Store Incentives and Retailer Inventory Performance under Asymmetric Demand Information and Unobservable Lost Sales*

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We study incentive issues in an inventory management setting in which high on-shelf availability is crucial. Headquarters of a retailer delegates inventory replenishment decisions to store managers in its various stores. Store manager has complete information of the local demand process, whereas headquarters has partial information and cannot observe unsatisfied demand. The problem is how to incentivize the manager to make an order quantity decision that minimizes the sum of headquarters' expected overage and underage costs. We propose two incentive schemes that explicitly incorporate excess inventory and stock-outs into the store manager's performance measurement. We prove that a perfect alignment of incentives is possible under certain conditions. Interestingly, perfect or near-perfect alignment requires the stock-out inspection *before* the end of the replenishment cycle. We validate our approach and assumptions on a retailer's actual data and show that the retailer may improve its profitability by using the proposed incentive scheme.

Key words: Incentive alignment, asymmetric information, unobservable shortages

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1. Introduction and Literature Review

In this paper, we study incentive issues in an inventory management setting in which attaining high on-shelf availability is crucial. In many industries, ramifications of stock unavailability can be very costly. For example, for fast-moving consumer goods, the retail stock-out rates are around 8% in developed countries world-wide (Gruen et al., 2002) and have not changed much in the past fifty years despite advances in information technology (Peckham, 1963, Coca-Cola Research

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Council, 1996). Gruen et al. (2002) estimate that on the average 30% of items that are out of stock are purchased from another store, costing retailers 4% of their sales annually. The authors also estimate that 72% of the stock-outs were caused by retail store practices as opposed to problems in the upstream of the supply chain. Nearly two-thirds of the store related stock-outs can be attributed to store ordering and forecasting (47% of all stock-outs) which results in not having the product anywhere in the store. The remaining one-third (25% of all stock-outs) can be attributed to store shelving which leads to having the product in the store warehouse but not on the shelf.

A key factor for the severity of the stock-out problem in many retail environments is the relative observability of obsolescence and stock-outs. When there is excess inventory that leads to shrinkage or needs to be salvaged in a secondary market, this is an easily observable and important event, whereas stock-outs are usually not easily observable and their adverse effects on immediate and future revenues are not well-understood (Anderson et al., 2006). Therefore, a core requirement to reduce stock-outs at the retailers is to develop an effective measurement system (Gruen and Corsten, 2008), quantify the effect of stock-outs on profitability, and bring the issue to the attention of the management (ECR Europe, 2003).

Given a large number of stores a typical retail chain owns, directly managing operations in each store by headquarters is difficult. Hence, it is imperative "to design appropriate incentives to motivate store managers to execute activities critical to the performance of the retail store" (DeHoratius and Raman, 2007). These incentives, including those to reduce stock-outs and excess inventory, can be tied to store managers' performance scorecards. The balanced scorecard is a common tool adopted for this purpose (Kaplan and Norton, 1992). Balanced scorecards may have several perspectives, each having multiple objectives. For example, Tesco used a five segment scorecard that includes customer, community, operations, people and finance-related dimensions (Witcher and Chou, 2008). Stock-out related measures reside in the operations segment. Despite the need for incentives, according to a survey by an alliance of food and consumer packaged goods manufacturers and retailers, only nine percent of the retailers include stock-outs as a factor in their incentives or rewards (FMI/GMA Trading Partner Alliance, 2015). Our communication with two large grocery retail chains, a North American giant grocery retailer that owns several chain brands and a European discount grocery chain, reveals that stock-outs are not explicitly contained in their performance scorecards. These two retail firms utilize only two key performance indicators (KPI) under the operations segment: "Sales Revenue" and "Inventory Shrinkage". DeHoratius and Raman (2007) also report that the same two KPIs are adopted by an American-based consumer electronics retail store. When sales generation is based on self-service as in grocery retail stores, on-shelf availability is one of the few factors that store managers can use to influence sales (DeHoratius and Raman, 2007). Hence, lowering stock-outs is only implicitly accounted for in the "Sales Revenue" KPI; lack of an explicit measure may not guarantee attaining the service levels desired by the headquarters.

There are two challenges with using incentives related to stock-outs in practice. First, measuring stock-outs may be difficult, if not impossible. Suggested approaches for measurement may either require substantial effort on behalf of retailers or lead to inaccurate predictions of lost sales (Gruen and Corsten, 2008). Second, store employees who are more heavily incentivised on availability explicitly or through sales performance implicitly may care less about inventories which are obviously also costly for the retailer. This leads to an incentive misalignment problem (see van Donselaar et al., 2010 for an example of store managers that are only rewarded for on-stock availability, order more or earlier than necessary). One option is to make the ordering decisions systematic and centralized through the use of a computer-aided ordering (CAO) system. For example, Tesco moved the ownership of ordering decisions from store managers to the supply chain director (Corsten and Gruen, 2004). However this leads to local information that is only available to store managers such as local events, variations in local demand, adjustments for spoilage or shrinkage not being used for these critical decisions. Therefore, many retailers that use CAO systems end up allowing stores to override the CAO recommendations (FMI/GMA Trading Partner Alliance, 2015). Both retailers that we had communication with also have CAO systems, with override privileges granted to their store managers. Hence, incentive schemes are still necessary in most retail settings when an information asymmetry exists between stores and headquarters.

In this paper, we propose an incentive and measurement scheme that attempts to include stock-outs in scorecards with an easy-to-implement measure and address information asymmetry issues mentioned above. We assume that a "principal" (a retailer or a manufacturer) needs to satisfy uncertain customer demand over a finite horizon. The principal incurs the typical underage and overage costs: for every unit of demand that is not satisfied there is a shortage cost; any inventory left over at the end of the horizon incurs a holding cost per unit. Replenishment can take place only before the horizon and that decision is delegated to an agent (such as a store manager or an inventory manager) who is better informed about the demand process. The principal can observe the inventory and sales throughout the horizon, but cannot observe unsatisfied demand. Therefore, an incentive scheme based on the amount of shortages (such as penalizing the agent for underage and overage in the same proportion of the principal's underage and overage costs) is not possible. The challenge for the principal is to design an incentive mechanism that induces the agent to make an ordering decision that minimizes the principal's expected overage and underage costs under unobservable shortages and incomplete demand information. We suggest to incorporate two measures into the store manager's scorecard: A lump-sum penalty if a stock-out is observed at a pre-specified instant in the horizon and a penalty proportional to the remaining stock at the end of the horizon, both to be deducted from the performance score of the store manager.

We study the case where the underlying demand process is a Wiener Process, and the principal only knows that the process is one of a finite number of such processes. We show that when the possible Wiener Processes share the same variance, it is possible to induce the agent to order the optimal quantity without revealing the exact demand information to the principal when the stock-out measure is enforced based on the inventory level at the end-of-the horizon. What is even more interesting is that checking and penalizing the stock-outs at an optimal time inside the horizon (i.e., an "early inspection" scheme), rather than at the end, leads to perfect or near-perfect alignment in more general cases. In particular, we show that the early inspection scheme can lead to a performance that is strictly better than penalizing the stock-outs only at the end of horizon. Our results also show that the proposed incentive schemes can be substantially more effective than policies in which the principal does not delegate the replenishment decision to the store or inventory manager, but instead relies on a CAO system.

The problem and our models are partially motivated by our interactions with a major European discount grocer with over 1000 stores. The company competes on low prices as well as high quality, which is ensured by a considerable weight of private label merchandise in its assortment that consists of approximately 1000 SKUs. Products are shipped to stores on a weekly schedule (certain products once a week, the others two or three times a week) from the company-owned distribution centers using the company's own fleet of trucks. The company uses a centralized CAO system to create replenishment orders for its stores. The store managers have limited authority to override the system. Actual order quantity recommendations of the system can be changed for only a fixed percentage of the SKUs. In addition, the store manager may include more SKUs in the replenishment, but the number of such new orders cannot exceed a certain percentage of SKUs that are already in the replenishment. The company pulls data from its ERP system and reports the number of SKUs without stocks at its stores and distribution centers to its senior management at the end of each day. This is used to estimate the stock-outs and lost sales at the store level and root causes are sought if the levels are unexpectedly high in a given day. The company acknowledges the fact that store employees may have local information that could lead to improved forecasts and replenishment, and perhaps to improved stock availability, but is unwilling to completely delegate the replenishment decision to its store employees. There are several reasons. First, as mentioned before, quantifying stock-outs and their effect on lost sales is difficult. Consequently, it is difficult to devise an incentive mechanism that specifically builds on the trade-off between shortage and inventory carrying costs. The company can only indirectly consider these in its current incentives;

store employees are rewarded for the sales and inventory shrinkage (loss of inventory due to spoilage, theft, shoplifting, etc.) below a specific target. Finally, the company perceives that stock-outs in its stores are not primarily due to store operations and they should focus more on problems at its distribution center operations, logistics and procurement. In Section 3.4, we performed an initial analysis for a limited number SKUs and store locations at this grocer. Our experiments with actual demand data show that the retailer may improve its profitability considerably by using the incentive schemes suggested in this paper. We do not have any data from the North American chain to validate our approach. However, the performance measurement and the oversight of store managers at this chain are similar, except that there are no restrictions on the override of CAO recommendations.

While this incentive problem is very relevant for many industries with products that are replenished periodically (such as fast-moving consumer goods, food and beverages, etc.), an alternative motivation comes from another application we faced in the banking industry. The headquarters of a bank delegates the nightly cash-loading decisions at its vast number of ATMs to local branches which are believed to have more information about the cash demand in their localities (at least more than what the headquarters may possess or can process). Excess cash carried in the ATMs is obviously costly due to possible interest charges. Unsatisfied cash withdrawals are also costly due to loss of goodwill. In addition, a good portion of withdrawals may lead to additional revenue for the bank due to withdrawal fees or interests charged to credit cards. However, all these costs or penalties are primarily borne by the headquarters. Since an ATM also serves the customers of different branches or different banks, the local branch may not directly associate the service level at the ATM to its own profitability or customer satisfaction. To prevent further customer dissatisfaction, ATMs usually warn the customers in case of a cash stock-out. Therefore, unsatisfied cash withdrawal requests are not observable. The problem for the headquarters is to design an incentive scheme for the branch managers such that they replenish ATMs with the amount of cash that minimizes the headquarters' expected inventory holding and shortage costs.

