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Elementary Teachers' Thinking (Beliefs) About Number Sense and Its Pedagogy

by

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ABSTRACT

The goal of this study was to investigate beliefs of inservice teachers about number sense in elementary school mathematics and the teaching and learning of that sense. Specifically it focused on what teachers' beliefs were about number sense, about students' development of number sense and about the teachers' roles in facilitating that development of number sense. The study compared the relationships between these teachers' beliefs about how the development of number sense can or should be facilitated in the mathematics classroom and the observed teachers' actions in instructional situations in the classroom.

Data were gathered through interviews of three elementary mathematics teachers regarding their beliefs about number sense followed by observations of the teachers instructing in their mathematics classes. Analyses included the categorization and comparison of these teachers' beliefs and practices.

Findings indicated that beliefs about number sense were unique and individual for each of the three teachers. For one teacher, number sense was pragmatic, for another teacher philosophical and for the third, both pragmatic and affective. Despite differences in the participant's beliefs about number sense, these teachers viewed development of number sense as giving student opportunities to share and talk about their ideas, to use objects and to be involved. They also viewed the facilitation of the development of number sense as providing students with appropriate situations and activities, by engaging students in classroom discourse and by assisting students to develop understanding of mathematical concepts in this environment. The research seemed to show a consistency between these teachers' beliefs about how development of number sense can or should be facilitated in the mathematics classroom and the observed teachers' actions in instructional situations in the classroom.

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DEDICATION

This thesis is dedicated to my maternal grandparents, Eng Suey Sang and Mok Seen, and my parents, Kathleen Suey Chow and Frank Chow, who possessed a wonderful sense of adventure driven by a spirit to see beyond the horizon. They are my history and from them I have that same sense of adventure and spirit that has helped me to explore the world as they did. They encouraged and valued learning and will always share my life's journey.

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CHAPTER 1

INTRODUCTION

1.1 Background and Rationale of the Study

In the mid 1980's I began my three year term as a mathematics consultant with elementary school mathematics teachers with the Calgary Board of Education. I made professional development presentations for textbook publishers and provided professional development workshops of mathematics instruction within the school system. In my role I was exposed to the thinking of some teachers that aroused my curiosity and eventually motivated me to examine more closely some of those teachers' beliefs, particularly in relation to basic aspects in developing mathematical concepts in students.

In 1982 a revised elementary school mathematics program of studies from Alberta Education began being implemented by teachers in Alberta. The curriculum encouraged instructional focuses on problem solving and the use of manipulatives in teaching elementary mathematics. The revised curriculum showed a major shift from the previous curriculum. This created new challenges for the teachers who had no previous hands-on experience using manipulatives as instructional tools. Hence, a focus in my work with elementary teachers that emerged was assisting teachers of mathematics to learn about manipulatives and their use in teaching specific concepts in mathematics. My workshops included more mathematical activities that allowed teachers to become familiar with manipulatives and to consider how they might be employed as instructional aides in the classroom. The workshops provided teachers with opportunities to develop an understanding of a variety of manipulatives and some unique applications of each to develop particular mathematical concepts. Teachers considered the preparation of lessons to develop mathematical concepts and how students might interact during those lessons. In summary, the teachers were being presented with a different approach to the teaching of mathematics, an approach with the goal of helping students more fully understand mathematics rather than memorizing it through drill and practice.

This recommended approach to teaching mathematics constituted a clear change from the structured, procedural presentations teachers used as a matter of course in the past. It produced mixed reactions from teachers. Some did not like it at all, others were uneasy with it, still others ambivalent about its value. Some teachers believed "students would never touch those blocks" or "toys were not part of a math class." However, a few who were initially doubtful about the value of the new approach began to accept it as they gained a greater understanding. These teachers reported that through the workshops they began to understand the mathematical concepts more easily themselves and to appreciate the utility of the activities in helping the students gain understanding of mathematical concepts more efficiently. Many teachers attended the workshops, returned to their classrooms and attempted to use the activities with their students.

Feedback from teachers on their successes and failures in using the approach and their negative feelings about the new approach when they felt it did not work raised questions for me both as a professional development leader and as a mathematics teacher. It seemed that there were serious obstacles for teachers to interpret the new approach as an effective means of instruction. I felt they were viewing it negatively as a means of justifying their traditional methods of teaching mathematics; they seemed reluctant to let go. It appeared that they were unwilling to undergo a change in their teaching philosophy and only willing to deal with the new approach on a surface or superficial level. Eventually I came to believe that something was missing in helping these teachers understand, something that required a deeper level of meaning or connection for them. As I began graduate studies, I became aware that understanding the thinking of teachers from the teachers' perspective can provide a means of making sense of this barrier to growth and change. A review of the literature on teacher thinking indicated that teachers' beliefs are important factors in predicting their teaching practice and professional development. The review provided the rationale for a study that focused specifically on teachers' beliefs as they related to instructional practice in mathematics.

My decision to focus on teachers' thinking in the teaching of number sense, in particular, was also influenced by my work with elementary school teachers, as previously noted, and my own experience as a teacher in making sense of primary number concept development as a teacher. As a mathematics consultant, some of the workshops I conducted for teachers focused on primary number concept development. The topic included a variety of concepts (classifying, patterning, seriating, number meaning, and number relationships) that are inter-related. It is important to provide students with opportunities to experience each of these concepts for them to gain a strong sense of number. Working with teachers in the workshops, I became aware that they seem to have a strong knowledge of algorithmic procedures for working with numbers and a limited view of number sense. This triggered my curiosity about what number sense really meant to a teacher and what guided his/her teaching of it. Thus, as I examined the literature on teacher thinking, my interest focused on teachers' beliefs and the teaching of number sense. Number sense is also at the heart of elementary school mathematics and is the foundation of later mathematics. Thus it was an important topic to study to understand teacher thinking as a means of gaining insights to enhance the teaching of mathematics.

1.2 Purpose of the Study

The goal of this study is to investigate beliefs of inservice teachers about number sense in elementary school mathematics and the teaching and learning of that sense. The primary focus is on the following two questions:

(i) What are teachers' beliefs about number sense, about students' development of number sense and about teachers' roles in facilitating that development of number sense?

(ii) What are the relationships between these teachers' beliefs about how this development of number sense can or should be facilitated in the mathematics classroom and the observed teachers' actions in instructional situations in the classroom?

1.3 Theoretical Orientation of the Study

The study is framed in the literature on mathematics teacher thinking, discussed in Chapter 3, and theoretical conceptions of beliefs and number sense. Number sense is associated with the understanding of number concepts, an idea which is developed in Chapter 4 as part of the literature review on number sense and related studies. A cognitive perspective of beliefs associated with this study is discussed in Chapter 2 in the context of teacher beliefs or thinking. The research process follows a descriptive qualitative approach in the investigation of three elementary school teachers.

