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HOMOTOPY FUNCTORS ON H-SEMIDIRECT PRODUCTS

BY

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Abstract

The concept of H-semidirect product structure on an H-group is introduced. We show that the loop space ΩX of any CW-complex X is the H-semidirect product of the identity path-component of ΩX with $\pi_1 X$.

The set of free homotopy classes of maps into an H-semidirect product inherits the structure of a semidirect product of groups. This leads to new insight concerning the nilpotency of homotopy classes of maps into an H-group.

In singular homology with suitable coefficients and suitable bordism theories the Pontryagin algebra of an H-semidirect product decomposes into a twisted tensor product of the Pontryagin algebras of the factors. The notion of a twisted tensor product of certain algebras is introduced and their universal properties are presented.

We make explicit the role played by the H-semidirect product structure of the loop space of a CW-complex X in the context of investigating X for nilpotency.

Contents

Introduction

There are various reasons why nilpotent CW-complexes enjoy special interest in Topology. This thesis evolved out of an attempt to understand the nature of nilpotency of CW-complexes.

A connected CW-complex X is called nilpotent, if its fundamental group $\pi_1 X$ is nilpotent and acts nilpotently on the higher homotopy groups of X or, equivalently, if the Postnikov system of X admits a principal refinement [B-K], [H-M-R]. It seems it was Roitberg [R] who was the first to consider the nature of nilpotency of X from the loop space point of view. He replaces ΩX by a homotopically equivalent topological group, a construction due to Milnor [M1] for the loop space of a countable CW-complex, and arrives at the characterization: X is nilpotent if and only if $\pi_1(X,*)$ is nilpotent and the action of $\pi_1(X,*)$ on $\pi_{n-1}(\Omega X,*)$ by loop conjugation is nilpotent.

Closer scrutiny reveals: ΩX is, up to homotopy, a semidirect product $(\Omega X)_0 \rtimes \pi_1 X$, where $(\Omega X)_0$ denotes the path component of ΩX containing the path that stays constant at the base point. The concept of an H-semidirect product $W_0 \rtimes_\phi \Pi$ is introduced in §1. Here Π denotes a group acting on a path connected H-group W_0 by classes of self homotopy equivalences of W_0 which are at the same time H-maps. We show that the loop space of every based CW-complex is an H-semidirect product.

Let Π be a group acting on another group G via a homomorphism $\phi : \Pi \longrightarrow \mathrm{Aut}G$. From group theory we know that the semidirect product

$G \rtimes_{\phi} \Pi$ is nilpotent if and only if Π is nilpotent and acts nilpotently on G [H], [V]. The corresponding notion of H-nilpotency of an H-group makes sense, and we show in §2 that an H-semidirect product $W_0 \rtimes_{\phi} \Pi$ is H-nilpotent if and only if Π is a nilpotent group and the action of Π on W_0 is H-nilpotent.

These considerations are linked up with the nilpotency of a space X because ΩX is the H-semidirect product $(\Omega X)_0 \rtimes \pi_1 X$. The group of free homotopy classes of maps $[S^{n-1}, \Omega X]$ inherits from ΩX the structure of a semidirect product, $[S^{n-1}, \Omega X] \cong [S^{n-1}, (\Omega X)_0] \rtimes \pi_1 X \cong \pi_{n-1}(\Omega X)_0 \rtimes \pi_1 X$, because $(\Omega X)_0$ is simple. The action of $\pi_1 X$ on $\pi_{n-1}(\Omega X)_0$ inherited from the H-semidirect product ΩX coincides with the action by loop conjugation used by Roitberg to characterize the nilpotency of X. Thus we arrive at the characterization: X is a nilpotent space if and only if $[S^{n-1}, \Omega X]$ is a nilpotent group for all $n \geq 2$ (cf. §5).

Conditions for the nilpotency of $[S^{n-1}, \Omega X]$ are contained in §2. E.g.: if ΩX is H-nilpotent, then $[S^{n-1}, \Omega X]$ is nilpotent for all $n \geq 2$. The difficult part of this approach is the problem: Does $\pi_1 X$ act H-nilpotently on $(\Omega X)_0$? An answer depends crucially on the group of classes of self homotopy equivalences of $(\Omega X)_0$ induced by H-conjugation in ΩX (cf. §5) or, more precisely, on the homomorphism ϕ from Π into this group. E.g. if this homomorphism takes everything in Π to the identity, then X is simple. Unfortunately, it seems like not much is yet known about this subgroup of the group of self homotopy equivalences of $(\Omega X)_0$.

We also study singular homology and suitable bordism functors on an H-semidirect product $W = W_0 \rtimes \Pi$. The resulting graded abelian group inherits from W a graded algebra structure which decomposes to a twisted tensor product reflecting the H-multiplication in W.

In §3, we lay the algebraic foundations. Given a commutative ring R with identity, we introduce the notion of a twisted tensor product $A \otimes_R^\psi RG$ of an R-algebra A with an R-group algebra RG and discuss its universal properties.

In §4, we show that $H_*(W;R) \cong H_*(W_0;R) \otimes_R^\psi H_*(\Pi;R)$ and in §6, we derive a similar result for certain bordism theories. To avoid confusion, $H_*(\Pi;R)$ denotes the singular homology of the discrete space Π with Pontryagin product coming from the group multiplication in Π.

The considerations in §4 enter again into the discussion of nilpotency of a space X. The actions of Π on $[S^{n-1},(\Omega X)_0]$ and on $H_{n-1}((\Omega X)_0)$ are both derived from the H-semidirect product structure on ΩX. It follows that the Hurewicz homomorphism is an operator homomorphism. Consequently, if $\pi_1 X$ acts nilpotently on $[S^{n-1},(\Omega X)_0]$, then $\pi_1 X$ also acts nilpotently on the image of $\pi_{n-1}(\Omega X)_0$ in $H_{n-1}((\Omega X)_0)$ (cf. §§4, 5).

A technical remark: This thesis is self contained in the sense that the relevant definitions are recalled and that facts not contained in a standard reference text like Spanier "Algebraic Topology" [Sp] are stated, without proof, as they are needed. To avoid confusion as to whether or not a statement labeled as "theorem", "proposition", "lemma" is taken from a source in print or is believed to be new, I have marked the known results with an asterisk. Thus *(5.2) Theorem ..., indicates a known result. Some of these results are of elementary nature and folklore, in which case no reference is given, others are implicitly contained in a source in print. A proof is given if such a result plays a key role in this thesis. All of the remaining stated facts are accompanied by a precise reference.

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Also I wish to thank the following mathematicians on the defense examining committee of this thesis: Dr. H. Farahat who pointed out that a concept equivalent to the "twisted tensor product", developed in §3, is known as a "crossed product algebra" [C-R]. Furthermore Dr. D. Sjerve and Dr. P. Zvengrowski for carefully reading the original version of this thesis and for offering a number of welcome editorial suggestions.

§0 Notation, technicalities, preliminaries on H-spaces

Throughout we shall be working in Steenrod's "Convenient category of topological spaces" [Ste1] or suitable subcategories or categories of pairs of it. We use the following symbols.

$\mathbb{N}$, $\mathbb{N}_0$	positive integers, non-negative integers
$\mathbb{Z}$	integers
$\mathbb{R}$	reals
I	$\{t \in \mathbb{R}: 0 \leq t \leq 1\}$
$(S^n,*)$	any pair homeomorphic to $(\{x \in \mathbb{R}^{n+1}: \|x\| = 1\}, (1,0,\dots,0))$
ΩX	loop space of a based space X
$f \approx g$	f is homotopic to g
$[f]$	homotopy class of f
$[X,Y]$	free homotopy classes of maps
$[(X,*),(Y,*)]$	based homotopy classes of maps
C_a	reduced mapping cone of a based map a
SA	reduced suspension of the based space $(A,*)$.

For the purposes of Homotopy Theory the concept of a topological group has been generalized to that of an H-space.

<u>(0.1) Definition</u> Let (W,e) be a space, $\mu: (W \times W,(e,e)) \longrightarrow (W,e)$ a based map.

(i) The pair (W,μ) is an H-space with homotopy identity e $:\Longleftrightarrow$ the following diagram commutes up to a homotopy which is constant on the base point (e,e) of $(W,e) \vee (W,e)$.

$$
\begin{tikzcd}
(W \times W, (e,e)) \arrow[drr, "\mu"] & & \\
(W,e) \vee (W,e) \arrow[u, hook] \arrow[rr, "\nabla"'] & & (W,e)
\end{tikzcd}
$$

Here ∇ denotes the folding map, $\nabla(e,w) = \nabla(w,e) = w$.

(ii) (W,μ) is homotopy associative $:\Longleftrightarrow$ the following diagram commutes up to a homotopy which is constant on the point (e,e,e).

$$
\begin{tikzcd}
W \times W \times W \arrow[r, "\mathrm{Id}_W \times \mu"] \arrow[d, "\mu \times \mathrm{Id}_W"'] & W \times W \arrow[d, "\mu"] \\
W \times W \arrow[r, "\mu"'] & W
\end{tikzcd}
$$

(iii) A map $i\colon (W,e) \longrightarrow (W,e)$ is a homotopy inverse in the H-space (W,μ) $:\Longleftrightarrow$ the following diagram commutes up to homotopies which are constant on the base point.

$$
\begin{tikzcd}
W \arrow[r, "(i,\mathrm{Id}_W)"] \arrow[d, "(\mathrm{Id}_W, i)"'] \arrow[dr, "c"] & W \times W \arrow[d, "\mu"] \\
W \times W \arrow[r, "\mu"'] & W
\end{tikzcd}
$$

Here $c\colon W \longrightarrow \{e\} \hookrightarrow W$ is the collapsing map.

(iv)' (W,μ) is homotopy commutative $:\Longleftrightarrow$ the following diagram commutes up to a homotopy which is constant on (e,e); $\tau(w,w') = (w',w)$.

$$\begin{tikzcd} W \times W \arrow[rr, "\tau"] \arrow[dr, "\mu"'] & & W \times W \arrow[dl, "\mu"] \\ & W & \end{tikzcd}$$

We shall write ww' for $\mu(w,w')$ and w^{-1} for $i(w)$ if there is no risk of confusion. By an H-group we mean a homotopy associative H-space with homotopy inverse i. Furthermore, W_0 will denote the path connected component of W containing the homotopy identity e.

Thus every topological group with identity element e is an H-group with strict homotopy identity e yielding strictly commutative diagrams (0.1)(i),(ii),(iii). The loop space ΩX of a based space X is an H-group with homotopy identity the loop staying constantly at the base point of X. The formation of loop spaces can be iterated; $\Omega^2 X := \Omega(\Omega X)$ is a homotopy commutative H-group.

The following theorem explains the role of H-structure in the connection with natural group structure in the set of homotopy classes of maps into a fixed space. [Wh], 116-121.

*<u>(0.2) Theorem</u> If (W,μ) is an H-space, then $[\_,W]$, respectively $[\_,(W,e)]$, is a contravariant functor from the category of compactly generated Hausdorff-spaces, respectively based compactly generated Hausdorff spaces into the category of non-associative monoids (monoid:

a set with multiplication satisfying $x(yz) = (xy)z$ and existence of a neutral element u with $xu = x = ux$ for all x). If (W,μ) also has any of the additional structures (0.1)(ii), (iii),(iv), then $[\_,W]$, respectively $[\_,(W,e)]$ inherits the strict analogues of these structures.

Conversely, if for a space $(V,*)$ the functor $[\_,(V,*)]$ from the category of based spaces to the category of based sets can be enriched to be a functor into the category of non-associative monoids, respectively associative monoids, respectively abelian monoids, respectively any of the above with inverse then $(V,*)$ has the structure of an H-space, respectively homotopy associative H-space, respectively homotopy abelian H-space, respectively with homotopy inverse. □

In particular $[X,W]$, $[(X,*),(W,e)]$ are groups if (W,μ) is an H-group.

<u>(0.3) Definition</u> Let (V,μ), (W,ν) be H-spaces, $f\colon (V,e) \longrightarrow (W,e)$ a map. We call f an H-map $:\Longleftrightarrow$ the following diagram commutes up to a homotopy which is constant on $(e,e) \in V \times V$.

$$\begin{tikzcd} V \times V \arrow[r, "f \times f"] \arrow[d, "\mu"'] & W \times W \arrow[d, "\nu"] \\ V \arrow[r, "f"'] & W \end{tikzcd}$$

*(0.4) Remark (a) An H-map $f: (V,e) \longrightarrow (W,e)$ yields natural transformations of non associative monoids $f_*: [\_,V] \overset{\cdot}{\longrightarrow} [\_,W]$ and $f_*: [\_,(V,e)] \overset{\cdot}{\longrightarrow} [\_,(W,e)]$. If in addition (V,μ), (W,ν) both have any of the properties (0.1)(ii),(iii),(iv) then f_* is a natural transformation of algebraic systems having as structures the strict analogues of (0.1)(ii),(iii),(iv).

(b) We shall frequently work with free homotopy classes of free maps into an H-group. In this context we may drop the conditions concerning base points from the definition of an H-space and of an H-map f and merely require $f: V \longrightarrow W$ to be a map which makes the diagram in Definition (0.3) commute. This still guarantees that $f_*: [\_,V] \overset{\cdot}{\longrightarrow} [\_,W]$ is a natural transformation with the properties stated in (a). □

The following observation will be fundamental on a technical level. Let (V,μ), (W,ν) be H-spaces.

*(0.5) Lemma If $f: V \longrightarrow W$ is a homotopy equivalence and also an H-map, let $g: W \longrightarrow V$ be a homotopy inverse of f. Then g is also an H-map.

Proof In the diagram

$$\begin{tikzcd}
V \times V \arrow[r, "f \times f"] \arrow[d, "\mu"'] & W \times W \arrow[r, "g \times g"] \arrow[d, "\nu"'] & V \times V \arrow[r, "f \times f"] \arrow[d, "\mu"] & W \times W \arrow[d, "\nu"] \\
V \arrow[r, "f"'] & W \arrow[r, "g"'] & V \arrow[r, "f"'] & W
\end{tikzcd}$$

we know that the left and right hand diagrams commute up to homotopy and need to verify homotopy commutativity of the middle diagram.

Commutativity of the right hand diagram yields

$$\begin{aligned} f \circ \mu \circ (g\times g) \approx \nu \circ (f\times f) \circ (g\times g) &= \nu((f\circ g) \times (f\circ g)) \\ &\approx \nu(\mathrm{Id}_W \times \mathrm{Id}_W) \\ &\approx \mathrm{Id}_W \circ \nu \\ &\approx f \circ g \circ \nu. \end{aligned}$$

Consequently, $(g\circ f) \circ \mu \circ (g\times g) \approx (g\circ f) \circ g \circ \nu$. Since $g\circ f \approx \mathrm{Id}_V$, homotopy commutativity of the middle diagram follows. □

An H-equivalence is an H-map which is also a homotopy equivalence. This terminology is justified by Lemma (0.5).

If (W,μ) is an H-space, the set of homotopy classes of self equivalences of W is a group with multiplication defined by $[f]\cdot[g] := [f\circ g]$. Denote this group by $\mathcal{E}(W)$. Lemma (0.5) says

<u>(0.6) Corollary</u> The set $H\mathcal{E}(W)$ of homotopy classes of self H-equivalences of W is a subgroup of $\mathcal{E}(W)$. □

Now let (W,μ) be an H-group. For $w \in W$, let $\overline{w}$ denote the path connected component of w in W. Let Π denote the set of all path components of W.

*<u>(0.7) Theorem</u> Π is a (discrete) group with multiplication $\overline{w}\,\overline{w'} = \overline{ww'}$. The identity element of Π is the path-component of the homotopy identity e of W and $(\overline{w})^{-1} = \overline{w^{-1}}$. □

Let $(X,*)$ be a based space. The adjointness homeomorphism

$$(X,*)^{(S^1,*)} \equiv (X,*)^{(S^1,*)\wedge(S^0,*)} \equiv ((X,*)^{(S^1,*)})^{(S^0,*)} = \Omega(X,*)^{(S^0,*)}$$

shows

(0.8) Remark   In the notation of (0.7), if $W = \Omega(X,*)$, then $\Pi \cong \pi_1(X,*)$. □

*(0.9) Lemma, Definition   Let Π be a group acting on another group G by a homomorphism $\psi:\Pi \longrightarrow \mathrm{Aut}G$. Then the set $G\times\Pi$ with multiplication $(g,p)(g',p') = (g\psi_p(g'),pp')$ is a group, called the semidirect product of G and Π with respect to ψ. Notation $G\rtimes_\psi\Pi$.

§1 Construction of H-semidirect products

Let (W_0,μ) be a path-connected H-group with H-inverse i and H-identity e. Let Π be a group and $\phi:\Pi \longrightarrow H\mathcal{E}(W_0)$ a group homomorphism. We define the H-semidirect product of W_0 with Π under ϕ and give conditions under which an H-group is an H-semidirect product.

For each $p \in \Pi$, fix an H-self homotopy equivalence $\varphi_p \in \phi(p) \in H\mathcal{E}(W_0)$. Define

$$m:(W_0\times\Pi) \times (W_0\times\Pi) \ni (w,p,w',p') \longmapsto (\mu(w,\varphi_p(w'),pp') \in W_0\times\Pi$$

$$j:W_0\times\Pi \ni (w,p) \longmapsto (\varphi_{p^{-1}}(i(w)),p^{-1}) \in W_0\times\Pi.$$

If we want m and j to be base point preserving, we take $\varphi_1 := \mathrm{Id}_{W_0}$.

<u>(1.1) Proposition</u> $(W,m) := (W_0\times\Pi,m)$ is an H-group with H-inverse map j.

<u>(1.2) Lemma</u> For $p \in \Pi$, let φ_p, $\varphi'_p \in \phi(p)$ and denote by $(W,m) := (W_0\times\Pi,m)$, $(W',m') := (W_0\times\Pi,m')$ the corresponding H-groups. Then, the identity map $W \longrightarrow W$ is an H-equivalence.

<u>Proof of (1.1)</u> <u>Step 1</u> m restricted to $W \vee W$ is homotopic to the folding map. For $\xi \in W \vee W$ (we write ww' for $\mu(w,w')$),

$$m(\xi) = \begin{cases} (e\varphi_1(w'),p') & \text{if } \xi = (e,1,w',p') \\ (w\varphi_p(e),p) & \text{if } \xi = (w,p,e,1). \end{cases}$$

Let F be a homotopy of φ_1 into Id_{W_0} and, for each $p \in \Pi$, let α_p be a path in W_0 joining $\varphi_p(e)$ to e (we must take $\alpha_1 := F_{|\{e\}\times I}$). This yields a homotopy

$$A:(W \vee W) \times I \ni (\xi,t) \mapsto \left\{ \begin{array}{ll} (eF(w',t),p') & \text{if } \xi = (e,1,w',p') \\ (w\alpha_p(t),p) & \text{if } \xi = (w,p,e,1) \end{array} \right\} \in W$$

satisfying

$$A(\xi,1) = \begin{cases} (\mu(e,w'),p') & \text{if } \xi = (e,1,w',p') \\ (\mu(w,e),p) & \text{if } \xi = (w,p,e,1) . \end{cases}$$

Thus a homotopy of $\mu_{|W_0 \vee W_0}$ induces a homotopy of $A(\cdot,1)$, and hence of $m_{|W\vee W}$, into the folding map of W.