This paper is related to the literature on incentive alignment problems in supply chains. These problems arise mainly due to hidden actions by the players in the chain, information asymmetries or badly designed incentive schemes (Narayanan and Raman, 2004). Incentive problems are relevant even for vertically integrated firms, as decisions at different echelons are often delegated to individuals whose performance measurement schemes are not well aligned with the overall profitability of the firm (Lee and Whang, 1999). Aligning or redesigning incentive schemes may yield significant increases in profitability of the supply chain whether it is within the boundaries of a single firm or consists of multiple independent firms. See Chen (2001) for a general review of the earlier literature in this area. Information asymmetry can exist for cost parameters and/or demand process. Asymmetric demand information is frequently observed in practice as the party closer to the customer will have more information about localities and past sales (Khanjari et al., 2014). Recent examples of asymmetric demand information considered in a supply chain context include papers by Babich et al. (2012), Akan et al. (2012), and Dai and Jerath (2013). Similar to our approach, these papers assume a finite set of states where each state has a corresponding and known demand distribution. One of the supply chain parties exactly knows the state, whereas the other party has a probabilistic knowledge. A related stream of literature is on accounting-based performance measures that lead to effective delegation or goal-congruent performance measures (e.g., Baldenius and Reichelstein, 2005). However, the emphasis of this literature on inventory management is cases where a product is manufactured and sold in different periods. In our setting, we assume a single period and focus on delegation of inventory decisions under demand uncertainty. We also note that there are studies on designing incentive contracts for store employees in retail settings (e.g., DeHoratius and Raman, 2007). However the focus in this strand of literature is usually on aligning incentives when the store employees need to allocate effort between multiple tasks.

Our paper is also related to the principal-agent problem which is a well-investigated topic in economics literature (see Laffont and Martimort, 2009, for a comprehensive review of this problem).

In its most general form, a principal delegates a certain task to an agent through a contract which induces the agent to act in alignment with the principal's objective. Van Ackere (1993) and Schenk-Mathes (1995) analyze this problem from an operations management perspective. In particular, the agent is the more informed salesperson who decides how much effort to exert to generate and flourish the demand, and the principal offers a corresponding incentive scheme (such as a sales target and bonus). Zhang and Zenios (2008) extend the basic model to multiple periods and dynamic information structures. Different from these studies, our focus is not the sales effort; we assume that the agent's (the store manager or the inventory manager) effort is fixed. Chu and Lai (2013), Chen (2000), and Dai and Jerath (2013) extend the basic principal-agent model to include inventory replenishment decisions which we also consider. However, unlike our setting, these studies assume that the inventory replenishment decisions are made by the principal, and exploit the interaction between product availability and the effort spent by the agent.

The incentive mechanism that we consider involves a fixed-penalty score deduction for stock-outs. Inventory management under lump-sum penalty costs for shortages has been studied before in the inventory literature, see, for example, papers by Bell and Noori (1984), Aneja and Noori (1987), Cetinkaya and Parlar (1989), and Benkherouf and Sethi (2010). In fact, the first two papers were also inspired by the examples from the banking industry which also partially motivated us for this study. Our models extend the models in this stream to inventory management under unobservable shortages and information asymmetry. Sieke et al. (2012) provide some examples from different industries where fixed-penalty costs are charged when a party cannot satisfy a preset service level. In particular, the authors consider the design problem of a "flat penalty contract" in which the supplier is penalized if she cannot satisfy a percentage of the orders placed by a manufacturer. They do not consider information asymmetry. Geng and Minutolo (2010) consider a similar problem under information asymmetry. The retailer allocates space for a manufacturer's product in return for a fixed slotting (or failure) fee. The retailer (which is the more powerful party in the chain) designs a contract which specifies a target sales volume and a failure fee which is charged to the manufacturer if the sales target cannot be met at the end of the planning horizon. The retailer also sets the ordering quantity and the sales price. Our problem setting is different as the owner of the product is the more powerful party and has no control over the demand.

In this paper, we make the following contributions:

- We propose easy-to-implement performance measurement schemes to align the incentives of store managers of multi-store retailers to those of their headquarters in settings where sales are heavily driven by on-shelf availability. These schemes utilize a lump-sum penalty for a stock-out occasion and a linear penalty for holding excess stock. Adopting such schemes may help retailers reduce stock-outs and attain desired service levels across their stores.

- Under demand information asymmetry and certain conditions, we show that these schemes lead to perfect incentive alignment.

- We show that "early inspection" schemes in which stock-outs are checked before the end of the horizon, may lead to better alignment than checking for stock-outs only at the end of the horizon. Early inspection unveils the severity of the lost demand with elevated precision, and this leads to better alignment.

- Under more general settings, we show through numerical studies that these schemes lead to near-perfect alignment of incentives.

- Even though our modeling approach predicates on a single item analysis, we propose an easy-to-implement heuristic method that extends the schemes we suggest to handle multiple items simultaneously. We show through numerical experiments that this method has an excellent performance.

- By using the historical sales data of a retail chain, we show that these schemes can be used in practice and lead to considerable revenue improvements.

The rest of the paper is organized as follows. In Section 2, we introduce the proposed incentive schemes and analyze them under complete and incomplete information. We propose two incentive schemes and show conditions where perfect alignment is possible. In Section 3, we conduct a numerical study to quantify the benefits of the proposed incentive mechanisms. We conclude the paper in Section 4.

2. Analysis of Alternative Incentive Schemes for Measuring Inventory Management Performance

We consider a single item which is subject to stochastic demand and offered to the market through several stores or sales channels. The principal, the owner or the main stakeholder of the item, hires agents (e.g., store or inventory managers) who are in charge of inventory replenishment decisions and sales operations. The stores could be different in size, located at distant regions, or have structurally different demand patterns. The agents are able to observe full demand information – the realized and lost demand – at their store, and have the most complete information for forecasting their future demand. We assume that the item observes exogenous demand, indicating that the sales effort exerted by the agents is fixed or does not influence the demand.

In this section, we propose and analyze alternative schemes used to evaluate the inventory management performance of the agents. These schemes will aid the principal, albeit having incomplete demand information, to incentivize the agents to make decisions in alignment with her objective. In the following analysis, we focus on the principal's interaction with one agent only. This analysis can be replicated for all other agents since their operations are independent from each other.

The planning horizon is finite with length T. The agent makes a single replenishment decision at $t = 0$, the start of the planning horizon. The item observes stochastic demand on a continuous basis until $t = T$, the end of the horizon. We assume that the accumulated demand at time t, $X(t)$, follows a Wiener Process with drift μ and variance σ^2 throughout the horizon. By definition of the Wiener Process, $X(0) = 0$ and the joint distribution of $X(t_0), X(t_1), ..., X(t_n)$ when $t_n > t_{n-1} > \cdots > t_1 > t_0 > 0$ satisfies the following conditions:

1. The differences $X(t_k) - X(t_{k-1})$ (total demand observed between t_{k-1} and t_k) are mutually independent normal distributed random variables.
2. The mean of the difference $X(t_k) - X(t_{k-1}) = (t_k - t_{k-1})\mu$.
3. $VAR\left[X(t_k) - X(t_{k-1})\right] = (t_k - t_{k-1})\sigma^2$.

Wiener Process or Brownian motion is frequently used in the inventory literature (e.g., Rudi et al., 2009, Rao, 2003). In order to ensure that the probability of negative demand is negligibly small, one can assume that the drift μ is sufficiently larger than variance σ (e.g., $\mu > 3.5\sigma$). By definition, the demand observed throughout the horizon, denoted by D, follows Normal distribution with mean $T\mu$ and variance $T\sigma^2$. The agent knows the exact values of these parameters whereas the principal has partial information. In particular, we assume that the principal knows that the demand is governed by one of N possible Wiener Processes with a parameter pair (μ_i, σ_i) with probability λ_i for $i = 1, ..., N$ where $\sum_{i=1}^{N} \lambda_i = 1$.

The principal incurs an overage cost, c_o, for each unit of excess inventory and an underage cost, c_u, for each unit of unmet demand at the end of the horizon at each of the stores. The excess inventory or an out-of-stock situation at the stores can easily be tracked by both the principal and the agent with an information system, however, the principal cannot observe the quantity of the lost demand, if any. Moreover, the principal has incomplete information of the demand distribution. Consequently, the principal cannot determine the optimal replenishment quantity that minimizes her expected total overage and underage cost. As a remedy to this, we devise two incentive schemes that will be imposed on the agent based on his inventory replenishment decisions. It is assumed that the agent will take the optimal course of action which maximizes his performance (equivalent to minimizing the his performance score in the imposed scheme). The principal will utilize this to incentivize the agent to take actions in favor of her objective.

We propose two incentive schemes, $[M]$ and $[M, t]$, each of which contains two key performance indicators: excess inventory and shortage. Let I_t be the on-hand inventory at time t and $1_{\{I_t=0\}}$ be an indicator function which is equal to 1 if $I_t = 0$, and 0 otherwise. The performance score of scheme $[M]$ is calculated by

$$PS_{[M]} = \hat{c}_o \times I_T + M \times 1_{\{I_T=0\}}$$

where $\hat{c}_o$ and M are the parameters to be set by the principal. Similarly, the performance score of the scheme $[M, t]$ is calculated by

$$PS_{[Mt]} = \hat{c}_o \times I_T + M \times 1_{\{I_t=0\}}$$

where $\hat{c}_o$, M, and t are the parameters to be set by the principal. Since both excess inventory and inventory shortage are costly for the principal, the lower values of the scores in these schemes indicate a better performance for the agent.

The agent decides on the number of items to order, denoted by Q_a, at the beginning of each planning horizon in a way that minimizes his performance score. Predicating on the prospective decision of the agent, the principal has the liberty to set the values of $\hat{c}_o$, M, and/or t, so that the Q_a value chosen by the agent is close to the order quantity that minimizes the principal's expected total overage and underage cost.

2.1. Scheme $[M]$

In this section, we analyze scheme $[M]$ from the principal's and agent's perspectives.

2.1.1. Agent's Problem Under scheme $[M]$, the expected performance score of the agent is given by the following expression:

$$EPS_{[M]}(Q_a) = \hat{c}_o E[\max\{Q_a - D, 0\}] + MP\{D \geq Q_a\} = \int_{-\infty}^{Q_a} \hat{c}_0 (Q_a - x) f(x) dx + \int_{Q_a}^{\infty} M f(x) dx$$

where f is the probability density function (pdf) of D, the total demand observed during the planning horizon. For the case of Wiener Process with parameters μ and σ, f is the density of a Normal random variable with mean $T\mu$ and standard deviation $\sqrt{T}\sigma$. We assume that the agent has complete demand information, i.e., these parameters are known by the agent.

We first provide a theorem which shows that the optimal order quantity that maximizes agent's performance is a function of the reversed hazard rate function of the demand distribution. Reversed hazard rate is defined by $\frac{f(x)}{F(x)}$ for any random variable with pdf, $f(x)$, and cumulative distribution function (cdf), $F(x)$. Marshall and Olkin (2007) show that this function is decreasing if the random variable has log-concave density. Many densities including uniform and Normal are log-concave (Bagnoli and Bergstrom, 2005).