CHAPTER 2

BELIEF AND TEACHER THINKING

2.1 Perspective of Belief

There are many perspectives of beliefs in the literature (cf. Pajares 1992). In education, the focus has been predominantly on a psychological perspective that deals with the nature of belief, its functions and structure. Within this perspective there are a variety of ways in which belief has been described. In the mathematics education literature, for example, beliefs are considered to be the same as concepts, meanings, propositions, rules, preferences or mental images (Thompson, 1992).

There are now numerous studies on "mathematics and beliefs" (for both teacher teaching and student learning) that considers belief as a primary construct by itself (Pehkonen & Torner, 1999). Examples of beliefs defined and used by mathematics education researchers in investigating mathematics teachers' thinking and their teaching include:

Thompson (1985) used Scheffler's (1965) definition of belief:

> Belief is rather a "theoretical" state characterizing in subtle ways the orientation of the person in the world [p. 282].

Ponte (1994) used Pajares' definition of belief:

> Beliefs are the incontrovertible personal "truths' held by everyone, deriving from experience or from fantasy with a strong affective and evaluative component [p. 199].

Finally, Chapman (2000) used Nespor's (1981) definition of beliefs:

> Beliefs are not so much sets of propositions or statements as they are conceptual systems which are functional or useful for explaining some domain of activity [p.183].

These descriptions of belief provide the perspective used in this study. The perspective is based on seeing these different descriptions of belief as related in as much as they, directly or indirectly, deal with the impact of beliefs on an individual's thinking and behaviour. Chapman (2000) provides a summary of the relevant aspects of this perspective:

> Beliefs, then, are not simply propositional, but are evaluative and judgmental. They reflect expectations and values. They form the bedrock of one's intentions, perceptions and interpretations of a given situation and the range of actions one considers. Beliefs, thus, provide a clustered set of structured expectations through which one organizes, not only one's knowledge of the world but one's behaviour in it [p.183].

In the context of this study, teachers' beliefs about number sense have to do with the teachers' perceptions, interpretations, expectations and values of what number sense is, how number sense is developed by the student, and how teachers facilitate this development.

2.2 Importance of Teacher Beliefs

The literature suggests that beliefs direct and predict behaviour and thus, it is important to study teachers' beliefs. The nature of teachers' beliefs is well documented in the literature, for example, Pajares (1992) offers the following fundamental assumptions about teachers' educational beliefs:

> 1. Beliefs are formed early and tend to self-perpetuate, persevering even against contradictions caused by reason, time, schooling or experience.
>
> . . .
>
> 4. Knowledge and beliefs are inextricably intertwined, but the potent affective, evaluative, and episodic nature of beliefs makes them a filter through which new phenomena are interpreted.
>
> . . .
>
> 10. The earlier a belief is incorporated into the belief structure, the more difficult it is to alter. Newly acquired beliefs are most vulnerable to change.
>
> 11. Belief change during adulthood is a relatively rare phenomenon, the most common cause being a conversion from one authority to another or a gestalt shift. Individuals tend to hold on to beliefs based on incorrect or

incomplete knowledge even after scientifically correct explanations are presented to them.

12. Beliefs are instrumental in defining tasks and selecting the cognitive tools with which to interpret, plan and make decisions regarding tasks; hence, they play a critical role in defining behaviour and organizing knowledge and information.

. . .

15. Beliefs must be inferred, and this inference must take into account the congruence among individuals' belief statements, the intentionality to behave in a predisposed manner, and the behaviour related to the belief in question. [pp. 325-326]

Specific to mathematics' teachers, Ernest (1989) discussed the nature of beliefs as reflected by a sample of teachers, in terms of the views of mathematics and teaching approaches that characterized the teachers' beliefs. These views are::

Mathematics:

instrumentalist view
- mathematics is an accumulation of facts, rules and skills to be used in the pursuance of some external end
- mathematics is a set of unrelated but utilitarian rules and facts

platonist view
- mathematics is a static but unified body of certain knowledge
- mathematics is discovered not created

problem solving view
- mathematics is a dynamic continually expanding field of human creation and invention, a cultural product

Teacher's Role:

instructor view
- skills mastery with correct performance
- strict following of a text

explainer view
- conceptual understanding with unified knowledge
- reception of knowledge model

facilitator view
- confident problem-posing and problem-solving
- active construction of understanding
- exploration and autonomous pursuit of own interests model [p. 251]

Ernest (1989) argued that, for example, a teacher with beliefs about mathematics that reflected an instrumental view, (in which mathematics is seen as an accumulation of facts, rules and skills,) was likely to have an instructor view of teaching that taught skill mastery. Similarly, a teacher with belief about mathematics that reflected a platonist view in which mathematics is discovered, was likely to have an explainer view of teaching following a conceptual model of knowledge. And finally, a teacher with a belief about mathematics that reflected a problem solving view seeing mathematics as a field of human creation and invention, was likely to have a facilitator view of teaching that would allow active construction of understanding.

Ernest (1989) concluded that "Mathematics teachers' beliefs have a powerful impact in the practice of teaching". He also argued that this impact has significant implications to bring about change in teaching. For example, he explained:

> [Deep changes in the teaching of mathematics] depends fundamentally on the teacher's system of beliefs. . . . Teaching reforms cannot take place unless teachers' deeply held beliefs about mathematics and its teaching and learning change. . . . Thus, the practice of teaching mathematics depends on a number of key elements, most notably the teacher's mental contents or schemes, particularly the system of beliefs concerning mathematics and its teaching and learning. [p. 249]

The next chapter will report on studies on the mathematics teacher and on number sense in order to locate this study in the field in terms of what has already been done.

CHAPTER 3

MATHEMATICS TEACHER AND BELIEF

3.1 Research on Mathematics Teachers

Research on the mathematics teacher began to emerge in the 1980's with a growing focus on the study of teacher thinking (Thompson, 1992; Hoyles, 1984). This focus was accompanied by a shift in paradigms from research on teaching methodology and moved from traditional, empirical models of studying teachers' behaviours to more humanistic, qualitative approaches of studying teachers' thinking and behaviour (Cooney, 1985).

Early studies treated teachers as objects to be analyzed based on universal criteria determined by the researcher. However, the personal/individual context of the teacher was ignored. Such studies generally offered prescriptions to effect teacher change or suggested aspects of defective teacher performance and methods and strategies to repair the performance of inservice teachers and student teachers (Ponte, 1994). Smith (1977) is an example of studies framed in this way. Smith concluded:

> Perhaps the single most relevant suggestion for teacher training is that trainees focus on manageable skills that can be observed, quantified, and objectively critiqued by supervisors. (p. 204).

