$. Since B may be of either sign, we may assume that X is positive. If $X = x + x_0$ for some natural number x_0, then

$$D(X) = A^2(x+x_0)^2 + 2B(x+x_0) + C = a^2x^2 + 2bx + c,$$

where $a = A$, $b = A^2x_0 + B$ and $c = A^2x_0^2 + 2Bx_0 + C$. Thus, $\delta = b^2 - a^2c = B^2 - A^2C = \Delta$, $\gcd(a^2,b) = \gcd(A^2,B)$ and $\gcd(a^2,2b,c) = \gcd(A^2,2B,C)$. If $x_0 > -B/A^2$, then b is positive. Also, since

$$c - G^4H^2 = A^2(x_0^2 - G^4H^2/A^2) + Bx_0 + C,$$

$c > G^4H^2$ if $x_0 > \max\{G^2H/A, -C/B\}$. Henceforth, X is assumed to be sufficiently large so that B is positive and $C > G^4H^2$.

The proofs of the main theorems in [206] made use of a technique involving 2×2 matrices described in [159] and [161]. Since our approach does not make use of the matrix method here, we will only provide the results, but not the proofs.

To avoid confusion of notation, for the remainder of this section, we denote the n-th convergent of a continued fraction by x_n/y_n.

Theorem 2.4.2 [206, Theorem 4.1] *Suppose that $G = 1$. If $|H| \geq 2$, then $\sqrt{D(X)}$ has the expansion*

$$\left(AX + c_0, \overline{\overrightarrow{w}, \frac{2(P+AX)}{|H|}, \overleftarrow{w}, 2(AX+c_0)}\right),$$

where B/A is the convergent $x_n/y_n = (c_0, c_1, \ldots, c_n)$, $\overrightarrow{w} = c_1, \ldots, c_n$, $\overleftarrow{w}$ is the reverse of $\overrightarrow{w}$, the subscript n is odd if $H < 0$ and is even if $H > 0$, and $P = (-1)^{n+1}(x_{n-1}x_n - y_{n-1}y_nC)$. If $|H| = 1$, then $\sqrt{D(X)}$ has the expansion

$$\left(AX + c_0, \overline{\overrightarrow{w}, 2(AX+c_0)}\right).$$

Theorem 2.4.3 [206, Theorem 4.2] *If $G = 2$, then $D(X) \equiv 5 \bmod 8$. If $|H| \geq 2$, then the expansion of $(1 + \sqrt{D(X)})/2$ is given by*

$$\left(\frac{AX + c_0 + 1}{2}, \overline{\overrightarrow{w}, \frac{2(P+AX)}{|H|}, \overleftarrow{w}, AX + c_0}\right),$$

where c_0 is an odd integer, $(B/A + 1)/2 = (x_n/y_n + 1)/2$ has the expansion $((c_0+1)/2, \overrightarrow{w})$ and $P = (-1)^{n+1}(x_{n-1}x_n - y_{n-1}y_nC)$. If $|H| = 1$, then $(1 + \sqrt{D(X)})/2$ has the expansion

$$\left(\frac{AX + c_0 + 1}{2}, \overline{\overrightarrow{w}, AX + c_0}\right).$$

We show that Theorem 2.4.2 may be written in simpler form in the following. Since we take sufficiently large X to guarantee that $G^4H^2 < C$, we have $G^2|H| < \sqrt{C}$. By Theorem 1.3.1, $(B/S)/(A/S)$ is a convergent of $\sqrt{C}$. Let P_i be the rational part of the numerator and Q_i the denominator of the i-th complete quotient of $\sqrt{C}$ for any non-negative integer i. Since $(B/S)^2 - (A/S)^2C = G^2H$, we have $G^2|H| = Q_{i+1}$ for some non-negative integer i. Thus, we may use the continued fraction expansion of $\sqrt{C}$ to get the continued fraction expansion of $\sqrt{D(X)}$.

From Theorem 2.4.2, in the case where $|H| > 1$, the middle term of the continued fraction expansion of $\sqrt{D(X)}$ is given by

$$\frac{2(P+AX)}{|H|}.$$

If $G = 1$, then $|H| = Q_{n+1}$. Moreover, $P = P_{n+1}$. By Lemma 2.4.1, we have $H \mid 2A$. Hence, $Q_{n+1} \mid 2A$ and therefore, $Q_{n+1} \mid 2P_{n+1}$. In fact, $c_{n+1}Q_{n+1} = 2P_{n+1}$. By Theorem 1.2.3, if $\ell = \operatorname{lp}(\sqrt{C})$, then ℓ is even and $n+1 \equiv \ell/2 \bmod \ell$. Now, Theorem 2.4.2 can be rewritten as

Theorem 2.4.4 [206, Theorem 4.3] *Suppose that $G = 1$ and let $\sqrt{C} = (c_0, c_1, \ldots, c_n, \ldots)$. Then $B/A = (c_0, c_1, \ldots, c_n)$. Put $\overrightarrow{w} = c_1, \ldots, c_n$ and $\overleftarrow{w} = c_n, \ldots, c_1$. Then for $|H| = Q_{n+1} \geq 2$, the expansion of $\sqrt{D(X)}$ is given by*

$$\left(AX + c_0, \overline{\overrightarrow{w}, \frac{2AX}{Q_{n+1}} + c_{n+1}, \overleftarrow{w}, 2(AX + c_0)}\right).$$

If $|H| = 1$, then the expansion of $\sqrt{D(X)}$ is given by

$$\left(AX + c_0, \overline{\overrightarrow{w}, 2(AX + c_0)}\right).$$

The period length of $\sqrt{D(X)}$ is given by

$$\operatorname{lp}(\sqrt{D(X)}) = \begin{cases} (2k+1)\ell & \text{if } |H| = Q_{n+1} > 1, \\ k\ell & \text{if } |H| = 1, \end{cases}$$

for some non-negative integer k. The value of k depends exclusively on A, B and C and is independent of X. In other words, $\operatorname{lp}(\sqrt{D(X)})$ remains constant as X varies as long as A, B and C are fixed.

In the example below, we illustrate the case where $G = 1 = H$.

Example 2.4.1 Consider $C = 14$. Then $\sqrt{14} = (3, \overline{1, 2, 1, 6})$. It can be checked that $A = 4$ and $B = 15$ is a solution of $B^2 - A^2C = 1$ and $B/A = (3, 1, 2, 1)$. Note that although B/A can be written as $(3, 1, 3)$, we write $B/A = (3, 1, 2, 1)$ to match a portion of the continued fraction expansion of $\sqrt{C} = (3, \overline{1, 2, 1, 6})$. Write $D(X) = 4^2X^2 + 2(15)X + 14$. Then

$$\sqrt{D(1)} = \sqrt{60} = (7, \overline{1, 2, 1, 14}), \qquad \sqrt{D(2)} = \sqrt{138} = (11, \overline{1, 2, 1, 22}),$$

$$\sqrt{D(3)} = \sqrt{248} = (15, \overline{1, 2, 1, 30}).$$

In general, we get

$$\sqrt{D(X)} = (4X + 3, \overline{1, 2, 1, 8X + 6}) \quad \text{and} \quad \operatorname{lp}\left(\sqrt{D(X)}\right) = 4.$$

Similarly, it can be checked that $A = 120$ and $B = 449$ is also a solution of $B^2 - A^2C = 1$. If we write $\sqrt{C} = (3, \overline{1, 2, 1, 6}) = (3, \overline{1, 2, 1, 6, 1, 2, 1, 6})$, then $B/A = (3, 1, 2, 1, 6, 1, 2, 1)$ matches a portion of the continued fraction expansion of $\sqrt{C}$. Write $D(X) = 120^2X^2 + 2(449)X + 14$. Then

$$\sqrt{D(1)} = \sqrt{15312} = (123, \overline{1, 2, 1, 6, 1, 2, 1, 246}),$$

$$\sqrt{D(2)} = \sqrt{59410} = (243, \overline{1, 2, 1, 6, 1, 2, 1, 486}),$$

$$\sqrt{D(3)} = \sqrt{132308} = (363, \overline{1, 2, 1, 6, 1, 2, 1, 726}).$$

In general, we get

$$\sqrt{D(X)} = (120X + 3, \overline{1, 2, 1, 6, 1, 2, 1, 240X + 6}) \quad \text{and} \quad \operatorname{lp}\left(\sqrt{D(X)}\right) = 8.$$

□

When $G = 2$, we can establish a similar result relating the continued fraction expansion of $(1 + \sqrt{C})/2$ to that of $(1 + \sqrt{D(X)})/2$ using Theorem 2.4.3.

Theorem 2.4.5 [206, Theorem 4.4] *Suppose that $G = 2$ and let $(1 + \sqrt{C})/2 = (c_0, c_1, \ldots, c_n, \ldots)$ and $|H| = Q_{n+1}$ be the denominator of the $(n+1)$-th complete quotient of $(1 + \sqrt{C})/2$. Then $(B/A + 1)/2 = (c_0, c_1, \ldots, c_n)$. Put $\overrightarrow{w} = c_1, \ldots, c_n$ and $\overleftarrow{w} = c_n, \ldots, c_1$. Then for $|H| \geq 2$, the expansion of $(1 + \sqrt{D(X)})/2$ is given by*

$$\left(\frac{AX}{2} + c_0, \overline{\overrightarrow{w}, \frac{AX}{Q_{n+1}} + c_{n+1}, \overleftarrow{w}, AX + 2c_0 - 1}\right).$$

If $|H| = 1$, then $(1 + \sqrt{D(X)})/2$ has the expansion

$$\left(\frac{AX}{2} + c_0, \overline{\overrightarrow{w}, AX + 2c_0 - 1}\right).$$

In the following example, we illustrate the above theorem for the case where $G = 2$ and $H = 1$.

Example 2.4.2 Consider $C = 109 \equiv 5 \bmod 8$. We compute $(1+\sqrt{109})/2 = (5, \overline{1,2,1,1,2,1,9})$. Since $(x, y) = (261, 25)$ is a solution of $x^2 - y^2C = -4$, we put $A = 2 \cdot 25 = 50$, $B = 2 \cdot 261 = 522$. It can be checked that $(B/A+1)/2$ can be written as $(5, 1, 2, 1, 1, 2, 1)$, which matches a portion of the continued fraction expansion of $(1+\sqrt{C})/2$. Write $D(X) = 50^2X^2 + 2(522)X + 109$. Then

$$\frac{1+\sqrt{D(1)}}{2} = \frac{1+\sqrt{3653}}{2} = (30, \overline{1,2,1,1,2,1,59}),$$

$$\frac{1+\sqrt{D(2)}}{2} = \frac{1+\sqrt{12197}}{2} = (55, \overline{1,2,1,1,2,1,109}),$$

$$\frac{1+\sqrt{D(3)}}{2} = \frac{1+\sqrt{25741}}{2} = (80, \overline{1,2,1,1,2,1,159}).$$

In general, we get

$$\frac{1+\sqrt{D(X)}}{2} = (25X+5, \overline{1,2,1,1,2,1,50X+9}).$$

Similarly, it can be checked that $(x, y) = (68123, 6525)$ is a solution of $x^2 - y^2C = 4$. Put $A = 2 \cdot 6525$ and $B = 2 \cdot 68123$. Then $(B/A+1)/2 = (5, 1, 2, 1, 1, 2, 1, 9, 1, 2, 1, 1, 2, 1)$. We write $D(X) = (2 \cdot 6525)^2X^2 + 2(2 \cdot 68123)X + 109$. Then

$$\frac{1+\sqrt{D(1)}}{2} = \frac{1+\sqrt{170575101}}{2} = (6530, \overline{1,2,1,1,2,1,9,1,2,1,1,2,1,13059}),$$

$$\frac{1+\sqrt{D(2)}}{2} = \frac{1+\sqrt{681755093}}{2} = (13055, \overline{1,2,1,1,2,1,9,1,2,1,1,2,1,26109}),$$

$$\frac{1+\sqrt{D(3)}}{2} = \frac{1+\sqrt{1533540085}}{2} = (19580, \overline{1,2,1,1,2,1,9,1,2,1,1,2,1,39159}).$$

In general, we get

$$\frac{1+\sqrt{D(X)}}{2} = (6525X+5, \overline{1,2,1,1,2,1,13050X+9}).$$

$\square$

There is a more general result [206, Theorem 5.6] concerning the continued fraction expansions of $\sqrt{C}$ and $\sqrt{D(X)}$ when $G = 2$. Since the statement of the theorem is rather involved, we will not provide the details here. We will, however, give an important consequence of the theorem, namely the period length of $\sqrt{D(X)}$. If we put $\ell = \operatorname{lp}(\sqrt{C})$, then for some $k \geq 0$,

$$\operatorname{lp}(\sqrt{D(X)}) = \begin{cases} (6k+1)\ell \text{ or } (6k+5)\ell & \text{if } |H| > 1, \\ (3k+1)\ell \text{ or } (3k+2)\ell & \text{if } |H| = 1. \end{cases}$$

Once again, the values of k here depend only on the values of A, B and C, but not on X.

Chapter 3

Symmetric Sequence Perspective of Continued Fractions

The aim of this chapter is to study certain parametric families of non-square natural numbers D in which every family is uniquely defined by a symmetric sequence of natural numbers. We will present a theorem of Perron in the first section; then use the theorem to define families of D in the ensuing section. Various properties of the families will be discussed therein. We will show that each family can be described by a quadratic polynomial of the form described in Theorems 2.4.4 or 2.4.5. Finally, we will establish a somewhat surprising result concerning the product of two numbers of a given family. When certain conditions are met, independent of the symmetric sequence that defines the family, the product is of RD-type and, thus, its square root has has a very short and predictable periodic continued fraction expansion.

3.1 A Theorem of Perron

If D is a non-square natural number, then from Theorems 1.2.1 and 1.2.4, we know that

$$\sqrt{D} = \left(a_0, \overline{a_1, a_2, \ldots, a_2, a_1, 2a_0}\right) .$$

and

$$\frac{1+\sqrt{D}}{2} = \left(a_0, \overline{a_1, a_2, \ldots, a_2, a_1, 2a_0 - 1}\right) \quad \text{when} \quad D \equiv 1 \bmod 4.$$

Each of the above two continued fractions has a palindromic string, $a_1, a_2, \ldots, a_2, a_1,$ of natural numbers in its period. We call this palindromic string the **symmetric part of the period.**

Recall from Section 2.2 that Kraitchik made a study of the period length of $\sqrt{D}$ and found explicit parametrizations of D for various cases of fixed period lengths. Perron [190, Satz 3.17] evidently used Kraitchik's method to obtain a parametrization of D for arbitrary period length. Perron also showed that when given a palindromic string of natural numbers, under certain conditions, it is possible to find a non-square natural number D such that the continued fraction expansion of $\sqrt{D}$ has this palindromic string as its symmetric part of the period. Similarly, he gave a result [190, Satz 3.34] concerning the case $(1+\sqrt{D})/2$. Friesen [54] gave a new proof of [190, Satz 3.17] and deduced that there is either no integer D or there are infinitely many squarefree D such that the continued fraction expansion of $\sqrt{D}$ has the given palindromic string as its symmetric part. Halter-Koch [66] improved Friesen's result by providing the probability of obtaining such D. He also gave a similar probability result regarding the case $(1+\sqrt{D})/2$.

Although Perron did not state explicitly that when given a palindromic string of natural numbers, there is a non-square natural number D such that either the continued fraction expansion of $\sqrt{D}$ or that of $(1+\sqrt{D})/2$ has the palindromic string as the symmetric part of its period, it is likely that he was aware of such a fact. This statement follows almost immediately by combining the proofs of [190, Satz 3.17] and [190, Satz 3.34] as we will show in the theorem below.

In the sequel, we write $\sqrt{D}$ and $(1+\sqrt{D})/2$ collectively by $(\tau-1+\sqrt{D})/\tau$ where $\tau=1$ or 2. When $\tau=2$, we require that $D\equiv 1 \bmod 4$.

Theorem 3.1.1 (Perron) *Given a palindromic string, $a_1, a_2, \ldots, a_2, a_1$, of $\ell-1\geq 0$ natural numbers, there are infinitely many non-square natural numbers D such that*

$$\left(a_0, \overline{a_1, a_2, \ldots, a_2, a_1, 2a_0-\tau+1}\right) = \frac{\tau-1+\sqrt{D}}{\tau} \tag{3.1}$$

where $a_0 = \lfloor(\tau-1+\sqrt{D})/\tau\rfloor$.

<u>Proof</u>: Let $a_1, a_2, \ldots, a_2, a_1$ be a palindromic string of $\ell-1$ natural numbers. Then there exists a

uniquely defined matrix $\mathcal{M}$ such that

$$\mathcal{M} = \prod_{j=1}^{\ell-1} \begin{pmatrix} a_j & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} u & v \\ v & w \end{pmatrix} \quad \text{and} \quad \det(\mathcal{M}) = uw - v^2 = (-1)^{\ell-1}. \tag{3.2}$$

If we have an empty string, the matrix $\mathcal{M}$ is simply the identity matrix. To establish the required result, it suffices to demonstrate that D and a_0 in (3.1) can be expressed in terms of u, v and w along with an appropriate choice of $\tau = 1$ or 2. Let

$$(a_0, \overline{a_1, a_2, \ldots, a_2, a_1, 2a_0 - \tau + 1}) = \frac{\tau - 1 + \sqrt{D}}{\tau} = \theta.$$

Let A_i/B_i be the i-th convergent and θ_i the i-th complete quotient of the above continued fraction expansion. Then, by (1.13),

$$\theta = \frac{A_{\ell-1}\theta_\ell + A_{\ell-2}}{B_{\ell-1}\theta_\ell + B_{\ell-2}}.$$

Since ℓ is the period length of θ, by Theorems 1.2.1 and 1.2.4, we have $\theta_\ell = a_0 - \tau + 1 + \theta$. Thus,

$$\theta = \frac{A_{\ell-1}(a_0 - \tau + 1 + \theta) + A_{\ell-2}}{B_{\ell-1}(a_0 - \tau + 1 + \theta) + B_{\ell-2}}.$$

We may rewrite the above equation as

$$B_{\ell-1}\theta^2 + (B_{\ell-1}(a_0 - \tau + 1) + B_{\ell-2} - A_{\ell-1})\,\theta - (A_{\ell-1}(a_0 - \tau + 1) + A_{\ell-2}) = 0. \tag{3.3}$$

By (1.11), we have

$$\frac{B_{\ell-2} - A_{\ell-1}}{B_{\ell-1}} = \frac{B_{\ell-2}}{B_{\ell-1}} - \frac{A_{\ell-1}}{B_{\ell-1}} = (0, a_{\ell-1}, \ldots, a_1) - (a_0, a_1, \ldots, a_{\ell-1}) = -a_0.$$

Hence,

$$B_{\ell-1} \mid (B_{\ell-1}(a_0 - \tau + 1) + B_{\ell-2} - A_{\ell-1}).$$

By (1.11), $A_{\ell-1} = a_0 B_{\ell-1} + B_{\ell-2}$. So,

$$\frac{B_{\ell-1}(a_0 - \tau + 1) + B_{\ell-2} - A_{\ell-1}}{B_{\ell-1}} = -\tau + 1.$$

Dividing (3.3) by $B_{\ell-1}$ yields

$$\theta^2 - (\tau-1)\theta - \frac{A_{\ell-1}(a_0-\tau+1)+A_{\ell-2}}{B_{\ell-1}} = 0.$$

This is a quadratic equation and its positive solution is given by

$$\theta = \frac{1}{2}\left((\tau-1)+\sqrt{(\tau-1)^2+4\left(\frac{A_{\ell-1}(a_0-\tau+1)+A_{\ell-2}}{B_{\ell-1}}\right)}\right). \tag{3.4}$$

Since $\tau = 1$ or 2, we may rewrite the above equation as

$$\theta = \frac{1}{\tau}\left((\tau-1)+\sqrt{(\tau-1)^2+\tau^2\left(\frac{A_{\ell-1}(a_0-\tau+1)+A_{\ell-2}}{B_{\ell-1}}\right)}\right). \tag{3.5}$$

Note that $(A_{\ell-1}(a_0-\tau+1)+A_{\ell-2})/B_{\ell-1}$ is an integer since $\theta = (\tau-1+\sqrt{D})/\tau$ and

$$D = (\tau-1)^2+\tau^2\left(\frac{A_{\ell-1}(a_0-\tau+1)+A_{\ell-2}}{B_{\ell-1}}\right). \tag{3.6}$$

Let $H = (A_{\ell-1}(a_0-\tau+1)+A_{\ell-2})/B_{\ell-1}$. Then

$$B_{\ell-1}H = A_{\ell-1}(a_0-\tau+1)+A_{\ell-2}. \tag{3.7}$$

By (1.11), $A_{\ell-1} = a_0u+v$, $B_{\ell-1} = u$ and $A_{\ell-2} = a_0v+w$, so (3.7) becomes

$$uH = (a_0u+v)(a_0-\tau+1)+a_0v+w.$$

This equation can be rewritten as

$$u(H-a_0^2+a_0(\tau-1)) - v(2a_0-\tau+1) = w. \tag{3.8}$$

We can think of (3.8) as a linear Diophantine equation with unknowns $(H-a_0^2+a_0(\tau-1))$ and $(2a_0-\tau+1)$. Since $uw-v^2 = (-1)^{\ell-1}$, we have

$$u\left((-1)^{\ell-1}w\right) - v\left((-1)^{\ell-1}v\right) = 1.$$

If we apply Theorem 1.1.2 to (3.8), then for some integer m,

$$H - a_0^2 + a_0(\tau - 1) = ((-1)^{\ell-1}w)w + mv = (-1)^{\ell-1}w^2 + mv$$

and

$$2a_0 - \tau + 1 = (-1)^{\ell-1}vw + mu.$$

Hence,

$$a_0 = \frac{\tau - 1 + mu - (-1)^{\ell}vw}{2} \quad \text{and} \quad H = a_0^2 - a_0(\tau - 1) + mv - (-1)^{\ell}w^2. \tag{3.9}$$

By (3.6),

$$D = (\tau - 1)^2 + \tau^2 H = (\tau - 1)^2 + \tau^2\left(a_0^2 - a_0(\tau - 1) + mv - (-1)^{\ell}w^2\right).$$

If $\tau = 1$, we get

$$D = a_0^2 + mv - (-1)^{\ell}w^2.$$

If $\tau = 2$, we get

$$D = (2a_0 - 1)^2 + 4mv - (-1)^{\ell}4w^2.$$

Hence, we can write

$$D = (\tau a_0 - \tau + 1)^2 + \tau^2 mv - \tau^2(-1)^{\ell}w^2. \tag{3.10}$$

The integers m and τ must be chosen to ensure that a_0 in (3.9) is a natural number. If u and vw are even, then to make a_0 a natural number, we need $\tau = 1$ along with an appropriate integer m. If u is even and vw is odd, then to make a_0 a natural number, we must have $\tau = 2$ and an appropriate integer m.

When u is odd, τ can be either 1 or 2. Note that in this case u and vw cannot be both odd by (3.2). Thus, vw is even. If we choose $\tau = 1$, then since u is odd, we need to pick an even m to make a_0 an integer. On the other hand, if $\tau = 2$, then we need to pick an odd m to ensure that a_0 is an integer. The appropriate values of m are given in the following table.

u even	vw even	$\tau = 1$	$m = m_0 + X$ for $X \geq 0$ where $m_0 = \lceil (-1)^\ell vw/u \rceil$ or $m_0 = 1$ if $vw = 0$
u even	vw odd	$\tau = 2$	$m = m_0 + X$ for $X \geq 0$ where $m_0 = \lceil (-1)^\ell (vw-1)/u \rceil$
u odd	vw even	$\tau = 1$	$m = m_0 + 2X$ for $X \geq 0$ where $m_0 = \lceil (-1)^\ell vw/u \rceil$ or $m_0 = \lceil (-1)^\ell vw/u \rceil + 1$ whichever is even or $m_0 = 2$ if $vw = 0$
		$\tau = 2$	$m = m_0 + 2X$ for $X \geq 0$ where $m_0 = \lceil (-1)^\ell (vw-1)/u \rceil$ or $m_0 = \lceil (-1)^\ell (vw-1)/u \rceil + 1$ whichever is odd

Table 3.1: Possible values of m.

By (3.9) and (3.10), the parameterizations of D and a_0 in terms of τ and m, we obtain all D that lead to the required continued fraction expansions.

3.2 Families $\mathcal{F}_\psi$

By Table 3.1, there are two main collections of palindromic strings, one with even u and the other with odd u. Within each of these two collections, there are two further sub-collections determined by the values of τ. We consider the four sub-collections separately and define them as individual families in the following.

Definition 3.2.1 (Families $\mathcal{F}_\psi$)

Let $\psi = a_1, a_2, \ldots, a_2, a_1$ be a palindromic string of $\ell - 1$ natural numbers and

$$\mathcal{M} = \prod_{j=1}^{\ell-1} \begin{pmatrix} a_j & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} u & v \\ v & w \end{pmatrix}.$$

Let m_0 be as defined in Table 3.1. We define families $\mathcal{F}_\psi$ as follows.

Type E_1: *When u is even and $\tau = 1$, we let $X \geq 0$ and $m = m_0 + X$ and write*

$$\begin{aligned}
a_0(X) &= (mu - (-1)^\ell vw)/2 = ((m_0 + X)u - (-1)^\ell vw)/2 \\
D(X) &= (a_0(X))^2 + (m_0 + X)v - (-1)^\ell w^2 \\
\mathcal{F}_\psi &= \{D(0), D(1), D(2), D(3), \ldots\}.
\end{aligned}$$

Type E_2: *When u is even and $\tau = 2$, we let $X \geq 0$ and $m = m_0 + X$ and write*

$$\begin{aligned}
a_0(X) &= (1 + mu - (-1)^\ell vw)/2 = (1 + (m_0 + X)u - (-1)^\ell vw)/2 \\
D(X) &= (2a_0(X) - 1)^2 + 4(m_0 + X)v - 4(-1)^\ell w^2 \\
\mathcal{F}_\psi &= \{D(0), D(1), D(2), D(3), \ldots\}.
\end{aligned}$$

Type O_1: *When u is odd and $\tau = 1$, we let $X \geq 0$ and $m = m_0 + 2X$ and write*

$$\begin{aligned}
a_0(X) &= (mu - (-1)^\ell vw)/2 = ((m_0 + 2X)u - (-1)^\ell vw)/2 \\
D(X) &= (a_0(X))^2 + (m_0 + 2X)v - (-1)^\ell w^2 \\
\mathcal{F}_\psi &= \{D(0), D(1), D(2), D(3), \ldots\}.
\end{aligned}$$

Type O_2: *When u is odd and $\tau = 2$, we let $X \geq 0$ and $m = m_0 + 2X$ and write*

$$\begin{aligned}
a_0(X) &= (1 + mu - (-1)^\ell vw)/2 = (1 + (m_0 + 2X)u - (-1)^\ell vw)/2 \\
D(X) &= (2a_0(X) - 1)^2 + 4(m_0 + 2X)v - 4(-1)^\ell w^2 \\
\mathcal{F}_\psi &= \{D(0), D(1), D(2), D(3), \ldots\}.
\end{aligned}$$

We will sometimes call the E_1 and E_2 types collectively the E-types and the O_1 and O_2 types the O-types.

From the above definition, it follows that the sequence $\{a_0(X)\}$ is an arithmetic progression in X. For instance, when $\mathcal{F}_\psi$ is of type E_1,

$$a_0(X) = \frac{mu - (-1)^\ell vw}{2} = \frac{(m_0 + X)u - (-1)^\ell vw}{2} = a_0(0) + \frac{uX}{2}.$$

Example 3.2.1 Consider the palindromic string $\psi = \{1, 2, 2, 2, 1\}$. Then

$$\mathcal{M} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 24 & 17 \\ 17 & 12 \end{pmatrix}.$$

Since $u = 24$ is even and $vw = 17 \cdot 12$ is even, $\tau = 1$ and we are in type E_1. Since the palindromic string has five terms, i.e. $\ell - 1 = 5$, we compute that $m_0 = \lceil (-1)^\ell vw/u \rceil = \lceil (-1)^6 (17 \cdot 12)/24 \rceil = 9$. Then $a_0(0) = (m_0 u - (-1)^\ell vw)/2 = (9 \cdot 24 - (-1)^6 17 \cdot 12)/2 = 6$ and $D(0) = (a_0(0))^2 + m_0 v - (-1)^\ell w^2 = 6^2 + 9 \cdot 17 - (-1)^6 12^2 = 45$. In general, for $X \geq 0$, we get $a_0(X) = ((m_0 + X)u - (-1)^\ell vw)/2 = 12X + 6$, $D(X) = (a_0(X))^2 + (m_0 + X)v - (-1)^\ell w^2 = 144X^2 + 161X + 45$ and

$$\sqrt{D(X)} = (12X + 6, \overline{1, 2, 2, 2, 1, 24X + 12}).$$

X	$m = m_0 + X$	$a_0(X)$	$D(X)$	C. F. expansion of $\sqrt{D(X)}$
0	$m_0 = 9$	6	45	$(6, \overline{1, 2, 2, 2, 1, 12})$
1	$m_0 + 1 = 10$	18	350	$(18, \overline{1, 2, 2, 2, 1, 36})$
2	$m_0 + 2 = 11$	30	943	$(30, \overline{1, 2, 2, 2, 1, 60})$
3	$m_0 + 3 = 12$	42	1824	$(42, \overline{1, 2, 2, 2, 1, 84})$
4	$m_0 + 4 = 13$	54	2993	$(54, \overline{1, 2, 2, 2, 1, 108})$
5	$m_0 + 5 = 14$	66	4450	$(66, \overline{1, 2, 2, 2, 1, 132})$
6	$m_0 + 6 = 15$	78	6195	$(78, \overline{1, 2, 2, 2, 1, 156})$
7	$m_0 + 7 = 16$	90	8228	$(90, \overline{1, 2, 2, 2, 1, 180})$
8	$m_0 + 8 = 17$	102	10549	$(102, \overline{1, 2, 2, 2, 1, 204})$
9	$m_0 + 9 = 18$	114	13158	$(114, \overline{1, 2, 2, 2, 1, 228})$
...	....	...	......	....

Table 3.2: $\psi = \{1, 2, 2, 2, 1\}$.

$\square$

Example 3.2.2 Consider the symmetric sequence of positive integers $\{1,2,2,1\}$. Then

$$\mathcal{M} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 10 & 7 \\ 7 & 5 \end{pmatrix}.$$

Since $u = 10$ is even and $vw = 7 \cdot 5 = 35$ is odd, we have $\tau = 2$ and we are in type E_2. Since the palindromic string has 4 terms, i.e. $\ell - 1 = 4$, we compute that

$$m_0 = \lceil (-1)^\ell (vw-1)/u \rceil = \lceil (-1)^5 (7\cdot 5 - 1)/10 \rceil = -3.$$

For $X \geq 0$, we compute $a_0(X)$ and $D(X)$ using formulas $a_0(X) = (1+(m_0+X)u-(-1)^\ell vw)/2 = 5X+3$ and $D(X) = (2a_0(X)-1)^2 + 4(m_0+X)v - 4(-1)^\ell w^2 = 100X^2 + 128X + 41$ and

$$\frac{1+\sqrt{D(X)}}{2} = (5X+3, \overline{1,2,2,1,10X+5}).$$

X	$m = m_0 + X$	$a_0(X)$	$D(X)$	C. F. expansion of $(1+\sqrt{D(X)})/2$
0	$m_0 = -3$	3	41	$(3, \overline{1,2,2,1,5})$
1	$m_0 + 1 = -2$	8	269	$(8, \overline{1,2,2,1,15})$
2	$m_0 + 2 = -1$	13	697	$(13, \overline{1,2,2,1,25})$
3	$m_0 + 3 = 0$	18	1325	$(18, \overline{1,2,2,1,35})$
4	$m_0 + 4 = 1$	23	2153	$(23, \overline{1,2,2,1,45})$
5	$m_0 + 5 = 2$	28	3181	$(28, \overline{1,2,2,1,55})$
6	$m_0 + 6 = 3$	33	4409	$(33, \overline{1,2,2,1,65})$
7	$m_0 + 7 = 4$	38	5837	$(38, \overline{1,2,2,1,75})$
8	$m_0 + 8 = 5$	43	7465	$(43, \overline{1,2,2,1,85})$
9	$m_0 + 9 = 6$	48	9293	$(48, \overline{1,2,2,1,95})$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 3.3: $\psi = \{1,2,2,1\}$

$\square$

Example 3.2.3 Consider the empty string, $\psi = \emptyset$. Then

$$\mathcal{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $u = 1$ is odd, we are in type O and may take $\tau = 1$ or 2. We first look at the case $\tau = 1$. By Table 3.1, we have $m_0 = 2$. Since $m = m_0 + 2X = 2(X+1)$ for non-negative integer X, $u = 1$, $v = 0$ and $w = 1$, we have $a_0(X) = ((m_0 + X)u - (-1)^\ell vw)/2 = X + 1$, $D(X) = (a_0(X))^2 + (m_0 + X)v - (-1)^\ell w^2 = (X+1)^2 + 1 = X^2 + 2X + 2$ and

$$\sqrt{D(X)} = (X+1, \overline{2X+2}).$$

In particular, $\sqrt{D(0)} = \sqrt{2} = (1, \overline{2})$, $\sqrt{D(1)} = \sqrt{5} = (2, \overline{4})$ and $\sqrt{D(2)} = \sqrt{10} = (3, \overline{6})$.

Now, take $\tau = 2$. By Table 3.1, we have $m_0 = 1$. Since $m = m_0 + 2X = 2X + 1$ for non-negative integers X, $u = 1$, $v = 0$ and $w = 1$, $a_0(X) = (1 + (m_0 + 2X)u - (-1)^\ell vw)/2 = X + 1$, $D(X) = (2a_0(X) - 1)^2 + 4(m_0 + 2X)v - 4(-1)^\ell w^2 = (2X+1)^2 + 4 = 4X^2 + 4X + 5$ and

$$\frac{1 + \sqrt{D(X)}}{2} = (X+1, \overline{2X+1}).$$

In particular,

$$\frac{1 + \sqrt{D(0)}}{2} = \frac{1 + \sqrt{5}}{2} = (1, \overline{1}) = (\overline{1}) \quad \text{and} \quad \frac{1 + \sqrt{D(1)}}{2} = \frac{1 + \sqrt{13}}{2} = (2, \overline{3}).$$

□

Example 3.2.4 Consider the symmetric sequence, $\psi = \{2,1,1,2\}$, of 4 natural numbers. Then

$$\mathcal{M} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 13 & 5 \\ 5 & 2 \end{pmatrix}.$$

We have $u = 13$, $v = 5$ and $w = 2$. Since u is odd, we are in type O and may have $\tau = 1$ or 2. First, we consider $\tau = 1$. By Table 3.1, m_0 is given by $\lceil (-1)^\ell vw/u \rceil$ or $\lceil (-1)^\ell vw/u \rceil + 1$, whichever is even. It is clear that $m_0 = \lceil (-1)^5 10/13 \rceil = 0$. For $X \geq 0$, $a_0(X) = ((m_0+2X)u - (-1)^\ell vw)/2 = 13X+5$, $D(X) = (a_0(X))^2 + (m_0+2X)v - (-1)^\ell w^2 = 169X^2 + 140X + 29$ and

$$\sqrt{D(X)} = (13X+5, \overline{2,1,1,2,26X+10}).$$

In particular,

$$\sqrt{D(0)} = \sqrt{29} = (5, \overline{2,1,1,2,10}) \quad \text{and} \quad \sqrt{D(1)} = \sqrt{338} = (18, \overline{2,1,1,2,36}).$$

Now, take $\tau = 2$. From Table 3.1, m_0 is given by $\lceil (-1)^\ell (vw-1)/u \rceil$ or $\lceil (-1)^\ell (vw-1)/u \rceil + 1$, whichever is odd. We find that $m_0 = 1$. For $X \geq 0$, then $a_0(X) = (1+(m_0+2X)u-(-1)^\ell vw)/2 = 13X+12$, $D(X) = (2a_0(X)-1)^2 + 4(m_0+2X)v - 4(-1)^\ell w^2 = 676X^2 + 1236X + 565$ and

$$\frac{1+\sqrt{D(X)}}{2} = (13X+12, \overline{2,1,1,2,26X+23}).$$

For instance,

$$\frac{1+\sqrt{D(0)}}{2} = \frac{1+\sqrt{565}}{2} = (12, \overline{2,1,1,2,23}) \quad \text{and} \quad \frac{1+\sqrt{D(1)}}{2} = \frac{1+\sqrt{2477}}{2} = (25, \overline{2,1,1,2,49}).$$

$\square$

Theorem 3.2.1 (Quadratic Structure of $\mathcal{F}_\psi$)

Let $D(X) \in \mathcal{F}_\psi$. If $\mathcal{F}_\psi$ is an E-type family, then

$$D(X) = (\tau B_{\ell-1})^2 \left(\frac{X}{2}\right)^2 + 2\tau G_{\ell-1}\left(\frac{X}{2}\right) + D(0), \quad X \geq 0; \tag{3.11}$$

whereas when $\mathcal{F}_\psi$ is an O-type family,

$$D(X) = (\tau B_{\ell-1})^2 X^2 + 2\tau G_{\ell-1} X + D(0), \quad X \geq 0, \tag{3.12}$$

where $\tau = 1$ or 2 depends on the types of $\mathcal{F}_\psi$, $G_i = \tau A_i - (\tau - 1)B_i$ and A_i/B_i is the i-th convergent of $(\tau - 1 + \sqrt{D(0)})/\tau$ for $i \geq 0$.

Proof: Suppose that $\mathcal{F}_\psi$ is an E_2-type. By Definition 3.2.1, when $\tau = 2$, we have

$$D(X) = (2a_0(X) - 1)^2 + 4(m_0 + X)v - 4(-1)^\ell w^2$$

for $X \geq 0$. By the discussion after Definition 3.2.1, $a_0(X) = a_0(0) + \frac{uX}{2}$ and the right side of the above equation becomes

$$\begin{aligned} D(X) &= \left(2\left(a_0(0) + \frac{uX}{2}\right) - 1\right)^2 + 4(m_0 + X)v - 4(-1)^\ell w^2 \\ &= (2a_0(0) - 1 + uX)^2 + 4vX + 4m_0 v - 4(-1)^\ell w^2 \\ &= (2a_0(0) - 1)^2 + 2\,(2a_0(0) - 1)\,(uX) + (uX)^2 + 4vX + 4m_0 v - 4(-1)^\ell w^2 \\ &= (uX)^2 + 2\,(2a_0(0) - 1)\,(uX) + 4vX + (2a_0(0) - 1)^2 + 4m_0 v - 4(-1)^\ell w^2 \\ &= (2u)^2 \left(\frac{X}{2}\right)^2 + 4\,(2\,(a_0(0)u + v) - u)\left(\frac{X}{2}\right) + D(0). \end{aligned}$$

Note that the continued fraction expansion of $\sqrt{D(0)}$ has period length ℓ and the numerator and the denominator of its $(\ell - 1)$-th convergent are $A_{\ell-1} = a_0(0)u + v$ and $B_{\ell-1} = u$, respectively. If we write $G_{\ell-1} = 2A_{\ell-1} - B_{\ell-1}$, then

$$D(X) = (2B_{\ell-1})^2 \left(\frac{X}{2}\right)^2 + 4G_{\ell-1}\left(\frac{X}{2}\right) + D(0).$$

Similarly, we can prove the remaining three cases. $\square$

Theorem 3.2.2 *Let $D(m) \in \mathcal{F}_\psi$ for some m, $G_i = \tau A_i - (\tau - 1)B_i$ and A_i/B_i be the i-th convergent of $(\tau - 1 + \sqrt{D(m)})/\tau$, where $\tau = 1$ or 2 depending on the types of $\mathcal{F}_\psi$. Also, let $X \geq 0$. If $\mathcal{F}_\psi$ is of E-type, then*

$$D(X+m) = (\tau B_{\ell-1})^2 \left(\frac{X}{2}\right)^2 + 2\tau G_{\ell-1}\left(\frac{X}{2}\right) + D(m); \tag{3.13}$$

whereas when $\mathcal{F}_\psi$ is O-type,

$$D(X+m) = (\tau B_{\ell-1})^2 X^2 + 2\tau G_{\ell-1} X + D(m). \tag{3.14}$$

Proof: A similar proof to that of Theorem 3.2.1 will secure the result. □

When we multiply two consecutive members of an E_1 type family, $\mathcal{F}_\psi$, regardless of the length of ψ, the product is of R-D type and the continued fraction expansion of the square root of the product has period length 2. Also, if we take any $D(X) \in \mathcal{F}_\psi$ and multiply it with $D(X+2) \in \mathcal{F}_\psi$ or $D(X+4) \in \mathcal{F}_\psi$, then the product is also of R-D type.

Lemma 3.2.1 *Let $\psi = a_1, a_2, \ldots, a_2, a_1$ be a palindromic string of $\ell - 1$ natural numbers with the corresponding matrix product*

$$\prod_{j=1}^{\ell-1} \begin{pmatrix} a_j & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} u & v \\ v & w \end{pmatrix}.$$

Then if u and w are even and v is odd, then $\ell - 1$ is odd.