Theorem 1 *The optimal order quantity of the agent that maximizes his performance under scheme* $[M]$ *is given by* $Q_a^* = r^{-1}\left(\frac{M}{\hat{c}_o}\right)$ *where* $r(x) = \frac{F(x)}{f(x)}$ *is the reciprocal of the reversed hazard rate function of the demand distribution.*

Proof: The first order condition of the function $EPS_{[M]}(Q_a)$ yields the following equality.

$$\frac{F(Q_a^*)}{f(Q_a^*)} = \frac{M}{\hat{c}_o}.$$

Since, the reversed hazard rate is a decreasing function when the demand is Normal (because of its log-concavity), $r(Q_a^*) = \frac{F(Q_a^*)}{f(Q_a^*)}$ is an increasing function and hence, $EPS_{[M]}(Q_a)$ is quasi–convex and its unique extremum is a minimum. □

If the principal uses scheme $[M]$ with parameters $\hat{c}_o$ and M as the performance measure of the agent and knows the demand distribution with certainty, then she can anticipate that the agent will order $r^{-1}\left(\frac{M}{\hat{c}_o}\right)$ units for each planning horizon.

2.1.2. Principal's Problem under Complete Information Suppose that there is no information asymmetry between the agent and the principal. This implies that $N = 1$ and $\lambda_1 = 1$. Principal's objective function can be stated as:

$$ETC(Q_p) = c_o E[\max\{Q_p - D, 0\} + c_u E[\max\{D - Q_p, 0\}]$$

where Q_p is the replenishment quantity. This is the well-studied Newsvendor problem and it is well-known that $Q_p^* = F^{-1}(\alpha)$ minimizes the above function where $\alpha = \frac{c_u}{c_u + c_o}$. The principal wants the agent to order $Q_a = Q_p^*$. The principal can achieve this by anticipating the optimal action of the principal stated in Theorem 1 and imposing the incentive scheme $[M]$ with the values of the parameters $\hat{c}_o$ and M that satisfy

$$Q_a^* = Q_p^* \Rightarrow \frac{M}{\hat{c}_o} = \frac{F(Q_a^*)}{f(Q_a^*)} = \frac{F(Q_p^*)}{f(Q_p^*)} = \frac{F(F^{-1}(\alpha))}{f(F^{-1}(\alpha))} = \frac{\alpha}{f(F^{-1}(\alpha))}.$$

Since the parameters of the incentive scheme do not correspond to any financial value, $\hat{c}_o$ can simply be set to 1 and the parameter M can be set to

$$M = \frac{\alpha}{f(F^{-1}(\alpha))}.$$

The function $s(\tau) = \frac{1}{f(F^{-1}(\tau))}$ is called the sparsity function by Tukey (1965) or the quantile density function by Parzen (1979). The sparsity function for Normal density with μ and σ is given by

$s_N(\tau) = \sqrt{2\pi}\sigma e^{\frac{\left(\Phi^{-1}(\tau)\right)^2}{2}}$ where Φ is the cdf of the standard Normal random variable. Consequently, a perfect alignment is possible under scheme $[M]$ and complete information, when the principal sets the parameters to $\hat{c}_o = 1$ and $M = \alpha s_N(\alpha)$. We note that this quantity is independent of the mean demand μ and is only a function of the standard deviation.

2.1.3. Principal's Problem under Incomplete Information Under incomplete information, we know by Theorem 1 that the agent orders

$$Q_i = r_i^{-1}\left(\frac{M}{\hat{c}_o}\right),$$

if the exact demand process is D_i with parameters μ_i and σ_i. Then the corresponding cost incurred for the principal is

$$ETC_i(Q_i) = c_o E[\max\{Q_i - D_i, 0\} + c_u E[\max\{D_i - Q_i, 0\}].$$

Since the exact demand distribution could be any of the N scenarios with probability λ_i, the principal's expected cost would be

$$\sum_{i=1}^{N} \lambda_i \cdot ETC_i(Q_i).$$

By presetting the parameter $\hat{c}_o = 1$, the principal solves the following optimization problem

$$\begin{aligned} PP^M(N) : \text{Min } & \sum_{i=1}^{N} \lambda_i ETC_i(Q_i) \\ \text{s.to } & Q_i = r_i^{-1}(M) \quad \forall i \end{aligned}$$

to find the value for the parameter M that will incentivize the agent to order in a way that minimizes the principal's expected total cost under $[M]$ scheme. In this problem, the decision variables are Q_i for $i = 1, \ldots, N$ and M. This problem can be restated as an unconstrained optimization problem with a single decision variable, M as

$$\text{Min } \sum_{i=1}^{N} \lambda_i ETC_i(r_i^{-1}(M)). \tag{1}$$

Clearly, the principal's cost under incomplete information is higher than or equal to her cost under complete information as a single M value may not lead the agent to order Q_p^* in each possible scenario. If this could be achieved, then a perfect alignment situation would be instated. This ideal situation can be achieved only if $r_1(Q_p^*) = r_2(Q_p^*) = \cdots = r_N(Q_p^*)$. In such a case, setting $M = r_i(Q_p^*) = r_i(F^{-1}(\alpha))$ for any demand scenario i would lead to perfect alignment. The following theorem characterizes a situation where this is attainable.

Theorem 2 *If the possible demand processes have the same standard deviation σ, then setting $M^* = \alpha s(\alpha)$ leads the agent to select Q_p^* in each possible demand scenario i.*

Proof: If all distributions have the same σ, then $r_i(Q_p^*) = \alpha s(\alpha)$ for all i as $s(\alpha)$ is only a function of σ. □

Theorem 2 states that perfect alignment is possible under scheme $[M]$ with a reasonable demand scenario[1]. This scenario is observed when the variability of demand is exogenous to the factors that differentiate alternative demand processes. In such cases, the differentiating factor will be the expected value of demand. In all cases other than this scenario, perfect alignment is not possible. For such cases, the following theorem provides a system of linear equations that solves Problem (1).

Theorem 3 *The following system of linear equations yields the optimal value of parameter M.*

$$\sum_{i=1}^{N} \frac{\lambda_i(\Phi(z_i) - \alpha)}{1 + z_i \frac{\Phi(z_i)}{\phi(z_i)}} = 0 \text{ and } \sigma_i \Phi(z_i) = M\phi(z_i), \ \ i = 1, \ldots, n$$

Proof : Recall that the agent's optimal ordering quantity satisfies $r(Q) = \frac{F(Q)}{f(Q)} = M$. Since the demand is Normal, this equation can be rewritten as $r(z) = \frac{\sigma\Phi(z)}{\phi(z)} = M$ with $Q = \mu + z\sigma$ transformation. Similarly, the expected total cost of the principal can be rewritten as follows (Porteus, 2002):

$$\begin{aligned} ETC(Q) \equiv L(z) &= \sigma\left(c_o z + (c_o + c_u)\left[\phi(z) - z(1 - \Phi(z))\right]\right) \\ &= \sigma\left((c_o + c_u) z\Phi(z) + (c_o + c_u)\phi(z) - c_u z\right). \end{aligned}$$

[1] Note that perfect alignment may be obtained for demand processes other than Wiener. For example, the sparsity function for uniform distribution between a and b is given by $s(\tau) = b - a$. Therefore, if the total demand is distributed with one of a number of uniform distributions with same interval length, the principal can also incentivize the agent to order its optimal quantity.

By using $z = \Phi^{-1}(\alpha)$, $L(\Phi^{-1}(\alpha)) = \sigma\left[(c_o + c_u)\Phi^{-1}(\alpha)\alpha + (c_o + c_u)\phi(\Phi^{-1}(\alpha)) - c_u\Phi^{-1}(\alpha)\right]$. Using $\alpha^* = c_u/(c_o + c_u)$ and multiplying this function with $1/(c_o + c_u)$, we obtain $\hat{L}(\Phi^{-1}(\alpha)) = \frac{1}{c_o+c_u}L(\Phi^{-1}(\alpha)) = \sigma\left[\alpha_i\Phi^{-1}(\alpha_i) + \phi(\Phi^{-1}(\alpha_i)) - \alpha^*\Phi^{-1}(\alpha_i)\right]$.

Using these expressions, (1) is equivalent to the following optimization problem:

$$\min_M \sum_{i=1}^{N} \lambda_i \hat{L}(r^{-1}(M/\sigma_i))$$

where we redefine $r(z)$ as $\frac{\Phi(z)}{\phi(z)}$ in the rest of this proof. Then, the first order condition is

$$\sum_{i=1}^{N} \lambda_i \hat{L}'(r^{-1}(M/\sigma_i))\frac{dr^{-1}(M/\sigma_i)}{dM} = 0.$$

Since $r(r^{-1}(x)) = x$, differentiating both sides, we get

$$r'(r^{-1}(x))\frac{dr^{-1}(x)}{dx} = 1.$$

But,

$$r'(z) = \frac{(\phi(z))^2 - \phi'(z)\Phi(z)}{(\phi(z))^2} = \frac{\phi(z) + z\Phi(z)}{\phi(z)} = 1 + zr(z) \Rightarrow \frac{dr^{-1}(M/\sigma_i)}{dM} = \frac{(1/\sigma_i)}{(1 + (M/\sigma_i)r(M/\sigma_i))}.$$

Then the first order condition becomes

$$\sum_{i=1}^{N} \lambda_i\sigma_i \frac{(\Phi(r^{-1}(M/\sigma_i)) - \alpha^*)}{\sigma_i(1 + r^{-1}(M/\sigma_i)(M/\sigma_i))} = \sum_{i=1}^{N} \frac{\lambda_i(\Phi(r^{-1}(M/\sigma_i)) - \alpha^*)}{(1 + r^{-1}(M/\sigma_i)(M/\sigma_i))} = 0.$$

We can also write this as a system of equations in z_i and M to obtain the desired result.

$$\sum_{i=1}^{N} \frac{\lambda_i(\Phi(z_i) - \alpha^*)}{1 + z_i\frac{\Phi(z_i)}{\phi(z_i)}} = 0 \text{ and } \sigma_i\Phi(z_i) = M\phi(z_i) \ \ \forall i = 1, \ldots, N.$$

□

2.2. Scheme $[M, t]$

Scheme $[M, t]$ is similar to scheme $[M]$ with the difference that the M value is contributed to the performance score if on-hand inventory at time $t < T$ is equal to zero, rather than at $t = T$. The scheme can be called as an "early inspection" scheme and builds on the fact that discovering a stock-out earlier in the horizon may lead to a better understanding of the actual amount of unsatisfied demand for the principal. Notice also that this scheme provides more information to the principal since a stock-out at given time $t < T$ also means a stock-out at time T, but not vice versa. First, we analyze the agent's problem in detail under this scheme, and then show that it outperforms scheme $[M]$ under certain conditions. Without loss of generality, we set $T = 1$ in the following discussion.