Smith's (1977) study examined discourse among high school teachers to identify verbal behaviours related to pupil achievement. The findings indicated that three variables (points awarded for lesson objectives, the percentage of relevant examples per lesson, and the average number of "OK's" per minute of teacher talk) were correlated positively with post test achievement. Based on the findings, Smith concluded that some payoff could be attained by requiring student teachers to list explicitly the objectives of their lesson and to plan carefully the sequencing of their lessons so that the objectives were addressed in hierarchical orders of succession.

Other studies that dealt with teachers' deficiencies focused on the teachers' knowledge of mathematics and pedagogical content knowledge, noting gaps, lack of understanding and misconceptions that should be considered in assisting to improve the teachers' understanding of teaching mathematics. The following examples of studies from the 1980's are illustrative of this.

Mayberry's (1983) study of Van Hiele levels of geometric thought in preservice teachers showed that the typical student in the study was not ready for a formal deductive geometry course. Students in the sample were on different levels for different geometric concepts considered in the study. Wheeler and Feghali (1983) investigated teacher's knowledge of zero and found that teachers in the study sample had inadequate knowledge about zero. They exhibited reluctance to accept zero as an attribute for classification. They also exhibited confusion about whether or not zero is a number and about stable patterns of computational error. Carpenter, Fennema, Peterson and Carey (1988) investigated teacher's pedagogical content knowledge of children's solution of addition and subtraction word problems. Most teachers in the study sample could identify a number of critical distinctions between problems and the primary strategies that children used to solve different types of problems. This knowledge, however, tended not to be organized into a coherent network that related distinctions between problems, children's solutions, and problem difficulty.

These studies on the mathematics teachers continued into the 1990's. For example, Ball (1990) probed preservice elementary and secondary teachers' understanding of division in three contexts. The study found that several teachers in the sample did not exhibit suitable understanding and few were able to give mathematical explanations for the underlying principles and their meanings. The knowledge of the prospective teachers was generally fragmented and each case of division was held as a separate bit of knowledge. Graeber and Tirosh (1990) investigated the misconception held by many preservice elementary teachers that in a division problem the quotient must be less than the dividend. The researchers found that the preservice teachers' reliance on information about the domain of whole numbers and their instrumental understanding of the division algorithm supported the misconception. Even (1993) investigated teachers' subject-matter knowledge and its interrelations with pedagogical content knowledge in the context of teaching the concept of function. She found that many of the subjects did not have a modern conception of function. Appreciation of the arbitrary nature of functions was lacking and very few could explain the importance and origin of the univalence requirement. This limited conception of function influenced the subjects' pedagogical thinking. Simon (1993) investigated prospective teachers' knowledge of division and found that the teachers' conceptual knowledge was weak in a number of areas including the conceptual underpinnings of familiar algorithms, the relationship between positive and quotitive division, the relationship between symbolic division and real-world problems, and the identification of units of quantities encountered in division computations.

The preceding types of studies have provided some useful information about mathematics teachers but have ignored the teachers' personal experience and thinking in shaping their knowledge and behaviour. However, recognition of the importance of the teacher's perspective in understanding teaching in mathematics education led to a growing number of studies focusing on humanistic aspects of the teacher including teachers' beliefs.

3.2 Research on Mathematics Teachers' Beliefs

Research on mathematics teachers' beliefs has focused on the nature of beliefs, change in beliefs, and the relationship between beliefs and teaching practice. Thompson (1985) provided one of the pioneering studies on mathematics teachers' beliefs. She completed case studies of three junior high school mathematics teachers to examine each teacher's conceptions of mathematics and mathematics teaching and their relation to the teacher's instructional practice. The findings indicated that the teacher's views, beliefs and preferences about mathematics and its teaching played a significant role in shaping the teachers' characteristic patterns of instructional behaviour. The teachers professed conceptions of mathematics were consistent with the manner in which they typically presented the content in their instructional practice. With respect to teachers' conceptions of mathematics teaching, the findings indicated that these consisted of views and beliefs that were specific to the teaching of mathematics as well as views about teaching in general.

Teacher subjects chosen for these studies included both preservice or prospective teachers and practicing teachers. Preservice teachers were chosen for study based on the view that by working with teachers who have not yet created their own methodology/pedagogy of teaching, their patterns of thinking at the outset might serve as a base point to begin change in the development of method courses. This provided a consideration for addressing the content area or the means by which teachers might be guided in improving the teaching and learning of mathematics. However, because preservice teachers have not had an opportunity to develop their practice and build up experience, research focusing on comparing and contrasting beliefs to practice have generally involved inservice teacher. As Ponte (1994) explained,

> Experience is certainly one major factor contributing to the development of teachers' knowledge. . . . Preservice teachers do not have a professional experience. . . . So, let us see in what ways may experience contribute to the development of teachers' professional knowledge. [p. 206].

The following studies reflect the nature of current research on preservice mathematics teacher's thinking. Bottino and Furinghetti (1994) studied the links between preservice mathematics teachers' beliefs about mathematics and the use of computers. The findings indicated that the preservice teachers' beliefs about the role of computers in mathematics was primarily a projection of their beliefs about mathematics teaching. Beliefs on the nature of mathematics were less influential in the acceptance or refusal of computers but played a role in the choice of the type of software tools used. Raymond (1996) investigated preservice teachers' beliefs about and knowledge of alternative mathematics assessment. The study explored a range of issues related to mathematics assessment. Findings reported on beliefs about mathematics assessment, the extent to which various alternative assessment techniques can provide important information about students' mathematical learning, and the role of Assessment Standards (NCTM, 1995) in mathematics assessment practice. The research described preservice teachers' thinking as developed from students of the mathematics teaching-learning-assessment process to student teachers who needed to confront teaching, learning, and assessment issues on a daily basis (p. 381). Thornton et al. (1996) found that enhanced knowledge and repeated reflection within a context of collaborative support influenced prospective teachers' beliefs and classroom practices in mathematics. Raymond (1997) investigated the relationships between a beginning elementary school teacher's beliefs and mathematics teaching practices. Her proposed model of relationships between beliefs and practice provided a conceptual framework for the examination of factors that influence beliefs, practice, and the level of inconsistency between them. Analyses of the case study included the categorization and comparison of the participant's beliefs and practice and the development of a revised model of relationships between beliefs and practice. Findings indicated that the participant's beliefs and practices were not wholly consistent. Rather, her practice was more closely related to her beliefs about mathematics content than to her beliefs about mathematics pedagogy. Her beliefs about mathematics content were highly influenced by her own experiences as a student and her beliefs about mathematics pedagogy were primarily influenced by her own teaching practice. However, the extent to which her teacher preparation program influenced either her beliefs or her practice were limited.