Proof: Suppose that $\ell - 1$ is even. Then we may write

$$\begin{pmatrix} u & v \\ v & w \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for some natural numbers a, b, c and d. It follows that $u = a^2 + c^2$, $v = ab + cd$ and $w = b^2 + d^2$. By the assumption that u is even, we have $a^2 + c^2 \equiv 0 \bmod 2$. This implies that $a \equiv c \bmod 2$. Since v is odd, we have $ab + cd \equiv 1 \bmod 2$, which can be written as $a(b+d) \equiv 1 \bmod 2$. This forces

a and $b+d$ to be odd. Since $b+d$ is odd, $(b+d)^2$ is odd. This means that $w = b^2 + d^2$ is odd, a contradiction to the assumption that w is even. Therefore, $\ell - 1$ is odd. □

Theorem 3.2.3 *Suppose that $\mathcal{F}_\psi$ is of type E_1 and $D(X), D(X+m) \in \mathcal{F}_\psi$ where $m = 1$, 2 or 4. Let $A_{\ell-1}/B_{\ell-1}$ be the $(\ell-1)$-th convergent of $\sqrt{D(X)}$. Then the product $D(X)D(X+m)$ is of R-D type. More explicitly, we have*

$$\sqrt{D(X)D(X+1)} = \left(\frac{2D(X) + A_{\ell-1} - 1}{2}, \overline{2, 2D(X) + A_{\ell-1} - 1}\right)$$

and

$$\sqrt{D(X)D(X+2)} = \left(D(X) + A_{\ell-1} - 1, \overline{1, 2\left(D(X) + A_{\ell-1} - 1\right)}\right).$$

Moreover, when $D(X)$ is even,

$$\sqrt{D(X)D(X+4)} = \left(D(X) + 2A_{\ell-1} - 1, \overline{1, \frac{D(X) + 2A_{\ell-1} - 4}{2}, 1, 2\left(D(X) + 2A_{\ell-1} - 1\right)}\right)$$

and when $D(X)$ is odd,

$$\sqrt{D(X)D(X+4)} = \left(D(X) + 2A_{\ell-1} - 1, \overline{1, \frac{D(X) + 2A_{\ell-1} - 3}{2}, 2, \frac{D(X) + 2A_{\ell-1} - 3}{2}, 1, 2\left(D(X) + 2A_{\ell-1} - 1\right)}\right).$$

Proof: By Theorem 3.2.2,

$$D(X+m) = {B_{\ell-1}}^2 \left(\frac{m}{2}\right)^2 + 2A_{\ell-1}\left(\frac{m}{2}\right) + D(X)$$

where $A_{\ell-1}/B_{\ell-1}$ is the $(\ell-1)$-th convergent of $\sqrt{D(X)}$, which has period length ℓ. Thus,

$$\begin{aligned} D(X)D(X+m) &= D(X)\left({B_{\ell-1}}^2 \left(\frac{m}{2}\right)^2 + 2A_{\ell-1}\left(\frac{m}{2}\right) + D(X)\right) \\ &= (D(X))^2 + 2A_{\ell-1}D(X)\left(\frac{m}{2}\right) + {B_{\ell-1}}^2 D(X)\left(\frac{m}{2}\right)^2. \end{aligned}$$

Since ${A_{\ell-1}}^2 - {B_{\ell-1}}^2 D(X) = (-1)^\ell$, ${B_{\ell-1}}^2 D(X) = {A_{\ell-1}}^2 - (-1)^\ell$. Since $\mathcal{F}_\psi$ is of type E_1, by Lemma 3.2.1, we have $\ell - 1$ odd. Thus, ${B_{\ell-1}}^2 D(X) = {A_{\ell-1}}^2 - 1$ and

$$\begin{aligned} D(X)D(X+m) &= (D(X))^2 + 2A_{\ell-1}D(X)\left(\frac{m}{2}\right) + ({A_{\ell-1}}^2 - 1)\left(\frac{m}{2}\right)^2 && (3.15) \\ &= \left(D(X) + A_{\ell-1}\left(\frac{m}{2}\right)\right)^2 - \left(\frac{m}{2}\right)^2. && (3.16) \end{aligned}$$

Clearly, when $m = 2$ or 4, we have $(m/2) \mid 4\,(D(X) + A_{\ell-1}\,(m/2))$. Hence, $D(X)D(X+m)$ is of R-D type. If $m = 1$, then

$$D(X)D(X{+}1) = \left(D(X) + A_{\ell-1}\left(\frac{1}{2}\right)\right)^2 - \left(\frac{1}{2}\right)^2 = \left(\frac{2D(X) + A_{\ell-1} - 1}{2}\right)\left(\frac{2D(X) + A_{\ell-1} + 1}{2}\right).$$

This product can be written as

$$\left(\frac{2D(X) + A_{\ell-1} - 1}{2}\right)^2 + \left(\frac{2D(X) + A_{\ell-1} - 1}{2}\right).$$

Since $A_{\ell-1} \pm 1 = a_0(X)u + v \pm 1$ is even, for u is even by assumption and v is odd since it is co-prime to u, $(2D(X) + A_{\ell-1} \pm 1)/2$ is an integer. Hence, the product $D(X)D(X+1)$ is of R-D type and by Theorem 2.1.1,

$$\sqrt{D(X)D(X+1)} = \left(\frac{2D(X) + A_{\ell-1} - 1}{2},\ \overline{2,\ 2D(X) + A_{\ell-1} - 1}\right).$$

If $m = 2$, then (3.16) becomes $D(X)D(X+2) = (D(X) + A_{\ell-1})^2 - 1$ and by Theorem 2.1.1,

$$\sqrt{D(X)D(X+2)} = \left(D(X) + A_{\ell-1} - 1,\ \overline{1,\ 2\,(D(X) + A_{\ell-1} - 1)}\right).$$

If $m = 4$, then (3.16) becomes $D(X)D(X+4) = (D(X) + 2A_{\ell-1})^2 - 4$. When $D(X)$ is even, by part (4) of Theorem 2.1.1, we get

$$\sqrt{D(X)D(X+4)} = \left(D(X) + 2A_{\ell-1} - 1,\ \overline{1,\ \frac{D(X) + 2A_{\ell-1} - 4}{2},\ 1,\ 2\,(D(X) + 2A_{\ell-1} - 1)}\right).$$

When $D(X)$ is odd, by part (6) of Theorem 2.1.1, we get

$$\sqrt{D(X)D(X+4)} = \left(D(X) + 2A_{\ell-1} - 1,\ \overline{1,\ \frac{D(X) + 2A_{\ell-1} - 3}{2},\ 2,\ \frac{D(X) + 2A_{\ell-1} - 3}{2},\ 1,\ 2\,(D(X) + 2A_{\ell-1} - 1)}\right).$$

$\square$

Example 3.2.5 Recall from Example 3.2.1 that when $\psi = 1, 2, 2, 2, 1$, $\mathcal{F}_\psi$ contains $D(0) = 45$, $D(1) = 350$, $D(2) = 943$, $D(3) = 1824$, $D(4) = 2993$, $D(5) = 4450$, $D(6) = 6195$, $D(7) = 8228$. We present the continued fraction expansion of $\sqrt{D(X)D(X+m)}$, where $m = 1$ or 2 in the following table.

m	$D(X)D(X+m)$	C. F. of $\sqrt{D(X)D(X+m)}$
1	$D(0)D(1)$	$(125, \overline{2, 250})$
	$D(1)D(2)$	$(574, \overline{2, 1128})$
	....	....
	$D(X)D(X+1)$	$(144X^2 + 305X + 125, \overline{2,\ 288X^2 + 610X + 250})$
2	$D(0)D(2)$	$(205, \overline{1, 410})$
	$D(1)D(3)$	$(798, \overline{1, 1596})$
	....	....
	$D(X)D(X+2)$	$(144X^2 + 449X + 205, \overline{2,\ 288X^2 + 898X + 410})$

Table 3.4: C. F. of $\sqrt{D(X)D(X+m)}$ when $\psi = 1, 2, 2, 2, 1$ and $m = 1$ or 2

When $m = 4$, we note that $D(X)$ is even if and only if X is odd. For the case where $D(X)$ is even, i.e, X is odd, we have

m	$D(X)$	$D(X)D(X+m)$	C. F. of $\sqrt{D(X)D(X+m)}$
4	even	$D(1)D(5)$	$(1247, \overline{1, 622, 1, 2494})$
		$D(3)D(7)$	$(3873, \overline{1, 1935, 1, 7746})$
		....	....
		$D(X)D(X+4)$	$(144X^2 + 737X + 366, \overline{1,\ (144X^2 + 737X + 363)/2,}$ $\overline{1, 288X^2 + 1474X + 732})$

Table 3.5: C. F. of $\sqrt{D(X)D(X+m)}$ when $\psi = 1, 2, 2, 2, 1$, $m = 4$ and $D(X)$ is even.

For the case where $D(X)$ is odd, i.e., X is even, we have

m	$D(X)$	$D(X)D(X+m)$	C. F. of $\sqrt{D(X)D(X+m)}$
4	odd	$D(0)D(4)$	$(366, \overline{1, 182, 2, 182, 1, 732})$
		$D(2)D(6)$	$(2416, \overline{1, 1207, 2, 1207, 1, 4832})$
		....	....
		$D(X)D(X+4)$	$(144X^2+737X+366, \overline{1,\ (144X^2+737X+364)/2,\ 2,\ (144X^2+737X+364)/2,\ 1,\ 288X^2+1474X+732})$

Table 3.6: C. F. of $\sqrt{D(X)D(X+m)}$ when $\psi = 1, 2, 2, 2, 1$, $m = 4$ and $D(X)$ is odd.

□

It is clear that when $m = 1$, 2 or 4, the $\mathrm{lp}(\sqrt{D(X)D(X+m)})$ is independent of the length of the palindromic string ψ and is bounded as X varies. In what follows, we show that if m is different from 1, 2 or 4, then $\mathrm{lp}(\sqrt{D(X)D(X+m)})$ is unbounded as X varies.

Theorem 3.2.4 *If $\mathcal{F}_\psi$ is of type E_1 and $D(X)$, $D(X+m) \in \mathcal{F}_\psi$, then the continued fraction expansion of $\sqrt{D(X)D(X+m)}$ has bounded period length for all non-negative integers X if and only if $m = 1$, 2 or 4.*

Proof: Recall from (3.15) that

$$D(X)D(X+m) = (D(X))^2 + 2A_{\ell-1}D(X)\left(\frac{m}{2}\right) + ({A_{\ell-1}}^2 - 1)\left(\frac{m}{2}\right)^2, \tag{3.17}$$

where $A_{\ell-1}/B_{\ell-1}$ is the $(\ell-1)$-th convergent of $\sqrt{D(X)}$. This is a quadratic function of $D(X)$. By Theorem 2.3.4, we have

$$\limsup_{X\to\infty} \mathrm{lp}\left(\sqrt{D(X)D(X+m)}\right) < \infty$$

if and only if $\Delta \mid 4\gcd\left(1, A_{\ell-1}(m/2)\right)^2$, where

$$\Delta = \left(A_{\ell-1}\left(\frac{m}{2}\right)\right)^2 - ({A_{\ell-1}}^2 - 1)\left(\frac{m}{2}\right)^2 = \left(\frac{m}{2}\right)^2 \text{ and } 4\gcd\left(1, A_{\ell-1}\left(\frac{m}{2}\right)\right)^2 = 4.$$

It follows that $m = 1$, 2 or 4.

If $(m/2)^2 \nmid 4$, then by (2.8), $(m/2)^2$ does not divide $4\left(D(X) + A_{\ell-1}\left(\frac{m}{2}\right)\right)^2 = (2D(X) + A_{\ell-1}m)^2$. If follows that $m^2 \nmid 4\left(2D(X) + A_{\ell-1}m\right)^2$ and, hence, $m^2 \nmid \left(2D(X) + A_{\ell-1}m\right)^2$. Write

$$\sqrt{(D(X))^2 + 2A_{\ell-1}\left(\frac{m}{2}\right)D(X) + ({A_{\ell-1}}^2 - 1)\left(\frac{m}{2}\right)^2} = \frac{1}{2}\sqrt{(2D(X) + A_{\ell-1}m)^2 - m^2}.$$

We are interested in knowing whether $\operatorname{lp}\left(\sqrt{D(X)D(X+m)}\right)$ is bounded. Theorem 2.3.2 says that if $\operatorname{lp}\left(\sqrt{D(X)D(X+m)}\right)$ is bounded, then the continued fraction expansion of any linear fractional transformation of $\sqrt{D(X)D(X+m)}$ with non-zero integer determinant also has bounded period length. Thus, it suffices to consider $\operatorname{lp}\left(\sqrt{(2D(X) + A_{\ell-1}m)^2 - m^2}\right)$. Since $m^2 \nmid (2D(X) + A_{\ell-1}m)^2$, Theorem 2.3.3 implies that

$$\lim_{X\to\infty} \operatorname{lp}\left(\sqrt{(2D(X) + A_{\ell-1}m)^2 - m^2}\right) = \infty.$$

Hence, by the contrapositive of Theorem 2.3.2,

$$\lim_{X\to\infty} \operatorname{lp}\left(\sqrt{D(X)D(X+m)}\right) = \infty.$$

$\square$

Using the same approach as above, we formulate a similar theorem for the O_1 type families as follows.

Theorem 3.2.5 *If $\mathcal{F}_\psi$ is type O_1 and $D(X), D(X+m) \in \mathcal{F}_\psi$, then $\operatorname{lp}\left(\sqrt{D(X)D(X+m)}\right)$ is bounded for all non-negative integers X if and only if $m = 1$ or 2.*

Proof: The proof for this theorem is similar to that of Theorem 3.2.4. We replace $m/2$ by m and obtain

$$\begin{aligned} D(X)D(X+m) &= (D(X))^2 + 2A_{\ell-1}mD(X) + \left({A_{\ell-1}}^2 - (-1)^\ell\right)m^2 && (3.18)\\ &= (D(X) + A_{\ell-1}m)^2 - (-1)^\ell m^2. && (3.19)\end{aligned}$$

The discriminant of (3.18) is $(A_{\ell-1}m)^2 - \left(A_{\ell-1}{}^2 - (-1)^\ell\right) m^2 = (-1)^\ell m^2$ and $4\gcd\left(1, A_{\ell-1}m\right)^2 = 4$. Thus, $(-1)^\ell m^2 \mid 4$ if and only if $m = 1$ or 2. If m^2 does not divide 4, then we can use a similar argument to the proof of Theorem 3.2.4 and establish that $\operatorname{lp}\left(\sqrt{D(X)D(X+m)}\right)$ is bounded for all non-negative integers X just in case m is 1 or 2. □

Theorem 3.2.6 *Suppose that $\mathcal{F}_\psi$ is type O_1 and $D(X)$, $D(X+m) \in \mathcal{F}_\psi$ where X is any non-negative integer and $m = 1$ or 2. Then $D(X)D(X+m)$ is of R-D type. Let $A_{\ell-1}/B_{\ell-1}$ be the $(\ell-1)$-th convergent of $\sqrt{D(X)}$. If $m = 1$, then*

$$\sqrt{D(X)D(X+1)} = \begin{cases} \left(D(X) + A_{\ell-1}, \overline{2\left(D(X) + A_{\ell-1}\right)}\right) & \text{if } \ell - 1 \text{ is even,} \\ \left(D(X) + A_{\ell-1} - 1, \overline{1, 2(D(X) + A_{\ell-1} - 1)}\right) & \text{if } \ell - 1 \text{ is odd.} \end{cases}$$

If $m = 2$, $\ell - 1$ is even and $D(X)$ is even, then

$$\sqrt{D(X)D(X+2)} = \left(D(X) + 2A_{\ell-1}, \overline{\frac{D(X) + 2A_{\ell-1}}{2}, 2(D(X) + 2A_{\ell-1})}\right).$$

If $m = 2$, $\ell - 1$ is even and $D(X)$ is odd, then

$$\sqrt{D(X)D(X+2)} = \left(D(X) + 2A_{\ell-1}, \overline{\frac{D(X) + 2A_{\ell-1} - 1}{2}, 1, 1, \frac{D(X) + 2A_{\ell-1} - 1}{2}, 2(D(X) + 2A_{\ell-1})}\right).$$

If $m = 2$, $\ell - 1$ is odd and $D(X)$ is even, then

$$\sqrt{D(X)D(X+2)} = \left(D(X) + 2A_{\ell-1} - 1, \overline{1, \frac{D(X) + 2A_{\ell-1} - 4}{2}, 1, 2\left(D(X) + 2A_{\ell-1} - 1\right)}\right).$$

If $m = 2$, $\ell - 1$ is odd and $D(X)$ is odd, then

$$\sqrt{D(X)D(X+2)} = \left(D(X) + 2A_{\ell-1} - 1, \overline{1, \frac{D(X) + 2A_{\ell-1} - 3}{2}, 2, \frac{D(X) + 2A_{\ell-1} - 3}{2}, 1, 2\left(D(X) + 2A_{\ell-1} - 1\right)}\right).$$

Proof: The proof is similar to that of Therorem 3.2.3. □

By Theorems 3.2.3 and 3.2.6, we see that when given two quadratics, it is possible to have a very short expansion for the square root of the product. We looked at some specific quadratics constructed earlier in this chapter. In the theorem below, we generalize Theorems 3.2.3 and 3.2.6.

Theorem 3.2.7 *Let $a, b, c, d, m \in \mathbb{Z}$ and $D(X) = (aX + c)(bX + d)$ with an integer variable X. Then $\mathrm{lp}(\sqrt{D(X)D(X+m)})$ is bounded if and only if $m(ad - bc) \mid 4$. Moreover, on writing $Y = abX^2 + (abm + ad + bc)X + bcm + cd$, we get $D(X)D(X+m) = Y(Y + m(ad - bc))$. Also,*

$$\sqrt{Y(Y+1)} = (Y, \overline{2,\ 2Y}), \qquad \sqrt{Y(Y+2)} = (Y, \overline{1,\ 2Y}),$$

$$\sqrt{Y(Y+4)} = \begin{cases} (Y+1, \overline{1,\ (Y-2)/2,\ 1, 2(Y+1)}) & \text{if } Y \text{ is even,} \\ (Y+1, \overline{1,\ (Y-1)/2,\ 2, (Y+1)/2,\ 1,\ 2(Y+1)}) & \text{if } Y \text{ is odd.} \end{cases}$$

<u>Proof</u>: If $D(X) = (aX + c)(bX + d)$, then $D(X+m) = (aX + am + c)(bX + bm + d)$ and

$$\begin{aligned} D(X)D(X+m) &= (aX+c)(bX+d)(aX+am+c)(bX+bm+d) \\ &= (aX+c)(bX+bm+d)(bX+d)(aX+am+c) \\ &= (abX^2 + (abm+ad+bc)X + bcm + cd)(abX^2 + (abm+ad+bc)X + adm + cd). \end{aligned}$$

Write $Y = abX^2 + (abm + ad + bc)X + bcm + cd$. Then

$$D(X)D(X+m) = Y(Y + m(ad-bc)) = Y^2 + m(ad-bc)Y.$$

Now, $\mathrm{lp}(\sqrt{Y^2 + m(ad-bc)Y})$ is bounded above as Y varies if and only if $Y^2 + m(ad-bc)Y$ satisfies the Schinzel condition, i.e. $m^2(ad-bc)^2/4$ divides $4\gcd(1, m(ad-bc))^2 = 4$. This implies that $m(ad-bc) \mid 4$.

Moreover, when $m(ad-bc) = 1$ or 2, the continued fraction expansion of $\sqrt{D(X)D(X+m)}$ is given by part (1) of Theorem 2.1.1. When $m(ad-bc) = 4$, we have $D(X)D(X+4) = Y^2 + 4Y = (Y+2)^2 - 4$. If Y is even, by part (4) of Theorem 2.1.1, we have

$$(Y+2)^2 - 4 = \left(Y+1,\ \overline{1,\ \frac{Y-2}{2},\ 1, 2(Y+1)}\right).$$

If Y is odd, then by part (6) of Theorem 2.1.1, we have

$$(Y+2)^2 - 4 = \left(Y+1,\ \overline{1,\ \frac{Y-1}{2},\ 2, \frac{Y+1}{2},\ 1,\ 2(Y+1)}\right).$$

□

Since $m(ad - bc) \mid 4$, it is not difficult to see that there are six possible cases in the above theorem. We illustrate them in the following.

Example 3.2.6

Case (1): $m = 1$ and $ad - bc = 1$. Let $a = 3$, $c = 2$ and $b = 1 = d$. Then $D(X) = (3X+2)(X+1)$, $Y = 3X^2 + 8X + 4$ and $D(X)D(X+1) = Y(Y+1)$. Thus, $D(1)D(2) = 10 \cdot 24 = 15 \cdot 17 =$ and $\sqrt{D(1)D(2)} = (15, \overline{1, 30})$. Similarly, $D(2)D(3) = 24 \cdot 44 = 32 \cdot 33$ and $\sqrt{D(2)D(3)} = (32, \overline{2, 64})$.

Case (2): $m = 1$ and $ad - bc = 2$. Choose $a = 4$, $c = 2$ and $b = 1 = d$. Then $D(X) = (4X + 2)(X+1)$, $Y = 4X^2 + 10X + 4$ and $D(X)D(X+1) = Y(Y+2)$. Thus, $D(1)D(2) = 12 \cdot 30 = 18 \cdot 20$ and $\sqrt{D(1)D(2)} = (18, \overline{1, 36})$. Similarly, $D(2)D(3) = 30 \cdot 56 = 40 \cdot 42$, $\sqrt{D(2)D(3)} = (40, \overline{1, 80})$.

Case (3): $m = 1$ and $ad - bc = 4$. Choose $a = 6$, $c = 2$ and $b = 1 = d$. Then $D(X) = (6X+2)(X+1)$, $Y = 6X^2 + 14X + 4$ and $D(X)D(X+1) = Y(Y+4)$. Since Y is even, we have $D(1)D(2) = 16 \cdot 42 = 24 \cdot 28$ and $\sqrt{D(1)D(2)} = (25, \overline{1, 11, 1, 50})$ and $D(2)D(3) = 42 \cdot 80 = 56 \cdot 60$ and $\sqrt{D(2)D(3)} = (57, \overline{1, 27, 1, 114})$.

Case (4): $m = 2$ and $ad - bc = 1$. Let $a = 3$, $c = 2$ and $b = 1 = d$. Then $D(X) = (3X+2)(X+1)$, $Y = 3X^2 + 11X + 6$ and $D(X)D(X+2) = Y(Y+2)$. Thus, $D(1)D(3) = 10 \cdot 44 = 20 \cdot 22$ and $\sqrt{D(1)D(3)} = (20, \overline{1, 40})$. Similarly, $D(2)D(4) = 24 \cdot 70 = 40 \cdot 42$ and $\sqrt{D(2)D(4)} = (40, \overline{1, 80})$.

Case (5): $m = 2$ and $ad - bc = 2$. Let $a = 4$, $c = 2$ and $b = 1 = d$. Then $D(X) = (4X+2)(X+1)$, $Y = 4X^2 + 14X + 6$ and $D(X)D(X+2) = Y(Y+4)$. Note that Y is even. So, $D(1)D(3) = 12 \cdot 56 = 24 \cdot 28$ and $\sqrt{D(1)D(3)} = (25, \overline{1, 11, 1, 50})$. Similarly, $D(2)D(4) = 30 \cdot 90 = 50 \cdot 54$ and $\sqrt{D(2)D(4)} = (51, \overline{1, 24, 1, 102})$.

Case (6): $m = 4$ and $ad - bc = 1$. Let $a = 2$, $c = 1$ and $b = 1 = d$. Then $D(X) = (2X+1)(X+1)$, $Y = 2X^2 + 11X + 5$ and $D(X)D(X+4) = Y(Y+4)$.

When Y is even, we have $D(1)D(5) = 6 \cdot 66 = 18 \cdot 22$ and $\sqrt{D(1)D(5)} = (19, \overline{1, 8, 1, 38})$.

Similarly, $D(3)D(7) = 28 \cdot 120 = 56 \cdot 60$ and $\sqrt{D(3)D(7)} = (57, \overline{1, 27, 1, 114})$.

When Y is odd, we have $D(2)D(6) = 15 \cdot 91 = 35 \cdot 39$ and $\sqrt{D(2)D(6)} = (36, \overline{1, 17, 1, 17, 1, 72})$. Similarly, $D(4)D(8) = 45 \cdot 153 = 81 \cdot 85$ and $\sqrt{D(4)D(8)} = (82, \overline{1, 40, 1, 40, 1, 164})$. $\square$

Chapter 4

The Continued Fraction Expansion of $\sqrt{D(X)}$

In this chapter, we present the major result of the thesis: the continued fraction expansion of $\sqrt{D(X)}$ for sufficiently large X, where

$$D(X) = A^2X^2 + 2BX + C \tag{4.1}$$

is a Schinzel sleeper, i.e., its discriminant $B^2 - A^2C$ divides $4\gcd(A^2, B)^2$. This result contributes to two mathematical disciplines. First, in the theory of continued fractions, we determine the actual continued fraction period of $\sqrt{D(X)}$. Indeed, knowing the continued fraction period of $\sqrt{D(X)}$ allows us to establish an upper bound for its length in the next chapter. Second, in the theory of real quadratic orders, we can easily compute the fundamental unit of the real quadratic order $[1, \sqrt{D(X)}]$ via the continued fraction expansion of $\sqrt{D(X)}$; we provide a simple formula for the fundamental unit of $[1, \sqrt{D(X)}]$ in the next chapter.

Our work here generalizes the results of van der Poorten and Williams [206] discussed in Section 2.3. They gave the continued fraction expansion of $\sqrt{D(X)}$ for sufficiently large X with the additional assumption that $\gcd(A^2, 2B, C)$ is squarefree. Here, we drop the restriction of squarefree $\gcd(A^2, 2B, C)$. Also, we will provide a lower bound for the sufficient size of X.

There are three sections in this chapter. In Section 4.1, we demonstrate the motivation for our approach and establish a crucial lemma for the work in the ensuing section. We will prove our main result by induction. There are several key components in our proof and they are presented as lemmas in Section 4.2. In Section 4.3, we combine all the lemmas in Section 4.2 and inductively construct the continued fraction expansion of $\sqrt{D(X)}$ in Theorem 4.3.1.

4.1 Embedding an Integer Sequence in a Continued Fraction

We start off this section with an observation. Consider $D(X) = 9X(X+2) = 3^2X^2 + 2(9)X$. The discriminant Δ of $D(X)$ is $9^2 = 81$, which divides $4\gcd(3^2, 9)^2 = 4 \cdot 9^2$. Hence, $D(X)$ is a Schinzel sleeper. We compute

$$\sqrt{D(1)} = \sqrt{27} = (5, \overline{10}), \qquad \sqrt{D(2)} = \sqrt{72} = (8, \overline{2, 16}),$$

$$\sqrt{D(3)} = \sqrt{135} = (11, \overline{1, 1, 1, 1, 1, 1, 1, 22}), \quad \sqrt{D(4)} = \sqrt{216} = (14, \overline{1, 2, 3, 2, 1, 28}),$$

$$\sqrt{D(5)} = \sqrt{315} = (17, \overline{1, 2, 1, 34}), \quad \sqrt{D(6)} = \sqrt{432} = (20, \overline{1, 3, 1, 1, 1, 3, 1, 40}),$$

$$\sqrt{D(7)} = \sqrt{567} = (23, \overline{1, 4, 3, 4, 1, 46}), \quad \sqrt{D(8)} = \sqrt{720} = (26, \overline{1, 4, 1, 52}),$$

$$\sqrt{D(9)} = \sqrt{891} = (29, \overline{1, 5, 1, 1, 1, 5, 1, 58}), \quad \sqrt{D(10)} = \sqrt{1080} = (32, \overline{1, 6, 3, 6, 1, 64}),$$

and

$$\sqrt{D(11)} = \sqrt{1287} = (35, \overline{1, 6, 1, 70}).$$

Notice that the continued fraction expansions

$$\sqrt{D(3)} = (11, \overline{1, 1, 1, 1, 1, 1, 1, 22}), \quad \sqrt{D(6)} = (20, \overline{1, 3, 1, 1, 1, 3, 1, 40})$$

and

$$\sqrt{D(9)} = (29, \overline{1, 5, 1, 1, 1, 5, 1, 58})$$

have the same period length. Also, the partial quotients in their periods exhibit a surprising pattern. It can be checked that if $X \equiv 0 \bmod 3$ and $X > 2$, then

$$\sqrt{D(X)} = (3X+2, \overline{1,\ (2X-3)/3,\ 1,\ 1,\ 1,\ (2X-3)/3,\ 1,\ 6X+4}).$$

A similar phenomenon can be seen with the continued fraction expansions

$$\sqrt{D(4)} = (14, \overline{1, 2, 3, 2, 1, 28}), \quad \sqrt{D(7)} = (23, \overline{1, 4, 3, 4, 1, 46}), \quad \sqrt{D(10)} = (32, \overline{1, 6, 3, 6, 1, 64}).$$

More generally, when $X \equiv 1 \bmod 3$ and $X > 2$, we get

$$\sqrt{D(X)} = (3X+2, \overline{1,\ 2(X-1)/3,\ 3,\ 2(X-1)/3,\ 1,\ 6X+4}).$$

For the case where $X \equiv 2 \bmod 3$ and $X > 2$, we have

$$\sqrt{D(5)} = (17, \overline{1,2,1,34}),\quad \sqrt{D(8)} = (26, \overline{1,4,1,52}),\quad \sqrt{D(11)} = (35, \overline{1,6,1,70}),$$

and, in general,

$$\sqrt{D(X)} = (3X+2, \overline{1,\ 2(X-2)/3,\ 1,\ 6X+4}).$$

Consider the quadratic $D(X) = 119^2X^2 + 2(2205)X + 343$. The discriminant of this quadratic is $\Delta = 2205^2 - (119)^2(343) = 4802 = 2 \cdot 7^4$. Since $4\gcd(119^2, 2205)^2 = 2^2 7^4$, we see that $D(X)$ is a Schinzel sleeper.

We find that

$$\sqrt{D(1)} = (137, \overline{1,1,8,2,1,2,4,1,1,1,2,5,4,3,1,136,1,3,4,5,2,1,1,1,4,2,1,2,8,1,1,274}),$$

$$\sqrt{D(8)} = (970, \overline{1,1,8,19,1,2,4,1,1,1,2,39,4,3,1,969,1,3,4,39,2,1,1,1,4,2,1,19,8,1,1,1940}),$$

$$\sqrt{D(15)} = (1803, \overline{1,1,8,36,1,2,4,1,1,1,2,73,4,3,1,1802,1,3,4,73,2,1,1,1,4,2,1,36,8,1,1,3606}).$$

In general, when $X \in \mathbb{N}$ and $X \equiv 1 \bmod 7$, the continued fraction expansion of $\sqrt{D(X)}$ is given by

$$(q_0(X), \overline{1,1,8,q_1(X),1,2,4,1,1,1,2,q_2(X),4,3,1,q_3(X),1,3,4,q_4(X),2,1,1,1,4,2,1,q_5(X),8,1,1,2q_0(X)})$$

for some integer-valued functions $q_i(X)$, where $0 \le i \le 5$. Hence, for all $X \equiv 1 \bmod 7$, the continued fraction expansions of $\sqrt{D(X)}$ are alike with exceptions at several fixed positions. In fact, if we compute the continued fraction expansion of $\sqrt{D(X)}$ for X lying in other residue classes modulo 7, we will see a similar phenomenon.

The above illustrations offer clues to the investigation of the continued fraction expansion of $\sqrt{D(X)}$. On restricting X to be sufficiently large and to belong to some appropriate residue class, we expect the continued fraction expansion of $\sqrt{D(X)}$ to have the form

$$(q_0(X),\ \mathcal{S}_0,\ q_1(X),\ \mathcal{S}_1,\ q_2(X),\ldots,\ \mathcal{S}_{i-1},\ q_i(X),\ldots),$$

where $i \in \mathbb{N}$, $q_i(X)$ is a linear function and $\mathcal{S}_i = \{s_0, s_1, \ldots, s_{m-1}\}$ is an ordered set of natural numbers.

We will discuss the determination of the sufficient size of X and of the appropriate residue classes in the next section. In what follows, we establish a lemma concerning the sequence $s_0,\ s_1,\ \ldots,\ s_{m-1}$.

Suppose that we have a quadratic irrational $\theta = (P + \sqrt{D})/Q$ as defined in Definition 1.2.2, a finite sequence of integers, $s_0, s_1, \ldots, s_{m-1}$, and $A_i/B_i = \langle s_0, s_1, \ldots, s_i\rangle$ for $0 \le i \le m-1$. The $\langle\ \rangle$ notation denotes a formal continued fraction expansion.

If $\theta^* = (P^* + \sqrt{D})/Q^*$ is defined by

$$\theta = \langle s_0, s_1, \ldots, s_{m-1}, \theta^*\rangle,$$

then by (1.14), we have

$$\theta^* = -\frac{B_{m-2}\,\theta - A_{m-2}}{B_{m-1}\,\theta - A_{m-1}}.$$

Thus,

$$\frac{P^* + \sqrt{D}}{Q^*} = -\frac{B_{m-2}\left((P+\sqrt{D})/Q\right) - A_{m-2}}{B_{m-1}\left((P+\sqrt{D})/Q\right) - A_{m-1}} = -\frac{(B_{m-2}P - A_{m-2}Q) + B_{m-2}\sqrt{D}}{(B_{m-1}P - A_{m-1}Q) + B_{m-1}\sqrt{D}}.$$

Put $Q' = (D - P^2)/Q \in \mathbb{Z}$. If we rationalize the denominator and simplify the resulting expression using (1.10), then we get

$$\frac{P^* + \sqrt{D}}{Q^*} = \frac{(-1)^m\left(B_{m-1}B_{m-2}Q' - A_{m-1}A_{m-2}Q + (A_{m-1}B_{m-2} + A_{m-2}B_{m-1})P\right) + \sqrt{D}}{(-1)^m\left(A_{m-1}^2 Q - B_{m-1}^2 Q' - 2A_{m-1}B_{m-1}P\right)}.$$

On cross-multiplying and comparing the coefficients of 1 and $\sqrt{D}$, we see that

$$P^* = (-1)^m \left(B_{m-1}B_{m-2}Q' - A_{m-1}A_{m-2}Q + (A_{m-1}B_{m-2} + A_{m-2}B_{m-1})P\right) \tag{4.2}$$

and

$$Q^* = (-1)^m \left(A_{m-1}^2 Q - B_{m-1}^2 Q' - 2A_{m-1}B_{m-1}P\right). \tag{4.3}$$

If for any $d \in \mathbb{N}$, we put $F = dA_{m-1}$ and $E = dB_{m-1}$, then since $\gcd(A_{m-1}, B_{m-1}) = 1$, we get $d = \gcd(E, F)$ and (4.3) becomes

$$Q^* = \frac{(-1)^m}{d^2}\left(F^2Q - E^2Q' - 2EFP\right) \tag{4.4}$$

and (4.2) becomes

$$P^* = \frac{(-1)^m}{d}\left(B_{m-2}EQ' - A_{m-2}FQ + (B_{m-2}F + A_{m-2}E)P\right). \tag{4.5}$$

Lemma 4.1.1 *Let $\theta = (P_0 + \sqrt{D})/Q_0$ be a quadratic irrational and put*

$$a_0 = \left\lfloor \frac{P_0 + \sqrt{D}}{Q_0} \right\rfloor, \quad P_1 = a_0Q_0 - P_0, \quad Q_1 = (D - P_1^2)/Q_0.$$

Let $L^2D = M^2 - N$, where $L, M \in \mathbb{N}$ and $N \in \mathbb{Z}$ and put $F = LQ_0$, $E = M - LP_1$ and $d = \gcd(E, F)$. Let

$$\frac{F}{E} = (s_0, s_1, \ldots, s_{m-1}),$$

where m is chosen to be odd if $N > 0$ and even if $N < 0$. If $(P^* + \sqrt{D})/Q^*$ is defined by

$$\theta = \left\langle a_0, s_0, s_1, s_2, \ldots, s_{m-1}, \frac{P^* + \sqrt{D}}{Q^*} \right\rangle,$$

then

$$P^* = \frac{M}{L} - \frac{H|N|}{dL} \quad \text{and} \quad Q^* = \frac{|N|Q_0}{d^2},$$

where $H(E/d) \equiv (-1)^{m-1} \bmod (F/d)$ and $0 \le H < F/d$. Note that P^* and Q^* are integers by (4.2) and (4.3).

Proof: Let $(P_1+\sqrt{D})/Q_1$ be the first complete quotient of the regular continued fraction expansion of $(P_0+\sqrt{D})/Q_0$. Then the values of P^* and Q^* are defined by

$$\frac{P_1+\sqrt{D}}{Q_1} = \left\langle s_0, s_1, s_2, \ldots, s_{m-1}, \frac{P^*+\sqrt{D}}{Q^*} \right\rangle.$$

If we substitute Q_1 for Q, P_1 for P, $Q_0 = (D-P_1^2)/Q_1$ for Q', $M - LP_1$ for E and LQ_0 for F in (4.4), then we get

$$\begin{aligned}
Q^* &= \frac{(-1)^m}{d^2}\left((LQ_0)^2Q_1 - (M-LP_1)^2Q_0 - 2(M-LP_1)(LQ_0)P_1\right) \\
&= \frac{(-1)^m}{d^2}Q_0\left(L^2(D-P_1^2) - (M-LP_1)^2 - 2(LP_1)(M-LP_1)\right) \\
&= \frac{(-1)^m}{d^2}Q_0\left(L^2D - \left(L^2P_1^2 + 2(LP_1)(M-LP_1) + (M-LP_1)^2\right)\right) \\
&= \frac{(-1)^m}{d^2}Q_0\left(L^2D - (LP_1 + M - LP_1)^2\right) \\
&= \frac{(-1)^m}{d^2}Q_0(-N) \\
&= \frac{|N|Q_0}{d^2}.
\end{aligned}$$

Similarly, by (4.5)

$$\begin{aligned}
LP^* &= \frac{(-1)^m}{d}L\left(B_{m-2}(M-LP_1)Q_0 - A_{m-2}(LQ_0)Q_1 + (B_{m-2}(LQ_0) + A_{m-2}(M-LP_1))P_1\right) \\
&= \frac{(-1)^m}{d}\left(B_{m-2}LMQ_0 - A_{m-2}L(LQ_0)Q_1 + A_{m-2}(M-LP_1)LP_1\right) \\
&= \frac{(-1)^m}{d}\left(B_{m-2}LMQ_0 + A_{m-2}LMP_1 - A_{m-2}L^2(Q_0Q_1 + P_1^2)\right) \\
&= \frac{(-1)^m}{d}\left(B_{m-2}LMQ_0 + A_{m-2}LMP_1 - A_{m-2}M^2 + A_{m-2}N\right) \\
&= \frac{(-1)^m}{d}\left((B_{m-2}LQ_0 - A_{m-2}(M-LP_1))M + A_{m-2}N\right).
\end{aligned}$$

Since $LQ_0 = A_{m-1}d$ and $M - LP_1 = B_{m-1}d$, we have

$$B_{m-2}LQ_0 - A_{m-2}(M-LP_1) = (A_{m-1}B_{m-2} - A_{m-2}B_{m-1})d = (-1)^m d.$$

Thus,

$$LP^* = \frac{(-1)^m(-1)^m dM}{d} + \frac{(-1)^m A_{m-2}N}{d} = M - \frac{A_{m-2}|N|}{d}.$$

Hence,

$$P^* = \frac{M}{L} - \frac{A_{m-2}|N|}{dL}.$$

If we put $H = A_{m-2}$, then by (1.10), $H(E/d) \equiv (-1)^m \bmod (F/d)$ and $0 \leq H < F$ and

$$P^* = \frac{M}{L} - \frac{H|N|}{dL}.$$

□

Remark 4.1.1 *Note that if $F > E$, i.e. $s_0 \in \mathbb{N}$ in the above lemma, and $(P^* + \sqrt{D})/Q^* > 1$, then by Remark 1.1.2, the regular continued fraction expansion of θ is given by*

$$\left(a_0, s_0, s_1, \ldots, s_{m-1}, (P^* + \sqrt{D})/Q^*\right).$$

4.2 Preliminary Lemmas

From the examples early on in this chapter, it appears that if $D(X) = A^2X^2 + 2BX + C$ is a Schinzel sleeper and X is sufficiently large, then the continued fraction expansion of $\sqrt{D(X)}$ is of the form

$$(q_0(X), \overline{\mathcal{S}_0,\ q_1(X),\ \mathcal{S}_1,\ q_2(X),\ \ldots,\ \mathcal{S}_{\kappa-1},\ q_\kappa(X)})$$

for some integer-valued polynomials $q_i(X)$, some natural number sequences $\mathcal{S}_i$, and some minimal subscript κ. The minimal subscript κ is the least number of insertions of $\mathcal{S}_i$ in the expansion of $\sqrt{D(X)}$. We will prove this result by induction. The inductive argument is separated into several key components and presented as lemmas here.

Note that for any quadratic $D(X) = A^2X^2 + 2BX + C$, not necessarily a Schinzel sleeper, we may write

$$D(X) = \left(\frac{A^2X + B}{A}\right)^2 - \frac{\Delta}{A^2}.$$

Hence, if we write $q = \lfloor B/A \rfloor$, then for sufficiently large X, $\lfloor \sqrt{D(X)} \rfloor$ will be of the shape $AX + q$ or $AX + q - 1$, where q is independent of X.

Lemma 4.2.1 *Write $B = Aq + r$ with $0 \le r < A$. Then*

$$\left\lfloor\sqrt{D(X)}\right\rfloor = \begin{cases} AX+q-1 & \text{if } \Delta > 0,\ \ r=0 \ \text{ and } \ X > \dfrac{\Delta}{2A^3} - \dfrac{2B-A}{2A^2}, \\[2ex] AX+q & \text{if } \Delta > 0,\ \ r>0 \ \text{ and } \ X > \dfrac{\Delta}{2A^3} - \dfrac{2B-A}{2A^2}, \\[2ex] AX+q & \text{if } \Delta < 0,\ \ r=0 \ \text{ and } \ X > \dfrac{-\Delta}{2A^3} - \dfrac{2B+A}{2A^2}, \\[2ex] AX+q & \text{if } \Delta < 0,\ \ r>0 \ \text{ and } \ X > \dfrac{-\Delta}{2A^2(A-r)} - \dfrac{2B+(A-r)}{2A^2}. \end{cases} \tag{4.6}$$

Proof: **Case (1):** If $\Delta > 0$ and $r = 0$, then $C < B^2/A^2 = q^2$. To have $\lfloor\sqrt{D(X)}\rfloor = AX + q - 1$, we need

$$(AX+q-1)^2 < A^2X^2 + 2BX + C < (AX+q)^2.$$

The latter inequality holds since $C < B^2/A^2 = q^2$. The term $A^2X^2 + 2BX + C$ is strictly greater than $(AX + q - 1)^2$ provided

$$A^2X^2 + 2BX + C > A^2X^2 + 2AqX - 2AX + \left(\frac{B-A}{A}\right)^2;$$

that is, $2AX > (B - A)^2/A^2 - C$. This means that

$$X > \frac{B^2 - 2AB + A^2 - A^2C}{2A^3} = \frac{\Delta}{2A^3} - \frac{2B-A}{2A^2}.$$

Case (2): If $\Delta > 0$ and $r > 0$, then $B^2/A^2 > C$ and $AX + q = AX + B/A - r/A$ is an integer. Since $0 \le r/A < 1$, $AX + q$ lies strictly between $AX + (B - A)/A$ and $AX + B/A$, which are not integers, but differ by 1. To obtain $\lfloor\sqrt{D(X)}\rfloor = AX + q$, it suffices to have

$$A^2X^2 + 2(B-A)X + \left(\frac{B-A}{A}\right)^2 < A^2X^2 + 2BX + C < A^2X^2 + 2BX + \frac{B^2}{A^2}.$$

The right inequality holds since $B^2/A^2 > C$. The left inequality holds provided

$$2AX > \left(\frac{B-A}{A}\right)^2 - C = \frac{\Delta - 2AB + A^2}{A^2};$$

that is,

$$X > \frac{\Delta - 2AB + A^2}{2A^3} = \frac{\Delta}{2A^3} - \frac{2B - A}{2A^2}.$$

For the remaining cases, we note that $\lfloor\sqrt{D(X)}\rfloor$ is of the form $AX + q$ if

$$A^2X^2 + 2AqX + q^2 < A^2X^2 + 2BX + C < A^2X^2 + 2A(q+1)X + (q+1)^2. \tag{4.7}$$

Case (3): If $\Delta < 0$ and $r = 0$, then $q^2 = B^2/A^2 < C$ and the left inequality of (4.7) holds. Also the right inequality of (4.7) holds if

$$2AX > C - (q+1)^2 = C - \left(\frac{B+A}{A}\right)^2;$$

that is,

$$X > \frac{A^2C - B^2 - 2AB - A^2}{2A^3} = \frac{-\Delta - 2AB - A^2}{2A^3} = \frac{-\Delta}{2A^3} - \frac{2B+A}{2A^2}.$$

Case (4): If $\Delta < 0$ and $r > 0$, then $q^2 < B^2/A^2 < C$ and the left inequality of (4.7) holds. The right inequality of (4.7) holds provided

$$A^2X^2 + 2BX + C < A^2X^2 + 2(A+B-r)X + \left(\frac{A+B-r}{A}\right)^2;$$

that is,

$$2(A-r)X > C - \left(\frac{A+B-r}{A}\right)^2 = \frac{-\Delta - 2(A-r)B - (A-r)^2}{A^2}.$$

In other words,

$$X > \frac{-\Delta - 2(A-r)B - (A-r)^2}{2A^2(A-r)} = \frac{-\Delta}{2A^2(A-r)} - \frac{2B + (A-r)}{2A^2}.$$

□

In Definition 2.3.1 where we define a Schinzel sleeper to be a quadratic $D(X) = A^2X^2+2BX+C$ with discriminant $\Delta = B^2 - A^2C \neq 0$ and $\Delta \mid 4\gcd(A^2, B)^2$, we never impose any conditions on the integer coefficients A, B and C other than $A > 0$ and $B^2 - A^2C \neq 0$. In other words, we do not restrict the signs of B and C. If $X = -|X|$ is a negative integer, then $D(X) = A^2(-|X|)^2 + 2B(-|X|) + C = A^2X^2 - 2B|X| + C$. The discriminant of $A^2X^2 - 2B|X| + C$ is $(-B)^2 - A^2C = B^2 - A^2C = \Delta$. Since there is no restriction on the sign of B, we may assume X to be non-negative. Henceforth, we consider only $X \geq 0$.

In what follows, we set up the notation that will be used in the sequel. Put

$$\sigma = \Delta/|\Delta| = \mathrm{sgn}(\Delta) \quad \text{and} \quad |\Delta| = \Delta_1\Delta_2^2\Delta_4^4,$$

where Δ_1 and Δ_2 are squarefree.

Since $\Delta = \Delta_1\Delta_2^2\Delta_4^4$ divides $4\gcd(A^2, B)^2$, we have $\Delta_1\Delta_2\Delta_4^2 \mid 2\gcd(A^2, B)$. Thus,

$$\Delta_1\Delta_2\Delta_4^2 \mid 2A^2 \quad \text{and} \quad \Delta_1\Delta_2\Delta_4^2 \mid 2B. \tag{4.8}$$

It follows that $\Delta_4 \mid A$ and $\Delta_1\Delta_2\Delta_4 \mid 2A$. This implies that $\Delta_2\Delta_4 \mid 2A$. If $\Delta_2\Delta_4 \mid A$, then put $\tau = 1$; otherwise, put $\tau = 2$. Thus, $(\Delta_2\Delta_4/\tau) \mid A$.

If $2 \mid \Delta_1$, then $(\Delta_1/2)\Delta_2\Delta_4^2 \mid A^2$. This implies that $\Delta_2\Delta_4 \mid A$ and, consequently, $\tau = 1$. Thus, as a contrapositive, if $\tau \neq 1$, i.e., $\tau = 2$, then $2 \nmid \Delta_1$. Hence,

$$\gcd(\Delta_1, \tau) = 1. \tag{4.9}$$

If $\tau = 2$, then since $\Delta_2\Delta_4 \nmid A$, we get $\Delta_2 \nmid (A/\Delta_4)$. Also, since $\Delta_2\Delta_4 \mid 2A$, we get $\Delta_2 \mid 2(A/\Delta_4)$. Hence, $2 \mid \Delta_2$. Moreover, since τ is either 1 or 2, we have

$$\tau \mid \Delta_2. \tag{4.10}$$

Moreover, since $\Delta_1\Delta_2\Delta_4 \mid 2A$, it follows that

$$\Delta_1\Delta_4\tau \mid 2A. \tag{4.11}$$

Since $\Delta_1\Delta_2^2\Delta_4^4 = |\Delta| = |B^2 - A^2C|$, $(\Delta_2\Delta_4/\tau)^2 \mid A^2$ and $(\Delta_2\Delta_4/\tau)^2 \mid \Delta_1\Delta_2^2\Delta_4^4$, it follows that $(\Delta_2\Delta_4/\tau)^2 \mid B^2$, i.e., $(\Delta_2\Delta_4/\tau) \mid B$. Put $A^* = A/(\Delta_2\Delta_4/\tau)$ and $B^* = B/(\Delta_2\Delta_4/\tau)$. Then $\tau^2\Delta_1\Delta_2^2\Delta_4^4 = |(\Delta_2\Delta_4B^*)^2 - (\Delta_2\Delta_4A^*)^2C|$ implies that $\Delta_1(\tau\Delta_4)^2 = |B^{*2} - A^{*2}C|$.