2.2.1. Agent's Problem Let Y and $\overline{Y}$ be the random variables denoting demand during $[0, t]$ and $[t, 1]$, respectively. Then, we have $X(t) - X(0) = Y \sim N(t\mu, \sqrt{t}\sigma)$ and $X(1) - X(t) = \overline{Y} \sim N((1 - t)\mu, \sqrt{1-t}\sigma)$ by the definition of Wiener Process. Let G (g) and H (h) denote the cdf (pdf) of the random variables Y and $\overline{Y}$, respectively. Then, the agent's performance score for any realization of y and $\overline{y}$, and order quantity of Q_a, is given by

$$PS_{[Mt]}(Q_a) = \begin{cases} M & \text{if } y \geq Q_a \text{ and } y + \overline{y} \geq Q_a \\ M + \hat{c}_o(Q_a - (y + \overline{y})) & \text{if } y \geq Q_a \text{ and } y + \overline{y} \leq Q_a \\ 0 & \text{if } y \leq Q_a \text{ and } y + \overline{y} \geq Q_a \\ \hat{c}_o(Q_a - (y + \overline{y})) & \text{if } y \leq Q_a \text{ and } y + \overline{y} \leq Q_a. \end{cases}$$

Then, the expected performance score for any order quantity Q_a is given by

$$\begin{aligned} EPS_{[Mt]}(Q_a) &= \int_{Q_a}^{\infty} \int_{Q_a - y}^{\infty} Mh(\overline{y})g(y)d\overline{y}dy + 0 + 0 + \int_{-\infty}^{Q_a} \int_{-\infty}^{Q_a - y} \hat{c}_o h\left(Q_a - (y + \overline{y})\right) h(\overline{y})g(y)d\overline{y}dy \\ &= \int_{Q_a}^{\infty} Mg(y)dy + \int_{-\infty}^{Q_a} \int_{-\infty}^{Q_a - y} \hat{c}_o h(Q_a - (y + \overline{y}))h(\overline{y})g(y)d\overline{y}dy \\ &= M(1 - G(Q_a)) + \int_{-\infty}^{Q_a} \int_{-\infty}^{Q_a - y} \hat{c}_o h(Q_a - (y + \overline{y}))h(\overline{y})g(y)d\overline{y}dy. \end{aligned}$$

A Wiener Process, $W(\mu, \sigma^2; t)$, has the following Markovian property (see Patel and Read, 1982):

Property 1 *The $W(\mu, \sigma^2; t)$ process has the Markov property that the conditional distribution of $X(t) - a$, given that $X(u) = a$ for $t > u$, is that of $\mu(t - u) + \sigma Z \sqrt{t - u}$ where Z is the standard Normal variate. This distribution is independent of the history of the process up to time u.*

Property 1 implies that the total demand observed during the planning horizon, $Y+\overline{Y}$, is independent of any particular realization of Y. Consequently, the agent's expected cost function can be re-written as

$$EPS_{[Mt]}(Q_a) = MP\{Y \geq Q_a\} + \hat{c}_o E[\max\{Q_a - D\}, 0] = M \int_{Q_a}^{\infty} g(x)dx + \hat{c}_o \int_{-\infty}^{Q_a} (Q_a - x) f(x)dx.$$

By letting $\hat{c}_o = 1$, $z = \frac{Q_a - \mu}{\sigma}$, $z_t = \frac{Q_a - t\mu}{\sqrt{t}\sigma}$, $k = \frac{\mu}{\sigma}$, $a = k(1-t)$, and noting that $z_t = \frac{a+z}{\sqrt{t}}$, we have

$$EPS'_{[Mt]}(Q_a) = F(Q_a) - Mg(Q_a) \equiv \omega(z) = \Phi(z) - \frac{M}{\sqrt{t}\sigma}\phi\left(\frac{a+z}{\sqrt{t}}\right), \tag{2}$$

$$EPS''_{[Mt]}(Q_a) = f(Q_a) - Mg'(Q_a) = \omega'(z) \equiv \delta(z) = \phi(z) + \frac{M(a+z)}{t\sqrt{t}\sigma}\phi\left(\frac{a+z}{\sqrt{t}}\right). \tag{3}$$

Lemma 1 *If* $M > \frac{e^{-\frac{1}{2}(k^2(1-t))}\sqrt{t}\sigma}{k}$ *then* $\omega(z)$ *has only one local finite minimum. Otherwise,* $\omega(-k) \leq \omega(z)$ *for all* $z \geq -k$.

Proof: First we show that the local optima of $\omega(z)$ for $z \geq -k$ are finite.

$$\lim_{z \to -k} \omega(z) = \Phi(-k) - \frac{M}{\sqrt{t}\sigma}\phi\left(\frac{k(1-t)-k}{\sqrt{t}}\right) = \Phi(-k) - \frac{M\phi(-k)}{\sqrt{t}\sigma}.$$

$$\lim_{z \to \infty} \omega(z) = \Phi(\infty) - \frac{M}{\sqrt{t}\sigma}\phi\left(\frac{k(1-t)+\infty}{\sqrt{t}}\right) = 1 - 0 = 1.$$

Moreover, $\omega(z)$ is a continuous function for all $z : z \in [-k, \infty)$. Hence, all local optima of $\omega(z)$ must be finite. Next, we show that there can be at most one local optimum of $\omega(z)$. We show this by showing that there can be at most one z that satisfies $\omega'(z) = \delta(z) = 0$. Equating (3) to 0, we get

$$\begin{aligned}
\phi(z) &= \frac{-M(a+z)}{t\sqrt{t}\sigma}\phi\left(\frac{a+z}{\sqrt{t}}\right), \text{or} \\
\frac{\phi(z)}{\phi\left(\frac{a+z}{\sqrt{t}}\right)} + \frac{Mz}{t\sqrt{t}\sigma} &= \frac{-Ma}{t\sqrt{t}\sigma}, \text{or} \\
e^{-\frac{1}{2}\left(z^2 - \frac{(a+z)^2}{t}\right)} + \frac{Mz}{t\sqrt{t}\sigma} &= \frac{-Ma}{t\sqrt{t}\sigma}.
\end{aligned} \tag{4}$$

Denote the left-hand side of (4) by $d(z)$. Next, we show that $d(z)$ is an increasing function. One can show that

$$d'(z) = -\frac{1}{2}\left(2z - \frac{2(a+z)}{t}\right)e^{-\frac{1}{2}\left(z^2 - \frac{(a+z)^2}{t}\right)} + \frac{M}{t\sqrt{t}\sigma} \geq 0,$$

since $a \geq 0$, $t \leq 1$, and $2z \leq \frac{2z}{t}$. Therefore, (4) can hold only for one value of z since its right-hand side is a constant. Consequently, if $d(-k) > \frac{-Ma}{t\sqrt{t}\sigma}$ then $\delta(z) \neq 0$ for all $z \geq -k$ and $w(z)$ has one local minimum. Since

$$d(-k) = e^{-\frac{1}{2}\left(k^2 - \frac{(k(1-t)-k)^2}{t}\right)} - \frac{Mk}{t\sqrt{t}\sigma},$$

$d(-k) > \frac{-Ma}{t\sqrt{t}\sigma}$ is equivalent to the condition

$$\frac{e^{-\frac{1}{2}(k^2(1-t))}\sqrt{t}\sigma}{k} < M. \;\square$$

We are now ready to present the main result of this section. The following theorem characterizes the optimal ordering quantities of the agent under different parameter ranges.

Theorem 4 *The optimal order quantity is characterized by the following rules:*

1. *If* $M > \frac{\Phi(-k)}{\phi(-k\sqrt{t})}\sqrt{t}\sigma$ *then* $z^* : \omega(z^*) = 0$ *is unique and corresponds to the optimal order quantity.*
2. *If* $M < \frac{\Phi(-k)}{\phi(-k\sqrt{t})}\sqrt{t}\sigma$ *and* $M < \frac{e^{-\frac{1}{2}(k^2(1-t))}\sqrt{t}\sigma}{k}$ *then* $z^* = -k$.
3. *If* $\frac{e^{-\frac{1}{2}(k^2(1-t))}\sqrt{t}\sigma}{k} < M < \frac{\Phi(-k)}{\phi(-k\sqrt{t})}\sqrt{t}\sigma$ *then there exist at most two values of z such that* $\omega(z) = 0$. *Either one of these two z values or* $z = -k$ *correspond to the optimal order quantity.*

Proof: First note that, $\lim_{z \to -k} \omega(z) = \Phi(-k) - \frac{M}{\sqrt{t}\sigma}\phi\left(\frac{k(1-t)-k}{\sqrt{t}}\right) = \Phi(-k) - \frac{M}{\sqrt{t}\sigma}\phi(-k\sqrt{t})$. So, $\omega(-k) < 0$ if $M > \frac{\Phi(-k)}{\phi(-k\sqrt{t})}\sqrt{t}\sigma$ and $\omega(-k) > 0$ otherwise.

(1) Under this condition $\omega(-k) < 0$. In this case, $\omega(z) = 0$ can hold only once for some $z \geq -k$ because of Lemma 1.

(2) Under this condition $\omega(-k) > 0$. When $\omega(-k)$ is positive and when $M < \frac{e^{-\frac{1}{2}(k^2(1-t))}\sqrt{t}\sigma}{k}$, $\omega(-k)$ cannot have any local minimum due to Lemma 1, and hence $\omega(z) > 0$ for all $z \geq -k$. Therefore, $z^* = -k$ must correspond to the optimal order quantity.

(3) Under this condition $\omega(-k) > 0$. When $M > \frac{e^{-\frac{1}{2}(k^2(1-t))}\sqrt{t}\sigma}{k}$, $\omega(z)$ has one local optimum at which either $\omega(z) > 0$ or $\omega(z) < 0$. If $\omega(z) > 0$, then $\omega(z)$ can never attain a value of 0 so $z^* = -k$ corresponds to the optimal order quantity. If $\omega(z) < 0$ then $\omega(z)$ must attain the value of zero twice so one of the z values that satisfy $\omega(z) = 0$ must correspond to the optimal order quantity. $\square$

2.2.2. Principal's Problem under Incomplete Information In this part, we assume that $M > \frac{\Phi(-k)}{\phi(-k\sqrt{t})}\sqrt{t}\sigma$ under which the optimal order quantity of the agent is the unique value that satisfies the first order condition (Theorem 4.1). From (2), the optimal order quantity of the agent satisfies the following equality when the agent's demand distribution assumes the ith parameters:

$$\frac{\Phi(z_i)}{\phi\left(\frac{k_i(1-t)+z_i}{\sqrt{t}}\right)}\sqrt{t}\sigma_i = M$$

where $k_i = \frac{\mu_i}{\sigma_i}$. Then, the principal solves the following problem:

$$PP^{Mt}(N) : \mathrm{Min}_{M,t,z_i} \sum_{i=1}^{N} \lambda_i L_i(z_i)$$
$$\text{s.to} \quad q_i(z_i, t) = M \;\; \forall i = 1, ..., N$$

where $L_i(z_i) = c_o z_i \sigma_i + (c_o + c_u)\sigma_i\left[\phi(z_i) - z_i(1 - \Phi(z_i))\right]$ and $q_i(z_i, t) = \frac{\Phi(z_i)}{\phi\left(\frac{k_i(1-t)+z_i}{\sqrt{t}}\right)}\sqrt{t}\sigma_i$. Note that $L_i(z_i) \equiv ETC_i(Q_i)$ when $Q_i = \mu_i + z_i\sigma_i$.

