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A GROUP THEORETICAL APPROACH TO MATERIAL UNIFORMITY

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled, "A Group Theoretical Approach to Material Uniformity" submitted by Mr. Nusret Uğurhan Aydemir in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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ABSTRACT

The problem of material uniformity for simple elastic bodies is studied and characterized. The body is conceived as an n-dimensional C^{∞} manifold possesing an extra structure through its constitutive law which is an open map $\Omega: \mathcal{B} \to \mathcal{A}$, $\mathcal{B}$ being the body manifold and $\mathcal{A}$ the space of response functions for the particular physical phenomenon considered. The concept of smooth material response is reviewed and its relation to smooth uniformity is clarified. The former is shown to be preferable due to its less restrictive underlying assumption. The continuous group of transformations $GL(r,\mathbb{R})$ of dimension r^2 is used to construct an open neighborhood of a point $\Omega_X = \Omega(X) \in \mathcal{A}$, $X \in \mathcal{B}$ and the action $A_{g \in GL}: \mathcal{A} \to \mathcal{A}$ is shown to be completely determined by Ω once the space $\mathcal{A}$ is fixed. The concept of tangential uniformity is introduced and a direct formulation of material uniformity using the Killing vectors of the action of GL at $\Omega_X \in \mathcal{A}$ is given. It is shown that this can be reduced to a condition on the determinant of a certain matrix whose entries are determined by the components of the Killing vectors and the components of a vector tangent to the curve $\Omega(c(t))$ at the point of interest where $c(t): I \subset \mathbb{R} \to \mathcal{B}$ is a smooth curve in $\mathcal{B}$ passing through $c(0) = X$.

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This work is dedicated to my parents.

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NOMENCLATURE

1. General

$\forall x$	for every x
$\exists$	there exists
$\in$	included in
$\cup$	union
$\cap$	intersection
$\subset$	subset
$\sim$	equivalence relation
iff	if and only if
$X \times Y$	direct product
$\mathbb{R}$	real line, set of real numbers
$\mathbb{R}^+$	set of positive real numbers
$f \circ g$	mapping composition
$\det[a_{ij}]$	determinant of the matrix $[a_{ij}]$
$T_x(X)$	tangent vector space to X at x
$T(X)$	tangent bundle
$tr(a_{ij})$	trace of the matrix $[a_{ij}]$
$\otimes$	tensor product
$\oplus$	material isomorphism

2. English letters

A	set of all possible stress tensors
A_g	group action of g
B	the body manifold

a_o, a_{ij},	
a_{ijkl},	scalar functions defined on the body
b_{ij}, c_{ijkl}	
$c(t)$	a curve with parameter t
$\mathcal{C}$	the set of all configurations of $\mathcal{B}$
C_1, C_2	constants
e	the identity element of a group
e_i	basis vectors
$\mathcal{E}$	Euclidean space
exp	exponential map
f	response function
f^R	response relative to a global reference
$\underline{F}, F, K, L$	invertible linear transformations
G	continuous group
G_X	isotropy group at X
G_X, K_X	local configuration gradients at a material point X
$G_{\alpha\beta}(X)$	symmetry element at X
g, h, l	elements of a group
$GL(r,\mathbb{R})$	the general linear group of dimension r^2
I	_ identity matrix
	_ open set on the real line
	_ set of integers
$\hat{k}$	vector tangent to a curve in the body
L	relaxation map
L_g	left translation
M	a differentiable manifold

$\mathcal{N}(x)$	open neighborhood about the point x
$\mathcal{O}_x$	orbit at x
$P_X(Y)$	material isomorphism in a reference configuration
P_1, P_2	parts of the body
Q	orthogonal transformation
r,t,s	scalar parameters
T	stress tensor
T^R	stress tensor relative to a global reference
u,v	vector fields
$\hat{u}$	vector tangent to a curve in the space of response functions
U	_ open set _ scalar field defined on the body
V	_ open set _ vector space
W	energy function of a hyperelastic body
x	deformation of the body
X	material point in a reference configuration
X,Y,Z	generators of orthogonal group
X,Y	material points in the body manifold
x^i, y^i	coordinate functions

3. Greek letters

$\alpha,\beta,\gamma,\kappa,\lambda$	_ global configurations of the body _ generators of one parameter subgroups _ indices
Γ	reference atlas

δ	kronecker delta
ς, η	local reference configuration
λ	_ global reference configuration
	_ scalar
Λ	space of response functions
$\xi(t)$	one parameter subgroup
$\rho(g)$	matrix representation of g
ρ_t	parallel transport
$\sigma(t,.)$	one parameter group of transformations
$\sigma(f,x)$	response function of a one dimensional body
Φ	_ global reference
	_ empty set
	_ curve
Ψ	_ invariant variety
	_ curve
Ω	constitutive function

CHAPTER 1

INTRODUCTION

1.1 Objectives

In this study we shall deal with the question of material uniformity in simple elastic and hyperelastic materials and the restrictions imposed on the constitutive laws describing such bodies.

1.2 Material Uniformity, Constitutive Laws and Locality: An Intuitive Discussion

A body is called materially uniform if all of its body points are of the same material. Clearly, a materially uniform body will remain so in all of its configurations.

In the general theory of Mechanics of Continua, it is known that the four fundamental axioms of Mechanics [1][1] namely, a) Principle of Conservation of Mass, b) Principle of Balance of Momentum, c) Principle of Balance of Moment of Momentum, d) Principle of Conservation of Energy are not sufficient to make a problem determinate and one has to introduce additional equations, called the laws of constitution or constitutive equations, for each

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[1] Numbers in square brackets are listed under References.

material. In general, they are functionals of the body points, history of deformation, temperature and time. The explicit dependence on the material points indicates that the body is possibly a non-uniform one; if it is uniform, then this dependence is due to the "inhomogeneities" within the body itself or else the reference configuration chosen is not homogenous. In the latter case one can always refer the body to a configuration such that the constitutive equation will not depend on the material points.

When further restrictions are imposed on the body, the law of constitution provides us with a purely geometrical theory for material inhomogeneities first proposed by W. Noll [2,3] and generalized by C.C. Wang [4]. These restrictions are:

a) The body is elastic, i.e. there is no dependence on history or time, b) The response of the body is completely determined by the geometry of the deformation, i.e. there is no dependence on temperature and the theory is purely a mechanical one, c) The physical characteristics of the body are local in the sense that they pertain to individual material points and their immediate neighborhoods, rather than the body as a whole. The theory of simple bodies deals only with such local characteristics. The concept of a simple material was axiomatized by Noll [5] along with several other concepts which serve as the foundations of rational continuum mechanics today.

The term "simple" expresses the assumption that deformations whose gradient is the identity at a given material point do not alter the physical response at that point with respect to the set of phenomena under consideration. To be precise, let the stress in an elastic material be given by

$$T(X) = f(x(X'), X) \tag{1.2.1}$$

where f is a tensor function of the material point $X \in B$ and the current global configuration $x\colon B \subset E \to E$. Here, B denotes the body in a given reference state and E is the Euclidean space. The axiom of neighborhood [1] states that the motions of material points that are not near the point X will not contribute appreciably to T at X. One way of formulating this is to rewrite (1.2.1.) as (smooth neighborhood) :

$$T(X) = f(x(X) + (X'_{K_1} - X_{K_1})\, x,_{K_1}\big|_X + \frac{1}{2!}(X'_{K_1} - X_{K_1})(X'_{K_2} - X_{K_2})\, x,_{K_1K_2}\big|_X + \dots, X) \tag{1.2.2}$$

where $x(X')$ is expanded at X with coordinates (X_1, X_2, X_3) and $x,_{K_1}\big|_X = \dfrac{\partial x(X)}{\partial X_{K_1}}$, etc.

Materials in which the function depends on terms of higher order than 1 are called non-simple materials. In a way, the range of the neighborhood is brought into play with the highest order deformation gradients involved. A further simplification is achieved when we employ the axiom of objectivity ([1],p. 157) resulting in the elimination of the dependence on x(X) in (1.2.2).

Now we state that a simple elastic response is given by:

$$T(X) = f(x,_{K}|_{X}, X) \qquad (1.2.3)$$

where the dependences on higher gradients of the deformation are dropped.

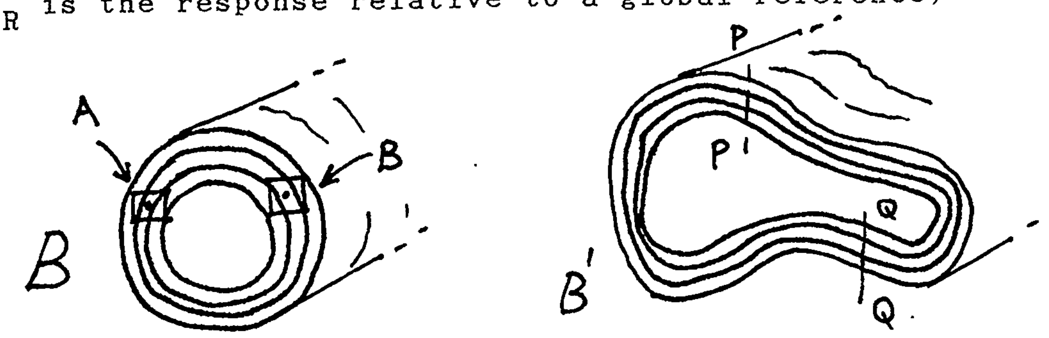
Given the response of a material point by (1.2.3) we can define a simple uniform body as the body whose physical response has the same form at all of its points. This was the approach taken by Noll [2] and Wang [4] where they have constructed a theory for inhomogeneities in materially uniform simple bodies. The connection between this theory and the problem of uniformity is discussed next.

1.3 Homogeneity and Uniformity

Let X denote a material point in the body manifold B where B is not necessarily embedded in E. In Eqn. (1.2.3), the dependence on the material point X (or X) may not necessarily stem from the non-uniformity of the body. Another source for this dependence is the inhomogeneity of the body itself. For example consider a cylindrical body made by gluing the opposite ends of many thin sheets of the same material (Fig.1.1a-1.1b). Obviously, this body is materially uniform but its response cannot be described by a simple relation like

$$T^R(X) = f_R(F)$$

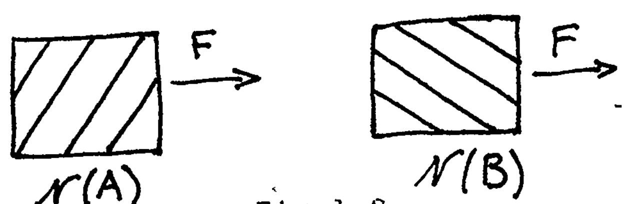
(f_R is the response relative to a global reference)



(a) (b)

Fig. 1.1

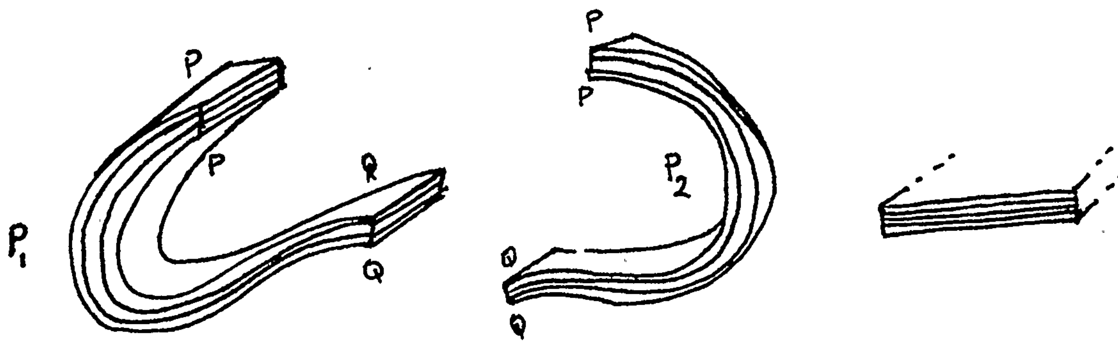
where $F \in GL(3)$. To see this, let us take two points, say A and B and imagine we cut out small neighborhoods $N(A)$, $N(B)$ (Fig.1.1a). They will look like (Fig.1.2):



$$T_A^R(F) \neq T_B^R(F)$$

Fig.1.2

Since N(A) and N(B) are not laminated in the same direction, they will not respond the same way. However, it is possible to "re-orient" one of them to get an identical response, say by rotating $N(B)$ rigidly clockwise so that the sheets will point in the same direction. But this body, because of its topology does not admit a smooth field of such rotations and the response is of the form $T(X) = f(x_K, X)$. Clearly, $T(X)$ can be defined on B locally so that the dependence on X is dropped in this case. Another point is that the above re-orientation maps need not always be rotations. They can be the members of the general linear group GL(3) when arbitrary pre-stresses are present in a body (eg. see Fig.1.1b). To illustrate this "localization" process intuitively, consider Fig.1.1b and the sections P-P, Q-Q. Divide B' into two parts defined by cutting through these sections. We have two pieces :



(a) (b) (c)

Fig. 1.3

Clearly, each part can be brought to the shape (c) by a <u>smooth</u> deformation (mathematically a homeomorphism).

The important conclusion in this example is that, given a uniform body, it is sometimes possible that its parts may admit reference configurations with respect to which the response

$T(X) = f(x,_K,X)$ can be written free from the dependence on X. Only inhomogenous bodies will not admit such parts. Now, let

$$\phi_1 : P_1 \to E \quad , \quad \phi_2 : P_2 \to E$$

be two such references for where P_1 and P_2 are two simply connected parts of B'. Then,

$$f_{P_1}(F \circ d\phi_1|_X) = f_{P_2}(F \circ d\phi_2|_Y) \qquad \forall X \in P_1 \ , \quad \forall Y \in P_2,$$

$F \in GL(3)$ where $(d\phi)^{\alpha}_{K} = \partial\phi^{\alpha}/\partial X_K$, (We also note here that the reference map ϕ manifests itself through its first gradient for simple bodies).

1.3.1 Definition

A uniform body B is called locally homogeneous when its "parts" admit reference configurations with respect to which the response function will take the same form.

From this, it follows that B is globally homogeneous when such maps are defined globally, i.e.

$f(Fd\phi|_X) \equiv g_{\phi}(F)$ $\quad \phi : B \to E, \ \forall X \in B$. (eg. P_1 or P_2 of B in Fig.2 are separately, globally homogeneous bodies).

1.3.2 Remark

In the light of the discussion above, when uniformity properties of B are considered it is seen that one should focus attention on local material uniformity because of the local structure of B induced by internal stresses. That is, the explicit dependence on X has its sources at a local level even when B is uniform. Hence, one is naturally led to a local investigation in a study of non-uniformity.

1.3.3 Remark

One must recognize the limitations of this definition of uniformity since it is dependent on the assumption of locality as pointed out by Kröner [6]. The most general definition is, of course, the one given at the beginning of section (1.2) but the geometrical structure that it imposes on the body is not clear. On the other hand, using locality, the geometry is a natural outcome of the laws of constitution and many useful results for simple materials can be obtained.