Put $\Gamma = \gcd(A^*, \tau\Delta_4)$ and $G = \gcd(A^*, B^*)$. Then $G^2 \mid \tau^2\Delta_1\Delta_4^2$ implies that $G \mid \tau\Delta_4$ and $G \mid \Gamma$. On the other hand, $\Gamma^2 \mid B^{*2}$ implies that $\Gamma \mid B^*$, which implies that $\Gamma \mid G$. Therefore, $G = \Gamma$. It follows that $\gcd(A^*/\Gamma, B^*/\Gamma) = 1$. Hence,

$$\gcd(A, B) = \gcd\left(A^*\frac{\Delta_2\Delta_4}{\tau}, B^*\frac{\Delta_2\Delta_4}{\tau}\right) = \frac{\Delta_2\Delta_4}{\tau}\gcd\left(\frac{A^*}{\Gamma}\Gamma, \frac{B^*}{\Gamma}\Gamma\right) = \frac{\Gamma\Delta_2\Delta_4}{\tau}. \tag{4.12}$$

Note that $\tau = 2$ and $\tau \mid A^*$ imply that $2 \mid 2A/(\Delta_2\Delta_4)$. This means that $\Delta_2\Delta_4 \mid A$, a contradiction. Hence, $\gcd(A^*, \tau) = 1$ and $\Gamma = \gcd(A^*, \tau\Delta_4) = \gcd(A^*, \Delta_4)$. Put

$$A' = \frac{A^*}{\Gamma} \quad \text{and} \quad \Delta' = \frac{\tau\Delta_4}{\Gamma}. \tag{4.13}$$

Then,

$$\gcd(A', \Delta') = 1. \tag{4.14}$$

If $A \mid B$, then

$$A = \gcd(A, B) = \frac{\Gamma\Delta_2\Delta_4}{\tau} \quad \text{and} \quad A' = 1. \tag{4.15}$$

Define

$$\eta = \begin{cases} 1 & \text{if } A \mid B \text{ and } \sigma = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.16}$$

For integers $a > r \geq 0$, define an ordered set

$$\mathcal{S}(a, r) = \begin{cases} \emptyset & \text{if } r = 0 \text{ and } \sigma = -1, \\ \{1\} & \text{if } r = 0 \text{ and } \sigma = 1, \\ \{s_0, s_1, \ldots, s_{m-1}\} & \text{otherwise.} \end{cases} \tag{4.17}$$

In the last case, $a/r = (s_0, s_1, \ldots, s_{m-1})$ and the subscript $m - 1$ is chosen such that $(-1)^{m-1} = \sigma$.

As we mentioned earlier, our result on the continued fraction expansion of $\sqrt{D(X)}$ requires X to be sufficiently large. Since we have established some lower bounds for X in Lemma 4.2.1, it may seem plausible that if X is greater those lowers bounds, then X is sufficiently large. However, we find that a larger lower bound is needed for the main result. Henceforth, we say X is sufficiently large if X is a non-negative integer such that

$$X > \frac{\Delta_1 \Delta'^2}{A} - \frac{2B-A}{2A^2}. \tag{4.18}$$

In the sequel, we assume (4.18). We now present the initial step of our inductive proof.

Lemma 4.2.2 *Put* $r_0 \equiv B\tau/(\Gamma\Delta_2\Delta_4) \bmod A'$, *where* $0 \le r_0 < A'$, *and* $\mathcal{S}_0 = \mathcal{S}(A', r_0)$. *When* $r_0 > 0$, *let* $H \in \mathbb{N}$ *such that* $Hr_0 \equiv \sigma \bmod A'$ *and* $1 \le H < A'$. *Then* $d_1 = \gcd(A', r_0) = 1$ *and*

$$\sqrt{D(X)} = \left(AX + q - \eta,\, \mathcal{S}_0,\, \frac{\mathbf{P}_1 + \sqrt{D(X)}}{\mathbf{Q}_1} \right),$$

where $q = \lfloor B/A \rfloor$, η *as defined in* (4.16),

$$g_1 = \begin{cases} 0 & \text{if } r_0 = 0 \text{ and } \sigma = -1, \\ \Delta' & \text{if } r_0 = 0 \text{ and } \sigma = 1, \\ H\Delta' & \text{if } r_0 > 0, \end{cases} \tag{4.19}$$

and

$$\mathbf{P}_1 = AX + \frac{B}{A} - \Delta_1 \frac{g_1}{A'} \frac{\Delta'}{d_1} \quad \text{and} \quad \mathbf{Q}_1 = \Delta_1 \left(\frac{\Delta'}{d_1} \right)^2. \tag{4.20}$$

<u>Proof</u>: Note that $\Delta/A^2 = \Delta_1\Delta'^2/A'^2$ and $\Delta/(A(A-r)) = \Delta_1\Delta'^2/(A'(A'-r_0))$. Hence, it follows from $X > \Delta_1\Delta'^2/A - (2B-A)/(2A^2)$ that X is greater than all four lower bounds in (4.6). Hence, on writing $B = Aq + r$ with $0 \le r < A$ and by Lemma 4.2.1, we get $\lfloor \sqrt{D(X)} \rfloor = AX + q - \eta$, where η is defined in (4.16).

We put

$$P_0 = 0, \quad Q_0 = 1 \quad \text{and} \quad a_0 = \lfloor \sqrt{D(X)} \rfloor = AX + q - \eta. \tag{4.21}$$

Then

$$P_1 = a_0 Q_0 - P_0 = a_0 = AX + q - \eta \quad \text{and} \quad Q_1 = \frac{D(X) - P_1^2}{Q_0}. \tag{4.22}$$

Case (1): Suppose that $r = 0$ and $\sigma = -1$. Then $r_0 = 0$,

$$P_1 = AX + \frac{B}{A} \quad \text{and} \quad Q_1 = \frac{-\Delta}{A^2} = \frac{\Delta_1 \Delta_2^2 \Delta_4^4}{A'^2(\Gamma \Delta_2 \Delta_4/\tau)^2} = \Delta_1 \left(\frac{\Delta'}{A'}\right)^2.$$

Since $A \mid B$, by (4.15), we have $A' = 1$. On putting $d_1 = \gcd(A', r_0) = A' = 1$, we write

$$\mathbf{P}_1 = AX + \frac{B}{A} \quad \text{and} \quad \mathbf{Q}_1 = \Delta_1 \Delta'^2. \tag{4.23}$$

Case (2): Suppose that $r = 0$ and $\sigma = 1$. Then $r_0 = 0$,

$$P_1 = AX + q - 1 \quad \text{and} \quad Q_1 = 2AX - \frac{\Delta}{A^2} + \frac{2AB - A^2}{A^2}.$$

Now,

$$a_1 = \left\lfloor \frac{P_1 + \sqrt{D(X)}}{Q_1} \right\rfloor = \left\lfloor \frac{P_1 + \lfloor \sqrt{D(X)} \rfloor}{Q_1} \right\rfloor = \left\lfloor \frac{2AX + 2B/A - 2}{2AX - \Delta/A^2 + (2AB - A^2)/A^2} \right\rfloor. \tag{4.24}$$

We claim that $a_1 = 1$ if X is sufficiently large, i.e. (4.18) holds. To see this, we only need to show that

$$1 < \frac{2AX + 2B/A - 2}{2AX - \Delta/A^2 + (2AB - A^2)/A^2} < 2.$$

Since $(P_1 + \sqrt{D(X)})/Q_1$ is a complete quotient, it is greater than 1. So we need only establish the right inequality, which holds provided

$$AX + \frac{B}{A} - 1 < 2AX - \frac{\Delta}{A^2} + \frac{2AB - A^2}{A^2}.$$

This is equivalent to

$$X > \frac{1}{A}\left(\frac{\Delta}{A^2} - \frac{B}{A}\right) = \frac{\Delta_1 \Delta'^2}{AA'^2} - \frac{B}{A^2}.$$

By (4.18), the above inequality holds and hence, $a_1 = 1$. We compute

$$\mathbf{P}_1 = P_2 = a_1 Q_1 - P_1 = AX + \frac{B}{A} - \frac{\Delta}{A^2} = AX + \frac{B}{A} - \frac{\Delta_1 \Delta_2^2 \Delta_4^4}{A'^2(\Gamma \Delta_2 \Delta_4/\tau)^2} = AX + \frac{B}{A} - \Delta_1 \left(\frac{\Delta'}{A'}\right)^2$$

and

$$\mathrm{Q}_1 = \frac{D - P_2^2}{Q_1} = \frac{\Delta}{A^2} = \Delta_1 \left(\frac{\Delta'}{A'}\right)^2.$$

By (4.15), we have $A' = 1$. On putting $d_1 = \gcd(A', r_0) = A' = 1$ and $g_1 = \Delta'$, we write

$$\mathrm{P}_1 = AX + \frac{B}{A} - g_1\Delta_1\Delta' \quad \text{and} \quad \mathrm{Q}_1 = \Delta_1\Delta'^2. \tag{4.25}$$

Case (3): When $r > 0$, we appeal to Lemma 4.1.1 with

$$L = A, \quad M = A^2X + B \quad \text{and} \quad N = \Delta \tag{4.26}$$

so that by (4.22),

$$E = M - LP_1 = A^2X + B - A\left(AX + \frac{B}{A} - \frac{r}{A}\right) = r \quad \text{and} \quad F = LQ_0 = A.$$

Let $d = \gcd(A, r)$ and the continued fraction expansion of A/r be given by $(s_0, s_1, \dots, s_{m-1})$, where $(-1)^{m-1} = \sigma$. By Lemma 4.1.1, we get

$$P^* = \frac{A^2X + B}{A} - \frac{H|\Delta|}{Ad} \quad \text{and} \quad Q^* = \frac{|\Delta|}{d^2}, \tag{4.27}$$

where $H(r/d) \equiv \sigma \bmod A/d$ and $0 \leq H < A/d$.

By (4.12), $d = \gcd(A, r) = \gcd(A, B) = \Gamma\Delta_2\Delta_4/\tau$. Put $r_0 = r/(\Gamma\Delta_2\Delta_4/\tau)$. Note that this definition of r_0 is equivalent to $r_0 \equiv B\tau/(\Gamma\Delta_2\Delta_4) \bmod A'$, where $0 \leq r_0 < A'$. Since $A' = A/(\Gamma\Delta_2\Delta_4/\tau)$, we have $d_1 = \gcd(A', r_0) = 1$. Now,

$$\frac{H|\Delta|}{Ad} = \frac{H\Delta_1\Delta_2^2\Delta_4^4}{A'(\Gamma\Delta_2\Delta_4/\tau)(\Gamma\Delta_2\Delta_4/\tau)} = \Delta_1\frac{H\Delta'^2}{A'}.$$

Also,

$$\frac{|\Delta|}{d^2} = \Delta_1\left(\frac{\Delta'}{d_1}\right)^2 = \Delta_1\Delta'^2.$$

On setting $g_1 = H\Delta'$, we get

$$P^* = \frac{A^2X + B}{A} - \Delta_1\frac{g_1}{A'}\Delta' \quad \text{and} \quad Q^* = \Delta_1\Delta'^2. \tag{4.28}$$

Also, the expressions $H(r/d) \equiv \sigma \bmod A/d$ and $0 \leq H < A/d$ can be written as

$$Hr_0 \equiv \sigma \bmod A' \quad \text{and} \quad 0 \leq H < A'.$$

Thus, H is as stated in the lemma, and $0 \leq g_1 < A'\Delta'$.

We now show that $(P^* + \sqrt{D(X)})/Q^* > 1$. It suffices to prove $P^* + \lfloor\sqrt{D(X)}\rfloor \geq Q^*$, i.e.,

$$2AX + \frac{2B}{A} - \frac{r}{A} - \Delta_1 \frac{g_1}{A'}\Delta' > \Delta_1 \Delta'^2. \tag{4.29}$$

By (4.18), we have

$$2AX > 2\Delta_1\Delta'^2 - \frac{2B}{A} + 1.$$

Since $0 \leq g_1 < A'\Delta'$, it follows that $\Delta_1\Delta'(g_1/A') < \Delta_1\Delta'^2$. Hence,

$$2AX + \frac{2B}{A} - \frac{r}{A} > 2\Delta_1\Delta'^2 > \Delta_1\Delta'^2 + \Delta_1\frac{g_1}{A'}\Delta';$$

that is, $2AX + 2B/A - r/A - \Delta_1(g_1/A')\Delta' > \Delta_1\Delta'^2$. Thus, $(P^* + \sqrt{D(X)})/Q^* > 1$.

Now, if we write

$$d_1 = 1 \quad \text{and} \quad g_1 = \begin{cases} 0 & \text{if } r = 0 \text{ and } \sigma = -1, \\ A' & \text{if } r = 0 \text{ and } \sigma = 1, \\ H\Delta' & \text{if } r > 0, \end{cases}$$

where A' and Δ' are defined in (4.13), $Hr_0 \equiv \sigma \bmod A'$ and $0 \leq H < A'$, then the continued fraction expansion of $\sqrt{D(X)}$ is given by

$$\left(AX + q - \eta, \mathcal{S}_0, \frac{\mathrm{P}_1 + \sqrt{D(X)}}{\mathrm{Q}_1} \right),$$

where $\mathcal{S}_0 = \mathcal{S}(A', r_0)$ as defined in (4.17),

$$\mathrm{P}_1 = AX + \frac{B}{A} - \Delta_1 \frac{g_1}{A'}\frac{\Delta'}{d_1} \quad \text{and} \quad \mathrm{Q}_1 = \Delta_1 \left(\frac{\Delta'}{d_1}\right)^2.$$

□

Note that in the above lemma, if we put $d_0 = \Delta'$, then $\mathcal{S}_0 = \mathcal{S}(A', r_0) = \mathcal{S}(A'\Delta'/d_0, r_0)$. We will show inductively on i that

$$\sqrt{D(X)} = (AX + q + \eta,\ \mathcal{S}_0,\ q_1(X),\ \mathcal{S}_1,\ q_2(X),\ \ldots,\ \mathcal{S}_i,\ q_{i+1}(X),\ \ldots),$$

where $\mathcal{S}_i = \mathcal{S}(A'\Delta'/d_i, r_i)$ and $q_{i+1}(X) = \lfloor (\mathbf{P}_{i+1} + \sqrt{D(X)})\mathbf{Q}_{i+1} \rfloor$ for some integers $d_i \geq 1$, $r_i \geq 0$, $\mathbf{P}_{i+1} > 0$ and $\mathbf{Q}_{i+1} > 0$.

For $i \in \mathbb{N}$, define

$$\varepsilon_i = \begin{cases} 1 & \text{if } i \text{ is odd}, \\ 0 & \text{if } i \text{ is even}. \end{cases} \tag{4.30}$$

It is easy to see that $\varepsilon_{i+1} = 1 - \varepsilon_i$. We will provide a method for calculating r_i, d_{i+1} and g_{i+1} inductively from r_0, d_1 and g_1 in which

$$\mathbf{P}_{i+1} = AX + \frac{B}{A} - \Delta_1^{\varepsilon_{i+1}} \frac{g_{i+1}}{A'} \frac{\Delta'}{d_{i+1}} \quad \text{and} \quad \mathbf{Q}_{i+1} = \Delta_1^{\varepsilon_{i+1}} \left(\frac{\Delta'}{d_{i+1}} \right)^2. \tag{4.31}$$

The purpose of the following four lemmas are as follows: the first one is to determine r_i, the second one is to find $q_i(X)$, which is needed in the third lemma for the computation of d_{i+1} and g_{i+1} when $r_i = 0$. The last lemma computes d_{i+1} and g_{i+1} when $r_i > 0$.

For the remainder of this chapter, we let

$$X \equiv K \bmod \Delta' \qquad \text{and} \qquad 0 \leq K < \Delta'. \tag{4.32}$$

Lemma 4.2.3 *If integers $d_i > 0$ and $g_i \geq 0$ are given such that $\mathbf{P}_i$ and $\mathbf{Q}_i$ are as defined in (4.31), then, in the language of Lemma 4.1.1,*

$$\frac{F'}{E'} = \frac{A\mathbf{Q}_i}{A^2X + B - AP'} = \frac{A'\Delta'/d_i}{r_i},$$

where $P' = q_i(X)\mathbf{Q}_i - \mathbf{P}_i$, $q_i(X) = \lfloor (\mathbf{P}_i + \sqrt{D(X)})/\mathbf{Q}_i \rfloor$ and r_i is defined by

$$r_i \equiv \frac{d_i(2A^2K + 2B)}{\Delta_1^{\varepsilon_i}\Delta_2\Delta_4^2} - g_i \bmod \frac{A'\Delta'}{d_i} \quad \textit{and} \quad 0 \leq r_i < \frac{A'\Delta'}{d_i}. \tag{4.33}$$

Note that we use F' and E' instead of F and E to avoid confusion with the notation in Lemma 4.2.2.

Proof: Let

$$q_i(X) = \left\lfloor \frac{\mathbf{P}_i + \sqrt{D(X)}}{\mathbf{Q}_i} \right\rfloor \quad \text{and} \quad \mathbf{R}_i = \left(\mathbf{P}_i + \lfloor\sqrt{D(X)}\rfloor\right) - q_i(X)\mathbf{Q}_i,$$

where $0 \le \mathbf{R}_i < \mathbf{Q}_i$. Then $\mathbf{R}_i \equiv \lfloor\sqrt{D(X)}\rfloor + \mathbf{P}_i \bmod \mathbf{Q}_i$. It follows that

$$A\mathbf{R}_i \equiv A\lfloor\sqrt{D(X)}\rfloor + A\mathbf{P}_i \bmod A\mathbf{Q}_i. \tag{4.34}$$

Put $P' = q_i(X)\mathbf{Q}_i - \mathbf{P}_i$. Then

$$P' = q_i(X)\mathbf{Q}_i - \mathbf{P}_i = \left(\mathbf{P}_i + \lfloor\sqrt{D(X)}\rfloor\right) - \mathbf{R}_i - \mathbf{P}_i = \lfloor\sqrt{D(X)}\rfloor - \mathbf{R}_i.$$

Write $B = Aq + r$, where $0 \le r < A$. Then by Lemma 4.2.1, $\lfloor\sqrt{D(X)}\rfloor = AX + q - \eta = AX + B/A - r/A - \eta$, where η is defined in (4.16). Since $0 \le r/A + \eta \le 1$, we have $0 \le \mathbf{R}_i + r/A + \eta \le \mathbf{Q}_i$, i.e.,

$$0 \le A\mathbf{R}_i + r + A\eta \le A\mathbf{Q}_i. \tag{4.35}$$

Note that if $r = 0$ and $\sigma = 1$, then $\eta = 1$ and the above inequalities become $0 < A\mathbf{R}_i + A \le A\mathbf{Q}_i$. If $r = 0$ and $\sigma = -1$, then $\eta = 0$ and we get $0 \le A\mathbf{R}_i < A\mathbf{Q}_i$. Also, if $r > 0$, then $0 < A\mathbf{R}_i + r < A\mathbf{Q}_i$.

Write $F' = A\mathbf{Q}_i$ and $E' = A^2X + B - AP'$ as in Lemma 4.2.2. Note that

$$\begin{aligned} A^2X + B - AP' &= A^2X + B - A(\lfloor\sqrt{D(X)}\rfloor - \mathbf{R}_i) \\ &= A^2X + B - A^2X - B + r + A\eta + A\mathbf{R}_i \\ &= A\mathbf{R}_i + r + A\eta. \end{aligned}$$

Hence, by (4.35), we have $0 \le A^2X + B - AP' \le A\mathbf{Q}_i$, i.e. $0 \le E' \le F'$. By (4.34), we get

$$A\mathbf{R}_i + r + A\eta \equiv A\lfloor\sqrt{D(X)}\rfloor + A\mathbf{P}_i + r + A\eta \bmod \mathbf{AQ}_i.$$

Since $A\lfloor\sqrt{D(X)}\rfloor = A^2X + B - r - A\eta$, $A\mathbf{P}_i = A^2X + B - A\Delta_1^{\varepsilon_i}(g_i/A')(\Delta'/d_i)$ and $A\mathbf{Q}_i = A\Delta_1^{\varepsilon_i}(\Delta'/d_i)^2$, we have

$$A\mathbf{R}_i + r + A\eta \equiv 2A^2X + 2B - A\Delta_1^{\varepsilon_i}\frac{g_i}{A'}\frac{\Delta'}{d_i} \bmod A\Delta_1^{\varepsilon_i}\left(\frac{\Delta'}{d_i}\right)^2.$$

By assumption, we have $X \equiv K \bmod \Delta'$, where $0 \le K < \Delta'$. Since $\Delta_1\Delta_4\tau \mid 2A$ by (4.11), we have $\Delta_1\Delta' \mid 2A$. Thus,

$$\frac{2A}{\Delta_1^{\varepsilon_i}\Delta'}d_i^2X \equiv \frac{2A}{\Delta_1^{\varepsilon_i}\Delta'}d_i^2K \mod \Delta'.$$

It follows that

$$2A^2X \equiv 2A^2K \mod A\Delta_1^{\varepsilon_i}\left(\frac{\Delta'}{d_i}\right)^2. \tag{4.36}$$

Hence,

$$A\mathbf{R}_i + r + A\eta \equiv 2A^2K + 2B - A\Delta_1^{\varepsilon_i}\frac{g_i}{A'}\frac{\Delta'}{d_i} \mod A\Delta_1^{\varepsilon_i}\left(\frac{\Delta'}{d_i}\right)^2. \tag{4.37}$$

Notice that the modulus in (4.37) is

$$A\Delta_1^{\varepsilon_i}\left(\frac{\Delta'}{d_i}\right)^2 = \frac{\Delta_1^{\varepsilon_i}\Delta_2\Delta_4^2}{d_i}\frac{A'\Delta'}{d_i},$$

the right-most term in (4.37) is

$$A\Delta_1^{\varepsilon_i}\frac{g_i}{A'}\frac{\Delta'}{d_i} = g_i\frac{\Delta_1^{\varepsilon_i}\Delta_2\Delta_4^2}{d_i},$$

and $(\Delta_1^{\varepsilon_i}\Delta_2\Delta_4^2/d_i)$ divides both $2A^2$ and $2B$. So we may choose an integer r_i such that

$$r_i \equiv \frac{d_i(2A^2K + 2B)}{\Delta_1^{\varepsilon_i}\Delta_2\Delta_4^2} - g_i \mod \frac{A'\Delta'}{d_i} \quad \text{and} \quad 0 \le r_i < \frac{A'\Delta'}{d_i},$$

and

$$\frac{F'}{E'} = \frac{A\mathbf{Q}_i}{A^2X + B - AP'} = \frac{A'\Delta'/d_i}{r_i}.$$

Further, we may write the above congruence as

$$\frac{d_i(2A^2K + 2B)}{\Delta_1^{\varepsilon_i}\Delta_2\Delta_4^2} - g_i = t_i\frac{A'\Delta'}{d_i} + r_i \tag{4.38}$$

for some $t_i \in \mathbb{Z}$. $\square$

Lemma 4.2.4 *Write $X = W\Delta' + K$ for some non-negative integer W. If*

$$\mathrm{P}_i = AX + \frac{B}{A} - \Delta_1^{\varepsilon_i}\frac{g_i}{A'}\frac{\Delta'}{d_i}, \qquad \mathrm{Q}_i = \Delta_1^{\varepsilon_i}\left(\frac{\Delta'}{d_i}\right)^2,$$

then $q_i(X) = \lfloor(\mathrm{P}_i + \sqrt{D(X)})/\mathrm{Q}_i\rfloor$ is given by

$$\frac{2AWd_i^2}{\Delta_1^{\varepsilon_i}\Delta'} + t_i + \left\lfloor \frac{r_i}{(A'\Delta'/d_i)} - \frac{A\eta + r}{A\Delta_1^{\varepsilon_i}(\Delta'/d_i)^2} \right\rfloor, \tag{4.39}$$

where $2AWd_i^2/(\Delta_1^{\varepsilon_i}\Delta') \in \mathbb{Z}$ because $\Delta_1\Delta' \mid 2A$, and t_i as defined in (4.38).

<u>Proof</u>: Note that

$$q_i(X) = \left\lfloor \frac{\mathrm{P}_i + \lfloor\sqrt{D(X)}\rfloor}{\mathrm{Q}_i} \right\rfloor = \left\lfloor \frac{2A^2X + 2B - A\Delta_1^{\varepsilon_i}(g_i/A')(\Delta'/d_i) - A\eta - r}{A\Delta_1^{\varepsilon_i}(\Delta'/d_i)^2} \right\rfloor.$$

Since $X \equiv K \bmod \Delta'$, where $0 \leq K < \Delta'$, we write $X = W\Delta' + K$ for some integer W. Then

$$\begin{aligned} q_i(X) &= \left\lfloor \frac{2A^2(W\Delta' + K) + 2B - A\Delta_1^{\varepsilon_i}(g_i/A')(\Delta'/d_i) - A\eta - r}{A\Delta_1^{\varepsilon_i}(\Delta'/d_i)^2} \right\rfloor \\ &= \frac{2AWd_i^2}{\Delta_1^{\varepsilon_i}\Delta'} + \left\lfloor \frac{2A^2K + 2B - A\Delta_1^{\varepsilon_i}(g_i/A')(\Delta'/d_i) - A\eta - r}{A\Delta_1^{\varepsilon_i}(\Delta'/d_i)^2} \right\rfloor. \end{aligned}$$

Note that $\Delta_1^{\varepsilon_i}\Delta' \mid 2A$ by (4.11). Thus, $(2AWd_i^2)/(\Delta_1^{\varepsilon_i}\Delta')$ is an integer. From (4.38), we compute

$$t_i = \left\lfloor \frac{d_i(2A^2K + 2B)/(\Delta_1^{\varepsilon_i}\Delta_2\Delta_4^2) - g_i}{A'\Delta'/d_i} \right\rfloor = \left\lfloor \frac{2A^2K + 2B - A\Delta_1^{\varepsilon_i}(g_i/A')(\Delta'/d_i)}{A\Delta_1^{\varepsilon_i}(\Delta'/d_i)^2} \right\rfloor.$$

Hence,

$$\begin{aligned} q_i(X) &= \frac{2AWd_i^2}{\Delta_1^{\varepsilon_i}\Delta'} + \left\lfloor t_i + \frac{r_i}{(A'\Delta'/d_i)} - \frac{A\eta + r}{A\Delta_1^{\varepsilon_i}(\Delta'/d_i)^2} \right\rfloor \\ &= \frac{2Ad_i^2}{\Delta_1^{\varepsilon_i}\Delta'}W + t_i + \left\lfloor \frac{r_i}{(A'\Delta'/d_i)} - \frac{A\eta + r}{A\Delta_1^{\varepsilon_i}(\Delta'/d_i)^2} \right\rfloor. \end{aligned}$$

□

Lemma 4.2.5 *If $r_i = 0$, then the continued fraction expansion of $(\mathrm{P}_i + \sqrt{D(X)})/\mathrm{Q}_i$ is given by*

$$\left(q_i(X),\ \mathcal{S}_i,\ \frac{\mathrm{P}_{i+1} + \sqrt{D(X)}}{\mathrm{Q}_{i+1}} \right),$$

where $q_i(X)$ is defined by Lemma 4.2.4, $\mathcal{S}_i = \emptyset$ or $\{1\}$ and $g_{i+1} = 0$ or d_i according as $\sigma = -1$ or 1, $d_{i+1} = A'\Delta'/d_i = \Delta'/d_i$,

$$\mathrm{P}_{i+1} = AX + \frac{B}{A} - \Delta_1^{\varepsilon_{i+1}} \frac{g_{i+1}}{A'} \frac{\Delta'}{d_{i+1}} \quad \textit{and} \quad \mathrm{Q}_{i+1} = \Delta_1^{\varepsilon_{i+1}} \left(\frac{\Delta'}{d_{i+1}} \right)^2. \tag{4.40}$$

<u>Proof</u>: Suppose that $r_i = 0$. Then from (4.33), we see that

$$\frac{A'\Delta'}{d_i} \;\Big|\; \frac{d_i(2A^2K + 2B)}{\Delta_1^{\varepsilon_i}\Delta_2\Delta_4^2} - g_i$$

The above expression can be written as

$$A\Delta_1^{\varepsilon_i} \left(\frac{\Delta'}{d_i} \right)^2 \;\Big|\; 2A^2K + 2B - \Delta_1^{\varepsilon_i} g_i \frac{\Delta_2\Delta_4^2}{d_i}.$$

By (4.36), $2A^2K \equiv 2A^2X \bmod A\Delta_1^{\varepsilon_i}\left(\Delta'/d_i\right)^2$, so we get

$$A\Delta_1^{\varepsilon_i} \left(\frac{\Delta'}{d_i} \right)^2 \;\Big|\; 2A^2X + 2B - \Delta_1^{\varepsilon_i} g_i \frac{\Delta_2\Delta_4^2}{d_i}.$$

This implies that

$$A \;\Big|\; 2(A^2X + B) - \Delta_1^{\varepsilon_i} g_i \frac{\Delta_2\Delta_4^2}{d_i}\,. \tag{4.41}$$

From (4.31), we have

$$\mathrm{P}_i = AX + \frac{B}{A} - \Delta_1^{\varepsilon_i} \frac{g_i}{A'} \frac{\Delta'}{d_i} = AX + \frac{B}{A} - \Delta_1^{\varepsilon_i} \frac{g_i}{A} \frac{\Delta_2\Delta_4^2}{d_i}.$$

Thus,

$$A\mathrm{P}_i = A^2X + B - \Delta_1^{\varepsilon_i} g_i \frac{\Delta_2\Delta_4^2}{d_i}$$

and, consequently, $A^2X + B \equiv \Delta_1^{\varepsilon_i}\left(\Delta_2\Delta_4^2/d_i\right) g_i \bmod A$. Hence, by (4.41), we have $A \mid A^2X + B$. It follows that $A \mid B$ and $r = 0$.

Case (1): If $\sigma = -1$, then by (4.39), we get

$$q_i(X) = \frac{2AWd_i^2}{\Delta_1^{\varepsilon_i}\Delta'} + t_i.$$

Put $\mathbf{P}_{i+1} = q_i(X)\mathbf{Q}_i - \mathbf{P}_i$ and $\mathbf{Q}_{i+1} = (D(X) - \mathbf{P}_i^{\,2})/\mathbf{Q}_i$. We compute

$$\mathbf{P}_{i+1} = AX + \frac{B}{A} \quad \text{and} \quad \mathbf{Q}_{i+1} = \Delta_1^{\varepsilon_{i+1}}\left(\frac{d_i}{A'}\right)^2.$$

Put $\mathcal{S}_i = \mathcal{S}(A'\Delta'/d_i, r_i) = \emptyset$, $d_{i+1} = A'\Delta'/d_i$ and $g_{i+1} = 0$. We may write

$$\mathbf{P}_{i+1} = AX + \frac{B}{A} - \Delta_1^{\varepsilon_{i+1}}\frac{g_{i+1}}{A'}\frac{\Delta'}{d_{i+1}} \quad \text{and} \quad \mathbf{Q}_{i+1} = \Delta_1^{\varepsilon_{i+1}}\left(\frac{\Delta'}{d_{i+1}}\right)^2. \tag{4.42}$$

Case (2): If $\sigma = 1$, then by (4.39), we get

$$q_i(X) = \frac{2AWd_i^2}{\Delta_1^{\varepsilon_i}\Delta'} + t_i - 1.$$

Also,

$$P' = q_i(X)\mathbf{Q}_i - \mathbf{P}_i = AX + \frac{B}{A} - \Delta_1^{\varepsilon_i}\left(\frac{\Delta'}{d_i}\right)^2$$

and

$$Q' = \frac{D(X) - P'^2}{\mathbf{Q}_i} = 2AX + \frac{2B}{A} - \Delta_1^{\varepsilon_i}\left(\frac{\Delta'}{d_i}\right)^2 - \Delta_1^{\varepsilon_{i+1}}\left(\frac{d_i}{A'}\right)^2.$$

Upon further computation, we get $\lfloor (P' + \sqrt{D(X)})/Q' \rfloor = 1$. Set $\mathbf{P}_{i+1} = 1 \cdot Q' - P'$ and $\mathbf{Q}_{i+1} = (D - P'^2)/Q'$. We find that

$$\mathbf{P}_{i+1} = AX + \frac{B}{A} - \Delta_1^{\varepsilon_{i+1}}\left(\frac{d_i}{A'}\right)^2 \quad \text{and} \quad \mathbf{Q}_{i+1} = \Delta_1^{\varepsilon_{i+1}}\left(\frac{d_i}{A'}\right)^2.$$

Put $\mathcal{S}_i = \mathcal{S}(A'\Delta'/d_i, r_i) = \{1\}$, $d_{i+1} = A'\Delta'/d_i$ and $g_{i+1} = d_i$. Then

$$\mathbf{P}_{i+1} = AX + \frac{B}{A} - \Delta_1^{\varepsilon_{i+1}}\frac{g_{i+1}}{A'}\frac{\Delta'}{d_{i+1}} \quad \text{and} \quad \mathbf{Q}_{i+1} = \Delta_1^{\varepsilon_{i+1}}\left(\frac{\Delta'}{d_{i+1}}\right)^2. \tag{4.43}$$

Therefore, the continued fraction expansion of $(\mathbf{P}_i + \sqrt{D(X)})/\mathbf{Q}_i$ is given by

$$\left(q_i(X),\ \mathcal{S}_i,\ \frac{\mathbf{P}_{i+1} + \sqrt{D(X)}}{\mathbf{Q}_{i+1}}\right).$$

□

Remark 4.2.1 *By the reasoning on page 108, we see that if $r > 0$, then $r_i > 0$ for all non-negative integers i. By the definition of d_{i+1}, it follows that when $r_i = 0$, we have $d_{i+1} \mid \Delta'$ since $A' = 1$.*

Lemma 4.2.6 *If $r_i > 0$, then the continued fraction expansion of $(\mathbf{P}_i + \sqrt{D(X)})/\mathbf{Q}_i$ is given by*

$$\left(q_i(X),\ \mathcal{S}_i,\ \frac{\mathbf{P}_{i+1} + \sqrt{D(X)}}{\mathbf{Q}_{i+1}}\right),$$

where $\mathcal{S}_i = \mathcal{S}\left(A'\Delta'/d_i, r_i\right)$, $d_{i+1} = \gcd\left(A'\Delta'/d_i, r_i\right)$, g_{i+1} is an integer such that

$$\frac{g_{i+1}r_i}{d_i d_{i+1}} \equiv \sigma \bmod \frac{A'\Delta'}{d_i d_{i+1}} \quad \text{and} \quad 0 \leq g_{i+1} < \frac{A'\Delta'}{d_{i+1}}, \tag{4.44}$$

$q_i(X) = \lfloor (\mathbf{P}_i + \sqrt{D(X)})/\mathbf{Q}_i \rfloor$,

$$\mathbf{P}_{i+1} = AX + \frac{B}{A} - \Delta_1^{\varepsilon_{i+1}} \frac{g_{i+1}}{A'} \frac{\Delta'}{d_{i+1}} \quad \text{and} \quad \mathbf{Q}_{i+1} = \Delta_1^{\varepsilon_{i+1}} \left(\frac{\Delta'}{d_{i+1}}\right)^2. \tag{4.45}$$

<u>Proof</u>: Suppose that $r_i > 0$. By Lemma 4.2.3, we let the continued fraction expansion of $F'/E' = (A'\Delta'/d_i)/r_i$ be given by $(s_0, s_1, \ldots, s_{m'-1})$, where the natural number m' is chosen such that $(-1)^{m'-1} = \sigma$. Note that since $F' \geq E'$, we have $s_0 \in \mathbb{N}$. Let $d' = \gcd(E', F')$ and $d_{i+1} = \gcd(A'\Delta'/d_i, r_i)$. Then $d' = (\Delta_1^{\varepsilon_i}\Delta_2\Delta_4^2/d_i)d_{i+1}$.

By Lemma 4.1.1, we get

$$\mathbf{P}_{i+1} = AX + \frac{B}{A} - \frac{H'|\Delta|}{Ad'} \quad \text{and} \quad \mathbf{Q}_{i+1} = \frac{|\Delta|\mathbf{Q}_i}{(d')^2}, \tag{4.46}$$

where $H'(E'/d') \equiv \sigma \bmod F'/d'$ and $0 \leq H' < F'/d'$. Notice that

$$\frac{|\Delta|}{d'A} = \frac{\Delta_1\Delta_2^2\Delta_4^4}{(\Gamma\Delta_2\Delta_4/\tau)A'(\Delta_1^{\varepsilon_i}\Delta_2\Delta_4^2/d_i)d_{i+1}} = \frac{d_i\Delta_1^{\varepsilon_{i+1}}\Delta'}{A'd_{i+1}}$$

and

$$\frac{|\Delta|\mathbf{Q}_i}{(d')^2} = \frac{\Delta_1\Delta_2^2\Delta_4^4}{(\Delta_1^{\varepsilon_i}\Delta_2\Delta_4^2/d_i)^2 d_{i+1}^2}\Delta_1^{\varepsilon_1}\left(\frac{\Delta'}{d_i}\right)^2 = \Delta_1^{\varepsilon_{i+1}}\left(\frac{\Delta'}{d_{i+1}}\right)^2.$$

Hence,

$$\mathbf{P}_{i+1} = AX + \frac{B}{A} - \Delta_1^{\varepsilon_{i+1}} \frac{d_i H'}{A'} \frac{\Delta'}{d_{i+1}} \quad \text{and} \quad \mathbf{Q}_{i+1} = \Delta_1^{\varepsilon_{i+1}}\left(\frac{\Delta'}{d_{i+1}}\right)^2.$$

Since $E'/d' = r_i/d_{i+1}$ and $F'/d' = (A'\Delta')/(d_id_{i+1})$, the expressions

$$H'\frac{E'}{d'} \equiv \sigma \bmod \frac{F'}{d'} \quad \text{and} \quad 0 \le H' < \frac{F'}{d'}$$

can be written as

$$H'\frac{r_i}{d_{i+1}} \equiv \sigma \bmod \frac{A'\Delta'}{d_id_{i+1}} \quad \text{and} \quad 0 \le H' < \frac{A'\Delta'}{d_id_{i+1}}.$$

Put

$$g_{i+1} = d_iH'. \tag{4.47}$$

Then

$$\frac{g_{i+1}r_i}{d_id_{i+1}} \equiv \sigma \bmod \frac{A'\Delta'}{d_id_{i+1}} \quad \text{and} \quad 0 \le g_{i+1} < \frac{A'\Delta'}{d_{i+1}}.$$

Also, we get

$$\mathbf{P}_{i+1} = AX + \frac{B}{A} - \Delta_1^{\varepsilon_{i+1}}\frac{g_{i+1}}{A'}\frac{\Delta'}{d_{i+1}} \quad \text{and} \quad \mathbf{Q}_{i+1} = \Delta_1^{\varepsilon_{i+1}}\left(\frac{\Delta'}{d_{i+1}}\right)^2.$$

We now show that $(\mathbf{P}_{i+1}+\sqrt{D(X)})/\mathbf{Q}_{i+1} > 1$. By (4.18), we have $2AX > 2\Delta_1\Delta'^2 - 2B/A + 1$. By the fact that $0 \le r < A$ and the definition of η, we have $1 \ge r/A + \eta$ and hence $2AX > 2\Delta_1\Delta'^2 - 2B/A + r/A + \eta$. Thus,

$$2AX + \frac{2B}{A} - r/A - \eta > 2\Delta_1\Delta'^2 \ge 2\Delta_1^{\varepsilon_{i+1}}\left(\frac{\Delta'}{d_{i+1}}\right)^2.$$

Since $0 \le g_{i+1} < A'\Delta'/d_{i+1}$, we have

$$2AX + \frac{2B}{A} - r/A - \eta > \Delta_1^{\varepsilon_{i+1}}\left(\frac{\Delta'}{d_{i+1}}\right)^2 + \Delta_1^{\varepsilon_{i+1}}\frac{g_{i+1}}{A'}\frac{\Delta'}{d_{i+1}};$$

that is

$$2AX + \frac{2B}{A} - r/A - \eta - \Delta_1^{\varepsilon_{i+1}}\frac{g_{i+1}}{A'}\frac{\Delta'}{d_{i+1}} > \Delta_1^{\varepsilon_{i+1}}\left(\frac{\Delta'}{d_{i+1}}\right)^2.$$

Therefore, $(\mathbf{P}_{i+1} + \sqrt{D(X)})/\mathbf{Q}_{i+1} > 1$ and if we put

$$q_i(X) = \left\lfloor \frac{\mathbf{P}_i + \sqrt{D(X)}}{\mathbf{Q}_i} \right\rfloor \quad \text{and} \quad \mathcal{S}_i = S\left(\frac{A'\Delta'}{d_i}, r_i\right),$$

then by Remark 4.1.1, the continued fraction expansion of $(\mathrm{P}_i + \sqrt{D(X)})/\mathrm{Q}_i$ is given by

$$\left(q_i(X),\ \mathcal{S}_i,\ \frac{\mathrm{P}_{i+1} + \sqrt{D(X)}}{\mathrm{Q}_{i+1}}\right).$$

$\square$

Remark 4.2.2 *By Lemma 4.1.1, $\mathrm{Q}_i \in \mathbb{N}$ for all $i \geq 0$. Thus, $d_i \mid \Delta'$ for all $i \geq 0$. Since $\gcd(A', \Delta') = 1$ by (4.14), we have $d_{i+1} = \gcd(A'\Delta'/d_i, r_i) = \gcd(\Delta'/d_i, r_i)$ for all $i \geq 0$.*

4.3 The Continued Fraction Expansion of $\sqrt{D(X)}$

Theorem 4.3.1 *Let $D(X) = A^2X^2+2BX+C$ be a Schinzel sleeper. Assume that X is sufficiently large, i.e., (4.18) holds. Let $X \equiv K \bmod \Delta'$; i.e., $X = W\Delta' + K$, where $0 \leq K < \Delta'$, and $W \geq 0$.*

Put $d_0 = \Delta'$, $r_0 \equiv B\tau/(\Gamma\Delta_2\Delta_4) \bmod A'$, where $0 \leq r_0 < A'$. For $i \geq 0$, define

$$d_{i+1} = \gcd\left(\frac{\Delta'}{d_i}, r_i\right) \quad \text{and} \quad \mathcal{S}_i = \mathcal{S}\left(\frac{A'\Delta'}{d_i}, r_i\right),$$

where the parity of $|\mathcal{S}_i|$ is even if $\sigma = -1$ and odd if $\sigma = 1$.

When $r_i > 0$, choose $g_{i+1} \in \mathbb{Z}$ according to (4.44). When $r_i = 0$, set $g_{i+1} = d_i$ if $\sigma = 1$ and $g_{i+1} = 0$ if $\sigma = -1$. Let $q_{i+1}(X)$ and r_{i+1} be as defined in (4.39) and (4.33), respectively.