<u>Step 2</u> Homotopy associativity of m. We must show that the maps $m(m\times Id)$, $m(Id\times m) : W\times W\times W \longrightarrow W$ are homotopic. Computing

$$m(m(w_1,p_1,w_2,p_2),w_3,p_3) = ((w_1\varphi_{p_1}(w_2))\varphi_{p_1p_2}(w_3),p_1p_2p_3)$$
$$m(w_1,p_1,m(w_2,p_2,w_3,p_3)) = (w_1\varphi_{p_1}(w_2\varphi_{p_2}(w_3)),p_1p_2p_3) ,$$

we see that such a homotopy can be obtained by going through the following succession of homotopies.

Since Φ is a homomorphism, we get $\varphi_{p_1p_2} \approx \varphi_{p_1}\circ\varphi_{p_2}$. This yields a homotopy between $m(m\times Id)$ and the map

$$(w_1,p_1,w_2,p_2,w_3,p_3) \mapsto ((w_1\varphi_{p_1}(w_2))\varphi_{p_1}\circ\varphi_{p_2}(w_3),p_1p_2p_3).$$

Using homotopy associativity in W_0, we see that this map is homotopic to

$$(w_1,p_1,w_2,p_2,w_3,p_3) \mapsto (w_1(\varphi_{p_1}(w_2)\varphi_{p_1}\circ\varphi_{p_2}(w_3)),p_1p_2p_3).$$

Since φ_{p_1} is an H-map, this latter map is homotopic to $m(\mathrm{Id}\times m)$.

Step 3 j is a homotopy inverse. We must show that the maps

$$W \xrightarrow{\Delta} W \times W \xrightarrow{\mathrm{Id}\times j} W \times W \xrightarrow{m} W$$
$$W \xrightarrow{\Delta} W \times W \xrightarrow{j\times \mathrm{Id}} W \times W \xrightarrow{m} W$$

are homotopic to the constant map $W \longrightarrow \{(e,1)\}$ (Δ denotes the diagonal map).

Let $(w,p) \in W$; then

$$m(\mathrm{Id}\times j)\Delta(w,p) = ((w\varphi_p(\varphi_{p^{-1}}(w^{-1})),1) = (w(\varphi_p\varphi_{p^{-1}}(w^{-1})),1).$$

Since $\varphi_p \circ \varphi_{p^{-1}} \approx \varphi_{pp^{-1}} \approx \varphi_1 \approx \mathrm{Id}_{W_0}$, we get a homotopy of $m(\mathrm{Id}\times j)\Delta$ with the map $(w,p) \mapsto (ww^{-1},1)$. This is homotopic to the constant map $W \longrightarrow \{(e,1)\}$ using the property of the H-inverse i in W_0.

The other way around we get

$$m(j \times \mathrm{Id})\ \Delta\ (w,p) = (\varphi_{p^{-1}}(w^{-1})\varphi_{p^{-1}}(w),1).$$

Since $\varphi_{p^{-1}}$ is an H-map, this map is homotopic to

$$(w,p) \mapsto (\varphi_{p^{-1}}(w^{-1}w),1).$$

Since W_0 is an H-group, this map is homotopic to

$$(w,p) \mapsto (\varphi_{p^{-1}}(e),1).$$

Since W_0 is path connected there is a path in W_0 joining $\varphi_{p^{-1}}(e)$ to e. Such a path induces a homotopy of the latter map with the constant map $W \longrightarrow \{(e,1)\}$. Hence $m(j\times \mathrm{Id})\Delta$ is homotopic to the constant map.

$\square$

Proof of (1.2) In view of Lemma (0.5) we need only check that Id_W is an H-map. For (w,p), $(w',p') \in W_0\times\Pi$, we get

$$\mathrm{Id}_W\circ m(w,p,w',p') = (w\varphi_p(w'),pp').$$

Since $\varphi_p \approx \varphi'_p$, this map is homotopic to the map

$$(w,p,w',p') \longmapsto (w\varphi'_p(w'),pp') = m'\circ(\mathrm{Id}_W\times \mathrm{Id}_W)(w,p,w',p'). \qquad \square$$

Proposition (1.1) and Lemma (1.2) suggest the following.

(1.3) Definition Let W_0 be a path-connected H-group, Π a discrete group, $\phi:\Pi \longrightarrow H\mathcal{E}(W_0)$ a group homomorphism. An H-group V is an H-semidirect product of W_0 and Π under ϕ if an only if V is H-equivalent to the space W constructed in (1.1). In this case, we write $V \approx W_0\rtimes_\phi\Pi$. The subscript "$\phi$" may be deleted if the context is clear.

Turning to the question as to whether or not a given H-group (W,M) is an H-semidirect product, let us denote by W_0 the identity path component of W and by Π the set of path-connected components of W with the canonical structure of a discrete group as explained in (0.7). Thus, if $\overline{x}$ denotes the path component of $x \in W$, then $\overline{x}\ \overline{y} = \overline{xy}$ in Π.

(1.4) Proposition Π acts on W_0 by classes of free H-equivalences.

Proof For each $\bar{x} \in \Pi$, let $\phi(\bar{x}) \in H\mathcal{E}(W_0)$ be represented by the map φ_x,

$$\varphi_x : W_0 \ni w \mapsto xwx^{-1} \in W_0 .$$

It is clear that φ_x takes values in W. Since $\varphi_x(e) \in W_0$ and W_0 is connected, φ_x actually takes values in W_0. Then φ_x is an H-map with H-inverse $\varphi_{x^{-1}}$, hence an H-equivalence.

To see that $\phi(\bar{x})$ is well defined, suppose $\bar{x} = \bar{x}'$. Then x and x' belong to the same path component of W. Take a path $\alpha: I \longrightarrow W$ joining x to x'. Then

$$I \times W_0 \ni (t,w) \mapsto \alpha(t)w\alpha(t)^{-1} \in W_0$$

is a homotopy between φ_x and $\varphi_{x'}$. Thus $\phi(\bar{x})$ is well defined.

The same technique shows that $\varphi_{xy} \simeq \varphi_x\varphi_y$. Hence, $\phi(\overline{xy}) = \phi(\bar{x})\phi(\bar{y})$ and $\phi: \Pi \longrightarrow H\mathcal{E}(W_0)$ is a homomorphism. □

Now fix an element x for each path-component $\bar{x} \in \Pi$. From the data in (1.4) we may then form the H-semidirect product $W_0 \rtimes_\phi \Pi$ in accordance with (1.1). There is a canonical continuous map

$$h: W_0 \rtimes_\phi \Pi \ni (w,\bar{x}) \mapsto wx \in W.$$

(1.5) Lemma (i) h is an H-map.

(ii) Taking different choices $\overline{x'} = \bar{x}$ in the various path components of W yields an H-map $h': W_0 \rtimes_\phi \Pi \longrightarrow W$ with $h \simeq h'$.

Proof (i) We claim that the following diagram commutes up to homotopy.

$$
\begin{tikzcd}
(W_0 \rtimes \Pi) \times (W_0 \rtimes \Pi) \arrow[ddd, "h \times h"'] & \ni & (w_1,\bar{x}_1,w_2,\bar{x}_2) \arrow[r, mapsto, "m"] \arrow[ddd, mapsto] & (w_1\varphi_{x_1}(w_2),\overline{x_1x_2}) \arrow[d, mapsto, "h"] & \in & W_0 \rtimes \Pi \arrow[ddd] \\
 & & & w_1\varphi_{x_1}(w_2)[x_1x_2] \arrow[d, mapsto, "\approx", "(1)"'] & & \\
 & & & w_1(x_1w_2x_1^{-1})x_1x_2 \arrow[d, mapsto, "\approx", "(2)"'] & & \\
W \times W & \ni & (w_1x_1,w_2x_2) \arrow[r, mapsto, "M"] & w_1x_1w_2x_2 & \in & W
\end{tikzcd}
$$

Here $[x_1x_2]$ denotes the fixed representative for the path component $\overline{x_1x_2} \in \Pi$. A homotopy (1) in the diagram above is obtained using any path joining $[x_1x_2]$ to x_1x_2 (same technique as in (1.4)). A homotopy (2) comes from homotopy associativity in W.

(ii) A homotopy of h into h' can be constructed using for each $\bar{x} \in \Pi$ a path $\alpha_{\bar{x}}$ joining x to x'. □

Representing the path component of the identity W_0 of Π by the homotopy identity e of W itself, we see that the restriction

$$h_{|W_0\times\{\bar{e}\}} : W_0\times\{\bar{e}\} \longrightarrow W_0$$

is homotopic to Id_{W_0} (if we identify $W_0\times\{\bar{e}\}$ with W_0). Therefore, for any $\bar{x} \in \Pi$,

$$h_{|W_0\times\{\bar{x}\}} : W_0\times\{\bar{x}\} \longrightarrow \bar{x}$$

is also a homotopy equivalence. Since h establishes a bijection between the path components of $W_0 \times \Pi$ and those of W, we would like to assert that the path componentwise homotopy inverses of h combine to a homotopy inverse k of h. We are then confronted with the question whether the topology of W is fine enough to admit this. Elementary point set topology yields

(1.6) Lemma The path componentwise homotopy inverses of h combine to a homotopy inverse k of h if and only if the path components of W are open in W. $\square$

This result can be utilized as follows. If X is a based CW-complex, Milnor [M2] shows that ΩX has the homotopy type of a CW-complex. The connected components of a CW-complex are open. Therefore, the path-components of ΩX are open. Thus

(1.7) Theorem Let X be a based CW-complex, $(\Omega X)_0$ the identity component of ΩX and $\Pi := \pi_1 X$. As in (1.4), let $\phi: \Pi \longrightarrow H\mathcal{E}(\Omega X)_0$ denote the action of Π on $(\Omega X)_0$ by classes of H-self homotopy equivalences. Then there is an H-equivalence $h: (\Omega X)_0 \rtimes_\phi \Pi \longrightarrow \Omega X$.

Proof Obvious from (1.5), (1.6) and the fact that the path-components of ΩX are open in ΩX. $\square$

Since the path components of a locally path connected space are open, we see that a locally path connected H-group is an H-semidirect product. Of particular interest here is the case where W is a locally path connected topological group with identity component W_0. In this case, W_0 is a normal subgroup of W and the quotient group

$\Pi := W/W_0$ is the same as the group constructed on the path components of W in accordance with (1.4).

<u>(1.8) Theorem</u> (i) The action of Π on W_0 as defined in (1.4) is by homotopy classes of inner automorphisms of W restricted to W_0.
(ii) There is an H-homeomorphism $h: W_0 \rtimes \Pi \longrightarrow W$.

<u>Proof</u> (i) is immediate from the definition of $\phi: \Pi \longrightarrow H\mathcal{E}(W_0)$: If $\bar{x} \in \Pi$, then $\phi(\bar{x})$ is represented by the map

$$\varphi_x : W_0 \ni w \longmapsto xwx^{-1} \in W_0.$$

(ii) As in (1.5), fix an element x for each path-component $\bar{x} \in \Pi$ and define

$$h: W_0 \rtimes_\phi \Pi \ni (w,\bar{x}) \longmapsto wx \in W.$$

From (1.5) and (1.6) we know that h is an H-equivalence. Furthermore, for each $\bar{x} \in \Pi$, the map $W_0 \times \{\bar{x}\} \ni (w,\bar{x}) \longmapsto wx \in \bar{x}$ has an inverse $\bar{x} \ni y \longmapsto yx^{-1} \in W_0 \times \{\bar{x}\}$. Thus h is even a homeomorphism.

□

§2 Nilpotency of mappings into H-groups

Throughout this section W will denote an H-group obtained by taking the H-semidirect product of a path-connected H-group W_0 with a discrete group Π under $\phi:\Pi \longrightarrow H\mathcal{E}(W_0)$.

In [Wh], [Wh1] G.W. Whitehead showed that for a path-connected space X the group $[X,W_0]$ is nilpotent if X has finite category. Here, we give conditions for $[X,W]$ (free homotopy classes) to be nilpotent. This goal will be approached by considering (a) suitable properties of the range space, (b) suitable properties of the domain space. We shall respectively show:

(a) If X is arbitrary and W itself is H-nilpotent, then $[X,W]$ will be nilpotent.

(b) If W is arbitrary and X is the mapping cone of a map $f:A \longrightarrow Y$, then the Puppe sequence of f can be used to give conditions for the nilpotency index of $[X,W]$ in terms of the nilpotency indices of $[Y,W]$ and $[SA,W]$.

We begin by stipulating some notation.

(i) Let $\gamma = \gamma^1: W\times W \ni (w_0,w_1) \mapsto w_0 w_1 w_0^{-1} w_1^{-1} \in W$ and, for $n \geq 1$,

$$\gamma^{n+1}: W\times W^{n+1} \ni (w_0,w_1,\ldots,w_{n+1}) \mapsto \gamma^1(w_0,\gamma^n(w_1,\ldots,w_{n+1}) \in W$$

denote the commutator maps of W.

(ii) Let $\gamma_\phi = \gamma_\phi^1: (W_0\times\Pi)\times W_0 \ni (w_0,p,w_1) \mapsto w_0\varphi_p(w_1)w_0^{-1}w_1^{-1} \in W_0$ and, for $n \geq 1$,

$$\gamma_\phi^{n+1}: (W_0\times\Pi)\times(W_0\times\Pi)^n\times W_0 \longrightarrow W_0$$

$$(w_0,p_0,w_1,\ldots,p_n,w_{n+1}) \longmapsto \gamma_\phi(w_0,p_0,\gamma_\phi^n(w_1,\ldots,p_n,w_{n+1}))$$

denote the ϕ-commutator maps of W_0.

(2.1) Definition (i) W is H-nilpotent of nilpotency index $\mathrm{nil}W \le c :\Longleftrightarrow \gamma^c$ is homotopic to a constant map.

(ii) W_0 is ϕH-nilpotent of nilpotency index $\mathrm{nil}_\phi W_0 \le c :\Longleftrightarrow \gamma_\phi^c$ is homotopic to a constant map.

Note that the above commutators, when formally applied to a (discrete) group G (resp. Π acting on G by $\psi:\Pi \longrightarrow \mathrm{Aut}G$), agree with the usual commutators (resp. ψ-commutators). Here for γ^c (resp. γ_ψ^c) to be homotopically trivial means that all c-fold commutators (resp. ψ-commutators) of G are equal to the identity element of G. Furthermore, the image set of γ^n (resp. γ_ψ^n) generates the usual n-th central series group $\Gamma^n G$ (resp. $\Gamma_\psi^n H$). In this sense the H-nilpotency indices of (2.1) agree with the usual group theoretic nilpotency indices.

For the reader's convenience, we include the following notions on ψ-nilpotent groups and ψ-central series. The knowledgeable reader may continue reading after Lemma (2.3).

Let G, Π be groups, $\psi:\Pi \longrightarrow \mathrm{Aut}G$ a homomorphism. Define the n-th ψ-central series group by

$$\Gamma_\psi^0 G := G$$
$$\Gamma_\psi^{n+1} G := \mathrm{grp}\{g\psi_p(h)g^{-1}h^{-1} : g \in G,\ p \in \Pi,\ h \in \Gamma_\psi^n G\} \qquad \text{for } n \in \mathbb{N}\ .$$

We say G is ψ-nilpotent of nilpotency index $\mathrm{nil}_\psi G \le c :\Longleftrightarrow$ $\Gamma_\psi^c G = \{1\}$. In case ψ is the homomorphism sending Π into $\{1\}$, the notion of ψ-nilpotency reduces to the standard notion of nilpotency.

*(2.2) Lemma (i) $\Gamma_\psi^n G$ is a ψ-invariant normal subgroup of G.
(ii) For all n, $\Gamma_\psi^{n+1}G \subset \Gamma_\psi^n G$.

Proof (i) The statement is true for $n = 0$. Suppose $\Gamma_\psi^n G$ is ψ-invariant. A generator of $\Gamma_\psi^{n+1}G$ is of the form $g\psi_p(h)g^{-1}h^{-1}$ with $g \in G$, $p \in \Pi$, $h \in \Gamma_\psi^n G$. Let $q \in \Pi$, then

$$\begin{aligned}\psi_q(g\psi_p(h)g^{-1}h^{-1}) &= \psi_q(g)\psi_q\psi_p(h)\psi_q(g^{-1})\psi_q(h^{-1}) \\ &= \psi_q(g)\psi_{qpq^{-1}}\psi_q(h)\psi_q(g)^{-1}\psi_q(h)^{-1} \in \Gamma_\psi^{n+1}G,\end{aligned}$$

because $\psi_q(h) \in \Gamma_\psi^n G$, by induction hypothesis.

Suppose that $\Gamma_\psi^n G$ is normal in G. We show that any conjugate of a generator of $\Gamma_\psi^{n+1}G$ is in $\Gamma_\psi^{n+1}G$. So let $g' \in G$, then

$$\begin{aligned}g'g\psi_p(h)g^{-1}h^{-1}g'^{-1} &= g'g\psi_p(g')^{-1}\psi_p(g'hg'^{-1})\psi_p(g')g^{-1}g'^{-1}g'h^{-1}g'^{-1} \\ &= a\psi_p(b)a^{-1}b^{-1}\end{aligned}$$

with $a = g'g\psi_p(g')^{-1}$ and $b = g'hg'^{-1} \in \Gamma_\psi^n G$ by induction hypothesis.
(ii) Follows from (i) by an obvious induction argument. □

*(2.3) Lemma If $\Gamma_\psi^n G$ is contained in the center of G, then $\Gamma_\psi^{n+a}G = \Gamma_\psi^a(\Gamma_\psi^n G)$ for all $a \in \mathbb{N}$.

Proof The statement is true for $a = 0$. So let us assume $\Gamma_\psi^{n+a}G = \Gamma_\psi^a(\Gamma_\psi^n G)$. Then

$$\Gamma_\psi^{n+a+1}G = \mathrm{grp}\{g\psi_p(h)g^{-1}h^{-1}:\ g \in G,\ h \in \Gamma_\psi^{n+a}G\}.$$

But (2.2) says that $\psi_p(h) \in \Gamma_\psi^{n+a}G \subset$ center of G. Hence, $\Gamma_\psi^{n+a+1}G$ is equal to

$$\mathrm{grp}\{l\cdot\psi_p(h)lh^{-1}:\ l \in \Gamma_\psi^n G,\ h \in \Gamma_\psi^a(\Gamma_\psi^n G)\} = \Gamma_\psi^{a+1}(\Gamma_\psi^n G). \qquad \square$$

Concerning $\mathrm{nil}W$, $\mathrm{nil}_\phi W_0$, we have the following results.

(2.4) Proposition Let $\phi:\Pi \longrightarrow H\mathcal{E}(W_0)$ and $\phi':\Pi' \longrightarrow H\mathcal{E}(W_0)$ be homomorphisms of groups. Suppose $\phi' = \phi\circ f$ for some homomorphism $f:\Pi' \longrightarrow \Pi$. Then $\mathrm{nil}_{\phi'}W_0 \le \mathrm{nil}_\phi W_0$.