The principal prefers the agent to calculate $z_i^* = \Phi^{-1}\left(\frac{c_u}{c_o+c_u}\right) = \hat{z}$ and set his order quantity as $Q_i = \mu_i + \hat{z}\sigma_i$. This will lead to perfect alignment. The following result shows that this is achievable when $N = 2$ under a mild condition.

Theorem 5 *When $N = 2$, if $k_2 > k_1$, and $\sigma_2 < \sigma_1$, then there exists a τ such that $0 < \tau < 1$ which satisfies*

$$e^{-\frac{1}{2}\left[\frac{(k_2(1-\tau)+\hat{z})^2-(k_1(1-\tau)+\hat{z})^2}{\tau}\right]} = \frac{\sigma_2}{\sigma_1}.$$

The scheme $[M, t]$ with parameters $\hat{c}_o = 1$, $t = \tau$, and $M = \frac{\Phi(\hat{z})}{\phi\left(\frac{k_1(1-\tau)+\hat{z}}{\sqrt{\tau}}\right)}\sqrt{\tau}\sigma_1 = \frac{\Phi(\hat{z})}{\phi\left(\frac{k_2(1-\tau)+\hat{z}}{\sqrt{\tau}}\right)}\sqrt{\tau}\sigma_2$ incentivizes the agent to order an amount that corresponds to $\hat{z}$, which is equal to the optimal order quantity for the principal.

Proof: $PP^{Mt}(2)$ can be written as:

$$\mathrm{Min}_{t,z_1,z_2} \sum_{i=1}^{2} \lambda_i L_i(z_i)$$
$$\text{s.to} \quad q_1(z_1, t) = q_2(z_2, t)$$

Given the values of M and t, we know that agent will pick a z value that satisfies $q_1(z_1, t) = M$ or $q_2(z_2, t) = M$ depending on whether the exact demand distribution is f_1 or f_2. We also know that

$z_1 = z_2 = \hat{z}$ is the ideal operating point of the principal. Hence, the question is whether there is a τ value that is feasible for $PP^{Mt}(2)$ where $z_1 = z_2 = \hat{z}$? For feasibility, we must have $q_1(\hat{z}, \tau) = q_2(\hat{z}, \tau)$. This is equivalent to:

$$\begin{aligned}
\frac{\Phi(\hat{z})}{\phi\left(\frac{k_1(1-\tau)+\hat{z}}{\sqrt{\tau}}\right)}\sqrt{\tau}\sigma_1 &= \frac{\Phi(\hat{z})}{\phi\left(\frac{k_2(1-\tau)+\hat{z}}{\sqrt{\tau}}\right)}\sqrt{\tau}\sigma_2, \text{or} \\
\frac{\phi\left(\frac{k_2(1-\tau)+\hat{z}}{\sqrt{\tau}}\right)}{\phi\left(\frac{k_1(1-\tau)+\hat{z}}{\sqrt{\tau}}\right)} &= \frac{\sigma_2}{\sigma_1}, \text{or} \\
e^{-\frac{1}{2}\left[\frac{(k_2(1-\tau)+\hat{z})^2-(k_1(1-\tau)+\hat{z})^2}{\tau}\right]} &= \frac{\sigma_2}{\sigma_1}. \qquad (5)
\end{aligned}$$

Denote the left-hand side of (5) as $l(\tau)$. Note that $\lim_{t\to 0} l(t) = 0$, $\lim_{t\to 1} l(t) = 1$ and the function $l(t)$ is continuous on $(0, 1]$. Hence, Equation (5) must be satisfied at some $\tau \in (0, 1)$ since $0 < \frac{\sigma_2}{\sigma_1} < 1$. □

Note that the condition of Theorem 5 is also satisfied when $\mu_1 = \mu_2$ and $\sigma_1 > \sigma_2$. This result has two implications. First, it is possible to incentivize the agent to order a quantity which is equal to the optimal order quantity of the principal under incomplete information. Second, the scheme $[M, t]$ outperforms scheme $[M]$ under certain parameter ranges. Next, we generalize the latter result for $N > 2$.

Lemma 2 *If $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{N-1} \geq \sigma_N$, then $z_1^* \leq z_2^* \leq \cdots \leq z_{N-1}^* \leq z_N^*$ and $z_1^* \leq \hat{z} \leq z_N^*$ in the optimal solution of $PP^M(N)$ where $\hat{z} = \Phi^{-1}(\alpha)$.*

Proof : $PP^M(N)$ can be stated as

$$\begin{aligned}
\text{Min}_{z_1,\ldots,z_N} \quad & \sum_{i=1}^{N} \lambda_i L_i(z_i) \\
\text{s.to} \quad & \frac{\Phi(z_i)}{\phi(z_i)}\sigma_i = \frac{\Phi(z_{i+1})}{\phi(z_{i+1})}\sigma_{i+1} \quad \forall i = 1, \ldots, N-1.
\end{aligned}$$

Since $\frac{\Phi(z)}{\phi(z)}$ is monotonically increasing function and $\sigma_i \geq \sigma_{i+1}$, the ith constraint can be satisfied only if $z_i^* \leq z_{i+1}^*$ for all $i = 1, \ldots, N-1$. This proves the first inequality.

Case 1. $z_1^* \leq \cdots \leq z_N^* < \hat{z}$.

Let z_1' be such that $\frac{\Phi(\hat{z})}{\phi(\hat{z})}\sigma_N = \frac{\Phi(z_1')}{\phi(z_1')}\sigma_1$. Since $\frac{\Phi(z)}{\phi(z)}$ is increasing in z and $\sigma_1 \geq \sigma_N$, we have $z_1' \leq \hat{z}$. Moreover, since $z_N^* \leq \hat{z}$, we have $\frac{\Phi(z_N^*)}{\phi(z_N^*)} \leq \frac{\Phi(\hat{z})}{\phi(\hat{z})}$. Noting that z_1^* and z_N^* satisfy the constraints of $PP^M(N)$, we can write

$$\frac{\Phi(z_1^*)}{\phi(z_1^*)}\sigma_1 = \frac{\Phi(z_N^*)}{\phi(z_N^*)}\sigma_N < \frac{\Phi(\hat{z})}{\phi(\hat{z})}\sigma_N = \frac{\Phi(z_1')}{\phi(z_1')}\sigma_1.$$

$$\Rightarrow \frac{\Phi(z_1^*)}{\phi(z_1^*)}\sigma_1 < \frac{\Phi(z_1')}{\phi(z_1')}\sigma_1 \Rightarrow z_1^* < z_1' < \hat{z} \Rightarrow L_1(z_1^*) > L_1(z_1').$$

Let $z_2', z_3', \ldots, z_{N-1}'$ be such that $\frac{\Phi(z_1')}{\phi(z_1')}\sigma_1 = \frac{\Phi(z_i')}{\phi(z_i')}\sigma_i$ for all $i = 2, \ldots, N-1$. Since $\frac{\Phi(z_1^*)}{\phi(z_1^*)}\sigma_1 = \frac{\Phi(z_i^*)}{\phi(z_i^*)}\sigma_i$ for $i = 2, \ldots, N-1$ and $z_1^* < z_1'$, we must have $z_i^* < z_i'$ and $L_i(z_i^*) > L_(z_i')$ for all $i = 2, \ldots, N-1$. Last inequality is due to convexity of function L. Hence,

$$\lambda_1 L_1(z_1^*) + \lambda_2 L_2(z_2^*) + \cdots + \lambda_{N-1} L_{N-1}(z_{N-1}^*) + \lambda_N L_N(z_N^*) \geq \lambda_1 L_1(z_1') + \lambda_2 L_2(z_2') + \cdots + \lambda_{N-1} L_{N-1}(z_{N-1}') + \lambda_N L_N(\hat{z}).$$

Last inequality violates the optimality of $\langle z_1^*, z_2^*, \ldots, z_{N-1}^*, z_N^* \rangle$ as $\langle z_1', z_2', \ldots, z_{N-1}', \hat{z} \rangle$ is also feasible to $PP^M(N)$ and produces a lower cost. Hence, $z_N^* \nleq \hat{z}$.

Case 2. $\hat{z} < z_1^* \leq \cdots \leq z_N^*$.

This case can be proven with similar arguments of Case 1. □

Theorem 6 *If $k_1 \leq k_2 \leq \cdots \leq k_N$ and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_N$, then there exists a scheme $[M, t]$ with $t < 1$ which outperforms scheme $[M]$.*

Proof: Suppose that $< z_1^*, z_2^*, \ldots, z_N^* >$ is an optimal solution to $PP(N)$. Due to Lemma 2, we have $z_1^* \leq z_2^* \leq \cdots \leq z_N^*$.

We can rewrite $PP^{Mt}(N)$ as

$$\begin{aligned} \text{Min}_{t, z_1, z_2, \ldots, z_N} \quad & f(z_1, z_2, \ldots, z_N, t) \\ \text{s.to} \quad & g_i(z_1, z_2, \ldots, z_N, t) = 0 \quad \forall i = \{1, 2, \ldots, N-1\} \end{aligned}$$

where

$$\begin{aligned} f(z_1, z_2, \ldots, z_N, t) &= \lambda_1 L_1(z_1) + \lambda_2 L_2(z_2) + \cdots + \lambda_N L_N(z_N), \\ g_1(z_1, z_2, \ldots, z_N, t) &= q_1(z_1, t) - q_2(z_2, t), \\ g_2(z_1, z_2, \ldots, z_N, t) &= q_1(z_1, t) - q_3(z_3, t), \\ &\vdots \\ g_{N-1}(z_1, z_2, \ldots, z_N, t) &= q_1(z_1, t) - q_N(z_N, t). \end{aligned}$$

Note that $\overline{x}_0 = \langle z_1^*, z_2^*, \ldots, z_N^*, 1\rangle$ is also a feasible solution to $PP^{Mt}(N)$ and satisfy $\frac{\Phi(z_1^*)}{\phi(z_1^*)}\sigma_1 = \frac{\Phi(z_2^*)}{\phi(z_2^*)}\sigma_2 = \frac{\Phi(z_3^*)}{\phi(z_3^*)}\sigma_3 = \cdots = \frac{\Phi(z_N^*)}{\phi(z_N^*)}\sigma_N = M$ for some value of M.