Studies on preservice teachers' beliefs continued along this path thoughout the late 1990's to the present. Cooney et al. (1998) conducted a study of the beliefs and belief structures of four preservice secondary mathematics teachers. They found that the various ways in which the preservice teachers structured their beliefs helped to account for the fact that some beliefs were permeable while others were not. The nature of the evidence supporting the teachers' beliefs was considered, particularly because that evidence related to the voices of significant others or to what the individuals valued. Siebert, Lobato and Brown (1998) examined the actions used by prospective secondary mathematics teachers to avoid restructuring their existing belief systems while they were enrolled in a mathematics course designed specifically to induce such restructuring. They found that the prospective secondary mathematics teacher was able to remove the potentially perturbatory effects of class activities designed to challenge his procedural view of mathematics through the compensation moves of seeing meaning in his procedures and proceduralizing the new meaning that he developed in class. Nesbitt-Vacc and Bright (1999) examined changes in preservice elementary school teachers' beliefs about teaching and learning mathematics and their abilities to provide mathematics instruction that was based on children's thinking. The thirty-four participants in this study were introduced to Cognitively Guided Instruction (CGI) as part of a mathematics method course. Belief-scale scores indicated that significant changes in their beliefs and perceptions about mathematics instruction occurred across the two-year sequence of professional course work and student teaching during their undergraduate program but that their use of knowledge of children's mathematical thinking during instructional planning and teaching was limited. Preservice teachers may acknowledge the tenets of CGI and yet be unable to use them in their teaching. The results raised several questions about factors that may influence success in planning instruction on the basis of children's thinking. Chapman (2000) reported on a study that investigated the effect of a reflective process involving the use of metaphor on preservice elementary teachers' beliefs about the problem solving process and teaching process for non-algorithmic, mathematical problems. Findings indicated that there were significant positive shifts in the participants' beliefs about problems, problem solving, and problem-solving instruction that were consistent with the constructivist perspective of the current reform movement in mathematics education. Chapman et al. (2000a, 2000b) investigated the nature of preservice high school teachers' beliefs about mathematical problem solving and its teaching and how this related to their actual teaching. They found consistency between the espoused beliefs and teaching when they were both of a traditional, teacher-centered, perspective and inconsistency when the espoused belief reflected a student-centered perspective of teaching and learning.

Studies involving inservice teachers are also growing in numbers since the Thompson (1985) study. Ponte (1994) discussed the place of the teacher in mathematics education research. He drew attention to teachers' beliefs that are espoused in relation to implementing a new curriculum that emphasizes problem solving, applications and calculations. Similarly, Chapman (2000) discussed some of the beliefs of inservice elementary teachers that influenced their teaching of problem solving. Chapman noted that:

> Mathematics teachers' beliefs that are important in framing their classroom behaviours in the teaching of problem solving are about:
> -self as problem solver
> -how to interpret problems and problem solving
> -how the learning of problem solving takes place in the context of the classroom
> -the limits placed by students upon what is thought to be useful or possible
> -the role of the teacher in problem solving instruction
> -priorities and constraints inherent in the professional and institutional context.
> [p. 185]

Pourdavood and Fleener (1996) examined the complex interplay among teachers' beliefs, mathematics instruction, sociomathematical norms, sociocultural norms, and the evolution of dialogic community among four elementary teachers. Their report addressed the relationship between the dialogic community and individual teacher's classroom practices. Findings suggested the dialogic community greatly facilitated the teachers' abilities to critically reflect on and make changes in their classroom practices and beliefs about mathematics instruction. Askew et al (1997) explored teachers' beliefs about what it means to be numerate, how pupils become numerate and the role of the teachers. Three sets of belief orientations were identified: connections, transmission and discovery. Results from pupil assessment suggested that there was a connection between teachers demonstrating strong orientation to one of these sets of beliefs and pupil numeracy gains. Lloyd and Wilson (1998) investigated the content conceptions of an experienced high school mathematics teacher and linked those conceptions to their role in the teacher's first implementation of reform-oriented curricular materials during a six-week unit on functions dominated by graphical representations and covariation notions. These themes played crucial roles in the teacher's practice when he emphasized the use of multiple representations to understand dependence patterns in data. The teacher's well-articulated ideas about features of a variety of relationships in different representations supported meaningful discussions with students during implementation of an unfamiliar classroom approach to functions.

Buzeika and Irwin (1999) studied two teachers who experimented with allowing students to invent ways of completing multidigit calculations. The students showed considerable success in completing the task and impressed their teachers as well. The methods that the students chose were closely related to their teachers' statements or examples. Despite seeing the value of this approach for the children, the teachers were uneasy about continuing to allow them the opportunity to continue using the strategy in the face of existing socio-historical practice and beliefs. Buzeika and Irwin argued that the strength of socially and institutionally sanctioned mathematical practice prevented the teachers from seeing their activity as central to children's developing number sense.

Chapman (1999) investigated the effect of a problem-solving inservice program based on a humanistic approach to teacher development. The program focused on self-understanding of teachers' personal meaning about problem solving as a basis for facilitating change in beliefs and practice. The findings indicated that the approach was very effective in bringing about meaningful changes in teachers's thinking and teaching. In particular, Chapman noted that the participants' beliefs reflected a more positive view of mathematical problem solving and their teaching of problem solving shifted from the show-and-tell, teacher-centered model to one that captured the active, social and constructivist nature of learning. She also suggested that there is a direct relationship between the teachers' experience with problem solving and beliefs about problem solving that influenced the teachers' teaching of problem solving.

Hart (in press) investigated teachers' beliefs about factors that influence change in their pedagogy. The findings indicated three factors believed to have been very helpful in the change process: collegiality, colleagues in the project, and modeling of thinking and behaviors advocated. Three factors believed to have had very little effect or may actually have hindered their change included the principal or school administration, colleagues in their school, and their day-to-day working conditions. Chapman (in press) discussed a study that considered three dimensions of beliefs: the belief content, the belief structure and the belief function, within the context of inservice high school mathematics teachers who changed their practice from a predominantly teacher-centered to a more student-centered perspective through their own initiative. The study focused on the relationship between the teachers' beliefs and changes in their teaching over their teaching career. Of particular interest was the relationship between beliefs about mathematics and changes to classroom behaviour and the possible influence of belief structure on the relationship. The findings of the study supported the view that belief about mathematics is a significant factor in how one teaches and in changing teaching. In particular, Chapman concluded that for change to occur, it is important for teachers to attend to how the belief about mathematics is held, pedagogical and experiential interpretations of mathematics, experiential/concrete contexts for generating and interpreting pedagogical conflicts related to the belief about mathematics and the generative aspect of beliefs. Lloyd (in press) investigated how teachers' beliefs can change on the basis of experiences with innovative curriculum materials. The findings discussed and illustrated two viable contexts for using innovative curriculum materials to help teachers develop reform-oriented beliefs about mathematics pedagogy.