Taking Π' to be the trivial group and ϕ', f the unique homomorphism, yields

(2.5) Corollary $\mathrm{nil}W_0 = \mathrm{nil}_{\phi'}W_0 \le \mathrm{nil}_\phi W_0$ for all $\phi:\Pi \longrightarrow H\mathcal{E}(W_0)$. □

(2.6) Theorem $W = W_0 \rtimes_\phi \Pi$ is H-nilpotent if and only if W_0 is ϕH-nilpotent and Π is nilpotent.

Theorem (2.6) is the H-analogue of a known result [H], [V] concerning nilpotency of a semi-direct product of discrete groups: $G\rtimes_\psi \Pi$ is nilpotent if and only if G is ψ-nilpotent and Π is nilpotent. The proof of Theorem (2.6) below yields also a proof for this result if one makes the necessary adaptations as explained immediately after Definition (2.1).

Proof of Proposition (2.4) We show that $\gamma^c_{\phi'}$ factors through γ^c_ϕ, so that a null-homotopy of γ^c_ϕ yields a null-homotopy of $\gamma^c_{\phi'}$. To factor $\gamma^c_{\phi'}$, let us assume that the actions of $p' \in \Pi'$ on W_0 via ϕ' and $f(p') \in \Pi$ on W_0 via ϕ are defined by the same element $\varphi_{p'} = \varphi_{f(p')} \in \phi'(p') = \phi f(p') \in H\mathcal{E}(W_0)$. Then consider the map

$$f^c:(W_0\times\Pi')\times(W_0\times\Pi')^{c-1}\times W_0 \longrightarrow (W_0\times\Pi)\times(W_0\times\Pi)^{c-1}\times W_0$$
$$(w_0,p_0',w_1,\dots,p_c',w_c) \longmapsto (w_0,f(p_0'),w_1,\dots,f(p_c'),w_c)$$

Visibly, $\gamma^c_{\phi'} = \gamma^c_\phi\circ f^c$. □

Proof of Theorem (2.6) Suppose $\mathrm{nil}W = \mathrm{nil}W_0 \rtimes_\phi \Pi = c$. Then γ^c is null homotopic so that the image of γ^c is contained in the single path-component of $W_0 \rtimes \Pi$ containing $\gamma^c(e,1,\ldots,e,1) \in W_0 \times \{1\}$. Inspection of the H-multiplication of W shows that for any $(w_0,p_0,\ldots,w_c,p_c) \in W^{c+1}$, the second coordinate of $\gamma^c(w_0,p_0,\ldots,w_c,p_c)$ is equal to $\gamma^c(p_0,\ldots,p_c) = 1$. Thus $\mathrm{nil}\Pi \le c$.

To see that W_0 is ϕH-nilpotent, observe that there is a homotopy $W_0 \times \Pi \times W_0 \times I \longrightarrow W_0$ deforming $\gamma^1(w_0,p,w_1,1)$ into $(\gamma^1_\phi(w_0,p,w_1),1)$. For

$$\begin{aligned}\gamma^1(w_0,p,w_1,1) &= (w_0,p)(w_1,1)(\varphi_{p^{-1}}(w_0^{-1}),p^{-1})(w_1^{-1},1)\\ &= (w_0\varphi_p(w_1),p)(\varphi_{p^{-1}}(w_0^{-1}),p^{-1})(w_1^{-1},1)\\ &= (w_0\varphi_p(w_1)(\varphi_p\varphi_{p^{-1}}(w_0^{-1})),1)(w_1^{-1},1).\end{aligned}$$

Since $\varphi_p\varphi_{p^{-1}}$ is homotopic to the identity map on W_0, there is a homotopy deforming the latter expression into

$$(w_0\varphi_p(w_1)w_0^{-1},1)(w_1^{-1},1) \approx (w_0\varphi_p(w_1)w_0^{-1}w_1^{-1},1) = (\gamma^1_\phi(w_0,p,w_1),1).$$

For $n \ge 1$, induction gives a homotopy $W_0 \times (\Pi \times W_0)^n \times I \longrightarrow W_0$ deforming $\gamma^n(w_0,p_0,\ldots,w_{n-1},p_{n-1},w_n,1)$ into $(\gamma^n_\phi(w_0,p_0,\ldots,w_{n-1},p_{n-1},w_n),1)$. Thus a null homotopy of γ^c yields a null homotopy of γ^c_ϕ showing that $\mathrm{nil}_\phi W_0 \le c$.

Conversely, suppose $\mathrm{nil}\Pi \le a$ and $\mathrm{nil}_\phi W_0 \le b$. Since the second coordinate of $\gamma^n(w_0,p_0,\ldots,w_n,p_n)$ is equal to $\gamma^n(p_0,\ldots,p_n)$, it follows that for $s \ge 0$, $\gamma^{a+s}(W_0 \rtimes \Pi)^{a+s+1} \subset W_0 \times \{1\}$. Hence, we may proceed as above and construct a homotopy (for $s \ge 1$),

$(W_0\rtimes\Pi)^s\times(W_0\rtimes\Pi)^{a+1}\times I \longrightarrow W_0$ deforming

$\gamma^{a+s}(w_0,p_0,\ldots,w_{s-1},p_{s-1},w_s,p_s,\ldots,w_{s+a},p_{s+a})$ into

$\gamma_\phi^s(w_0,p_0,\ldots,w_{s-1},p_{s-1},\alpha(w_s,p_s,\ldots,w_{s+a},p_{s+a}))$, where

$\alpha(w_s,p_s,\ldots,w_{s+a},p_{s+a})$ denotes the first coordinate of

$\gamma^a(w_s,p_s,\ldots,w_{s+a},p_{s+a})$.

Consequently, γ^{a+b} is homotopic to γ_ϕ^b following α, so that a null homotopy of γ_ϕ^b yields a null homotopy of the composite and hence, of γ^{a+b}. Thus $\mathrm{nil} W_0\rtimes\Pi \le a + b$. □

Let $(X,*)$ be a path connected well pointed space (i.e. the inclusion $\{*\} \hookrightarrow X$ has the homotopy extension property). Then the canonical map $[(X,*),(W_0,e)] \longrightarrow [X,W_0]$ is an isomorphism of groups. The homotopy group $[X,W]$ inherits information from the H-semidirect product structure on W and its H-nilpotency in the following way.

<u>(2.7) Proposition</u> If $\mathrm{nil}W \le c$, then $\mathrm{nil}[X,W] \le c$.

Furthermore, Π acts on $[X,W_0]$ by composition, $\psi:\Pi \longrightarrow \mathrm{Aut}[X,W_0]$ being defined by

$$\psi_p[f] := \phi(p)\circ[f] \qquad \text{for } p \in \Pi,\ [f] \in [X,W_0].$$

<u>(2.8) Theorem</u> The function

$$R: [X,W_0]\rtimes_\psi\Pi \ni ([f],p) \longmapsto [(f,p)] \in [X,W_0\rtimes_\phi\Pi]$$

is an isomorphism.

<u>Proof of Proposition (2.7)</u> Let $f_0,\ldots,f_c \in [X,W]$ be represented by maps $\alpha_0,\ldots,\alpha_c: X \longrightarrow W$. The commutator of $f_0,\ldots,f_c$ is represented by the composite

$$X \xrightarrow{\Delta} X^{c+1} \xrightarrow{\alpha_0 \times \cdots \times \alpha_c} W^{c+1} \xrightarrow{\gamma^c} W$$

which is homotopically trivial, because γ^c is homotopically trivial.

□

Proof of Theorem (2.8) To see that R is a homomorphism, take $([f],p),([f'],p') \in [X,W_0]\rtimes\Pi$. Then

$$\begin{array}{ccccc} ([f],p) & ([f'],p') & = & ([\mu(f,\varphi_p f')],pp') \\ \downarrow & \downarrow & & \downarrow \\ [(f,p)] & [(f',p')] & = & [\mu(f,\varphi_p f'),pp'] \end{array} .$$

If (f,p) is homotopic to the map $X \longrightarrow \{(e,1)\}$, then $p = 1$ and f is homotopic to the map $X \longrightarrow \{e\}$. Hence $R([f],p) = [(e,1)]$ implies $([f],p) = ([e],1)$, showing that R is mono. It is obvious that R is onto. □

We shall now enter a discussion of the nilpotency of the groups $[C,W_0\rtimes_\phi\Pi] \cong [C,W_0]\rtimes_\psi\Pi$, where C is a mapping cone. Since $[C,W_0]\rtimes\Pi$ can only be nilpotent if Π is, we shall from now on require Π to be nilpotent. Let us agree that all spaces denoted by symbols A, Y are based path connected compactly generated Hausdorff spaces. In order to get a canonical isomorphism $[(Y,*),(W_0,e)] \longrightarrow [Y,W_0]$, we require $(Y,*)$ to be well pointed. Furthermore, we stipulate for the sequel that mappings and homotopies $A \longrightarrow Y$ are to be based and mappings and homotopies into W_0 or W are to be free.

So let $a: A \longrightarrow Y$ be a map and consider the exact sequence

(2.9) $$[Y,W_0] \xleftarrow{\epsilon = i^*} [C,W_0] \xleftarrow{\nu = q^*} [SA,W_0]$$

arising from the Puppe sequence

$$A \xrightarrow{a} Y \xrightarrow{i} C := C_a \xrightarrow{q} SA.$$

We know that Π acts on $[Y,W_0]$, $[C,W_0]$, $[SA,W_0]$. Denote the corresponding homomorphisms into the respective automorphism groups by $\psi(Y)$, $\psi(C)$, $\psi(SA)$.

*<u>(2.10) Proposition</u> Let $\alpha: Y \longrightarrow Y'$ be a map, then $\alpha^*: [Y',W_0] \longrightarrow [Y,W_0]$ is an operator homomorphism.

<u>Proof</u> Let $[f] \in [Y',W_0]$, $p \in \Pi$. Then

$$\alpha^*(\psi(Y')_p[f]) = \alpha^*(\psi(Y')_p \circ [f]) = \psi(Y')_p \circ [f] \circ [\alpha] = \psi(Y')_p(\alpha^*[f]). \quad \square$$

We need the following folklore Lemma. The proof is inspired by [Wh] Theorem X.3.10.

*<u>(2.11) Lemma</u> The sequence (2.9) is a central extension.

<u>Proof</u> Let $[f]$, $[g] \in [C,W_0]$ be such that $\epsilon[f] = 1$ in $[Y,W_0]$ (i.e. $i \circ f$ is null homotopic). We show that the commutator $\gamma([f],[g]) = 1$ in $[C,W_0]$.

Recall that the mapping cone C is obtained by identifying the points $(x,0)$ of the reduced cone over A with $a(x) \in Y$. Denote by

$$Y' := \{\xi \in C : \xi \in Y \ \text{ or } \ (\xi = (a,t) \text{ and } t \le \tfrac{1}{2})\}$$

$$C'A := \{(a,t) \in C : t \ge \tfrac{1}{2}\}$$

and observe that

(i) Y is a strong deformation retract of Y' and $\{A \times \{1\}\}$ is a strong deformation retract of B.

(ii) The above deformation retractions can be extended to homotopies $S,T\colon C\times I \longrightarrow C$ (respectively).

It follows that $f\circ S(\cdot,1)$ restricted to Y' is null homotopic and $g\circ T(\cdot,1)$ restricted to $C'A$ is a constant map. Now consider the following composition of maps

$$C \xrightarrow{\Delta} C\times C \xrightarrow{f\circ S(\cdot,1)\,\times\, g\circ T(\cdot,1)} W_0\times W_0 \longrightarrow W_0.$$

From (i), (ii) above we see that $\{f\circ S(\cdot,1)\times g\circ T(\cdot,1)\}\circ\Delta$ takes values in $W_0\vee W_0$. But γ restricted to $W_0\vee W_0$ is null homotopic. Thus $\gamma([f],[g]) = 1$ in $[C,W_0]$ showing that $\ker \epsilon$ is contained in the center of $[C,W_0]$. □

When investigating $[C,W]$ for nilpotency, the following Lemma plays a key role.

<u>(2.12) Lemma</u> $[C,W]$ is nilpotent if and only if there exists $s \in \mathbb{N}$ such that

(i) $\Gamma^{s}_{\psi(C)}[C,W_0] \subset \operatorname{im} \nu$, and

(ii) there exists $t \in \mathbb{N}$ such that $\Gamma^{t}_{\psi(SA)}(\nu^{-1}\Gamma^{s}_{\psi(C)}[C,W_0]) \subset \ker \nu$.

(2.13) Corollary Suppose $[Y,W]$ is nilpotent and there exists $t \in \mathbb{N}$ such that $\Gamma^{t}_{\psi(SA)}[SA,W_0] \subset \ker \nu$. Then $[C,W]$ is nilpotent.

An immediate consequence of (2.13) is

(2.14) Corollary If $[Y,W]$ and $[SA,W]$ are nilpotent, then $[C,W]$ is nilpotent. □

Our previous discussion can be extended to iterated mapping cones as follows. Let $\mathcal{A} = \{A_\lambda\}_{\lambda \in \Lambda}$ be a family of spaces. Following G.W. Whitehead [Wh], let us call a sequence $Y = S_0 \subset \ldots \subset S_k = K$ an $\mathcal{A}$-stratification of K if, for $n \geq 1$, S_i is obtained from S_{i-1} as the mapping cone of a wedge of spaces in $\mathcal{A}$.

(2.15) Theorem Let $[Y,W]$ be nilpotent and let $Y = S_0 \subset \ldots \subset S_k = K$ be an $\mathcal{A}$-stratification for K. Suppose there exists a positive integer N such that for all $\lambda \in \Lambda$, $\operatorname{nil}[SA_\lambda,W] \leq N$. Then $[K,W]$ is nilpotent.

Letting $\mathcal{A}$ be the family of n-spheres S^n $(n \geq 0)$ and Y the 1-point space, we obtain

(2.16) Corollary Let $Y = S_0 \subset \ldots \subset S_k = K$ be a stratification of a connected CW-complex K by connected subcomplexes S_i. Suppose there exists a positive integer N such that for every $n \geq 0$, $\operatorname{nil}[S^n W] \leq N$. Then $[K,W]$ is nilpotent. □

Using the stratification of finite dimensional CW-complexes K by their skeleta, we obtain

*(2.17) Corollary $[K,W]$ is nilpotent for all finite dimensional CW-complexes K if and only if $[S^n,W]$ is nilpotent for all $n \in \mathbb{N}_0$.

Corollary (2.17) has also been obtained by Scheerer [Sch] by direct inspection of the homotopy groups in question.

Proof of Lemma (2.12) If $\mathrm{nil}[C,W] \le a$, the choices $s:= a$ and $t:= 0$ clearly satisfy conditions (i), (ii).

Conversely, suppose there exist $s,t \in \mathbb{N}$ such that conditions (i) and (ii) are satisfied. By (i), $\Gamma^s_{\psi(C)}[C,W_0]$ is contained in the image of ν and, by Lemma (2.11), in the center of $[C,W_0]$. By Lemma (2.3), we get for $k \ge 0$

$$\Gamma^{k+s}_{\psi(C)}[C,W_0] = \Gamma^k_{\psi(C)}\Gamma^s_{\psi(C)}[C,W_0].$$

Using that ν is an operator homomorphism, this implies

$$\begin{aligned}\Gamma^{t+s}_{\psi(C)}[C,W_0] &= \Gamma^t_{\psi(C)}\Gamma^s_{\psi(C)}[C,W_0] \\ &\subset \nu\, \Gamma^t_{\psi(SA)}(\nu^{-1}\Gamma^s_{\psi(C)}[C,W_0]) \subset \nu(\ker \nu) = \{1\}.\end{aligned}$$

Thus $[C,W_0]$ is $\psi(C)$-nilpotent. By (2.8), $[C,W] \cong [C,W_0]\rtimes_{\psi(C)}\Pi$ which is nilpotent by the remark following (2.6). □

Proof of Corollary (2.13) Suppose $\mathrm{nil}_{\psi(Y)}[Y,W_0] \le s$, then (because ϵ is an operator homomorphism)

$$\epsilon\, \Gamma^s_{\psi(C)}[C,W_0] \subset \Gamma^s_{\psi(Y)}[Y,W_0] = \{1\}.$$

Thus $\Gamma^s_{\psi(C)}[C,W_0] \subset \ker \epsilon = \mathrm{im}\, \nu$ and conditions (i) and (ii) of Lemma (2.12) are satisfied. □

Proof of Theorem (2.15) Using Corollary (2.14), we show by induction on the stages of the stratification of K that $[K,W_0]$ is $\psi(K)$-nilpotent.

$[S_0,W_0]$ is $\psi(S_0)$ nilpotent by hypothesis. So suppose $0 \le i < k$ and $[S_i,W_0]$ is $\psi(S_i)$-nilpotent. Let Λ_{i+1} be an indexing set for the $(i+1)$-st wedge corresponding to the $\mathcal{A}$-stratification of K. By distributivity of wedge and suspension, we get

$$P := [S(\bigvee_{\lambda\in\Lambda_{i+1}} A_\lambda),W_0] = [\bigvee_{\lambda\in\Lambda_{i+1}} (SA_\lambda),W_0] \cong \prod_{\lambda\in\Lambda_{i+1}} [SA_\lambda,W_0] =: Q.$$

The isomorphism above is an operator isomorphism. Hence $\mathrm{nil}_\psi P = \mathrm{nil}_\psi Q \le N$. By (2.14), $\mathrm{nil}_{\psi(S_{i+1})}[S_{i+1},W_0] \le \mathrm{nil}_{\psi(Y)}[Y,W_0] + (i+1)N$, which completes the induction. Summing up, we obtain

$$\mathrm{nil}_{\psi(K)}[K,W_0] \le \mathrm{nil}_{\psi(Y)}[Y,W_0] + kN.$$ □

Proof of Corollary (2.17) Since S^n is a finite dimensional CW-complex, the direction "$\Longrightarrow$" is trivial. To see that the converse is also true, let $\mathcal{J}$ be an indexing set for the connected components $\{K_i\}_{i\in\mathcal{J}}$. By (2.16), $[K_i,W_0]$ is $\psi(K_i)$ nilpotent for every $i \in \mathcal{J}$. Furthermore,

$$\mathrm{nil}_{\psi(K_i)}[K_i,W_0] \le \mathrm{nil}_{\psi(S^1)}[S^1,W_0] + \dots + \mathrm{nil}_{\psi(S^{\dim K})}[S^{\dim K},W_0] =: n.$$

Since K is the topological sum of its connected components, we get

$$[K,W] \cong \prod_{i\in\mathcal{J}} [K_i,W] \cong \prod_{i\in\mathcal{J}} ([K_i,W_0]\rtimes_{\psi(K_i)}\pi).$$

Since $\mathrm{nil}[K_i,W_0]\rtimes_{\psi(K_i)}\Pi \le \mathrm{nil}_{\psi(K_i)}[K_i,W_0] + \mathrm{nil}\Pi \le n + \mathrm{nil}\Pi$,

$\mathrm{nil}[K,W] \le n + \mathrm{nil}\Pi$. □

Analogous to Theorem (X.3.10) in [Wh], in the situation of Theorem (2.15), we may derive the following explicit $\psi(K)$-central series for $[K,W_0]$.

<u>(2.18) Proposition</u> Let F_i be the set of homotopy classes of maps $K \longrightarrow W$ whose restriction to S_i is null homotopic. Let $\mathrm{nil}_{\psi(S_0)}[S_0,W_0] \le c$, $e_i : S_i \longrightarrow K$ the inclusion map and $X_i := \bigvee_{\lambda\in\Lambda_i} SA_\lambda$.