$\overline{x}_0$ can be an optimal solution only if it satisfies the KKT necessary conditions

$$\nabla f(\overline{x}_0) + v_1 \cdot \nabla g_1(\overline{x}_0) + v_2 \cdot \nabla g_2(\overline{x}_0) + \cdots + v_{N-1} \cdot \nabla g_{N-1}(\overline{x}_0) = \overrightarrow{0} \tag{6}$$

for some values of $v_1, v_2, \ldots, v_{N-1}$. First we note that

$$\begin{aligned}
\frac{\partial q(z,t)}{\partial t} &= \frac{\frac{1}{2}\Phi(z)t^{-1/2}\sigma\phi\left(\frac{a+z}{\sqrt{t}}\right) + \Phi(z)\sqrt{t}\sigma\frac{a+z}{\sqrt{t}}\phi\left(\frac{a+z}{\sqrt{t}}\right)\frac{-k\sqrt{t}-(a+z)\frac{1}{2}t^{-1/2}}{t}}{\phi^2\left(\frac{a+z}{\sqrt{t}}\right)} \\
\left.\frac{\partial q(z,t)}{\partial t}\right|_{t=1} &= \frac{\Phi(z)}{\phi(z)}\sigma\left[\frac{1}{2} - z\left(k + \frac{z}{2}\right)\right].
\end{aligned}$$

Further, we can write the following partial derivatives:

$$\begin{aligned}
\frac{\partial f}{\partial z_1}(\overline{x}_0) &= \lambda_1 L_1'(z_1^*), \quad \frac{\partial f}{\partial z_2}(\overline{x}_0) = \lambda_2 L_2'(z_2^*), \ldots, \quad \frac{\partial f}{\partial z_N}(\overline{x}_0) = \lambda_N L_N'(z_N^*), \quad \frac{\partial f}{\partial t}(\overline{x}_0) = 0, \\
\frac{\partial g_1}{\partial z_1}(\overline{x}_0) &= \frac{\sigma_1(\phi(z_1^*) + z_1^*\Phi(z_1^*))}{\phi(z_1^*)} = \sigma_1 + z_1^* M, \\
\frac{\partial g_1}{\partial z_2}(\overline{x}_0) &= \frac{-\sigma_2(\phi(z_2^*) + z_2^*\Phi(z_2^*))}{\phi(z_2^*)} = -(\sigma_2 + z_2^* M), \\
\frac{\partial g_1}{\partial z_3}(\overline{x}_0) &= \cdots = \frac{\partial g_1}{\partial z_N}(\overline{x}_0) = 0, \\
\frac{\partial g_1}{\partial t}(\overline{x}_0) &= \frac{\Phi(z_1^*)\sigma_1}{\phi(z_1^*)}\left[\frac{1}{2} - z_1^*\left(k_1 + \frac{z_1^*}{2}\right)\right] - \frac{\Phi(z_2^*)\sigma_2}{\phi(z_2^*)}\left[\frac{1}{2} - z_2^*\left(k_2 + \frac{z_2^*}{2}\right)\right] = Mz_2^*\left(k_2 + \frac{z_2^*}{2}\right) - Mz_1^*\left(k_1 + \frac{z_1^*}{2}\right), \\
\frac{\partial g_2}{\partial z_1}(\overline{x}_0) &= \sigma_1 + z_1^* M, \quad \frac{\partial g_2}{\partial z_2}(\overline{x}_0) = 0, \quad \frac{\partial g_2}{\partial z_3}(\overline{x}_0) = -(\sigma_3 + z_3^* M), \quad \frac{\partial g_2}{\partial z_4}(\overline{x}_0) = \cdots = \frac{\partial g_2}{\partial z_N}(\overline{x}_0) = 0, \\
\frac{\partial g_2}{\partial t}(\overline{x}_0) &= \frac{\Phi(z_1^*)\sigma_1}{\phi(z_1^*)}\left[\frac{1}{2} - z_1^*\left(k_1 + \frac{z_1^*}{2}\right)\right] - \frac{\Phi(z_3^*)\sigma_3}{\phi(z_3^*)}\left[\frac{1}{2} - z_3^*\left(k_3 + \frac{z_3^*}{2}\right)\right] = Mz_3^*\left(k_3 + \frac{z_3^*}{2}\right) - Mz_1^*\left(k_1 + \frac{z_1^*}{2}\right), \\
&\vdots \\
\frac{\partial g_{N-1}}{\partial z_1}(\overline{x}_0) &= \sigma_1 + z_1^* M, \frac{\partial g_{N-1}}{\partial z_2}(\overline{x}_0) = \cdots = \frac{\partial g_{N-1}}{\partial z_{N-1}}(\overline{x}_0) = 0, \frac{\partial g_{N-1}}{\partial z_N}(\overline{x}_0) = -(\sigma_N + z_N^* M), \\
\frac{\partial g_{N-1}}{\partial t} &= Mz_N^*\left(k_N + \frac{z_N^*}{2}\right) - Mz_1^*\left(k_1 + \frac{z_1^*}{2}\right).
\end{aligned}$$

We can rewrite (6) as follows:

$$\lambda_1 L_1'(z_1^*) + v_1 \cdot (\sigma_1 + z_1^* M) + v_2(\sigma_1 + z_1^* M) + v_{N-1}(\sigma_1 + z_1^* M) = 0$$

$$\lambda_2 L_2'(z_2^*) - v_1(\sigma_2 + z_2^* M) = 0 \tag{7}$$

$$\lambda_3 L_3'(z_3^*) - v_2(\sigma_3 + z_3^* M) = 0 \tag{8}$$

$$\vdots$$

$$\lambda_N L_N'(z_N^*) - v_{N-1}(\sigma_N + z_N^* M) = 0 \tag{9}$$

$$0 + v_1 M\left(z_2^*\left(k_2 + \frac{z_2^*}{2}\right) - z_1^*\left(k_1 + \frac{z_1^*}{2}\right)\right) + v_2 M\left(z_3^*\left(k_3 + \frac{z_3^*}{2}\right) - z_1^*\left(k_1 + \frac{z_1^*}{2}\right)\right) + \cdots$$
$$+ v_{N-1} M\left(z_N^*\left(k_N + \frac{z_N^*}{2}\right) - z_1^*\left(k_1 + \frac{z_1^*}{2}\right)\right) = 0 \tag{10}$$

From (7) - (9): $v_i = \frac{\lambda_{i+1} L'_{i+1}(z^*_{i+1})}{\sigma_{i+1} + z^*_{i+1} M}$ for $i = 1, \ldots, N-1$. Plug $v_1, v_2, \ldots, v_{N-1}$ into (10) to obtain

$$\sum_{i=1}^{N-1} \frac{\lambda_{i+1} L'_{i+1}(z^*_{i+1})}{\sigma_{i+1} + z^*_{i+1} M} M\left[z^*_{i+1}\left(k_{i+1} + \frac{z^*_{i+1}}{2}\right) - z_1^*\left(k_1 + \frac{z_1^*}{2}\right)\right] = 0$$

$$\Rightarrow \sum_{i=1}^{N-1} \frac{\lambda_{i+1} L'_{i+1}(z^*_{i+1})}{\sigma_{i+1} + z^*_{i+1} M} z^*_{i+1}\left(k_{i+1} + \frac{z^*_{i+1}}{2}\right) = \sum_{i=1}^{N-1} \frac{\lambda_{i+1} L'_{i+1}(z^*_{i+1})}{\sigma_{i+1} + z^*_{i+1} M} z_1^*\left(k_1 + \frac{z_1^*}{2}\right)$$

However, this last equality cannot hold because $k_i \geq k_1$ and $z_i^* \geq z_1^*$ for all $i = 2, \ldots, N$, and hence $\overline{x}_0$ cannot be an optimal solution. □

2.3. Computer Aided Ordering

As discussed in Section 1, many multi-store retailers have embedded CAO method in their ordering systems. This computerized method suggests the store manager an order quantity for each item in the store, based on historical demand/sales data. The main pitfall is that the historical data might include demand streams originating from distinct populations and the store managers might have a better idea of the temporal realization of the streams. To remedy this pitfall, some retailers let the store managers override the proposed quantities to a certain extent.

Let $\hat{X}$ denote the random variable corresponding to the aggregated demand observed by the retailer. Then, along with the lines of our modeling approach, we have $\hat{X} = \sum_{i=1}^{N} \lambda_i X_i$. In this case, the CAO system will propose an order quantity, Q_p, which solves the following problem::

$$PP^{CAO} : \text{Min}_{Q_p} \quad \sum_{i=1}^{N} \lambda_i \left(c_o \int_{-\infty}^{Q_p} (Q_p - x) f_i(x)dx + c_u \int_{Q_p}^{\infty} (x - Q_p) f_i(x)dx\right)$$

where f_i is the pdf of the ith demand stream. The problem is equivalent to solving a Newsvendor problem when the demand follows a mixture of N Normal random variables, with ith variable having a mean $\mu_i T$ and standard deviation $\sigma_i \sqrt{T}$. In Section 3, we show through numerical analysis that a pure CAO system is inferior to our proposed incentive schemes.

3. Numerical Study

In this section, we present the results of a numerical study we conducted (i) to investigate the value of the proposed schemes in aligning incentives of both parties and (ii) to identify the parameter ranges for which these schemes bring a superior alignment. We experiment with two problem sets. The first set contains demand streams with the same coefficient of variations and the other with randomized parameters. In the first problem set, the number of possible demand distributions, N, is set to 3 with drift parameters $\mu_1 = 20, \mu_2 = 30$, and $\mu_3 = 40$. The standard deviation of each demand process is set to

$$\sigma_i = CoV \cdot \mu_i \;\; \forall i = 1,2,3 \text{ with } CoV \in \{0.1, 0.15, 0.2, 0.25\}.$$

Note that CoV corresponds to the coefficient of variation of the demand process and is assumed to be constant across all possible demand processes in a given problem instance. The overage cost, c_o, is set to 1 and the underage cost, c_u, takes one of the following values: $\{2, 5, 10, 20\}$. Finally, the weights of the potential demand processes are set to

$$\lambda_1 \in \{1/3, 0.5, 0.75, 0.9\}, \lambda_2 = \lambda_3 = \frac{1-\lambda_1}{2}.$$

Note that as λ_1 increases, the level of information asymmetry decreases. In total, we generate $4 \times 4 \times 4 = 64$ different problem instances by enumerating all possible combinations of the four different values for each of the parameters c_u (overage cost), CoV (demand variation), and λ_1 (level of information asymmetry).

Let ETC_{PI} denote the minimum expected total cost of the principal under perfect demand information. This value is given by the unconstrained solution of problem $PP^M(N)$ and is the principal's minimum possible cost value. Let ETC_M, ETC_{Mt}, and ETC_{CAO} denote the optimal cost obtained by solving $PP^M(N)$, $PP^{Mt}(N)$, and PP^{CAO}, respectively. We define Inc_M and Inc_{Mt} to denote the cost increment for schemes $[M]$ and $[M,t]$, respectively, relative to the ideal but hypothetical case of perfect information. We define these measures as

$$Inc_M = \frac{ETC_M - ETC_{PI}}{ETC_{PI}},$$

$$Inc_{Mt} = \frac{ETC_{Mt} - ETC_{PI}}{ETC_{PI}}.$$

Lower values of Inc_M and Inc_{Mt} correspond to better alignment between the principal and the agent. We quantify the potential savings obtained by the $[M,t]$ scheme rather than relying on CAO system by

$$Sav_{CAO} = \frac{ETC_{CAO} - ETC_{Mt}}{ETC_{CAO}}.$$

We conjecture that the delegation of inventory replenishment to an agent through $[M,t]$ scheme would outperform CAO system. In this case, SAV_{CAO} returns a positive value.