Then the regular continued fraction expansion of $\sqrt{D(X)}$ is given by

$$(AX + q - \eta,\ \overline{\mathcal{S}_0,\ q_1(X),\ \mathcal{S}_1,\ q_2(X),\ \mathcal{S}_2,\ q_3(X),\ \ldots,\ \mathcal{S}_{\kappa-1}(X),\ q_\kappa(X)}),$$

where $q = \lfloor B/A \rfloor$, η is defined by (4.16) and κ is the least natural number such that

$$d_\kappa = \Delta' \quad \text{and} \quad \Delta_1^{\epsilon_\kappa} = 1.$$

Proof: We prove the theorem by induction on i. If $i = 0$, then by Lemma 4.2.2, we get d_1, g_1 and

$$\sqrt{D(X)} = (AX + q - \eta,\ \mathcal{S}_0,\ (\mathrm{P}_1 + \sqrt{D(X)})/\mathrm{Q}_1).$$

By Lemma 4.2.3, we get r_1. By Lemma 4.2.4, we get $q_1(X)$. By Lemma 4.2.5 or Lemma 4.2.6 depending on whether $r_1 = 0$ or $r_1 > 0$, we get $\mathcal{S}_1 = \mathcal{S}\left(A'\Delta'/d_1, r_1\right)$, a non-negative integer g_2, and natural numbers d_2, P_2 and Q_2 such that

$$\sqrt{D(X)} = \left(AX + q - \eta,\ \mathcal{S}_0,\ q_1(X),\ \mathcal{S}_1,\ \frac{\mathrm{P}_2 + \sqrt{D(X)}}{\mathrm{Q}_2}\right).$$

By Remark 4.2.2, $d_2 = \gcd\left(A'\Delta'/d_1, r_1\right) = \gcd\left(\Delta'/d_1, r_1\right)$

Suppose that the theorem holds for some $i > 0$. Let d_i and g_i be as in the statement of the theorem. We use Lemma 4.2.3 to compute r_i. By Lemma 4.2.4, we obtain $q_i(X)$. By Lemma 4.2.5 or Lemma 4.2.6 depending on whether $r_i = 0$ or $r_i > 0$, we get $\mathcal{S}_i = \mathcal{S}\left(A'\Delta'/d_i, r_i\right)$, a non-negative integer g_{i+1}, and natural numbers d_{i+1}, P_{i+1} and Q_{i+1} such that

$$\sqrt{D(X)} = \left(AX + q - \eta,\ \mathcal{S}_0,\ q_1(X),\ \mathcal{S}_1,\ \ldots,\ q_i(X),\ \mathcal{S}_i,\ \frac{\mathrm{P}_{i+1} + \sqrt{D(X)}}{\mathrm{Q}_{i+1}}\right).$$

Hence, by induction, the continued fraction expansion of $\sqrt{D(X)}$ is given by

$$(AX + q - \eta,\ \mathcal{S}_0,\ q_1(X),\ \mathcal{S}_1,\ q_2(X),\ \mathcal{S}_2,\ \ldots,\ q_{i+1}(X),\ \ldots).$$

Since $\sqrt{D(X)}$ is a quadratic irrational, its continued fraction expansion is periodic. The end of the period is signaled by $Q_\ell = 1$ for some minimal $\ell \in \mathbb{N}$ and the corresponding partial quotient takes the form $2AX + 2q - 2\eta$, twice the initial partial quotient. Since the partial quotients in the sequences $\mathcal{S}_i$ are all less than $A'\Delta'$, the only partial quotient that can be as large as $2AX + 2q - 2\eta$ must be $q_\kappa(X)$ for some κ. Hence, if ℓ is the period length, then

$$Q_\ell = 1 \quad \text{and} \quad P_\ell = AX + q - \eta.$$

Suppose that

$$Q_\ell = \Delta_1^{\varepsilon_\kappa}\left(\frac{\Delta'}{d_\kappa}\right)^2 \quad \text{and} \quad P_\ell = AX + \frac{B}{A} - \Delta_1^{\varepsilon_\kappa}\frac{g_\kappa}{A'}\frac{\Delta'}{d_\kappa}.$$

Then, we must have $\Delta_1^{\varepsilon_\kappa} = 1$ and $d_\kappa = \Delta'$. In other words, our computation for the continued fraction expansion of $\sqrt{D(X)}$ is complete when we get the minimal $\kappa > 0$ such that $\Delta_1^{\varepsilon_\kappa} = 1$ and $d_\kappa = \Delta'$. $\square$

Remark 4.3.1 *We note that if $\Delta_1 > 1$, then κ is even. Also, it is easy to see that when $D(X)$ is given and X_1 and X_2 are two sufficiently integers that belong to the same residue class modulo Δ', then the expansions of $\sqrt{D(X_1)}$ and $\sqrt{D(X_2)}$ will have the same sequence $\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_\kappa$.*

Example 4.3.1 Consider $D(X) = 119^2X^2 + 2(2205)X + 343$. Then $A = 119 = 7\cdot 17$, B$= 2205 = 3^2\cdot 5\cdot 7^2$ and $C = 343 = 7^3$. The discriminant $\Delta = B^2 - A^2C = 4802 = 2\cdot 7^4$. Thus, $\Delta_1 = 2$, $\Delta_2 = 1$, $\Delta_4 = 7$, $\Delta_2\Delta_4 = 7$, which divides $7\cdot 17 = A$, so that $\tau = 1$. Then $A^* = A/(\Delta_2\Delta_4/\tau) = (7\cdot 17)/7 = 17$ and $\Gamma = \gcd(A^*, \Delta_4) = \gcd(17, 7) = 1$. Hence, $A' = A^*/\Gamma = 17$ and $\Delta' = (\tau\Delta_4)/\Gamma = 7$. Moreover, $d_0 = \Delta' = 7$, $r_0 = r/(\Delta_2\Delta_4) = 9$ and $d_1 = 1$.

Since X is sufficiently large when $X > \Delta_1\Delta'^2/A - B/A^2 + 1/(2A) \approx 0.67$, we first consider $X = 1$. We compute $r = B - A\lfloor B/A\rfloor = 63$ and $\lfloor\sqrt{D(1)}\rfloor = q_0(1) = 119\cdot 1 + 18 = 137$.

The fraction $A'/r_0 = 17/9$ may be represented by $(1, 1, 8)$. Although we may write $A'/r_0 = (1, 1, 7, 1)$, we insist on writing $A'/r_0 = (1, 1, 8)$ since $\sigma = 1$, which means that we need an odd number of terms. Therefore, $\mathcal{S}_0 = \mathcal{S}(A'\Delta'/d_0, r_0) = \{1, 1, 8\}$.

We find that $q_1(1) = 2$, $g_1 = 14$ and $r_1 = 82$. It can be checked that $A'\Delta'/d_1r_1 = 119/82 = (1, 2, 4, 1, 1, 1, 2)$. Again, we pick $(1, 2, 4, 1, 1, 1, 2)$ instead of $(1, 2, 4, 1, 1, 1, 1, 1)$ because we need an odd number of terms. Put $\mathcal{S}_1 = \{1, 2, 4, 1, 1, 1, 2\}$.

We list $q_i(X)$, $\mathcal{S}_i$, g_i, d_i and r_i in the following.

$$\begin{array}{lllll}
q_0(1) = 137, & \mathcal{S}_0 = \{1,1,8\}, & d_0 = \Delta' = 7, & & r_0 = 9, \\
q_1(1) = 2, & \mathcal{S}_1 = \{1,2,4,1,1,1,2\}, & d_1 = 1, & g_1 = 14, & r_1 = 82, \\
q_2(1) = 5, & \mathcal{S}_2 = \{4,3,1\}, & d_2 = 1, & g_2 = 45, & r_2 = 28, \\
q_3(1) = 136, & \mathcal{S}_3 = \{1,3,4\}, & d_3 = 7, & g_3 = 13, & r_3 = 13, \\
q_4(1) = 5, & \mathcal{S}_4 = \{2,1,1,1,4,2,1\}, & d_4 = 1, & g_4 = 28, & r_4 = 45, \\
q_5(1) = 2, & \mathcal{S}_5 = \{8,1,1,\}, & d_5 = 1, & g_5 = 82, & r_5 = 14, \\
q_6(1) = 274, & \mathcal{S}_0 = \{1,1,8\}, & d_6 = \Delta' = 7, & & r_6 = 9.
\end{array}$$

Since $\varepsilon_6 = 0$ by the definition of ε_i, we have $\Delta_1^{\varepsilon_6} = 1$. Also, since $d_6 = \Delta'$, the computation of $\mathcal{S}_i$ of the continued fraction expansion of $\sqrt{D(1)}$ is complete and $\kappa = 6$. In other words,

$$\sqrt{D(1)} = (137, \overline{1,1,8,\mathbf{2},1,2,4,1,1,1,2,\mathbf{5},4,3,1,\mathbf{136},1,3,4,\mathbf{5},2,1,1,1,4,2,1,\mathbf{2},8,1,1,\mathbf{274}}).$$

When $X = 7W + 1$ for $W \in \mathbb{N}$, we find that

$$\sqrt{D(7W+1)} = (833W + 137, \overline{1,1,8,q_1(W),1,2,4,1,1,1,2,q_2(W),4,3,1,q_3(W),}$$

$$\overline{1,3,4,q_4(W),2,1,1,1,4,2,1,q_5(W),8,1,1,2(833W+137)}),$$

where $q_1(W) = q_5(W) = 17W + 2$, $q_2(W) = q_4(W) = 34W + 5$ and $q_3(W) = 833W + 136$. For instance, when $X = 8 = 7 \cdot 1 + 1$, we have

$$\sqrt{D(8)} = (970, \overline{1,1,8,\mathbf{19},1,2,4,1,1,1,2,\mathbf{39},4,3,1,\mathbf{969},1,3,4,\mathbf{39},2,1,1,1,4,2,1,\mathbf{19},8,1,1,\mathbf{1940}}).$$

We compute the continued fraction expansion of $\sqrt{D(2)}$. We find that

$$\begin{array}{lllll}
q_0(2) = 256, & \mathcal{S}_0 = \{1,1,8\}, & d_0 = \Delta' = 7, & & r_0 = 9, \\
q_1(2) = 5, & \mathcal{S}_1 = \{8,1,1\}, & d_1 = 1, & g_1 = 14, & r_1 = 14, \\
q_2(2) = 512, & \mathcal{S}_2 = \{1,1,8\}, & d_2 = 7. & &
\end{array}$$

Since $\Delta_1^{\varepsilon_2} = 1$ and $d_2 = 7 = \Delta'$, the computation is done and we have

$$\sqrt{D(2)} = (256, \overline{1, 1, 8, 5, 8, 1, 1, 512}).$$

Again, we notice that if $X \equiv 2 \bmod 7$, i.e., $X = 7W + 2$ for some non-negative integer W, then

$$\sqrt{D(7W+2)} = (833W + 256, \overline{1, 1, 8, 17W + 5, 8, 1, 1, 2(833W + 256)}).$$

For instance, we have $\sqrt{D(9)} = (1089, \overline{1, 1, 8, 22, 8, 1, 1, 2178})$. □

From the above example, we suspect $d_i = d_{\kappa-i}$ and $g_{i+1} = r_{\kappa-i-1}$ for $0 \leq i \leq \kappa$ when $r > 0$. Indeed, this is always the case as we prove below.

Theorem 4.3.2 *If $r > 0$, then $d_i = d_{\kappa-i}$ for $0 \leq i \leq \kappa$.*

Proof: Since $r > 0$, $r_i > 0$ for all $i \geq 0$ by Remark 4.2.1. By symmetry of the continued fraction expansion of $\sqrt{D(X)}$, we have $A'\Delta'/(d_i r_i) = (s_0, s_1, \ldots, s_{m-1})$ and $A'\Delta'/(d_{\kappa-i-1} r_{\kappa-i-1}) = (s_{m-1}, s_{m-2}, \ldots, s_0)$ for some natural number m and $0 \leq i \leq \kappa$. Let A_j/B_j and A'_j/B'_j be the j-th convergents of $A'\Delta'/(d_i r_i)$ and $A'\Delta'/(d_{\kappa-i-1} r_{\kappa-i-1})$, respectively. Then

$$\begin{pmatrix} \dfrac{A'\Delta'}{d_i d_{i+1}} & A_{m-2} \\ \dfrac{r_i}{d_{i+1}} & B_{m-2} \end{pmatrix} = \begin{pmatrix} \dfrac{A'\Delta'}{d_{\kappa-i-1} d_{\kappa-i}} & A'_{m-2} \\ \dfrac{r_{\kappa-i-1}}{d_{\kappa-i}} & B'_{m-2} \end{pmatrix}^T, \tag{4.48}$$

where $d_{i+1} = \gcd(A'\Delta'/d_i, r_i)$ and $d_{\kappa-i} = \gcd(A'\Delta'/d_{\kappa-i-1}, r_{\kappa-i-1})$. It follows that $d_i d_{i+1} = d_{\kappa-i-1} d_{\kappa-i}$ for all $i \geq 0$. Since $d_0 = \Delta' = d_\kappa$, we have $d_1 = d_{\kappa-1}$. By induction, we get $d_i = d_{\kappa-i}$ for $i \geq 0$. □

Theorem 4.3.3 *If $r > 0$, then*

$$\frac{r_i}{d_{i+1}} \frac{r_{\kappa-i-1}}{d_i} \equiv \sigma \bmod \frac{A'\Delta'}{d_i d_{i+1}} \quad \textit{and} \quad g_{i+1} = r_{\kappa-i-1} \tag{4.49}$$

for $i \geq 0$.

Proof: Since $r > 0$, we have $d_{\kappa-i} = d_i$ by Theorem 4.3.2. We may rewrite (4.48) as

$$\begin{pmatrix} \frac{A'\Delta'}{d_i d_{i+1}} & A_{m-2} \\ \frac{r_i}{d_{i+1}} & B_{m-2} \end{pmatrix} = \begin{pmatrix} \frac{A'\Delta'}{d_{\kappa-i-1} d_{\kappa-i}} & \frac{r_{\kappa-i-1}}{d_i} \\ A'_{m-2} & B'_{m-2} \end{pmatrix}. \tag{4.50}$$

It follows that $A_{m-2} = r_{\kappa-i-1}/d_i$. Hence, $A'\Delta'/d_i d_{i+1} \cdot B_{m-2} - r_i/d_{i+1} \cdot r_{\kappa-i-1}/d_i = (-1)^m$, i.e.,

$$\frac{r_i}{d_{i+1}} \cdot \frac{r_{\kappa-i-1}}{d_i} \equiv -(-1)^m \equiv \sigma \mod \frac{A'\Delta'}{d_i d_{i+1}}.$$

By (4.47), we have $g_{i+1} = d_i H'$ for some H' such that

$$\frac{r_i}{d_{i+1}} H' \equiv \sigma \mod \frac{A'\Delta'}{d_i d_{i+1}} \quad \text{and} \quad 1 \leq H' \leq \frac{A'\Delta'}{d_i d_{i+1}}.$$

This implies that $H' = r_{\kappa-i-1}/d_i$. Hence, $g_{i+1} = d_i\, r_{\kappa-i-1}/d_i = r_{\kappa-i-1}$. $\square$

Chapter 5

Results Pertaining to the Continued Fraction Expansion of $\sqrt{D(X)}$

In this chapter, we use the continued fraction expansion of $\sqrt{D(X)}$ given in Theorem 4.3.1 to establish three results: the number of different patterns of the continued fraction period of $\sqrt{D(X)}$ as X varies, an explicit upper bound for the continued fraction period length of $\sqrt{D(X)}$ and the fundamental unit of the real quadratic order $[1, \sqrt{D(X)}]$.

By Theorem 4.3.1, the continued fraction expansion of $\sqrt{D(X)}$ not only depends on the coefficients A, B and C, but also on the residue class of X modulo Δ', which is a function of A, B, C. By a **pattern** of the continued fraction expansion of $\sqrt{D(X)}$ we mean the sequence $\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_{\kappa-1}$, where κ is as defined in Theorem 4.3.1. By Remark 4.3.1, we know that if positive integers X_1 and X_2 are sufficiently large and belong to the same residue class modulo Δ', the expansions of $\sqrt{D(X_1)}$ and $\sqrt{D(X_2)}$ have the same pattern. However, if X_1 and X_2 belong to different residue classes, the expansions of $\sqrt{D(X_1)}$ and $\sqrt{D(X_2)}$ may have different patterns. In Section 5.1, we study the number of different patterns and show that this number is a divisor of Δ'.

In Section 5.2, we establish an upper bound for the period length of the expansion of $\sqrt{D(X)}$. More explicitly, we follow Schinzel's notation in denoting the period length of $\sqrt{D(X)}$ by $\mathrm{lp}(\sqrt{D(X)})$ and show that

$$\mathrm{lp}(\sqrt{D(X)}) < \begin{cases} 3\Delta' \cdot \lfloor \log_\varphi(\sqrt{5} \cdot A'\Delta') \rfloor & \text{if } \Delta' \text{ is even,} \\ 2\Delta' \cdot \lfloor \log_\varphi(\sqrt{5} \cdot A'\Delta') \rfloor & \text{if } \Delta' \text{ is odd,} \end{cases}$$

where $\varphi = (1+\sqrt{5})/2$ is the golden ratio and $\log_\varphi(x)$ is the logarithm of x with base φ.

In Section 5.3, we show that the fundamental unit of the quadratic order $[1, \sqrt{D(X)}]$ is given by

$$|\Delta|^{\kappa/2}\left(\frac{A^2X + B + A\sqrt{D(X)}}{|\Delta|}\right)^{\kappa},$$

and the norm of it is σ^{κ}, where κ is defined in Theorem 4.3.1.

5.1 Number of Patterns

In Example 4.3.1, we see that $\sqrt{119^2X^2 + 2(2205)X + 343}$ with $X \in \mathbb{N}$ has two different patterns of expansion depending on whether $X \equiv 1$ or $X \equiv 2 \bmod 7$. As we run through the other five residue classes of 7, we find that the continued fraction expansion of $\sqrt{D(X)}$ exhibits another five different patterns depending on which residue class X belongs to. They are represented by

$$\begin{aligned}
\sqrt{D(3)} &= (375, \overline{1,1,8,\mathbf{7},1,1,4,1,10,\mathbf{15},4,3,1,\mathbf{374},1,3,4,\mathbf{15},10,1,4,1,1,\mathbf{7},8,1,1,\mathbf{750}}),\\
\sqrt{D(4)} &= (494, \overline{1,1,8,\mathbf{9},1,38,1,1,1,\mathbf{19},1,1,11,2,2,\mathbf{9},1,2,4,1,1,1,2,\mathbf{19},}\\
&\qquad \overline{1,4,5,1,3,\mathbf{9},1,4,1,18,1,\mathbf{19},4,3,1,\mathbf{493},1,3,4,\mathbf{19},1,18,1,4,1,\mathbf{9},3,1,5,4,1,}\\
&\qquad \overline{\mathbf{19},2,1,1,1,4,2,1,\mathbf{9},2,2,11,1,1,\mathbf{19},1,1,1,38,1,\mathbf{9},8,1,1,\mathbf{988}}),\\
\sqrt{D(5)} &= (613, \overline{1,1,8,\mathbf{12},2,2,11,1,1,\mathbf{24},1,1,11,2,2,\mathbf{12},8,1,1,\mathbf{1226}}),\\
\sqrt{D(6)} &= (732, \overline{1,1,8,\mathbf{14},1,4,1,18,1,\mathbf{28},1,18,1,4,1,\mathbf{14},8,1,1,\mathbf{1464}}),\\
\sqrt{D(7)} &= (851, \overline{1,1,8,\mathbf{17},3,1,5,4,1,\mathbf{33},1,18,1,4,1,\mathbf{16},1,1,4,1,10,\mathbf{34},1,1,1,38,1,\mathbf{16},}\\
&\qquad \overline{2,2,11,1,1,\mathbf{34},4,3,1,\mathbf{850},1,3,4,\mathbf{34},1,1,11,2,2,}\\
&\qquad \overline{\mathbf{16},1,38,1,1,1,\mathbf{34},10,1,4,1,1,\mathbf{16},1,4,1,18,1,\mathbf{33},1,4,5,1,3,\mathbf{17},8,1,1,\mathbf{1702}}).
\end{aligned}$$

This observation may suggest that the number of different patterns exhibited by the continued fraction expansion of $\sqrt{D(X)}$ is Δ'. This is not always true as we shall see below.

By Lemma 4.2.2, when the coefficients A, B and C are fixed, the terms r_0, d_1 and g_1 are independent of X. By Lemma 4.2.3, the first quantity that depends on X, or K, is r_1. By our inductive approach in Theorem 4.3.1, we see that for $i \geq 1$, the terms d_i, g_i and r_{i+1} depend on r_1, which depends on K by Lemma 4.2.3. By (4.33) and since $d_1 = 1$, we write

$$r_1(K) \equiv \frac{2A^2K + 2B}{\Delta_1\Delta_2\Delta_4^2} - g_1 \mod A'\Delta', \quad \text{where} \quad 0 \leq r_1(K) < A'\Delta'.$$

Lemma 5.1.1 *If $r_1(0) = r_1(K)$ for some $K \geq 0$, then $r_1(i) = r_1(K+i)$ for all $i \geq 0$.*

Proof: Suppose that for some K, we have $r_1(0) = r_1(K)$. Then

$$r_1(0) \equiv \frac{2A^2(0) + 2B}{\Delta_1\Delta_2\Delta_4^2} - g_1 \equiv \frac{2B}{\Delta_1\Delta_2\Delta_4^2} - g_1 \mod A'\Delta'$$

and

$$r_1(K) \equiv \frac{2A^2K + 2B}{\Delta_1\Delta_2\Delta_4^2} - g_1 \equiv r_1(0) \equiv \frac{2B}{\Delta_1\Delta_2\Delta_4^2} - g_1 \mod A'\Delta'.$$

It follows that

$$\frac{2A^2K}{\Delta_1\Delta_2\Delta_4^2} \equiv 0 \mod A'\Delta'.$$

Now, for any integer i, we have

$$\begin{aligned} r_1(K+i) &\equiv \frac{2A^2(K+i) + 2B}{\Delta_1\Delta_2\Delta_4^2} - g_1 \equiv \frac{(2A^2K)}{\Delta_1\Delta_2\Delta_4^2} + \frac{(2A^2i + 2B)}{\Delta_1\Delta_2\Delta_4^2} - g_1 \mod A'\Delta' \\ &\equiv r_1(i) \mod A'\Delta'. \end{aligned}$$

Since $r_1(K+i)$ and $r_1(i)$ are residues of modulo $A'\Delta'$, we have $r_1(K+i) = r_1(i)$. □

By (4.8), we have $\Delta_4 \mid A$ and $\Delta_1\Delta_2\Delta_4 \mid 2A$. Thus, $2A^2\Delta'/(\Delta_1\Delta_2\Delta_4^2) \equiv 0 \mod A'\Delta'$. It follows that

$$r_1(\Delta') \equiv \frac{2A^2\Delta' + 2B}{\Delta_1\Delta_2\Delta_4^2} - g_1 \equiv \frac{2B}{\Delta_1\Delta_2\Delta_4^2} - g_1 \equiv r_1(0) \mod A'\Delta'.$$

This implies that $r_1(0) = r_1(\Delta')$. Hence, by Lemma 5.1.1, we see that $r_1(i) = r_1(\Delta' + i)$ for all $i \geq 0$, i.e., the pattern repeats modulo Δ'.

We define the **number of patterns** ρ to be the minimal number of distinct values among $r_1(0), r_1(1), \ldots, r_1(\Delta' - 1)$. In other words, we have $r_1(0) = r_1(\rho)$ with $\rho \geq 0$ minimal. By Lemma 5.1.1, it follows that $r_1(i) = r_1(\rho + i)$ for all $i \geq 0$. Also, we have

Theorem 5.1.1 $\rho \mid \Delta'$.

Proof: Write $\Delta' = \rho\lfloor \Delta'/\rho \rfloor + \gamma$, where $0 \leq \gamma < \rho$. Then,

$$r_1(\Delta') \equiv \frac{(2A^2\Delta' + 2B)}{\Delta_1\Delta_2\Delta_4^2} - g_1 \equiv \frac{(2A^2\gamma + 2B)}{\Delta_1\Delta_2\Delta_4^2} - g_1 \equiv r_1(\gamma) \mod A'\Delta'.$$

Since $r_1(\Delta') = r_1(0) = r_1(\rho)$, we have $r_1(\gamma) = r_1(\rho)$. Since ρ is minimal, $\gamma = 0$. □

Note that in Example 4.3.1, we have $\Delta' = 7$, which is a prime. So, the possible divisors are 1 and 7. In that particular example, we have 7 different patterns.

Example 5.1.1 Consider $D(X) = 833^2X^2 + 2 \cdot 114562X + 18914$. We have $A = 833 = 7^2 \cdot 17$, $B = 114562 = 2 \cdot 7^3 \cdot 167$, $C = 18914 = 2 \cdot 7^2 \cdot 193$ and $\Delta = 235298 = 2 \cdot 7^6 = 2 \cdot 7^2 \cdot 7^4$, so that $\Delta_1 = 2$, $\Delta_2 = 7$ and $\Delta_4 = 7$. Also, we find that $\Delta' = 7$. By Theorem 5.1.1, we expect the continued fraction expansion of $\sqrt{D(X)}$ to have either 1 or 7 different patterns. Moreover, by (4.18), the sufficient size of X is $X > \Delta_1\Delta'^2/A - B/A^2 + 1/(2A) \approx -0.047$. Here, we consider $X = 1$.

We find that

$$\sqrt{D(1)} = (970, \overline{1,1,8,\mathbf{19},1,2,4,1,1,1,2,\mathbf{39},4,3,1,\mathbf{969},1,3,4,\mathbf{39},2,1,1,1,4,2,1,\mathbf{19},8,1,1,\mathbf{1940}}),$$

$$\sqrt{D(2)} = (1803, \overline{1,1,8,\mathbf{36},1,2,4,1,1,1,2,\mathbf{73},4,3,1,\mathbf{1802},1,3,4,\mathbf{73},2,1,1,1,4,2,1,\mathbf{36},8,1,1,\mathbf{3606}}),$$

and for all $X \in \mathbb{N}$, we get

$$\sqrt{D(X)} = (833X + 137, \overline{1,1,8,q_1(X),1,2,4,1,1,1,2,q_2(X),4,3,1,q_3(X),}$$
$$\overline{1,3,4,q_4(X),2,1,1,1,4,2,1,q_5(X),8,1,1,1666X+274}),$$

where $q_1(X) = q_5(X) = 17X + 2$, $q_2(X) = q_4(X) = 34X + 5$ and $q_3(X) = 833X + 136$. Therefore, there is only one pattern. Note that 1 divides $7 = \Delta'$. $\square$

Since the continued fraction expansion of a quadratic irrational is unique and the expansions in the above example are the same as those in Example 4.3.1 with $X \equiv 1 \bmod 7$, we are led to think that $D(X) = 833^2X^2 + 2 \cdot 114562X + 18914$ is a special case of $D(X) = 119^2X^2 + 2(2205)X + 343$ for some $X \in \mathbb{N}$. Indeed, if we substitute $X = 7Y + 1$ into $D(X) = 119^2X^2 + 2(2205)X + 343$, then $D(7Y + 1) = 119^2(7Y + 1)^2 + 2(2205)(7Y + 1) + 343 = 833^2Y^2 + 2 \cdot 114562Y + 18914$.

5.2 Upper Bound for $\mathrm{lp}(\sqrt{D(X)})$

In the previous chapter, we established the continued fraction expansion of $\sqrt{D(X)}$ for any quadratic $D(X)$ that satisfies the Schinzel condition. In this section, we focus on the period length of $\sqrt{D(X)}$, $\mathrm{lp}(\sqrt{D(X)})$. When Schinzel established the result concerning the boundedness of $\mathrm{lp}(\sqrt{D(X)})$, he did not provide an explicit upper bound for $\mathrm{lp}(\sqrt{D(X)})$. Here, we give an explicit upper bound on $\mathrm{lp}(\sqrt{D(X)})$.

By Theorem 4.3.1, we know that the calculation of the continued fraction expansion of $\sqrt{D(X)}$ ends when $d_\kappa = \Delta'$ and $\Delta_1^{\epsilon_\kappa} = 1$ for some minimal κ. Hence, if we write

$$|\mathcal{S}_i| = \left|\mathcal{S}\left(\frac{A'\Delta'}{d_i}, r_i\right)\right| = |\mathcal{S}\left(A'\Delta, d_i r_i\right)|$$

as the cardinality of the set $\mathcal{S}_i$, then

$$\mathrm{lp}(\sqrt{D(X)}) = \sum_{i=0}^{\kappa-1}(1 + |\mathcal{S}_i|). \tag{5.1}$$

We are interested in obtaining an upper bound for $|\mathcal{S}_i|$. Recall that $|\mathcal{S}_i|$ is the length of the continued fraction expansion of $A'\Delta'/d_i r_i$ when $r_i > 0$, and $|\mathcal{S}_i| = 0$ or 1 when $r_i = 0$. We henceforth consider $r_i > 0$.

It is easy to see that the length of the continued fraction expansion of a rational number a/r is determined by the number of division steps required when applying the Euclidean algorithm to a/r. Note that the length is not necessarily the same as the number of division steps since the finite continued fraction expansion of a/r has two representations, depending on whether the last partial quotient is 1 or greater than 1. For our purpose, we need to consider the maximum length, i.e, the last partial quotient is 1. In 1845, G. Lamé gave an interesting result concerning the number of division steps of the Euclidean algorithm applied to a rational number. We restate it as follows: for fixed $n \geq 1$, let a, r be integers with $0 < r < a$ and a minimal such that the continued fraction expansion of a/r has length exactly n. Then $a = F_{n+1}$ and $r = F_n$, where F_i is the i-th Fibonacci number.

If $0 \leq r < a < N$ for some $N \in \mathbb{N}$, then by Lamé's theorem, the maximum length occurs when $a = F_{n+1}$ and $r = F_n$, where n is as large as possible with $F_{n+1} < N$. Denote the golden ratio by $\varphi = (1+\sqrt{5})/2$. It is well-known that $F_i = (\varphi^i - \bar{\varphi}^i)/\sqrt{5}$ for all $i \in \mathbb{Z}$. Since $\bar{\varphi}^i$ gets very small in absolute value as i increases, it is not difficult to see that $F_{n+1} < N$ implies $\varphi^{n+1}/\sqrt{5} < N$. It follows that $n+1 < \log_\varphi(\sqrt{5}\cdot N)$. Hence, the maximum length for the continued fraction expansion of a/r is at most $\lfloor \log_\varphi(\sqrt{5} \cdot N) \rfloor - 1$. Interested readers in this area are referred to Knuth [91, p. 343] and Shallit [220].

On applying the above reasoning, we get $|S_i| \leq \lfloor \log_\varphi(\sqrt{5} \cdot A'\Delta') \rfloor - 1$ and

$$\operatorname{lp}(\sqrt{D(X)}) \leq \sum_{i=0}^{\kappa-1}(1 + \lfloor \log_\varphi(\sqrt{5} \cdot A'\Delta') \rfloor - 1) = \kappa \cdot \lfloor \log_\varphi(\sqrt{5} \cdot A'\Delta') \rfloor. \tag{5.2}$$

Now it remains to determine an upper bound on the value of κ. We will show that if $\Delta' \mid 2$, then $\kappa = 1$, 2, 3, 4 or 6. Also, we will establish two general results regarding upper bounds for κ: $\kappa \mid 2\,\omega(\Delta')$ and

$$\omega(\Delta') \leq \begin{cases} \Delta' & \text{if } \Delta' \text{ is odd,} \\ (3/2)\Delta' & \text{if } \Delta' \text{ is even,} \end{cases}$$

where $\omega(\Delta')$ is the rank of apparition of Δ' in a certain Lucas function U_n.

Lemma 5.2.1 *If $D(X)$ satisfies the Schinzel condition, then $\Delta_1 \mid 2D(X)$.*

Proof: Let $d = \gcd(A, B)$. Then

$$\left(\frac{B}{d}\right)^2 - \left(\frac{A}{d}\right)^2 C = \pm\Delta_1 \left(\frac{\Delta_2\Delta_4^2}{d}\right)^2. \tag{5.3}$$

Since $\Delta = B^2 - A^2C$ divides $4\gcd(A^2, B)^2$, it follows that

$$\Delta_1 \left(\frac{\Delta_2\Delta_4^2}{d}\right)^2 \Bigg| \gcd\left(\frac{2A^2}{d}, \frac{2B}{d}\right)^2.$$

Since Δ_1 is squarefree, it follows that Δ_1 divides both $2A^2/d$ and $2B/d$. In view of (5.3), we get $\Delta_1 \mid (2A/d)^2C$. Again, since Δ_1 is squarefree, we have $\Delta_1 \mid (2A/d)C$. Since A/d and B/d are relatively prime, Δ_1 divides $2C$. Therefore, Δ_1 divides $2(A^2X^2 + 2BX + C)$, which is $2D(X)$. □

Lemma 5.2.2 *If $d_k = \Delta'$ and $\Delta_1^{\varepsilon_k} \neq 1$ for some $k \in \mathbb{N}$ minimal, then $\kappa = 2k$.*

Proof: Suppose that $d_k = \Delta'$ but $\Delta_1^{\varepsilon_k} \neq 1$ for some $k \in \mathbb{N}$. Then $\varepsilon_k = 1$, $\Delta_1 > 1$ and $\mathbf{Q}_k = \Delta_1(\Delta'/d_k) = \Delta_1$. Since $\Delta_1 = \mathbf{Q}_k \mid D(X) - \mathbf{P}_k{}^2$ and $\Delta_1 \mid 2D(X)$ by Lemma 5.2.1, we have $\Delta_1 \mid 2\mathbf{P}_k{}^2$. Since Δ_1 is squarefree by assumption, we have $\Delta_1 \mid 2\mathbf{P}_k$. Hence, $\mathbf{Q}_{2k} = 1$ by Theorem 1.2.3. Now, if k is minimal, then $\kappa = 2k$. □

In the sequel, we put

$$T = \frac{2A^2K + 2B}{\Delta_2\Delta_4^2}. \tag{5.4}$$

The following theorem recaptures the result of van der Poorten and Williams discussed in Section 2.4. Assuming $\gcd(A^2, 2B, C)$ is squarefree, the two authors showed that, in our notation, κ is 1, 2, 3 or 6. It is not difficult to see that when $\gcd(A^2, 2B, C)$ is squarefree, $\Delta_4 = 1$, and consequently $\Delta' \mid 2$. Indeed, we have

Theorem 5.2.1 *If $\Delta' \mid 2$, then $\kappa = 1$, 2, 3, 4 or 6. More explicitly,*

Δ'	1	1	2	2	2	2
Δ_1	1	>1	*any*	1	>1	*any*
T/Δ_1	*any*	*any*	*even*	*odd*	*odd*	*odd*
T	*any*	*any*	*any*	*odd*	*odd*	*even*
κ	1	2	2	3	6	4

Table 5.1: Values of κ when $\Delta' \mid 2$.

Proof: By Lemma 4.2.2, we have $d_1 = 1$. If $\Delta' \mid 2$, then $\Delta' = 1$ or 2.

Case (1): If $\Delta' = 1$, then $Q_1 = \Delta_1(\Delta'/d_1)^2 = \Delta_1$. If $\Delta_1 = 1$, then $\kappa = 1$. If $\Delta_1 > 1$, then by Lemma 5.2.2, we are half-way through the period of the continued fraction of $\sqrt{D(X)}$ and $\kappa = 2$.

Case (2): Suppose that $\Delta' = 2$. Then

$$r_1 \equiv \frac{2A^2K + 2B}{\Delta_1\Delta_2\Delta_4^2} - g_1 \equiv \frac{T}{\Delta_1} - g_1 \bmod 2A'.$$

Since $\Delta' = 2$, by (4.19), we see that g_1 is even. Hence, $r_1 \equiv T/\Delta_1 \bmod 2$.

If T/Δ_1 is even, then r_1 is even. Hence, $d_2 = \gcd(A'\Delta'/d_1, r_1) = \gcd(2A', r_1)$ is even. Since $d_2 \mid \Delta'$, it follows that $d_2 = 2$. Hence, $Q_2 = (\Delta'/d_2)^2 = 1$ and $\kappa = 2$.

If T/Δ_1 is odd, then r_1 is odd. This implies that d_2, being a divisor of r_1, must also be odd. Since $d_2 \mid \Delta'$, $d_2 = 1$. Hence, the congruence $g_2r_1/(d_1d_2) \equiv \sigma \bmod A'\Delta'/(d_1d_2)$ can be written as $g_2r_1 \equiv \sigma \bmod 2A'$. Thus, g_2 is odd. Since $d_1 = 1$, $d_2 = 1$, $\varepsilon_2 = 0$ and g_2 is odd, the congruence $r_2 \equiv d_2T/\Delta_1^{\varepsilon_2} - g_2 \equiv T - g_2 \bmod 2A'$ implies that $r_2 \equiv T - 1 \bmod 2$.

If T is odd, i.e., r_2 is even, then $d_3 = \gcd(2A', r_2)$ is even. Thus, $d_3 = 2$ and $Q_3 = \Delta_1^{\varepsilon_3}(\Delta'/d_3)^2 = \Delta_1$. If $\Delta_1 = 1$, then $\kappa = 3$. If $\Delta_1 > 1$, then by Lemma 5.2.2, $\kappa = 6$.

If T is even, i.e., r_2 is odd, then $d_3 = 1$ and $g_3r_2 \equiv \sigma \bmod 2A'$ implies that g_3 is odd. Hence, $r_3 \equiv T/\Delta_1 - g_3 \equiv 0 \bmod 2$. This implies that $d_4 = 2$ and $Q_4 = (\Delta'/d_4)^2 = 1$. Hence, $\kappa = 4$. □

Remark 5.2.1 van der Poorten and Williams obtain their results by making use of the continued fraction expansion of $\sqrt{C}$. In terms of our notation, they show that when X is sufficiently large, $\mathcal{S}_i$ of $\sqrt{D(X)}$ are segments of the continued fraction expansion of $\sqrt{C}$. Although we do not pay any particular attention to $\sqrt{C}$, it is clear that we arrive at the same conclusion when $\gcd(A^2, 2B, C)$.

We illustrate the six cases in Table 5.1 of Theorem 5.2.1 in the following.

Example 5.2.1 Case (1): When $\Delta' = 1$ and $\Delta_1 = 1$, take $D(X) = 4^2X^2 + 2(15)X + 14$ as in Example 2.4.1. Then $\Delta = B^2 - A^2C = 15^2 - 16 \cdot 14 = 1$. So, $\Delta_1 = \Delta_2 = \Delta_4 = 1$ and $\Delta' = 1$. For sufficiently large X, we have $\sqrt{D(X)} = (4X+3, \overline{1, 2, 1,\ 8X+6})$. It is not difficult to see that $\kappa = 1$ in this case.

Case (2): When $\Delta' = 1$ and $\Delta_1 > 1$, take $D(X) = 36^2X^2 + 2(168)X + 22$. Then $A = 3 \cdot 12$, $B = 12 \cdot 14$, $C = 22$ and $|\Delta| = |-288| = \Delta_1\Delta_2^2\Delta_4^4 = 2 \cdot 3^2 \cdot 2^4$. So, $\Delta_1 > 1$. Also, $\tau = 1$ and $A^* = A/(\Delta_2\Delta_4/\tau) = 6$, $\Gamma = \gcd(A^*, \Delta_4) = 2$, and $\Delta' = \tau\Delta_4/\Gamma = 1$. Thus, by Theorem 5.1.1, we have $\rho = 1$. Further, we compute

$$\sqrt{D(X)} = (36X+4, \overline{1, 2,\ 36X+4,\ 2, 1,\ 72X+8}) \quad \text{and} \quad \kappa = 2.$$

Case (3): When $\Delta' = 2$ and T/Δ_1 is even, consider $D(X) = 119^2X^2 + 2(2037)X + 293$. Then, $A = 119$, $B = 2037$, $C = 293$ and $\Delta = 2^2 \cdot 7^2$. Also, $\Delta_1 = 1$, $\Delta_2 = 14$, $\Delta_4 = 1$, $\tau = 2$, $A^* = 17$, $\Gamma = 1$, and $\Delta' = \tau\Delta_4/\Gamma = 2$.

When $X \equiv 1 \mod 2$, i.e., $K = 1$ and $X = 2W + 1$, where $W \geq 0$, we have $T = 2314 = T/\Delta_1$, which is even. Now,

$$\sqrt{D(2W+1)} = (238W+136, \overline{8, 1, 1,\ 119W+67,\ 1, 1, 8,\ 476W+272}) \quad \text{and} \quad \kappa = 2.$$

Case (4): When $\Delta' = 2$, $\Delta_1 = 1$, T is odd and T/Δ_1 is odd, consider $D(X) = 119^2X^2 + 2(2037)X + 293$ with $X \equiv 0 \mod 2$, i.e., $K = 0$ and $X = 2W$, where $W \geq 1$. We find that $T = 4337 = T/\Delta_1$,

$$\sqrt{D(2W)} = (238W+17, \overline{8, 1, 1,\ 119W+8,\ 34,\ 119W+8,\ 1, 1, 8, 476W+34}) \quad \text{and} \quad \kappa = 3.$$

Case (5): When $\Delta' = 2$, $\Delta_1 > 1$, T is odd and T/Δ_1 is odd, consider $D(X) = 30^2X^2+2(50)X+5$, then $A = 30$, $B = 50$, $C = 5$ and $\Delta = -2000 = -2^4 \cdot 5^3$. Also, $\Delta_1 = 5$, $\Delta_2 = 5$, $\Delta_4 = 2$, $\tau = 1$, $A^* = 3$, $\Gamma = 1$, and $\Delta' = \tau\Delta_4/\Gamma = 2$.

When X is an odd natural number, i.e, $X = 2W + 1$ for some $W \geq 0$, we have $T = 95$, $T/\Delta_1 = 19$,

$$\begin{aligned}\sqrt{D(2W+1)} &= (60W+31, \overline{1,2,\ 6W+2,\ 1,5,\ 30W+15,\ 1,2,\ 24W+12,} \\ &\qquad \overline{2,1,\ 30W+15,\ 5,1,\ 6W+2,\ 2,1,\ 120W+62})\end{aligned}$$

and $\kappa = 6$.

When X is an even natural number, i.e, $X = 2W$ for some $W \geq 1$, we find $T = 185$ and $T/\Delta_1 = 37$. Further, we compute

$$\sqrt{D(2W)} = (60W+1, \overline{1,2,\ 6W-1,\ 1,5,\ 30W,\ 1,2,\ 24W,\ 2,1,\ 30W,\ 5,1,\ 6W-1,\ 2,1,\ 120W+2})$$

and $\kappa = 6$.

It is easy to see that the above two expansions have the same pattern. In fact, for any natural number X, we have

$$\sqrt{D(X)} = (30X+1, \overline{1,2,\ 3X-1,\ 1,5,\ 15X,\ 1,2,\ 12X,\ 2,1,\ 15X,\ 5,1,\ 3X-1,\ 2,1,\ 60X+2}).$$

Case (6): If $D(X) = 24^2X^2 + 2(80)X + 12$, then $A = 24$, $B = 80$, $C = 12$ and $\Delta = -512 = -2^9$. Also, $\Delta_1 = 2$, $\Delta_2 = 1$, $\Delta_4 = 4$, $\tau = 1$, $A^* = 6$, $\Gamma = 2$, and $\Delta' = \tau\Delta_4/\Gamma = 2$.

When X is an odd natural number, i.e, we have $X = 2W + 1$, where $W \geq 0$, we find that $T = 82$ is even and $T/\Delta_1 = 41$ is odd. Now,

$$\sqrt{D(2W+1)} = (48W+27, \overline{2,1,\ 12W+6,\ 5,1,\ 24W+12,\ 1,5,\ 12W+6,\ 1,2,96W+54})$$

and $\kappa = 4$.

When X is an even natural number, i.e, we have $X = 2W$, where $W \geq 1$, we find that $T = 154$ is even, $T/\Delta_1 = 77$ is odd,

$$\sqrt{D(2W)} = (48W+3, \overline{2,1,\ 12W,\ 5,1,\ 24W,\ 1,5,\ 12W,\ 1,2,96W+6})$$

and $\kappa = 4$. In fact, for $X \in \mathbb{N}$, we have

$$\sqrt{D(X)} = (24X+3, \overline{2,1,\ 6X,\ 5,1,\ 12X,\ 1,5,\ 6X,\ 1,2,48X+6}).$$

□

In what follows, we use the Lucas function U_n to establish an upper bound for κ. Put

$$P = \frac{T^2}{\Delta_1} - 2\sigma. \tag{5.5}$$

Let α and β be roots of $x^2 - Px + 1 = 0$. Then

$$\alpha + \beta = P \qquad \text{and} \qquad \alpha\beta = 1. \tag{5.6}$$

Let $U_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ as in Section 1.7. Then $U_0 = 0$, $U_1 = 1$, $U_2 = P$, and $U_{i+1} = PU_i - U_{i-1}$ for $i \in \mathbb{N}$. We will show that if ω is the rank of apparition of Δ' in the Lucas function U_n, then

$$\kappa \mid 2\,\omega. \tag{5.7}$$

Before we proceed to the theorem, we need several results.

Lemma 5.2.3 *For all $i \in \mathbb{N}$, we have*

$$r_i r_{i+1} \equiv d_{i+1} r_i \frac{T}{\Delta_1^{\varepsilon_{i+1}}} - \sigma d_i d_{i+1} \mod A'\Delta'. \tag{5.8}$$

Proof: If $r_i = 0$, then by Theorem 4.3.1, $d_{i+1} = \gcd(A'\Delta'/d_i, r_i)$. So we get $d_{i+1} = A'\Delta'/d_i$. Hence, $d_i d_{i+1} \equiv 0 \bmod A'\Delta'$ and (5.8) holds trivially.