Then

$$\begin{aligned}
[K,W_0] &\supset (e_0^*)^{-1}\Gamma^1_{\psi(S_0)}[S_0,W_0] \supset \dots \supset (e_0^*)^{-1}\Gamma^c_{\psi(S_0)}[S_0,W_0] = F_0 \supset \\
&\supset (e_1^*)^{-1}(\nu_1\Gamma^1_{\psi(X_1)}[X_1,W_0]) \supset \dots \supset (e_1^*)^{-1}(\nu_1\Gamma^N_{\psi(X_1)}[X_1,W_0]) = F_1 \supset \\
&\quad \vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\
&\supset (e_k^*)^{-1}(\nu_k\Gamma^1_{\psi(X_k)}[X_k,W_0]) \supset \dots \supset (e_k^*)^{-1}(\nu_k\Gamma^N_{\psi(X_k)}[X_k,W_0]) = F_k \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \{1\}
\end{aligned}$$

is a $\psi(K)$-central series for $[K,W_0]$.

<u>Proof</u> In the commuting diagram,

$$\begin{tikzcd}
 & X_1 & X_2 & & X_k \\
S_0 \arrow[r] \arrow[rrrrd, "e_0"'] & S_1 \arrow[u] \arrow[r] \arrow[rrrd, "e_1"] & S_2 \arrow[u] \arrow[r] \arrow[rrd, "e_2"] & \cdots \arrow[r] & S_k \arrow[u] \arrow[d, equal, "e_k"] \\
 & & & & K
\end{tikzcd}$$

every sequence $S_{i-1} \to S_i \to X_i$ is a cofibration. Applying the functor $[\cdot,W_0]$ to this diagram yields a commuting diagram of groups

$$\begin{tikzcd}
 & {[X_1,W_0]} \arrow[d, "\nu_1"] & {[X_2,W_0]} \arrow[d, "\nu_2"] & & {[X_k,W_0]} \arrow[d, "\nu_k"] \\
{[S_0,W_0]} & {[S_1,W_0]} \arrow[l, "\epsilon_1"'] & {[S_2,W_0]} \arrow[l, "\epsilon_2"'] & \cdots \arrow[l] & {[S_k,W_0]} \arrow[l, "\epsilon_k"'] \\
 & & & & {[K,W_0]} \arrow[ullll, "e_0^*"] \arrow[ulll, "e_1^*"] \arrow[ull, "e_2^*"] \arrow[u, equal, "e_k^*"']
\end{tikzcd}$$

in which every sequence $[S_{i-1},W_0] \leftarrow [S_i,W_0] \leftarrow [X_i,W_0]$ is exact and a central extension (by Lemma (2.11)). Furthermore, every homomorphism in this diagram is an operator homomorphism. The claim now follows by applying the proof of Lemma (2.12) successively to the sequences $[S_{i-1},W_0] \leftarrow [S_i,W_0] \leftarrow [X_i,W_0]$, beginning with $[S_0,W_0]$, and taking inverse images with respect to suitable e_i^*'s. □

Of particular interest is the question whether or not the groups $[S^k,\Omega X]$ are nilpotent, where X is a based CW-complex (cf. §5). For now, we offer a brief discussion of the H-nilpotency of the orthogonal groups O_n. This discussion is based on Corollary (3.5) in [V].

*<u>(2.19) Proposition</u> (i) If n is odd, $O_n \cong SO_n \times \mathbb{Z}_2$. Consequently, $[S^k,O_n] \cong [S^k,SO_n] \oplus \mathbb{Z}_2$ is abelian.

(ii) If $n > 0$ is even, $O_n \cong SO_n \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ acts nontrivially on SO_n. Furthermore, $[S^{n-1},O_n] \cong [S^{n-1},SO_n] \rtimes \mathbb{Z}_2$ is not nilpotent. Consequently, O_n is not H-nilpotent.

Proof (i) Let I_n denote the identity map on $\mathbb{R}^n$. Then $\mathbf{Z}_2$ is isomorphic to the multiplicative group $\{I_n,-I_n\}$. Since n is odd, $-I_n$ is orientation reversing. Therefore, the inclusion $s:\mathbf{Z}_2 \longrightarrow O_n$ is a section of the exact sequence $0 \to SO_n \to O_n \to \mathbf{Z}_2 \to 0$. Hence, $O_n \cong SO_n \rtimes \mathbf{Z}_2$. But the action of $\mathbf{Z}_2$ on SO_n via s is trivial because conjugation in SO_n by $-I_n$ is the identity on SO_n. Thus $O_n \cong SO_n \times \mathbf{Z}_2$.

(ii) We use results contained in Steenrod's book on fibre bundles [Ste2], §23. The tangent bundle of S^n is the vector bundle of n-planes associated with the principal fibre bundle $SO_n \to SO_{n+1} \xrightarrow{p} S^n$, where $p(u) = u(e_{n+1})$, $u \in SO_{n+1}$, e_i the i-th canonical basis vector in $\mathbb{R}^{n+1}$. Since S^n is a suspension, $S^n = SS^{n-1}$, this principal SO_n-fibre bundle has a 2-chart atlas giving rise to a single transition map $T_{n+1}:S^{n-1} \to SO_n$ whose homotopy class completely characterizes the isomorphism class of p.

Let r_n denote the reflection of $\mathbb{R}^n$ at the $\mathbb{R}^{n-1}\times\{0\}$ hyperplane. Then $\mathbf{Z}_2$ is isomorphic to the multiplicative group $\{I_n,r_n\}$. As in (i), the inclusion $\{I_n,r_n\} \longrightarrow O_n$ is a section of the exact sequence $0 \to SO_n \to O_n \to \mathbf{Z}_2 \to 0$. Hence, $O_n \cong SO_n \rtimes \mathbf{Z}_2$ so that $[S^{n-1},O_n] \cong [S^{n-1},SO_n] \rtimes \mathbf{Z}_2$.

Now, Lemma (23.4) in [Ste2] says that the homotopy class of T_{n+1} generates an infinite cyclic subgroup of $[S^{n-1},SO_n]$ and Lemma (23.11) in [Ste2] says that $\psi(SO_n)_{r_n}[T_{n+1}] = -[T_{n+1}]$. It follows that $\Gamma^c_{\psi(SO_n)}[S^{n-1},SO_n]$ contains a copy of $2^c\mathbf{Z}$ and, consequently, is not 0 for all $c \in \mathbb{N}_0$. Consequently, $[S^{n-1}, SO_n]$ is not $\psi(SO_n)$-nilpotent. Hence $[S^{n-1}, O_n]$ is not nilpotent. By Proposition (2.7), O_n is not H-nilpotent. $\square$

§3 Twist on the product of groups and the tensor product of certain algebras

In this chapter we develop the algebraic concepts that are needed to appropriately express the structure of certain functors in Algebraic Topology when applied to an H-semidirect product.

Specifically, let R be a commutative ring with 1, Π a group acting on an R-algebra A via a group homomorphism $\psi:\Pi \longrightarrow \mathrm{Aut}A$. Then the R-module $A\otimes_R R\Pi$ can be endowed with an R-algebra structure which reflects the action of Π on A. We shall show that this twisted R-algebra structure solves a certain universal problem generalizing the universal problem for the standard tensor product of R-algebras.

The multiplication in the twisted R-algebra structure on $A\otimes_R R\Pi$ formally resembles the multiplication in the semidirect product of groups. In order to clarify this resemblance, we shall begin our development by approaching the semidirect product of groups as the solution of a universal problem resembling the universal problem for the twisted tensor product of an R-algebra with a group algebra. We remark that the concept of a "crossed product algebra" [C-R] is equivalent to that of a "twisted tensor product".

Let G, Π, H be groups, Π acting on G via a homomorphism $\psi:\Pi \longrightarrow \mathrm{Aut}G$.

<u>(3.1) Definition</u> A function $f:G\times\Pi \longrightarrow H$ is ψ-twisted $:\Longleftrightarrow$ for all (g,p), $(g',p') \in G\times\Pi$, $f(g,p)f(g',p') = f(g\psi_p(g'),pp')$.

<u>(3.2) Definition</u> A group T is called the ψ-twisted product of G and Π $:\Longleftrightarrow$ there exists a ψ-twisted function $i:G\times\Pi \longrightarrow T$ such that for every group H and every ψ-twisted homomorphism $f:G\times\Pi \longrightarrow H$, there exists a unique group homomorphism $F:T \longrightarrow H$ making the following diagram commute.

$$\begin{tikzcd} T \arrow[r, dashed, "F"] & H \\ G \times \Pi \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

<u>(3.3) Theorem</u> The universal problem in Definition (3.2) has a solution, namely the usual semidirect product $G\rtimes_\psi\Pi$.

<u>(3.4) Remark</u> The terminology "ψ-twisted product" is merely an invention for use in this chapter. It is motivated by the program suggested above. We shall in general use the standard terminology "semidirect product".

<u>Proof of Theorem (3.3)</u> It is clear that the function $i:G\times\Pi \ni (h,p) \longmapsto (h,p) \in G\rtimes_\psi\Pi$ is a ψ-twisted function. Let $f:G\times\Pi \longrightarrow H$ be a ψ-twisted function. In order to have the diagram

$$\begin{tikzcd} G\rtimes_\psi\Pi \arrow[r, dashed, "F"] & H \\ G \times \Pi \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

commute, we are forced to set $F(g,p) = f(g,p)$ for all $(g,p) \in G\rtimes_\psi\Pi$.

Therefore, if F satisfying the conditions in (3.2) exists, it is unique. To see that F is indeed a group homomorphism, we check

$$\begin{aligned} F((g,p)\cdot(g',p')) &= F(g\psi_p(g'),pp') \\ &= F\circ i(g\psi_p(g'),pp') \\ &= f(g\psi_p(g'),pp') \\ &= f(g,p)f(g',p') \\ &= F(g,p)F(g',p'). \end{aligned}$$ □

Now let R be a commutative ring with 1, Π a group and A an R-algebra. Let $\mathrm{Aut}A$ denote the group of R-algebra automorphisms and suppose Π acts on A via a group homomorphism $\psi:\Pi \longrightarrow \mathrm{Aut}A$.

<u>(3.5) Definition</u> Let B be an R-algebra and $f:A\times R\Pi \longrightarrow B$ a function. We call f R-balanced with ψ-twist $:\Longleftrightarrow$

(i) f is R-balanced as a function of R-modules (i.e. f is additive in each variable and $f(ar,x) = f(a,rx)$ for all $a \in A$, $r \in R$, $x \in R\Pi$).

(ii) For all $a,a' \in A$, $x \in R\Pi$, $p \in \Pi$

$$f(a,1p)f(a',x) = f(a\psi_p(a'),1px).$$

If there is no risk of confusion we shall refer to an R-balanced function with ψ-twist merely as a balanced ψ-map.

<u>(3.6) Definition</u> An R-algebra T is called the ψ-twisted tensor product of A and $R\Pi$ over R $:\Longleftrightarrow$ there exists an R-balanced ψ-twisted function $i:A\times R\Pi \longrightarrow T$ such that for every R-algebra B and for every balanced ψ-map $f:A\times R\Pi \longrightarrow B$, there exists a unique

R-algebra homomorphism $F:T \longrightarrow B$ with the following diagram commutative.

$$\begin{tikzcd} T \arrow[r, "F"] & B \\ A \times R\Pi \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

<u>(3.7) Theorem</u> The universal problem in Definition (3.6) has a solution, namely the tensor product $A\otimes_R R\Pi$ endowed with the multiplication, defined by

$$(a\otimes_R 1p)(a'\otimes_R x) := a\psi_p(a')\otimes_R 1p\cdot x$$

where $a,a' \in A$, $p \in \Pi$, $x \in R\Pi$.

We shall denote the R-algebra described in Theorem (3.7) by $A \otimes_R^{\psi} R\Pi$. As usual, if an R-algebra T also solves the universal problem in Definition (3.6), then $T \cong A\otimes_R^{\psi} R\Pi$ as R-algebras.

<u>Proof of Theorem (3.7)</u> <u>Part 1</u> $A\otimes_R^{\psi} R\Pi$ is an R-algebra. Let us first spell out how the multiplication of certain elements in $A\otimes_R^{\psi} R\Pi$ extends to arbitrary elements in $A\otimes_R^{\psi} R\Pi$. So let

$$\xi = \sum_{i=1}^{n} a_i \otimes y_i , \qquad \eta = \sum_{j=1}^{m} b_j \otimes x_j \in A\otimes_R^{\psi} R\Pi.$$

Then $y_i = \sum_{p\in\Pi} r_p(y_i)p$, with $r_p(y_i) \in R$ and for all, but at most finitely many elements $p \in \Pi$, $r_p(y_i) = 0$. Thus

$$\xi = \sum_{i=1}^{n} a_i \otimes \left(\sum_{p\in\Pi} r_p(y_i)p\right) = \sum_{i=1}^{n} \sum_{p\in\Pi} (a_i r_p(y_i))\otimes 1p$$

and we set

$$\xi\cdot\eta = \Big(\sum_{i=1}^{n}\sum_{p\in\Pi}(a_i r_p(y_i))\otimes 1p\Big)\Big(\sum_{j=1}^{m} b_j\otimes x_j\Big)$$

$$:= \sum_{i=1}^{n}\sum_{p\in\Pi}\sum_{j=1}^{m}(a_i r_p(y_i)\otimes 1p)(b_j\otimes x_j)$$

$$= \sum_{i=1}^{n}\sum_{p\in\Pi}\sum_{j=1}^{m} a_i r_p(y_i)\psi_p(b_j)\otimes(1p)x_j .$$

This renders the multiplication in $A\otimes_R^{\psi}R\Pi$ biadditive. We must check the compatibility of this definition with the relations of the tensor product. Furthermore, the behaviour of the multiplication with respect to the scalar multiplication in the R-module structure needs to be checked.

To check compatibility with additive tensor relations of factors on the right hand side, we compute firstly,

$$\begin{aligned}(a\otimes 1p)(a'\otimes x_1 + a'\otimes x_2) &= a\psi_p(a')\otimes(1p)x_1 + a\psi_p(a')\otimes(1p)x_2\\ &= a\psi_p(a')\otimes(1p)(x_1+x_2)\\ &= (a\otimes 1p)(a'\otimes(x_1+x_2)) ,\end{aligned}$$

and secondly

$$\begin{aligned}(a\otimes 1p)(a_1\otimes x + a_2\otimes x) &= a\psi_p(a_1)\otimes(1p)x + a\psi_p(a_2)\otimes(1p)x\\ &= a(\psi_p(a_1)+\psi_p(a_2))\otimes(1p)x\\ &= a\psi_p(a_1+a_2)\otimes(1p)x\\ &= (a\otimes 1p)((a_1+a_2)\otimes x) .\end{aligned}$$

To check compatibility with additive tensor relations of factors on the left hand side, we compute

$$\begin{aligned}(a_1\otimes 1p + a_2\otimes 1p)(a\otimes x) &= a_1\psi_p(a)\otimes(1p)x + a_2\psi_p(a)\otimes(1p)x \\ &= (a_1+a_2)\psi_p(a)\otimes(1p)x \\ &= ((a_1+a_2)\otimes 1p)(a\otimes x)\ .\end{aligned}$$

On the other hand, $(a \otimes (1p+1p'))(a'\otimes x) = (a\otimes 1p + a\otimes 1p')(a'\otimes x)$ according to our definition of the multiplication in $A\otimes_R^{\psi}R\Pi$. For the same reason scalar multiple tensor relations of factors on the left are preserved.

To check compatibility of scalar multiple tensor relations of factors on the right, we compute

$$\begin{aligned}(a\otimes 1p)(a'\otimes rx) &= a\psi_p(a')\otimes(1p)(rx) \\ &= ar\psi_p(a')\otimes(1p)x \\ &= a\psi_p(ra')\otimes(1p)x \\ &= (a\otimes 1p)(a'r\otimes x)\ .\end{aligned}$$

As for the scalar multiplication in the R-module structure of $A\otimes_R^{\psi}R\Pi$ we have

$$\begin{aligned}r[(a\otimes 1p)(a'\otimes x)] &= r[a\psi_p(a')\otimes(1p)x] \\ &= r(a\psi_p(a'))\otimes(1p)x \\ &= (ra)\psi_p(a')\otimes(1p)x \\ &= (ra\otimes 1p)(a'\otimes x) \\ &= [r(a\otimes 1p)](a'\otimes x)\ .\end{aligned}$$

On the other hand,

$$(ra)\psi_p(a')\otimes(1p)x = ar\psi_p(a')\otimes(1p)x$$
$$= a\psi_p(ra')\otimes(1p)x$$
$$= (a\otimes 1p)(ra'\otimes x)$$
$$= (a\otimes 1p)[r(a'\otimes x)] \ .$$

Furthermore, the multiplication in $A\otimes_R^{\psi}R\Pi$ is associative. It suffices to check this for the following particular case. The case of arbitrary factors follows from this particular case and the compatibility results already proven.

$$[(a_1\otimes 1p)(a_2\otimes 1p')](a_3\otimes x) = (a_1\psi_p(a_2)\otimes 1pp')(a_3\otimes x)$$
$$= (a_1\psi_p(a_2)\psi_{pp'}(a_3)\otimes(1pp')x$$
$$= a_1(\psi_p(a_2)\psi_{pp'}(a_3))\otimes 1p(p'x)$$
$$= a_1\psi_p(a_2\psi_{p'}(a_3))\otimes 1p(p'x)$$
$$= (a_1\otimes 1p)(a_2\psi_{p'}(a_3)\otimes 1p'x)$$
$$= (a_1\otimes 1p)[(a_2\otimes 1p')(a_3\otimes x)] \ .$$

This completes the proof that $A\otimes_R^{\psi}R\Pi$ is an associative R-algebra.

Part 2 As a balanced ψ-map $i: A\times R\Pi \longrightarrow A\otimes_R^{\psi}R\Pi$, we take the map $i: A\times R\Pi \longrightarrow A\otimes_R^{\psi}R\Pi$ arising from the R-module construction. Then i is already known to be R-balanced and the computation

$$i(a,1_p)i(a',x) = (a\otimes 1p)(a'\otimes x)$$
$$= a\psi_p(a')\otimes(1p)x$$
$$= i(a\psi_p(a'),(1p)x)$$

confirms that i is also ψ-twisted.