1.4 Geometry of a Simple Body, General Considerations

1.4.1 Parallelism: An Informal Exposition

In abstract terms a geometry is defined on a manifold when one has the means of comparing vectors at different points unambiguously, i.e. in a way free from the coordinate system (or sytems) chosen about these points. For example, if we have a three dimensional Euclidean manifold, a single coordinate chart is sufficient to map this space onto our ordinary 3-D world. In particular, we can choose this chart to be cartesian and the comparison of vectors is trivial. Namely,vectors at different points are parallel if their components are the same. For non-Cartesian charts, the local bases are different at each point. In order to compare vectors at different points, we must first refer them to a standard basis, say an orthonormal triad. The process of transformation into a standard basis is defined globally on a Euclidean manifold which enables us to define a field of vectors parallel to a vector at a fixed point. We call this the Euclidean parallelism. When we have a Riemannian manifold such global parallelism cannot be found in general but it can be defined in a coordinate free way along a smooth curve on the manifold. In most simple terms, this is equivalent to solving a system of ordinary differential equations (first degree) along a curve and the property follows from the uniqueness of its solution. It provides one with a field of parallel vectors along the chosen curve and depends on the metric chosen (since the "coefficients of connection" are defined by this metric). This is called the Levi-Civita parallelism of vectors [7]. When a general connected manifold is considered, there is no means of measurement of distances, hence, no metric structure; one has no means of defining the connection functions. As a remedy, those functions are assigned a' priori with the requirement that they satisfy appropriate transformation laws when coordinate charts are changed. The connection coefficients defined in this way are said to form an affine connection on the manifold and one can again construct a parallel field of vectors along a given curve with given initial conditions [8].

In all of the above type of parallelism structures, it is seen that the objective is to define a field of parallel vectors to a given vector at a point either globally or locally, depending on the type of the manifold. This is equivalent to defining a field of isomorphisms, called parallel transports which are one-to-one mappings of different tangent spaces along a given curve.

The constitutive law of a simple elastic body (uniform) determines an affine geometry on B and is discussed next.

1.4.2 The Parallelism Structure on a Uniform Simple Body

Defining a geometry on B is equivalent to specify the above parallel transports. Of course, physical distances can be measured on B (hence we have a metric) which will induce a parallelism; but this type of geometry does not bring into play the physical properties of the body. However a parallel transport can be defined in a very natural way for uniform simple bodies [2] using their constitutive law. Let $X, Y \in U \subset B$; the response functions at X and Y are given by $f(d\Phi, X)$ and $f(d\Phi, Y)$ where $\Phi : U \subset B \to E$. Let $N(X)$ and $N(Y)$ be open neighborhoods of X, Y in U. If the body is uniform two local configurations γ, β of $N(X)$ and $N(Y)$ can be defined such that

$$f(Fd\gamma|_X, X) = f(Fd\beta|_Y, Y) \qquad (1.4.2.1)$$

where $\gamma : N(X) \to E$, $\beta : N(Y) \to E$, $F = d\lambda : V \to V$ (V is the translation space of E), and λ is a "local deformation" of E (see Chap.2). Let $F = d\alpha|_X(d\gamma|_X)^{-1}$ and substitute in

Eqn.(1.4.2.1) to get

$$f(d\alpha|_X, X) = f(d\alpha|_X(d\gamma|_X)^{-1}d\beta|_Y, Y)$$

The quantity $(d\gamma|_X)^{-1}d\beta|_Y$ is called a "material isomorphism" and is a linear map of $T_Y B$ onto $T_X B$.

Physically, this process is equivalent to cutting out a small neighborhood of Y and gluing it back after applying a certain deformation to it. This deformation is such that the materials at X and Y are indistinguishable (with respect to the response f) after the process is completed. (A formal definition is given in Chap.2). The maps $\Phi(X,Y) = (d\gamma|_X)^{-1}d\beta|_Y$ when defined smoothly on $U \subset B$ play the role of the parallel transports discussed before. The connection induced by Φ is an affine connection and is completely determined by the constitutive law of the body. The torsion of this connection is shown [2] to characterize the material inhomogeneity of B and in general need not vanish.

The question of geometrical structures on B, when B is not known to be uniform has not been discussed in literature until the recent expositions of H. Cohen - M. Epstein [9] and M. Elżanowski - M. Epstein [10], where they study hyperelastic uniformity. These approaches are given briefly in the next chapter after introducing some formal definitions that are also necessary to discuss the theory given in this thesis.

1.5 Suggested Approach to Uniformity

In the present theory, we consider a wider class of simple bodies, namely those which are uniform along a curve in an open neighborhood of a point $X \in B$. This property is referred to as the "directional" or "tangential" uniformity. This is a weaker structure than that introduced in [9,10] and therefore covers a wider class of bodies.

The following identifications and assumptions are made:

a) The body B is an n dimensional differentiable manifold of class $C^p (p \geq 1)$.

b) The constitutive law is viewed as a mapping $\Omega : B \to \Lambda$ (definition 2.2.4) where Λ is a finite dimensional linear vector space. The nature of Λ depends on the type of physical phenomena under consideration.

c) The general linear group GL(n=3) acts as a Lie Group of transformations on Λ. Its action is determined by the constitutive law.

d) B is a crystal body, that is, its isotropy group consists of discrete points in GL(3). This is done for the sake of simplicity in the mathematical manipulations.

e) B is a smoothly non-uniform body (definition 2.2.19).

With this set up, it will be shown that the study of uniformity can be reduced to the study of the orbits of

$GL(3)$ on $\mathcal{A}$ in the neighborhood of $\Omega(X) \in \mathcal{A}$.

To summarize, we conclude with a classification of simple bodies in an order of decreasing generality:

Non-uniform

Tangentially uniform

Curvewise uniform

Locally uniform

Uniform

Inhomogeneous

Locally homogeneous

Globally homogeneous

(All uniformities are smooth in the sense explained in Chapter 2).

The strongest structure is found in a globally homogeneous body. Its behaviour under external effects is easier to determine. However, in a physically meaningful situation, inhomogeneities in the form of internal stresses or non-uniformities due to material property differences exist in a body. Therefore, it is natural to seek new forms of the equations of motion to take such effects into account.

CHAPTER 2

SOME CONCEPTS IN THE THEORY OF A SIMPLE BODY

The ideas discussed in chapter 1 are stated here in a precise form. The theories for uniform [2,4] and non-uniform [9,10] bodies are given with comparisons to the present approach. Mathematical concepts are explained when they are needed. This chapter also fixes the notation that is to be used in the remaining parts of this work. Different expositions of this material can be found in [2,11,12,13,14,15] addressing to readers with different backgrounds. The present discussion is unique, however, due to its treatment of the subject from a different point of view: the possibility of non-uniformities in the body.

2.1 The Need for a Consistent Geometrical Model

When large-scale plastic phenomena are considered, such as the plastic flow above the yield stress, it is known that a large number of dislocations are involved, of the order of 10^{11} to 10^{12} lines/cm^2. In such a situation a reasonable point of view is to treat the aggregate of dislocations. Such a view leads to the theory of continuous distributions of dislocations. This theory was originated by Bilby-Bullough and Smith [16] based on the concept of an atomistic and crystalline structure of the dislocated materials. It is known that if one takes a closed curve in a dislocated crystal and maps it on a perfect reference crystal, its image would not close under this map. The closure failure is known as the Burgers vector of the dislocations threading the original curve. It is also known that the local Burgers vectors can be defined in terms of a second order tensor, called the dislocation density [12, p.244]. Now, since the line integral on the <u>perfect crystal</u> is given in terms of local deformations from the <u>real crystal</u>, we obtain a relationship between the dislocation density tensor and the local deformations. Motivated by this, in [16] it is shown that the torsion tensor of a connection on the real crystal and the dislocation density tensor are equivalent. This is done by taking advantage of the crystalline structure of the body and two vectors at different points (of the original crystal) are called parallel if they correspond to the same number of local lattice steps. The curvature of the connection defined by this parallelism vanishes and its torsion has the meaning mentioned above.

In the theory of Noll [2], although the end result is the same, no crystalline structure is assumed and the geometry of the body is determined once a constitutive assumption is made.

2.2 Theory of Inhomogeneities in Simple Elastic Bodies

A one-to-one mapping $\lambda : W \subset E \to E$ is called a deformation of class $C^r (r \geq 1)$ if it is not only of class C^r but if also the values of its gradient are invertible, i.e. $d\lambda(x) \in GL(n, \mathbb{R})$ for all $x \in W$.

2.2.1 Definition

A body is a set whose members $X, Y, \ldots$ are called material points and which is endowed with a structure defined by a class C of mappings $\kappa : B \to E$ called the configurations of B (in the space E). The point $\kappa(X)$ is called the place of the material point $X \in B$ in the configuration κ.

B is a continuous body of class $C^p (p \geq 1)$ if the class C of configurations satisfies the following axioms:

1) Every $\kappa \in C$ is one-to-one and $\kappa(B)$ is open in E,

2) If $\gamma, \kappa \in C$ then $\lambda = \gamma \circ \kappa^{-1} : \kappa(B) \to \gamma(B)$ is a deformation of class C^p called the deformation of B from the configuration κ into the configuration γ

3) If $\kappa \in C$ and $\lambda : \kappa(B) \to E$ is a deformation, then $(\lambda \circ \kappa) \in C$.

2.2.2 Definition Manifolds, charts and atlas of a C^p Manifold (see [17] for a detailed exposition).

A (topological) manifold is a Hausdorf topological space such that every point has a neighborhood homeomorphic to $\mathbb{R}^n$. A chart (U, ς) of a manifold M is an open set U of M together with homeomorphism $\varsigma : U \to V$ of U onto an open set $V \subset \mathbb{R}^n$. The coordinates $(x^1,\ldots,x^n)$ of the image $\varsigma(x)\in\mathbb{R}^n$ of the point $x\in M$ are called the <u>coordinates</u> of x in the chart (U,ς). An <u>atlas</u> of class C^p on a manifold M is a set $\{U_\alpha, \varsigma_\alpha\}$ of charts of M such that the domains $\{U_\alpha\}$ cover M and the homeomorphisms satisfy the following compatibility conditions: The maps $\varsigma_\beta \circ \varsigma_\alpha^{-1} : \varsigma_\alpha(U_\alpha \cap U_\beta) \to \varsigma_\beta(U_\alpha \cap U_\beta)$ are maps of open sets of $\mathbb{R}^n$ into $\mathbb{R}^n$ of class C^p. In other words, the mapping $\varsigma_\beta \cdot \varsigma_\alpha^{-1}$ is given by n real valued C^p functions of n variables: $(x^i) \to y^i = f^i(x^j)$ where (x^i) and (y^i) are the coordinates of x in $(U_\alpha, \varsigma_\alpha)$ and $(U_\beta, \varsigma_\beta)$ respectively. Two C^p atlases are <u>equivalent</u> if their union is again a C^p atlas. A topological manifold M together with an equivalence class of C^p atlases is a C^p structure on M; we say that M is a C^p manifold. From this point of view, it is easily seen that B is a C^p manifold with an atlas of the form (B,ς), that is, it can be covered by a single open set and mapped on E homeomorphically.

2.2.3 <u>Definition</u>

Two global configurations κ,γ at X are said to be equivalent at X and we write $\kappa \sim \gamma$ if

$$d(\kappa \circ \gamma^{-1})\ |_{\gamma(X)} = I\in GL(n,\mathbb{R})$$

where $GL(n,\mathbb{R})$ is the general linear group of dimension n^2. The resulting partition of C into equivalence classes at X is a set C_X whose members K_X, G_X, ... are called the <u>local</u>

configurations at X. Clearly, if K_X, $G_X \epsilon C_X$, then $G_X K_X{}^{-1} \epsilon GL$ and for $L \epsilon GL$, LK_X defines a new local configuration in C_X by $LK_X = \{ \lambda \circ \kappa | \, d\lambda|_{\kappa(X)} = L, \, d\kappa(X) = K_X \}$.

Now a standart definition of the concept of a tangent vector space $T_x M$ of a manifold M at a point $x \epsilon M$ will be given. Intuitively, $T_x M$ generalizes the notion of tangent plane to a surface in $\mathbb{R}^3$. One way of defining tangent vectors at $x \epsilon M$ is as a triple (x, ξ, v) such that (x, ξ, v) and (x, ξ', v') define the same vector if

$$v' = d(\varsigma' \circ \varsigma^{-1})|_x v, \qquad \text{where } v, v' \epsilon V \text{ holds.}$$

If $\varsigma^i : x \to x^i$ is a chart at x, the vectors of $T_x M$ that are represented by

$$(\partial/\partial x^1, \ldots, \partial/\partial x^n)$$

form a basis for the tangent vector space which is called the natural basis. Since $T_x M$ and M have the same dimension, a chart (U, ς) at x thus induces an isomorphism of $T_x M$ onto $\mathbb{R}^n$.

2.2.4 Definition

Let A be the set of mathematical objects whose nature depends on the particular physical phenomena to be described. For example, in the theory of elasticity A consists of all possible stress tensors, i.e., of all symmetric linear transformations of V into V, where V is the translation space of E. In theories that include non-mechanical effects, A consists of functions or functionals whose independent and dependent variables have interpretations as local temperatures, energy or entropy densities, heat fluxes, electric or magnetic field strengths, polarizations, electric currents, etc. In this work (chapter 5), A will be assumed to be the set of all possible stress tensors for reasons of explicit calculation.

A continuous body of class C^p will be called a <u>simple body with respect to A</u> if it is endowed with a structure by a function Ω which assigns to each material point $X \in B$ a mapping

$$\Omega_X: C_X \to A$$

The value $\Omega_X(G_X)$ is the <u>response descriptor</u> of the material at X in any configuration γ of B such that $d\gamma(X) = G_X$.

2.2.5 <u>Remark</u>

The function Ω acts as follows

$$\Omega: B \to \mathcal{A} \quad \text{by} \quad X \mapsto \Omega_X,$$

i.e. it assigns a <u>response function</u> Ω_X to each point $X \in B$.

In the theory given in Chapter 4, the map Ω will be assumed to define a C^1 function in an open neighborhood $N(X)$ of $X \in B$. If κ is a configuration of B, then the representation $\bar{\Omega}_\kappa$ of Ω defined by $\bar{\Omega}_\kappa: U \to \mathcal{A}$, $U \subset E$ is also a C^1 function.

We say X and Y are of the same material if their response functions are identical. However, since Ω_X and Ω_Y act on different domains (C_X and C_Y respectively) one first should "re-write" these response functions referred to a common domain. This can be done if an isomorphism of $T_Y B$ and $T_X B$ is given.

2.2.6 Definition

The linear transformation $\circledast_{XY}: T_Y B \to T_X B$ is called a material isomorphism from $T_Y B$ onto $T_X B$ if

$$\Omega_X(K_X) = \Omega_Y(K_X \circledast_{XY}) \qquad \forall\ K_X \in C_X.$$

The physical meaning of $\circledast_{XY}$ was explained in Chapter 1.

2.2.7 Definition

A simple body B is materially uniform if the material at any two of its points is the same.

2.2.8 Definition

The tensor function $K: B \to L(T_X B, V)$ is called a reference for B and furthermore if

$$\circledast(X,Y) = K(X)^{-1}K(Y): T_Y B \to T_X B \qquad \forall\ X,Y \in B$$

is a material isomorphism, then K is called a uniform reference for B.