3.1. Performances of $[M]$ and $[M,t]$ Incentive Schemes

Out of the 64 problem instances solved, the minimum, maximum, and average values of Inc_M are 0.97%, 4.77%, and 2.26%, respectively. The same values for Inc_{Mt} are 0.04%, 0.15%, and 0.09%, respectively. These numbers reveal the strength of the scheme $[M,t]$ for aligning incentives of the agent and the principal. Table 1 depicts the average Inc_{Mt} values for each combination of (CoV, λ_1) and (c_u, λ_1) pairs. We observe that as the demand variability increases, the scheme $[M,t]$ better aligns the two parties' incentives. The same effect is also observed under lower underage costs. We might expect to have a better alignment as the information asymmetry weakens, which is captured by the increasing value of λ_1. However, our numerical results show that this is not the case. Better alignment is observed when λ_1 value is the highest (which means that one of the demand processes is significantly more likely than the others) and when all three demand processes are equally likely. However, this result is due to the special structure of the demand

Table 1 Average Inc_{Mt} Values

	λ_1					λ_1			
CoV	1/3	0.5	0.75	0.9	c_u	1/3	0.5	0.75	0.9
0.1	0.11%	0.13%	0.13%	0.07%	2	0.07%	0.09%	0.08%	0.05%
0.15	0.10%	0.12%	0.11%	0.07%	5	0.09%	0.11%	0.10%	0.06%
0.2	0.09%	0.11%	0.10%	0.06%	10	0.10%	0.12%	0.11%	0.07%
0.25	0.07%	0.09%	0.09%	0.05%	20	0.11%	0.13%	0.12%	0.08%

variance (having the same CoV in all processes) because a contrary observation is made under randomized demand variances.

Table 2 depicts the average t^*, Inc_{Mt}, and Inc_M values for each of the changing problem parameters. We first observe that the scheme $[M, t]$ produces significantly lower costs for the principal, in any of the given problem parameters. The average t^* values are not close to 1 (the end-of-horizon). The minimum t^* obtained is 0.38 and is observed when $c_u = 2, CoV = 0.25$, and $\lambda_1 = 0.5$. As the demand variability increases, it is better to review the inventory status at an earlier time. This is true because, when there is more variability, earlier stock-out information is more valuable. As the underage cost increases, the principal will set a higher penalty for the stock-out and the agent will have a larger order quantity. In this case, it is more likely that the retailer will not face any stock-out throughout the horizon. Therefore the relative benefit of earlier stock-out information goes down and it is better to review the inventory status at a later time.

Table 2 Average t^*, Inc_{Mt}, Inc_M, and Sav_{CAO} Values

CoV	t^*	Inc_{Mt}	Inc_M	Sav_{CAO}	c_u	t^*	Inc_{Mt}	Inc_M	Sav_{CAO}	λ_1	t^*	Inc_{Mt}	Inc_M	Sav_{CAO}
0.1	0.72	0.11%	2.26%	68.93%	2	0.51	0.07%	3.89%	48.49%	1/3	0.59	0.09%	1.91%	50.65%
0.15	0.62	0.10%	2.26%	57.78%	5	0.58	0.09%	2.32%	54.10%	0.5	0.59	0.11%	2.48%	56.94%
0.2	0.54	0.09%	2.26%	48.85%	10	0.62	0.10%	1.63%	56.78%	0.75	0.59	0.11%	2.77%	59.75%
0.25	0.48	0.08%	2.26%	41.77%	20	0.65	0.11%	1.19%	57.95%	0.9	0.59	0.06%	1.87%	49.99%

With our second experimental set, we investigate the impact of the number of potential demand processes by letting N take one of the following values: $\{2,3,5\}$. For each N value, we generated 50 random problem instances by varying μ_i and CoV parameters by picking a random value from the following ranges:

$$\mu_i \in [20,100],\ \forall i = 1,...,N; CoV \in [0.1,0.25].$$

As noted above, $\sigma_i = CoV \cdot \mu_i$. We fix the underage cost $c_u = 10$ and vary the information asymmetry through the value of λ_1 as follows:

$$\lambda_1 \in \{1/N, 0.5, 0.75, 0.9\}, \lambda_2 = \cdots = \lambda_N = \frac{1-\lambda_1}{N-1}.$$

Table 3 Average Measures for Random Problem Instances when $c_u = 10$

	t^*			Inc_{Mt}			Inc_M		
λ_1	$N=2$	$N=3$	$N=5$	$N=2$	$N=3$	$N=5$	$N=2$	$N=3$	$N=5$
$1/N$	0.85	0.88	0.89	0.14%	0.54%	0.91%	1.47%	2.04%	2.53%
0.5	0.85	0.88	0.89	0.14%	0.48%	0.76%	1.47%	1.93%	2.53%
0.75	0.86	0.88	0.88	0.08%	0.34%	0.57%	1.32%	1.62%	2.38%
0.9	0.86	0.88	0.88	0.03%	0.19%	0.31%	1.15%	1.08%	1.91%

Table 3 depicts the average Inc_{Mt} values for different levels of information asymmetry. As expected, the Inc_{Mt} value deteriorates as N increases, however, the percentage deviation is below 1% on the average. Contrary to our previous observation, the average percentage deviation has a non-increasing structure as λ_1 increases.

3.2. Performance of CAO

In this section, we report potential savings obtained by using $[M,t]$ scheme, rather than relying on a CAO system. In our first experimental set, we confirm our conjecture that the CAO system is inferior to delegating replenishment decisions to the local store manager, as the minimum, maximum, and average values of Sav_{CAO} are 21.88%, 76.44%, and 54.33%, respectively. The reason for this is that individual demand processes cannot be reflected separately in CAO protocol as they are averaged out in a single replenishment quantity; whereas this is not the case under our proposed incentive approach because the agent would be ordering a different quantity based on the exact demand distribution that is known by him. More detailed results on this set are given in Table 2.

A more detailed analysis is conducted with our second experimental set which contains 50 random problem instances for each value of $N \in \{2,3,5\}$. Collectively, these problem instances can represent the overall product portfolio that should be managed by a retailer. First, we summarize the minimum, maximum, and average percentage savings for different values of N and λ_1 in Table 4. Even for a modest case where there are two demand streams for each of 50 items with a dominating stream ($N = 2, \lambda_1 = 0.9$), the CAO system yields about 20% more inventory related costs than $[M,t]$ scheme. The minimum SAV_{CAO} values are all positive for all problem instances; indicating that none of the items are better off under the CAO system. We observe that the value of the $[M,t]$ scheme amplifies as N and the information asymmetry level increases.

Table 4 Minimum, Maximum, and Average % Sav_{CAO} Values for Random Problem Instances

	$N=2$			$N=3$			$N=5$		
λ_1	Min	Ave	Max	Min	Ave	Max	Min	Ave	Max
1/N	0.89%	35.02%	69.91%	0.89%	43.45%	78.75%	10.25%	52.36%	81.88%
0.5	0.89%	35.02%	69.91%	1.03%	41.32%	72.38%	10.65%	49.48%	85.85%
0.75	0.65%	28.66%	82.91%	0.82%	34.13%	80.37%	5.40%	41.84%	85.48%
0.9	0.32%	20.05%	87.32%	0.42%	23.45%	81.61%	2.18%	30.60%	81.96%

3.3. Identical Scheme Parameters for Multiple Items

Even though the percentage savings reported in Table 4 are quite appealing for the principal, implementation of $[M,t]$ scheme may be cumbersome in practice for some retailers. The values presented in Table 4 predicates on imposing a different combination of t and M values for each item being monitored and included in the agent's performance evaluation. Figure 1 shows the dispersion of t^* and M^* values for 50 instances from the second experimental set for different λ_1 values when $N = 2$. Each instance shown in this graph has a different t and M combination. Coping with a different (t, M) pair for each item would be very distracting and difficult for the store manager. In order to make the new scheme easy to understand and implementable, the principal might prefer to set a single (t, M) pair across the board for all items.

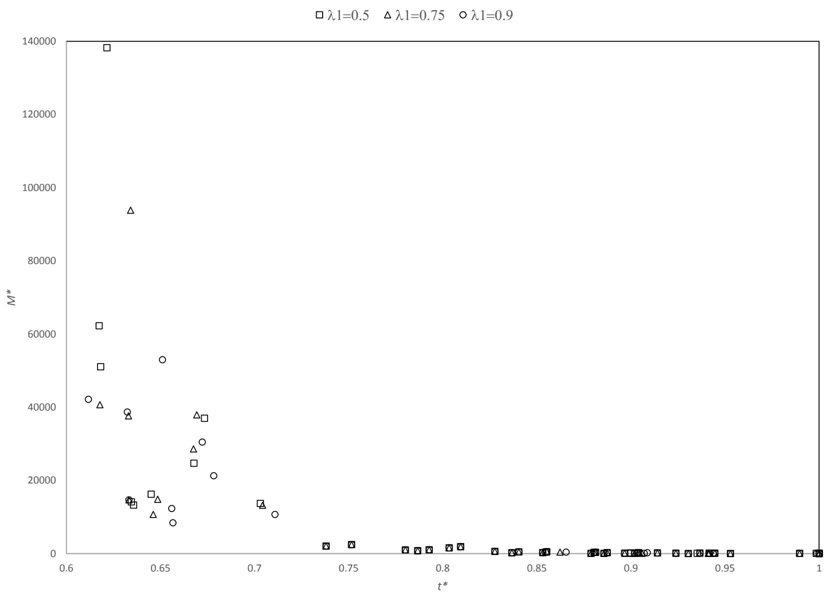

Figure 1 t^* vs M^* in $[M, t]$ scheme when $N = 2$

From Figure 1, we observe that the t^* and M^* values are strongly skewed. For example, when $N = 2$ and $\lambda_1 = 0.5$, the average t^* and M^* values are 0.8546 and 7752, whereas their medians are 0.8835 and 255, respectively. This observed discrepancy between the average and median values are typical for all problem instances considered with different N and λ_1 values. Hence, if a single (t, M) pair is sought, the median values, rather than averages, would better represent the overall central tendency in the distribution. Consequently, we analyze the impact of adopting the

Table 5 Assessment of $[M^*_{Median}, t^*_{Median}]$ scheme with respect to CAO

	$N=2$			$N=3$				$N=5$			
λ_1:	0.5	0.75	0.9	0.33	0.5	0.75	0.9	0.2	0.5	0.75	0.9
Average SAV_{CAO} in %:	33.57	27.19	18.48	42.94	40.8	33.43	22.50	52.17	49.25	41.43	29.95
# of Instances with Savings:	46	46	43	50	50	50	49	50	50	50	50
Minimum SAV_{CAO} in %:	-6.20	-5.20	-5.97	0.64	0.72	0.13	-0.58	9.57	13.60	5.03	1.74

$[M^*_{Median}, t^*_{Median}]$ scheme to all 50 items for each value of $N \in \{2, 3, 5\}$ in our second experimental set and compare it to the CAO system.