Collectively, the studies reviewed show the importance placed on teachers' beliefs in mathematics education. One factor underscoring this notion is the current reform movement in mathematics education (e.g. NCTM 1991, 2000) and the expressed need for teachers to change their practice accordingly. While studies involving inservice teachers have examined mathematics in general, problem solving and, to a lesser degree, assessment, gaps exist in the literature in specific areas of mathematics. Teachers' beliefs in relation to number sense and its teaching and learning are only now beginning to be explored in the literature despite the importance of number sense in the reform movement. The next chapter suggests this may be a matter of timing because of the relatively new emphasis on number sense in an explicit way in the school curriculum. However, one study that has considered the teacher and number sense is Kaminski (1997). Kaminski argued that the use of number sense can assist learners in their understanding of and calculating in mathematics. Reports on those aspects that were investigated with primary preservice education students at the commencement of a semester unit in mathematics education concluded that the students had at least a limited development of number sense.

CHAPTER 4

NUMBER SENSE: WHAT AND HOW

4.1 Introduction

Historically, studies related to number sense, focused on learners and their learning of number concepts, particularly at the elementary school level. However, such studies did not explicitly focus on number sense or make specific reference to number sense. Rather, they dealt with constructs/concepts such as students' error patterns and mastery of computational skills. This line of research continues but with a shift in focus to investigating students' understanding of constructs including the number system (Thomas, 1996), place value (Hiebert and Werne, 1992), and approaches to counting (Thomas, Mulligan and Goldin, 1996; Wiegel, 1998). Such studies may indirectly deal with number sense but the findings are not presented in specific relation to number sense. One reason for this is that number sense is a relatively recent consideration in the mathematics curriculum.

Number sense was not a topic identified as important until the late 1980's when it emerged as a primary objective of elementary school mathematics in the document *Everybody Counts* (National Research Council, 1989) and when it was emphasized as a standard in the National Council of Teachers of Mathematics Curriculum Standards (NCTM, 1989). As Sowder (1994) explained:

> A decade ago the term "number sense" was rarely heard, although many good teachers were undoubtedly teaching in ways that led their students to develop a sound, intuitive "feel" for numbers. Recent documents have brought number sense to the forefront. . . framers of curriculum documents from various parts of the world list number sense an essential component of curriculum (e.g., National Council of Teachers of Mathematics, 1989; Australian Education Council, 1990). [p. 4]

Since that time there has been a growing interest in number sense in the mathematics education community of classroom teachers, curriculum and resource writers and researchers. For example, Reys et al. (1995) recently formed a "Number Sense Research Group" (website: http://tiger.coe.missouri.edu/ barb/number/html) but no actual studies have been reported in relation to it. However, there is one study that involves Reys. Reys and Yang (1998) conducted a study that provided information on the number sense of Taiwanese students in Grades 6 and 8. Data were collected with separate tests on written computation and number sense. Seventeen students were interviewed to learn more about their knowledge of number sense. Taiwanese students' overall performance on number sense was lower than their performance on written computation. Student performance on questions requiring written computation was significantly better than on similar questions relying on number sense. There was little evidence that identifiable components of number sense, such as use of benchmarks, were naturally used by Taiwanese students in their decision-making. The research supported the need to look beyond the correct answers when computational test results are reported.

This chapter focuses on theory involving the definition of number sense and the nature of teaching and learning to acquire number sense in the mathematics education literature.

4.2 Defining Number Sense

Various researchers/theorists, professional organizations, curriculum documents and resources have provided a variety of definitions of number sense. These definitions overlap in various ways but no single definition appears to be universally accepted. The following examples illustrate the range of definitions in the literature from four categories of sources.

(i) Professional Organizations

The professional organization that influences mathematics education in North America, the National Council of Teachers of Mathematics (NCTM), as previously alluded to, has explicitly dealt with number sense in its Curriculum Standards (1989). These standards state:

> Number sense is an intuition about numbers that is drawn from all the varied meanings of number. It has five components:
>
> 1. Developing number meanings. This includes the cardinal and ordinal meanings of number.
> 2. Exploring number relationships with manipulatives.
> 3. Understanding the relative magnitudes of numbers.
> 4. Developing intuitions about the relative effect of operating on numbers.
> 5. Developing referents for measures of common objects and situation in their environments. [NCTM 1989, pp. 39-40]

(ii) Curriculum Documents

The Alberta Education mathematics curriculum is heavily influenced by the NCTM standards. Thus, number sense stands out implicitly as one of the mathematics processes, *Estimation and Mental Mathematics* (Alberta Education, 1996, p.8) and explicitly as one of nine components of the nature of mathematics (i.e., numbers). The Alberta curriculum describes number as follows:

> Number, number systems and the operations on numbers are vital to all mathematics learning. The use of number must go beyond procedure and accuracy to include what is called number sense. Number sense includes:
>
> - an intuitive feeling about numbers and their multiple relationships
> - construction of meaning of number through a variety of experiences, and development of an appreciation of the need for numbers beyond whole numbers (NCTM, p. 38)
> - an appreciation and ability to make quick order of magnitude approximations (Steen, p. 79) with emphasis on quick and accurate estimations for computation and measurement
> - the ability to detect arithmetic errors
> - knowledge of place value and the effects of arithmetic operations [Alberta Education, 1995, p. 13]

(iii) Curriculum Resources

New textbooks and teacher resources to support the NCTM Standards and curricula like the new Alberta curriculum also explicitly provide their perspective of number sense. For example, one of the textbooks approved for use in Alberta Schools, *Interactions* (Hope and Small, 1994) explained:

> In *Interactions*, number sense is fostered and presented not as another topic for teachers to include; rather it is presented as a different perspective from which to view the learning of mathematics. Number sense is promoted by many of the teaching/learning activities, particularly in the activities pertaining to the following topics:
>
> - operations
> - mental calculation
> - numeration
> - estimation
> - data management [p. 18]

This way of viewing number sense is representative of other new elementary school mathematics textbooks.

(iv) Researchers

Researchers have also defined number sense based on their work in the area. Some of these definitions are presented here.

Reys et al. (1995) offers the following definition:

> Number sense refers to a person's general understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgments and to develop useful strategies for solving complex problems. It reflects an inclination and an ability to use numbers and quantitative methods as a means of communicating, processing and interpreting information. It results in an expectation that numbers are useful and that mathematics has a certain regularity.
> (Number Sense Research Group; website:http://tiger.coe.missouri.edu/barb/number/html)

Greeno (1994) identified three properties of number sense that include:

1. flexible numerical computation: number sense involves recognition of equivalences in order to regroup numbers and perform mental operation of numbers.
2. number estimation: involves recognition of approximate numerical values in the context of computation.
3. quantitative judgment and inference: evidence of number sense involves judging and making inferences about quantities with numerical value.

[p. 170-173]

Sowder (1988) described number sense as a well-organized conceptual network that enables one to relate number and operation properties and to solve problems in flexible and creative ways.