2.2.9 Remark

In the generalized theory of Wang [4], the function K is defined as locally smooth on B, whereas in Noll's theory as given above, K is defined to be globally smooth on B leading to a distant parallelism. An example to justify Wang's generalization is the Moebius crystal. It is known that there is no global reference K which will map the crystalline axes on to the crystalline axes of a rectangular rod. In other words, the rod, twisted into a Moebius crystal defines a material tangent bundle which is not trivial (see remark 2.2.15).

From the general point of view of Wang, instead of a global function K on B, we define a <u>reference chart</u> for B to be a pair (U_α, K^α), $U_\alpha \subset B$ is open in B and K^α is a smooth field of <u>uniform</u> local reference configurations on U_α. Choosing $K_X = FK^\alpha(X)$ where $F \in GL(3)$, it follows from definition (2.2.6) and (2.2.8) that

$$\Omega_X(FK^\alpha(X)) = \Omega_Y(FK^\alpha(Y)) \quad \text{for } X, Y \in U^\alpha.$$

U^α is called a reference neighborhood, K^α a reference map. A <u>reference atlas</u> for B is a set $\Gamma = \{(U_\alpha, K^\alpha), \alpha \in I\}$ of reference charts such that two charts (U_α, K^α), (U_β, K^β) must obey the compatibility condition: if $U_\alpha \cap U_\beta \neq \phi$ and $X \in U_\alpha \cap U_\beta$, then $\Omega_X(FK^\alpha(X)) = \Omega_X(FK^\beta(X))$. When B possesses a reference atlas Γ, it is called a <u>smooth materially uniform</u> elastic body.

2.2.10 <u>Remark</u>

We note that there does not necessarily exist a configuration κ: $U_\alpha \to E$ such that $d\kappa|_X = K^\alpha(X)$. Hence we give:

2.2.11 <u>Definition</u>

A simple elastic body B is called <u>locally homogeneous</u> if it can be equipped with a reference atlas $\Gamma = \{(U_\alpha, K^\alpha), \alpha \in I\}$ with the property that to each $\alpha \in I$, there corresponds a configuration $\kappa^\alpha : U_\alpha \to E$ such that $K^\alpha(X) = d\kappa^\alpha|_X$ for all $X \in U_\alpha$.

2.2.12 <u>Definition</u>

B is homogeneous if it can be equipped with a global reference chart (B, K) such that $K(X) = d\kappa|_X$ for some $\kappa: B \to E \quad \forall X \in B$.

Noll [2] in his paper, assuming a reference atlas of the form $\Gamma = (B, K)$, defined a distant parallelism using the parallel transports $\oplus(X,Y) = K^{-1}(X)K(Y)$. He showed that the torsion of the connection induced by these maps characterized the inhomogeneity of B and he also proved the fact that a connection with a vanishing torsion implies B is locally homogeneous. Since the curvature is also zero (due to the distant parallelism) it is seen that a flat material connection completely characterizes a locally homogeneous body in Noll's theory.

2.2.13 <u>Remark</u>

The appearance of Lie Groups in Wang's theory and in this work have different motivations and they define different structures as summarized below:

2.2.14 <u>Definition</u>

A bundle chart (U_α, ϕ_α) of the fibre bundle $T(B)$, the material tangent bundle, is called a <u>material chart</u> if the transformations (see [18] for an exposition of fibre bundles):

$$\Psi_\alpha(X,Y) = \phi_{\alpha,Y} \circ \phi^{-1}_{\alpha,X}: T_X(B) \to T_Y(B)$$

are material isomorphisms $\forall\ X, Y \in U_\alpha$. In other words, if $K^\alpha = \phi^{-1}_\alpha$, then (U_α, K^α) is a reference chart on B. Two material charts (U_α, ϕ_α), (U_β, ϕ_β) are compatible if

$$\Psi_{\alpha\beta}(X,Y) \equiv \phi_{\alpha,X} \circ \phi^{-1}_{\beta,Y}: T_Y(B) \to T_X(B)$$

is a material isomorphism $\forall\ X \in U_\alpha$ and $Y \in U_\beta$. A <u>material atlas</u> is a maximal collection of such pairwise compatible charts. The fields

$$G_{\alpha\beta}(X) = \phi^{-1}_{\alpha,X} \circ \phi_{\beta,X} \stackrel{\text{def}}{\equiv} K^\alpha_X \circ (\ K^\beta_X\ )^{-1}$$

are smooth and take their values in the isotropy group of B <u>relative</u> to the material atlas $\{(U_\alpha, \phi_\alpha), \alpha \in I\}$ (this is so, if we recall the compatibility condition for reference charts (U_α, K^α), (U_β, K^β) namely $\Omega_X(FK^\alpha(X)) = \Omega_X(FK^\beta(X))$. Letting $F = \underset{\sim}{F}G^{-1}_{\alpha\beta}(X)$, we get

$$\Omega_X(\underset{\sim}{F}G^{-1}_{\alpha\beta}(X)K^\alpha(X)) = \Omega_X(\underset{\sim}{F}K^\beta(X)) = \Omega_X(\underset{\sim}{F}G^{-1}_{\alpha\beta}(X)K^\beta(X))$$

From the last two terms, we observe that $G^{-1}_{\alpha\beta}(X)$ is a member of the isotropy group at X and obviously so is $G_{\alpha\beta}(X)$).

2.2.15 <u>Remark</u>

In Wang's theory, Lie Groups arise as the structure group of the material tangent bundle and correspond to the isotropy group of B at a point relative to a given chart. In the present theory (Chapter 4) Lie Groups appear as the continuous group of transformations on the space of response functions.

Although most of the definitions that are needed in this work are given now, for completeness we briefly discuss Wang's approach to material connections.

If Γ is a reference atlas for B, the material atlas which corresponds to Γ (the one generated by the maps : $\Phi^{-1}_{\alpha,X} = K^{\alpha}(X)\colon T_X B \to \mathbb{R}^3$) is denoted by $\Phi(\Gamma)$. The structure group is given by the $G_{\alpha\beta}$ above and the sub-bundle of TB whose atlas is taken to be $\Phi(\Gamma)$ and whose structure group is $G(\Phi)$ will be denoted by $T(B,\Gamma)$. It is called the <u>material tangent bundle</u> of B relative to Γ. The associated principal bundle of $T(B,\Gamma)$ will be denoted by $E(B,\Gamma)$ and is called the <u>bundle of reference frames</u>, relative to Γ.

2.2.16 <u>Definition (Wang)</u>

A <u>material connection</u> on B is a "G" connection on $E(B,\Gamma)$. A "G" connection is a distribution of <u>horizontal subspaces</u> H on $E(B,\Gamma)$, which satisfy the following condition: If $\lambda(t)$ is a smooth curve in $U_\alpha \subset B$, and $\rho_t\colon TE_{\lambda(0)} \to TE_{\lambda(t)}$ are the parallel transports along λ relative to H, then the maps

$\rho_{t,\alpha}\colon G(\Gamma) \to G(\Gamma)$ defined by

$$\rho_{t,\alpha} \equiv \eta^{-1}_{\alpha,\lambda(t)} \circ \rho_t \circ \eta_{\alpha,\lambda(0)}$$

must be elements of the structure group of $E(B,\Gamma)$. Here $\eta(\Gamma) = \{(U_\alpha,\eta_\beta), \alpha\epsilon I\}$ is the bundle atlas of $E(B,\Gamma)$.

2.2.17 Conclusion

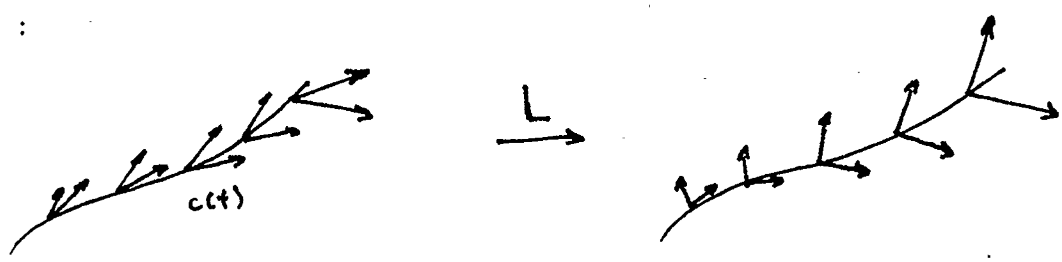
As the main conclusion of this section, we state that the difference in Noll's and Wang's theories is the assumption made on the reference atlas $\Gamma = \{(U_\alpha, K^\alpha), \alpha\epsilon I\}$. Noll assumes this to be of the form $\Gamma = (B,K)$, i.e., the tensor function $K_X\colon T_XB \to \mathbb{R}^3$ is globally smooth leading to an integrable connection whose curvature is zero. On the other hand, no such global smoothness assumption is made in Wang's theory for there are bodies violating it. The common assumption of both theories is the smoothness of the reference maps K^α on their domain (globally in [2], locally in [4]). This is a physical assumption which requires the mechanical response of the particles to vary smoothly over the body manifold. In both theories, a flat connection characterizes a locally homogeneous body.

2.2.18 Remark

In the theory of inhomogeneities, the assumption of smoothness (note that smoothness refers to the differential characteristics of a map rather than just continuity) is necessary to exclude sudden changes in the response of B along a given curve, or on an open set in B. In the continuum theory of dislocations it has been previously stated that this is equivalent to assume the dislocations are distributed continuously within the body. In the theory of uniformity, an analogous assumption is going to be made. H. Cohen and M. Epstein [9] have defined a body to be <u>smoothly uniform</u> along a curve $c: I \subset \mathbb{R} \to B$ if the composition $K_c: K_X \circ c: \mathbb{R} \to L(T_X B, V)$ (where K is a uniform reference <u>along</u> c) is smooth. K, of course, need not be a gradient of an embedding of c. This assumption is a restriction of Noll and Wang [2,4] smoothness hypothesis along a curve c. It requires the smoothness of the uniformity maps along a curve c, whose points are of the same material. Clearly, it looses its meaning if one desires to construct a theory without referring to the uniformity of B. Therefore, we look for a different type of smoothness which would make sense even if B is not uniform. Hence we give:

2.2.19 <u>Definition</u>

A body B is said to enjoy a <u>smooth material response</u> along a curve $c: I \subset \mathbb{R} \to B$ if $\Omega(Fd\gamma|_{c(t)}, c(t))$ varies smoothly along c(t) for all $F \in GL(3)$ and for some $\gamma: B \to E$ of class C^r via $c(t) \mapsto \gamma(c(t))$ (see remark 2.2.5 also). Obviously, this definition is not dependent on B being uniform along c. It states the assumption that if B is not uniform, the <u>material properties</u> along c vary smoothly. It therefore , excludes non-uniformities in B due to isolated material impurities, abrupt temperature changes, etc. In this definition, it is implicitly assumed that smoothness in the sense of Noll/Wang holds if the body happens to be uniform along c which can be illustrated as follows: Let $\mathcal{B}$ be a non-uniform body and $\Omega(G_X, X)$ denote its mechanical response. In a reference configuration $\gamma\colon \mathcal{B} \to \mathcal{E}$, the values $\Omega(Fd\gamma|_{c(t)}, c(t))$ of the function Ω is smooth in GL(3) by definition (2.2.19) above. Now let the tensor field $L\colon \mathcal{B} \to L(T\mathcal{B}, V)$ be such that $\Omega(QL(X),X) \equiv 0$ (a null response field) for any $Q \in \mathcal{O} \subset GL$ such that $QQ^T = I$. It will be assumed that <u>L is locally smooth</u>. Here we call L a <u>relaxation state</u> and it is analogous to the uniform reference field K for materially uniform bodies. An intuitive picture can be given as :



In the figure above, the length of the arrows symbolically denote <u>material property</u> (which is assumed to vary smoothly along c) and their orientation denote the distorsion of the material point (which is also assumed to be smooth by stating that L is smooth). Now in order to demonstrate that the assumption of smooth material response reduces to the smoothness hypothesis for uniform bodies, all that is needed is to assume that $\mathcal{B}$ is uniform along the curve c. It is easily seen that if $\mathcal{B}$ is uniform, the relaxation maps $\bar{L}$ can be chosen in the form $\bar{L}(c(t)) = Q(c(t))L(c(t))$ for some smooth field $Q: V \to V$ along c so that $\Omega(Q(c(t))L(c(t)), c(t))$ will not depend on the material point c(t) explicitly. That is, $\bar{L}(c(t))$ is the same as a uniform reference along c(t).

In the next section, a brief discussion of two papers on uniformity is given. Both studies consider uniformity of hyperelastic bodies, but with different approaches. In case of [9] a possible extension to elastic bodies is given here, where the response is in the form of a symmetric second order tensor.

2.3 Studies on Hyperelastic Uniformity

2.3.1 Curvewise Smooth Uniformity [9]

The response of a hyperelastic body is given by a scalar function $W(K_X, X)$ where $K_X: T_X B \to V$ as before. Letting $\Omega_X = W(., X)$ and $\Theta_{XY} \doteq K^{-1}(X)K(Y)$ in definition (2.2.6) we get

$$W(FK_X, X) + C_1 = W(FK_Y, Y) + C_2 \qquad (2.3.1.1)$$

as the condition for X and Y to be of the same material. C_1 and C_2 are scalars introduced since the reference energy level is arbitrary. If B is uniform, then Eqn. (2.3.1.1) holds for all $X \in B$, i.e. there must exist a function $\hat{W}$ such that

$$W(FK_X, X) = \hat{W}(F) + U(X)$$

$$F: V \to V, \qquad U: B \to \mathbb{R}$$

Let $F = \bar{F}K_X^{-1}$, $\qquad \bar{F}: T_X B \to V \qquad$ to get

$$W(\bar{F}, X) = \hat{W}(\bar{F}K_X^{-1}) + U(X) \qquad (2.3.1.2)$$

In [9] it is assumed that the field

$$K_X^{-1} \circ c: I \subset \mathbb{R} \to L(V, T_X B)$$

is smooth along the curve c on B. Reference [9] then proceed to derive a condition on W for B to be uniform along c. We outline this when Ω is the stress tensor. From Eqn. (2.3.1.2)

$$T(\bar{F}, X) = \hat{T}(\bar{F}K^{-1}(X))$$

If B is uniform along c and $K_c^{-1} = K^{-1}(X) \circ c$, then

$$T(\bar{F}, c(t)) = \hat{T}(\bar{F}K_c^{-1}(t))$$

Differentiating with respect to t:

$$dT^{\alpha\beta}/dt = \partial\hat{T}^{\alpha\beta}/\partial(FK_c^{-1})^{ij} \cdot d(FK_c^{-1})^{ij}/dt$$

$$= \partial\hat{T}^{\alpha\beta}/\partial(FK_c^{-1})^{ij} \cdot F^{i\rho} \cdot (dK_c^{-1}/dt)^{\rho j} \qquad (2.3.1.3)$$

and with respect to F:

$$\partial T^{\alpha\beta}/\partial F^{i\rho} = \partial\hat{T}^{\alpha\beta}/\partial(FK_c^{-1})^{mn} \cdot \partial(FK_c^{-1})^{mn}/\partial F^{i\rho}$$

$$= \partial\hat{T}^{\alpha\beta}/\partial(FK_c^{-1})^{mn}\, \delta_i^m\, \delta_\rho^\lambda\, (K_c^{-1})^{\lambda n}$$

or

$$\partial T^{\alpha\beta}/\partial F^{i\rho} = \partial\hat{T}^{\alpha\beta}/\partial(FK_c^{-1})^{in} \cdot (K_c^{-1})^{\rho n} \qquad (2.3.1.4)$$

$$(\text{symbolically } \partial T/\partial F = \partial\hat{T}/\partial(FK_c^{-1}) \cdot (K_c^{-1})^T)$$

In order to find a condition on T, we are naturally led to eliminate the $\partial\hat{T}/\partial(FK_c^{-1})$ terms.