Part 3 $i: A\times R\Pi \longrightarrow A\otimes_R^{\psi} R\Pi$ solves the universal problem of Definition (3.6). So let B be an R-algebra and $f: A\times R\Pi \longrightarrow B$ a balanced ψ-map. The universal property of the R-module $A\otimes_R R\Pi$ yields a unique R-module homomorphism $F: A\otimes_R R\Pi \longrightarrow B$ which makes the following diagram commute as R-modules.

$$\begin{tikzcd} A \otimes_R R\Pi \arrow[r, "F"] & B \\ A \times R\Pi \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

The following computation on generators shows that F is also a homomorphism of R-algebras.

$$\begin{aligned} F(a\otimes 1p)F(a'\otimes x) &= Fi(a,1p)Fi(a',x) \\ &= f(a,1p)f(a',x) \\ &= f(a\psi_p(a'),(1p)x) \\ &= Fi(a\psi_p(a'),(1p)x) \\ &= F(a\psi_p(a')\otimes(1p)x) \\ &= F((a\otimes 1p)(a'\otimes x)). \end{aligned}$$

This completes the proof of Theorem (3.7) □

(3.8) Remark One might attempt to generalize the concept of a ψ-twisted product to the situation where A and S are R-algebras and S acts on A by R-algebra homomorphisms via an R-algebra homomorphism $\psi: S \longrightarrow \mathrm{End}A$, where $\mathrm{End}A$ denotes the endomorphism algebra of A. One would then define $A\otimes_R^{\psi} S$ as the R-module $A\otimes_R S$

with multiplication defined on generators by

$$(a\otimes s)(a'\otimes s') := a\psi_s(a')\otimes ss'.$$

This, however, does not always work because, for $r \in R$,

$$\begin{aligned} r(a\otimes s)(a'\otimes s') &= (a\otimes rs)(a'\otimes s') \\ &= a\psi_{rs}(a')\otimes rss' \\ &= r(ar\psi_s(a')\otimes ss') \\ &= r^2(a\psi_s(a')\otimes ss') \\ &= r^2(a\otimes s)(a'\otimes s'). \end{aligned}$$

This is, in general, a contradiction. When defining $A\otimes_R^\psi R\Pi$ on particular elements, our approach was, from a technical point of view, motivated by the observation above.

We can now make the resemblance between the ψ-twisted product of groups and the ψ-twisted tensor product of an R-algebra A with a group algebra $R\Pi$ explicit. Observe that the balanced ψ-map $i: A\times R\Pi \longrightarrow A\otimes_R^\psi R\Pi$ of Theorem (3.7) factors through $A\otimes_R R\Pi$ with the R-balanced map $j: A\times R\Pi \longrightarrow A\otimes_R R\Pi$ followed by the identity function $\mathrm{Id}: A\otimes_R R\Pi \longrightarrow A\otimes_R^\psi R\Pi$.

$$\begin{tikzcd} A \otimes_R R\Pi \arrow[r, "\mathrm{Id}"] & A \otimes_R^\psi R\Pi \\ & A \times R\Pi \arrow[ul, "j"] \arrow[u, "i"'] \end{tikzcd}$$

Give $A\otimes_R R\Pi$ the usual R-algebra structure. Then Id is an isomorphism of R-modules which behaves with respect to multiplication (formally) like a ψ-twisted function of groups (cf. Definition (3.1)).

The following lemma helps to recognize a ψ-twisted tensor product of an R-algebra with a group algebra.

<u>(3.9) Lemma</u> Let A, B be R-algebras and suppose

$$A \xrightarrow{\ \alpha\ } B \underset{s}{\overset{q}{\rightleftarrows}} R\Pi$$

is a (not necessarily exact) diagram of R-algebras and R-algebra homomorphisms such that $qs = \mathrm{Id}_{R\Pi}$. In this situation, the function

$$t:A\times R\Pi \ni (a,x) \longmapsto \alpha(a)s(x) \in B$$

is R-balanced and, therefore, induces a unique homomorphism of R-modules $T':A\otimes_R R\Pi \longrightarrow B$. Suppose that

(i) T' is an isomorphism of R-modules.

(ii) There exists a homomorphism $\psi:\Pi \longrightarrow \mathrm{Aut}A$ such that

$$t(a,1p)t(a',x) = t(a\psi_p(a'),(1p)x)$$

for all $a,a' \in A$, $x \in R\Pi$, $p \in \Pi$.

Then $B \cong A\otimes_R^{\psi} R\Pi$.

<u>Proof</u> Property (ii) and the universal properties of ψ-twisted tensor products provide us with a homomorphism $T:A\otimes_R^{\psi} R\Pi \longrightarrow B$ of R-algebras making the following diagram commute.

$$\begin{tikzcd}
A \otimes_R^{\psi} R\Pi \arrow[rr, "T"] & & B \\
A \times R\Pi \arrow[u, "i"] \arrow[urr, "t"'] & &
\end{tikzcd}$$

This diagram may be extended as follows.

$$\begin{tikzcd}
A \otimes_R R\Pi \arrow[rrrr, "T'"] & & & & B \\
& & A \otimes_R^{\psi} R\Pi \arrow[ull, "Id"'] \arrow[urr, "T"] & & \\
& & A \times R\Pi \arrow[uull, "j"] \arrow[u, "i"] \arrow[uurr, "t"'] & &
\end{tikzcd}$$

We already know that the functions Id, T' are isomorphisms of R-modules. Consequently, T is also an isomorphism of R-modules. But T is also an R-algebra homomorphism and, therefore, an R-algebra isomorphism. □

The following Lemma (3.10) gives conditions under which a ψ-twisted tensor product $A\otimes_R^{\psi}R\Pi$ allows for a sequence of R-algebra homomorphisms as in (3.9).

<u>(3.10) Lemma</u> Let A be an R-algebra with 1. Suppose $R \ni r \mapsto r1 \in A$ is a monomorphism and that there is an ideal A' of A such that $A = R1 \oplus A'$ (internal direct sum) as an R-module. Thus $a \in A$ can be written uniquely as $a = (r,a')$ with $r \in R$, $a' \in A'$. Let Π be a group acting on A via $\psi:\Pi \longrightarrow \mathrm{Aut}A$ by automorphisms having $R1$ and A' as invariant submodules. Then

(i) $\tilde{q}: A \times R\Pi \ni ((r,a'),x) \mapsto rx \in R\Pi$ is ψ-balanced and, hence, induces a unique homomorphism $q: A \otimes_R^{\psi} R\Pi \longrightarrow R\Pi$ of R-algebras.

(ii) The sequence $A \xrightarrow{\alpha} A\otimes_R^{\psi} R\Pi \underset{s}{\overset{q}{\rightleftarrows}} R\Pi$ with $\alpha(a) = a \otimes 1_R 1_{\Pi}$ and $s(x) = 1 \otimes x$ satisfies the requirements of Lemma (3.9).

(iii) The action of Π on A can be recovered from the multiplication in $A \otimes_R^{\psi} R\Pi$ by the identity $\psi_p(a) = \alpha^{-1}(s(p)\alpha(a)s(p^{-1}))$ for all $a \in A$, $p \in \Pi$.

<u>Proof</u> (i) $\tilde{q}$ is obviously R-balanced. To see that $\tilde{q}$ is ψ-twisted, we check

$$\tilde{q}((r_1,a_1'),1p_1)\tilde{q}((r_2,a_2'),sp_2) = (r_1p_1)(r_2sp_2) = r_1r_2s(p_1p_2)$$

and

$$\begin{aligned}\tilde{q}((r_1,a_1')\psi_{p_1}(r_2,a_2'),1p_1sp_2) &= \tilde{q}((r_1,a_1')(r_2,\psi_{p_1}(a_2'),sp_1p_2) \\ &= \tilde{q}((r_1r_2,r_1\psi_{p_1}(a_2')+r_2a_1'+a_1\psi_{p_1}(a_2')),sp_1p_2) \\ &= r_1r_2s(p_1p_2)\ .\end{aligned}$$

Visibly, the results are the same. For more general factors we use this result and the fact that $\tilde{q}$ is R-balanced.

(ii) We need only check that $qs = \mathrm{Id}_{R\Pi}$. But for $x \in R\Pi$,

$$qs(x) = q(1\otimes x) = \tilde{q}((1,0),x) = x.$$

(iii) Follows from the computation

$$\begin{aligned} s(p)\alpha(a)s(p^{-1}) &= (1_A\otimes 1_R p)(a\otimes 1_R 1_\Pi)(1_A\otimes 1_R p^{-1}) \\ &= (\psi_p(a)\otimes 1_R p)(1_A \otimes 1_R p^{-1}) \\ &= \psi(a)\psi_p(1_A)\otimes 1_R(pp^{-1}) \\ &= \psi_p(a)1_A\otimes 1_R 1_\Pi \\ &= \alpha(\psi_p(a)) . \end{aligned} \qquad \square$$

The reader may now compare the statements of Lemmas (3.9) and (3.10) with the following fact for groups.

A group extension $G \longrightarrow E \longrightarrow \Pi$ has a section $s:\Pi \longrightarrow E$ so that Π acts on G via $p\cdot g = s(p)gs(p^{-1})$ if and only if $E \cong G\rtimes_\psi \Pi$ and $\psi_p(g)$ = first coordinate of $(1,p)(g,1)(1,p^{-1})$ for all $g \in G$, $p \in \Pi$.

§4 The Pontryagin homology algebra of an H-semidirect product

If X is an H-space, the graded singular homology group H_*X inherits a product structure from X turning H_*X into a graded ring, called the Pontryagin homology ring of X. If X has several path-connected components, additivity of singular homology says that H_*X, as a graded group, is isomorphic to the direct sum of the graded homology groups of the various path components of X. In general, however, H_*X as a graded ring will not decompose in any obvious way.

If $W = W_0 \rtimes_\phi \Pi$ is an H-semidirect product, Π acts on W_0 by classes of H-self homotopy equivalences and, therefore, on H_*W_0 by graded ring isomorphisms. We shall show that H_*W, as a graded ring, is isomorphic to a twisted tensor product of H_*W_0 with $H_*\Pi$, where $H_*\Pi$ is the Pontryagin ring of Π viewed as a discrete topological group (not to be confused with group homology; note that $H_*\Pi \cong R\Pi$ viewed as a graded R-algebra concentrated at degree 0).

So let Π be a group acting on a path connected H-group (W_0,μ_0) via a homomorphism $\phi:\Pi \longrightarrow H\mathcal{E}(W_0)$, and denote by (W,μ) the corresponding H-semidirect product $W_0 \rtimes_\phi \Pi$ (cf. (1.3)). Let R be a commutative ring with 1. All homology groups, unless stated otherwise, will be with coefficients in R. Consequently, H_*X is a graded R-module for any space X. In particular H_*W is a graded R-module.

Let us recall the definition of the multiplication in H_*W. The conceptually easiest approach uses cubical singular homology.

The key construction is the cross product in singular homology. Let X, Y be spaces and let $1u \in C_p(X;R)$, $1v \in C_q(Y;R)$ be generators with $u:I^p \longrightarrow X$, $v \in I^q \longrightarrow Y$ singular cubes. Then the map $u\times v:I^p\times I^q \longrightarrow X\times Y$ is a singular $(p + q)$-cube in $X\times Y$ and, hence, yields a generator $(1\otimes_R 1)u\times v \in C_{p+q}(X\times Y;R\otimes_R R) \cong C_{p+q}(X\times Y;R)$. This construction turns out to be compatible with the formation of homology classes and yields an R-balanced function $H_*X\times H_*Y \longrightarrow H_*(X\times Y)$. This function induces the cross product homomorphism $\times:H_*X\otimes_R H_*Y \longrightarrow H_*(X\times Y)$.

If (X,m) is an H-space, we get an R-homomorphism $H_*X\otimes_R H_*X \xrightarrow{\times} H_*(X\times X) \xrightarrow{m_*} H_*X$ turning H_*X into a graded R-algebra. Furthermore, if X,Y are H-spaces and $f:X \longrightarrow Y$ is an H-map, f induces a homomorphism of graded R-algebras.

<u>(4.1) Lemma</u> π acts on H_*W_0 by automorphisms of the graded R-algebra via $\psi:\pi \ni p \longmapsto \phi(p)_* \in \mathrm{Aut}\, H_*W_0$.

The proof is trivial. $\square$

<u>(4.2) Theorem</u> $H_*W \cong H_*W_0\otimes_R^{\psi} H_*\pi$.

<u>Proof</u> Consider the "exact" sequence of H-maps

$$\{(e,1)\} \longrightarrow W_0\times\{1\} \xhookrightarrow{\propto} W_0\rtimes_\phi \pi \underset{s}{\overset{q}{\rightleftarrows}} \pi \longrightarrow \{1\}$$

admitting a section $s:\pi \ni p \longmapsto (e,p) \in W_0\rtimes\pi$. The computation

$$\begin{tikzcd}
p \cdot p' \arrow[d, mapsto] \arrow[d, mapsto] \arrow[rrr, equal] & & & (pp') \arrow[d, mapsto] \\
(e,p)(e,p') \arrow[r, equal] & (e\varphi_p(e),pp') \arrow[rr, "\approx"] & & (e,pp')
\end{tikzcd}$$

shows that s is also an H-map. The homotopy in the above diagram can be constructed from any path joining $e\varphi_p(e)$ to e.

An H-map between two H-spaces induces a homomorphism of the Pontryagin homology algebras. Therefore, the "exact" sequence of H-maps above induces the following diagram of homology algebras.

$$\begin{tikzcd}
H_*W_0 \arrow[r, "\alpha_*"] & H_*W_0 \rtimes_\phi \Pi \arrow[r, shift left, "q_*"] & H_*\Pi \arrow[l, shift left, dashed, "s_*"]
\end{tikzcd}$$

Since α_* is induced by the map identifying W_0 with the path-connected component $W_0 \times \{1\}$ of $W_0 \rtimes \Pi$, α_* is a monomorphism by additivity of singular homology. Since $qs = Id_\Pi$, $q_*s_* = Id_{H_*\Pi}$. We shall now invoke Lemma (3.9) to complete the proof of Theorem (4.2).

First of all, let us view $R\Pi$ as a graded R-algebra whose only non zero term sits at dimension 0. Then $R\Pi \cong H_*\Pi$ under the isomorphism induced by the function on generators $\Pi \ni p \mapsto 1\tilde{p} \in H_0\Pi$. Here $\tilde{p}$ denotes the unique map $I^0 \longrightarrow \{p\}$ and, by slight abuse of notation, $1\tilde{p}$ the homology class represented by the (properly denoted) element $1\tilde{p}$ of $C_0\Pi$. This function establishes a bijection between the canonical bases of R-modules $R\Pi$ and $H_0\Pi$ and, hence, induces an isomorphism $\rho: R\Pi \longrightarrow H_0\Pi$. To see that ρ also preserves multiplication, consider the following computation on generators

$$\begin{array}{ccc} 1p \cdot 1p' & = & 1p \\ \Big\downarrow \rho \Big\downarrow & & \Big\downarrow \\ 1\tilde{p} \cdot 1\tilde{p} & = & 1\widetilde{pp} \end{array}$$

where $\widetilde{pp}'$ is identified with $I^0 \times I^0 \xrightarrow{\tilde{p} \times \tilde{p}'} \Pi \times \Pi \xrightarrow{m} \Pi$ via the unique homeomorphism $I^0 \equiv I^0 \times I^0$. Since the higher homology groups of Π are all 0, ρ is indeed an isomorphism of graded R-algebras.

Following the set up of Lemma (3.9) we get a homomorphism $T': H_*W_0 \otimes_R H_*\Pi \longrightarrow H_*W$ of R-modules defined on generators by $\tilde{a} \otimes \tilde{x} \longmapsto \alpha_*(\tilde{a}) s_*(\tilde{x})$. We are left to verify

(i) T' is an isomorphism of graded R-modules.

(ii) For all $\tilde{a}, \tilde{a}' \in H_*W_0$, $\tilde{x} \in H_*\Pi$, $p \in \Pi$ we have the identity

$$T'(\tilde{a} \otimes 1\tilde{p}) T'(\tilde{a}' \otimes \tilde{x}) = T'[\tilde{a} \psi_p(\tilde{a}') \otimes (1\tilde{p}) \tilde{x}] .$$

Both statements appear plausible upon inspection of the underlying situation on the level of function values of maps for singular simplices. The following observation will help us to transform this idea into a formal proof of (i) and (ii).

<u>(4.3) Observation</u> Let $c_1: C_*W_0 \otimes_R C_0\Pi \longrightarrow C_*(W_0 \rtimes \Pi)$ denote the chain map defined on generators by $1u \otimes 1p \longmapsto 1(u,p)$. Here (u,p) denotes the singular n-cube $(u,p): I^n \ni t \longmapsto (u(t), p(t)) \in W_0 \rtimes \Pi$, where $p: I^n \longrightarrow \{p\}$ denotes the unique map. Identifying I^n with $I^n \times I^0$, we see that c_1 coincides with the cross-product map sending $1u \otimes 1p \longmapsto 1u \times p$ where $u \times p$ denotes the singular n-cube $u \times p: I^n \times I^0 \ni (t,0) \longmapsto (u(t), p) \in W_0 \rtimes \Pi$.

Let $\tau_e : W_0 \ni w \mapsto w \cdot e \in W_0$ and suppose that in the definition of μ, $\phi(1)$ is represented by Id_{W_0}. Now consider $c_2 := c_1(\tau_{e\#} \otimes_R Id)$: $C_* W_0 \otimes C_0 \Pi \longrightarrow C_*(W_0 \rtimes \Pi)$. Then $c_2(1u \otimes 1p)$ is represented by the map

$$\begin{aligned} I^n \ni t \mapsto (u(t)e,p) &= (u(t)\psi_1(e),p) \\ &= (u(t),1)(e,p) \\ &= \mu(\alpha u(t),sp) \in W_0 \rtimes \Pi . \end{aligned}$$

Since τ_e is homotopic to Id_{W_0}, c_1 and c_2 are chain homotopic and, therefore, induce the same map in homology.

<u>Verification of (i)</u> c_1 gives rise to the Künneth-map $k_* : H_* W_0 \otimes_R H_* \Pi \longrightarrow H_*(W_0 \rtimes \Pi)$. Because of the particular structure of $H_* \Pi$, k_* is an isomorphism of graded R-modules. On the other hand, c_2 induces T'. Since c_1 and c_2 are chain homotopic, this shows that $T' = k_*$. Hence T is an isomorphism of graded R-modules.