McIntosh, Reys and Reys (1992) proposed a model for characterizing basic number sense that is organized not by instructional "topics" but by the following collection of "understandings" a learner is likely to exhibit/utilize:

1. Knowledge of and facility with numbers
   • sense of orderliness of number
   • multiple representations for numbers
   • sense of relative and absolute magnitude of numbers
   • system of benchmarks

2. Knowledge and facility with operations
   • understanding the effect of operations
   • understanding mathematical properties
   • understanding the relationship between operations

3. Applying knowledge of and facility with numbers and operations to computational settings
   • understanding the relationship between problem context and the necessary computation
   • awareness that multiple strategies exist
   • inclination to utilize an efficient representation and/or method

[p. 4]

They explained that their proposed framework was not intended to be accepted as a definitive model but rather to provide a useful starting point for continued dialogue.

Reys (1994) suggested,

Perhaps number sense, like common sense, is best described by looking at specific behavioral characteristics of those who value and use it. A student with number sense will:

• look at a problem holistically before confronting details.

- look for relationships among numbers and operations and will consider the context in which a question is posed.
- choose or invent a method that takes advantage of his or her own understanding. [p. 115]

Thompson and Rathmell (1989) explained that

> . . . good number sense involves the development of understanding of the following: (1) number meanings and relationships, (2) relative magnitudes of numbers, (3) the relative effects of operations on numbers, and (4) referents for quantities and measures as numbers used in everyday situations. [p. 3]

Finally, Howden (1989) described number sense by presenting classroom scenarios of how children show their number sense in various activities. For example, in one instance, she asked students what came to their minds when she said twenty-four. Students responded by tracing in the air, going to the calendar and by finding 24 on a digital watch. She continued to describe her experiences: "the students exhibited a "feel" for the number 24 in many ways." [p. 6] For Howden, children with good number sense have well understood number meanings, have developed multiple relationships among numbers, can recognize the relative magnitudes of numbers, and know the relative effect of operating on numbers.

The preceding examples of definitions of number sense in the literature highlight what are considered the essential components of number sense. In this study, it is not necessary to have a specific definition of number sense because it will be determined based on the thinking of the participants.

4.3 Facilitating Students' Development of Number Sense

Several researchers cited earlier made suggestions about how to facilitate students' development of number sense that will be presented in this section. First, the textbook, *Interactions* (Hope and Small, 1994), provides the following list of factors as an overview of general suggestions to facilitate number sense development:

Interactions is based on the belief that children of all ages develop number sense in environments where they are encouraged to:

- work with concrete materials and familiar ideas
- discuss and share their discoveries and solutions
- compose and recompose different arrangements and representations of numbers
- investigate the realistic uses of numbers in their everyday world
- explore number patterns and number relationships
- create alternative methods of calculation and estimation
- solve realistic problems using a variety of approaches
- calculate for the purpose rather than for the sake of calculating
- gather, organize, display, and interpret quantitative information [p. 18]

Reys' et al. (2000) way of thinking about students' development of number sense and effective ways to facilitate that development in the classroom is reflected in the following:

> Number sense exhibits itself in a variety of ways as the learner engages in mathematical thinking. In essence, it is an important underlying theme as the learner chooses, develops and uses computational methods, including written computation, mental computation, calculators and estimation. The invention and application of an invented algorithm calls upon facets of number sense including decomposition/recomposition and understanding of number properties. As learned/acquired paper/pencil algorithms and calculator algorithms are used, number sense is important because answers are reflected upon by learners and judged for reasonableness.
>
> The acquisition of number sense is a gradual evolutionary process, commencing long before formal schooling begins. Although evidence of number sense can be demonstrated early, growing older does not necessarily ensure either the development or the utilization of even the most primitive notions of number sense. Indeed, although many young children exhibit creative and sometimes efficient strategies for operating with numbers, attention to formal algorithms may, in fact, deter use of formal methods.
> (Reys et al., 2000, website: http:// tiger.coe.missouri.edu/ barb/number/html.)

Greeno (1991) proposed that students work in a conceptual environment, searching and finding resources about numbers and applying their understanding in a variety of activities which might include finding patterns in numbers, solving routine problems about numbers and quantities, and allowing students to see the relationships between and among numbers. Students would have opportunity to use physical objects, for example, blocks, beans and popsicle sticks and concrete representational items such as coins, bills and money. From his perspective, Greeno sees teachers developing and negotiating the meaning of terms for learners and making sense of numbers by engaging students in conversations about number. Rather than having students mimic procedures outlined by the teacher, the learning environment was to be fostered by the teacher assisting students to be curious about numbers; teachers presentations and activities would allow students to construct and discuss methods of solving problems. Teachers and students work together to understand mathematical concepts, notations and procedures. As Greeno wrote, "The capabilities we associate with number sense go beyond knowing facts and procedures; they involve participation in activities." [p. 211]

Sowder (1992) suggested that as teachers deal with the topic of number sense, they need to understand what characterizes number sense and need to prepare activities that present students with opportunities to explore within that framework. As students develop their "intuitive" feeling about number sense, teachers also need to know and recognize the dispositions that indicate the presence of number sense within the learner. Estimation and mental computation were two topics that are part of Sowder's conceptual framework that allowed learners to demonstrate an understanding of numbers and the structure of number systems. For Sowder (1992), estimation consists of three categories: computational estimation, measurement estimation, and numerosity estimation. Sowder's research has shown that good estimators have good number sense which they exhibit by using a variety of estimation strategies; they are flexible in their thinking and demonstrate a deep understanding of numbers and operations. Conversely, poor estimators apply algorithms, fail to value estimation and often equate estimation with guessing. Sowder suggested that mental computation is an important component of estimation as well as an important skill associated with understanding the structure of the number system. Students skilled in mental computation use their understanding to their advantage rather than relying on paper-and-pencil algorithms.

Hope (1989) argued that number sense develops through meaningful and purposeful activities involving calculating, measuring, and estimating. He suggested the following activities, of these three types, to assist learners in developing number sense:

<u>Calculating and Number Sense</u>

1. Ensure that calculation done in school is not separate from practical work. Students should be calculating for the purpose of solving practical problems.
2. Students need to understand how the practical context of a calculation influences their choice of a procedure to carry out the calculation
3. Children should always be encouraged to look for ways to simplify a calculation before carrying it out. Thinking about numbers and numerical relationships rather than about digits and book-keeping rules is an important aspect of number sense.
4. The practical context also supplies important cues to judge the reasonableness of a calculated answer. By providing a context for a calculation, teachers can help students identify the implicit and explicit circumstances of the situation that can be used to evaluate the reasonableness of the answer produced at school.
5. Students should also learn to make decisions about how to interpret the results of a calculation. Teachers help children by presenting exercises that give opportunities to interpret the answers they produce when calculating. (e.g., Georgia calculated that the amount of landscaping gravel needed for a garden was 12.357876 cubic yards. How much gravel should be ordered? Defend your answer.)

<u>Measuring and Number Sense</u>

Teachers need to ensure that students are afforded regular opportunities to work with measurements that occur in real life. It is important that students acquire the following understandings about measurement.