In (2.3.1.4), K_c^{-1} has one free index. Hence, multiplying both sides by K_c^T,

$$\partial T/\partial F \cdot K_c^T = \partial\hat{T}/\partial(FK_c^{-1}) \to \partial\hat{T}^{\alpha\beta}/\partial(FK_c^{-1})^{ij}$$

$$= \partial T^{\alpha\beta}/\partial F^{i\rho} \cdot (K_c^T)^{\rho j}$$

substituting

$\partial\hat{T}/\partial(FK_c^{-1})$ in Eqn. (2.3.1.3):

$$dT^{\alpha\beta}/dt = \partial T^{\alpha\beta}/\partial F^{i\rho} \cdot K^{j\rho}_{c}\, F^{i\lambda} \cdot (dK_c^{-1}/dt)^{\lambda j}$$

or

$$dT^{\alpha\beta}/dt = \partial T^{\alpha\beta}/\partial F^{i\rho} \cdot (F \cdot dK^{-1}_{c}/dt \cdot K_c)^{i\rho} \qquad (2.3.1.5)$$

Let $\Gamma_c = dK^{-1}_{c}/dt \cdot K_c$ (In [9], $\Gamma_c = dP_c/dt \cdot P^{-1}$ where $P_{c(t)}: V \to T_{c(t)}B$ and the present treatment of the stress tensor results in the same meaning for Γ.) Rewriting Eqn. (2.3.1.5) as:

$$dT^{\alpha\beta}/dt = \partial T^{\alpha\beta}/\partial F^{i\rho} \cdot (F\Gamma)^{i\rho} \qquad (2.3.1.6)$$

the condition for B to be uniform along c is seen to be:

2.3.1.7 Proposition (Cohen-Epstein)

A necessary condition for an elastic body B with constitutive law $T(F, X)$ to be smoothly uniform along c is that the functions $\Gamma_c(t)$ can be found such that Eqns. (2.3.1.6) are satisfied.

Once Γ is given, with proper initial conditions on P_c, a smooth field of uniform reference can be generated.

It should be noted that if B has a continuous isotropy group, say of r-parameters, then Γ is not unique. If G_X is the isotropy group at X and B is uniform, it can be shown that

$G_X = K(X)^{-1}GK(X)$ where $G \subset Sl$ ($\equiv$ volume preserving subgroup of GL), and $r = \dim G \le n^2-1$ if $\dim B = n$. By definition,

$$T(F, X) = T(FK^{-1}(X)G(\alpha_1,\alpha_2,\ldots,\alpha_r)K(X), X)$$

or

$$0 = dT^{\alpha\beta}/d\alpha_k \,|_{\boldsymbol{\alpha} = 0}$$
$$= \partial T^{\alpha\beta}/\partial F^{i\rho} \cdot (FK_c^{-1}(X) \cdot \partial G/\partial\alpha_k|_{\boldsymbol{\alpha}=0} \cdot K_c(X))^{\rho i}$$

where $\boldsymbol{\alpha} = (0,0,\ldots,0)$ and $G(\alpha_1,\ldots\alpha_r)\,|_{\boldsymbol{\alpha}=0} = I$.

Comparing with Eqn. (2.3.1.6) we conclude that

$$\hat{\Gamma}_c = \Gamma_c + \sum_{k=1}^{r} K_c^{-1}\, \lambda_k\, (\partial G/\partial\alpha_k)|_{\boldsymbol{\alpha}=0}\, K_c, \qquad \{\lambda_k\}\in\mathbb{R}$$

also satisfies Eqn. (2.3.1.6)

In [9] this approach is generalized later for local uniformity of hyperelastic bodies in which case, the functions Γ appear with three free indices actually defining a connection on B. In this instance, the field K is assumed to be smooth on an open neighborhood of X and the directional derivative in Eqn. (2.3.1.3) is replaced by a gradient at X. Since the parallelism defined by Γ need not be path independent (the only restriction on Γ is that it should satisfy Eqn. (2.3.1.6)), it is concluded that Γ can be called a material connection in the sense of Wang [4], for it is required that K(X) be defined on B locally.

2.3.1.8 <u>Remark</u>

As can be seen, proposition (2.3.1.7) is the statement of an inverse problem and does not give a direct condition on the constitutive law itself. However, for a one-dimensional body, as a special case it does so and in Chapter 5 it is used to construct an example for comparison purposes with the approach taken in this thesis.

2.3.2 A Constitutive Parallelism Structure on Hyperelastic Bodies [10]

When B is not known to be uniform, the isomorphisms of definition (2.2.6) do not exist. That is, no parallelism structure can be defined on B induced by its constitutive law. However, a constitutive connection can be defined on a certain fibre bundle of which B is the base space. Let W denote the hyperelastic response of B, clearly if B is not uniform, Eqn. (2.3.1.1) does not hold.

However, for a fixed $\underline{F}$, Eqn. (2.3.1.1) holds for some $\underline{K}_X$ and $\underline{K}_Y$ i.e.,

$$W(\underline{F}\ \underline{K}_X(\underline{F}),\ X) = W(\underline{F}\ \underline{K}_Y(\underline{F}),\ Y) + C_2 - C_1$$

where we have assumed that the range of the functions $W(.,\ X)$ and $W(.,\ Y)$ have a non-empty intersection. The maps $\underline{K}_X$ and $\underline{K}_Y$ depend on $\underline{F}$. Now let $\underline{F} = \underline{F}_X K_X^{-1}(\underline{F})$ to get

$$W(\underline{F}_X,\ X) = W(\underline{F}_X\ \oplus(X,Y,\underline{F}),\ Y) + C_2 - C_1$$

where

$$\oplus(X,Y,\underline{F}) = K^{-1}_X(\underline{F})\ \underline{K}_Y(\underline{F})\ .$$

Fix $X = X_o$ and we have

$$W(\underline{F}_X\ \oplus(X_o,X,\underline{F}),\ X) + U(X) = W(\underline{F}_{X_o},\ X_o)$$

or

$$W(\underline{F}\ \underline{K}(X,\ \underline{F}),\ X) + U(X) = W(\underline{F}\ \underline{K}(X_o,\ \underline{F}),\ X_o)$$

For a fixed X_o, the right hand side is only a function of $\underline{F}$ and we get:

$$W(FK(X,F),\ X) = \hat{W}(F) + U(X)$$

replacing F by $\underline{F}$ and K by $\underline{K}$, for some function $\hat{W}$ whose range contains the range of W. In [10] it is assumed that the above "reference" maps K: $\mathcal{B} \times GL \rightarrow L(T\mathcal{B}, V)$ are smooth on $\mathcal{B} \times GL$. Then it follows that a smooth vector field $K^{-1}(X,F)$ $\{e_i\}$ can be defined as the horizontal frame for $T_{(X,H)}(\mathcal{B} \times GL)$ if $\{e_i\} \in V$ <u>is fixed</u>. The vertical frame is a constant frame $\{\underline{e}_i \otimes \underline{e}_j\}$ of $L(V,V)$ for a fixed basis of V. Thus, a vector field on $\mathcal{B} \times GL$ is a parallel field if and only if its vertical part is <u>constant</u> since the horizontal part is parallel by definition. This is called an L(V)-trivial parallelism structure.

It is proved in theorem 1 of [10] that an L(V) trivial parallelism structure on $\mathcal{B} \times GL$ and a parallelism structure <u>on</u> $\mathcal{B}$ generate each other if the torsion operator of the above constitutive connection on $T(\mathcal{B} \times GL)$ vanishes. In other words, if the torsion is zero, a constitutive parallelism is guaranteed, and this implies $\mathcal{B}$ is uniform (since it would require the isomorphisms $K(X,F)$ be independent of F).

2.4 <u>Conclusion</u>

The motivation for the study of the geometry of a simple body is discussed, the definitions in the analysis of response of a non-uniform body have been given. The theories of inhomogeneities in a simple body are reviewed and compared; the concept of smooth uniformity is explained. The role played by Lie Groups in Wang's theory and the present one is explained and distinguished. The assumption of smooth material response is introduced. Finally, two existing theories on the response of a non-uniform hyperelastic body are discussed and one of them has been extended to elastic bodies.

CHAPTER 3

CONTINUOUS GROUPS OF TRANSFORMATIONS

The objective of this chapter is to summarize the basic definitions, notations and results of the mathematical theory of a Lie Group and its one-parameter subgroups that will be used in later chapters. Complete and systematic treatment of the subjects discussed here can be found in the references cited.

Section 3.1- 3.2 are devoted to the topological groups and continuous groups of transformations where the underlying axioms are stated in a precise form.

Lie Groups and finite dimensional Lie Groups of transformations are discussed in section 3.3, the concepts of the integral curve and the flow of a vector field, a local one-parameter subgroup as generated by a vector field are given.

Attention is focused on the local structure of a group of transformations in section 3.4 by means of studying the properties of its one-parameter subgroups. The notion of a Killing vector field on a manifold is given and its relation to the generators of the one-parameter subgroups of G is explained. In general, references [17,21,22,23] are suggested for analytical and topological terminology.

3.1 General Considerations

The original ideas related to transformation groups and their <u>infinitesimal generators</u> were introduced by Sophus Lie and are given also in [19]. Lie's motivation was to determine all r-term transformation groups of an n-dimensional manifold and classify them.

Explicitly, a family of transformations

$$\bar{x}^i = f^i(x^j;\ a^1,\ldots a^r), \qquad i,j = 1,\ldots,n$$

where $\bar{x}$ denotes the transformed point and a^i the parameters, forms a transformation group if the composition of two transformations of the family is a transformation of the family, i.e. when from the equations

$$\bar{x} = f(x,\ a^1,\ldots,a^r)$$

$$\bar{\bar{x}} = f(\bar{x},\ b^1,\ldots,b^r)$$

there follows

$$\bar{\bar{x}} = f(x,\ c^1,\ldots,c^r)$$

where c^i are functions of the a's and b's alone.

A topological group or a continuous group has two distinct kinds of structure on it. It has a <u>topological structure</u> and it also has an <u>algebraic structure</u>. Algebraically, it is a group; it therefore obeys the axioms of a group [20]. Topologically, it is a manifold [21]. The algebraic and topological properties are combined by the following continuity requirement:

Let $g, \bar{g} \in G$ and $g^{-1} \in G$ denote the inverse of g. Then the maps

$$g \cdot \bar{g} \rightarrow g\bar{g} \qquad \text{and} \qquad g \rightarrow g^{-1}$$

are <u>continuous</u>. Graphically this is illustrated in Figure 3.1

The significance of the above continuity requirement is that the product of any group element near g with any group element near $\bar{g}$ is a group element near $g\bar{g}$. Similarly, if $\bar{g}$ is a group element near g, then $(\bar{g})^{-1}$ is a group element near g^{-1}. The end result of these two assumptions is a rich structure - the theory of Lie Groups, studied extensively in literature.

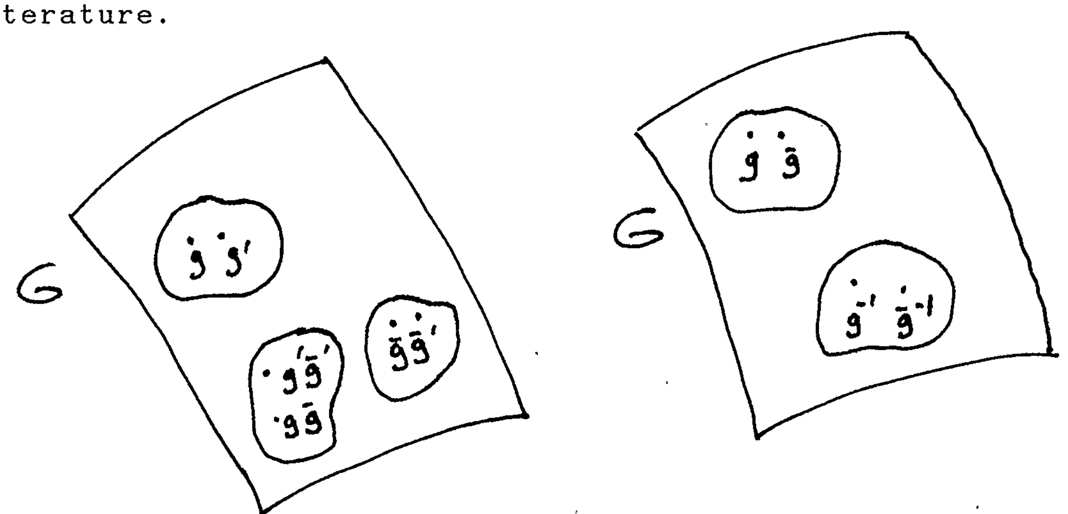


Figure 3.1

3.2 Topological Groups

Before studying the local structure of G near e, the general properties of topological groups will be summarized here. For a detailed study, reference [22] is recommended.

3.2.1 Definition

A topological group or a continuous group consists of:

1. An underlying r-dimensional manifold G,
2. An operation Φ mapping each pair of points (g,h) in the

manifold into another point l in the manifold,

3. In terms of coordinate systems around the points l,g,h we write

$$l^{\mu} = \phi^{\mu}(g^{1},\ldots,g^{r};\ h^{1},\ldots,h^{r})$$

$$e^{\mu} = \phi^{\mu}(g^{1},\ldots,g^{r};\ (g^{-1})^{1},\ldots,(g^{-1})^{r});\quad \mu = 1,\ldots,r$$

The functions

$$\Phi:\ (g,h) \to l = g \cdot h$$

$$\Psi:\ g \to g^{-1}$$

must be <u>continuous</u>. The group multiplication properties may be transcribed into conditions on Φ:

1. Closure

$$l^{\mu} = \phi^{\mu}(g,h), \qquad l,g,h \in G$$

2. Associativity

$$\phi^{\mu}(l,\phi(g,h)) \equiv \phi^{\mu}(\phi(l,g),h)$$

3. Identity : There exists $e \in G$ such that

$$\phi^{\mu}(e,g) = g^{\mu} = \phi^{\mu}(g,e)$$

4. Inverse : For each $g \in G$, there exists $g^{-1} \in G$ such that

$$\phi^{\mu}(g,g^{-1}) = e^{\mu} = \phi(g^{-1},g)$$

3.2.2 <u>Definition</u>

A <u>continuous group of transformations</u> consists of:

a) An underlying topological space G^{r}, which is an r dimensional manifold, together with a binary mapping Φ:
$\Phi:\ G \times G \to G$

b) A geometric space M which is an n-dimensional manifold, and a mapping $f:\ G \times M \to M$ which obey:

<u>Postulate (a'):</u> G, ϕ obey the postulates of a topological group.