Table 5 summarizes the results for all random problem instances considered. Under $[M^*_{Median}, t^*_{Median}]$ scheme, the average savings compared to a CAO system are quite high, ranging between 18 and 52%. The worst performance is observed when $N = 2$ and $\lambda_1 = 0.9$. In this case, 43 of the 50 items are still better off under the proposed incentive scheme; indicating that higher costs are incurred only for seven items in the product portfolio. In the rest of the problem instances, $[M^*_{Median}, t^*_{Median}]$ scheme performs better for a great majority of all items, if not all.

The average savings reported in Table 5 are only slightly lower than the average savings reported in Table 4 when the $[M, t]$ scheme is optimally customized for each item. Considering its simplicity, the $[M^*_{Median}, t^*_{Median}]$ scheme can be a viable option for the principal.

3.4. Application using the Discount Retailer's Demand Data

We are provided with a dataset by the European discount grocery chain that we explained in Section 1. The dataset has daily demand data for 50 item-store pairs for the past 55 weeks prior to the fourth week of 2017 and fixed replenishment frequencies of these items at these stores. We have specifically requested the company to include stores that have been in operation for a considerable amount of time and items that have stable and non-seasonal demand patterns. As discussed before, each item at a store is replenished based on a weekly schedule. We removed the item-store pairs that are replenished more frequently than once a week and for which we have no sales consecutively for more than a week, indicating a long-term stock-out. For each item-store pair,

Table 6 Parameters of the Four Items Selected from Retailer's Data Set

Item	λ_1	μ_1	σ_1	k_1	λ_2	μ_2	σ_2	k_2	λ_3	μ_3	σ_3	k_3
1	0.89	92.9	19.8	4.69	0.11	161.3	19.8	8.15	-	-	-	-
2	0.78	81.6	16.4	4.98	0.22	187.5	129.1	1.45	-	-	-	-
3	0.9	33.9	9.7	3.49	0.1	71.0	32.9	2.16	-	-	-	-
4	0.56	16.7	4.3	3.88	0.44	29.4	10.6	2.77	-	-	-	-
5	0.78	130.4	23.0	5.67	0.12	41.1	15.2	2.70	0.1	305.3	148.2	2.10

we also removed weeks for which there were no sales for more than one day to avoid censoring due to short-term stock-outs. For each of the remaining item-store pairs, using the total weekly demand, we seek to group weeks into different clusters, each cluster corresponding to a different Normal variate. For this purpose, we used model-based clustering which clusters data based on Normal mixture modeling (Fraley and Raftery, 2002). The clustering algorithm developed in R for this purpose (Fraley et. al, 2012) left us with eleven item-store pairs with more than one cluster: ten with two clusters and one with three clusters. We then carried out routine tests for the fitness of Wiener Process using the daily demand data. These tests are used to verify that daily demand fits the Normal distribution, is stationary (using augmented Dickey-Fuller test), and does not exhibit auto-regression. In the end, we are left with five item-store pairs. Four of these item-store pairs had two Gaussian density clusters and one had three clusters. There were four distinct items in three stores. These items were bottled water, UHT processed milk, pasta and margarine. Table 6 lists the mixture probabilities as well as the mean and standard deviation of the best fit clusters provided by the algorithm.

Figure 2 shows the daily and weekly sales of the first item-store pair in Table 6. The item is a 1/2 liter UHT processed milk. There was no significant seasonality or trend pattern (within week or within year) in the daily sales. The model based clustering algorithm identified two clusters for the weekly sales. Six of the weeks belonged to the second cluster (higher sales), while the remaining weeks belonged to the first cluster. There was no clear indication (such as national holidays or other calendar effects) that can be used by the retail chain headquarters to explain why there was a higher demand in these six weeks. Our communication with the retail chain, however, suggests that the store representatives may possess some information to identify the weeks in which their stores will face higher than usual demand.

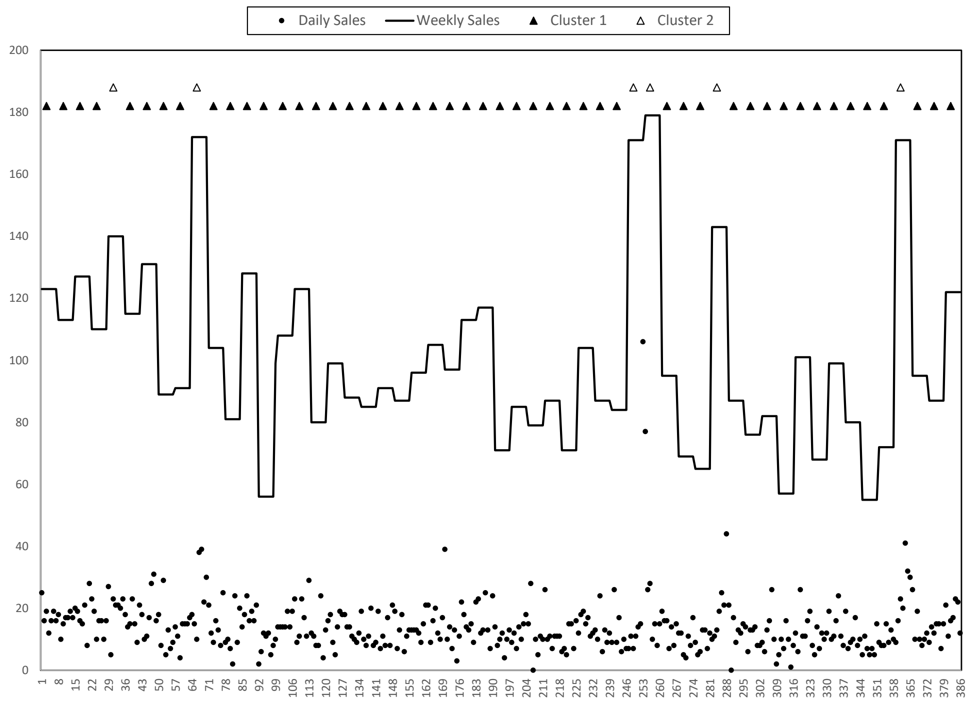

Figure 2 Daily and Weekly Sales for Item-Store 1 with Two Demand Clusters

The results of our analysis with these five item-store pairs are presented in Table 7. We use a c_u value of 50 and 100. The latter value is a more realistic scenario for the chain, corresponding to roughly 20% annual inventory carrying cost per unit and 35% shortage cost per unit. Perfect alignment is achieved for the first four selected item-store pairs; their demand parameters listed in Table 6 satisfy the conditions of Theorem 5. In particular, for each of these item-store pairs, if $k_1 > k_2$ ($k_1 < k_2$) then $\sigma_1 \leq \sigma_2$ ($\sigma_2 \leq \sigma_1$). The fifth selected item has three demand clusters and perfect alignment is not possible. Despite this and even though the parameters of this item-store pair does not satisfy the conditions of Theorem 6, we observe that $[M, t]$ scheme is preferred over

Table 7 Results of the $[M,t]$ Scheme

c_u	Item	Inc_{Mt}	t^*	M^*	SAV_{CAO}	t^*_{Median}	M^*_{Median}	SAV_{CAO}
50	1	0	1	408	46.49%	0.82	6,249	44.39%
	2	0	0.82	7,798	68.36%			68.28%
	3	0	0.75	6,249	63.43%			61.85%
	4	0	0.78	2,388	41.75%			37.01%
	5	0.61%	0.84	9,612	73.45%			72.58%
100	1	0	1	742	45.30%	0.83	13,564	43.30%
	2	0	0.83	15,844	68.99%			68.95%
	3	0	0.77	13,564	64.86%			63.48%
	4	0	0.79	5,127	41.59%			37.32%
	5	0.53%	0.85	19,291	68.63%			67.78%

$[M]$. For this last item-store pair, the retailer can obtain near-perfect alignment; complete demand information will further reduce the costs only slightly for the retailer, for about 0.5%.

As mentioned in Section 1, this retailer utilizes a CAO system to generate replenishment orders for each item in each store with some limited override privileges given to store representatives. We compare the optimal proposed incentive schemes to the CAO system recommendations in Table 7. The first SAV_{CAO} column reports these savings. We observe that such savings can be substantial (up to 73%) when c_u is set to 50 or 100. The median of the t^* and M^* values when $c_u = 50$ are 0.82 and 6249, respectively. The same values for $c_u = 100$ are 0.83 and 13564, respectively. The last column of Table 7 reports SAV_{CAO} values when these median values are used for all five items for ease of implementation. Even under this case, substantial savings can still be achieved. The results show that this chain can have significant savings if the headquarters can use a proper scheme for incentive alignment and delegate replenishment decisions to more informed store representatives.

4. Conclusion

We consider a problem faced by a principal who delegates the replenishment decisions of a periodically ordered item to an agent. The principal has incomplete demand information and cannot observe lost sales and needs an incentive scheme so that the agent orders a quantity that minimizes the principal's overage and underage costs. The demand process is assumed to be a Wiener Process; the agent knows which particular Wiener Process will take place prior to ordering whereas the principal only knows the set of possible Wiener Processes and their probabilities. We propose a scheme where the performance score of the agent depends on how much inventory remains at the end of period and whether there is any stock-out at a pre-specified instant prior to or at the end of the period. We show that when the Wiener Processes share the same variance, the principal can perfectly align the agent's incentives by inspecting the stock-out at the end of the period and setting a proper penalty for a potential stock-out to be deducted from the agent's performance score. Under some mild conditions and when there are only two possible processes, perfect alignment is still possible, but interestingly requires the inspection of stock-out before the period ends. In general, such early inspection schemes may lead to strictly better results than only relying on stock-out information at the end of the period. Our numerical results on synthetic and real data show that the scheme we suggest leads to near-perfect alignment and significant savings over centralized ordering based on incomplete demand information.

There may be many avenues for further research in this area. First, it may be interesting to study more general demand processes and investigate settings where the incentive schemes proposed here still lead to perfect or near-perfect alignment. Second, new schemes may be developed under the assumptions of this paper. For example, in some settings it may be possible to detect the exact time the store runs out of stock and design an incentive scheme based on the stock-out time. However, one may argue that these schemes may be complex and hard to implement in practice. Finally, this paper handles the multiple items case in a heuristic manner by solving the problem separately for each item and using the median values of the incentive parameters for all items. While the results for this heuristic are impressive, determining the common $[t, M]$ pair that minimizes the principal's underage and overage costs over all items can be posed as an interesting and challenging optimization problem.

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