<u>Verification of (ii)</u> Since $\tilde{x} = r_1\tilde{p}_1 + \ldots + r_k\tilde{p}_k$, we get

$$\tilde{a}' \otimes \tilde{x} = r_1 \tilde{a}' \otimes 1\tilde{p}_1 + \ldots + r_k \tilde{a}' \otimes 1\tilde{p}_k .$$

Using distributivity in $H_* W$ it suffices to prove (ii) when $\tilde{x} = 1\tilde{p}'$. Suppose now that $a = r_1 u_1 + \ldots + r_k u_k$ and that $a' = r_1' u_1' + \ldots + r_\ell' u_\ell'$. Since $T' = k_*$, $T'(\tilde{a} \otimes 1\tilde{p})$ is represented by $r_1(u_1 \times p) + \ldots + r_k(u_k \times p)$ and $T'(\tilde{a}' \otimes 1_{\tilde{p}'})$ is represented by $r_1'(u_1' \times p) + \ldots + r_k'(u_k' \times p')$. Consequently, $T'(\tilde{a} \otimes 1\tilde{p})T'(\tilde{a}' \otimes 1\tilde{p}')$ is represented by

$$\begin{aligned}\sum_{i=1}^{k}\sum_{j=1}^{\ell} r_i r_j' \mu^\circ((u_i\times p)\times(u_j'\times p')) &= \sum_{i=1}^{k}\sum_{j=1}^{\ell} r_j r_j' \mu(u_i\times p, u_j'\times p') \\ &= \sum_{i=1}^{k}\sum_{j=1}^{\ell} r_i r_j' (\mu_0(u_i, \varphi_p u_j'), pp') \\ &= \sum_{i=1}^{k}\sum_{j=1}^{\ell} r_i r_j' (\mu_0{}^\circ(u_i\times\varphi_p u_j)\times pp') \\ &= c_1\left[\sum_{i=1}^{k}\sum_{j=1}^{\ell} r_i r_j' \mu_0{}^\circ(u_i\times\varphi_p u_j)\otimes 1pp'\right] \\ &= c_1\left\{\mu_{0\#}\left[\left[\sum_{i=1}^{k} r_i u_i\right]\times\left[\sum_{j=1}^{\ell} r_j'\varphi_p u_j'\right]\otimes 1pp'\right]\right. \\ &= c_1\left\{\mu_{0\#}\left[\sum_{i=1}^{k} r_i u_i\times\left\{\varphi_{p\#}\left[\sum_{j=1}^{\ell} r_j' u_j'\right]\right]\right]\otimes 1pp'\right].\end{aligned}$$

The argument of c_1 is just a representative of $\tilde{a}\varphi_p(\tilde{a}')\otimes 1\widetilde{\widetilde{pp}}'$ and T' is induced by c_1 on the chain level. Hence (ii) is verified which completes the proof of Theorem (4.2). □

Let $R = \mathbf{Z}$ from now on and let us look at the Hurewicz homomorphism $h_n:\pi_n(W_0,e) \longrightarrow H_nW_0$. To recall the construction, denote by σ_0 the generator of $H_0(S^0,\{1\})$ represented by the $1u$, where $u:\Delta_0 \longrightarrow \{-1\}$ is the unique map. Then the n-fold suspension $\sigma_n = S^n\sigma_0$ determines a generator of H_nS^n. If $(X,*)$ is a based space we get a natural transformation

$$h_n:\pi_n(X,*) \ni [f] \mapsto f_*\sigma_n \in H_nX.$$

<u>(4.4) Proposition</u> Let $W = W_0\rtimes_\phi\Pi$ be an H-semidirect product. Then $h_n:\pi_n(W_0,e) \longrightarrow H_nW_0$ $(n \geq 0)$ is an operator homomorphism with respect to the actions of Π on $\pi_n(w_0,e)$ and H_nW_0 induced by $\phi:\Pi \longrightarrow H\mathcal{E}(W_0)$.

Proof For $p \in \Pi$ and $[f] \in \Pi_n(W_0,e)$ we get

$$\begin{aligned} h_n(p\cdot[f]) &= h_n(\phi(p)[f]) \\ &= (\phi(p)[f])_*\sigma_n \\ &= \phi(p)_*([f]_*\sigma_n) \\ &= p\cdot h_n[f]. \end{aligned}$$ □

(4.5) Corollary If $\pi_n(W_0,e)$ is nilpotent with respect to the action of Π on $\pi_n(W_0,e)$ then $\operatorname{im} h_n$ is nilpotent with respect to the action of Π on H_nW_0. □

Finally we remark that, if desired, the question whether or not Π acts nilpotently on $\operatorname{im} h_n$ can be considered completely within H_*W. The transition is accomplished by making use of Lemma (3.10) after observing that H_*W_0 has a multiplicative identity in H_0W_0.

§5. Nilpotency of CW-complexes and H-semidirect products

Bousfield, Kan [B-K] and Hilton, Mislin, Roitberg [H-M-R] give the following definition of nilpotency of a connected based CW-complex.

(5.1) Definition X is nilpotent $:\Longleftrightarrow \pi_1(X,*)$ is nilpotent and acts nilpotently on $\pi_n(X,*)$ for $n \geq 2$.

Under the adjointness isomorphism $\pi_n(X,*) \cong \pi_{n-1}(\Omega X,*)$, this requirement for nilpotency of X translates into the following equivalent one.

(5.1)' Definition X is nilpotent $:\Longleftrightarrow \pi_1(X,*)$ is nilpotent and the action of $\pi_1(X,*)$ on $\pi_{n-1}(\Omega X,*)$ by loop conjugation is nilpotent.

Denoting the path component of the constant path $* \in \Omega X$ by $(\Omega X)_0$, we get canonical isomorphisms $\pi_{n-1}(\Omega X,*) \cong \pi_{n-1}((\Omega X)_0,*) \cong [S^{n-1},(\Omega X)_0]$ and the action of $\pi_1(X,*)$ on $[S^{n-1},(\Omega X)_0]$ of Definition (5.1)' coincides with the action of $\pi_1(X,*)$ on $[S^{n-1},(\Omega X)_0]$ arising from the H-semidirect product structure on ΩX (denoted by $\psi(S^{n-1})$ in §2). Thus our results in §2 lead to yet another characterization of nilpotency of X: X is nilpotent if and only if $[S^n,\Omega X]$ is nilpotent for all $n \geq 1$.

The main objective of this section is to give an explicit development of this characterization and then to use results in the previous chapters to provide additional tools for investigating a connected CW-complex for nilpotency.

It should be noted, however, that this approach to nilpotency of CW-complexes is not entirely new. Roitberg [R] utilized the characterization (5.1)' of nilpotency in the following way.

Given a countable connected based simplicial complex X, a result of Milnor's [M1] guarantees the existence of a topological group T having the homotopy type of ΩX. Furthermore, T is the fiber of a principal T-fiber bundle map $E \longrightarrow X$, where T and E are countable CW-complexes. Also E is contractible so that X is, up to homotopy equivalence, the classifying space of T. Based on these facts, Roitberg uses T for ΩX in Definition (5.1)' to characterize nilpotency of countable connected based simplicial complexes.

At this level of consideration, Roitberg benefits from his approach in the following result.

*<u>(5.2) Theorem</u> Classifying spaces of nilpotent Lie groups are nilpotent.

<u>Proof</u> If T is nilpotent as a group, the requirements of (5.1)' are satisfied (compare Proposition (2.7) of this thesis). □

We shall now embark on an explanation why (5.1) and (5.1)' are equivalent. This is implicit in [R]. We shall actually prove a more general result.

Let $(D,*)$ be a based well pointed space (i.e. the inclusion $* \hookrightarrow D$ has the homotopy extension property) and let $(X,*)$ be a based space. Then $\pi_1(X,*)$ acts on $[(D,*),(X,*)]$ on the left as follows. If $f:(D,*) \longrightarrow (X,*)$ is a based map and $\psi:(I,\{0,1\}) \longrightarrow (X,*)$ is a loop, we obtain a map $\gamma.f$ out of the following process.

(i) Let $F: D \times I \longrightarrow X$ be a homotopy extension of the data $F(d,0) = f(d)$ for all $d \in D$ and $F(*,t) = \gamma^{-1}(t) = \gamma(1-t)$ for all $t \in T$.

(ii) $\gamma.f(d) := F(d,1)$.

It turns out that choosing different homotopy extensions of the data in (i) and varying γ and f in their homotopy classes only varies $\gamma.f$ in its homotopy class. Consequently, we obtain a well defined function

$$\theta : \pi_1(X,*) \times [(D,*),(X,*)] \ni ([\gamma],[f]) \longmapsto [\gamma.f] \in [(D,*),(X,*)] .$$

Furthermore, $\theta([\gamma_1],[\gamma_2]) = \theta[\gamma_1]\theta[\gamma_2]$ as a function from $[(D,*),(X,*)]$ into itself. Hence, θ describes an action of $\pi_1(X,*)$ on $[(D,*),(X,*)]$.

Now let $(A,*)$ be a connected CW-complex with a 0-cell as base point. Then $(SA,*) =: (D,*)$ is well pointed. Let θ denote the action of $\pi_1(X,*)$ on $[(SA,*),(X,*)]$ explained above.

On the other hand, using adjointness of suspension and loop functors we have the action,

$$\begin{tikzcd}
\theta' : \pi_1(X,*) \times [(SA,*),(X,*)] \arrow[r] \arrow[d, "\cong"'] & {[(SA,*),(X,*)]} \\
\pi_1(X,*) \times [(A,*),(\Omega X,*)_0] \arrow[r] & {[A,(\Omega X)_0] \cong [(A,*),(\Omega X,*)_0]} \arrow[u, "\cong"']
\end{tikzcd}$$

defined by going through the bottom part of the diagram above. Explicitly, let $([\gamma],[f]) \in \pi_1(X,*) \times [(SA,*),(X,*)]$. Let

$\overline{f}:(A,*) \longrightarrow (\Omega X,*)$ denote the adjoint of f. For $g:(A,*) \longrightarrow (\Omega X,*)$, let $\underline{g}:(SA,*) \longrightarrow (X,*)$ denote the adjoint of g. Observe that $\underline{(\overline{f})} = f$ and $\overline{(\underline{g})} = g$. Then $\theta'([\gamma],[f])$ is represented by the composite of maps

$$(\gamma,f) \longmapsto (\gamma,\overline{f}) \longmapsto \gamma\overline{f}\gamma^{-1} \longmapsto g \longmapsto \underline{g}\ ,$$

where $g:(A,*) \longrightarrow (\Omega X,*)$ is defined as follows.

The map $\gamma\overline{f}\gamma^{-1}:A \ni a \mapsto \gamma\overline{f}(a)\gamma^{-1} \in (\Omega X)_0$ is given by loop conjugation and will in general not be based. Using the homotopy extension property of the inclusion $* \rightarrowtail A$, we get a based map $g:(A,*) \longrightarrow (\Omega X,*)_0$ which is freely homotopic to $\gamma\overline{f}\gamma^{-1}$. Since $(\Omega X)_0$ is a simple space, $[g]$ and subsequently $[\underline{g}] =: \theta'([\gamma],[f])$ are uniquely determined by $\gamma\overline{f}\gamma^{-1}$.

<u>(5.3) Lemma</u> $\theta = \theta':\pi_1(X,*)\times[(SA,*),(X,*)] \longrightarrow [(SA,*),(X,*)]$.

<u>Proof</u> In the notation of the introductory explanations above, we shall show that $[\underline{g}] = [\overline{\gamma.f}]$ proceeding along the following steps.

<u>Step 1</u> is based on the following observation. If $(B,*)$ is a well pointed space, then the data $F(b,0) = (b,0)$, $F(*,t) = (*,t)$ have a homotopy extension $F:B \times I \longrightarrow (B \times 0) \cup (* \times I)$. Given F, we have a uniform way of constructing homotopy extensions for the inclusion $\{*\} \rightarrowtail B$. If $f:(B,*) \longrightarrow (Z,*)$, $a:(* \times I,(*,0)) \longrightarrow (Z,*)$ are maps, then (∇ denotes the folding map)

$$B\times I \xrightarrow{\quad F \quad} (B\times 0) \cup (*\times I) \xrightarrow{\quad f\vee a \quad} Z\vee Z \xrightarrow{\quad \nabla \quad} Z$$

is a homotopy extension of the data f, a.

Therefore, we shall fix suitable homotopies

$$R: A\times I \longrightarrow (A\times 0)\cup(*\times I)\ , \qquad R(a,t) = (\rho_1(a,t),\rho_2(a,t))$$
$$S: S^1\times I \longrightarrow (S^1\times 0)\cup(*\times I)\ , \qquad S(s,t) = (\sigma_1(s,t),\sigma_2(s,t))$$

to construct a homotopy

$$T: SA\times I \longrightarrow (SA\times 0)\cup(*\times I)\ , \quad T(s\wedge a,t) = (\tau_1(s\wedge a,t),\tau_2(s\wedge a,t))$$

and construct homotopy extensions of A, S^1, SA using R, S, T as explained above.

<u>Step 2</u> Compute g and $\overline{\gamma.f}$.

<u>Step 3</u> Show that g is homotopic to $\overline{\gamma.f}$.

<u>On step 1</u> For every $n \in \mathbb{N}_0$, the inclusion $S^n \hookrightarrow B^{n+1}$ has the homotopy extension property. Using the cell structure of A, this is the key to constructing a homotopy $R: A\times I \longrightarrow (A\times 0)\cup(*\times I)$ such that $R(*,t) = (*,t)$ for all $t \in I$ and $\rho_2(a,t) < \rho_2(*,t)$ for all $a \in A = \{*\}$ and all t, $0 < t \le 1$, and $R(a,0) = a$ for all $a \in A$.

For $(S^1,*)$ we exhibit a homotopy $S: S^1\times I \longrightarrow (S^1\times 0)\cup(*\times I)$ with these properties explicitly. Using the homeomorphism $S^1 \equiv I/\{0,1\}$, define

$$S(s,t) := \begin{cases} (*,t-(1+2t)s) & 0 \le s \le \dfrac{t}{1+2t} \\[2ex] ((1+2t)s-t,0) & \dfrac{t}{1+2t} \le s \le \dfrac{1+t}{1+2t} \\[2ex] (*,(1+2t)s-1-t) & \dfrac{1+t}{1+2t} \le s \le 1 \end{cases} .$$

$S(.,t)$ pulls the base point of S^1 out to $(*,t)$ under uniform stretching of S^1. In fact, the construction of S shows how to construct R on the 1-skeleton of A.

Now define $T: SA\times I \longrightarrow (SA\times 0)\cup(*\times I)$ by

$$T(s\wedge a,t) := (\sigma_1(s,t)\wedge\rho_1(a,t), \max\{\sigma_2(s,t),\rho_2(a,t)\}).$$

To see that τ_2 is continuous, we show that the function $\tau_2': S^1\times A\times I \ni (s,a,t) \longmapsto \max\{\sigma_2(s,t),\rho_2(a,t)\} \in I$ is continuous and factors through $(S^1\wedge A)\times I$ with τ_2. Continuity of τ_2' can be seen by using the continuous difference function $\delta: S^1\times A\times I \ni (s,a,t) \longmapsto \sigma_2(s,t) - \rho_2(a,t) \in \mathbb{R}$ which yields two closed subsets $\delta^{-1}(\mathbb{R}^{\geq 0})$, $\delta^{-1}(\mathbb{R}^{\leq 0})$ of $S^1\times A\times I$ over which τ_2' is defined by σ_2, ρ_2 respectively.

<u>On Step 2</u> Let $\gamma \in \Omega X$ and $f:(SA,*) \longrightarrow (X,*)$ be as above. Then $\gamma\cdot f:(SA,*) \longrightarrow (X,*)$ is given by

$$\gamma\cdot f(s\wedge a) = \begin{cases} f\tau_1(s\wedge a,1) & \text{if } T(s\wedge a,1) \in SA\times 0 \\ \gamma(1-\tau_2(s\wedge a,1)) & \text{if } T(s\wedge a,1) \in *\times I \end{cases}.$$

Consequently, if $a \in A$, then $\overline{\gamma\cdot f}(a)$ is the loop

$$S^1 = I/\{0,1\} \ni s \longmapsto \left\{ \begin{array}{ll} \gamma(3s) & 0 \leq s \leq \frac{1}{3} \\ \bar{f}R(a,1)(3(s-\frac{1}{3})) & \frac{1}{3} \leq s \leq \frac{2}{3} \\ \gamma(1-3(s-\frac{2}{3})) & \frac{2}{3} \leq s \leq 1 \end{array} \right\} \in X$$

in case $R(a,1) \in A\times 0$. If $R(a,1) = (*,t) \in *\times I$, then $\overline{\gamma.f}(a)$ is the loop

$$S^1 = I/\{0,1\} \ni s \mapsto \left\{ \begin{array}{ll} \gamma(3s) & 0 \le s \le \frac{1}{3}(s-t) \\ \gamma(1-t) & \frac{1}{3}(1-t) \le s \le 1 - \frac{1}{3}(1-t) \\ \gamma(1-3s) & 1 - \frac{1}{3}(1-t) \le s \le 1 \end{array} \right\} \in X .$$

The function values of the loop conjugated map $\gamma\overline{f}\gamma^{-1}$ actually depend on how we bracket this product of three elements. Regardless of how this product is bracketed, there is an easily constructed homotopy $A\times I \longrightarrow (\Omega X)_0$ which allows us to assume that for $a \in A$, $\gamma\overline{f}(a)\gamma^{-1}$ is the loop

$$S^1 \ni s \mapsto \left\{ \begin{array}{ll} \gamma(3s) & 0 \le s \le \frac{1}{3} \\ \overline{f}(a)(3(s - \frac{1}{3})) & \frac{1}{3} \le s \le \frac{2}{3} \\ \gamma(1 - 3(s - \frac{2}{3})) & \frac{2}{3} \le s \le 1 \end{array} \right\} \in X .$$

In particular, $\gamma\overline{f}(*)\gamma^{-1}$ is the loop γ at triple speed for $s \in \left[0 , \frac{1}{3} \right]$, constant at $*$ for $s \in \left[\frac{1}{3} , \frac{2}{3} \right]$, γ backwards at triple speed for $s \in \left[\frac{2}{3} , 1 \right]$.

The path $\nu\colon I \longrightarrow (\Omega X)_0$ joins $\gamma f(*)\gamma^{-1}$ to the constant loop.

$$\nu(t)\colon S^1 \ni s \mapsto \left\{ \begin{array}{ll} (3s) & 0 \le s \le \frac{1}{3}(1-t) \\ (1-t) & \frac{1}{3}(1-t) \le s \le 1 - \frac{1}{3}(1-t) \\ (1-3(s - \frac{2}{3})) & 1 - \frac{1}{3}(1-t) \le s \le 1 \end{array} \right\} \in X .$$

Using the homotopy R, we get $g:(A,*) \longrightarrow (\Omega X,*)_0$ as the homotopy extension of the data $\gamma\bar{f}\gamma^{-1}:A \longrightarrow (\Omega X)_0$ and $\nu:I \longrightarrow (\Omega X)_0$. Specifically, we get for $a \in A$, $g(a) = \gamma\bar{f}(R(a,1))\gamma^{-1}$ if $R(a,1) \in A\times 0$. Thus

$$g(a):S^1 \ni s \mapsto \left\{ \begin{array}{ll} \gamma(3s) & 0 \le s \le \frac{1}{3} \\ \bar{f}R(a,1)(3(s-\frac{1}{3})) & \frac{1}{3} \le s \le \frac{2}{3} \\ \gamma(1-3(s-\frac{2}{3})) & \frac{2}{3} \le s \le 1 \end{array} \right\} \in X .$$

If $R(a,1) = (*,t) \in *\times I$, we get $g(a) = \nu R(a,1) = \nu(t)$.

<u>Step 3</u> is now trivial because we see from Step 2 that $\overline{\gamma.f}(a) = g(a)$ for all $a \in A$.

This completes the proof of Lemma (5.3). $\square$

In §1, we have seen that $\Omega X \approx (\Omega X)_0 \rtimes_\phi \pi_1 X$, where $\phi:\pi_1 X \longrightarrow H\mathcal{E}(\Omega X)_0$ is defined by loop conjugation. Explicitly, if $[\gamma] \in \pi_1 X$, $\phi[\gamma]$ is represented by the map $(\Omega X)_0 \ni \alpha \mapsto \gamma\alpha\gamma^{-1} \in (\Omega X)_0$; cf. (1.4), (1.7).

In §2, we have seen that ϕ induces an action $\psi(A)$ on the free homotopy groups $[A,(\Omega X)_0]$ by composition. Explicitly, $\psi(A)_{[\gamma]}[g] = \phi([\gamma])\circ[g]$ and $\phi([\gamma])\circ[g]$ is represented by the map $A \ni a \mapsto \gamma g(a)\gamma^{-1} \in (\Omega X)_0$. Hence, we have shown

<u>(5.4) Corollary</u> For all $[\gamma] \in \pi_1 X$, $[f] \in [(SA,*),(X,*)]$, $\theta([\gamma],[f]) = \theta'([\gamma],[f]) = \underline{\psi(A)_{[\gamma]}\overline{[f]}}$. $\square$

Applying Corollary (5.4) to n-spheres ($n \geq 1$), we get the following equivalent conditions for the nilpotency of CW-complexes.