<u>Postulate (b'):</u> The function

$$y^i = f^i(g^1,\ldots,g^r;\ x^1,\ldots,x^n)$$

is <u>continuous</u> and in addition has the properties:

1. <u>Closure:</u> $g \epsilon G,\ x \epsilon M \rightarrow (gx) \epsilon M$

   i.e. $y^i = f^i(g^1,\ldots,g^r;\ x^1,\ldots,x^n) \epsilon M$

2. <u>Associativity:</u> $g(hx) = (g \cdot h)x$

   i.e. $f^i(g;f(h,x)) = f^i(\phi(g,h);x)$

3. <u>Identity:</u> $ex = x$

   i.e. $f^i(e;x) = x^i$

4. <u>Inverse:</u> $g^{-1}(gx) = g(g^{-1}x) = (g \cdot g^{-1})x = x$

   i.e. $f^i(g^{-1};f(g,x)) = f^i(g;f(g^{-1},x)) = f^i(\phi(g,g^{-1});x) = x^i$

Now the main definition is stated:

3.2.3 <u>Definition:</u>

A <u>Lie Group</u> is the connected component of a continuous group in which the composition function ϕ is analytical on its domain of definition.

i.e. the group operations

$$G \times G \xrightarrow{\text{into}} G \qquad (g,h) \rightarrow g \cdot h$$

and

$$G \rightarrow G \qquad g \rightarrow g^{-1}$$

are C^∞ maps and ϕ can be expanded in power series. (Lie Groups of transformations are defined in an analagous manner).

3.3 Lie Groups and Lie Groups of Transformations

A) Lie Groups

In this part, the necessary background on transformation groups is given. The proofs of the propositions can be found in any standart text and will not be repeated here [see 18,21,22,23].

3.3.1 Proposition

The (real) general linear group of degree n

$GL(n,\mathbb{R}) = \{a \in M(n,\mathbb{R}) \mid \det(a) \neq 0\}$

which is the group of all non-singular nxn-matrices is a Lie group under matrix multiplication.

3.3.2 Remark

It should be noted that only the local structure of G near e is relevant to this study, i.e. constructing the elements of G near e, the identity element, will be sufficient. Analogously, when we examine the action of G on a manifold M as a transformation group at $x \in M$, we shall only be interested in constructing an open neighborhood instead of the whole orbit (definition (3.3.17)) of G at x.

Let M be a differentiable manifold of dimension n with points $x, y, \ldots \in M$, let $\sigma: I \subset \mathbb{R} \to M$ be a curve such that the tangent to the curve σ at $x = \sigma(t)$ is the vector $v(x)$, where v is a vector field on M.

3.3.3 <u>Definition</u>

The map

$\sigma: I \to M$

is called an integral curve of the vector field v if it satisfies

$$\frac{d\sigma(t)}{dt} = v(\sigma(t)).$$

3.3.4 <u>Theorem</u>

If v is a C^r vector field on M, then for every $x \in M$, there exists an integral curve of v, $t \mapsto \sigma(t,x)$ such that

1. $\sigma(t,x)$ is defined for t belonging to some interval $I(x) \subset \mathbb{R}$ containing $t = 0$ and is of <u>class</u> C^{r+1} there,
2. $\sigma(0,x) = x$ for every $x \in M$,
3. <u>Uniqueness:</u> Given $x \in M$, <u>there is only one</u> C^1 integral curve of v defined on an interval properly containing $I(x)$ and passsing through x.

The property of uniqueness allows us to state:

3.3.5 <u>Theorem</u>

If $t, s, t+s \in I(x)$ then

$\sigma(t,\sigma(s,x)) = \sigma(t+s,x)$

(compare definition 3.2.2, (2) associativity)

3.3.6 <u>Definition</u>

The mapping $\sigma: (x,t) \to \sigma(t,x)$ is called the <u>flow</u> of v. If M and v are C^∞, the flow is C^∞.

3.3.7 Definition

A local transformation of M at x_o is defined on $N(x_o) \subset M$ for $t \in I(x_o)$ and is given by

$$\sigma(t, .) \equiv \sigma_t : x \to \sigma(t,x).$$

3.3.8 Remark

For σ_t to be a global transformation of M, one should normally require M to be compact, which then assures

$$I = \bigcap_{i=1}^{i=k} I(x_i) \neq \Phi$$

such that

$$M = \bigcup_{i=1}^{i=k} N(x_i) \quad \text{and } x_i \in M, \text{ (k is finite since M is}$$

compact)

In other words, I is never empty when M is compact since it is then given by the intersection of finitely many intervals. It will be assumed that M is compact.

3.3.9 Remark

Theorem 3.3.5 implies the relation

$$\sigma_{t+s} = \sigma_t \cdot \sigma_s$$

for the maps σ_t. Also, it follows from the above that for $s = -t$, $\sigma_t^{-1} = \sigma_{-t}$.

3.3.10 Definition

The set of mappings σ_t is called a one parameter

<u>local pseudo group</u> and if M is compact it is called a <u>one parameter local group</u> .When I = $\mathbb{R}$, then one is said to have a <u>one parameter group</u>.

It is the converse of the preceding arguments that we are interested here:

3.3.11 <u>Theorem</u>

Any one parameter local group $\{\sigma_t\}$ of transformations σ_t: $x \to \sigma(t,x)$ can be generated by a vector field v defined uniquely by the equation

$$v(x) = \frac{d\sigma}{dt}(t,x)\big|_{t=0}$$

(The proof is attained by showing $\dfrac{d(\sigma(t,x))}{dt} = v(\sigma(t,x))$).

3.3.12 <u>Corollary</u>

Let $\bar{V}(x)$ be the Lie algebra of all vector fields on M at x. If $u, v \in \bar{V}(x)$, then the one parameter group of local transformations $\sigma_{t,u}$, $\sigma_{t,v}$ generated by u, v are <u>identical</u> if $u = v$, i.e., there is a one to one correspondence between $\bar{V}(x)$ and the set of all one parameter local groups of transformations at x.

The members of $\bar{V}(x)$ will be called the <u>infinitesimal generators</u> (of the one parameter subgroups).

Now the finite dimensional <u>group of transformations</u> will be discussed.

B) Lie Transformation Groups

3.3.13 Definition

The set $\{\sigma_g\}$, $g \in G$ is a Lie group of transformations if the mapping

$$\sigma\colon G \times M \to M \quad \text{by} \quad (g,x) \mapsto \sigma(g,x)$$

is differentiable and if the set of transformations $\{\sigma_g\colon M \to M;\ \sigma_g(x) = \sigma(g,x)\}$ together with the composition mapping follow the group property:

$$\sigma_{gh} = \sigma_g \cdot \sigma_h \text{ and } \sigma_e \text{ is the identity}$$

transformation. It follows that $\sigma_{g^{-1}} = \sigma_g^{-1}$ (see definition (3.2.3), (1), (4)).

The map σ_g is said to define an action of g on M.

3.3.14 Definition

G is said to act effectively, if $\sigma_g(x) = x$, $\forall\, x \in M$ implies that $g = e$; G acts without fixed point (or freely) if the stronger condition holds: If $\sigma_g(x) = x$ for some $x \in M$, then $g = e$.

G acts transitively on M if for every $x \in M$ and $y \in M$, there exists a $g \in G$ such that $\sigma_g(x) = y$.

3.3.15 Proposition

The set

$$G_{x_o} = \{g \in G \,|\, \sigma_g(x_o) = x_o\}$$

is a subgroup of G.

3.3.16 Definition

The subgroup G_{x_o} is called the isotropy subgroup of G at $x_o \in M$.

3.3.17 Definition

The set of points of M that can be reached by applying elements of G to a single point $x_o \in M$ is called the orbit of G at x_o; symbolically:

$$G \cdot x_o = \{\sigma_g(x_o) \,|\, g \in G\} \equiv \mathcal{O}_{x_o} \subseteq M.$$

3.3.18 Remark

In this study, only the structure of orbits at $x_o \in M$ generated by an open neighborhood $\mathcal{N}(e) \subset G$ of e is of interest.

3.3.19 Remark

If $G \cdot x_o = M$ for at least one $x_o \in M$, then G acts transitively on M. Conversely: G acts transitively on every orbit.

3.4 One Parameter Groups and the Local Structure of G

3.4.1 Definition

A one parameter subgroup of a Lie group G is a differentiable curve

$$g: \mathbb{R} \to G \quad \text{by} \quad t \mapsto g(t)$$

such that

$$g(t)g(s) = g(t+s) \text{ and } g(0) = e$$

The concepts developed for the group $\{\sigma_t\}$ apply to the group

$\{\sigma_{g(t)}\}$.

3.4.2 Remark

The curve generated by the transformations $\{\sigma_{g(t)};\ t\in\mathbb{R}\}$ operating on x is the image of the one parameter subgroup $\{g(t);\ t\in\mathbb{R}\}$ by σ_x where

$$\sigma_x(g(t)) = \sigma(g(t),x) = \sigma_{g(t)}(x)$$

Graphically,

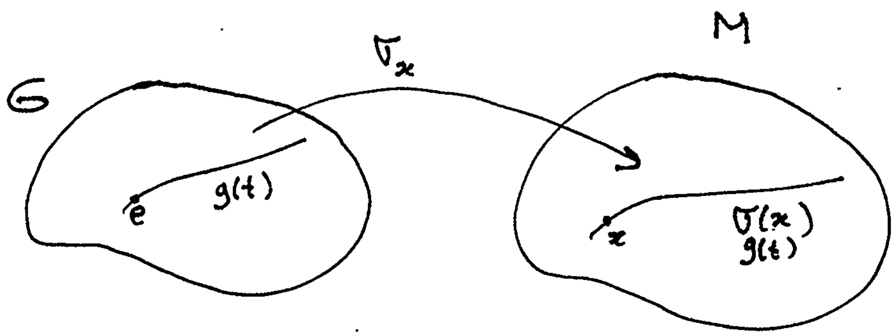


Figure 3.2

Note that σ_x: $G \to M$ whereas $\sigma_{g(t)}$: $M \to M$ and σ: $G \times M \to M$. The representation $\sigma_{g(t)}$ is going to be used and once g(t) is specified as the one parameter subgroup $t \to g(t)$, the curves $\sigma_{g(t)}(x)$ and $\sigma(g(t),x)$ become identical.

3.4.3 Definition

The vector field which generates the group of transformations $\{\sigma_{g(t)};\ t\in\mathbb{R}\}$ is called a Killing vector field on M relative to the group G.

3.4.4 Remark

The Killing vector fields play an important role in the concept of directional uniformity. It will be seen that the value of this vector field at $x \in M$ determines the nature of the material (associated with the corresponding curve in $\mathcal{B}$), M being the space of response functions, denoted by Λ in Chapter 2.

3.4.5 Proposition

The integral curve going through x of the Killing vector field v satisfies the equations:

$$\frac{d\sigma_x(g(t))}{dt} = v(\sigma_x(g(t)), \qquad \sigma_x(e) = x$$

This simply states that $\sigma_x(g(t))$ is an integral curve of v and all the previous properties for integral curves on M naturally apply to $\sigma_x(g(t))$.

3.4.6 Remark

By the uniqueness theorem 3.3.4(3), the action of a given one parameter subgroup of G on M can be described by only one Killing vector field v.

3.4.7 Definition

The set of vector fields invariant under the left (right) translations are called the left (right) invariant vector fields on G, where a left translation is given by L_g: $G \to G$ via $L_g(h) = g.h$ (similarly $R_g(h) = h \cdot g$). Thus, if v satisfies $dL_g v(h) = v(L_g h) = v(g \cdot h)\ \forall\ g,h \in G$, it is left invariant.

3.4.8 Theorem

There is a one to one correspondence between the set of left invariant vector fields and $T_e(G)$, the tangent space of G at the identity.

3.4.9 Theorem

The one parameter subgroups of G are the integral curves passing through the origin e of the left (right) invariant vector fields.

3.4.10 Corollary

It follows that for each $\gamma \epsilon T_e G$ there is a unique solution g(t) of

$$dL_{g(t)}\gamma = \frac{dg(t)}{dt} \quad \text{obeying } g(t)g(s) = g(t+s).$$

3.4.11 Remark

Denoting the Killing vector field which generates $\{\sigma_{g(t)} \mid \left.\frac{dg(t)}{dt}\right|_{t=0} = \gamma,\ t \epsilon \mathbb{R}\}$ by v^γ,

we have:

$$v^\gamma(x) = \left.\frac{d\sigma_{g(t)}(x)}{dt}\right|_{t=0} = \frac{d\sigma_x(g(t))}{dt} = d\sigma_x(e)\gamma$$

where

$d\sigma_x$: $T_e G \to T_x M$ is a vector space isomorphism $\forall\ x \epsilon M$ if G acts transitively and freely on M.

3.4.12 Remark

For G with a discrete isotropy group (as it acts on M; definition 3.3.16), it is possible to find an open neighborhood of $e \in G$ such that $d\sigma_x(e)$ is an isomorphism.

3.4.13 Proposition

1. For a finite dimensional group G, the one parameter subgroups fill in a neighborhood of the identity,
2. Every element of a connected group G can be constructed by multiplication of elements in an arbitrary neighborhood $N(e)$ (this is a direct consequence of G being continuous group: let $g \in G$ be a finite operator not in $N(e)$. Select any curve joining g and e and points on this curve, say g_i, $i = 0,1,2,\ldots,\infty$, $g_0 = e$ such that $g_{i+1} \cdot g_i^{-1} \in N(e)$. Examining Figure 3.1 shows that this selection is always possible. Then $g = g_\infty \cdot \ldots \cdot (g_{i+1} \cdot g_i^{-1}) \cdot \ldots \cdot (g_2 \cdot g_1^{-1}) \cdot (g_1 \cdot e)$. Thus, g is the product of operators - or elements - in $N(e)$).

So one has the following fundamental result:

3.4.14 Theorem

The necessary and sufficient condition for a tensor field on M to be invariant under $\{\sigma_g; g \in G\}$ is that it is invariant under $\{\sigma_{g(t)}; g(t) \in N(e)\}$. The proof is easy since it has already been shown that G can be constructed by the points in $N(e)$.

3.4.15 Remark

Theorem 3.4.14 is used when constructing the invariant variety of M under G, simply by considering the Killing vector fields of the one parameter subgroups of G.

3.4.16 Definition

The map $T_eG \to G$ given by $t\gamma \mapsto \exp(t\gamma)$ where $t \in \mathbb{R}$, $\gamma \in T_eG$ is called the exponential mapping and is onto $g_\gamma(t)$. It is defined by the following operation:

$$\exp: T_eG \to G \text{ by } \quad \gamma \mapsto \exp(\gamma) = g_\gamma(1)$$

It can be shown that

$$\exp(t\gamma)\exp(s\gamma) = g_\gamma(t)g_\gamma(s) = g_\gamma(t+s) = \exp(t+s)\gamma$$

which justifies the name "exponential mapping".