1. Measuring should be done in school primarily for the purpose of gathering data needed for the solution of a problem or the interpretation of a quantitative situation.
2. Children should be given experience with a wide variety of meaning tools. They should learn that the selection of a measuring tool is guided by the demands of the particular measurement situation. Students should learn how to read and reason with measurement tools and to apply what can be called the "tricks of the trade."
3. Students need to learn that all measurements are approximate and that in measuring continuous quantities, upper and lower limits have to be established. They should learn that more precise instruments are not necessarily superior to less precise instruments and that standards of accuracy and precision are subjective decisions guided by practical circumstances.
4. Measurement sense requires a knowledge of everyday equivalences. A knowledge of a wide variety of everyday measurement referents such as that doorways are about two meters in height, is the foundation of good measurement sense as well as good number sense.

5. Children also need to learn that numbers used in the everyday world are not always labeled with a unit and the meaning attributed to these numbers depends entirely on one's understanding of the context in which the numbers are seen. Students should be exposed regularly to numbers found in newspapers, magazines, shopping catalogs, advertising flyers, government and business publications, recipes, transportation schedules, utility rate tables, and so on.

<u>Estimation and Number Sense</u>
Besides its obvious practical use, producing reasonable numerical estimates can also contribute to one's number sense.

1. Estimation involves comparing quantities. Children need to develop a feeling for what people mean when they use the relative terms *between, about, near, close* and so on.
2. As soon as children learn the meaning of an operation and have memorized some number combinations, they can be taught to estimate sums, differences, products, and quotients.
3. Because no clear-cut rules can be invoked, children need considerable help and practice in determining the desired precision of an estimate. Children need to learn that no single estimate is correct; rather, the adequacy of an estimate depends entirely on the practical circumstances.

Number sense can develop only when children are exposed to the "messier" aspects of everyday problem solving. More emphasis needs to be placed on thinking about the various procedures that can be used to solve a problem and on interpreting the answers that these procedures produce. [pp.13-15]

Thornton and Tucker (1989) suggested that teachers provide instruction which allows students to construct number meanings through realistic experiences, particularly with the support of physical materials. They explained that in planning such lessons the teacher:

- recognizes the importance of developing number sense
- creates a positive climate for students to grow in their understanding and application of number
- constructs situations that stimulate the development of number sense. [p. 18]

They continued by noting that:

Teachers create opportunities for developing number sense by planning and looking at all phases of the lesson. These opportunities include the following:
1. Warm ups. As teachers plan for the warm-ups, different opportunities come to mind: a problem-solving skill activity, reviewing yesterday's work, looking ahead to review or reteach a prerequisite for next week's lesson, or a short mental-computation or estimation exercise.
2. Lesson Development. In planning, the teacher needs to think about the following:
    - tying the new topics to previous knowledge

- using manipulatives
- incorporating real-world experiences
- promoting discussion through appropriate questioning techniques
- checking for understanding

3. Independent Student Work Time. During the period of independent practice under the supervision of the teacher, the students work carefully, building their confidence. The teacher reinforces and reteaches as necessary, assessing the seatwork and making decisions whether to continue with the next phase of the lesson.
4. Time Outside Planned Lessons. When students finish early to ensure academic learning continues teachers can use sponge activities. Sponge activities can be used on arrival as ongoing activities or during dismissal. These activities furnish students with opportunities, for example, to help students develop feel for the size of numbers, for operations on numbers, or for relationships between numbers. At times students could experience the opportunity to extend learning to other subject areas and into the real world. [pp. 19-21]

Reys (1994) pointed out that teaching for the development of number sense requires conscious, coordinated effort to build connections and meaning on the part of the teacher. Teachers play an important role in building number sense in the type of classroom environment they create, in the teaching practices they employ and in the activities they select. Some strategies teachers might consider when teaching number sense are:

1. Use process questions. Process questions, those that require more than a simple factual response, can stimulate discussions of an idea, which can lead to further exploration and the use of oral language to explain and justify a thought.
2. Use writing assignments. Having students summarize their thinking in written form is an effective method for helping students nurture their sense of number.
3. Encourage invented methods. Creating and exploring their own methods for calculating and solving problems prepare students to consider traditional methods as a later stage and view standard algorithms as yet another means of producing sensible answers.
4. Use appropriate calculation tools. Number sense can be promoted by ensuring that students learn to calculate in various ways including written, mental, approximate, and electronic methods.
5. Help students establish benchmarks. Approximate computation or estimation is another important tool for encouraging students to use what they already know about numbers to make sense of new numerical situations. Oftentimes this tactic means that students use their own benchmarks to judge the reasonableness of a situation.
6. Promote internal questioning. An important role for teachers in the development of number sense is helping students to learn to ask themselves key questions before, during and after the solution process.

> By establishing a classroom atmosphere that encourages exploration, thinking and discussion, and by selecting appropriate problems and activities, teachers can cultivate number sense during all mathematical experiences. [pp.116-119]

Ross (1989) discussed the use of numerical part-whole relationships and place value to perform mental computations and numerical estimates and the importance of having good number sense to do this. She pointed out the need for teachers to assist students in developing such number sense and to apply it to problem solving involving computation. To accomplish her goal, she recommended a question and discussion approach to instruction.

Finally, Kastner (1989) discussed the use of measurement applications to aide students development of number sense. She explained:

> In this connection real-world applications can make a significant contribution to the development of number sense in elementary school students, since they offer an opportunity to test numbers that result from computational processes against observations. Measurement experiences, in particular, can promote and illustrate number work in an intuitively satisfying way for common and decimal fractions as well as for whole numbers. [p. 40]

She further explained:

> The role of applications in the development of number sense can be strengthened considerably if measurement-based curricular applications are revised to take actual measurement practices into account. [p. 46]

Thus, her focus was on actual measurements as part of the process of developing number sense.

This chapter has summarized a range of studies and theories involving the nature of number sense and how to facilitate students' development of it. from my experience, many teachers are not aware of this information because they tend to avoid formal theories and focus on experience. The participants of this study (i.e. the thesis) fall in this category. Thus, investigating their beliefs on number sense and its pedagogy provides another avenue of gaining insights about these topics.

CHAPTER 5

RESEARCH METHODOLOGY

5.1 Research Perspective

Qualitative research is the approach of inquiry used in this study. As Cresswell (1998) noted, qualitative research is characterized by a reliance on few cases. The format of qualitative research follows the traditional research approach by presenting a problem, asking a question, collecting data to answer the question, analyzing the data and answering the question. However, the problems or issues to be explored are selected to present a detailed view of a topic and, in using qualitative research, the questions ask how or what rather than why. Interviews, observations, documents and audio-visual materials are methods of data collection.