<u>(5.5) Theorem</u> Let $(X,*)$ be a connected CW-complex. Then the following are equivalent:

(i) $\pi_1(X,*)$ is nilpotent and acts nilpotently on $\pi_n(X,*)$ for $n \geq 2$.

(ii) $\pi_1(X,*)$ is nilpotent and acts nilpotently on $\pi_{n-1}(\Omega X,*)$ by loop conjugation for $n \geq 2$.

(iii) $\pi_1(X,*)$ is nilpotent and $[S^{n-1},(\Omega X)_0]$ is $\psi(S^{n-1})$ nilpotent for $n \geq 2$.

(iv) $[S^{n-1},\Omega X]$ is nilpotent for $n \geq 2$.

<u>Proof</u> (i) $\Longleftrightarrow$ (ii) follows from (5.3).

(ii) $\Longleftrightarrow$ (iii) follows from (5.4).

(iii) $\Longleftrightarrow$ (iv) $[S^{n-1},\Omega X] \cong [S^{n-1},(\Omega X)_0] \rtimes_{\psi(S^{n-1})} \pi_1(X,*)$, by (2.8) and the semidirect product on the right hand side is nilpotent if and only if (iii) is true, for purely group theoretic reasons (cf. §2). □

The material in §2 can now be exploited in the following way. The action $\psi(S^{n-1})$ in (5.5) (iii), (iv) is induced by the homomorphism $\phi: \pi_1 X \longrightarrow H\mathcal{E}(\Omega X)_0$, determining the H-semidirect product structure of ΩX.

<u>(5.6) Proposition</u> If ϕ is trivial ($\ker\phi = \pi_1 X$), then X is simple.

□

This calls our attention to homomorphisms of $\pi_1 X$ into the following tower of groups.

$\pi_1 X \longrightarrow C\mathcal{E}(\Omega X)_0$; group of homotopy classes of H-conjugations of $(\Omega X)_0$

$\subset H\mathcal{E}(\Omega X)_0$; group of H-equivalences of $(\Omega X)_0$

$\subset \mathcal{E}(\Omega X)_0$; group of self homotopy equivalences of $(\Omega X)_0$.

<u>(5.7) Corollary</u> If the set of homomorphisms from $\pi_1 X$ into one of the groups $C\mathcal{E}(\Omega X)_0$, $H\mathcal{E}(\Omega X)_0$, $\mathcal{E}(\Omega X)_0$ is the 1-element set, then X is a simple space. □

In Proposition (2.7) we have shown that $\mathrm{nil}[S^n,\Omega X] \le c$ for all $n \ge 1$ if ΩX has H-nilpotency index $\le c$. Together with (5.5)(iv), we get

<u>(5.8) Corollary</u> If ΩX is H-nilpotent, then X is nilpotent. □

As for (5.5)(iii), the material in §4 can be utilized in the following way.

<u>(5.9) Proposition</u> Let $h_n:\pi_n(\Omega X)_0 \longrightarrow H_n(\Omega X)_0$ denote the Hurewicz homomorphism. If X is nilpotent, then $\pi_1 X$ acts nilpotently on im h_n. In particular, $\pi_1 X$ acts nilpotently on $H_1(\Omega X)_0$.

<u>Proof</u> Use (4.5) and the Hurewicz isomorphism theorem.

§6 Bordism rings of H-semidirect products

Conceptually, bordism theories have a flavour of generalizing aspects of free homotopy sets as well as aspects of singular homology groups. It is therefore not too surprising that the graded bordism group of an associative H-space has the natural structure of a graded ring arising in very much the same way as the Pontryagin ring structure in singular homology. Indeed, in this chapter we shall show that the bordism ring of an H-semidirect product decomposes into a twisted tensor product in a way resembling the decomposition of the Pontryagin algebra (cf. §4).

Let us begin by introducing the necessary concepts related to (co-)bordism theories. We are interested in bordism having a description by singular manifolds via Thom's theorem. Therefore, we shall assume that the extra structure on a manifold comes from an extra requirement concerning the normal bundle of this manifold with respect to a fixed imbedding in some $\mathbb{R}^s$. To keep this chapter to a certain extent self contained, we shall explain the underlying formalities and state, without proof, the results invoked. This material is extracted from Stong "Notes on cobordism theory" [St] and Switzer "Algebraic Topology" [Sw], which are the general references.

We shall refer to a smooth compact manifold ("smooth": it has a C^∞ atlas), with or without boundary, simply as a manifold. If we use the symbol M for a manifold, then m will denote the dimension of M. We agree that $\emptyset$ is a manifold of arbitary dimension. Whitney's imbedding theorem says that M can be imbedded in the half space

$\mathbb{H}^{s+1} := \{(x_1,\dots,x_{s+1})\colon x_1 \geq 0\} \subset \mathbb{R}^{s+1}$ $(s \geq 2m+1)$ such that ∂M gets mapped to $\partial\mathbb{H}^{s+1} = \{(0,x_2,\dots,x_{s+1}) \in \mathbb{R}^{s+1}\}$ and $(M - \partial M)$ gets mapped into the open half space $(x_1 > 0)$ of $\mathbb{H}^{s+1}$. We require all imbeddings in later considerations to have these properties.

<u>Notation and preparations</u> Let $\mathbb{R}^\infty := \varinjlim \mathbb{R} \hookrightarrow \mathbb{R}^2 \hookrightarrow \mathbb{R}^3 \hookrightarrow \dots$ with the limit topology. For $r \geq 1$, it will be advantageous to use the isomorphism $\mathbb{R}^\infty \cong \varinjlim \mathbb{R}^r \hookrightarrow \mathbb{R}^{2r} \hookrightarrow \mathbb{R}^{3r} \hookrightarrow \dots$ induced by deleting the vector spaces $\mathbb{R}^n$, n not divisible by r, from the first limiting system above.

For $r, r' \geq 1$, we get an isomorphism $\pi_{r,r'}\colon \mathbb{R}^\infty \oplus \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$ given by the following self explanatory picture.

$$\left.\begin{matrix} \mathbb{R}^\infty \cong \\ \oplus \\ \mathbb{R}^\infty \cong \end{matrix}
\begin{bmatrix}\mathbb{R}^r\\ \oplus\\ \mathbb{R}^{r'}\end{bmatrix} \begin{matrix}\oplus\\ \\ \oplus\end{matrix}
\begin{bmatrix}\mathbb{R}^r\\ \oplus\\ \mathbb{R}^{r'}\end{bmatrix} \begin{matrix}\oplus\\ \\ \oplus\end{matrix}
\begin{bmatrix}\mathbb{R}^r\\ \oplus\\ \mathbb{R}^{r'}\end{bmatrix} \begin{matrix}\oplus \ \dots\\ \\ \oplus \ \dots\end{matrix}\right\} \cong \mathbb{R}^\infty$$

If rr' is even, $\pi_{r,r'} \approx \pi_{r',r}$ by a homotopy through orthogonal maps interchanging the summands $\mathbb{R}^r \oplus \mathbb{R}^{r'}$ pointwise in each of the blocks $\mathbb{R}^{r+r'}$. For $n \geq 1$, let $\rho_n\colon \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$ be the isomorphism induced blockwise by the map $\mathbb{R}^n \ni (x_1,x_2,\dots,x_n) \longmapsto (-x_1,x_2,\dots,x_n) \in \mathbb{R}^n$. If rr' is odd, $\pi_{r,r'} \approx \rho_{r+r'}\pi_{r',r}$ by a homotopy through orthogonal maps interchanging the summands $\mathbb{R}^r \oplus \mathbb{R}^{r'}$ pointwise in each of the blocks $\mathbb{R}^{r+r'}$. Evidently, the diagram

$$\begin{array}{ccc}
\mathbb{R}^\infty \oplus \mathbb{R}^\infty \oplus \mathbb{R}^\infty & \xrightarrow{\pi_{r,r'} \oplus \mathrm{Id}} & \mathbb{R}^\infty \oplus \mathbb{R}^\infty \\
{\scriptstyle \mathrm{Id} \oplus \pi_{r',r''}} \downarrow & & \downarrow {\scriptstyle \pi_{r+r',r''}} \\
\mathbb{R}^\infty \oplus \mathbb{R}^\infty & \xrightarrow[\pi_{r,r'+r''}]{} & \mathbb{R}^\infty
\end{array}$$

commutes strictly. Furthermore, the inclusion $i_r\colon \mathbb{R}^r \ni (x_1,\dots,x_r) \longmapsto (x_1,\dots,x_r,0) \in \mathbb{R}^{r+1}$ induces an inclusion $j_r\colon \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$ given by

$$\begin{array}{ccccccccc}
\mathbb{R}^\infty & \cong & \mathbb{R}^r & \oplus & \mathbb{R}^r & \oplus & \mathbb{R}^r & \oplus & \dots \\
{\scriptstyle j_r}\downarrow & & {\scriptstyle i_r}\downarrow & & {\scriptstyle i_r}\downarrow & & {\scriptstyle i_r}\downarrow & & \\
\mathbb{R}^\infty & \cong & \mathbb{R}^{r+1} & \oplus & \mathbb{R}^{r+1} & \oplus & \mathbb{R}^{r+1} & \oplus & \dots .
\end{array}$$

For $0 \le r \le n$, denote by $G_{n,r}$ the Grassmann manifold of unoriented r-planes in $\mathbb{R}^n$ and by $BO_r := \lim G_{r,r} \hookrightarrow G_{r+1,r} \hookrightarrow \dots$. On the set level, there is an immediate identification between BO_r and $G_{\infty,r}$, the set of r-planes in $\mathbb{R}^\infty$. We give $G_{\infty,r}$ the topology rendering this identification a homeomorphism. Let γ_r denote the canonical r-plane bundle over BO_r. Note that γ_r is a universal r-plane bundle, i.e. if (E,p,B) is a numerable r-plane bundle, then there exists a unique homotopy class of maps $[f]\colon B \longrightarrow BO_r$ such that the pull back bundle $f^*\gamma_r$ is vector bundle isomorphic to (E,p,B).

Let O_r denote the orthogonal group of $\mathbb{R}^r$. The inclusion $i_r\colon \mathbb{R}^r \cong \mathbb{R}^r \times \{0\} \hookrightarrow \mathbb{R}^{r+1}$ induces the inclusion $i_r\colon O_r \cong O_r \times \{\mathrm{Id}_{\mathbb{R}}\} \hookrightarrow O_{r+1}$, giving rise to the inclusion $Bi_r\colon BO_r \ni P \longmapsto \mathrm{span}(j_r(P) \cup \{e_{r+1}\}) \in BO_{r+1}$. Here $e_n \in \mathbb{R}^\infty$ denotes the vector having 0-entries everywhere but for a 1 at the n-th position.

For $1 \le r \le n$, denote by $V_{n,r}$ the Stiefel manifold of ordered orthogonal r-frames in $\mathbb{R}^n$, and by $EO_r := \varinjlim V_{r,r} \hookrightarrow V_{r+1,r} \hookrightarrow \cdots$. On the set level, there is an immediate identification between EO_r and $V_{\infty,r}$, the set of orthogonal r-frames in $\mathbb{R}^\infty$. We give $V_{\infty,r}$ the topology rendering this identification a homeomorphism. Let $f^1_r\colon EO_r \ni (v_1,\dots,v_r) \longmapsto \mathrm{span}\{v_1,\dots,v_r\} \in BO_r$. Then (EO_r, f^1_r, BO_r) is a universal principal O_r-bundle.

The inclusion $j_r\colon \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$ induces an inclusion $Ei_r\colon EO_r \ni (v_1,\dots,v_r) \longmapsto (v_1,\dots,v_r,e_{r+1}) \in EO_{r+1}$. Now, the direct sum decomposition $\mathbb{R}^\infty \cong \mathbb{R}^r \oplus \mathbb{R}^r \oplus \cdots$ gives us a "diagonal" action of O_r on $\mathbb{R}^\infty$ inducing an action of O_r on EO_r. The inclusion $i_r\colon O_r \cong O_r \times \{Id_{\mathbb{R}}\} \longrightarrow O_{r+1}$ gives us an action of O_r on EO_{r+1}. It is immediate that Ei_r is an equivariant map with respect to this O_r-action. In particular, we get the commuting diagram

$$\begin{tikzcd} EO_r \arrow[r, "Ei_r"] \arrow[d, "f^1_r"'] & EO_{r+1} \arrow[d, "f^1_{r+1}"] \\ BO_r \arrow[r, "Bi_r"'] & BO_{r+1} \end{tikzcd} .$$

<u>(6.1) Definition</u> (i) A (B,f)-system is an infinite ladder

$$\begin{tikzcd}
B_b \arrow[r, "g_b"] \arrow[d, "f_b"'] & B_{b+1} \arrow[r] \arrow[d, "f_{b+1}"'] & \cdots \arrow[r] & B_r \arrow[r, "g_r"] \arrow[d, "f_r"'] & B_{r+1} \arrow[r] \arrow[d, "f_{r+1}"] & \cdots \\
BO_b \arrow[r, "Bi_b"'] & BO_{b+1} \arrow[r] & \cdots \arrow[r] & BO_r \arrow[r, "Bi_r"'] & BO_{r+1} \arrow[r] & \cdots
\end{tikzcd}$$

in which every square is strictly commutative and every vertical map is a Hurewicz fibration.

(ii) A (B,f)-structure on a manifold M is a pair (ϵ_r, λ_r), $r \geq b$, where $\epsilon_r\colon M \longrightarrow \mathbb{R}^{m+r}$ is an imbedding. ϵ_r gives rise to a canonical map $c_r\colon M \longrightarrow G_{m+r,r} \hookrightarrow BO_r$ which classifies the normal bundle of $\epsilon_r(M)$ in $\mathbb{R}^{m+r}$ and we require $\lambda_r\colon M \longrightarrow B_r$ to be a lift of c_r (i.e. $f_r\lambda_r = c_r$).

Actually a pair (ϵ_r, λ_r) induces more pairs $(\epsilon_{r+k}, \lambda_{r+k})$, $k \geq 1$ by composition: $\epsilon_{r+k} := M \xrightarrow{\epsilon_r} \mathbb{R}^{m+r} \hookrightarrow \mathbb{R}^{m+r+k}$, which yields $c_{r+k} = Bi_{r+k-1} \circ \cdots \circ Bi_{r+1} \circ Bi_r \circ c_r$ and $\lambda_{r+k} = g_{r+k-1} \circ \cdots \circ g_{r+1} \circ g_r \circ \lambda_r$. For our purposes we do not need to favour any of the (ϵ_s, λ_s), $s \geq r$, above. We shall, therefore, write a (B,f)-structure on M as a pair (ϵ, λ) and specify subscripts only when necessary.

Note also that a (B,f)-structure on M induces a (B,f)-structure on ∂M by taking restrictions of the maps involved.

(iii) Two (B,f)-structures (ϵ^1, λ^1), (ϵ^2, λ^2) on M are equivalent $:\Longleftrightarrow$ there exists $r \geq b$ and a translation $T\colon \mathbb{R}^{m+r} \longrightarrow \mathbb{R}^{m+r}$ such that $T\epsilon_r^1 = \epsilon_r^2$ and $\lambda_r^1 \approx \lambda_r^2$ by a vertical homotopy. The latter requirement makes sense because $T\epsilon_r^1 = \epsilon_r^2$ implies $c_r^1 = c_r^2$.

(iv) A (B,f)-manifold is a manifold M together with an equivalence class of (B,f)-structures.

(v) Let M_1, M_2 be (B,f) manifolds of dimensions $m_2 - 1 \le m_1 \le m_2$ with representing (B,f)-structures (ϵ^1,λ^1), (ϵ^2,λ^2). A (B,f)-imbedding of M_1 in M_2 is an imbedding $a\colon M_1 \longrightarrow M_2$ such that

(a) $a(\partial M_1) \subset \partial M_2$ if $m_1 = m_2$

(b) $\partial M_1 = \emptyset$ and $a(M_1) \subset \partial M_2$ if $m_1 = m_2 - 1$.

Furthermore, there is to exist $r \ge b$ and a translation $T\colon \mathbb{R}^{m_2+r} \longrightarrow \mathbb{R}^{m_2+r}$ such that $T\epsilon^1_r = \epsilon^2_r a$, and $\lambda^1_r \approx \lambda^2_r a$ by a vertical homotopy. The latter requirement makes sense because $T\epsilon^1_r = \epsilon^2_r a$ implies $c^1_r = c^2_r a$.

If C is a (B,f)-manifold with boundary M, and M is given the induced (B,f)-structure, then the inclusion $M \hookrightarrow C$ is a (B,f)-imbedding with T the translation by the 0-vector, i.e. $T = \mathrm{Id}$.

A (B,f)-diffeomorphism between M_1 and M_2 is an invertible (B,f)-imbedding of M_1 in M_2. We denote this situation by $M_1 \cong M_2$ (assuming that the underlying (B,f)-system is fixed).

(vi) Let M_1, M_2 be two (B,f)-manifolds of the same dimension m, without boundary. M_1 is (B,f)-cobordant to M_2 $:\Longleftrightarrow$ there exist (B,f)-manifolds C_1, C_2 of the same dimension $c = m+1$ such that ("$\dot{\cup}$" denotes disjoint union)

$$M_1 \,\dot{\cup}\, \partial C_1 \cong M_2 \,\dot{\cup}\, \partial C_2 \,.$$

*<u>(6.2) Lemma</u> (i) "(B,f)-diffeomorphism" is an equivalence relation. Denote the collection of equivalence classes of closed (B,f)-manifolds of dimension m by $\mathcal{M}_m^{(B,f)}$.

(ii) $\mathcal{M}_m^{(B,f)}$ is a proper set for all $m \in \mathbb{N}_0$.

(iii) Under the operation of disjoint union (on representing manifolds, imbeddings, classifying maps for normal bundles and lifts into B_r), the set $\mathcal{M}_m^{(B,f)}$ becomes an abelian monoid with the empty manifold (and the unique paraphernalia of maps) as the neutral element.

(iv) (B,f)-cobordism is an equivalence relation on $\mathcal{M}_m^{(B,f)}$. Denote by $\Omega_m^{(B,f)}$ the set of (B,f)-cobordism classes of $\mathcal{M}_m^{(B,f)}$ and by [M] the cobordism class of a (B,f)-manifold M.