3.4.17 Remark

For a finite dimensional vector space V and $G = GL(V)$, the Lie algebra $\overline{V}(GL(V))$ constitutes a set of $\mathbb{R}$-linear endomorphisms of V. Let $X \in \overline{V}(GL(V))$, $\{\xi_X(t)\}$ the corresponding one parameter subgroup. It can be shown that in this case

$$\exp(tX) = \xi_X(t) = e^{tX} = \sum_{n=0}^{\infty} \frac{1}{n!} (tX)^n.$$

CHAPTER 4

TANGENTIAL MATERIAL UNIFORMITY

4.1 Motivation and General Considerations

The assumption of smooth material response introduced in definition (2.2.19), enables one to construct a theory for the characterization of material uniformity, regardless of the body being uniform or not. Based on this assumption, the image of a smooth curve c(t) in B is again a smooth curve in $\mathit{\Lambda}$ under the mapping (remark 2.2.5)

$$\Omega\colon B \to \mathit{\Lambda} \qquad \text{via} \qquad c(t) \mapsto \Omega_{c(t)} \in \mathit{\Lambda}$$

i.e. the curve $\Omega_{c(t)}$ is smooth in the space of response functions. On the other hand, the material isomorphisms (if they exist) are members of the group GL when referred to a reference configuration. That is, let $\gamma\colon B \to E$ be an embedding of B, $P_X(Y)$ be the representation of $\Theta(X,Y)\colon T_Y B \to T_X B$ in γ. Clearly,

$$P_X(Y) = d\gamma|_X \circ \Theta(X,Y) \circ d\gamma^{-1}|_Y$$

and since γ is an embedding, Θ a vector space isomorphism, then $P_X(Y) \in GL$.

$$\begin{tikzcd}
T_X B \arrow[d, "d\gamma|_X"'] & T_Y B \arrow[l] \arrow[d, "d\gamma|_Y"] \\
V & V \arrow[l, "P_X(Y)"']
\end{tikzcd}$$

It is possible to construct a connected neighborhood of $\Omega_X \in \mathit{\Lambda}$ if we let P vary continuously in $\Omega_X(P d\gamma|_X)$, $P \in GL$, where P takes values in $N(I)$, a neighborhood of the identity element of GL. This defines the local action of GL at Ω_X; the neighborhood generated this way is an open set in the orbit of GL at Ω_X. This open set includes all possible forms of response, that are "close" to Ω_X and even now one can intuitively conclude that if B is uniform locally at X, it is possible to find an open set $W \subset \Lambda$ such that $\Omega(U) \subset W$ where U is an open neighborhood of $X \in B$. The set W is generated by the action of $N(I)$ at Ω_X. (See Figure 4.1).

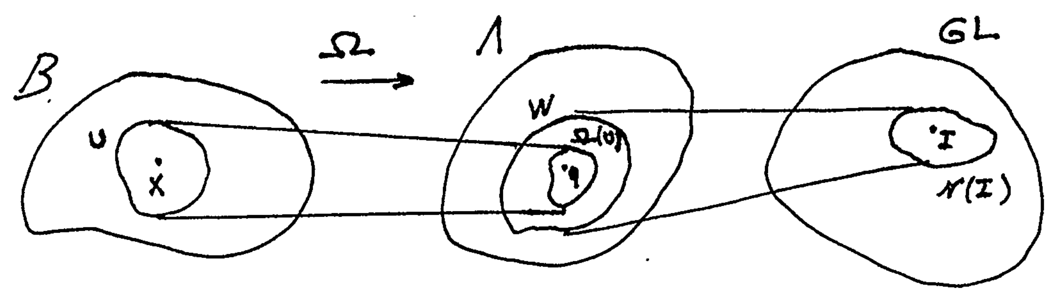


Figure 4.1

$$W = \{p \in \Lambda \mid p = \sigma_g(q),\ g \in N(I),\ q = \Omega(X)\}$$

However, when B is not uniform at X, it is clear that no such W completely containing the image of U under Ω will exist.

In proposition 3.4.13 it has been stated that the one parameter subgroups fill in a neighborhood of the identity element e of G. Therefore, by theorem 3.3.11, the local action is completely determined by the generators of these subgroups. On the other hand, it is known that when G acts transitively on Λ, there is a one to one correspondence between these generators and the Killing vectors induced by the group action. We recall that Killing vectors at a point

$p \in \Lambda$ are the tangents to the curves at p which are the images of one parameter subgroups under the group action. To illustrate this we refer to Figure 4.2.

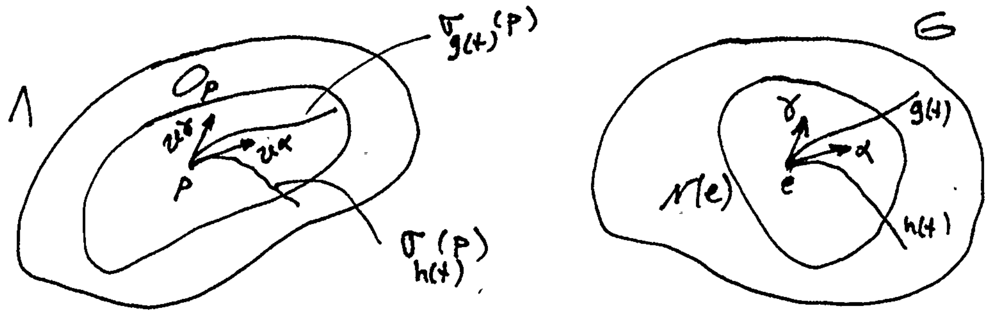


Figure 4.2

The map $d\sigma_p(e)$: $T_eG \to T_p\Lambda$ is a vector space isomorphism if and only if G acts freely on the orbit O_p.(See remark 3.4.11).

It is clear that in order to be able to perform explicit calculations, the precise form of the action of G must be known. This is accomplished once the constitutive law and a representation in a function space is given. In this study, it will be assumed that this is a finite dimensional vector space (for illustrations, see Chapter 5).

Obviously, all the points on an orbit of G at Ω_X are the response functions belonging to the same material but they are different in form. It is also obvious that the image of a curve c in B under Ω need not lie on this orbit nor be tangent to it at the point of interest. If it completely lies on the orbit, it will be said that B is tangentially uniform at Ω_X along c. A possible method of formulating this, is then checking to see if the tangent of the curve $\Omega_{c(t)}$ at $\Omega_{c(0)}$ is in the linear span of the Killing vectors (of the action of G at $\Omega_{c(0)}$). These ideas are now going to be put in a precise form.

4.2 The Constitutive Law as a Map and its Properties

For a uniform body, it will be assumed that the map $\Omega: \mathcal{B} \to \mathcal{A}$ is open relative to the orbits of G in $\mathcal{A}$, i.e. it carries open sets in $\mathcal{B}$ to relatively open sets in $\mathcal{A}$ in the usual topologies of $\mathcal{B}$ and $\mathcal{A}$ (The image of an open set $U(X) \in \mathcal{B}$ under Ω may not be open in $\mathcal{A}$, however, if $U(X)$ happens to be a materially uniform set, it will be open relative to the set $G \cdot \Omega_X \equiv$ the orbit of G at Ω_X).

4.2.1 Proposition

The linear group $GL(r,\mathbb{R})$ of dimension r^2 acts on $\mathcal{A}$ as a Lie Group of transformations, i.e. the operation

$\sigma_g: \mathcal{A} \to \mathcal{A}$, $\quad g \in GL$ is differentiable.

(In what follows, $GL(r,\mathbb{R})$ will be denoted by G^r).

The action of G^r on $\mathcal{A}$ is completely determined by Ω as follows:

$$\sigma_g(\Omega(X)(F)) = \sigma_g(\Omega_X(F)) = \Omega_X(FL).$$

Here, L is the matrix representation of the operator g as it acts on $\mathcal{A}$.

4.2.2 Proposition

The group action $\sigma_g: \Lambda \to \Lambda$ is transitive on a subset U of Λ if $U = G^r \cdot p = O_p$, $p \in U$.

Proof: If $q, r \in U$, then $q = gp$ and $r = hp$ for some $g, h \in G^r$. Since $p = g^{-1}q$, it follows that $r = hg^{-1}q$. Obviously $hg^{-1} \in G^r$. Therefore the group action is transitive on U since q,r are arbitrary points of U.

4.2.3 Definition

$U \subset \Lambda$ is called an orbit of G^r at p if $U = G^r \cdot p$.

4.2.4 Proposition

Let $V \subset B$ be an open neighborhood of the material point X and U be the orbit at $\Omega(X) \in \Lambda$. Then B is locally uniform at X if and only if the set

$$U' = \{\Omega(Y) \mid Y \in V\}$$

is open in Λ relative to U.

Proof: Let $Y \in V$, then $\Omega(Y) \in U$. So,

$$\Omega(Y)(F) = \sigma_g(\Omega(X)(F)) = \Omega(FL) \quad \text{for some } g \in G^r$$

with representation $L \in GL$, i.e. $\Omega_Y(F) = \Omega_X(FL)$. It follows from the definition (2.2.6) and (2.2.7) that $Y \in B$ is of the same material as X. Thus, V is a uniform neighborhood of X. Conversely, if B is uniform in the open set V, then $\Omega_X(F) = \Omega_Y(FP) \in U$ $\forall Y \in V$ and for some $P \in GL$. Hence, the proof is complete.

(see diagram for illustration)

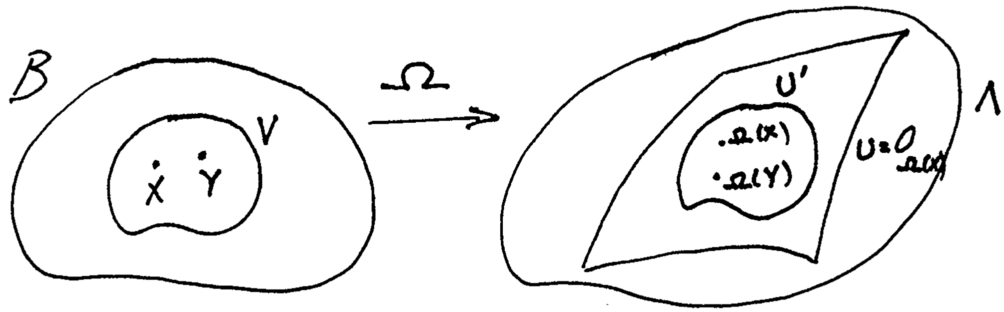


Figure 4.3

In general, however, B is not uniform even locally at X. The concept of tangential uniformity is introduced next.

4.3 Tangential Uniformity and its Characterization

The fact that $U' \subset U$ may not hold for bodies which are not locally uniform, motivates the question of "tangency" of the submanifold U to U' at the point of interest, say Ω_X. Here, $U = O_{\Omega_X}$ and $U' = \Omega(V)$ (see proposition (4.2.4)).

Let ϕ: $I \subset \mathbb{R} \rightarrow B$ be a curve in B passing through X and let

$$\hat{k} = \left.\frac{d\phi}{dt}\right|_{t=0} \quad \text{where } \phi(0) = X.$$

Thus, $\hat{k}$ defines the direction of $\phi(t)$ at X.

4.3.1 Definition

B is called tangentially uniform at X in the direction of $\hat{k}$ if

$$d\Omega|_X \colon T_X B \rightarrow T_{\Omega(X)}\Lambda \quad \text{is such that } d\Omega(\hat{k}) \in T_{\Omega(X)}U.$$

4.3.2 Remark

Assuming B is uniform in the direction of $\hat{k}$, it is worthwhile to examine its consequences. From the above definition, $d\Omega(\hat{k}) = \hat{u}$ is the vector in $T_{\Omega(X)}\Lambda$, tangent to the submanifold U, which is the orbit of G at Ω_X. Therefore, G is transitive on U and for any p,q$\in$U, $\exists$ g$\in$G such that $p = \sigma_g(q)$. The particular group action is a material isomorphism, i.e. if Ω_X, $\Omega_Y \in U$, then $\Omega_X(F) = \Omega_Y(FP)$, P,F$\in$GL. Therefore, any two points in U are materially isomorphic by construction. Now, if Y is a point on $\Phi(t)$ in a neighborhood X (remark 2.2.5) then $\Omega(Y)$ lies in a neighborhood of $\Omega(X)$ on the curve $\Omega(\Phi(t))$ with direction $d\Omega|_X(\hat{k})$. The map Ω_X restricted to $\Phi(t)$ is one-to-one in a neighborhood $V_\epsilon(\bar{X}) = \{\bar{Y} | \ ||\bar{X}-\bar{Y}|| < \epsilon\} \subset V$, i.e. if $Y \neq X$ then $\Omega(\bar{Y}) - \Omega(\bar{X}) = d\bar{\Omega}|_X(\bar{Y} - \bar{X})$ and therefore $\Omega(Y) \neq \Omega(X)$ by virtue of the mean value theorem in $\mathbb{R}^n$.

Here $\bar{X}$, $\bar{Y}$ and $d\bar{\Omega}|_X : V \to T_{\Omega_X}\Lambda$ are the images and representations of X, $Y\in B$ and $d\Omega|_X$ respectively, in a local reference configuration.

To illustrate this situation, the following diagram will be useful:

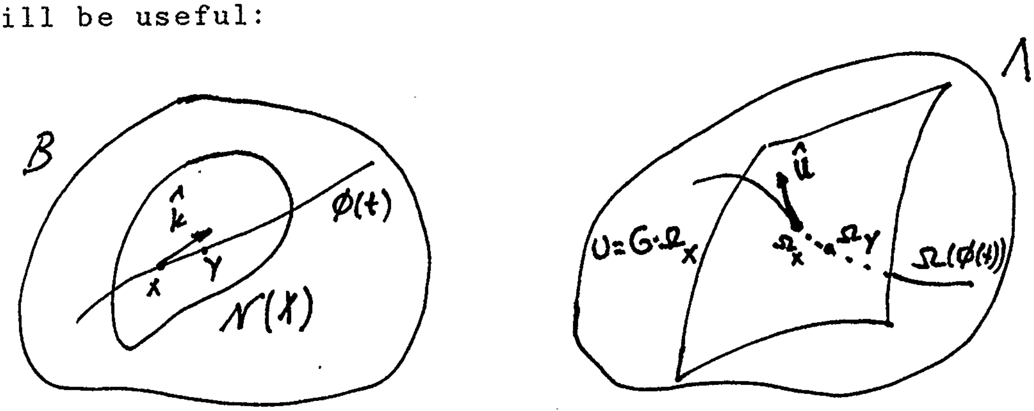


Figure 4.4

The action of a neighborhood of e$\in$G^r can be represented by the action of its one parameter subgroups (proposition (3.4.13)) completely. From remark (3.4.11), we define the corresponding Killing vector space at $p \in U \subset \mathcal{A}$. For transitive action of G^r it has already been stated that this correspondence is a vector space isomorphism.

4.3.3 Proposition

The dimension of the orbit U is the same as the dimension of G^r.

Proof: The dimension of the isotropy group at any body point is zero by assumption for $\mathcal{B}$ is a crystalline body (see Chp.1, sec.1.5 , assumption (d) and remark 4.3.5). Therefore G^r acts transitively and freely on U, so the map $T_e G^r \to T_p U$ is a vector space isomorphism. From the inverse function theorem, then there exists a map $\mathcal{N}(e) \to \mathcal{N}(p)$ which is one-to-one and onto U, its range being an open set $\mathcal{N}(p)$. Thus the dimension of G^r and U are equal. We also note that all

$v^{\alpha_i} \in T_p U$ such that $v^{\alpha_i} = (d\sigma_p(e))\alpha_i$, $\alpha_i \in T_e G^r$

are linearly independent $(i = 1,2,\ldots,r)$. As a consequence we have:

4.3.4 Theorem

$\mathcal{B}$ is uniform at X in the direction given by the vector $\hat{k}$ if

$$d\Omega(\hat{k}) = c_i v_i, \quad \{c_i\} \in \mathbb{R}, \quad v_i \in T_p U, \quad i = 1,2,\ldots,r.$$

Also, if the above holds for any $\hat{k} \in T_X \mathcal{B}$ for some $\{c_i\}$, then $\mathcal{B}$ is locally uniform at X.