Suppose that $r_i \neq 0$. By (4.33), we have

$$r_{i+1} \equiv d_{i+1}\frac{T}{\Delta_1^{\varepsilon_{i+1}}} - g_{i+1} \mod \frac{A'\Delta'}{d_{i+1}}.$$

Thus,

$$r_i r_{i+1} \equiv d_{i+1} r_i \frac{T}{\Delta_1^{\varepsilon_{i+1}}} - g_{i+1} r_i \mod \frac{A'\Delta' r_i}{d_{i+1}}.$$

Since $d_{i+1} \mid r_i$, we have

$$r_i r_{i+1} \equiv d_{i+1} r_i \frac{T}{\Delta_1^{\varepsilon_{i+1}}} - g_{i+1} r_i \mod A'\Delta'.$$

By (4.44), we get

$$r_i r_{i+1} \equiv d_{i+1} r_i \frac{T}{\Delta_1^{\varepsilon_{i+1}}} - \sigma d_i d_{i+1} \mod A'\Delta'.$$

□

Lemma 5.2.4 $\gcd(d_i, d_{i+1}) = 1$ *for all* $i \geq 1$.

Proof: We prove this lemma by induction on $i \in \mathbb{N}$. By Lemma 4.2.2, we have $d_1 = 1$. Hence, the result holds for $i = 1$. Suppose that $\gcd(d_i, d_{i+1}) = 1$ for some $i \geq 1$. Let p be a prime dividing d_{i+1}, then $p \nmid d_i$. By Theorem 4.3.1, we have $d_{i+1} = \gcd(A'\Delta'/d_i, r_i)$. It follows that $p \mid r_i$ and $p \mid A'\Delta'/d_i$. Let $n \in \mathbb{N}$ be such that $p^n \mid\mid A'\Delta/d_i$.

Case (1): If $p^n \mid r_i$, then $p^n \mid\mid d_{i+1}$. Since $\gcd(A', \Delta') = 1$ by (4.14), we have $p \nmid A'\Delta'/d_{i+1}$. Hence, $p \nmid d_{i+2}$ and $\gcd(d_{i+1}, d_{i+2}) = 1$.

Case (2): Suppose that $p^n \nmid r_i$. Then $r_i \neq 0$ and there exists some $m \in \mathbb{N}$ such that $m < n$ and $p^m \mid\mid r_i$. Hence, $p^m \mid\mid d_{i+1}$. By (5.8), we may write

$$\frac{r_i}{p^m} r_{i+1} \equiv d_{i+1} \frac{r_i}{p^m} \frac{T}{\Delta_1^{\varepsilon_{i+1}}} - \sigma d_i \frac{d_{i+1}}{p^m} \mod p^{n-m}.$$

If $p \mid r_{i+1}$, then since $p \mid d_{i+1}$ and $p \mid p^{n-m}$, we get $p \mid \sigma d_i d_{i+1}/p^m$, a contradiction. Hence, $p \nmid r_{i+1}$ and it follows that $p \nmid \gcd(A'\Delta'/d_{i+1}, r_{i+1})$, which is d_{i+2}. Therefore, $\gcd(d_{i+1}, d_{i+2}) = 1$. □

Put

$$Z_{-1} = 0, \;\; Z_0 = 1 \;\text{ and }\; Z_{i+1} = \frac{T}{\Delta_1^{\varepsilon_{i+1}}} Z_i - \sigma Z_{i-1} \;\;\text{ for }\;\; i \geq 0. \tag{5.9}$$

It follows that for $i \in \mathbb{N}$,

$$Z_{2i} = T Z_{2i-1} - \sigma Z_{2i-2} \quad \text{ and } \quad Z_{2i+1} = \frac{T}{\Delta_1} Z_{2i} - \sigma Z_{2i-1}.$$

We compute further to get from (5.5)

$$Z_{2i} = P Z_{2i-2} - Z_{2i-4} \quad \text{ and } \quad Z_{2i+1} = P Z_{2i-1} - Z_{2i-3},$$

where the first equation holds for $i \geq 2$ and the second one holds for $i \geq 1$.

If we use the Lucas function U_n in (5.6), then we get

$$\begin{aligned}
Z_2 &= \frac{T^2}{\Delta_1} - \sigma = \left(\frac{T^2}{\Delta_1} - \sigma\right) \cdot 1 - 0 = \left(\frac{T^2}{\Delta_1} - \sigma\right) U_1 - U_0, \\
Z_3 &= P\left(\frac{T}{\Delta_1}\right) = \left(\frac{T}{\Delta_1}\right) U_2, \\
Z_4 &= P Z_2 - Z_0 = P\left(\frac{T^2}{\Delta_1} - \sigma\right) - 1 = \left(\frac{T^2}{\Delta_1} - \sigma\right) U_2 - U_1, \\
Z_5 &= P Z_3 - Z_1 = P\left(\frac{T}{\Delta_1}\right) U_2 - \frac{T}{\Delta_1} = \left(\frac{T}{\Delta_1}\right) U_3.
\end{aligned}$$

By induction on $i \in \mathbb{N}$, we get

$$Z_{2i} = \left(\frac{T^2}{\Delta_1} - \sigma\right) U_i - U_{i-1} = U_{i+1} + \sigma U_i \quad \text{ and } \quad Z_{2i+1} = \frac{T}{\Delta_1} U_{i+1}. \tag{5.10}$$

Lemma 5.2.5 $\gcd(Z_i, Z_{i+1}) = 1$ *for all* $i \geq 0$.

<u>Proof</u>: Let p be a prime that divides $\gcd(Z_i, Z_{i+1})$. It follows from (5.9) that $p \mid Z_{i-1}$. By induction, we find p divides $\gcd(Z_0, Z_1) = 1$, hence, $\gcd(Z_i, Z_{i+1}) = 1$. □

Lemma 5.2.6 $d_i Z_i \equiv r_i Z_{i-1} \bmod \Delta'$ *for all* $i \geq 0$.

Proof: We prove by induction on i. The statement holds trivially if $\Delta' = 1$, so we assume $\Delta' > 1$. Since $Z_{-1} = 0$ and $d_0 = \Delta'$, it is clear that $d_0 Z_0 \equiv 0 \equiv r_0 Z_{-1} \bmod \Delta'$. Further, since $Z_0 = 1$ and $Z_1 = T/\Delta_1$ and by Lemma 4.2.2, $d_1 = 1$ and $\Delta' \mid g_1$, it follows that

$$d_1 Z_1 = \frac{T}{\Delta_1} \equiv \frac{T}{\Delta_1} - g_1 \equiv r_1 \equiv r_1 Z_0 \bmod \Delta'.$$

Let p be a prime such that $p^n \mid\mid \Delta'$, where $n \in \mathbb{N}$. Suppose that for some $i \geq 1$ and all j such that $0 \leq j \leq i$,

$$d_j Z_j \equiv r_j Z_{j-1} \bmod p^n. \tag{5.11}$$

Note that

$$\begin{aligned} d_{i+1} r_i Z_{i+1} &= d_{i+1} r_i \left(\frac{T}{\Delta_1^{\varepsilon_{i+1}}} Z_i - \sigma Z_{i-1} \right) && (5.12) \\ &= d_{i+1} r_i \frac{T}{\Delta_1^{\varepsilon_{i+1}}} Z_i - \sigma d_{i+1} r_i Z_{i-1} \\ &\equiv d_{i+1} r_i \frac{T}{\Delta_1^{\varepsilon_{i+1}}} Z_i - \sigma d_i d_{i+1} Z_i \mod d_{i+1} p^n \\ &\equiv \left(d_{i+1} r_i \frac{T}{\Delta_1^{\varepsilon_{i+1}}} - \sigma d_i d_{i+1} \right) Z_i \mod d_{i+1} p^n. && (5.13) \end{aligned}$$

So, by Lemma 5.2.3, we have

$$d_{i+1} r_i Z_{i+1} \equiv r_i r_{i+1} Z_i \mod p^n. \tag{5.14}$$

Case (1): If $p \nmid r_i$, then

$$d_{i+1} Z_{i+1} \equiv r_{i+1} Z_i \mod p^n.$$

For the remaining cases, we assume $p \mid r_i$.

Case (2): If $p \mid d_i$, then $p^n \nmid d_{i-1}$ and $p \nmid d_{i+1}$ by Lemma 5.2.4. Since $d_{i+1} = \gcd(\Delta'/d_i, r_i)$, it follows that $p^n \mid\mid d_i$. Since $d_i \mid r_{i-1}$, we have $p^n \mid r_{i-1}$. By (5.11) with $j = i - 1$, we have $p^n \mid Z_{i-1}$.

Hence, by (5.9),

$$Z_{i+1} = \frac{T}{\Delta_1^{\varepsilon_{i+1}}} Z_i - \sigma Z_{i-1} \equiv \frac{T}{\Delta_1^{\varepsilon_{i+1}}} Z_i \bmod p^n.$$

Since $p^n \mid \Delta'$, by (4.33),

$$r_{i+1} \equiv d_{i+1} \frac{T}{\Delta_1^{\varepsilon_{i+1}}} - g_{i+1} \mod p^n.$$

If $r_i = 0$, then $g_{i+1} \equiv 0 \bmod d_i$ by Lemma 4.2.5. If $r_i > 0$, then $g_{i+1} \equiv 0 \bmod d_i$ by (4.47). Since $p^n \mid\mid d_i$, we have

$$r_{i+1} \equiv d_{i+1} \frac{T}{\Delta_1^{\varepsilon_{i+1}}} \mod p^n.$$

Thus,

$$d_{i+1} Z_{i+1} \equiv d_{i+1} \frac{T}{\Delta_1^{\varepsilon_{i+1}}} Z_i \equiv r_{i+1} Z_i \bmod p^n.$$

For Cases (3) and (4), we assume $p \nmid d_i$.

Case (3): If $p^n \mid r_i$, then by (5.11) with $j = i$, we have $p^n \mid Z_i$. Since $p \nmid d_i$, it follows that p^n divides $d_{i+1} = \gcd(\Delta'/d_i, r_i)$. Hence, trivially,

$$d_{i+1} Z_{i+1} \equiv 0 \equiv r_{i+1} Z_i \mod p^n.$$

Case (4): If $p^n \nmid r_i$, then $r_i \neq 0$ and there exists some $m \in \mathbb{N}$ such that $n > m$ and $p^m \mid\mid r_i$. Since $p \nmid d_i$, we have $p^m \mid\mid d_{i+1}$. Now, $p^m \mid\mid Z_i$ by (5.11) with $j = i$. Now, by (5.8), we may write

$$r_i r_{i+1} Z_i \equiv \left(d_{i+1} r_i \frac{T}{\Delta_1^{\varepsilon_{i+1}}} - \sigma d_i d_{i+1} \right) Z_i \mod p^{m+n}.$$

Thus, by (5.13), we have $d_{i+1} r_i Z_{i+1} \equiv (r_i r_{i+1}) Z_i \mod p^{m+n}$. Since $p^m \mid\mid r_i$, it follows that $d_{i+1} Z_{i+1} \equiv r_{i+1} Z_i \mod p^n$.

Therefore, the statement holds for all $i \geq 0$ by induction. □

Theorem 5.2.2 *Let ω be the rank of apparition of Δ' in U_n. Then $\kappa \mid 2\omega$, which implies that $\kappa \leq 2\omega$.*

Proof: By Lemmas 5.2.5 and 5.2.6, it follows that $\Delta' \mid Z_i$ implies $\Delta' \mid d_{i+1}$ for $i \geq 0$. If ω is the rank of apparition of Δ' in the Lucas function U_n, then $\Delta' \mid U_\omega$. By (5.10), we have $\Delta' \mid Z_{2\omega-1}$ and hence, $\Delta' \mid d_{2\omega}$. Since $d_{2\omega} \mid \Delta'$ by Remark 4.2.2, it follows that $d_{2\omega} = \Delta'$. Since $Q_i = \Delta_1^{\varepsilon_i}(\Delta'/d_i)^2$ by (4.40) or (4.45), we have

$$Q_{2\omega} = \Delta_1^{\varepsilon_{2\omega}} \left(\frac{\Delta'}{d_{2\omega}}\right)^2 = 1.$$

Since κ by definition is the least natural number such that $Q_\kappa = 1$, we have $\kappa \leq 2\omega$. Also, since the continued fraction expansion of $\sqrt{D(X)}$ is periodic, we have $\kappa \mid 2\omega$. □

Example 5.2.2 Consider $D(X) = 119^2X^2 + 2(833)X + 245$. Then $A = 119$, $B = 833$, $C = 245$, $\Delta = -2^2 \cdot 7^4 \cdot 17^2$ and $\sigma = -1$. Also, $\Delta_1 = 1$, $\Delta_2 = 34$, $\Delta_4 = 7$ and $\Delta' = 14$. For $X \geq 2$, as K varies from 0 to 13, we compute κ and $\omega(\Delta')$ and arrive at Table 5.2:

K	0	1	2	3	4	5	6	7	8	9	10	11	12	13
κ	24	8	**6**	8	24	6	6	8	24	**2**	24	8	6	6
$\omega(\Delta')$	12	4	**21**	4	12	6	3	4	12	**14**	12	4	3	6

Table 5.2: Values of κ and $\omega(\Delta')$ of $\sqrt{119^2X^2 + 2(833)X + 245}$.

Observe that aside from the exceptional cases at $K = 2$ and 9, in which $\Delta' = 7$ divides $\omega(\Delta')$, we see that $\kappa = 2\omega(\Delta')$ or $\kappa = \omega(\Delta')$. This phenomenon can be further demonstrated.

Example 5.2.3 Let $D(X) = 2002^2X^2 + 2(44044) + 605$. Then, $A = 2002$, $B = 44044$, $C = 605$ and $\Delta = -2^2 \cdot 7^2 \cdot 11^4 \cdot 13^2$. Also, $\Delta_1 = 1$; $\Delta_2 = 2 \cdot 7 \cdot 13$, $\Delta_4 = 11$, $\tau = 1$, $A^* = 1$ and $\Delta' = 11$. Let $X \geq 1$. For $0 \leq K < \Delta' = 11$, we compute κ as well as $\omega(\Delta')$ in the Lucas function U_n and arrive at Table 5.3.

K	0	1	2	3	4	5	6	7	8	9	10
κ	10	12	12	10	4	12	10	**2**	10	12	4
$\omega(\Delta')$	5	6	6	5	4	6	5	**11**	5	6	4

Table 5.3: Values of κ and $\omega(\Delta')$ of $\sqrt{2002^2X^2+2(44044)+605}$.

Again, excluding the exceptional case at $K=7$, where $\Delta'=11$ divides $\omega(\Delta')$, we see that κ is either $\omega(\Delta')$ or $2\,\omega(\Delta')$.

Example 5.2.4 Consider $D(X)=119^2X^2+2(2023)X+1445$. We have $\Delta=-2^2\cdot 7^2\cdot 17^4$ and $\Delta'=34$. For $X\geq 9$,

K	0	1	2	3	4	5	6	7	8	9	10	11	**12**	13	14	15	16
κ	9	34	24	18	9	8	51	18	9	6	24	4	**6**	4	24	6	9
$\omega(\Delta')$	9	34	12	18	9	4	51	18	9	6	12	4	**51**	4	12	6	9
K	17	18	19	20	21	22	23	24	25	26	27	28	**29**	30	31	32	33
κ	18	51	8	9	18	24	34	9	18	3	8	12	**2**	12	8	3	18
$\omega(\Delta')$	18	51	4	9	18	12	34	9	18	3	4	6	**34**	6	4	3	18

Table 5.4: Values of κ and $\omega(\Delta')$ of $\sqrt{119^2X^2+2(2023)X+1445}$.

We see that κ is either $\omega(\Delta')$ or $2\,\omega(\Delta')$ when K is not 12 or 29.

It is not difficult to see from the above tables that $\kappa \mid 2\,\omega(\Delta')$ as asserted by Theorem 5.2.2. More importantly, Tables 5.2, 5.3 and 5.4 suggest that κ is either the rank of apparition of Δ' in the Lucas function U_n or twice that in most cases. Indeed, from our other computations, we find that is so.

As we have seen above, the use of $\omega(\Delta')$ gives us a good upper bound for κ. However, it may be time consuming to find $\omega(\Delta')$. It will be more practical to have an upper bound that does not require any additional computations. For this purpose we have

Theorem 5.2.3 *$\omega(\Delta') \leq \Delta'$ if Δ' is odd and $\omega(\Delta') \leq (3/2)\Delta'$ if Δ' is even.*

Proof: Case (1): Let Δ' be odd and p be an odd prime. If $p \mid P^2 - 4$, where P is as defined in (5.5) and $P^2 - 4$ is the discriminant of the quadratic $x^2 - Px + 1 = 0$, then $p \mid U_p$ by (1.59). This implies that $\omega(p)$ divides p by Theorem 1.7.3. Since p is an odd prime, $\omega(p) = p$. It follows by induction that $\omega(p^n) = p^n$ for all $n \in \mathbb{N}$.

For the case where $p \nmid P^2 - 4$, we write $\epsilon = \epsilon(p)$ as the Legendre symbol $(\frac{P^2-4}{p})$. Since p is an odd prime, $p \nmid P^2 - 4$ and $(\frac{1}{p}) = 1$, by Theorem 1.7.1, we have $p \mid U_{(p-\epsilon)/2}$. By Theorem 1.7.4, the law of repetition, we have $p^n \mid U_{p^{n-1}(p-\epsilon(p))/2}$ for any natural number n.

Define

$$\Lambda'(p^n) = \begin{cases} p^n & \text{if } p \mid P^2 - 4, \\ p^{n-1}(p-\epsilon)/2 & \text{if } p \nmid P^2 - 4. \end{cases} \tag{5.15}$$

Let $m = \prod_{i=1}^{k} p_i^{n_i}$, where p_i are distinct primes and $n_i \in \mathbb{N}$, and define

$$\Lambda'(m) = \text{lcm}\,\{\Lambda'(p_i^{n_i}) : i = 1, 2, \ldots, k\}.$$

If we write $\omega(m)$ as the rank of apparition of m in U_n, then $\omega(m) \mid \Lambda'(m)$. Also, it follows from our construction of $\Lambda'(m)$ that $\Lambda'(m) \leq m$. Hence, $\omega(\Delta') \mid \Lambda(\Delta')$ and $\omega(\Delta') \leq \Delta'$.

Case (2): Suppose that Δ' is even. By (1.61), we have $2 \mid U_{2-\epsilon(2)}$, where $\epsilon(2)$ is defined in (1.60). By the law of repetition, we have $2^n \mid U_{2^{n-1}(2-\epsilon(2))}$ for $n \geq 1$. Hence, $2^n \mid U_{2^n}$ if $2 \mid P$ and $2^n \mid U_{3 \cdot 2^{n-1}}$ if $2 \nmid P$. Thus, $\omega(2^n) < 3 \cdot 2^{n-1}$. Let $2^n \parallel \Delta'$ for some $n \in \mathbb{N}$. Then

$$\omega(\Delta') \leq \omega(2^n)\frac{\Delta'}{2^n} \leq 3 \cdot 2^{n-1}\frac{\Delta'}{2^n} = \frac{3}{2}\Delta'.$$

$\square$

Corollary 5.2.1 *$\kappa \leq 2\Delta'$ if Δ' is odd and $\kappa \leq 3\Delta'$ if Δ' is even.*

Example 5.2.5 Using Example 4.3.1 and the discussion concerning the number of different patterns of continued fraction expansion of $\sqrt{119^2X^2+2(2205)X+343}$ in Section 5.1, we have $\Delta'=7$ and

K	0	1	2	3	4	5	6
κ	14	6	2	6	14	4	4

Table 5.5: Values of κ for $\sqrt{119^2X^2+2(2205)X+343}$.

We see that the upper bound $2\Delta'$ for κ is attainable at $K=0$ or 4. □

Note that in the above example Δ' is an odd prime. When $\Delta'=2$, in view of Case (5) on page 127, we see that $\kappa=6=3\Delta'$. So, $3\Delta'$ is also attainable as an upper bound for κ.

5.3 Fundamental Unit of $[1, \sqrt{D(X)}]$

In Section 2.3, we presented Stender's result (2.11) on the fundamental unit of the order $[1, \sqrt{D(X)}]$, where the radicand $D(X)$ is assumed to be squarefree for some sufficiently large integer X. In this section, we make use of Theorem 4.3.1 to get the fundamental unit of the real quadratic order $[1, \sqrt{D(X)}]$ for integers $A>0$ and $X>(2A\Delta_1\Delta'^2-2B+A)/(2A^2)$. Here, we do not require $D(X)$ to be squarefree.

It is well-known that if ε is the fundamental unit of $[1, \sqrt{D}]$ for some non-square natural number D, then

$$\varepsilon=\prod_{i=1}^{\ell}\theta_i,$$

where $\ell=\mathrm{lp}(\sqrt{D(X)})$ and θ_i is the i-th complete quotient of $\sqrt{D}$. For the proof of the above statement, see [133, Theorem 2.1.3].

Lemma 5.3.1 *Let $\theta(X) = (P+\sqrt{D(X)})/Q$ be a complete quotient of $\sqrt{D(X)}$, $q(X) = \lfloor (P+\sqrt{D(X)})/Q \rfloor$, and E and F as defined in Lemma 4.1.1 so that $F/E = (s_0,\ s_1, \ldots, s_{m-1})$ and*

$$\frac{P+\sqrt{D(X)}}{Q} = (q(X),\ s_0,\ s_1, \ldots, s_{m-1}, \theta'),$$

where the parity of m is chosen according to the sign of Δ. If $\theta_j(X)$ is the j-th complete quotient of $(P+\sqrt{D(X)})/Q$, then

$$\prod_{j=1}^{m+1} \theta_j(X) = \frac{d(A^2X + B + A\sqrt{D(X)})}{|\Delta|},$$

where $d = \gcd(E, F)$.

Proof: Let A_j/B_j be the j-th convergent of $\theta(X)$. Then by (1.14), we have

$$\theta_j(X) = -\frac{B_{j-2}\,\theta(X) - A_{j-2}}{B_{j-1}\,\theta(X) - A_{j-1}}$$

for $j \geq 0$. Thus,

$$\prod_{j=1}^{m+1} \theta_j(X) = \frac{(-1)^{m+1}}{B_m\,\theta(X) - A_m}.$$

Since $A_m/B_m = (q(X),\ s_0,\ s_1, \ldots, s_{m-1})$, we have

$$\frac{A_m}{B_m} = q(X) + \frac{1}{(s_0,\ s_1, \ldots, s_{m-1})} = q(X) + \frac{E}{F} = \frac{q(X)F + E}{F}.$$

Recall from Lemma 4.1.1 that $E = A^2X + B - AP'$ and $F = AQ$, where $P' = q(X)Q - P$. If $d = \gcd(E, F)$, then $d = \gcd(q(X)F + E, F)$. Hence,

$$dA_m = q(X)F + E = q(X)AQ + A^2X + B - AP' \quad \text{and} \quad dB_m = AQ.$$

So,

$$B_m\,\theta(X) - A_m = \frac{1}{d}\left(AQ\theta(X) - q(X)AQ - A^2X - B + AP'\right).$$

Since

$$AQ\theta(X) - q(X)AQ + AP' = AQ\left(\frac{P+\sqrt{D(X)}}{Q}\right) - q(X)AQ + A(q(X)Q - P) = A\sqrt{D(X)},$$

we have

$$B_m\,\theta(X) - A_m = -\frac{A^2X + B - A\sqrt{D(X)}}{d}.$$

Hence,

$$\prod_{j=1}^{m+1}\theta_j(X) = \frac{(-1)^{m+1}}{B_m\theta(X) - A_m} = \frac{(-1)^{m+1}d(A^2X + B + A\sqrt{D(X)})}{\Delta} = \frac{d(A^2X + B + A\sqrt{D(X)})}{|\Delta|};$$

the last equality follows by the parity of $l(i)$ and the sign of Δ. $\square$

Theorem 5.3.1 *The fundamental unit ε of $[1, \sqrt{D(X)}]$ is given by*

$$|\Delta|^{\kappa/2}\left(\frac{A^2X + B + A\sqrt{D(X)}}{|\Delta|}\right)^{\kappa} = \left(\frac{A^2X + B + A\sqrt{D(X)}}{\sqrt{|\Delta|}}\right)^{\kappa}. \tag{5.16}$$

Moreover, $\mathcal{N}(\varepsilon) = \sigma^{\kappa}$. Note that by Remark 4.3.1, *κ is even when $\Delta_1 > 1$, so $|\Delta|^{\kappa/2} \in \mathbb{N}$.*

Proof: By Theorem 4.3.1,

$$\sqrt{D(X)} = (q_0(X), \overline{\mathcal{S}_0,\ q_1(X),\ \mathcal{S}_1, \ldots,\ \mathcal{S}_{\kappa-1}, q_\kappa(X)}),$$

where $\mathcal{S}_i$ and $q_i(X)$ are defined in the previous chapter.

For $i \geq 0$, let $\theta_i(X) = (q_i(X),\ \mathcal{S}_i,\ \theta_{i+1}(X))$, $\theta'_j(X)$ be the j-th complete quotient of $\theta_i(X)$ and

$$\vartheta_i(X) = \prod_{j=1}^{1+|\mathcal{S}_i|}\theta'_j(X).$$

Further, let $\theta_k(X)$ be the k-th complete quotient of $\sqrt{D(X)}$ and $\ell = \sum_0^{\kappa-1}(1 + |\mathcal{S}_i|)$. Then by the remark in the beginning of this section, the fundamental unit ϵ of $[1, \sqrt{D(X)}]$ is given by

$$\varepsilon = \prod_{k=1}^{\ell}\theta_i(X) = \prod_{i=0}^{\kappa-1}\vartheta_i(X).$$

By Lemma 5.3.1, we have

$$\vartheta_i(X) = \frac{\delta_{i+1}(A^2X + B + A\sqrt{D(X)})}{|\Delta|},$$

where $\delta_1 = (\Gamma\Delta_2\Delta_4/\tau)d_1$, $\delta_i = (\Delta_1^{\varepsilon_{i-1}}\Delta_2\Delta_4^2/d_{i-1})d_i$ for $i \geq 2$. Thus,

$$\varepsilon = \prod_{i=0}^{\kappa-1}\vartheta_i = \prod_{i=0}^{\kappa-1}\frac{\delta_{i+1}(A^2X + B + A\sqrt{D(X)})}{|\Delta|} = \left(\frac{A^2X + B + A\sqrt{D(X)}}{|\Delta|}\right)^{\kappa}\prod_{i=1}^{\kappa}\delta_i.$$

Now,

$$\prod_{i=1}^{\kappa}\delta_i = \frac{\Gamma\Delta_2\Delta_4}{\tau}d_1 \cdot \frac{\Delta_1^{\varepsilon_1}\Delta_2\Delta_4^2}{d_1}d_2 \cdot \frac{\Delta_1^{\varepsilon_2}\Delta_2\Delta_4^2}{d_2}d_3 \cdot \cdots \cdot \frac{\Delta_1^{\varepsilon_{\kappa-1}}\Delta_2\Delta_4^2}{d_{\kappa-1}}d_\kappa.$$

Since ε_i is either 0 or 1 according as i being even or odd, and since κ is even if $\Delta_1 > 1$ by Remark 4.3.1, we may write

$$\prod_{i=1}^{\kappa}\delta_i = \Delta_1^{\kappa/2}\,\frac{\Gamma\Delta_2\Delta_4}{\tau}\,(\Delta_2\Delta_4^2)^{\kappa-1}\,d_\kappa.$$

By Theorem 4.3.1, we have $d_\kappa = \Delta'$ and, consequently, $(\Gamma\Delta_2\Delta_4/\tau)d_\kappa = (\Gamma\Delta_2\Delta_4/\tau)(\tau\Delta_4/\Gamma) = \Delta_2\Delta_4^2$. Thus,

$$\prod_{i=1}^{\kappa}\delta_i = |\Delta|^{\kappa/2} \quad \text{and} \quad \varepsilon = |\Delta|^{\kappa/2}\left(\frac{A^2X + B + A\sqrt{D(X)}}{|\Delta|}\right)^{\kappa}.$$

Now,

$$\mathcal{N}(\varepsilon) = \frac{|\Delta|^\kappa}{|\Delta|^{2\kappa}} \cdot \mathcal{N}\left(A^2X + B + A\sqrt{D(X)}\right)^{\kappa} = \frac{|\Delta|^\kappa}{|\Delta|^{2\kappa}}\Delta^\kappa = \frac{\Delta^\kappa}{|\Delta|^\kappa} = \sigma^\kappa.$$

□

Corollary 5.3.1 *If* $\Delta_1 > 1$, *then* $\mathcal{N}(\varepsilon) = 1$.

Example 5.3.1 Consider $D(X) = 119^2X^2 + 2(2205)X + 343$ with $X = 1$. As we have seen in Example 4.3.1, $A = 119 = 7\cdot 17$, $B = 2205 = 3^2\cdot 5\cdot 7^2$, $C = 343 = 7^3$, $\Delta = B^2 - A^2C = 4802 = 2\cdot 7^4$, $\sigma = 1$, $\Delta_1 = 2$, $\Delta_2 = 1$, $\Delta_4 = 7$ and $\kappa = 6$. Also, $D(1) = 18914 = 2\cdot 7^2\cdot 193$ and

$$\sqrt{D(1)} = (137, \overline{1,1,8,2,1,2,4,1,1,1,2,5,4,3,1,136,1,3,4,5,2,1,1,1,4,2,1,2,8,1,1,274}).$$

By Theorem 1.4.1, we use the above continued fraction expansion and find the fundamental unit of $[1, \sqrt{18914}]$ to be

$$\varepsilon = 5552992832780835 + 40377127625446\sqrt{18914}.$$

Now, since $\kappa = 6$, by Theorem 5.3.1 , we have

$$\begin{aligned}
\varepsilon &= 4802^3 \left(\frac{119^2 \cdot 1 + 2205 + 119\sqrt{18914}}{4802} \right)^6 \\
&= 5552992832780835 + 40377127625446\sqrt{18914},
\end{aligned}$$

which agrees with the value given by Theorem 1.4.1. We also find $\mathcal{N}(\varepsilon) = 1 = \sigma^6$. $\square$

Chapter 6

Final Comments

The theme of this thesis is the continued fraction expansion of the $\sqrt{D(X)}$, where $D(X)$ is a quadratic polynomial that satisfies the Schinzel condition. We established two main results. The first result in Chapter 3 showed that the continued fraction expansion of $\sqrt{D}$ can be determined by a palindromic string of natural numbers by means of using a quadratic polynomial. On arriving at the appropriate quadratic $D(X)$, we also obtained a family of non-square D such that all members of the family correspond to the given palindromic string. Also, we found that when certain conditions are met, independent of the symmetric sequence that defines the family, the product of two members of a family is of R-D type. More importantly, the quadratic $D(X)$ satisfies the Schinzel condition.

The second result, and the more significant one, gives the continued fraction expansion of $\sqrt{D(X)}$ for any Schinzel sleeper $D(X)$, i.e, a quadratic $D(X)$ that satisfies the Schinzel condition. By this result, we were able to construct an upper bound for $\mathrm{lp}(\sqrt{D(X)})$ via the Fibonacci numbers and Lucas function U_n. We get

$$\mathrm{lp}(\sqrt{D(X)}) < \begin{cases} 3\Delta' \cdot \lfloor \log_\varphi(\sqrt{5} \cdot A'\Delta') \rfloor & \text{if } \Delta' \text{ is even,} \\ 2\Delta' \cdot \lfloor \log_\varphi(\sqrt{5} \cdot A'\Delta') \rfloor & \text{if } \Delta' \text{ is odd,} \end{cases}$$

where $\varphi = (1+\sqrt{5})/2$ is the golden ratio. Also, knowing the continued fraction expansion of $\sqrt{D(X)}$ allows us to compute the fundamental unit ε of the order $[1, \sqrt{D(X)}]$; we find

$$\varepsilon = \left(\frac{A^2X + B + A\sqrt{D(X)}}{\sqrt{|\Delta|}} \right)^\kappa$$

when $X > (2A\Delta_1\Delta'^2 - 2B - A)/(2A^2)$.

There are several possible research directions that can be followed: symmetries of the continued fraction expansions of $\sqrt{D(X)}$ as X varies, behaviour of r_i, and upper bound and expected value for $\mathrm{lp}(\sqrt{D(X)})$.

6.1 Symmetries of the Continued Fraction Expansions of $\sqrt{D(X)}$

Since the period of $\sqrt{D(X)}$ is symmetric when we exclude the last partial quotient, $q_\kappa(X)$, it is clear that the continued fraction expansion of $\sqrt{D(X)}$ has a horizontal symmetry.

There is also a vertical symmetry. Recall that $\sqrt{D(X)}$ has a number of different patterns depending on which residue class modulo Δ' that X belongs to. We write $X \equiv K \mod \Delta'$, where $0 \leq K < \Delta'$. For each K, we compute the corresponding values of d_ir_i and list them in a table. We observe a symmetry pertaining to the values of d_ir_i of different K. Take Case (5) of Appendix C.1, for instance. The quadratic here is $D(X) = 119^2X^2 + 2(1666)X + 98$ with $A' = 1$ and $\Delta' = 7$. We list the values of d_ir_i for different values of K in the table below.

K	d_0r_0	d_1r_1	d_2r_2	d_3r_3	d_4r_4	d_5r_5	d_6r_6	d_7r_7	d_8r_8	d_9r_9	$d_{10}r_{10}$	$d_{11}r_{11}$	$d_{12}r_{12}$	$d_{13}r_{13}$
1	7	5	7	7	3	7								
2	7	1	1	7										
3	7	4	6	5	5	1	7	7	1	3	3	6	2	7
4	7	7												
5	7	3	1	2	2	6	7	7	6	4	4	1	5	7
6	7	6	6	7										
0	7	2	7	7	4	7								

It is easy to see that the sum of $d_1(1)r_1(1) = 5$ and $d_1(0)r_1(0) = 2$ is congruent to 0 modulo $\Delta' = 7$. Indeed, for $0 \leq K, K' \leq 6$, if $K + K' \equiv 1 \mod 7$, then

$$d_i(K)r_i(K) + d_i(K')r_i(K') \equiv 0 \mod 7 \tag{6.1}$$

for $i \geq 0$. In this particular example, the pairs (K, K') are $(0,1)$, $(2,6)$, $(3,5)$ and $(4,4)$. So, it is clear that there is a symmetry defined by (6.1).

In general, we find that for some fixed $n \in \mathbb{Z}$ and $0 \leq K, K' \leq \Delta' - 1$, if $K + K' \equiv n \mod \Delta'$, then

$$d_i(K)r_i(K) + d_i(K')r_i(K') \equiv 0 \mod \Delta' \tag{6.2}$$

for $i \geq 0$. Numerical evidence for this claim is available in Appendix C; in particular, see Table C.15 of Example C.2.6.

There is more to learn in regard to the values of $d_i r_i$. For example, the values of $d_i r_i$ when viewed in a certain way exhibit an implicit group structure. In the multiplicative group $\mathbb{Z}_7^*$, there are 2 proper subgroups: $H_1 = \{1, 6\}$ and $H_2 = \{1, 2, 4\}$. Also, the primitive roots are 3 and 5.

Now, we realign the above table by matching the entries of the shorter expansions to those of the longer expansions. For example, the first entry in the first row is 7, which is the same as the first entry in the third row. So, 7 remains where it is. The second entry in the first row is 5. The first occurrence of 5 in the third row is at the fourth entry. So, we move 5 to the fourth entry on the first row. Continuing in this fashion, we arrive at the following table:

K	$d_i r_r$													
1	7			5			7	7			3			7
2	7					1			1					7
3	7	4	6	5	5	1	7	7	1	3	3	6	2	7
4	7													7
5	7	3	1	2	2	6	7	7	6	4	4	1	5	7
6	7					6			6					7
0	7			2			7	7			4			7

It is easy to see that except for the columns that contain $\Delta' = 7$, all other columns on the upper or lower half of the table contain entries that either belong to a particular proper subgroup or have the same order in $\mathbb{Z}_7^*$. As it turns out, when Δ' is prime and there are only two long expansions, we can usually arrange the values of $d_i r_i$ so that the columns correspond to a proper subgroup or a particular order. However, when there are more than two long expansions, there is no clear method of rearranging the entries so that a similar correspondence between columns and subgroups can be achieved.

When Δ' is composite, the situation gets more complicated. However, in some cases, we can mimic the above method. Consider Table C.15 where $A' = 1$ and $\Delta' = 22$. We have

K	d_0r_0	d_1r_1	d_2r_2	d_3r_3	d_4r_4	d_5r_5	d_6r_6	d_7r_7	d_8r_8	d_9r_9	$d_{10}r_{10}$	$d_{11}r_{11}$	$d_{12}r_{12}$	$d_{13}r_{13}$	$d_{14}r_{14}$	$d_{15}r_{15}$	$d_{16}r_{16}$	$d_{17}r_{17}$	$d_{18}r_{18}$	$d_{19}r_{19}$	$d_{20}r_{20}$	$d_{21}r_{21}$
5	22	12	22	22	4	22																
6	22	14	18	4	22	22	12	10	16	22												
7	22	16	6	8	14	22																
8	22	18	18	8	22	22	6	10	10	22												
9	22	20	16	6	6	12	10	2	12	16	22	22	14	4	2	18	4	8	8	14	20	22
10	22	22																				
0	22	2	6	16	16	10	12	20	10	6	22	22	8	18	20	4	18	14	14	8	2	22
1	22	4	4	14	22	22	16	12	12	22												
2	22	6	16	14	8	22																
3	22	8	4	18	22	22	10	12	6	22												
4	22	10	22	22	18	22																

Since all entries in the main body of the table are even and they are denominators of the ratio $A'\Delta'/(d_i r_i) = 22/(d_i r_i)$, we can remove the common factor 2 from the ratio and rewrite the above table as

K	d_0r_0	d_1r_1	d_2r_2	d_3r_3	d_4r_4	d_5r_5	d_6r_6	d_7r_7	d_8r_8	d_9r_9	$d_{10}r_{10}$	$d_{11}r_{11}$	$d_{12}r_{12}$	$d_{13}r_{13}$	$d_{14}r_{14}$	$d_{15}r_{15}$	$d_{16}r_{16}$	$d_{17}r_{17}$	$d_{18}r_{18}$	$d_{19}r_{19}$	$d_{20}r_{20}$	$d_{21}r_{21}$
5	11	6	11	11	2	11																
6	11	7	9	2	11	11	6	5	8	11												
7	11	8	3	4	7	11																
8	11	9	9	4	11	11	3	5	5	11												
9	11	10	8	3	3	6	5	1	6	8	11	11	7	2	1	9	2	4	4	7	10	11
10	11	11																				
0	11	1	3	8	8	5	6	10	5	3	11	11	4	9	10	2	9	7	7	4	1	11
1	11	2	2	7	11	11	8	6	6	11												
2	11	3	8	7	4	11																
3	11	4	2	9	11	11	5	6	3	11												
4	11	5	11	11	9	11																

Since each of the two long expansions has four occurrences of 11 and they are positioned in the middle or on the extreme ends of the table, we use them as outlines to partition the table into four quadrants. Now that 11 is the numerator, we work in the multiplicative group $\mathbb{Z}_{11}^*$. the proper subgroups of $\mathbb{Z}_{11}^*$ are $H_2 = \{1, 10\}$ and $H_5 = \{1, 3, 4, 5, 9\}$, and the primitive roots are $2, 6, 7, 8$.

In each of the four quadrants, we allot elements of the same order to the same column except the orders in the top part of the table are always half or double those in the lower part.

K	reduced d_ir_i with numerator 11																					
5	11					6					11	11					2					11
6	11		7		9				2		11	11		6				5		8		11
7	11		8		3													4		7		11
8	11			9	9		4				11	11				3		5	5			11
9	11	10	8	3	3	6	5	1	6	8	11	11	7	2	1	9	2	4	4	7	10	11
10	11																					11
0	11	1	3	8	8	5	6	10	5	3	11	11	4	9	10	2	9	7	7	4	1	11
1	11			2	2		7				11	11				8		6	6			11
2	11		3		8													7		4		11
3	11		4		2				9		11	11		5				6		3		11
4	11					5					11	11					9					11

It is easy to see that the third column of the top left quadrant consists of $6, 7, 8$, which correspond to the primitive roots; the fourth column consists of 3 and 9, which correspond to $\{1,3,4,5,9\}$. Similarly, we see this correspondence in other quadrants as well. This observation leads us to believe that there is an implicit group structure involved in the computation of $d_i r_i$.

From the table above (6.1), we see that $r_0(K)$ remains fixed as K varies and $r_1(K) = 3K + 2 \bmod 7$ for $0 \leq K \leq 6$. By Lemma 4.2.2, it is clear that $r_0(K)$ is constant. By (4.33), since $d_1 = 1$ and g_1 are independent of K, it follows that $r_1(K)$ is a linear function in K modulo $A'\Delta'$. Since $r_0(K)$ is constant and $r_1(K)$ is a linear function modulo $A'\Delta'$, it is natural to guess that $r_2(K)$ is a quadratic function. This is not so. We have $r_2(K) = 7,1,6,7,1,6,7$ for $K = 0,1,\ldots,6$, and this sequence of natural numbers is not given by a quadratic. So it remains to determine whether $r_2(K)$ can be represented by some function in K. In fact, we are interested in whether $r_i(K)$ is representable by a function in K for $i \geq 2$.

When we inspect the entries in the above tables horizontally, we see a repetition of a particular $d_i r_i$ in a row. For instance, in the table above (6.1), we have $d_3(3)r_3(3) = 5 = d_4(3)r_4(3)$ and $d_3(5)r_3(5) = 2 = d_4(5)r_4(5)$. From our other computations, we did not find any instances in which a value of r_i appears more than twice in a period, except when $r_i \equiv 0 \bmod A'\Delta'/d_{i-1}$. It will be of interest to know if this is the case in general.

6.2 Estimates for $\mathrm{lp}(\sqrt{D(X)})$

Although we have a formula, namely (5.1), to determine the actual value of $\mathrm{lp}(\sqrt{D(X)})$, this formula requires us to compute the continued fraction expansion of $\sqrt{D(X)}$ using Theorem 4.3.1, which can be a significant computational effort. Hence, we would like to find a simple formula that gives a good estimate of $\mathrm{lp}(\sqrt{D(X)})$, yet does not require a substantial amount of calculation. Since $\mathrm{lp}(\sqrt{D(X)})$ is determined by $|S_i|$ and κ, we need to improve the estimates for these two quantities.

With respect to $|\mathcal{S}_i|$, there are relevant results available in the literature. For example, an empirical result of Knuth [91, pp. 316−333] can be restated as follows: Assuming the last partial quotient to be strictly greater than 1, the average length of the continued fraction expansion of a/r with $\gcd(a,r)=1$ as r varies from 1 to a is approximately

$$\frac{12\ln 2}{\pi^2}\ln a + 0.47, \tag{6.3}$$

where ln is the natural logarithm. This empirical result was proved by J. W. Porter [192] and later improved by D. Hensley [71]. When the condition $\gcd(a,r)=1$ is removed, Knuth deduced the average length to be approximately

$$\frac{12\ln 2}{\pi^2}\left(\ln a - \sum_{d|a}\frac{\Lambda(d)}{d}\right) + 0.47, \tag{6.4}$$

where d runs through all the proper divisors of a and $\Lambda(d)$ is von Mangoldt's function defined by

$$\Lambda(d) = \begin{cases} \ln p & \text{if } d = p^n \text{ for } p \text{ prime,} \\ 0 & \text{otherwise.} \end{cases}$$

To make use of Knuth's result for our purpose, we need to make an adjustment, since the parity of $|\mathcal{S}_i|$ is restricted by the sign of Δ, i.e., our last partial quotient can be 1 or greater than 1. It is known that when assuming the last partial quotient to be strictly greater than 1, half of the expansions of a/r have an even number of terms and the other half have an odd number of terms, unless a is odd, in which case the numbers of even terms and odd terms differ by 1. So if we separate $|\mathcal{S}_i|$ into two cases, odd length and even length, we expect the averages in the two cases to be fairly close. Indeed, this statement seems to be supported by our data in Tables D.1 and D.2 in the Appendix. Now, to apply Knuth's result, we only need to add 1/2 to his estimate, i.e.,

$$\frac{12\ln 2}{\pi^2}\left(\ln a - \sum_{d|n}\frac{\Lambda(d)}{d}\right) + 0.97. \tag{6.5}$$

For example, when $a = 15$, the above formula gives an estimate of 2.67; the actual values are 2.71 and 2.60 by row F of Tables D.1 and D.2.

As we have seen in Chapter 5, the use of the rank of apparition $\omega(\Delta')$ provides us with a very good upper bound for κ. In fact, in most cases, κ, depending on K, is either $\omega(\Delta')$ or $2\omega(\Delta')$, where $\omega(\Delta')$ is the rank of apparition of Δ' in the Lucas Function U_n defined by $x^2 - Px + 1 =$ and $P = (2A^2K + 2B)^2/|\Delta| - 2\sigma$. Using (6.5), we claim that $\mathrm{lp}(\sqrt{D(X)})$ is approximately

$$\left\lfloor \kappa \cdot \left(\frac{12 \ln 2}{\pi^2} \left(\ln A'\Delta' - \sum_{d | A'\Delta'} \frac{\Lambda(d)}{d} \right) + 0.97 \right) \right\rfloor \tag{6.6}$$

where d is a proper divisor of $A'\Delta'$. Indeed, by the data in Table E.1 in the Appendix, when $\kappa = \omega(\Delta')$ or $2\omega(\Delta')$, this claim seems to be true.