(v) The adddition on $\mathcal{M}_m^{(B,f)}$ is compatible with the formation of (B,f)-cobordism classes and induces on $\Omega_m^{(B,f)}$ the structure of an abelian group. Its neutral element is the (B,f)-cobordism class consisting of all closed m-manifolds arising as the boundary of some closed (B,f)-manifold C (c = m+1) with the induced (B,f)-structure. Inverses are obtained as follows. If M with (B,f)-structure (ϵ,λ) represents a certain (B,f)-cobordism class, for r sufficiently large, let $T\colon \mathbb{R}^{m+r} \longrightarrow \mathbb{R}^{m+r}$ be a translation such that $\epsilon_r(M)$ is disjoint from $T\cdot\epsilon_r(M)$. Then there exists an imbedding $\epsilon'_r\colon M\times I \longrightarrow \mathbb{H}^{r+m+1}$ such that $\epsilon'_{|M\times\{0\}} = \epsilon_r$ and $\epsilon'_{|M\times\{1\}} = T\epsilon_r$. Then the canonical classifying map $c'_r\colon M\times I \longrightarrow BO_r$ is a homotopy of c_r. Hence the lift $\lambda_r\colon M \longrightarrow B_r$ has a homotopy extension $\lambda'_r\colon M\times I \longrightarrow B_r$ over c'_r. Give $M\times\{1\}$ the induced (B,f)-structure. It follows that

$$(M\times\{0\} \cup M\times\{1\})\ \dot{\cup}\ \partial\emptyset \cong \emptyset\ \dot{\cup}\ \partial(M\times I),$$

which means that $M\times\{1\}$ with the (B,f)-structure constructed above represents the inverse of M in $\Omega_m^{(B,f)}$. $\square$

We indicate a source for many (B,f)-systems. Let $b \in \mathbb{N}$, and suppose we have an infinite ladder of closed subgroups G_r of O_r

$$\begin{array}{ccccccccccc}
G_b & \hookrightarrow & G_{b+1} & \hookrightarrow & \cdots & \hookrightarrow & G_r & \hookrightarrow & G_{r+1} & \hookrightarrow & \cdots \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
O_b & \hookrightarrow & O_{b+1} & \hookrightarrow & \cdots & \hookrightarrow & O_r & \hookrightarrow & O_{r+1} & \hookrightarrow & \cdots
\end{array}$$

where all arrows are inclusions, so that each square in this ladder automatically commutes. If for all $r \geq 1$, $G_r = \{Id_{\mathbb{R}^r}\}$ we take $B_r := EO_r$, $f_r := f_r^1$, $g_r := Ei_r$. In the other cases, observe that G_r acts on EO_r by restricting the O_r-action. The quotient space is a classifying space for principal G_r-bundles. Furthermore f_r^1 is the composite of two fibrations $q_r\colon EO_r \longrightarrow BG_r = B_r$, a principal G_r-bundle, and $f_r\colon B_r \longrightarrow BO_r$ with fibre O_r/G_r. Since Ei_r is equivariant and $i_r\colon O_r \hookrightarrow O_{r+1}$ induces an inclusion $G_r \hookrightarrow G_{r+1}$, we get the double ladder below in which each square commutes strictly and each vertical map is a Hurewicz-fibration (cf. [B], Chapter III).

$$\begin{array}{ccccccccccc}
EO_b & \xrightarrow{Ei_b} & EO_{b+1} & \longrightarrow & \cdots & \longrightarrow & EO_r & \xrightarrow{Ei_r} & EO_{r+1} & \longrightarrow & \cdots \\
{\scriptstyle q_b}\downarrow & & \downarrow & & & & {\scriptstyle q_r}\downarrow & & \downarrow & & \\
BG_b & \xrightarrow{g_b} & BG_{b+1} & \longrightarrow & \cdots & \longrightarrow & BG_r & \xrightarrow{g_{r+1}} & BG_{r+1} & \longrightarrow & \cdots \\
{\scriptstyle f_b}\downarrow & & \downarrow & & & & {\scriptstyle f_r}\downarrow & & \downarrow & & \\
BO_b & \xrightarrow{Bi_b} & BO_{b+1} & \longrightarrow & \cdots & \longrightarrow & BO_r & \xrightarrow{Bi_r} & BO_{r+1} & \longrightarrow & \cdots
\end{array}$$

The bottom ladder is a (B,f)-system associated with the system of groups $\{G_r \subset O_r\}$.

In essence, a (B,f)-structure on a manifold M comes from an imbedding $\epsilon_r: M \longrightarrow \mathbb{R}^{m+r}$ so that the associated normal bundle of M in $\mathbb{R}^{m+r}$ is a G_r-bundle. Furthermore, (B,f)-cobordism is the classical cobordism idea applied to the class of manifolds allowing for a (B,f)-structure. In particular the choice

$G_r := O_r$ yields standard unoriented cobordism

$G_r := SO_r$ yields standard oriented cobordism

$G_{2r} := G_{2r+1} := U_r$ yields complex cobordism

$G_r := \{Id_{\mathbb{R}^r}\}$ yields Pontryagin's framed cobordism.

We shall briefly explain the method of constructing (B,f)-bordism theories. For systematic reasons, we present now the formalities needed to state conditions which will insure that the graded (B,f)-bordism group of a point will turn out to be a graded associative commutative ring with 1.

*<u>(6.3) Lemma</u> (i) The map $\pi_{r,r'}: \mathbb{R}^\infty \oplus \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$ induces a map $k_{r,r'}: BO_r \times BO_{r'} \longrightarrow BO_{r+r'}$ by taking direct sums. Note that $k_{r,r'}$ classifies the product bundle $\gamma_r \times \gamma_{r'}$.

(ii) The diagram

$$\begin{array}{ccc}
BO_r \times BO_{r'} \times BO_{r''} & \xrightarrow{Id \times k_{r',r''}} & BO_r \times BO_{r'+r''} \\
\Big\downarrow {\scriptstyle k_{r,r'} \times Id} & & \Big\downarrow {\scriptstyle k_{r,r'+r''}} \\
BO_{r+r'} \times BO_{r''} & \xrightarrow[k_{r+r',r''}]{} & BO_{r+r'+r''}
\end{array}$$

commutes.

(iii) If rr' is even, a homotopy of $\pi_{r,r'}$ to $\pi_{r',r}$ through rotations renders the diagram

$$\begin{tikzcd} BO_r \times BO_{r'} \arrow[r, "k_{r,r'}"] \arrow[d, "\tau"'] & BO_{r+r'} \\ BO_{r'} \times BO_r \arrow[ur, "k_{r',r}"'] & \end{tikzcd}$$

commutative up to homotopy, where $\tau(P,P') = (P',P)$. Let $\tilde{\rho}_n\colon BO_n \longrightarrow BO_n$ be the map induced by $\rho_n\colon \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$. If rr' is odd, a homotopy of $\pi_{r,r'}$ to $\rho_{r+r'}\pi_{r',r}$ through rotations renders the diagram

$$\begin{tikzcd} BO_r \times BO_{r'} \arrow[r, "k_{r,r'}"] \arrow[d, "\tau"'] & BO_{r+r'} \\ BO_{r'} \times BO_r \arrow[ur, "\tilde{\rho}_{r+r'}k_{r',r}"'] & \end{tikzcd}$$

commutative up to homotopy. □

(iv) BO_r has the canonical base point $\mathbb{R}^r = *_r$ (the first $\mathbb{R}^r$ block of $\mathbb{R}^\infty$). Bi_r is a based map, $k_{r,r'}(*_r,*_{r'}) = *_{r+r'}$ and $k_{r,r'}(x,*_{r'}) = Bi_{r+r'-1}\circ\cdots\circ Bi_r(x)$.

We shall from now on work with (B,f)-systems satisfying the following additional requirements.

(C1) For all $r, r' \geq b$, there exists a map $\ell_{r,r'}\colon B_r \times B_{r'} \longrightarrow B_{r+r'}$ such that the diagram

$$\begin{tikzcd} B_r \times B_{r'} \arrow[r, "\ell_{r,r'}"] \arrow[d, "f_r \times f_{r'}"'] & B_{r+r'} \arrow[d, "f_{r+r'}"] \\ BO_r \times BO_{r'} \arrow[r, "k_{r,r'}"'] & BO_{r+r'} \end{tikzcd}$$

commutes. Since $k^*_{r+r'}\gamma_{r+r'} = \gamma_r \times \gamma_{r'}$, this implies that $f_{r+r'}\ell_{r+r'}$ classifies $f^*_r\gamma_r \times f^*_{r'}\gamma_{r'}$.

(C2) The diagram

$$\begin{tikzcd} B_r \times B_{r'} \times B_{r''} \arrow[r, "\mathrm{Id} \times \ell_{r',r''}"] \arrow[d, "\ell_{r,r'} \times \mathrm{Id}"'] & B_r \times B_{r'+r''} \arrow[d, "\ell_{r,r'+r''}"] \\ B_{r+r'} \times B_{r''} \arrow[r, "\ell_{r+r',r''}"'] & B_{r+r'+r''} \end{tikzcd}$$

commutes up to a vertical homotopy over the diagram (6.3)(ii).

(C3) If rr' is even, the homotopy between $k_{r,r'}$ and $k_{r',r}\tau$ in (6.3)(iii) lifts to a homotopy between $\ell_{r,r'}$ and $\ell_{r',r}\tau$ over the diagram in (6.3)(iii).

(C4) In each fiber $f_r^{-1}\{*_r\} \subset B_r$, there is a distinguished point $\tilde{*}_r$ such that $\ell_{r,r'}(\tilde{*}_r,\tilde{*}_{r'})$ is in the same path connected component of $f_r^{-1}\{*_r\}$ as $\tilde{*}_{r+r'}$ and $\ell_{r,r'}(\tilde{x},\tilde{*}_{r'})$ is in the same path component of $f_{r+r'}^{-1}(k_{r,r'}(f_r(x),*_{r'}))$ as $g_{r+r'-1} \cdots g_r(\tilde{x})$.

*<u>(6.4) Remark</u> The (B,f)-systems arising from a system of subgroups $\{G_r \subset O_r\}$ as explained above, satisfy the requirements (C1),...,(C4).

<u>Proof sketch</u> The map $\pi_{r,r'}\colon \mathbb{R}^\infty \oplus \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$ induces a map $\ell^1_{r,r'}\colon EO_r \times EO_{r'} \longrightarrow EO_{r+r'}$ satisfying the requirements (C1),...,(C4). For the other systems, we get maps $\ell_{r,r'}\colon BG_r \times BG_{r'} \longrightarrow BG_{r+r'}$ induced by $\ell^1_{r,r'}$ by taking quotients. For the distinguished point $\tilde{*}_r$, we take the quotient class of the r-frame $(e_1,\ldots,e_r)$. □

A (B,f)-system gives rise to a homology theory as follows. Let X be a space, $M \in \mathcal{M}_m^{(B,f)}$ a closed (B,f)-manifold, $u\colon M \longrightarrow X$ a continuous map. The pair (M,u) is called a singular (B,f)-manifold in X.

<u>(6.5) Definition</u> Two singular (B,f)-manifolds (M,u), (M',u') are called (B,f)-bordant :⟺ there exists an (m+1)-dimensional (B,f)-manifold C such that

$$M \dot{\cup} (-M') = \partial C,$$

where $-M'$ represents $-[M']$ in $\Omega_m^{(B,f)}$, and there exists a continuous map $a\colon C \longrightarrow X$ such that $a_{|M} = u$ and $a_{|M'} = u'$.

*<u>(6.6) Lemma</u> (i) (B,f)-bordism on singular (B,f)-manifolds is an equivalence relation. Let {M,u} denote the (B,f)-bordism class of the singular (B,f)-manifold (M,u).

(ii) The (B,f)-bordism classes of dimension m in X form an abelian group under the operation disjoint union of domain manifolds and singular maps. Denote this group by $\Omega_m^{(B,f)}X$.
(iii) $\Omega_*^{(B,f)}\_$ is a generalized homology functor from the category of compactly generated Hausdorff spaces to the category of $\mathbb{N}_0$-graded abelian groups.

Next let us construct a cross-product for $\Omega_*^{(B,f)}\_$. Let X, Y be spaces and let (M,u) be a singular (B,f)-manifold in X, (N,v) a singular (B,f)-manifold in Y. Let (ϵ,λ), (ϵ',λ') represent (B,f)-structures for M, N. The cross product of (M,u) with (N,v) is $(M\times N, u\times v)$ with (B,f)-structure $(\epsilon_r\times\epsilon'_{r'}, \ell_{r,r'}(\lambda_r\times\lambda_{r'}))$. That this makes sense follows from the commutativity of the diagram below and the fact that $k_{r,r'}(c_r\times c'_{r'})$ classifies the normal bundle of $M\times N$ in $\mathbb{R}^{m+r+n+r'}$ because $k_{r,r'}$ classifies $\gamma_r\times\gamma_{r'}$.

$$\begin{tikzcd}
 & B_r \times B_{r'} \arrow[r, "\ell_{r,r'}"] \arrow[d, "f_r \times f_{r'}"] & B_{r+r'} \arrow[d, "f_{r+r'}"] \\
M \times N \arrow[ur, "\lambda_r \times \lambda'_{r'}"] \arrow[r, "c_r \times c'_{r'}"'] & BO_r \times BO_{r'} \arrow[r, "k_{r,r'}"'] & BO_{r+r'}
\end{tikzcd}$$

*<u>(6.7) Lemma</u> (i) The cross product on singular (B,f)-manifolds defined above is natural, biadditive and associative.
(ii) The cross product on singular (B,f)-manifolds is compatible with the (B,f)-bordism relation and, therefore, induces a natural

homomorphism $\times\colon \Omega_*^{(B,f)}X \otimes_{\mathbb{Z}} \Omega_*^{(B,f)}Y \longrightarrow \Omega_*^{(B,f)}X{\times}Y$. The cross product for three spaces is associative.

(iii) Let $P = \{*\}$ denote the 1-point space and identify $P{\times}X = X = X{\times}P$ by the obvious homeomorphisms. Then "$\times$" induces on $\Omega_*^{(B,f)}P$ the structure of an $\mathbb{N}_0$-graded associative, commutative ring with identity element 1, the cobordism class represented by (P,Id) where P has (B,f)-structure $\epsilon_b\colon P \longrightarrow \mathbb{R}^b$ any map and $\lambda_b\colon P \ni * \longmapsto \tilde{*}_b \in B_r$. Note also that $\Omega_0^{(B,f)}P$ is a subring of $\Omega_*^{(B,f)}P$ containing 1.

(iv) The cross product induces on $\Omega_*^{(B,f)}X$ a natural $\Omega_*^{(B,f)}P$-bimodule structure.

(v) The biadditive map $\Omega_*^{(B,f)}X \times \Omega_*^{(B,f)}Y \longrightarrow \Omega_*^{(B,f)}X{\times}Y$ is $\Omega_*^{(B,f)}P$-balanced and, therefore, induces a homomorphism

$$\times\colon \Omega_*^{(B,f)}X \otimes_{\Omega_*^{(B,f)}P} \Omega_*^{(B,f)}Y \longrightarrow \Omega_*^{(B,f)}X{\times}Y$$

of $\Omega_*^{(B,f)}P$-modules.

The proof is obvious at the level of singular manifolds. For the related (B,f)-structures it follows from properties C1,...,C4 above.

□

Now let (X,μ) be an H-space. We get a Pontryagin algebra structure on $\Omega_*^{(B,f)}X$ out of the composite

$$\Omega_*^{(B,f)}X \otimes_{\Omega_*^{(B,f)}P} \Omega_*^{(B,f)}X \longrightarrow \Omega_*^{(B,f)}X{\times}X \xrightarrow{\mu_*} \Omega_*^{(B,f)}X.$$

The algebra structure is associative, respectively commutative if (X,μ) is H-associative, respectively H-commutative.

Now let Π be a group acting on a path connected H-group (W_0,μ_0) by a group homomorphism $\phi\colon \Pi \longrightarrow H\mathcal{E}W_0$ and denote by $W = W_0 \rtimes_\phi \Pi$ the corresponding H-semidirect product with H-multiplication μ. Then Π acts on $\Omega_*^{(B,f)}W_0$ by $\Omega_*^{(B,f)}P$-algebra automorphisms via $\psi\colon \Pi \ni p \longmapsto \phi(p)_* \in \operatorname{Aut} \Omega_*^{(B,f)}W_0$.

<u>(6.8) Theorem</u> $\Omega_*^{(B,f)}W \cong \Omega_*^{(B,f)}W_0 \otimes_{\Omega_0^{(B,f)}P} \Omega_0^{(B,f)}\Pi$ as graded $\Omega_0^{(B,f)}P$-algebras.

We are working with a fixed (B,f)-system. Thus we can, without risk of confusion, write "Ω_*" instead of "$\Omega_*^{(B,f)}$". Also denote $\Omega_0 P$ by R. All tensor products here are over R.

<u>Proof of (6.8)</u> Ω_* is an additive (unreduced) homology theory. Thus $\Omega_0\Pi \cong R\Pi$, so that the ψ-twisted tensor product above fits in the frame work of §3.

We know from (6.7)(v) that the cross product

$$\times\colon \Omega_* W_0 \times \Omega_0\Pi \ni (\{M,u\},\{A,v\}) \longmapsto \{M\times A, u\times v\} \in \Omega_* W$$

is R-balanced. It remains to show

(1) "$\times$" is ψ-twisted

(2) the R-algebra homomorphism $\times\colon \Omega_* W_0 \otimes_R^{\psi} \Omega_0\Pi \longrightarrow \Omega_* W$ is an isomorphism.

Verification of (1). We must show that for all $\{M,u\} \in \Omega_m W_0$, $\{N,v\} \in \Omega_n W_0$, $\{n,t\} \in \Omega_0\Pi$, $p \in \Pi$, the identity

$$(\{M,u\}\times\{1,p\})(\{N,v\}\times\{n,t\}) = \{M\times N,\mu_0\circ(u\times(\varphi_p\circ v))\} \times \{n,pt\}$$

holds. Here, $p\colon 1 \longrightarrow \{p\} \subset \Pi$ denotes the unique map and $pt\colon n \longrightarrow \Pi$ is given by multiplication of function values: $pt(x) = p\cdot t(x)$. Now

$$\begin{aligned}(\{M,u\}\times\{1,p\})(\{N,v\}\times\{n,t\}) &= \{M\times 1,u\times p\}\{N\times n,v\times t\}\\ &= \{M\times 1\times N\times n,u\times p\times v\times t\}\\ &= \{M\times N\times n,(\mu_0(u\times(\varphi_p\circ v))) \times pt\}\\ &= \{M\times N,\mu_0(u\times(\varphi_p\circ v))\} \times \{n,pt\} ,\end{aligned}$$

where φ_p is defined as in §1. Thus "$\times$" is ψ-twisted.

Verification of (2). We know that Ω_* is additive. Thus $\Omega_0\Pi \cong \bigoplus_{r\in\Pi} \Omega_0 P$ as an R-module so that we get the isomorphisms of R-modules

$$\begin{aligned}\Omega_* W_0 \otimes_R \Omega_0\Pi &\cong \bigoplus_{p\in\Pi} \Omega_* W_0\times\{p\}\\ &\cong \Omega_* W .\end{aligned}$$

To make this isomorphism explicit, define for $p \in \Pi$

$i_p\colon W_0\times\{p\} \hookrightarrow W$ the inclusion

$\tau_p\colon W_0 \ni w \hookrightarrow (w,p) \in W\times\{p\}$ the translation map.

Then, the above isomorphism takes an element $\{M,u\}\otimes\{1,p\}$ to $i_*\tau_{p_*}\{M,u\}$ $= \{M,i\tau_p u\} = \{M\times I,u\times p\} = \{M,u\} \times \{1,p\}$. Thus "$\times$" is a monomorphism. Since the elements of the form $\{M,u\}\otimes\{1,p\}$ generate $\Omega_* W_0 \otimes \Omega_0\Pi$, "$\times$" is onto. $\square$

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