4.3.5 Remark

If $\mathcal{B}$ is assumed to be a crystal body, then its isotropy group will consist of only finitely many discrete points in G^r. Therefore, one can always choose an open set W of G^r about e to exclude any of these points. The action of W at $p \in \mathcal{A}$ is then such that the generators γ_i of the one parameter groups at e and the Killing vectors v^{γ_i} at p are in one -to-one correspondence. This implies that $c_i v^{\gamma_i} = 0$ if and only if $\{c_i\} = 0$, $i = 1,2,\ldots,r$, i.e. v^{γ_i} form a set of linearly independent vectors. On the other hand, if the action of G^r is such that it possesses an m-parameter continuous subgroup of transformations as the isotropy group, then only r-m+1 of the v^{γ_i}'s are independent.

The following is a property of transformation groups.

4.3.6 Proposition

Let $f\colon G^r \times \mathcal{A}$ define the action of G^r on $\mathcal{A}$. In coordinates

$$p^i = f^i(q^1,\ldots,q^n;\ a_1,\ldots,a_r) \quad p,q \in \mathcal{A}^n$$

Then

$$v^i_{\alpha}(q) = \left.\frac{\partial f^i(q;a)}{\partial a_\alpha}\right|_{a=0} \qquad \begin{array}{l}\alpha = 1,\ldots,r\\ i = 1,\ldots,n\end{array}$$

where $a = (a_1,\ldots,a_r) \in G^r$ and $a = 0$ corresponds to the identity element e of G^r (the coordinate chart on G about e can always be selected such that $a = (0,\ldots,0)$ where a is the image of e in this chart). v_α are the r-independent infinitesimal generators of the action of G^r at q.

4.3.7 Proposition

The matrix

$$\left[\frac{\partial f^i}{\partial a_\alpha}\right] \qquad \text{has a rank of } r.$$

This is obvious since $\left[\frac{\partial f^i}{\partial a_\alpha}\right]$ is an $n\times r$ matrix and we have r linearly independent column vectors as entries.

4.3.8 Proposition

The vector $\hat{u}$ at $p\epsilon\mathcal{M}$ belong to T_pU (U being the orbit of G^r at p) if the augmented matrix

$\left[\frac{\partial f^i}{\partial a_\alpha}\middle| u^i\right]$ given by

$$\left[\begin{array}{cccc|c} \dfrac{\partial f^1}{\partial a_1} & \dfrac{\partial f^1}{\partial a_2} & \cdots & \dfrac{\partial f^1}{\partial a_r} & \hat{u}^1 \\ \cdot & & & & \cdot \\ \dfrac{\partial f^n}{\partial a_1} & \dfrac{\partial f^n}{\partial a_2} & \cdots & \dfrac{\partial f^n}{\partial a_r} & \hat{u}^n \end{array}\right]$$

is of rank r.

The proof is as follows: If $\hat{u}\epsilon T_pU$, then $\hat{u} = c_\alpha v_\alpha$ where

$$v_\alpha = \frac{\partial f}{\partial a_\alpha}, \quad \{c_\alpha\} \in \mathbb{R}.$$

Then the $(r+1)^{th}$ column of the above matrix is expressible in terms of the remaining r columns. A basic property of matrices then implies that the augmented matrix must be of rank r.

4.3.9 Remark

The map f is identified by the map Ω: $\mathcal{B} \to \mathcal{A}$ and the components of f^i are replaced by Ω^i in proposition (4.3.8). The simplest example for this, is when Ω is the scalar potential of a hyperelastic body and a polynomial representation is possible, such as in

$$\Omega_X(F) = a_o + a_{ij}F_{ij} + a_{ijkl}F_{ij}F_{kl} + \ldots$$ where the

action.

$A_g(\Omega_X(F)) = \Omega_X(FL)$ is given by

$a_{ij} \to a_{ij}L^{kj}$

$a_{ijkl} \to a_{ijkl}L^{mj}L^{nj}$, etc. Now if

$$\hat{u} = d\Omega_X(\hat{k}), \text{ i.e. } u^i = \frac{d\Omega^i}{dt}{}_X(X + t\hat{k})\Big|_{t=0},$$

then for $\mathcal{B}$ to be tangentially uniform in the direction of $\hat{k}$ at X, all (r+1) minors of the above matrix must vanish.

The preceding results can also be reformulated in a concise form as follows:

4.4 A Geometric Approach

4.4.1 Definition

Let f: $M^n \to \mathbb{R}$ be a C^∞ function on M^n. f is said to be invariant under the action of G^r on M^n if

$$f(x) = f(\sigma_g x), \quad \forall\ g \in G^r.$$

In particular, f is invariant under the transformations generated by every vector v which is a generator of the action of G^r on M^n. Since

$$f(\bar{x}^i) = f(x^i + v_\alpha{}^i t) = f(x^i) + \frac{\partial f}{\partial x^j} v^j{}_\alpha t$$

Then,

$$v^j{}_\alpha \frac{\partial f}{\partial x^j} = 0 \rightarrow v(f) = 0$$

if we identify $\partial/\partial x^j$ as the canonical basis of v at x. Now let us construct the invariant variety under G^r. The coordinates of x^i after an infinitesimal transformation become

$$\bar{x}^i = x^i + \frac{\partial f^i}{\partial a_\alpha}(x;a)\, \lambda_\alpha \qquad \alpha = 1,\ldots,r,\ i = 1,\ldots,n$$

where λ_α are constants. Now in the equations given above we have an $(n\times r)$ system of equations in r unknowns. Solving for the α_α in the first r equations (it can be uniquely solved since rank $\left[\frac{\partial f^i}{\partial a_\alpha}\right] = r$) in terms of x and substituting in the remaining (n-r) equations we obtain

$$\Psi_\rho(\bar{x};\ x) = 0, \quad \rho = 1,\ldots,n-r$$

defining the invariant variety at x. It follows that

$$v_\alpha(\Psi_\rho) = 0 \quad \text{for} \quad \begin{array}{l}\alpha = 1,\ldots,r\\ \rho = 1,\ldots,n-r\end{array}$$

Hence we have:

4.4.2 <u>Proposition</u>

If Ψ_ρ defines the equations of an invariant variety at Ω_X, then B is uniform in the direction $\hat{k}$ if $d\Omega_X(\hat{k}) = \hat{u}$ is such that

$$\hat{u}(\varphi_\rho) = 0 \quad \text{for} \quad \rho = 1,\dots,n-r.$$

4.4.3 Remark

Invariance under a neighbourhood of $e\epsilon G$ is sufficient for the variety to be invariant under G (theorem (3.4.14)).

As an example consider the generators of the orthogonal group

$$X = z\,\partial/\partial y - y\,\partial/\partial z, \quad Y = -z\,\partial/\partial x + x\,\partial/\partial z \text{ and}$$

$$Z = y\,\partial/\partial x - x\,\partial/\partial y.$$

Then,

$$\begin{aligned} \bar{x} &= x - sz + ty \\ \bar{y} &= y + rz - tx \\ \bar{z} &= z - ry + sx \end{aligned} \tag{4.4.4}$$

define an arbitrary infinitesimal transformation

$\bar{\underset{\sim}{x}} = \underset{\sim}{x} + rX + sY + tZ$ where $\underset{\sim}{x} = (x,y,z)$ and $\bar{\underset{\sim}{x}} = (\bar{x},\bar{y},\bar{z})$

Solving for s, r in Eqs. (4.4.4) :

$$s = \frac{ty - a}{z} \qquad r = \frac{tx + b}{z} \; .$$

where $a = \bar{x} - x$ and $b = \bar{y} - y$. Substituting in $\bar{z}$,we get $\bar{x}x + \bar{y}y + \bar{z}z = x^2 + y^2 + z^2$, the equation of the invariant variety at x with $\bar{x}$, $\bar{y}$, $\bar{z}$ as variables. That is $\Phi(\bar{x};x) = \bar{x}x + \bar{y}y + \bar{z}z - x^2 - y^2 - z^2 = 0$ is the set on which the orthogonal group acts transitively (here, the independent variables are $\bar{x},\bar{y},\bar{z}$ defining the orbit at the point (x,y,z)). Since Φ defines the orbit, it is expected that $X(\Phi) = Y(\Phi) = Z(\Phi) = 0$.

i.e.,

$$X(\phi) = z \left.\frac{\partial\phi}{\partial\bar{y}}\right|_{(x,y,z)} - y \left.\frac{\partial\phi}{\partial\bar{z}}\right|_{(x,y,z)} = zy-yz = 0$$

Similarly, $Y(\phi) = Z(\phi) = 0$. Clearly, $(\alpha X + \beta Y + \gamma Z)(\phi) = 0$ for any set $\{\alpha, \beta, \gamma\} \in \mathbb{R}$. This is what we mean in proposition (4.4.2) where the orbit ϕ is identified with a surface in Λ whose points represent the same constitutive function differing only within a material isomorphism (the surface in Λ is generated by the group GL).

CHAPTER 5

EXAMPLES

In this chapter examples are given for those constitutive laws, where the response functions admit a representation in a vector space of finite dimension.

5.1 Example 1

First we consider a one dimensional body and derive the condition of uniformity for its constitutive law. Unfortunately, this derivation cannot be extended for bodies of finite dimension to obtain a condition of similar nature as in the one dimensional case.

Let $\sigma(f, x)$ denote te response of B. If B is uniform then,

$$\sigma(f, x) = \hat{\sigma}(f\, p^{-1}(x)) \quad \forall\ f \in \mathbb{R} \tag{5.1.1}$$

where

$$p: B \to L(TB, \mathbb{R})$$

is a smooth function on B. Clearly, p has the meaning of a uniform reference in the sense that

$$p_x^{-1} : \mathbb{R} \to T_x B \qquad \text{and}$$

$$\sigma(fp(x), x) = \sigma(fp(y), y) \quad \forall\ x, y \in B.$$

Now

$$\frac{\partial \sigma}{\partial f} = \frac{\partial \hat{\sigma}}{\partial z}\, r$$

$$\frac{\partial\sigma}{\partial x} = \frac{\partial\hat{\sigma}}{\partial z} fr'$$

where $r = p^{-1}$ and $z = fp^{-1}$.

Eliminating $\frac{\partial\hat{\sigma}}{\partial z}$ from the above, we get:

$$r'f \frac{\partial\sigma}{\partial f} = \frac{\partial\sigma}{\partial x} r$$

$$\frac{r'(x)}{r(x)} = \left.\frac{\partial\sigma/\partial x}{f \frac{\partial\sigma}{\partial f}}\right|_x$$

so that we have

$$\left(\frac{\sigma_x}{f\sigma_f}\right)_{,f} = 0 \qquad (5.1.2)$$

Equation (5.1.2) is a condition on $\sigma(f,x)$ for B to be a uniform body.

In order to find the conditions imposed on σ by the present theory, we will assume that $\sigma(f,x)$ has a polynomial representation of the form

$$\sigma(f,x) = a_o(x) + a_1(x)f + \ldots + a_n(x)f^n \qquad (5.1.3)$$

Now, if B is uniform, then eqn.(5.1.3) must satisfy eqn.(5.1.2):

$$(\sigma_{x,f})(f\sigma_f) - (\sigma_f + f\sigma_{f,f})\sigma_x = 0 \qquad (5.1.4)$$

where

$$\sigma_x = \dot{a}_i f^i$$
$$\sigma_{x,f} = i\ \dot{a}_i f^{i-1} \qquad (5.1.5)$$
$$\sigma_f = (j+1)a_{j+1} f^i$$

where $i,j = 0,1,\ldots,n$ and $\dot{a}_i = \frac{da_i}{dx}$. Substituting eqn.(5.1.5) in eqn (5.1.4), we get:

$$i(j+1)\dot{a}_i \, a_{j+1} \, f^{i+j} - (j+1)^2 \, \dot{a}_i \, a_{j+1} \, f^{i+j} = 0$$

$i,j = 0,1,\ldots,n$. Here, the summation convention on the indices i and j have been assumed. Let $\alpha = i + j$ and evaluate the above for like coefficients f^{i+j} by setting $\alpha = 0,1,\ldots,$ for example,

$\alpha = 0 \quad i = j = 0 \quad \rightarrow \quad \dot{a}_o a_1 = 0 \rightarrow a_o(x) = \text{constant}$

$\alpha = 1 \quad i = 1,\ j = 0 \quad \rightarrow \quad \dot{a}_1 a_1 - \dot{a}_1 a_1 + \underbrace{4\dot{a}_o a_2}_{0} = 0$

$\qquad\quad i = 0,\ j = 1$

is identically satisfied.

$$\alpha = 2 \quad \rightarrow \quad \underbrace{-9\dot{a}_o a_3}_{0} - 4\dot{a}_1 a_2 - \dot{a}_2 \, a_1 + 2\dot{a}_1 \, a_2 + 2\dot{a}_2 \, a_1 = 0$$

or $$\dot{a}_2 \, a_1 - 2\dot{a}_1 \, a_2 = 0 \rightarrow \frac{\dot{a}_1}{a_1} = \frac{\dot{a}_2}{2a_2}$$

$$\alpha = 3 \quad 3\dot{a}_1 \, a_3 + 4\dot{a}_2 \, a_2 + 3\dot{a}_3 \, a_1 - 16\dot{a}_o \, a_4 - 4\dot{a}_2 \, a_2 - \dot{a}_3 \, a_1 = 0$$

or $$-6\dot{a}_1 \, a_3 + 2\dot{a}_3 \, a_1 = 0 \rightarrow \frac{\dot{a}_1}{a_1} = \frac{1}{3}\frac{\dot{a}_3}{a_3} \quad \text{etc.}$$

In general, we have

$$a_o(x) = \text{constant and}$$

$$\frac{\dot{a}_1}{a_1} = \frac{\dot{a}_2}{2a_2} = \ldots\ldots = \frac{\dot{a}_n}{na_n}$$

for eqn.(5.1.3) to represent a uniform one-dimensional body. Thus $\sigma(f,x)$ must be of the form

$$\sigma(f,x) = a_o + a_1(x)f + c_2 a_1(x)^2 f^2 + \ldots + c_n a_1(x)^n f^n$$

$$= a_o + a_1(x)[f + c_2 a_1(x) f^2 + \ldots + c_n a_1(x)^{n-1} f^n].$$

5.1.6 Remark

Clearly, the uniform reference p(x) is given by

$$p(x) = \frac{1}{a_1(x)}$$

and

$$\sigma(fp(x),\ x) = a_o + f + f^2 + \ldots + f^n = \hat{\sigma}(f)$$

is independent of x.

Now let us find the conditions imposed on $\sigma(f,x)$ by the present theory. For the sake of completeness, the necessary mathematical and physical identifications will be made for this example only. Generalizations to higher dimensional bodies are obvious and omitted in the other examples.