Note that $\omega(\Delta')$ may be hard to calculate. So it will be advantageous to get an upper bound or expected value for $\omega(\Delta')$ that does not require extra computation. Indeed, we demonstrated in Example 5.2.5 and case (5) of Example 5.2.1 that the upper bound: $2\Delta'$ when Δ' is odd and $3\Delta'$ when Δ' is even, is optimal when Δ' is prime. However, when Δ' is composite, our calculations in Section 5.2 suggest that there is a sharper bound.

Ultimately, we want a good estimate for κ. We know $\kappa \mid 2\omega(\Delta')$ by Theorem 5.2.2. Also, the calculations in Section 5.2 suggests that κ is either $\omega(\Delta')$ or $2\omega(\Delta')$ in most cases. However, when given K, we do not know if $\kappa = \omega(\Delta')$ or $2\omega(\Delta')$ or neither. It will be of interest to determine the particular values of K such that $\kappa = \omega(\Delta')$ or $2\omega(\Delta')$. Working in this direction may help our search for a good estimate of κ.

In conclusion, we have found the continued fraction expansion of $\sqrt{D(X)}$ for any Schinzel sleeper $D(X)$, but there is still much to learn. The above suggested research topics are just the tip of the iceberg; there appear to be a great deal of interesting directions for further research.

Bibliography

[1] J.-P. Allouche, M. Mendès France and A. J. van der Poorten, *An infinite product with bounded partial quotients*, Acta Arith., **59** (1991), No. 2, 171–182; MR **92k**:11074.

[2] J.-P. Allouche, A. Lubiw, M. Mendès France, A. J. van der Poorten and J. Shallit, *Convergents of folded continued fractions*, Acta Arith., **77** (1996), No. 1, 77–96; MR **97g**:11077.

[3] R. André-Jeannin, *Divisibility of generalized Fibonacci and Lucas numbers by their subscripts*, Fibonacci Quart., **29** (1991) 364–366; MR **92i:11018**.

[4] T. Azuhata, *On the fundamental units and the class numbers of real quadratic fields.* Nagoya Math. J., **95** (1984), 125–135; MR **86b**:11069.

[5] T. Azuhata, *On the fundamental units and class numbers of real quadratic fields II*, Tokyo J. Math., **10** (1987), 259–270; MR **86b**:11069.

[6] *E. Bach[1], *Explicit bounds for primality testing and related problems*, Math. Comp., **55** (1990), No. 191, 355–380; MR **91m**:11096.

[7] B. D. Beach and H. C. Williams, *Some computer results on periodic continued fractions*, Proceedings of the Second Louisiana Conference on Combinatorics, Graph Theory and Computing (Louisiana State Univ., Baton Rouge, LA, 1971), 133–146. Louisiana State Univ., Baton Rouge, LA, 1971; MR **48** #245.

[1]The entries with * have been referred to in the thesis.

[8] B. D. Beach and H. C. Williams, *A computer algorithm for determining the solution of the Diophantine equation* $x^4 - dy^4 = 1$, Proceedings of the Manitoba Conference on Numerical Mathematics (Univ. Manitoba, Winnipeg, Man., 1971), pp. 663–670. Dept. Comput. Sci., Univ. Manitoba, Winnipeg, Man., 1971; MR **48** #10955.

[9] B. D. Beach and H. C. Williams, *A numerical investigation of the Diophantine equation* $x^2 - dy^2 = -1$, Proceedings of the Third Southeastern Conference on Combinatorics, Graph Theory and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1972), pp. 37–68. Florida Atlantic Univ., Boca Raton, Fla., 1972; MR **50** #231.

[10] L. Bernstein, *Fundamental units and cycles in the period of real quadratic number fields I*, Pacific J. Math., **63** (1976), No. 1, 37–61; MR **53** #13166.

[11] L. Bernstein, *Fundamental units and cycles in the period of real quadratic fields II*, Pacific J. Math., **63** (1976), 63–78; MR **53** #13167.

[12] L. Bernstein, *Fundamental units and cycles*, J. Number Theory, **8** (1976), 446–491; MR **54** #7427.

[13] E. Bombieri and A. J. van der Poorten, *Continued fractions of algebraic numbers*, Computational algebra and number theory (Sydney, 1992), 137–152, Math. Appl., **325**, Kluwer Acad. Publ., Dordrecht, 1995; MR **96g**:11079.

[14] D. Bowman and J. McLaughlin, *Polynomial continued fractions*, Acta Arith., **103** (2002), No. 4, 329–342; MR **2003d**:11019.

[15] *D. Byeon and H. K. Kim *Class number* 1 *criteria for real quadratic fields of Richaud-Degert type*, J. Number Theory, **57** (1996), No. 2, 328–339; MR **97h**:11134.

[16] R. P. Brent, A.J. van der Poorten and H. J. J. te Riele, *A comparative study of algorithms for computing continued fractions of algebraic numbers*, Algorithmic number theory (Talence, 1996), 35–47, Lecture Notes in Comput. Sci., **1122**, Springer, Berlin, 1996; MR **98c**:11144.

[17] T. C. Brown, *A characterization of the quadratic irrationals*, Canad. Math. Bull., **34** (1991), No. 1, 36–41; MR **92c**:11020.

[18] J. Buchmann and H. C. Williams, *On the computation of the class number of an algebraic number field*, Math. Comp., **53** (1989), No. 188, 679–688; MR **90a**:11128.

[19] J. Buchmann and H. C. Williams, *On the existence of a short proof for the value of the class number and regulator of a real quadratic field*, Number theory and applications (Banff, AB, 1988), 327–345, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., **265**, Kluwer Acad. Publ., Dordrecht, 1989; MR **92h**:11089.

[20] N. Buck and K. S. Williams, *Comparisons of the lengths of the continued fractions of* $\sqrt{D}$ *and* $\frac{1}{2}(1+\sqrt{D})$, Proc. Amer. Math. Soc., **120** (1994), 995–1002; MR **94f**:11062.

[21] *R. D. Carmichael, *Note on a new number theory function*, Bull. Amer. Math. Soc., **16** (1910), 232-238.

[22] J. W. Cassels and A. Fröhlich, **Algebraic Number Theory**, Academic Press, London, New York, 1967.

[23] P. Chowla and S. Chowla, *Problems on periodic simple continued fractions*, Proc. Nat. Acad. Sci. U.S.A., **69** (1972), 37–45; MR **47** #8483.

[24] P. Chowla and S. Chowla, *On Hirzebruch sums and a theorem of Schinzel*, Collection of articles dedicated to Carl Ludwig Siegel on the occasion of his seventy-fifth birthday, II. Acta Arith., **24** (1973), 223–224; MR **48** #11043.

[25] L, E. Clemens, K. D. Merrill and D. W. Roeder, *Continued fractions and series*, J. Number Theory, **54** (1995), No. 2, 309–317; MR **96j**:11010.

[26] H. Cohen, *Multiplication par un entier d'une fraction continue périodique*, C. R. Acad. Sci. Paris Sr. A-B **276** (1973), A595–A598; MR **47** #4936.

[27] H. Cohen, *Multiplication par un entier d'une fraction continue périodique*, Acta Arith., **26** (1974/75), 129–148; MR **50** #12892.

[28] *H. Cohn, **A Second Course in Number Theory**, Wiley, New York, 1962.

[29] T. W. Cusick, *Integer multiples of periodic continued fractions*, Pacific J. Math., **78** (1978), No. 1, 47–60; MR **80b**:10009.

[30] *H. Davenport, **Multiplicative Number Theory**, Markham Publishing Company, Chicago, 1967.

[31] *H. Davenport, **The Higher Arithmetic**, Cambridge Univeristy Press, Cambridge, 1985.

[32] *G. Degert, *Über die Bestimmung der Grundeinheit gewisser reell-quadratischer Zahlkörper*, Math. Sem. Univ. Hamburg **22** (1958), 92–97; MR **19**,1164d.

[33] I. G. deMille, **The Continued Fraction of $(1+\sqrt{D})/2$ for Certain Infinite Classes of D with Applications to Units and Classnumbers**, MSc thesis, Carleton University, 1988.

[34] J. E. Desmond, *On the existence of the rank of apparition of m in the Lucas sequence*, Fibonacci Quart., **16** (1978), 7–10; MR **57** #12363.

[35] L. E. Dickson, **History of the Theory of Numbers**, Vol. I: Divisibility and primality. Chelsea Publishing Co., New York.

[36] *L. E. Dickson, **History of the Theory of Numbers**, Vol. II: Diophantine analysis. Chelsea Publishing Co., New York.

[37] *L. E. Dickson, **History of the Theory of Numbers**, Vol. III: Quadratic and higher forms. Chelsea Publishing Co., New York.

[38] *L. G. Dirichlet, *Beweis eines Satzes über die arithmetische Progression*, Bericht über die Verhandlungen der Königl. Preuss. Akademie der Wissenschaften. Jahrg. 1836, 108–110.

[39] *L. G. Dirichlet, *Applications de L'Analyse infinitésimale a la théorie des nombres*, J. reine angew. Math., **19** (1839), 324–369, **21** (1840), 1-12, 134-155.

[40] *L. G. Dirichlet, *Sur une propriété des formes quadratiques a déterminant positif*, Comptes rendus de l'Académie de Berlin, (1855), 493-495.

[41] *J. D. Dixon, *The number of steps in the Euclidean algorithm*, J. Number Theory, **2** (1970), 414–422; **42** #1791.

[42] J. D. Dixon, *A simple estimate for the number of steps in the Euclidean algorithm*, Amer. Math. Monthly, **78** (1971), 374–376; **44** #2697.

[43] E. Dubois and C. , *On determining certain real quadratic fields with class number one and relating this property to continued fractions and primality properties*, Nagoya Math. J., **124** (1991), 157–180; MR **92k**:11117.

[44] E. Dubois and R. Paysant–Le Roux, *Sur la longueur du développement en fraction continue de* $\sqrt{f(n)}$, Journées Arithmétiques, 1989 (Luminy, 1989). Astérisque **8-200** (1991), 107–119 (1992); MR **93c**:11006.

[45] H. J. A. Duparc, **Divisibility Properties of Recurring Sequences**, Thesis, University of Amsterdam, 1953. 96 pp.; MR **15**, 200c.

[46] C. Elsner, J. W. Sander and J. Steuding, *Kettenbrüche als Summen ebensolcher*, Math. Slovaca **51** (2001), No. 3, 281–293; MR **2002i**:11008.

[47] N. Elezović, *A note on continued fractions of quadratic irrationals*, Math. Commun., **2** (1997), No. 1, 27–33; MR **98h**:11011.

[48] *L. Euler, *Correspondance Mathématique et Physique*, St. Petersburg, 1843, P. H. Fuss (ed.), **T1**, 35–39.

[49] C. Faivre, *On calculating a continued fraction expansion from a decimal expansion*, Acta Sci. Math. (Szeged), **67** (2001), No. 3-4, 505–519; MR **2002j**:11088.

[50] A. Farhane, **Spécialisation de points extrémaux applications aux fraction continues et aux unités d'une famille de corps cubiques**, PhD thesis, Univ. de Caen, 1992.

[51] A. Farhane, *Sur la longueur de la période du développement de* $\sqrt{n^2+h}$ *en fraction continue*, C. R. Acad. Sci. Paris Sér. I Math., **316** (1993), No. 6, 537–540; MR **94b**:11008.

[52] * A. Farhane, *Minoration de la période du développement de* $\sqrt{a^2n^2+bn+c}$ *en fraction continue*, Acta Arith., **67** (1994), No. 1, 63–67; MR **95j**:11007.

[53] *D. H. Fowler, **The Mathematics of Plato's Academy: A New Reconstruction**, Oxford University Press, 1987; MR **2000c**:01004.

[54] *C. Friesen, *On continued fractions of given period*, Proc. Amer. Math. Soc., **103** (1988), No. 1, 9–14; MR **89d**:11004.

[55] C. Friesen, *Legendre symbols and continued fractions*, Acta Arith., **59** (1991), No. 4, 365–379; MR **92m**:11007.

[56] *C. F. Gauss, **Disquisitiones Arithmeticae** (English translation), Springer, Berlin, 1985.

[57] I. J. Good, *Complex Fibonacci and Lucas numbers, continued fractions, and the square root of the golden ratio*, Fibonacci Quart., **31** (1993), No. 1, 7–20; MR **93k**:11010.

[58] G. Grisel, *Sur la fraction continue de α^n*, C. R. Acad. Sci. Paris Sr. I Math., **319** (1994), No. 7, 659–664; MR **95i**:11006.

[59] G. Grisel, *Sur la longueur de la fraction continue de α^n*, Acta Arith., **74** (1996), No. 2, 161–176; MR **96m**:11004.

[60] G. Grisel, *Longueur de la fraction continue des puissances d'une fraction rationnelle*, C. R. Acad. Sci. Paris Sér. I Math., **322** (1996), No. 4, 301–306; MR **97b**:11094.

[61] G. Grisel, *Length of the powers of a rational fraction*, J. Number Theory, **62** (1997), No. 2, 322–337; MR **98a**:11088.

[62] G. Grisel, *Length of continued fractions in principal quadratic fields*, Acta Arith., **85** (1998), No. 1, 35–49; MR **99g**:11015.

[63] F. Halter-Koch, *Über Pellsche Gleichungen und Kettenbrüche*, Arch. Math. (Basel), **49** (1987), No. 1, 29–37, MR **88f**:11012.

[64] F. Halter-Koch, *Einige periodische Kettenbruchentwicklungen und Grundeinheiten quadratischer Ordnungen*, Abh. Math. Sem. Univ. Hamburg., **59** (1989), 157–169; MR **91h**:11115.

[65] F. Halter-Koch, *Reell–quadratische Zahlkörper mit großer Grundeinheit*, Abh. Math. Sem. Univ. Hamburg., **59** (1989), 171–181; MR **91h**:11116.

[66] *F. Halter-Koch, *Continued fractions of given symmetric period*, Fibonacci Quart., **29** (1991), No. 4, 298–303; MR **92k**:11008.

[67] *H. Hasse, *Über mehrklassige, aber eingeschlechtige reell-quadratische Zahlkörper*. Elem. Math., **20** (1965) 49–59; MR **33** #116 .

[68] *E. Hecke, **Vorlesungen über die Theorie der Algebraische Zahlen**, Second edition of the 1923 original, Chelsea Publishing Company, NY, 1948, **Lectures on the theory of algebraic numbers**, (English translation by G. U. Brauer, J. R. Goldman and R. Kotzen) Graduate Texts in Mathematics, **77** (1981) Springer-Verlag, New York-Berlin; MR **83m**:12001.

[69] D. Hensley, *The largest digit in the continued fraction expansion of a rational number*, Pacific J. Math., **151** (1991), 237–255; MR **92i**:11009.

[70] D. Hensley, *The distribution mod n of fractions with bounded partial quotients*, Pacific J. Math., **166** (1994), No. 1, 43–54; MR **95i**:11083.

[71] *D. Hensley, *The number of steps in the Euclidean algorithm*, J. Number Theory, **49** (1994), No. 2, 142–182; MR **96b**:11131 .

[72] H. Heilbronn, *On the average length of a class of finite continued fractions*, Number Theory and Analysis (Papers in Honor of Edmund Landau) (1969) pp. 87–96 Plenum, New York; **41** #3406.

[73] *J. Hoffstein, *On the Siegel-Tatuzawa theorem*, Acta Arith., **38** (1980/81), No. 2, 167–174; MR **82k**:10050.

[74] *A. Hurwitz, *Über eine besondere Art der Kettenbruchentwicklung Reeller Grössen* Acta Math., **42** (1899), 367–405.

[75] *A. Hurwitz, *Über die angenäherte Darstellung der Zahlen durch rationale Brüche*, Math. Ann., 44 (1894), 417–436.

[76] K. Ireland and M. Rosen, A Classical Introduction to Modern Number Theory, Springer, Berlin, Heidelberg, New York, 1982.

[77] N. Ishii, P. Kaplan and K. S. Williams, *On Eisenstein's problem*, Acta Arith., **54** (1990), No. 4, 323–345; MR **91f**:11024.

[78] *C. G. J. Jacobi, *Observatio arithmetica de numero classium divisorum quadraticorum formae $yy + Azz$, designante A numerum primum formae* $4n+3$, J. reine angew. Math., **10** (1832) 189–192.

[79] M. J. Jacobson Jr., R. F. Lukes and H. C. Williams, *An investigation of bounds for the regulator of quadratic fields*, Experiment. Math., **4** (1995), No. 3, 211–225; MR **97d**:11173.

[80] M. J. Jacobson Jr. and H. C. Williams, *The size of the fundamental solutions of consecutive Pell equations*, Experiment. Math., **9** (2000), No. 4, 631–640; MR **2002f**:11142.

[81] P. Kaplan and K. S. Williams, *Pell's equation* $X^2 - mY^2 = -1, -4$ *and continued fractions*, J. Number Theory., **23** (1986), 169–182; MR **87j**:11114.

[82] P. Kaplan and Y. Mimura, *Développement en fraction continue à l'entier supérieur, idéaux 0-réduits et un problème d'Eisenstein*, Acta Arith., **78** (1997), No. 3, 275–285; MR **98e**:11125.

[83] *S. Katayama, *On the class numbers of real quadratic fields of Richaud-Degert type*, J. Math. Tokushima Univ., **31** (1997), 1–6; MR **99c**:11136.

[84] A. Ya. Khinchin, *Continued Fractions*, with a preface by B. V. Gnedenko, translated from the third (1961) Russian edition, reprint of the 1964 translation, Dover Publications, Inc., Mineola, NY, 1997.

[85] P. Kiss, *On second order recurrences and continued fractions*, Bull. Malaysian Math. Soc. (2), **5** (1982) 33-41, MR **84e**:10012.

[86] P. Kiss, *Primitive divisors of Lucas numbers*, Applications of Fibonacci numbers (San Jose CA 1986) 29–38, Kluwer Acad. Publ., Dordrecht 1988; MR **89j**:11016.

[87] P. Kiss, *On rank of apparition of primes in Lucas sequences*, Publ. Math. Debrecen, **36** (1989) 147–151 (1990); MR **91e**:11019.

[88] P. Kiss, *On primitive prime power divisors of Lucas numbers*, Number theory, Vol. II (Budapest, 1987) 773–786, Colloq. Math. Soc. János Bolyai, **51**, North-Holland, Amsterdam, 1990; MR **91g**:11014.

[89] P. Kiss, *Results on the ratios of the terms of second order linear recurrences*, Math. Slovaca, **41** (1991) 257–260; MR **92g**:11014.

[90] P. Kiss and M.-P. Bui, *On a function concerning second-order recurrences*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math., **21** (1978) 119–122 (1979); MR **80h**:10015.

[91] *D. Knuth, **The Art of Computer Programming, Vol. II: Seminumerical algorithms**, 1st edition, Addison-Wesley, 1969; 2nd edition, Addison-Wesley, Reading, Mass.,1981.

[92] *M. Kraitchik, **Théorie des Nombres**, T. II. Gauthier-Villars, Paris, 1926.

[93] H. Koch, **Algebraic Number Theory**, Springer, Berlin, Heidelberg, 1997.

[94] T. Komatsu, *An infinite family of pairs of quadratic fields* $\mathbb{Q}(\sqrt{D})$ *and* $\mathbb{Q}(\sqrt{mD})$ *whose class numbers are both divisible by 3*, Acta Arith., **104** (2002), No. 2, 129–136.

[95] J. Kühner, *On a family of generalized continued fraction expansions with period length going to infinity*, J. Number Theory, **53** (1995) 1–12; MR **96i**:11010.

[96] J. C. Lagarias and J. O. Shallit, *Linear fractional transformations of continued fractions with bounded partial quotients*, J. Théor. Nombres Bordeaux **9** (1997), No. 2, 267–279; MR **99c**:11088.

[97] S. Lang, **Cyclotomic fields**, Graduate Texts in Mathematics, Vol. 59. Springer, New York-Heidelberg, 1978.

[98] S. Lang and H. Trotter, *Continued fractions for some algebraic numbers*, J. reine angew. Math., **255** (1972), 112–134; MR **46** #2086.

[99] S. Lang and H. Trotter, *Addendum to: "Continued fractions for some algebraic numbers"* , (J. reine angew. Math., **255** (1972), 112–134). J. reine angew. Math., **267** (1974), 219–220; MR **50** #5258.

[100] *D. H. Lehmer, *On the Multiple Solutions of the Pell Equation*, Ann. Math., **30** (1928), 66–72.

[101] *D. H. Lehmer, *An Extended Theory of Lucas' functions*, Ann. Math., **31** (1930), 419–448.

[102] *D. H. Lehmer, *On the Factorization of Lucas' functions*, Tohoku Math. J., **34** (1931), 1–7.

[103] D. H. Lehmer, *Factorization of certain cyclotomic functions*, Ann. Math., **34** (1933), 461–479.

[104] D. H. Lehmer, *Continued fractions containing arithmetic progressions*, Scripta Math., **29**, 17–24. (1973); MR **48** #5979.

[105] E. Lehmer, *On the cubic character of quadratic units*, J. Number Theory, **5** (1973), 385–389; MR **48** #271.

[106] *H. W. Lenstra, Jr., *On the calculation of regulators and class number of quadratic fields*, London Math. Soc. Lecture Note Ser., **56**, (1982), 123–150; MR **86g**:11080.

[107] H. W. Lenstra Jr. and J. O. Shallit, *Continued fractions and linear recurrences*, Math. Comp., **61** (1993), No. 203, 351–354; MR **1993k**:11004.

[108] C. Levesque, *Continued fraction expansions and fundamental units*, J. Math. Phys. Sci., **22** 1988, No. 1, 11–44; MR **89f**:11021.

[109] C. Levesque and G. Rhin, *A few classes of periodic continued fractions*, Utilitas Math., **30** (1986), 79–107; MR **88b**:11072.

[110] J. Lewittes, *Quadratic irrationals and continued fractions*, Number theory (New York, 1991–1995), 253–268, Springer, New York, 1996; MR **97m**:11014.

[111] *J. E. Littlewood, *On the class number of the corpus* $P(\sqrt{-k})$, Proc. London Math. Soc., **27** (1928), 358-372.

[112] *S. Louboutin, *Continued fractions and real quadratic fields*, J. Number Theory, **30** (1988), No. 2, 167–176; MR **90a**:11119.

[113] *S. Louboutin, *Une version effective d'un théorèm de A. Schinzel sur les longueurs des périodes de certains développements en fractions continues*, C. R. Acad. Sci. Paris Sér. I Math., **308** (1989), No. 17, 511–513; MR **90e**:11019.

[114] S. Louboutin, *Explicit bounds for residues of Dedekind zeta functions, values of L-functions at* $s = 1$*, and relative class numbers*, J. Number Theory, **85** (2000), No. 2, 263–282; MR **2002i**:11111.

[115] S. Louboutin, *Explicit upper bounds for residues of Dedekind zeta functions and values of L-functions at* $s = 1$, *and explicit lower bounds for relative class numbers of CM-fields*, Canad. J. Math., **53** (2001), No. 6, 1194–1222; MR **20003d**:11167.

[116] S. Louboutin, *Majorations explicites de* $|L(1,\chi)|$. *IV*, C. R. Math. Acad. Sci. Paris, **334** (2002), No. 8, 625–628; MR **2003f**:11123.

[117] *S. Louboutin, *Explicit upper bounds for* $|L(1,\chi)|$ *for primitive even characters*, Acta. Arith., **101** (2002), No. 1, 1–18; MR **2003b**:11089.

[118] S. Louboutin, *Explicit upper bounds for values at* $s = 1$ *of Dirichlet L-series associated with primitive even characters*, to appear in J. Number Theory.

[119] S. Louboutin and R. A. Mollin, *Solutions to* $x^2 - Dy^2 = Q$, Number theory (Ottawa, ON, 1996), 265–280, CRM Proc. Lecture Notes, **19**, Amer. Math. Soc., Providence, RI, 1999; MR **2000f**:11138.

[120] S. Louboutin, R. A. Mollin and H. C. Williams, *Class numbers of real quadratic fields, continued fractions, reduced ideals, prime-producing quadratic polynomials, and quadratic residue covers*, Canad. J. Math., **44** (1992), 824–842; MR **93h**:11138.

[121] *H. W. Lu, *Real quadratic fields with class number* 1, J. China Univ. Sci. Tech., **10** (1980) No. 4, 133–135; MR **84j**:12004.

[122] H. W. Lu, *The continued fractions, class number and the others*, Sci. Sinica Ser. A, **26** (1983), No. 12, 1275–1284; MR **85i**:11091.

[123] H. W. Lu, *The length of the period of the simple continued fraction of a real quadratic irrational number*, (Chinese) Acta Math. Sinica, **29** (1986), No. 4, 433–443; MR **88a**:11063.

[124] H. W. Lu, *Hirzebruch sum and class number of the quadratic fields*, Chinese Sci. Bull., **36** (1991), No. 14, 1145–1147; MR **92k**:11129.

[125] É. Lucas, *Théorie des fonctions numériques simplement périodiques*, Amer. J. Math., **1** (1878), 184-240, 289-321.

[126] *D. A. Marcus, **Number Fields**, Springer-Verlag, New York, 1972.

[127] D. J. Madden, *Constructing families of long continued fractions*, Pacific J. Math., **8** (2001), No. 1, 123–147; MR **2002b**:11014.

[128] B. Mazur and A. Wiles, *Analogies between function fields and number fields*, Amer. J. Math., **105** (1983), No. 2, 507–521; MR **84g**:12003.

[129] *D. McCarthy, **Selected Papers of D. H. Lehmer**, V. I–III, The Charles Babbage Research Centre, Box 370, St. Pierre, Manitoba, Canada, 1981.

[130] M. Mendès France, *Sur les fractions continues limitées*, Acta Arith., **23** (1973), 207–215; MR **48** #2083.

[131] M. Mendès France and A. J. van der Poorten, *Some explicit continued fraction expansions*, Mathematika, **38** (1991), No. 1, 1–9; MR **92g**:11072.

[132] M. Mendès France, A. J. van der Poorten and J. Shallit, *On lacunary formal power series and their continued fraction expansion*, Number theory in progress, Vol. **1** (Zakopane-Kościelisko, 1997), 321–326, de Gruyter, Berlin, 1999; MR **2000d**:11036.

[133] *R. A. Mollin, **Quadratics**, CRC Press LLC, Boca Raton, New York, 1995.

[134] *R. A. Mollin, **Fundamental Number Theory and Applications**, CRC Press LLC, Boca Raton, New York, 1997.

[135] R. A. Mollin, Algebraic Number Theory, Chapman & Hall/CRC Press LLC, Boca Raton, London, New York, Washington D.C, 1999.

[136] R. A. Mollin, *Diophantine equations and class numbers*, J. Number Theory, **24** (1986), No. 1, 7–19; MR **1987i**:11155.

[137] R. A. Mollin, *Lower bounds for class numbers of real quadratic fields*, Proc. Amer. Math. Soc., **96** (1986), No. 4, 545–550; MR **88e**:11113.

[138] R. A. Mollin, *On class numbers of quadratic extensions of algebraic number fields*, Proc. Japan Acad. Ser. A Math. Sci., **62** (1986), No. 1, 33–36; MR **87i**:11156.

[139] R. A. Mollin, *Class numbers of quadratic fields determined by solvability of Diophantine equations*, Math. Comp., **48** (1987), No. 177, 233–242; MR **88a**:11105.

[140] *R. A. Mollin, *On the insolubility of a class of Diophantine equations and the nontriviality of the class numbers of related real quadratic fields of Richaud-Degert type*, Nagoya Math. J., **105** (1987), 39–47; MR **88d**:11105.

[141] R. A. Mollin, *Necessary and sufficient conditions for the class number of a real quadratic field to be one, and a conjecture of S. Chowla*, Proc. Amer. Math. Soc., **102** (1988), No. 1, 17–21; MR **89d**:11098.

[142] R. A. Mollin, *On the divisor function and class numbers of real quadratic fields I*, Proc. Japan Acad. Ser. A Math. Sci., **66** (1990), No. 5, 109–111; MR **91g**:11121.

[143] R. A. Mollin, *On the divisor function and class numbers of real quadratic fields II* Proc. Japan Acad. Ser. A Math. Sci., **66** (1990), No. 9, 274–277; MR **92d**:11117.

[144] R. A. Mollin, *On the divisor function and class numbers of real quadratic fields IV*, Proc. Japan Acad. Ser. A Math. Sci., **68** (1992), No. 1, 15–17; MR **93c**:11092.

[145] R. A. Mollin, *A survey of Jacobi symbols, ideals, and continued fractions*, Far East J. Math. Sci., 6 (1998), No. 1, 355-368; MR **99h**:11119.

[146] R. A. Mollin, *Continued fraction gems*, Nieuw Arch. Wisk. (4), **17** (1999), No. 3, 383–405; MR **2000m**:11008.

[147] R. A. Mollin, *Fundamental units and continued fractions*, Util. Math., **57** (2000), 165–170; MR **2001d**:11106.

[148] *R. A. Mollin, *Polynomials of pellian type and continued fractions*, Serdica Math. J., **27** (2001), 317–342; MR **2003c**:11139.

[149] R. A. Mollin, *Quadratic Diophantine equations determined by continued fractions*, JP J. Algebra Number Theory Appl., **1** (2001), No. 1, 57–75; MR **2003b**:11020.

[150] R. A. Mollin, *Simple continued fraction solutions for Diophantine equations*, Expo. Math., **19** (2001), No. 1, 55–73; MR **2003a**:11022.

[151] R. A. Mollin, *Continued fractions and class number two*, Int. J. Math. Math. Sci., **27** (2001), No. 9, 565–571; MR **2002k**:11190.

[152] R. A. Mollin, *The Diophantine equation $ax^2 - by^2 = c$ and simple continued fractions*, Int. Math. J., **2** (2002), No. 1, 1–6; MR **2002j**:11022.

[153] R. A. Mollin and K. H. F. Cheng, *Palindromy and ambiguous ideals revisited*, J. Number Theory, **74** (1999), 98–110; MR **99m**:11123.

[154] R. A. Mollin and K. H. F. Cheng, *Beepers, Creepers, and Sleepers*, Int. Math. J., **2** (2002), No. 9, 951–956; MR **2003f**:11011.

[155] R. A. Mollin and K. H. F. Cheng, *Continued fractions beepers and Fibonacci numbers*, C. R. Math. Rep. Acad. Sci. Soc. R. Can., **2** (2002), No. 3, 102–108; MR **2003e**:11006.

[156] R. A. Mollin and K. H. F. Cheng, *Matrices and continued fractions*, Int. Math. J., **3** (2003), No. 1, 41–58; MR **2003i**:11018.

[157] R. A. Mollin, K. H. F. Cheng and B. Goddard, *The Diophantine equation $AX^2 - BY^2 = C$ solved via continued fractions*, Acta Math. Univ. Comenian. (N.S.), **71** (2002), No. 2, 121–138.

[158] R. A. Mollin, K. H. F. Cheng and B. Goddard, *Pellian polynomials and period lengths of continued fractions*, JP J. Algebra Number Theory Appl., **2** (2002), No. 1, 47–60; MR **2003i**:11017.

[159] *R. A. Mollin and A. J. van der Poorten, *A note on symmetry and ambiguity*, Bull. Austral. Math. Soc., **51** (1995), No. 2, 215–233; MR **96b**:11051.

[160] R. A. Mollin and A. J. van der Poorten, *Continued fractions, Jacobi symbols, and quadratic Diophantine equations*, Canad. Math. Bull., **43** (2000), No. 2, 218–225; MR **2001a**:11049.

[161] *R. A. Mollin, A. J. van der Poorten and H. C. Williams, *Halfway to a solution of* $X^2 - DY^2 = -3$, J. Théor. Nombres Bordeaux, **6** (1994), No. 2, 421–457; MR **96j**:11030.

[162] R. A. Mollin and H. C. Williams, *Prime producing quadratic polynomials and real quadratic fields of class number one*, Théorie des nombres (Quebec, PQ, 1987), 654–663, de Gruyter, Berlin, 1989; MR **90m**:11153.

[163] R. A. Mollin and H. C. Williams, *On prime valued polynomials and class numbers of real quadratic fields*, Nagoya Math. J., **112** (1988), 143–151; MR **90c**:11080.

[164] R. A. Mollin and H. C. Williams, *Real quadratic fields of class number one and continued fraction period less than six*, C. R. Math. Rep. Acad. Sci. Canada **11** (1989), No. 2, 51–56; MR **90b**:11105.

[165] R. A. Mollin and H. C. Williams, *Class number problems for real quadratic fields*, Number theory and cryptography (Sydney, 1989), 177–195, London Math. Soc. Lecture Note Ser., **154**, Cambridge Univ. Press, Cambridge, 1990; MR **91f**:11074.

[166] R. A. Mollin and H. C. Williams, *Period four and real quadratic fields of class number one*, Proc. Japan Acad. Ser. A Math. Sci., **65** (1989), No. 4, 89–93; MR **90i**:11129.

[167] R. A. Mollin and H. C. Williams, *Quadratic nonresidues and prime-producing polynomials*, Canad. Math. Bull., **32** (1989), No. 4, 474–478; MR **91a**:11009.

[168] R. A. Mollin and H. C. Williams, *Class number one for real quadratic fields, continued fractions and reduced ideals*, Number theory and applications (Banff, AB, 1988), 481–496, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., **265**, Kluwer Acad. Publ., Dordrecht, 1989; MR **92f**:11143.

[169] R. A. Mollin and H. C. Williams, *Continued fractions of period five and real quadratic fields of class number one*, Acta Arith., **56** (1990), No. 1, 55–63; MR **91g**:11126.

[170] R. A. Mollin and H. C. Williams, *On a solution of a class number two problem for a family of real quadratic fields*, Computational number theory (Debrecen, 1989), 95–101, de Gruyter, Berlin, 1991; MR **93d**:11118.

[171] R. A. Mollin and H. C. Williams, *On a determination of real quadratic fields of class number one and related continued fraction period length less than* 25, Proc. Japan Acad. Ser. A Math. Sci., **67** (1991), No. 1, 20–25; MR **92c**:11113.

[172] R. A. Mollin and H. C. Williams, *On the divisor function and class numbers of real quadratic fields. III*, Proc. Japan Acad. Ser. A Math. Sci., **67** (1991), No. 10, 338–342; MR **93c**:11091.

[173] R. A. Mollin and H. C. Williams, *Powers of 2, continued fractions, and the class number one problem for real quadratic fields $Q(\sqrt{d})$, with $d \equiv 1 \pmod 8$*, The mathematical heritage of C. F. Gauss, 505–516, World Sci. Publishing, River Edge, NJ, 1991; MR **92m**:11119.

[174] R. A. Mollin and H. C. Williams, *Affirmative solution of a conjecture related to a sequence of Shanks*, Proc. Japan Acad. Ser. A Math. Sci., **67** (1991), No. 3, 70–72; MR **92c**:11122.

[175] R. A. Mollin and H. C. Williams, *On the period length of some special continued fractions*, Sém. Théor. Nombres Bordeaux (2) **4** (1992), No. 1, 19–42; MR **93j**:11005.

[176] R. A. Mollin and H. C. Williams, *Computation of the class number of a real quadratic field*, Utilitas Math., **41** (1992), 259–308; MR **93d**:11134.

[177] *R. A. Mollin and H. C. Williams, *Solution of the class number one problem for real quadratic fields of extended Richaud-Degert type (with one possible exception)*, Number theory (Banff, AB, 1988), 417–425, de Gruyter, Berlin, 1990; MR **92f**:11144.

[178] R. A. Mollin and H. C. Williams, *On real quadratic fields of class number two*, Math. Comp., **59** (1992), No. 200, 625–632; MR **93a**:11089.

[179] R. A. Mollin and H. C. Williams, *Consecutive powers in continued fractions*, Acta Arith., **61** (1992), No. 3, 233–264; MR **93f**:11073.

[180] R. A. Mollin and H. C. Williams, *Quadratic residue covers for certain real quadratic fields*, Math. Comp., **62** (1994), No. 206, 885–897; MR **94g**:11092.

[181] H. L. Montgomery and P. J. Weinberger, *Real quadratic fields with large class number*, Math. Ann., **225** (1977), 173-176; MR **55** #306.

[182] L. J. Mordell, **Diophantine Equations**, Pure and Applied Mathematics, Vol. 30, Academic Press, London and New York, 1969; MR **40** #2600.

[183] Y. Motoda, *On units of a real quadratic field*, Mem. Fac. Gen Ed. Kumamoto Univ., Ser. Natur. Sci., **13** (1977), 9–13; MR **58** #578.

[184] Y. Motoda and S. Isogawa, *On the divisibility properities of second order recurrences*, II. Mem. Fac. Gen Ed. Kumamoto Univ.. Natur. Sci., **30** (1995), 65–74; MR **97h**:11009.

[185] T. Nagell, *Zur Arithmetik der Polynome*, Abh. Math. Sem. Univ. Hamburg, **1** (1922), 179–194.

[186] F. Nemenzo and H. Wada, *An elementary proof of Gauss' genus theorem*, Proc. Japan Acad. Ser. A Math. Sci., **68** (1992), No. 4, 94–95; MR **93c**:11093.

[187] M. Nyberg, *Culminating and almost culminating continued fractions*, (Norwegian) Norsk Mat. Tidsskr., **31** (1949). 95–99; MR **11** 329d.

[188] T. Okano, *Explicit continued fractions with expected partial quotient growth*, Proc. Amer. Math. Soc., **130** (2002), No. 6, 1603–1605; MR **2003a**:11096.

[189] C. D. Patterson and H. C, Williams, *Some periodic continued fractions with long periods*, Math. Comp., **44** (1985), No. 170, 523–532; MR **86h**:11113.

[190] O. Perron, **Die Lehre von den Kettenbrüchen**, Bd I. Elementare Kettenbrüche. 3te Aufl, B. G. Teubner Verlagsgesellschaft, Stuttgart, 1954.

[191] C. G. Pinner, A. J. van der Poorten and N. Saradha, *Some infinite products with interesting continued fraction expansions*, J. Théor. Nombres Bordeaux, **5** (1993), No. 1, 187–216; MR **95d**:11087.

[192] *J. W. Porter, *On a theorem of Heilbronn*, Mathematika, **22** (1975), No. 1, 20–28; MR **58** #16567.

[193] A. J. van der Poorten, *Remarks on the continued fractions of algebraic numbers*, Groupe de travail en théorie analytique et élémentaire des nombres, 1986–1987, 89–90, Publ. Math. Orsay, 88-01, Univ. Paris XI, Orsay, 1988; MR 950 952 .

[194] A. J. van der Poorten, *An introduction to continued fractions*, Diophantine Analysis (Kensington, 1985), London Math. Soc. Lecture Note Ser., **109** (1986), 99–138, Cambridge Univ. Press, Cambridge, 1986; MR **88d**:11006.

[195] A. J. van der Poorten, *Notes on continued fractions and recurrence sequences*, Number theory and cryptography (Sydney, 1989), London Math. Soc. Lecture Note Ser., **154**, 86–97, Cambridge Univ. Press, Cambridge, 1990; MR **91e**:11079.

[196] A. J. van der Poorten, *Fractions of the period of the continued fraction expansions of quadratic integers*, Bull. Austral. Math. Soc., **44** (1991), No. 1, 155–169; MR **92g**:11007.

[197] A. J. van der Poorten, *Explicit formulas for units in certain quadratic number fields*, Algorithmic Number Theory Symposium, ANTS I, (Ithaca, NY, 1994), 194–208, Lecture Notes in Comput. Sci., **877**, Springer, Berlin, 1994; MR **96b**:11146.

[198] A. J. van der Poorten, *Continued fraction expansions of values of the exponential function and related fun with continued fractions*, Nieuw Arch. Wisk. (4), **14** (1996), No. 2, 221–230; MR **97f**:11011.

[199] A. J. van der Poorten, *Continued fractions of formal power series*, Advances in number theory (Kingston, ON, 1991), 453–466, Oxford Sci. Publ., Oxford Univ. Press, New York, 1993; MR **97a**:11017.

[200] A. J. van der Poorten, *Beer and continued fractions with periodic periods*, CRM proceedings and Lecture Notes, **19** (1999), 309–314; MR **2000e**:11007.

[201] A. J. van der Poorten, *Formal power series and their continued fraction expansion*, Algorithmic Number Theory Symposium, ANTS III, (Portland, OR, 1998), 358–371, Lecture Notes in Comput. Sci., **1423**, Springer, Berlin, 1998; MR **2000m**:11009.

[202] A. J. van der Poorten and J. Shallit, *Folded continued fractions*, J. Number Theory, **40** (1992), No. 2, 237–250; MR **93a**:11008.

[203] A. J. van der Poorten and J. Shallit, *A specialised continued fraction*, Canad. J. Math., **45** (1993), No. 5, 1067–1079; MR **94h**:11001.

[204] A. J. van der Poorten and X. C. Tran, *Quasi-elliptic integrals and periodic continued fractions*, Monatsh. Math., **131** (2000), No. 2, 155–169; MR **2002b**:11093.

[205] A. J. van der Poorten and P. G. Walsh, *A note on Jacobi symbols and continued fractions*, Amer. Math. Monthly, **106** (1999), No. 1, 52–56; MR **2000e**:11004.

[206] *A. J. van der Poorten and H. C. Williams, *On certain continued fraction expansions of fixed period length*, Acta Arith., **89** (1999), no, 1, 23–35; MR **2000m**:11010.

[207] E. Preissmann, *On the length of the period of* $\sqrt{d}$, Forum Math., **6** (1994), No. 2, 137–138; MR **95b**:11010.

[208] *P. Ribenboim, **The New Book of Prime Number Records**, Springer-Verlag, New York, 1996.

[209] *C. Richaud, *Sur la résolution des équations* $x^2 - Ay^2 = \pm 1$, Atti Accad. pontif. Nuovi Lincei (1866), 177–182.

[210] A. M. Rockett and P. Szüsz, *On the lengths of the periods of the continued fractions of square roots of integers*, Forum Math., **2** (1990), No. 2, 119–123; MR **91c**:11045.

[211] *K. H. Rosen, **Elementary Number Theory and its Applications**, Addision–Wesley Publishing Company, New York, 1993; MR **93i**:11002.

[212] H. J. A. Sallé, *A maximum value for the rank of apparition of integers in recursive sequences*, Fibonacci Quart., **13** (1975), 159–161; MR **50** #12897.

[213] *A. Schinzel, *On some problems of the arithmetical theory of continued fractions*, Acta Arith., **6** (1961), 393–413; MR **23** #A3111.

[214] *A. Schinzel, *On some problems of the arithmetical theory of continued fractions II*, Acta Arith., **7** (1962), 187–298; MR **25** #2998.

[215] A. Schinzel, *On two conjectures of P. Chowla and S. Chowla concerning continued fractions*, Ann. Mat. Pura Appl. (4), **98** (1974), 111-117; MR **49** #4943.

[216] A. Schinzel, *Corrigendum: "On some problems of the arithmetical theory of continued fractions II"* [Acta Arith., **7** (1961/62), 287–298; MR 25 #2998], Acta Arith., **47** (1986), No. 3, 295; MR **88b**:11007.

[217] *H. Schmidt, *Zur Approximation und Kettenbruchentwicklung quadratischer Zahlen*, Math. Z., **52** (1950), 168–190; MR **11** 331b.

[218] *R. G. Schoof, *Quadratic fields and factorization*, Computational Methods in Number Theory (H. W. Lenstra, Jr. and R. Tijdeman, eds.), Math. Centrum Tracts. No. 155, Part II, Amsterdam, (1983), 235–286; MR **85g**:11118b.

[219] P. Seeling, *Über die Auflösung der Gleichung $x^2 - Ay^2 = \pm 1$ in ganzen Zahlen, wo A positiv und kein vollständiges Quadrat sein muss*, Archiv der Mathematik und Physik, **52** (1887), 40–49.

[220] *J. Shallit, *Origins of the analysis of the Euclidean algorithm*, Historia Math., **21** (1994), No. 4, 401–419; MR **95h**:01015.

[221] D. Shanks, *On Gauss's class number problems*, Math. Comp., **23** (1969), 151–163; MR **41** #6814.

[222] D. Shanks, *Class number, a theory of factorization and genera*, Proc. Sympos. Pure Math., **20** (1971), 415–440; MR **47** #4932 .

[223] D. Shanks, *The infrastructure of a real quadratic field and its applications*, Proc. 1972 Number Theory Conference, Boulder, Colorado. 1972, 217–224; MR **52** #10672.

[224] B. M. Shirokov, *On a length of the continued fraction's period*, Tr. Petrozavodsk. Gos. Univ. Ser. Mat. No. **1**, (1993), 85–90; MR **99f**:11013.

[225] N. A. Sidorov, *"Sum-of-digits" function for some nonstationary number systems*, (Russian) Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **240** (1997), Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 2, 257–267, 295; translation in J. Math. Sci. (New York), **96** (1999), No. 5, 3609–3615; MR **2000c**:11017.

[226] *C. L. Siegel, *Über die Klassenzahl quadratischer Zahlköper*, Acta Arith., **1** (1935), 83-86.

[227] L. Somer, *The divisibility properties of primary Lucas recurrences with respect to primes*, Fibonacci Quart., **18** (1980), 316–334; MR **82g**:10023.

[228] L. Somer, *Possible periods of primary Fibonacci-like sequences with respect to a fixed odd prime*, Fibonacci Quart., **20** (1982), 311–333; MR **84g**:10022.

[229] L. Somer, *Divisibility of terms in Lucas sequences by their subscripts*, Applications of Fibonacci numbers (St. Andrews), **5** (1992), 515–525, Kluwer Acad. Publ., Dordrecht, 1993; MR **94m**:11022,

[230] A. Srinivasan, *Prime producing polynomials, proof of a conjecture by Mollin and Williams*, Acta Arith., **89** (1999), No. 1, 1–7; MR **2000h**:11112.

[231] R. G. Stanton and H. C. Williams, *An application of combinatorics in number theory*, Ars Combinatoria, **1** (1976), No. 1, 321–330; **54** #2575

[232] R. G. Stanton, C. Sudler Jr. and H. C. Williams, *An upper bound for the period of the simple continued fraction for* $\sqrt{D}$, Pacific J. Math., **67** (1976), No. 2, 525–536; MR **55** #2735.

[233] *H.-J. Stender, *Über die Grundeinheit der reell-quadratischen Zahlkörper* $Q(\sqrt{A^2N^2+BN+C})$, J. reine angew. Math., **311/312** (1979), 302–306; MR **80k**:12008.

[234] H.-J. Stender, *Einheitenbasen für parametrisierte Zahlkörper vierten und achten Grades mit beliebigem reell-quadratischen Teilkörper*, J. Number Theory, **17** (1983), No. 2, 246–269; MR **85a**:11020.

[235] A. J. Stephens and H. C. Williams, *Computation of real quadratic fields with class number one*, Math. Comp., **51** (1988), No. 184, 809–824; MR **90b**:11106.

[236] *M. A. Stern, *Theorie der Kettenbrüche und ihre Anwendung*, J. reine angew. Math., **11** (1834), 327–341.

[237] *G. W. Tenner, *Einige Bemerkungen über die Gleichung* $ax^2 \pm 1 = y^2$, Progr., Merseburg, 1841.

[238] *T. Tatuzawa, *On a theorem of Siegel*, Japan. J. Math., **21** (1951), 163–178; MR **14** #452c.

[239] D. S. Thakur, *Continued fraction for the exponential for $F_q[T]$*, J. Number Theory, **41** (1992), No. 2, 150–155; MR **98d**:11145.

[240] D. S. Thakur, *Exponential and continued fractions*, J. Number Theory, **59** (1996), No. 2, 248–261; MR **97h**:11145.

[241] D. S. Thakur, *Patterns of continued fractions for the analogues of e and related numbers in the function field case*, J. Number Theory, **66** (1997), No. 1, 129–147; MR **93d**:1113.

[242] A. Verdoodt, *Continued fractions for finite sums*, Ann. Math. Blaise Pascal **1** (1994), No. 2, 71–84 (1995); MR **95m**:11072.

[243] K. Wada, *Construction of quadratic irrational numbers with the continued fraction expansions of fixed periods and Eisenstein's problem*, Math. Japan, **36** (1991), No. 4, 747–767; MR **92f**:11043.

[244] M. Ward, *The linear p-adic recurrence of order two*, Illinois J. Math., **6** (1962), 40–52; MR **25** #2028.

[245] *A. Weil, **Number Theory. An Approach Through History**, Birkäuser, Boston, 1984.

[246] *H. C. Williams, **Édouard Lucas and Primality Testing**, Canadian Mathematical Society Series of Monographs and Advanced Texts, **22**. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York.

[247] H. C. Williams, *Note on a diophantine equation*, Elem. Math., **25** (1970), 123–125; MR **43** #3207.

[248] H. C. Williams, *A generalization of the Fibonacci numbers*, 1970 Proc. Louisiana Conf. on Combinatorics, Graph Theory and Computing (Louisiana State Univ., Baton Rouge, LA, 1970) pp. 340–356 Louisiana State Univ., Baton Rouge, LA; MR **43** #4758.

[249] H. C. Williams, *Some properties of the general Lucas polynomials*, Matrix Tensor Quart., **21** (1971), 91–93, 82; MR **44** #2767.

[250] H. C. Williams, *On a generalization of the Lucas functions*, Acta Arith., **20** (1972), 33–51; MR **45** #1827.

[251] *H. C. Williams, *Some results concerning the nearest integer continued fraction expansion of* $\sqrt{D}$, J. reine angew. Math., **315** (1980), 1–15; MR **81f**:10015.

[252] H. C. Williams, *A numerical investigation into the length of the period of the continued fraction expansion of* $\sqrt{D}$, Math. Comp., **36** (1981), No. 154, 593–601; MR **82f**:10011.

[253] H. C. Williams, *A note on the period length of the continued fraction expansion of certain* $\sqrt{D}$, Utilitas Math., **28** (1985), 201–209; MR **87e**:11016.

[254] H. C. Williams, *Continued fractions and number-theoretic computations*, Number theory (Winnipeg, Man., 1983). Rocky Mountain J. Math., **15** (1985), No. 2, 621–655; MR **87h**:11129.

[255] H. C. Williams, *Eisenstein's problem and continued fractions*, Utilitas Math., **37** (1990), 145–158; MR **1991h**:11018.

[256] H. C. Williams, *Some generalizations of the* S_n *sequence of Shanks.* Acta Arith., **69** (1995), No. 3, 199–215; MR **96a**:11118.

[257] H. C. Williams, *A number theoretic function arising from continued fractions*, Fibonacci Quart., **38** (2000), No. 3, 201–211; MR **2002e**:11010.

[258] H. C. Williams, *Some formulas concerning the fundamental unit of a real quadratic field*, Discrete Math., **92** (1991), No. 1-3, 431–440; MR **92j**:11126.

[259] H. C. Williams and P. A. Buhr, *Calculation of the regulator of* $\mathbb{Q}(\sqrt{D})$ *by use of the nearest integer continued fraction algorithm*, Math. Comp., **33** (1979), No. 145, 369–381; MR **80e**:12003.

[260] *H. C. Williams and M. C. Wunderlich, *On the parallel generation of the residues for the continued fractions factoring algorithm*, Math. Comp., **48** (1987), No. 177, 405–423; MR **88i**:11099.

[261] Y. Yamamoto, *Real quadratic number fields with large fundamental units*, Osaka Math. J., **7** (1970), 57–76; MR **45** #5107.

[262] H. Yokoi, *On real quadratic fields containing units with norm* -1, Nagoya Math. J., **33** (1968), 139–152; MR **38** #2124.

[263] H. Yokoi, *A note on the class-number of real quadratic fields with prime discriminants*, Proc. Japan Acad. Ser. A Math. Sci., **67** (1991), No. 9, 308–312; MR **93e**:11121.

[264] D. Zagier, **Zetafunktionen und quadratische Körper, Eine Einführung in die höhere Zahlentheorie**, Hochschultext, [University Text], Springer, Berlin-New York, 1981.

[265] D. Zagier, *A Kronecker limit formula for real quadratic fields*, Math. Ann., **213** (1975), 153–184; MR **51** #3123.

[266] X. K. Zhang, *Ideal class groups and their subgroups of real quadratic fields*, Sci. China Ser. A, **40** (1997), No. 9, 909–916; MR **98j**:11100.

Appendix A

Supplementary Examples for Kraitchik's work

In Section 2.2, we presented the work of Kraitchik [92] on the parametrization of continued fractions of fixed period length. Here, we give a more detailed account of his work for period lengths 1 through 5. Also, we state his work for period lengths 6 and 7 with proof.

Case (1): It is easy to see that when the period length is 1, D is of the form X^2+1, where $X \in \mathbb{N}$. Examples of $\sqrt{D}$ having period length 1 are easy to find, such as $\sqrt{1^2+1} = \sqrt{2} = (1, \overline{2})$, $\sqrt{2^2+1} = \sqrt{5} = (2, \overline{4})$, $\sqrt{3^2+1} = \sqrt{10} = (3, \overline{6})$, $\sqrt{4^2+1} = \sqrt{17} = (4, \overline{8})$, $\sqrt{5^2+1} = \sqrt{26} = (5, \overline{10})$, etc.

Case (2): For period length 2, i.e., $\sqrt{D} = (a_0, \overline{a_1, 2a_0})$, we compute $D = a_0^2 + 2a_0/a_1$. Since D needs to be an integer, it follows that $a_1 \mid 2a_0$. So, there are two cases: $\sqrt{k^2X^2+2k} = (kX, \overline{X, 2kX})$ and $\sqrt{k^2X^2+k} = (kX, \overline{2X, 2kX})$, where k is a non-zero integer and $X \in \mathbb{N}$.

For the first case, when $k=1$, $\sqrt{X^2+2} = (X, \overline{X, 2X})$. In particular, when $X=1$, we get $\sqrt{3} = (1, \overline{1, 2})$. Similarly, when $X=2$, we have $\sqrt{8} = (2, \overline{3, 4})$.

When $k=2$, we get $\sqrt{4X^2+4} = (2X, \overline{X, 4X})$. In particular, when $X=1$, we have $\sqrt{8} = (2, \overline{2, 4})$. When $X=2$, we have $\sqrt{20} = (4, \overline{2, 8})$.

For the second case, since we are interested in the cases where the period length is two, i.e, $a_1 \neq 2a_0$, we exclude $k=1$.

When $k=2$, we get $\sqrt{4X^2+2} = (2X, \overline{2X, 4X})$. When $X=1$, we get $\sqrt{6} = (2, \overline{2, 4})$. When $X=2$, we have $\sqrt{18} = (4, \overline{4, 8})$.

Case (3): For period length 3, i.e., $\sqrt{D} = (a_0, \overline{a_1, a_1, 2a_0})$, we have $D = a_0^2 + (2a_0a_1+1)/(a_1^2+1)$. Since $2a_0a_1+1$ is odd, a_1^2 must be even. Since we are interested in period length 3, we assume that $a_1 \neq 2a_0$. On writing $a_1 = 2k$, we get

$$2a_0a_1 + 1 = 4a_0k + 1 \equiv 0 \mod 4k^2 + 1.$$

This implies that $a_0 \equiv k \mod 4k^2+1$, i.e, $a_0 = (4k^2+1)X+k$ for some integer X. We need $X \neq 0$ so that $a_1 \neq 2a_0$. Thus, $(2a_0a_1+1)/(a_1^2+1) = 4kX+1$ and $D = (4k^2+1)^2X^2+2k(4k^2+3)X+k^2+1$. We find

$$\sqrt{D(X)} = ((4k^2+1)X+k, \overline{2k,\ 2k,\ (8k^2+2)X+2k}).$$

For example,

$$\sqrt{25X^2+14X+2} = (5X+1, \overline{2,\ 2,\ 10X+2}),$$

$$\sqrt{289X^2+76X+5} = (17X+2, \overline{4, 4, 34X+4}),$$

$$\sqrt{1369X^2+234X+10} = (37X+3, \overline{6, 6, 74X+6}).$$

In particular, when $X = 1$, $25(1)^2 + 14(1) + 2 = 41$ and $\sqrt{41} = (6, \overline{2, 2, 12})$. Similarly, $289(1)^2 + 76(1) + 5 = 370$ and $\sqrt{370} = (19, \overline{4, 4, 38})$. Further, $1369(1)^2 + 234(1) + 10 = 1613$, and $\sqrt{1613} = (40, \overline{6, 6, 80})$.

Case (4): For period length 4, i.e., $\sqrt{D} = (a_0, \overline{a_1, a_2, a_1, 2a_0})$, we find

$$D = a_0^2 + \frac{2a_0(a_1a_2+1)+a_2}{a_1(a_1a_2+2)}.$$

Note that if a_1 is even, then a_2 is even. Also, when a_1 is odd, there is no restriction on the parity of a_2, and hence, there are three possible cases here, namely, a_1 even and a_2 even, a_1 odd and a_2 even, and a_1 odd and a_2 odd. It suffices to examine the cases according to the parity of a_2.

Since D is a natural number, we have $2a_0(a_1a_2+1)+a_2 \equiv 0 \mod a_1(a_1a_2+2)$. It is easy to see that $(a_1a_2+1)^2 \equiv 1 \mod a_1(a_1a_2+2)$. Hence, $2a_0 \equiv -a_2(a_1a_2+1) \mod a_1(a_1a_2+2)$. If a_2 is even, then

$$a_0 \equiv \frac{-a_2(a_1a_2+1)}{2} \mod \frac{a_1(a_1a_2+2)}{2}.$$

When a_2 is odd, by the fact that odd a_2 implies odd a_1, $a_1(a_1a_2+2$ is odd, and hence,

$$a_0 \equiv \frac{-a_2(a_1a_2+1)}{2} \mod a_1(a_1a_2+2).$$

For instance, when $a_2=2$ and $a_1=3$, we have $a_0 \equiv -7 \equiv 5 \mod 12$. If we take $a_0=5$, then $D=28$ and $\sqrt{28}=(5,\overline{3,2,3,10})$. Similarly, if we take $a_0=17$, then $D=299$ and $\sqrt{299}=(17,\overline{3,2,3,34})$

When $a_2=2$ and $a_1=4$, we get $a_0 \equiv -9 \mod 20$. If we pick $a_0=11$, then $D=126$ and $\sqrt{126}=(11,\overline{4,2,4,22})$. Similarly, if we take $a_0=31$, then $D=975$ and $\sqrt{975}=(31,\overline{4,2,4,31})$.

When $a_2=5$ and $a_1=3$, we get $a_0 \equiv -40 \mod 51$. If we pick $a_0=11$, then $D=128$ and $\sqrt{128}=(11,\overline{3,5,3,22})$. Similarly, if we take $a_0=62$, then $D=3883$ and $\sqrt{3883}=(62,\overline{3,5,3,124})$.

When $a_1=5$, $a_2=5X+2$, we get $D(X)=(40X+19)^2+(16X+7)$ and

$$\sqrt{D(X)}=(40X+19,\overline{5,5X+2,5,80X+38}).$$

In particular, when $X=1$, we have $\sqrt{59^2+23}=(59,\overline{5,7,5,118})$. Similarly, when $X=2$, we have $\sqrt{99^2+39}=(99,\overline{5,12,5,198})$.

When $a_1=7$, $a_2=7X+2$, we get $D(X)=(126X+41)^2+(36X+11)$ and

$$\sqrt{D(X)}=(126X+41,\overline{7,7X+2,7,252X+82}).$$

In particular, when $X=1$, we have $\sqrt{167^2+47}=(167,\overline{7,9,7,334})$. Similarly, when $X=2$, we have $\sqrt{293^2+83}=(293,\overline{7,16,7,586})$.

Case (5): For period length 5, i.e., $\sqrt{D}=(a_0,\overline{a_1,a_2,a_2,a_1,2a_0})$, we find

$$D=a_0^2+\frac{2a_0(a_1a_2^2+a_1+a_2)+a_2^2+1}{a_1^2a_2^2+2a_1a_2+a_1^2+1}.$$

It is not difficult to see the case where a_1 is odd and a_2 is even is impossible. Otherwise, since

$$\begin{aligned}(a_1a_2^2+a_1+a_2)^2 &= a_2^2(a_1^2a_2^2+2a_1a_2+a_1^2+1)+(a_1^2a_2^2+2a_1a_2+a_1^2+1)-1\\ &\equiv -1 \mod a_1^2a_2^2+2a_1a_2+a_1^2+1,\end{aligned}$$

we have

$$2a_0 \equiv (a_1a_2^2 + a_1 + a_2)(a_2^2 + 1) \mod a_1^2a_2^2 + 2a_1a_2 + a_1^2 + 1.$$

Since we cannot have a_1 odd and a_2 even at the same time, the right term in the above congruence is even, and the modulus is odd. Hence,

$$a_0 \equiv \frac{(a_1a_2^2 + a_1 + a_2)(a_2^2 + 1)}{2} \mod a_1^2a_2^2 + 2a_1a_2 + a_1^2 + 1.$$

When $a_1 = 1$ and $a_2 = 3$, we get $a_0 \equiv 65 \equiv 14 \mod 17$. If we pick $a_0 = 14$, then $D = 218$ and $\sqrt{218} = (14, \overline{1, 3, 3, 1, 28})$. If we pick $a_0 = 31$, then $D = 1009$ and $\sqrt{1009} = (31, \overline{1, 3, 3, 1, 62})$.

When $a_1 = 2$ and $a_2 = 3$, we find $a_0 \equiv 115 \equiv 9 \mod 53$. If we pick $a_0 = 9$, then $D = 89$ and $\sqrt{89} = (9, \overline{2, 3, 3, 2, 18})$. If we pick $a_0 = 62$, then $D = 3898$ and $\sqrt{3898} = (62, \overline{1, 3, 3, 1, 124})$.

When $a_1 = 2$ and $a_2 = 4$, we find $a_0 \equiv 323 \equiv 68 \mod 85$. If we pick $a_0 = 68$, then $D = 4685$ and $\sqrt{4685} = (68, \overline{2, 4, 4, 2, 136})$. If we pick $a_0 = 153$, then $D = 23546$ and $\sqrt{23546} = (153, \overline{2, 4, 4, 2, 306})$.

Case (6): For period length 6, i.e., $\sqrt{D} = (a_0, \overline{a_1, a_2, a_3, a_2, a_1, 2a_0})$. Kraitchik found

$$2a_0 = -P(a_3a_2^2 + 2a_2) + Qm \quad \text{and} \quad D = a_0^2 - (a_3a_2^2 + 2a_2)^2 + Pm,$$

where $P = a_1a_2^2a_3 + 2a_1a_2 + a_2a_3 + 1$, $Q = a_1^2a_2^2 + 2a_1a_2a_3 + 2a_1^2a_2 + 2a_1 + a_3$ and m is some integer.

For example,

$$\sqrt{(3X+1)^2 + (4X+2)} = (3X + 1, \overline{1, 2, 3X + 1, 2, 1, 6X + 2})$$

and $\sqrt{(3 \cdot 1 + 1)^2 + (4 \cdot 1 + 2)} = \sqrt{22} = (4, \overline{1, 2, 4, 2, 1, 8})$. Also,

$$\sqrt{(3X+1)^2 + (2X+2)} = (3X + 1, \overline{2, 1, 3X, 1, 2, 6X + 2})$$

and $\sqrt{(3 \cdot 1 + 1)^2 + (2 \cdot 1 + 2)} = \sqrt{20} = (4, \overline{2, 1, 3, 1, 2, 8})$.

Case (7): For period length 7, i.e., $\sqrt{D} = (a_0, \overline{a_1, a_2, a_3, a_3, a_2, a_1, 2a_0})$. Kraitchik found

$$2a_0 = PR - Qm \quad \text{and} \quad D = a_0^2 + R^2 + Pm,$$

where

$$R = a_2^2a_3^2 + 2a_2a_3 + a_2^2 + 1,$$

$$P = a_1a_2^2a_3 + 2a_1a_2a_3 + a_1a_2^2 + a_1 + a_2a_3^2 + a_2 + a_3,$$

$$Q = a_1^2a_2^2a_3^2 + a_3^2a_1a_2 + 2a_1^2a_2a_3 + a_1^2a_2^2 + 2a_1a_2 + 2a_1a_3 + a_1^2 + a_1a_2a_3^2 + a_3^2 + 1,$$

and m is some integer.

For example, when $a_1 = 1 = a_2 = a_3$, $D = (13X+7)^2 + 16X + 9$ and

$$\sqrt{(13X+7)^2 + 16X + 9} = (13X + 7, \overline{1, 1, 1, 1, 1, 1, 26X + 14}).$$

In particular, $\sqrt{(13 \cdot 0 + 7)^2 + 16 \cdot 0 + 9} = \sqrt{7^2 + 9} = (7, \overline{1, 1, 1, 1, 1, 1, 14})$ and $\sqrt{(13 \cdot 1 + 7)^2 + 16 \cdot 1 + 9} = \sqrt{20^2 + 25} = \sqrt{425} = (20, \overline{1, 1, 1, 1, 1, 1, 40})$.

Appendix B

Letters from Kaplansky

In the following two letters, the first contains Kaplansky's reference to the term *sleeper* and the second one is his permission to the author for the use of the term sleeper.

In Kaplansky's letter, "A memo on creepers", we only reproduce an excerpt containing his reference to sleepers.

Letter 1

A memo on creepers

Irving Kaplansky

This memo, by an ex-Canadian, is being sent to three Canadians: Richard Mollin (with whom I have been corresponding about continued fractions) and two Williams's, HC and KS, with whom (I believe) this is my first contact......

"Creepers" ? Along with "Sleepers" these are my silly nicknames. A sleeper is a family of continued fractions bounded in length; Schinzel pretty well wrapped these up. In a creeper the lengths go to infinity, but gently, forming one or more arithmetic progressions when sorted out into residue classes; there may be a waiting period before the arithmetic progressions begin.........

Letter 2

Dear Mr. Kell Cheng,

Of course you may use the term "sleeper". (Also, "creeper").

Irving Kaplansky

P.S. I hope this reaches you safely. I did not find an e-mail address in your message.

Appendix C

Examples for the Continued Fraction Expansion of $\sqrt{D(X)}$

C.1 Eight Possible Cases

At this end, we illustrate the different cases of Theorem 4.3.1. We consider different combinations of the parity of $|\mathcal{S}_i|$, the values of $\eta = 1$ or 0, $\sigma = 1$ or -1, $\tau = 1$ or 2 and the cases $r = 0$ or $r > 0$. So, there are 32 possible cases. We note that the parity of $|\mathcal{S}_i|$ is decided by σ and η is determined by σ, in conjunction with whether $r = 0$. So we only need to look at the different combinations of $\sigma = -1$ or 1, $\tau = 1$ or 2, and $r = 0$ or $r > 0$, i.e., 8 cases.

Besides illustrating the eight cases, we also use the examples to provide numerical evidence for the symmetry of the continued fraction expansion of $\sqrt{D(X)}$ discussed in Chapter 6.

Case (1): $\sigma = -1$, $\tau = 1$ and $r = 0$. Consider $D(X) = 119^2X^2 + 2(833)X + 98$, where $A = 119$, $B = 833$, and $C = 98$. Since $833 = 7 \cdot 119$, we have $r = 0$. We find that $\Delta = -7^4 \cdot 17^2$ and $\sigma = -1$. So, $\Delta_1 = 1$, $\Delta_2 = 17$, $\Delta_4 = 7$ and $\Delta_2\Delta_4 = 119$. Clearly, $\Delta_2\Delta_4$ divides $A = 119$, it follows that $\tau = 1$. Further, $A^* = A/(\Delta_2\Delta_4/\tau) = 1$, $\Gamma = \gcd(A^*, \Delta_4) = 1$ and $\Delta' = \tau\Delta_4/\Gamma = 7$.

For $X > (\Delta_1\Delta'^2)/A - (2B-A)/(2A^2) \approx 0.35$, there are $\Delta' = 7$ different patterns corresponding to the seven residue class modulo 7. Let $K = 0, 1, 2, \ldots, 6$ and write $X = 7W + K$, where $W \geq 0$.

Subcase (1.0): When $K = 0$, we get

$$\sqrt{D(7W)} = (833W+7, \overline{34W,\ 3,2,\ 34W-1,\ 1,6,\ 34W,\ 6,1,\ 34W-1,\ 2,3,\ 34W,\ 1666W+14}).$$

Subcase (1.1): When $K = 1$, we get

$$\begin{aligned}\sqrt{D(7W+1)} \;=\; (833W+126, &\ \overline{34W+5,\ 6,1,\ 34W+4,\ 3,2,\ 34W+4,\ 1,2,1,1,\ 34W+4,} \\ &\ \overline{1,1,2,1,\ 34W+4,\ 2,3,\ 34W+4,\ 1,6,\ 34W+5,\ 1666W+252}).\end{aligned}$$

Subcase (1.2): When $K = 2$, we get

$$\sqrt{D(7W+2)} = (833W + 245, \overline{34W + 10,\ 1666W + 490}).$$

Subcase (1.3): When $K = 3$, we get

$$\begin{aligned}\sqrt{D(7W+3)} = (833W + 364, &\overline{34W + 14,\ 1,6,\ 34W + 14,\ 1,2,1,1,\ 34W + 14,\ 3,2,\ 34W + 14} \\ &\overline{2,3,\ 34W + 14,\ 1,1,2,1,\ 34W + 14,\ 6,1,\ 34W + 14,\ 1666W + 728}).\end{aligned}$$

Subcase (1.4): When $K = 4$, we get

$$\begin{aligned}\sqrt{D(7W+4)} = (833W + 483, &\overline{34W + 19,\ 1,2,1,1,\ 34W + 19,\ 6,1,\ 34W + 18,} \\ &\overline{1,6,\ 34W + 19,\ 1,1,2,1,\ 34W + 19,\ 1666W + 966}).\end{aligned}$$

Subcase (1.5): When $K = 5$, we get

$$\begin{aligned}\sqrt{D(7W+5)} = (833W + 602, &\overline{34W + 24,\ 1,1,2,1,\ 34W + 23,\ 1,6,\ 34W + 24,\ 2,3,\ 34W + 24} \\ &\overline{3,2,\ 34W + 24,\ 6,1,\ 34W + 23\ 1,2,1,1,\ 34W + 24,\ 1666W + 1204}).\end{aligned}$$

Subcase (1.6): When $K = 6$, we get

$$\begin{aligned}\sqrt{D(7W+6)} = (833W + 721, &\overline{34W + 29,\ 2,3,\ 34W + 29,\ 6,1,\ 34W + 28,\ 1,1,2,1,\ 34W + 28} \\ &\overline{1,2,1,1,\ 34W + 28,\ 1,6,\ 34W + 29,\ 3,2,\ 34W + 29,\ 1666W + 1442}).\end{aligned}$$

We compute the values of the denominator $d_i r_i$ of the fraction $A'\Delta'/(d_i r_i)$ and list them in the following table. For $K = 0, 1, 2, \ldots, 6$, we list the corresponding $d_i r_i$ in a row.

K	d_0r_0	d_1r_1	d_2r_2	d_3r_3	d_4r_4	d_5r_5	d_6r_6	d_7r_7
0	0	2	6	1	3	0		
1	0	1	2	5	4	3	6	0
2	0	0						
3	0	6	5	2	3	4	1	0
4	0	5	1	6	4	0		
5	0	4	6	3	2	1	5	0
6	0	3	1	4	5	6	2	0

Table C.1: $D(X) = 119^2X^2 + 2(833)X + 98$

Instead of starting with $K = 0$, we start with $K = 6$ in the following

K	d_0r_0	d_1r_1	d_2r_2	d_3r_3	d_4r_4	d_5r_5	d_6r_6	d_7r_7
6	0	3	1	4	5	6	2	0
0	0	2	6	1	3	0		
1	0	1	2	5	4	3	6	0
2	0	0						
3	0	6	5	2	3	4	1	0
4	0	5	1	6	4	0		
5	0	4	6	3	2	1	5	0

Table C.2: $D(X) = 119^2X^2 + 2(833)X + 98$

Now, the sum of $r_1(6) = 3$ and $r_1(5) = 4$ is congruent to 0 modulo $\Delta' = 7$. In fact, it is easy to see that when $K + K' \equiv 4 \mod 7$, $r_i(K) + r_i(K') \equiv 0 \mod 7$ for $i \geq 0$. In this manner, we see a vertical symmetry in the above table.

Case (2): $\sigma = -1$, $\tau = 1$ and $r > 0$. Consider $D(X) = 119^2X^2 + 2(1176)X + 98$, where $A = 119$, $B = 1176$ and $C = 98$. Since $119 \nmid 1176$, we have $r > 0$. Now, $\Delta = -2 \cdot 7^4$ tells us that $\sigma = -1$. Also, we see that $\Delta_1 = 2$, $\Delta_2 = 1$ and $\Delta_4 = 7$. Moreover, $\Delta_2\Delta_4 = 7$, which divides $A = 119$, means that $\tau = 1$. So, $A^* = A/(\Delta_2\Delta_4/\tau) = 17$, $\Gamma = \gcd(A^*, \Delta_4) = 1$ and $\Delta' = \tau\Delta_4/\Gamma = 7$. When $X > (\Delta_1\Delta'^2)/A - (2B - A)/(2A^2) \approx 0.75$, there are seven patterns. Let $K = 0, 1, \ldots, 6$ and write $X = 7W + K$.

Subcase (2.0): When $K = 0$, we get

$$\begin{aligned}\sqrt{D(7W)} &= (833W + 9, \overline{1,7,1,1,\ 17W-1,\ 1,2,19,2,\ 34W-1,\ 1,10,1,9,\ 17W,} \\ &\quad \overline{9,1,10,1,\ 34W-1,\ 2,19,2,1,\ 17W-1,\ 1,1,7,1,\ 1666W+18}).\end{aligned}$$

Subcase (2.1): When $K = 1$, we get

$$\begin{aligned}\sqrt{D(7W+1)} &= (833W + 128, \overline{1,7,1,1,\ 17W+2,\ 9,1,10,1,\ 34W+4,\ 2,1,9,4,\ 17W+2,} \\ &\quad \overline{2,1,1,2,2,1,1,1,\ 34W+4,\ 1,1,1,2,2,1,1,2,} \\ &\quad \overline{17W+2,\ 4,9,1,2,\ 34W+4,\ 1,10,1,9,\ 17W+2,\ 1,1,7,1,1666W+256}).\end{aligned}$$

Subcase (2.2): When $K = 2$, we get

$$\sqrt{D(7W+2)} = (833W + 247, \overline{1,7,1,1,\ 17W+4,\ 1,1,7,1,\ 1666W+494}).$$

Subcase (2.3): When $K = 3$, we get

$$\begin{aligned}\sqrt{D(7W+3)} &= (833W + 366, \overline{1,7,1,1,\ 17W+6,\ 1,22,1,4,\ 34W+14,\ 1,3,2,2,4,1,} \\ &\quad \overline{17W+6,\ 1,2,19,2,\ 34W+14,\ 2,19,2,1,17W+6,} \\ &\quad \overline{1,4,2,2,3,1,\ 34W+14,4,1,22,1,\ 17W+6,\ 1,1,7,1,1666W+732}).\end{aligned}$$

Subcase (2.4): When $K = 4$, we get

$$\begin{aligned}\sqrt{D(7W+4)} &= (833W+485, \overline{1,7,1,1,\ 17W+9,\ 2,1,1,2,2,1,1,1,\ 34W+19,}\\ &\quad \overline{4,1,22,1,\ 17W+8,\ 1,22,1,4,}\\ &\quad \overline{34W+19,\ 1,1,1,2,2,1,1,2,\ 17W+9,\ 1,1,7,1,\ 1666W+970}).\end{aligned}$$

Subcase (2.5): When $K = 5$, we get

$$\begin{aligned}\sqrt{D(7W+5)} &= (833W+604, \overline{1,7,1,1,\ 17W+11,\ 1,4,2,2,3,1,\ 34W+23,\ 1,10,1,9,}\\ &\quad \overline{17W+12,\ 4,9,1,2,\ 34W+24,\ 2,1,9,4,\ 17W+12,}\\ &\quad \overline{9,1,10,1,\ 34W+23,\ 1,3,2,2,4,1,\ 17W+11,\ 1,1,7,1,\ 1666W+1208}).\end{aligned}$$

Subcase (2.6): When $K = 6$, we get

$$\begin{aligned}\sqrt{D(7W+6)} &= (833W+723, \overline{1,7,1,1,\ 17W+14,\ 4,9,1,2,\ 34W+29,\ 4,1,22,1,}\\ &\quad \overline{17W+13,\ 1,4,22,3,1,\ 34W+28,\ 1,3,22,4,1,\ 17W+13,}\\ &\quad \overline{1,22,1,4,\ 34W+29,\ 2,1,9,4,\ 17W+14,\ 1,1,7,1,\ 1666W+1446}).\end{aligned}$$

The values of the product $d_i r_i$ are given in the table below.

K	d_0r_0	d_1r_1	d_2r_2	d_3r_3	d_4r_4	d_5r_5	d_6r_6	d_7r_7
6	105	29	24	97	92	114	41	63
0	105	80	109	12	58	63		
1	105	12	41	46	75	29	109	63
2	105	63						
3	105	114	92	80	58	97	24	63
4	105	46	24	114	75	63		
5	105	97	109	29	41	12	92	63

Table C.3: $D(X) = 119^2X^2 + 2(1176)X + 98$

It is easy to check that when $K + K' \equiv 4 \mod 7$, $d_i(K)_i(K) + d_i(K')r_i(K') \equiv 0 \mod 7$. Also, the sum is 126 or 133 according as i is odd or even.

Case (3): $\sigma = -1$, $\tau = 2$ and $r = 0$. Consider $D(X) = 119^2X^2 + 2(1666)X + 224$, where $A = 119$, $B = 1666$ and $C = 224$. Since $1666 = 14 \cdot 199$, $r = 0$. We find that $\Delta = -2^2 \cdot 7^3 \cdot 17^2$ which means that $\sigma = -1$, $\Delta_1 = 7$, $\Delta_2 = 2 \cdot 7 \cdot 17$ and $\Delta_4 = 1$. Since $\Delta_2\Delta_4 = 2 \cdot 7 \cdot 17 = 238$ does not divide $A = 119$, we have $\tau = 2$. Also, $A^* = A/(\Delta_2\Delta_4/\tau) = 1$, $\Gamma = \gcd(A^*, \Delta_4) = 1$ and $\Delta' = \tau\Delta_4/\Gamma = 2$.

When $X > (\Delta_1\Delta'^2)/A - (2B - A)/(2A^2) \approx 0.12$, there are two patterns. Let $K = 0$ or 1 and write $X = 2W + K$.

Subcase (3.0): When $K = 0$, we get

$$\sqrt{D(2W)} = (238W + 14, \overline{17W + 1,\ 476W + 28}).$$

Subcase (3.1): When $K = 1$, we get

$$\begin{aligned}\sqrt{D(2W+1)} &= (238W + 133, \overline{17W + 9,\ 1, 1,\ 119W + 66,\ 68W + 38,} \\ &\qquad \overline{119W + 66,\ 1, 1,\ 17W + 9,\ 476W + 266}).\end{aligned}$$

The values of d_ir_i are given in the following table.

K	d_0r_0	d_1r_1	d_2r_2
0	0	0	
1	0	1	0

Table C.4: $D(X) = 119^2X^2 + 2(1666)X + 224$

It may seem that there is no symmetry in the above table. However, we see that $2d_i(K)r_i(K) \equiv 0 \mod 2$ for $i = 0, 1, 2$ and $K = 0, 1$.

Case (4): $\sigma=-1$, $\tau=2$ and $r>0$. Consider $D(X)=119^2X^2+2(2555)X+461$, where $A=119$, $B=2555$ and $C=461$. Since $A\nmid B$, we have $r>0$. We find that $\Delta=-2^2\cdot 7^2$, $\sigma=-1$, $\Delta_1=1$, $\Delta_2=2\cdot 7$ and $\Delta_4=1$. Since $\Delta_2\Delta_4=2\cdot 7=14$ which does not divide $A=119$, we have $\tau=2$. So, $A^*=A/(\Delta_2\Delta_4/\tau)=17$, $\Gamma=\gcd(A^*,\Delta_4)=1$ and $\Delta'=\tau\Delta_4/\Gamma=2$.

When $X>(\Delta_1\Delta'^2)/A-(2B-A)/(2A^2)\approx -0.14$, there are two patterns. Let $K=0$ or 1 and write $X=2W+K$.

Subcase (4.0): When $K=0$, we get

$$\sqrt{D(2W)} = (238W+21,\ \overline{2,8,\ 119W+10,\ 1,1,1,1,1,1,1,1,\ 119W+10,\ 8,2,\ 476W+42}).$$

Subcase (4.1): When $K=1$, we get

$$\sqrt{D(2W+1)} = (238W+140,\ \overline{2,8,\ 119W+70,\ 8,2,\ 476W+280}).$$

The values of d_ir_i are given in the following table.

K	d_0r_0	d_1r_1	d_2r_2
0	16	21	16
1	16	16	

Table C.5: $D(X)=119^2X^2+2(2555)X+461$

It is clear that $2d_i(K)r_i(K)\equiv 0 \mod 2$ for $i=0,1,2$ and $K=0,1$.

Case (5): $\sigma = 1$, $\tau = 1$ and $r = 0$. Consider $D(X) = 119^2X^2 + 2(1666)X + 98$. Then $A = 119$, $B = 1666$ and $C = 98$. Since $119 \mid 1666$, we have $r = 0$. Since $\Delta = 2 \cdot 7^4 \cdot 17^2$, we get $\sigma = 1$, $\Delta_1 = 2$, $\Delta_2 = 17$ and $\Delta_4 = 7$. Also, since $\Delta_2\Delta_4 = 7 \cdot 17 = 119$, which clearly divides $A = 119$, we get $\tau = 1$. Thus, $A^* = A/(\Delta_2\Delta_4/\tau) = 1$, $\Gamma = \gcd(A^*, \Delta_4) = 1$ and $\Delta' = \tau\Delta_4/\Gamma = 7$.

When $X > (\Delta_1\Delta'^2)/A - (2B - A)/(2A^2) \approx 0.71$, there are seven patterns. Let $K = 0, 1, \ldots, 6$ and write $X = 7W + K$.

Subcase (5.0): When $K = 0$, we get

$$\begin{aligned}\sqrt{D(7W)} &= (833W + 13, \overline{1,\ 17W - 1,\ 3, 1, 1,\ 34W - 1,\ 1,\ 833W + 12,} \\ &\quad \overline{1,\ 34W - 1,\ 1, 1, 3,\ 17W - 1,\ 1, 1666W + 26})\end{aligned}$$

Subcase (5.1): When $K = 1$, we get

$$\begin{aligned}\sqrt{D(7W+1)} &= (833W + 132, \overline{1,\ 17W + 1,\ 1, 2, 2,\ 34W + 4,\ 1,\ 833W + 131,} \\ &\quad \overline{1,\ 34W + 4,\ 2, 2, 1,\ 17W + 1,\ 1, 1666W + 264})\end{aligned}$$

Subcase (5.2): When $K = 2$, we get

$$\sqrt{D(7W+2)} = (833W + 251, \overline{1,\ 17W + 4,\ 7,\ 34W + 10,\ 7,\ 17W + 4,\ 1,\ 1666W + 502})$$

Subcase (5.3): When $K = 3$, we get

$$\begin{aligned}\sqrt{D(7W+3)} &= (833W + 370, \overline{1,\ 17W + 6,\ 1, 1, 3,\ 34W + 14,\ 1, 5, 1,\ 17W + 6,} \\ &\quad \overline{1, 2, 2,\ 34W + 14,\ 1, 2, 2,\ 17W + 7,\ 7,\ 34W + 14,\ 1,\ 238W + 369} \\ &\quad \overline{1,\ 34W + 14,\ 7,\ 17W + 7,\ 2, 2, 1\ 34W + 14,\ 2, 2, 1} \\ &\quad \overline{17W + 6,\ 1, 5, 1,\ 34W + 14,\ 3, 1, 1,\ 17W + 6,\ 1,\ 1666W + 740}).\end{aligned}$$

Subcase (5.4): When $K = 4$, we get

$$\sqrt{D(7W+4)} = (833W + 489, \overline{1,\ 17W + 8,\ 1,\ 1666W + 978}).$$

Subcase (5.5): When $K = 5$, we get

$$\begin{aligned}\sqrt{D(7W+5)} &= (833W+608, \overline{1,\ 17W+11,\ 2,2,1,\ 34W+24,\ 7,\ 17W+12,}\\ &\quad \overline{3,1,1,\ 34W+24,\ 3,1,1,\ 17W+11,\ 1,5,1,\ 34W+23,\ 1,\ 833W+607}\\ &\quad \overline{1,\ 34W+23,\ 1,5,1,\ 17W+11,\ 1,1,3,\ 34W+24,\ 1,1,3}\\ &\quad \overline{17W+12,\ 7,\ 34W+24,\ 1,2,2,\ 17W+11,\ 1,\ 1666W+1216}).\end{aligned}$$

Subcase (5.6): When $K = 6$, we get

$$\begin{aligned}\sqrt{D(7W+6)} &= (833W+727, \overline{1,\ 17W+13,\ 1,5,1,\ 34W+28,}\\ &\quad \overline{1,5,1,\ 17W+13,\ 1,\ 1666W+1454}).\end{aligned}$$

The values of $d_i r_i$ are given in the following table.

K	d_0r_0	d_1r_1	d_2r_2	d_3r_3	d_4r_4	d_5r_5	d_6r_6	d_7r_7	d_8r_8	d_9r_9	$d_{10}r_{10}$	$d_{11}r_{11}$	$d_{12}r_{12}$	$d_{13}r_{13}$
1	7	5	7	7	3	7								
2	7	1	1	7										
3	7	4	6	5	5	1	7	7	1	3	3	6	2	7
4	7	7												
5	7	3	1	2	2	6	7	7	6	4	4	1	5	7
6	7	6	6	7										
0	7	2	7	7	4	7								

Table C.6: $D(X) = 119^2X^2 + 2(1666)X + 98$

It is easy to check that when $K + K' \equiv 1 \mod 7$, $d_i(K)r_i(K) + d_i(K')r_i(K') \equiv 0 \mod 7$.

Case (6): $\sigma = 1$, $\tau = 1$ and $r > 0$. Consider $D(X) = 119^2X^2 + 2(2890)X + 578$, where $A = 119$, $B = 2890$ and $C = 578$. Since $119 \nmid 2890$, we get $r > 0$. Since $\Delta = 2 \cdot 17^4$, we get $\sigma = 1$, $\Delta_1 = 2$, $\Delta_2 = 1$ and $\Delta_4 = 17$. Also, since $\Delta_2\Delta_4 = 17$, which divides $A = 119$, we get $\tau = 1$. Thus, $A^* = A/(\Delta_2\Delta_4/\tau) = 7$, $\Gamma = \gcd(A^*, \Delta_4) = 1$ and $\Delta' = \tau\Delta_4/\Gamma = 17$.

When $X > (\Delta_1\Delta'^2)/A - (2B - A)/(2A^2) \approx 4.66$, there are seventeen patterns. Let $K = 0, 1, 2, \ldots, 16$ and write $X = 17W + K$.