Let M be a linear vector space and $N^{n+1} \subset M$ a finite dimensional subspace, to be identified with the space of polynomials of degree n. Clearly, the response function $\sigma(f,x)$ is represented by the (n+1) component vector $(a_o(x),\ a_1(x),\ \ldots\ ,a_n(x))$. The action of the Lie Group of transformations as defined in Chapter 4 is such that it will have the same dimension as the body itself. That is, the action is defined by (for a general body where n > 1)

$$A_g(\sigma_\kappa(F,\ X) = \sigma_\kappa(f\rho(g),\ X)$$

where ρ: $G \rightarrow GL(n)$ is a linear representation of G, σ_κ is $\sigma(F \circ d\kappa,\ X)$, $F: V \rightarrow V$, $\rho(g): V \rightarrow V$. (Recalling from Chapter 4 that $\rho(g(t)) = d\kappa|_{\kappa(c(t))} \circ \oplus(c(t),\ c(0)) \circ d\kappa^{-1}|_{\kappa(c(0))}$, i.e., the representation of the material isomorphisms on a curve c, in the chart $\kappa: U\subset B \to E$). Since $d\kappa$ is a diffeomorphism, V and $T_X B$ $\forall$ $X\in B$ have the same dimension. This implies

$\dim(v) = \dim(B)$. Since

$\dim \text{Range } \rho(g) = \dim V = \dim \text{Domain } (\rho(g))$,

we have the basic result that $\rho(G) \equiv GL$ is the Lie Group of Transformations having the same dimension as B.

For the one dimensional body symbolically, $\{\rho(G^1)\} = \mathbb{R}^+ - \{0\}$, that is, the set of positive scalars which is a group under ordinary multiplication. The identity element is unity and inverse of $a\in\{\mathbb{R}^+ - \mathbb{R}\{0\}\}$ is simply $1/a$. Clearly, $\dim T_e(\rho(G^1)) = 1$ and it is seen that

$$A_{g(t)}(\sigma(f,x)) = \sigma(f\rho(g(t)), x)$$

is a curve in N^n, which is also the invariant variety under the action of $\mathbb{R}^+ - \{0\}$ at $\sigma(f,x)$.

5.1.7 Remark

Only for a one dimensional body the orbit of G^r in N^n takes the form of a curve. In general, the orbits define hypersurfaces in N^n.

Also, note that $\rho: I \subset \mathbb{R} \to G^1$ by $t \to g(t)$ is a curve in G^1 whose image under ρ is the group of transformations $\rho(g(t))$ under consideration. For one dimensional cases, different parametrizations of G^1 lead to the same result but for higher dimensional bodies, one distinguishes between $g_i: I \subset \mathbb{R} \to G^r$, $i = 1,2,\ldots,r$

representing r one parameter subgroups of G^r.

Now let

$c: I \subset \mathbb{R} \to \kappa(B)$

(again for dim $B = 1$, different parametrizations of c will lead to the same result) be a curve given by the identity map $t \mapsto t$ where κ is a configuration of B, in this case a map into $\mathbb{R}$. The image of the map ϕ defined by

$\phi: B \to N$

takes the form $\phi(c(t)) = \sigma(f, c(t))$ on the curve $c(t)$. Clearly, $\phi(c(0)) = \sigma(f,x_o)$, $x_o = c(0)$. Denote

$\sigma(f,x) = \sigma_x(f) \in N^n$ and

consider the curves

$\sigma_{c(t)}(f) \doteq \phi$ and $\sigma_x(f\rho(g(t)) = \psi$.

According to the criterion, for B to be tangentially uniform, the tangent vectors of the curves $\phi(t)$ and $\psi(t)$ (with a slight abuse of notation) must point in the same direction <u>at the point</u> $\sigma_{c(0)}(f) \in N^n$, i.e.

$$\frac{d}{dt}[\sigma_{c(t)}(f)]\Big|_{t=0} = \lambda \frac{d}{dt}[\sigma_{x_o}(f\rho(g(t)))]\Big|_{t=0},$$

where $\lambda \in \mathbb{R}$. Using the polynomial representation of σ, the above can also be written as

$$\frac{d}{dt}[(a_o(t), a_1(t), \ldots, a_n(t))]_{t=0}$$

$$= \lambda \frac{d}{dt}[(a_o(x_o), a_1(x_o)\rho(g(t)), \ldots, a_n(x_o)\rho(g(t))^n]_{t=0}$$

$$\to (\dot{a}_o(0), \ldots, \dot{a}_n(0)) =$$

$$\lambda \frac{d}{dt}(\rho(g(t)))\Big|_{t=0}(0, a_1(x_o), 2a_2(x_o), \ldots, na_n(x_o))$$

$$= \beta(0, a_1(x), 2a_2(x), \ldots, na_n(x))$$

which leads to the obvious restrictions on $a_i(x)$:

$$a_o(x) = \text{constant}$$

$$\frac{\dot{a}_1}{a_1} = \frac{\dot{a}_2}{2a_2} = \ldots = \frac{\dot{a}_n}{na_n} \qquad (5.1.8)$$

These restrictions on $a_i(x)$ are valid <u>at all $x \in B$</u> although the computation is made at x_o. The reason is that we were able to factor out the term

$$\frac{d}{dt}\,(\rho(g(t))).$$

For dim $B > 1$, dim $G > 1$ and the tangent vector to the curve $\varphi(t) = T(F\rho(g(t)), X_o)$ can only be expressed as a linear combination of the generators of the orbit at $T_{X_o}(F)$. In general this varies from point to point in N^n (i.e. the linear dependence is different at different points).

Finally, we note that Eqn. (5.1.8) defines the same restrictions on $a_i(x)$ as the one obtained with the previous method.

5.2 Example 2

Consider the hyperelastic response of B (dim $B > 1$) given by:

$$W(G_X, X) = a(X) + tr(G_X)$$

where

$$G_X\colon T_X B \to V, \qquad a\colon B \to \mathbb{R}$$

In continuum mechanics, the laws of constitution are almost always given in a reference configuration which is most of the time a diffeomorphism of B into $U \subset E$. Let

$$G_X = F\,d\kappa|_X, \quad F \in GL(n), \quad \kappa\colon B \to E \text{ is}$$

a local configuration of an open set S in B, $X \in S$. In coordinate form

$$F = F^j{}_k \, e_j \otimes e^k \quad \text{and} \quad d\kappa|_X = (d\kappa)^i{}_\alpha \, e_i \otimes d^\alpha(X)$$

where the set $\{e_i\}$ is a fixed basis of V, $i = 1,\ldots,n$. The basis $\{d^\alpha\}|_X \in T^*_X B$ form a smooth field of cotangent vectors on S and each set $\{d^\alpha(X)\}$, $\alpha = 1,\ldots,n$ span $T^*_X B$. It is not relevant here to which configuration the field $\{d^\alpha\}$ is referred to. For instance, let us assume that $\{d^\alpha\} = (d\gamma)^{-1}\{e^\alpha\}$, the set $\{e^\alpha\}$ being the dual of the set $\{e_\alpha\}$. The induced basis $\{d_\alpha\} \in T_X B$ should not be confused with the uniformity basis introduced in [9] for we do not require the maps $(d\gamma)|_S$ to be a uniform on S.

$$\text{So,} \quad G_X = F^j{}_i (d\kappa|_X)^i{}_\alpha \, e_j \otimes d^\alpha(X) \quad \text{and}$$

$$\text{tr}(G_X) = F^j{}_i (d\kappa|_X)^i{}_j$$

Therefore,

$$W(G_X, X) = W_\kappa(F, X) = a(\bar{X}) + b_{ij}(\bar{X}) F_{ij}$$

where $b_{ij}(\bar{X}) = (d\kappa(X))^i{}_j$, $\bar{X}$ is the <u>place</u> of X in the reference configuration κ. (Note that we do not differentiate between the place of upper and lower indices anymore).

Now it is observed that if given

$$W_\kappa(F, X) = a(\bar{X}) + b_{ij}(\bar{X}) F_{ij} \; ,$$

the $b_{ij}(\bar{X})$ must be interpreted as the gradient of the reference configuration with respect to which $W(G_X, X)$ is given.

As before,

$$W_\kappa(FP(t), X_o) = a_o(\bar{X}_o) + b_{ij}(\bar{X}_o) F_{ik} P_{kj}(t)$$

is the curve in N^n (n = 10, for dim B = 3) generated by the one parameter subgroup of transformations P(t). Its action is given by:

$$b_{ij} \to b_{ik}P_{jk}$$
$$a_o \to a_o$$

i.e. the point (a_o, b_{ij}) is mapped on to the point $(a_o, b_{ik}P_{kj})$. Let $c: I \subset \mathbb{R} \to B$ be a curve in B and $W(F, X(c(t)))$ be its image in N^n. Denote

$$\Psi(t) = W_\kappa(FP(t), X_o) \quad \text{and} \quad \Phi(t) = W(F, X(c(t))),$$

such that $P(0) = I \in GL(3)$, $X_o = X(c(0))$ for dim B = 3. The curves Ψ and Φ do not coincide unless B is uniform along c(t). The criterion for this is given in [9]. The weaker condition is the tangency of these curves at $W(F, X(c(0))) \in N$, which can be checked by comparing their tangent vectors.

$$\frac{d\Psi}{dt}\Big|_{t=0} = \lambda \frac{d\Phi}{dt}\Big|_{t=0}, \quad \lambda \in \mathbb{R} \quad \text{leading to:}$$

$$a_o(\bar{X}_o) + b_{ik}(\bar{X}_o)\dot{P}_{jk}(t)\Big|_{t=0} F_{ij}$$

$$= \lambda[\dot{a}_o(\bar{X})\Big|_{\bar{X}=\bar{X}_o} + \dot{b}_{ij}(\bar{X})\Big|_{\bar{X}=\bar{X}_o} F_{ij}]$$

$$\to \quad a_o(X) = \text{constant}$$

$$\dot{P}^T(t)\Big|_{t=0} = \lambda b^{-1}(\bar{X}_o)\dot{b}(\bar{X}(t))\Big|_{t=0}$$

(Note that b is invertible since κ is one-to-one).

The linear maps P(t) are the material isomorphisms with respect to the configuration κ and the chosen frame in V and the field $\{d_\alpha\}$. Obviously, for all $W_\kappa(F, X)$ of the given form, namely linear in F, we can find material isomorphisms locally along a given curve and therefore, such bodies are always tangentially uniform.

The material isomorphisms in the configuration κ are given by the exponential mapping

$$P(t) = \exp(t\dot{b}^T b^{-T}|_{\bar{X}_o}) \cong I + t(b^{-1}\dot{b})^T|_{\bar{X}_o}$$

recalling that (denote $X(c(t))$ by $X(t)$ for short).

$(d\kappa(X(t)))^{-1}P(t)d\kappa(X(0)) = \oplus(X_o, X(t)): T_{X_o}B \to T_{X(t)}B$

is the material isomorphism of $T_{X_o}B$ and $T_{X(t)}B$ defined for $X(t)$ close to $X(0)$ along $c(t)$.

Note that $P(t_2 - t_1)$ is the representative of $\oplus(X(t_1), X(t_2))$ with respect to κ.

5.3 Example 3

For non-linear hyperelastic response functions, we obtain additional requirements on the coefficients, unlike the linear case treated in the previous example.

Let $W_\kappa(F, X)$ be given by:

$W_\kappa(F, X) = a_o(\bar{X}) + b_{ij}(\bar{X})F_{ij} + c_{ijkl}(\bar{X})F_{ij}F_{kl}$

Assuming W is tangentially uniform along $c: I \subset \mathbb{R} \to B$, we again attempt to find the conditions on W_κ imposed by this requirement:

Let:

$\Psi(t) = W_\kappa(FP(t), \bar{X}_o)$

and $\Phi(t) = W_\kappa(F, X(c(t)))$.

The action of the transformation group is defined by

$$W_\kappa(FP(t), X_o) = a_o(\bar{X}) + b_{ik}P_{jk}F_{ij} + \underbrace{c_{imkn}P_{jm}P_{nl}}_{\bar{c}_{ijkl}}F_{ij}F_{kl}$$

where $\bar{c}_{ijkl} = c_{imkn}\, P^T_{mj}\, P^T_{nl}$.

The tangency condition of the curve $\Phi(t)$ to the variety generated by GL(3) at $W(F, X_o)$ is given by:

(a) $\dot{a}(\bar{X}(t))|_{t=0} = 0$

(b) $\lambda \dot{b}_{ij}(\bar{X}(t))|_{t=0} = b_{ik}(\bar{X}(0))\dot{P}^T_{kj}(0)$

(c) $\lambda \dot{c}_{ijkl}(\bar{X}(t))|_{t=0} = c_{imkl}\, \dot{P}_{jm}(0) + c_{ijkn}\dot{P}_{ln}(0)$

where $P(0) = I$ by definition. From condition (b) we get as before,

$$\dot{P}^T(0) = \lambda\ b^{-1}\ \dot{b}|_{\bar{X}_o}$$

Substituting this in the third condition (for compatibility),

$$\dot{c}_{ijkl}(\bar{X}(t))\ |_{t=0} = [c_{imkl}(b^{-1})_{mp}\ \dot{b}_{pj} + c_{ijkn}(b^{-1})_{nq}\dot{b}_{ql}]|_{\bar{X}_o}$$

Therefore, for the tangential uniformity, these relations between c_{ijkl} and b_{ij} must be satisfied along the given curve at the point under consideration.

5.3.1 <u>Remark</u>

We check this condition to see if the one dimensional form of the compatibility is obtained. In that case we have $b^{-1} = 1/b$ and all tensors reduce to scalars, i.e., $b_{ij} \equiv b$, $c_{ijkl} = c$, etc.

$$\dot{c} = c \frac{1}{b} \dot{b} + c \frac{1}{b} \dot{b} = 2c \frac{\dot{b}}{b}$$

or

$$\frac{\dot{c}}{2c} = \frac{\dot{b}}{b} \text{ as expected.}$$

For polynomials of higher degree than two, we shall expect relations among the coefficients similar to the above.

CHAPTER 6

CONCLUSIONS

Considering the initial objectives of this study, it seems that there is still some work needed to be done. This mainly includes removing the following mathematical restrictions on:

1. The type of the representation space for the constitutive function, and
2. The type of the isotropy group of B (which was taken to be discrete in this thesis).

The former enables us to do computations as in Chapter 5 since it defines the action of G explicitly; it must be noted, however, that the ideas presented here are invariant. Therefore once the representation space is fixed, the results are valid as they are given irrespective of the space chosen. The second restriction allows one to construct a one-to-one correspondence between the generators of the one parameter subgroups in $T_e G$ and the Killing vectors in $T_{\Omega(X)}\Lambda$ at Ω_X leading to the proposition (4.3.8). This assumption, however, can be removed if one has the information on the type of the isotropy group of B (say an m-parameter continuous group) at X. We will have (r-m+1) linearly independent Killing vectors. In this instance, the criterion in proposition (4.3.8) must be modified so that the augmented matrix will be required to be of rank (r-m+1) for B to be tangentially uniform at X.

The assumption of smooth material response given in definition (2.2.19), contrary to the above mathematical assumptions is purely a physical one and it essentially excludes non-smooth changes in the material properties of the body. It allows one to formulate the problem without any reference to the material uniformity and all that is required is that, the image of a smooth curve in $\dot{B}$ be smooth under the mapping $\Omega\colon B \to \Lambda$. With this restriction, it was possible to construct a theory that includes a wider class of elastic bodies, namely those which are tangentially uniform along a smooth curve in $\dot{B}$.

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