Subcase (6.0): When $K = 0$, we get

$$\begin{aligned}\sqrt{D(17W)} &= (2023W + 24, \overline{3,1,1,\ 7W-1,\ 1,1,19,2,1,\ 14W-1,} \\ &\quad \overline{2,58,1,\ 7W-1,\ 9,1,11,\ 14W,\ 11,1,9,\ 7W-1,\ 1,58,2,} \\ &\quad \overline{14W-1,\ 1,2,19,1,1,\ 7W-1,\ 1,1,3,\ 4046W+48}).\end{aligned}$$

Subcase (6.1): When $K = 1$, we get

$$\begin{aligned}\sqrt{D(17W+1)} &= (2023W + 143, \overline{3,1,1,\ 7W-1,\ 1,12,4,1,1,\ 14W,\ 2,3,2,6,1,\ 7W-1,} \\ &\quad \overline{1,1,1,2,2,1,1,1,1,\ 14W,\ 2,1,1,1,4,2,1,\ 7W-1,\ 1,4,5,1,3,} \\ &\quad \overline{14W,\ 1,2,1,2,1,1,4,\ 7W,\ 3,1,1,1,1,5,1,\ 14W,\ 7,\ 2023W+143,} \\ &\quad \overline{7,\ 14W,\ 1,5,1,1,1,1,3,\ 7W,\ 4,1,1,2,1,2,1,\ 14W,} \\ &\quad \overline{3,1,5,4,1,\ 7W-1,\ 1,2,4,1,1,1,2,\ 14W,\ 1,1,1,1,2,2,1,1,1,} \\ &\quad \overline{7W-1,\ 1,6,2,3,2,\ 14W,\ 1,1,4,12,1,\ 7W-1,\ 1,1,3,\ 4046W+286}).\end{aligned}$$

Subcase (6.2): When $K = 2$, the continued fraction expansion of $\sqrt{D(17W+2)}$ is given by

$$(2023W + 262, \overline{3,1,1,\ 7W,\ 2,1,39,\ 14W+1,\ 1,3,1,3,6,\ 7W,\ 1,2,1,28,1,}$$
$$\overline{14W,\ 1,5,1,1,1,1,3,\ 7W,\ 1,1,1,2,2,1,1,1,1,\ 14W+1,\ 4,1,23,\ 7W,}$$
$$\overline{1,6,2,3,2,\ 14W+1,\ 2,1,1,1,4,2,1,\ 7W,\ 4,1,1,2,1,2,1,\ 14W+1,\ 11,1,9,}$$
$$\overline{7W,\ 1,4,5,1,3,\ 14W+1,\ 1,1,4,12,1,\ 7W-1,\ 1,58,2,\ 14W+1,\ 3,7,1,1,2,}$$
$$\overline{7W,1,1,19,2,1,\ 14W+1,\ 7,\ 2023W+262,\ 7,\ \ 14W+1,\ 1,2,19,1,1,\ 7W,}$$
$$\overline{2,1,1,7,3,\ 14W+1,\ 2,58,1,\ 7W-1,\ 1,12,4,1,1,\ 14W+1,\ ,3,1,5,4,1,\ 7W,)}$$
$$\overline{9,1,11,\ 14W+1\ 1,2,1,2,1,1,4,\ 7W,\ 1,2,4,1,1,1,2,\ 14W+1,\ 2,3,2,6,1,}$$
$$\overline{7W,\ 23,1,4,\ 14W+1,\ 1,1,1,1,2,2,1,1,1,\ 7W,\ 3,1,1,1,1,5,1,\ 14W,}$$
$$\overline{1,28,1,2,1,\ 7W,\ 6,3,1,3,1,\ 14W+1,\ 39,1,2,\ 7W,\ 3,1,1,\ 4046W+524}).$$

Subcase (6.3): When $K = 3$, we get

$$\begin{aligned}
\sqrt{D(17W+3)} &= (2023W+381, \overline{3,1,1,\ 7W,\ 1,2,1,28,1,\ 14W+1,\ 1,2,19,1,1,\ 7W,} \\
&\quad \overline{1,4,5,1,3,\ 14W+2,\ 2,1,1,1,4,2,1,\ 7W,\ 1,1,1,2,2,1,1,1,1,} \\
&\quad \overline{14W+2,\ 39,1,2,\ 7W,\ 1,58,2,\ 14W+2,\ 7,\ 2023W+381,} \\
&\quad \overline{7,\ 14W+2,\ 2,58,1,\ 7W,\ 2,1,39,\ 14W+2,} \\
&\quad \overline{1,1,1,1,2,2,1,1,1,\ 7W,\ 1,2,4,1,1,1,2,\ 14W+2,\ 3,1,5,4,1,} \\
&\quad \overline{7W,\ 1,1,19,2,1,\ 14W+1,\ 1,28,1,2,1,\ 7W,\ 1,1,3,\ 4046W+762}).
\end{aligned}$$

Subcase (6.4): When $K = 4$, we get

$$\begin{aligned}
\sqrt{D(17W+4)} &= (2023W+500, \overline{3,1,1,\ 7W+1,\ 6,3,1,3,1,\ 14W+2,\ 1,2,19,1,1,} \\
&\quad \overline{7W+1,\ 4,1,1,2,1,2,1,\ 14W+2,\ 1,2,1,2,1,1,4,\ 7W+1,} \\
&\quad \overline{1,1,19,2,1,\ 14W+2,\ 1,3,1,3,6,\ 7W+1,\ 1,1,3,\ 4046W+1000}).
\end{aligned}$$

Subcase (6.5): When $K = 5$, we get

$$\sqrt{D(17W+5)} = (2023W+619, \overline{3,1,1,\ 7W+1,\ 1,1,3,\ 4046W+1238}).$$

Subcase (6.6): When $K = 6$, we get

$$\begin{aligned}\sqrt{D(17W+6)} = (2023W+738, &\overline{3,1,1,\ 7W+1,\ 1,58,2,\ 14W+4,\ 1,1,1,1,2,2,1,1,1,} \\ &\overline{7W+1,\ 1,12,4,1,1,\ 14W+4,\ 1,1,4,12,1,\ 7W+1,} \\ &\overline{1,1,1,2,2,1,1,1,1,\ 14W+4,\ 2,58,1,\ 7W+1,\ 1,1,3,\ 4046W+1476}).\end{aligned}$$

Subcase (6.7): When $K = 7$, the continued fraction expansion of $\sqrt{D(17W+7)}$ is given by

$$\begin{aligned}(2023W+857, &\overline{3,1,1,\ 7W+2,\ 2,1,1,7,3,\ 14W+5,\ 1,1,1,1,2,2,1,1,1,\ 7W+2,} \\ &\overline{2,1,39,\ 14W+5,\ 1,9,1,4,2,\ 7W+2,\ 1,1,19,2,1,\ 14W+5,\ 3,1,5,4,1,\ 14W+2,} \\ &\overline{6,3,1,3,1,\ 14W+5,\ 7,\ 2023W+857,\ 7,\ 14W+5,\ 1,3,1,3,6,} \\ &\overline{14W+2,\ 1,4,5,1,3,\ 14W+5,\ 1,2,19,1,1,\ 7W+2,\ 2,4,1,9,1,\ 14W+5,\ 39,1,2,} \\ &\overline{7W+2,\ 1,1,1,2,2,1,1,1,1,14W+5,\ 3,7,1,1,2,\ 7W+2,\ 1,1,3,\ 4046W+1714}).\end{aligned}$$

Subcase (6.8): When $K = 8$, the continued fraction expansion of $\sqrt{D(17W+8)}$ is given by

$$\begin{aligned}(2023W+976, &\overline{3,1,1,\ 7W+2,\ 1,4,5,1,3,\ 14W+6,\ 2,58,1,\ 7W+2,\ 2,1,1,7,3,\ 14W+6,} \\ &\overline{2,3,2,6,1,\ 7W+2,\ 1,1,19,2,1,\ 14W+6,\ 11,1,9,\ 7W+3,\ 3,1,1,1,1,5,1,\ 14W+5,} \\ &\overline{1,9,1,4,2,\ 7W+2,\ 1,12,4,1,1,\ 14W+6,\ 4,1,23,\ 7W+3,\ 2,1,39,\ 14W+6,} \\ &\overline{1,2,1,2,1,1,4,\ 7W+3,\ 6,3,1,3,1,\ 14W+5,\ 1,28,1,2,1,\ 7W+2,\ 1,1,1,2,2,1,1,1,1,} \\ &\overline{14W+6,\ 7,\ 2023W+976,\ 7,\ 14W+6,} \\ &\overline{1,1,1,1,2,2,1,1,1,\ 7W+2;\ 1,2,1,28,1,\ 14W+5,\ 1,3,1,3,6,\ 7W+3,\ 4,1,1,2,1,2,1,} \\ &\overline{14W+6,\ 39,1,2,\ 7W+3,\ 23,1,4,\ 14W+6,\ 1,1,4,12,1,\ 7W+2,\ 2,4,1,9,1} \\ &\overline{14W+5,\ 1,5,1,1,1,1,3,\ 7W+3,\ 9,1,11,\ 14W+6,\ 1,2,19,1,1,\ 7W+2,\ 1,6,2,3,2} \\ &\overline{14W+6,\ 3,7,1,1,2,7W+2,\ 1,58,2,\ 14W+6,\ 3,1,5,4,1,\ 7W+2,\ 1,1,3,\ 4046W+1952}).\end{aligned}$$

Subcase (6.9): When $K = 9$, the continued fraction expansion of $\sqrt{D(17W+9)}$ is given by

$$(2023W+1095, \overline{3,1,1,\ 7W+3,\ 4,1,1,2,1,2,1,\ 14W+6,\ 1,5,1,1,1,1,3,\ 7W+3,\ 1,1,19,2,1,}$$
$$\overline{14W+6,\ 1,9,1,4,2,\ 7W+3,\ 2,1,39,\ 14W+7,\ 1,1,4,12,1,\ 7W+2,}$$
$$\overline{1,6,2,3,2,\ 14W+7,\ 7,\ 2023W+1095,\ 7,\ 14W+7,\ 2,3,2,6,1}$$
$$\overline{7W+2,\ 1,12,4,1,1,\ 14W+7,\ 39,1,2,\ 7W+3,\ 2,4,1,9,1,\ 14W+6,}$$
$$\overline{1,2,19,1,1,\ 7W+3,\ 1,1,1,1,5,1,\ 14W+6,\ 1,2,1,2,1,1,4,\ 7W+3,\ 1,1,3,\ 4046W+2190}).$$

Subcase (6.10): When $K = 10$, we have

$$\begin{aligned}\sqrt{D(17W+10)} &= (2023W+1214, \overline{3,1,1,\ 7W+3,\ 1,1,1,2,2,1,1,1,1,\ 14W+7,}\\ &\quad \overline{1,3,1,3,6,\ 7W+4,\ 23,1,4,\ 14W+8,\ 4,1,23,\ 7W+3,\ 6,3,1,3,1,}\\ &\quad \overline{14W+7,\ 1,1,1,1,2,2,1,1,1,\ 7W+3,\ 1,1,3,\ 4046W+2428}).\end{aligned}$$

Subcase (6.11): When $K = 11$, the continued fraction expansion of $\sqrt{D(17W+11)}$ is

$$(2023W+1333, \overline{3,1,1,\ 7W+4,\ 23,1,4,\ 14W+9,\ 39,1,2,\ 7W+4,\ 3,1,1,1,1,5,1,}$$
$$\overline{14W+8,\ 2,1,1,1,4,2,1,\ 7W+3,\ 1,12,4,1,1,\ 14W+8,\ 1,2,19,1,1,\ 7W+4,\ 9,1,11,}$$
$$\overline{14W+9,\ 7,\ 2023W+1333,\ 7,\ 14W+9,}$$
$$\overline{11,1,9,\ 7W+4,\ 1,1,19,2,1,\ 14W+8,\ 1,1,4,12,1,\ 7W+3,\ 1,2,4,1,1,1,2,\ 14W+8,}$$
$$\overline{1,5,1,1,1,1,3,\ 7W+4,\ 2,1,39,\ 14W+9,\ 4,1,23,\ 7W+4,\ 1,1,3,\ 4046W+2666}).$$

Subcase (6.12): When $K = 12$, we have

$$\begin{aligned}\sqrt{D(17W+12)} &= (2023W+1452, \overline{3,1,1,\ 7W+4,\ 2,4,1,9,1,\ 14W+9,\ 7,\ 2023W+1452,}\\ &\quad \overline{7,\ 14W+9,\ 1,9,1,4,2,\ 7W+4,\ 1,1,3,\ 4046W+2904}).\end{aligned}$$

Subcase (6.13): When $K = 13$, we have

$$\sqrt{D(17W+13)} = (2023W+1571, \overline{3,1,1,\ 7W+4,\ 1,6,2,3,2,\ 14W+10,} \\ \overline{2,3,2,6,1,\ 7W+4,\ 1,1,3,\ 4046W+3142}).$$

Subcase (6.14): When $K = 14$, we have

$$\sqrt{D(17W+14)} = (2023W+1690, \overline{3,1,1,\ 7W+5,\ 3,1,1,1,1,5,1,\ 14W+10,} \\ \overline{1,5,1,1,1,1,3,\ 7W+5,\ 1,1,3,\ 4046W+3380}).$$

Subcase (6.15): When $K = 15$, we have

$$\sqrt{D(17W+15)} = (2023W+1809, \overline{3,1,1,\ 7W+5,\ 1,2,4,1,1,1,2,\ 14W+12,\ 7,\ 2023W+1809,} \\ \overline{7,\ 14W+12,\ 2,1,1,1,4,2,1,\ 7W+5,\ 1,1,3,\ 4046W+3618}).$$

Subcase (6.16): When $K = 16$, the continued fraction expansion of $\sqrt{D(17W+16)}$ is

$$(2023W+1928, \overline{3,1,1,\ 7W+6,\ 9,1,11,\ 14W+13,\ 3,1,5,4,1,\ 7W+5,\ 1,6,2,3,2,} \\ \overline{14W+12,\ 1,9,1,4,2,\ 7W+6,\ 4,1,1,2,1,2,1,\ 14W+12,\ 1,1,1,1,2,2,1,1,1,} \\ \overline{7W+6,\ 23,1,4,\ 14W+13,\ 7,\ 2023W+1928,\ 7,\ 14W+13,\ 4,1,23,\ 7W+6,} \\ \overline{1,1,1,2,2,1,1,1,1,\ 14W+12,;1,2,1,2,1,1,4,\ 7W+6,\ 2,4,1,9,1,\ 14W+12,} \\ \overline{2,3,2,6,1,\ 7W+5,\ 1,4,5,1,3,\ 14W+13,\ 11,1,9,\ 7W+6,\ 1,1,3,\ 4046W+3856}).$$

We present the values of $d_i r_i$ in the tables below. There are 17 patterns here and we will see that for $0 \le K, K' \le 16$, if $K + K' \equiv 10 \mod 17$, then $d_i(K)r_i(K) + d_i(K')r_i(K') \equiv 0 \mod 17$.

K	d_0r_0	d_1r_1	d_2r_2	d_3r_3	d_4r_4	d_5r_5	d_6r_6	d_7r_7	d_8r_8	d_9r_9	$d_{10}r_{10}$	$d_{11}r_{11}$	$d_{12}r_{12}$	$d_{13}r_{13}$	$d_{14}r_{14}$	$d_{15}r_{15}$	$d_{16}r_{16}$
14	34	33															
15	34	82	17														
16	34	12	31	103	108	26	73	5	17								
0	34	61	59	12													
1	34	110	52	75	45	96	87	33	17								
2	34	40	94	89	101	75	24	103	45	26	10	96	66	117	38	61	17
3	34	89	80	96	45	75	3	117	17								
4	34	19	80	26													
5	34																
6	34	117	73	110													
7	34	47	73	40	108	61	31	19	17								
8	34	96	59	47	52	61	10	33	108	110	24	40	87	19	115	75	17
9	34	26	101	61	108	40	66	103	17								
10	34	75	94	5													
11	34	5	3	33	45	110	80	12	17								
12	34	54	17														
13	34	103															

K	$d_{17}r_{17}$	$d_{18}r_{18}$	$d_{19}r_{19}$	$d_{20}r_{20}$	$d_{21}r_{21}$	$d_{22}r_{22}$	$d_{23}r_{23}$	$d_{24}r_{24}$	$d_{25}r_{25}$	$d_{26}r_{26}$	$d_{27}r_{27}$	$d_{28}r_{28}$	$d_{29}r_{29}$	$d_{30}r_{30}$	$d_{31}r_{31}$	$d_{32}r_{32}$	$d_{33}r_{33}$
14	101	68															
15	17	45	68														
16	17	24	75	87	54	52	96	10	68								
0	10	117	80	68													
1	17	101	26	31	82	73	103	66	68								
2	17	80	47	59	110	31	12	87	82	52	5	73	33	115	19	3	68
3	17	59	40	73	82	31	61	115	68								
4	87	61	94	68													
5	68																
6	66	73	59	68													
7	17	94	96	80	54	3	75	38	68								
8	17	73	89	94	26	3	5	66	54	101	12	80	103	38	117	31	68
9	17	52	110	3	54	80	33	87	68								
10	24	19	73	68													
11	17	10	61	66	82	101	40	24	68								
12	17	108	68														
13	52	68															

Table C.7: $D(X) = 119^2X^2 + 2(2890)X + 578$

Case (7): $\sigma = 1$, $\tau = 2$ and $r = 0$. Consider $D(X) = 119^2X^2 + 2(1666)X + 168$, where $A = 119$, $B = 1666$ and $C = 168$. Since $119 \mid 1666$, $r = 0$. Since $\Delta = 2^2 \cdot 7^3 \cdot 17^2$, we have $\sigma = 1$, $\Delta_1 = 7$, $\Delta_2 = 2 \cdot 7 \cdot 17$ and $\Delta_4 = 1$. Since $\Delta_2\Delta_4 = 2 \cdot 7 \cdot 17 = 238$, which does not divide $A = 119$, we get $\tau = 2$. Hence, $A^* = A/(\Delta_2\Delta_4/\tau) = 1$, $\Gamma = \gcd(A^*, \Delta_4) = 1$ and $\Delta' = \tau\Delta_4/\Gamma = 2$.

When $X > (\Delta_1\Delta'^2)/A - (2B - A)/(2A^2) \approx 0.12$, there are two patterns. Let $K = 0, 1$ and write $X = 2W + K$.

Subcase (7.0): When $K = 0$,

$$\sqrt{D(2W)} = (238W + 13, \overline{1,\ 17W - 1,\ 1,\ 476W + 26}).$$

Subcase (7.1): When $K = 1$,

$$\begin{aligned}\sqrt{D(2W+1)} = (238W + 132, &\overline{1,\ 17W + 8,\ 2,\ 119W + 65,\ 1,\ 68W + 36,} \\ &\overline{1,\ 119W + 65,\ 2,\ 17W + 8,\ 1,\ 476W + 264}).\end{aligned}$$

The values of d_ir_i given in the following table.

K	d_0r_0	d_1r_1	d_2r_2	d_3r_3	d_4r_4	d_5r_5
0	0	0				
1	0	1	0	0	1	0

Table C.8: $D(X) = 119^2X^2 + 2(1666)X + 168$

Case (8): $\sigma = 1$, $\tau = 2$ and $r > 0$. Consider $D(X) = 119^2X^2 + 2(1700)X + 204$. Then $A = 119$, $B = 1700$ and $C = 204$. If is not difficult to see that $A \nmid B$, so $r > 0$. Now, $\Delta = 2^2 \cdot 17^2$ means that $\sigma = 1$, $\Delta_1 = 1$, $\Delta_2 = 2 \cdot 7 \cdot 17$ and $\Delta_4 = 1$. Since $\Delta_2\Delta_4 = 2 \cdot 17 = 34$, which does not divide $A = 119$, we get $\tau = 2$. Hence, $A^* = A/(\Delta_2\Delta_4/\tau) = 7$, $\Gamma = \gcd(A^*, \Delta_4) = 1$ and $\Delta' = \tau\Delta_4/\Gamma = 2$.

When $X > (\Delta_1\Delta'^2)/A - (2B - A)/(2A^2) \approx -0.82$, there are two patterns. Let $K = 0, 1$ and write $X = 2W + K$.

Subcase (8.0): When $K = 0$,

$$\sqrt{D(2W)} = (238W + 14, \overline{3, 1, 1, \ 119W + 6, \ 1, 1, 3, \ 476W + 28}).$$

Subcase (8.1): When $K = 1$,

$$\sqrt{D(2W + 1)} = (238W + 133, \overline{3, 1, 1, \ 119W + 66, \ 14, \ 119W + 66, \ 1, 1, 3, \ 476W + 266}).$$

The values of d_ir_i given in the following table.

K	d_0r_0	d_1r_1	d_2r_2
0	4	4	
1	4	1	4

Table C.9: $D(X) = 119^2X^2 + 2(1700)X + 204$

C.2 More Computations

In this section, we provide more numerical evidence for the vertical symmetry of the continued fraction expansion of $\sqrt{D(X)}$ discussed in Chapter 6 and in the preceding section.

Example C.2.1 Consider $3^2X^2 + 2(9)X$. We find $\Delta = 3^4$, $\Delta' = 3$ and $A' = 1$. Let $X > (\Delta_1\Delta'^2)/A - (2B - A)/(2A^2) \approx 2.17$.

Case (0): When $X \equiv 0 \mod 3$, i.e., $X = 3W$, we get

$$\sqrt{D(3W)} = (9W + 2, \overline{1, 2W - 1, 1, 1, 1, 2W - 1, 1, 18W + 4}).$$

Case (1): When $X \equiv 1 \mod 3$, i.e., $K = 1$ and X=3W+1, we get

$$\sqrt{D(3W + 1)} = (9W + 5, \overline{1, 2W, 3, 2W, 1, 18W + 10}).$$

Case (2): When $X \equiv 2 \mod 3$, i.e., $K = 2$,, we get

$$\sqrt{D(3W + 2)} = (9W + 8, \overline{1, 2W, 1, 18W + 16}).$$

In the table below, we list values of d_ir_i.

K	d_0r_0	d_1r_1	d_2r_2
1	3	1	3
2	3	3	
0	3	2	3

Table C.10: $D(X) = 3^2X^2 + 2(9)X$

For $i = 0, 1, 2$, it is clear that $d_i(0)r_i(0) + d_i(1)r_i(1) \equiv 0 \mod 3$ and trivially, $2d_i(2)r_i(2) \equiv 0 \mod 3$. $\square$

Example C.2.2 Consider $180^2X^2 + 2(4950)X + 750$. We find $\Delta = 2^2 \cdot 3^4 \cdot 5^4$, $\Delta' = 5$ and $A' = 2$. Let $X > (\Delta_1\Delta'^2)/A - (2B - A)/(2A^2) \approx -0.11$.

Case (0): When $X = 5W$, the continued fraction expansion of $\sqrt{D(5W)}$ is given by

$$(900W + 27, \overline{2,\ 72W + 1,\ 1, 2, 3,\ 72W + 1,\ 1, 8, 1,\ 72W + 1,\ 3, 2, 1,\ 72W + 1,\ 2,\ 1800W + 54}).$$

Case (1): When $X = 5W + 1$,

$$\sqrt{D(5W + 1)} = (900W + 207, \overline{2,\ 72W + 16,\ 10,\ 72W + 16,\ 2,\ 1800W + 414}).$$

Case (2): When $X = 5W + 2$, we get

$$\sqrt{D(5W + 2)} = (900W + 387, \overline{2,\ 72W + 30,\ 2,\ 1800W + 774}).$$

Case (3): When $X = 5W + 3$, we get

$$\sqrt{D(5W + 3)} = (900W + 567, \overline{2,\ 72W + 44,\ 1, 8, 1,\ 72W + 44,\ 2,\ 1800W + 1134}).$$

Case (4): When $X = 5W + 4$, the continued fraction expansion of $\sqrt{D(5W + 4)}$ is given by

$$(900W{+}747, \overline{2,\ 72W + 59,\ 3, 2, 1,\ 72W + 59,\ 10,\ 72W + 59,\ 1, 2, 3,\ 72W + 59,\ 2,\ 1800W + 1494}).$$

The values of d_ir_i are given in the following table.

K	d_0r_0	d_1r_1	d_2r_2	d_3r_3	d_4r_4
0	5	7	9	3	5
1	5	1	5		
2	5	5			
3	5	9	5		
4	5	3	1	7	5

Table C.11: $D(X) = 180^2X^2 + 2(4950)X + 750$

We see that for $0 \le K, K' \le 4$ such that $K + K' \equiv 0 \mod 4$, $d_i(K)r_i(K) + d_i(K')r_i(K') \equiv 0 \mod 5$. $\square$

Example C.2.3 Consider $5^2X^2 + 2(25)X$. We find $\Delta = 5^4$, $\Delta' = 5$ and $A' = 1$. Let $X > (\Delta_1\Delta'^2)/A - (2B-A)/(2A^2) \approx 4.10$.

Case (0): When $X = 5W$, we get

$$\sqrt{D(5W)} = (25W+4, \overline{1,\ 2W-1,\ 2,1,1,\ 2W-1,\ 1,3,1,\ 2W-1,\ 1,1,2,\ 2W-1,\ 1,\ 50W+8}).$$

Case (1): When $X = 5W+1$, we get

$$\sqrt{D(5W+1)} = (25W+9, \overline{1,\ 4W-1,\ 1,3,1,\ 4W-1,\ 1,\ 50W+18}).$$

Case (2): When $X \equiv 2 \mod 5$, we get

$$\sqrt{D(5W+2)} = (25W+14, \overline{1,\ 2W,\ 5,\ 2W,\ 1,\ 50W+28}).$$

Case (3): When $X \equiv 3 \mod 5$, we get

$$\sqrt{D(5W+3)} = (25W+19, \overline{1,\ 2W,\ 1,1,2,\ 2W+1,\ 5,\ 2W+1,\ 2,1,1,\ 2W,\ 1,\ 50X+38}).$$

Case (4): When $X \equiv 4 \mod 5$, we get

$$\sqrt{D(5W+4)} = (25W+24, \overline{1,\ 2W,\ 1,\ 50W+48}).$$

The values of d_ir_i are given in the following table.

K	d_0r_0	d_1r_1	d_2r_2	d_3r_3	d_4r_4
2	5	1	5		
3	5	3	1	2	5
4	5	5			
0	5	2	4	3	5
1	5	4	5		

Table C.12: $D(X) = 5^2X^2 + 2(25)X$

We see that when $0 \le K, K' \le 3$ such that $K + K' = 3$, $d_i(K)r_i(K) + d_i(K')r_i(K') \equiv 0 \mod 5$ for $i = 0,1,2,3,4$. Also, $2d_i(4)r_i(4) \equiv 0 \mod 5$. □

Example C.2.4 Consider $2^2X^2 + 2(8)X$. We find $\Delta = 2^6$, $\Delta' = 4$ and $A' = 1$. Let $X > (\Delta_1\Delta'^2)/A - (2B - A)/(2A^2) \approx 6.25$.

Case (0): When $X = 4W$, we get

$$\sqrt{D(4W)} = (8W+3, \overline{1,\ W-1,\ 2,\ 4W+1,\ 2,\ W-1,\ 1,\ 16W+6}).$$

Case (1): When $X = 4W+1$, we get

$$\sqrt{D(4W+1)} = (8W+5, \overline{1,\ W-1,\ 1,2,1,\ W-1,\ 1,\ 16W+10}).$$

Case (2): When $X = 4W+2$, we get

$$\sqrt{D(4W+2)} = (4W+7, \overline{1,\ W-1,\ 1,\ 8W+14}).$$

Case (3): When $X = 4W+3$, we get

$$\sqrt{D(4W+3)} = (4W+9, \overline{1,\ W,\ 4,\ W,\ 4W+18}).$$

The values of d_ir_i are given in the following table.

K	d_0r_0	d_1r_1	d_2r_2	d_3r_3
0	4	2	2	4
1	4	3	4	
2	4	4		
3	4	1	4	

Table C.13: $D(X) = 2^2X^2 + 2(8)X$

We see that for $0 \le K, K' \le 3$ such that $K + K' \equiv 0 \mod 4$, $d_i(K)r_i(K) + d_i(K')r_i(K') \equiv 0 \mod 4$. □

Example C.2.5 Consider $3^2X^2 + 2(18)X$. We find $\Delta = 2^2 \cdot 3^4$, $\Delta' = 6$ and $A' = 1$. Let $X > (\Delta_1\Delta'^2)/A - (2B-A)/(2A^2) \approx 10.17$.

Case (0): When $X = 6W$,

$$\sqrt{D(6W)} = (18W+5, \overline{1,\ W-1,\ 3,\ 4W,\ 1,\ 9W+1,\ 1,\ 4W,\ 3,\ W-1,\ 1,\ 36W+10}).$$

Case (1): When $X = 6W+1$,

$$\begin{aligned}\sqrt{D(6W+1)} &= (18W+8, \overline{1,\ W-1,\ 2,\ 9W+3,\ 1,\ X,} \\ &\quad \overline{1,\ 9W+3,\ 2,\ W-1,\ 1,\ 36W+16}).\end{aligned}$$

Case (2): When $X = 6W+2$,

$$\begin{aligned}\sqrt{D(6W+2)} &= (18W+11, \overline{1,\ W-1,\ 1,1,1,\ 4W+1,\ 1,\ 9W+4,} \\ &\quad \overline{1,\ 4W+1,\ 1,1,1,\ W-1,\ 1,\ 36W+22}).\end{aligned}$$

Case (3): When $X = 6W+3$,

$$\sqrt{D(6W+3)} = (18W+14, \overline{1,\ W-1,\ 1,4,1,\ W-1,\ 1,\ 36W+28}).$$

Case (4): When $X = 6W+4$,

$$\sqrt{D(6W+4)} = (18W+17, \overline{1,\ W-1,\ 1,\ 36W+34}).$$

Case (5): When $X = 6W+5$,

$$\sqrt{D(6W+5)} = (18W+20, \overline{1,\ W,\ 6,\ W,\ 1,\ 36W+40}).$$

The values of d_ir_i are given in the following table.

K	d_0r_0	d_1r_1	d_2r_2	d_3r_3	d_4r_4	d_5r_5
0	6	2	6	6	2	6
1	6	3	6	6	3	6
2	6	4	6	6	4	6
3	6	5	6			
4	6	6				
5	6	1	6			

Table C.14: $D(X) = 3^2X^2 + 2(18)X$

We see that for $0 \leq K, K' \leq 5$ such that $K + K' \equiv 2 \mod 6$, $d_i(K)r_i(K) + d_i(K')r_i(K') \equiv 0 \mod 6$. □

Example C.2.6 Consider $D(X) = 22^2X^2 + 2(484)X$. Then $A = 22 = 2 \cdot 11$, $B = 484 = 2^2 \cdot 11^2$, $C = 0$ and $\Delta = B^2 - A^2C = 2^4 \cdot 11^4$. So, $\Delta_1 = 1$, $\Delta_2 = 1$, $\Delta_4 = 2 \cdot 11 = 22$, $\tau = 1$, $\Gamma = 1$, $A' = 1$ and $\Delta' = 22$. Let $X > (\Delta_1 \Delta'^2)/A - (2B - A)/(2A^2) \approx 21.02$. We find that there are 11 patterns, i.e., $\rho = 11$. So, we write $X = 11W + K$ for $K = 0, 1, 2, \ldots 10$.

Case (0): $X = 11W$. The continued fraction expansion of $\sqrt{D(11W)}$ is given by

$$\begin{aligned}
&(484W + 21, \overline{1,\ 2W-1,\ 11,\ 4W,\ 3,1,2,\ 2W-1,\ 1,2,1,1,1,\ 4W-1,\ 1,2,1,1,1,} \\
&\overline{2W-1,\ 2,4,1,\ 4W-1,\ 1,1,5,\ 2W-1,\ 1,9,1,\ 4W-1,\ 2,4,1,} \\
&\overline{2W-1,\ 3,1,2,\ 4W-1,\ 1,\ 242W+9,1,\ 4W-1,\ 2,1,3,\ 2W-1,} \\
&\overline{1,4,2,\ 4W-1,\ 1,9,1,\ 2W-1,\ 5,1,1,\ 4W-1,\ 1,4,2,\ 2W-1,} \\
&\overline{1,1,1,2,1,\ 4W-1,\ 1,1,1,2,1,\ 2W-1,\ 2,1,3,\ 4W,\ 11,\ 2W-1,\ 1,\ 968W+42}).
\end{aligned}$$

Case (1): $X = 11W + 1$. We have

$$\begin{aligned}\sqrt{D(11W+1)} &= (484W+21, \overline{1,\ 2W-1,\ 5,1,1,\ 4W,\ 5,1,1,\ 2W-1,\ 1,1,1,2,1,}\\ &\quad \overline{4W-1,\ 1,\ 484W+20,\ 1,\ 4W-1,}\\ &\quad \overline{1,2,1,1,1,\ 2W-1,\ 1,1,5,\ 4W,\ 1,1,5,\ 2W-1,\ 1,\ 968W+42}).\end{aligned}$$

Case (2): $X = 11W + 2$.

$$\begin{aligned}\sqrt{D(11W+2)} &= (484W+65, \overline{1,\ 2W-1,\ 3,1,2,\ 4W,\ 1,2,1,1,1,}\\ &\quad \overline{2W-1,\ 1,1,1,2,1,\ 4W,\ 2,1,3,\ 2W-1,\ 1,\ 968W+130}).\end{aligned}$$

Case (3): $X = 11W + 3$.

$$\begin{aligned}\sqrt{D(11W+3)} &= (484W+87, \overline{1,\ 2W-1,\ 2,1,3,\ 4W+1,\ 5,1,1,\ 2W-1,}\\ &\quad \overline{1,4,2,\ 4W,\ 1,\ 242W+42,\ 1,\ 4W,\ 2,4,1,}\\ &\quad \overline{2W-1,\ 1,1,5,\ 4W+1,\ 3,1,2,\ 2W-1,\ 1,\ 968W+174}).\end{aligned}$$

Case (4): $X = 11W + 4$.

$$\begin{aligned}\sqrt{D(11W+4)} &= (484W+109, \overline{1,\ 2W-1,\ 2,4,1,\ 4W,\ 1,\ 242W+53,}\\ &\quad \overline{1,\ 4W,\ 1,4,2,\ 2W-1,\ 1,\ 968W+218})\end{aligned}$$

Case (5): $X = 11W + 5$.

$$\begin{aligned}\sqrt{D(11W+5)} &= (484W+131, \overline{1,\ 2W-1,\ 1,1,5,\ 4W+1,\ 1,\ 242W+64,}\\ &\quad \overline{1,\ 4W+1,\ 5,1,1,\ 2W-1,\ 1,\ 968W+262}).\end{aligned}$$

Case (6): $X = 11W + 6$.

$$\begin{aligned}\sqrt{D(11W+6)} &= (484W+153, \overline{1,\ 2W-1,\ 1,1,1,2,1,\ 4W+1,\ 1,4,2,\ 2W,}\\ &\quad \overline{5,1,1,\ 4W+1,\ 1,\ 242W+75,\ 1,\ 4W+1,\ 1,1,5,}\\ &\quad \overline{2W,\ 2,4,1,\ 4W+1,\ 1,2,1,1,1,\ 2W-1,\ 1,\ 968W+306}).\end{aligned}$$

Case (7): $X = 11W + 7$.

$$\begin{aligned}\sqrt{D(11W+7)} &= (484W+175, \overline{1,\ 2W-1,\ 1,2,1,1,1,\ 4W+2,\ 3,1,2,\ 2W,} \\ &\quad \overline{2,1,3,\ 4W+2,\ 1,1,1,2,1,\ 2W-1,\ 1,\ 968W+350}).\end{aligned}$$

Case (8): $X = 11W + 8$

$$\begin{aligned}\sqrt{D(11W+8)} &= (484W+197, \overline{1,\ 2W-1,\ 1,4,2,\ 4W+2,\ 1,4,2,\ 2W,\ 2,1,3,} \\ &\quad \overline{4W+2,\ 1,\ 242W+97,\ 1,\ 4W+2,} \\ &\quad \overline{3,1,2,\ 2W,\ 2,4,1,\ 4W+2,\ 2,4,1,\ 2W-1,\ 1,\ 968W+394}).\end{aligned}$$

Case (9): $X = 11W + 9$.

$$\begin{aligned}\sqrt{D(11W+9)} &= (484W+219, \overline{1,\ 2W-1,\ 1,9,1,\ 4W+2,\ 1,2,1,1,1,\ 2W,} \\ &\quad \overline{3,1,2,\ 4W+3,\ 3,1,2,\ 2W,\ 1,1,5,\ 4W+3,\ 2,4,1,} \\ &\quad \overline{2W,\ 11,\ 4W+3,\ 1,1,5,\ 2W,} \\ &\quad \overline{1,2,1,1,1,\ 4W+2,\ 1,\ 242W+108, 1,\ 4W+2,\ 1,1,1,2,1,} \\ &\quad \overline{2W,\ 5,1,1,\ 4W+3,\ 11,\ 2W,} \\ &\quad \overline{1,4,2,\ 4W+3,\ 5,1,1,\ 2W,\ 2,1,3,\ 4W+3,\ 2,1,3,} \\ &\quad \overline{2W,\ 1,1,1,2,1,\ 4W+2,\ 1,9,1,\ 2W-1,\ 1,\ 968W+438}).\end{aligned}$$

Case (10): $X = 11W + 10$.

$$\sqrt{D(11W+10)} = (484W+241, \overline{1,\ 2W-1,\ 1,\ 968W+482}).$$

The values of $d_i r_i$ are given in the following table.

K	d_0r_0	d_1r_1	d_2r_2	d_3r_3	d_4r_4	d_5r_5	d_6r_6	d_7r_7	d_8r_8	d_9r_9	$d_{10}r_{10}$	$d_{11}r_{11}$	$d_{12}r_{12}$	$d_{13}r_{13}$	$d_{14}r_{14}$	$d_{15}r_{15}$	$d_{16}r_{16}$	$d_{17}r_{17}$	$d_{18}r_{18}$	$d_{19}r_{19}$	$d_{20}r_{20}$	$d_{21}r_{21}$
5	22	12	22	22	4	22																
6	22	14	18	4	22	22	12	10	16	22												
7	22	16	6	8	14	22																
8	22	18	18	8	22	22	6	10	10	22												
9	22	20	16	6	6	12	10	2	12	16	22	22	14	4	2	18	4	8	8	14	20	22
10	22	22																				
0	22	2	6	16	16	10	12	20	10	6	22	22	8	18	20	4	18	14	14	8	2	22
1	22	4	4	14	22	22	16	12	12	22												
2	22	6	16	14	8	22																
3	22	8	4	18	22	22	10	12	6	22												
4	22	10	22	22	18	22																

Table C.15: $D(X) = 22^2X^2 + 2(484)X$

We see that for $0 \leq K, K' \leq 10$ such that $K + K' \equiv 9 \mod 11$, $d_i(K)r_i(K) + d_i(K')r_i(K') \equiv 0 \mod 22$. $\square$

Appendix D

Tables for the Average of $l(a/r)$

In the following two tables, we consider even and odd lengths of $l(a/r)$ separately. In order to represent every expansion of a/r properly and compactly, we denote 10 by A, 11 by B and so forth, i.e., we write $(1, 12, 1)$ as 1C1. The left-most column lists the values of a and the top row lists the values of r. Every entry of the main body of the table gives the continued fraction expansion of a/r. The third right-most column lists the total number of partial quotients. The second right-most column lists the average lengths of continued fraction expansions of a/r with a fixed. The last column gives the expected value of the average lengths using (6.5).

$a\backslash r$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Total	Avg.	E. V.
2	11															1	1	1.55
3	21	12														4	2	1.90
4	31	11	13													6	2	1.70
5	41	22	1111	14												10	2.5	2.33
6	51	21	11	12	15											10	2	1.88
7	61	32	23	1121	1211	16										16	2.67	2.61
8	71	31	2111	11	1112	13	17									18	2.57	1.98
9	81	42	21	24	1131	12	1311	18								20	2.5	2.51
A	91	41	33	22	11	1111	1221	14	19							22	2.44	2.35
B	A1	52	3111	2121	25	1141	1113	1212	1411	1A						26	2.6	2.99
C	B1	51	31	21	2211	11	1122	12	13	15	1B					26	2.36	2.32
D	C1	62	43	34	2112	26	1151	111111	1231	1321	1511	1C				38	3.17	3.13
E	D1	61	4111	32	2131	23	11	1121	1114	1211	1312	16	1D			38	2.92	2.67
F	E1	72	41	3121	21	22	27	1161	1111	12	1213	14	1611	1E		38	2.71	2.67
G	F1	71	53	31	35	2111	2311	11	1132	1112	1241	13	1421	17	1F	40	2.67	2.80

Table D.1: Even Length

$a\backslash r$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Total	Avg.	E. V.
1	1															1	1	
2	2	1														2	1	1.55
3	3	111	1													5	1.67	1.90
4	4	2	121	1												6	1.5	1.70
5	5	211	112	131	1											11	2.2	2.33
6	6	3	2	111	141	1										10	1.67	1.88
7	7	311	221	113	122	7	1									15	2.14	2.61
8	8	4	212	2	11111	121	161	1								19	2.38	1.98
9	9	411	3	231	114	111	132	171	1							21	2.33	2.51
A	A	5	321	211	2	112	123	121	181	1						22	2.20	2.35
B	B	511	312	213	241	115	11121	12111	142	191	1					33	3	2.99
C	C	6	4	3	222	2	11211	111	121	141	1A1	1				26	2.17	2.32
D	D	611	421	331	21111	251	116	11112	124	133	152	1B1	1			39	3	3.13
E	E	7	412	311	214	221	2	113	11131	122	13111	151	1C1	1		38	2.71	2.67
F	F	711	5	313	3	211	261	117	112	111	12121	131	162	1D1	1	39	2.60	2.67

Table D.2: Odd Length

Appendix E

Some Estimates for $\mathrm{lp}(\sqrt{D(X)})$

Consider $D(X) = 4927230^2X^2 + 2(12138809742675)X + 6069410874450$. In this case, $A' = 385$ and $\Delta' = 158$.

In the table below, we list the actual value of $\mathrm{lp}(\sqrt{D(158W+K)})$ for some values of K and an estimate obtained by

$$\left\lfloor \kappa \cdot \left(\frac{12 \ln 2}{\pi^2} \left(\ln A'\Delta' - \sum_{d \mid A'\Delta'} \frac{\Lambda(d)}{d} \right) + 0.97 \right) \right\rfloor,$$

where κ is either $\omega(\Delta')$ or $2\omega(\Delta')$, d is a proper divisor of $A'\Delta'$ and $\Lambda(d)$ is von Mangoldt's function given in Chapter 6. When $\kappa = 2\omega(\Delta')$, the estimates are **boldfaced**.

K	1	3	4	5	6	7	10	12	13	14	15	17
Actual	910	324	596	294	294	552	914	136	224	532	353	3766
Est.	968	322	**553**	304	304	**553**	968	138	**276**	553	**387**	3551
K	19	20	21	24	25	26	27	29	31	32	34	35
Actual	118	102	500	326	92	44	1478	1952	918	164	118	532
Est.	138	**138**	**553**	322	**110**	55	1522	**1937**	968	**184**	138	**553**

Table E.1: Some estimates for $\mathrm{lp}(\sqrt{4927230^2X^2 + 2(12138809742675)X + 6069410874450})$

In the above example, we did not include entries for $K = 2, 8, 9, 11, 16, 18, \ldots$ They are omitted because the corresponding κ is not $\omega(\Delta')$ or $2\omega(\Delta')$.

Appendix F

Constructing $D(X)$

When we study the continued fraction expansion of $\sqrt{D(X)}$, where $D(X) = A^2X^2 + 2BX + C$ and $\Delta = B^2 - A^2C$ divides $4\gcd(A^2, B)^2$, it is imperative to have examples of such $D(X)$. It is not difficult to find certain $D(X)$ via a trial and error method using a computer, i.e, testing whether an integer triple $(A^2, 2B, C)$ satisfies the Schinzel condition. However, there is a severe drawback to this approach: we cannot predict the important values, such as A' and Δ'. In particular, if we want to have a large A' or Δ', the trial and error method could take a good deal of time. Here, we provide a simple method to find $D(X)$ that allows us to choose the values of A' and Δ'.

Recall from Section 4.2 that $|\Delta| = \Delta_1\Delta_2^2\Delta_4^4$, where Δ_1 and Δ_2 are squarefree. By (4.12), $\gcd(A, B) = \Gamma\Delta_2\Delta_4/\tau$, where Γ and τ are defined on page 99 and τ is either 1 or 2. Also, we defined $A' = A/(\Gamma\Delta_2\Delta_4/\tau)$ and $\Delta' = \tau\Delta_4/\Gamma$ in Section 4.2. Now, we write $B' = B/(\Gamma\Delta_2\Delta_4/\tau)$. Note that $B' = B^*/\Gamma$, where B^* was defined in Section 4.2. It is clear that $\gcd(A', B') = 1$.

The Schinzel condition says that $\Delta \mid 4\gcd(A^2, B)^2$, i.e., $\Delta_1\Delta_2^2\Delta_4^4 \mid 4\gcd(A^2, B)^2$. Since $\Delta_1\Delta_2^2\Delta_4^4 = \Delta_1(\tau\Delta_4/\Gamma)^2(\Gamma\Delta_2\Delta_4/\tau)^2$, the Schinzel condition can be written as

$$\Delta_1\Delta'^2 \mid 4\gcd(A \cdot A', B')^2.$$

Since $\gcd(A', B') = 1$, we have

$$4\gcd(A \cdot A', B')^2 = 4\gcd(A, B')^2 = 4\gcd(A'(\Gamma\Delta_2\Delta_4/\tau), B')^2 = 4\gcd(\Gamma\Delta_2\Delta_4/\tau, B')^2.$$

Thus, the Schinzel condition becomes

$$\Delta_1\Delta'^2 \mid 4\gcd(\Gamma\Delta_2\Delta_4/\tau, B')^2. \tag{F.1}$$

This implies that $\Delta' \mid 2B'$. For the sake of simplicity, we assume that $\Delta' \mid B'$, i.e, $B' = k\Delta'$ for non-zero integer k.

Now, $B^2 - A^2C = \Delta$ implies that $B'^2 - A'^2C = \Delta_1\Delta'^2$. Since $\Delta' \mid B'$ and $\gcd(A', B') = 1$, we have $\Delta'^2 \mid C$. Thus, we may write $C = m\Delta'^2$ for some non-zero integer m. Now we rewrite $B'^2 - A'^2C = \Delta_1\Delta'^2$ as

$$(k\Delta')^2 - A'^2m\Delta'^2 = \Delta_1\Delta'^2,$$

which is equivalent to

$$k^2 - A'^2m = \Delta_1$$

since Δ' is non-zero. Hence,

$$A'^2 = \frac{k^2 - \Delta_1}{m}.$$

To find solutions to the above equation, we first fix Δ_1, say $\Delta_1 = 1, 2$, and then vary k to compute $k^2 - \Delta_1$. We seek square factors in $k^2 - \Delta_1$ as k varies. When we find some square factor in $k^2 - \Delta_1$ that we want, we put it as A'^2 and set $m = (k^2 - \Delta_1)/A'^2$. Now that we have A'^2, m and k, we have a solution to $k^2 - A'^2m = \Delta_1$. We may now choose any Δ', co-prime to A', to get

$$(k\Delta')^2 - A'^2m\Delta'^2 = \Delta_1\Delta'^2.$$

For simplicity's sake, we set $\Delta_2 = 1$, $\tau = 1$ and $\Gamma = 1$. Then $\Delta_4 = \Delta'$, $A = A'\Delta_4$, $B = B'\Delta_4$ and $C = m\Delta'^2$.

We note that the above method can be modified to accommodate the cases where $\tau = 2$, $\Delta_2 = 2$.