Cresswell (1998) also notes that researchers have opportunities to emphasize their roles as active learners who can tell the story from the participant's view rather than as experts. To assist the process, the researcher poses a problem and researches the issue using open-ended questions with the intention of listening to the participants. Data gathered in this manner allows participants to talk about their meanings. Thus they can choose to be both emotional and practical. Analysis is begun by organizing and sorting the data. Qualitative data is examined through working from general to particular perspectives using possible categories such as themes, dimensions, codes or classifications. The data is presented partly from a participant's perspective and partly as a researcher's interpretation.

5.2 Selection of Participants

The following criteria were used to select the participants for this study:

I) The teacher be drawn from kindergarten to Grade 3, i.e. the primary level. The choice of this level is based on the fact that it is the foundation to developing number sense and, thus, is of importance to gain insight into teachers' thinking relating to it.

ii) The teachers be experienced in terms of the number of years they have been teaching elementary mathematics. This is considered to be useful to the study because they will likely have a rich history across several curricula to draw from in establishing their beliefs and practice.

iii) The teachers be willing o participate in the study and to openly share their thinking, relating to number sense and the way they facilitated its development by students. This is important to make it more likely to access the teachers' thinking with sufficient depth and collect meaningful information from them during the interviews.

To satisfy this criteria, the three participants selected for this study were professional colleagues with whom I have had working relationships and an on-going professional dialogue over my teaching career up to about 10 years prior to this study. Based on this experience, I felt that the teachers were open to reflecting on their practice in a research setting. They were also confident enough to not be intimidated by probing questions about their practice. Finally, they would also be comfortable with me as researcher and likely to be forthright in what they shared about themselves.

Karen

Karen holds a Bachelor of Education, Bachelor of Arts and Master of Education degrees. She has been an elementary teacher for 30 years and at the time of the study taught Grade 3 in an inner city school with students that were immigrants from the Eastern European block and Southeast Asia. . She has interest and experience in the field of language development working both in the classroom and in a school system as a writing project coordinator and language learning consultant. Some ideas from her previous experiences working with teachers, and her continued interest in language development and writing, encouraged her to consider changes in how she taught mathematics. She applied ideas and strategies she used in language development in her approach to the teaching of mathematics.

Louise

Louise has taught for 20 years and holds a Bachelor of Education degree. She is married and is a mother three grown children. She is an active traveler often venturing on cruises and holidays to Mexico and the Caribbean. At the time of the study she worked with students at the Grade 1 and 2 levels and in her mathematics program focused on students' understanding of mathematics. She has participated in a program called 'looping' whereby she worked with one class of students for two years. This allows Louise to begin with the students in Grade 1 and work with them the following year in Grade 2 then begin with a new class the third year. Louise advised and consulted with a reputable publishing firm on a new mathematics program, assisting in writing parts of the program.

Mary

Mary is a primary elementary teacher with 18 years of teaching experience interrupted by time out of the classroom to raise her family. She is married and has two grown sons. With her family she moved to Australia for a year. She earned a Bachelor of Education degree majoring in social studies. For the majority of her career, Mary worked with young children often teaching split grade classes. At the time of the study, Mary taught Grade 2. Like Louise, her classes were involved in the "looping" program where she began with a class of Grade 1's and followed them through Grade 2. She continued working as a classroom teacher while changes in math education were emerging and being implemented.

5.3 Data Collection

Data collection consisted of personal interviews individually with each of the three participants and three classroom observations. The interviews with the participants were completed and transcribed prior to planning the classroom observations to discourage influencing the collection of data during classroom observations.

5.3.1 The Interview Process

Each participant was interviewed on two occasions. Each interview was scheduled at the participant's convenience and conducted in a private, uninterrupted site at the participant's home or at the school and lasted from one and a half to two hours. The interviews were tape-recorded and transcribed into hardcopy. Participants were encouraged to describe their thinking, actions and ideas during the interview process; to respond frankly and openly; and to provide authentic and genuine descriptions of their practice and experiences. No time limits were imposed on the interviews.

Interview questions were open-ended to allow the participants to provide full descriptions of their thoughts about: 1) their personal beliefs about number sense; 2) how they believed their students developed a sense of number; 3) how they facilitated the development of number sense in their students by their teaching.

The initial interview developed a rapport with each of the participants and introduced questions that called for participants to reflect on their responses in preparation for the second interview. Following time for reflection, participants appeared more relaxed and more comfortable in the second interview. They responded frankly and expressed their ideas and articulated their thoughts often with deep emotion and a sense of commitment

5.3.2 Classroom Observations

Dates and times for observations in the classrooms were established with each of the three participants. The observed lessons were part of the participant's regular schedule and included the topic of number sense. All students in each class were well informed as to the purpose of the researcher's visit in the classroom. Most students followed classroom routine and ignored my presence.

A variety of lessons were observed including whole class activities incorporating operations of numbers into a measurement activity, presentation of eight mathematics center activities, follow-up lessons and rounds of the center activities, and small group preparation lessons for the upcoming provincial achievement tests. The participants were not asked to plan or conduct any special lessons or make any alterations to normal routines for the study. Participants were encouraged to arrange consecutive lessons that were part of their schedule.

A sequence of three lessons was arranged by each of the participants. Observations were made on consecutive days. Class periods ranged from 30 to 45 minutes in length. The researchers' purpose in observing classes was to note and record the participants' actual teaching practices in class, to observe their interaction with students and to observe the activities being developed by the participant for use in teaching students and in student learning.

While observing the class, the field notes consisted of recordings of conversations between the teacher and class members in large groups and small groups or between the teacher and individuals. The actions of both teacher and students and the activities they were involved in were also recorded in the field notes. A secretary notepad with a two-column page was used in order to record the dialogue between teacher/students and the description of the activity. The descriptions were recorded in the left-hand column and the matching conversations in the classroom were recorded in the right-hand column.

5.4 Analysis of the Data

Analysis noted general characteristics drawn from the participants' viewpoint specific to: 1) how each understood number sense, 2) how each believed students' developed number sense and 3) how each believed they facilitated students' number sense development. The categories were used throughout the analysis because the research was initiated by a search for what teachers knew about number sense and its' teaching and learning. Therefore, the analysis was prepared to consider any unique or personal characteristics the participants exhibited in their discussion relative to these three categories.

Subsequent readings of the transcripts considered a search for indicators important to the three focus areas. The objective was an attempt to determine what there was about these selected sections of transcript that could be identified and considered meaningful about the teachers' beliefs about number sense, students' development of number sense and facilitation of the development of number sense. Repeated readings of the transcripts began to reveal more detail about the participants' beliefs and allow me to see separation of the three focus areas. Each of these significant sections was highlighted in separate colours. To form a complete perspective of the participants' beliefs and illustrate their depth of understanding in relation to number sense, a summary for each of the focus areas was collected for each participant and this data was synthesized into a table for each participant.

Field notes from the classroom observations were recorded as each participant and the students were engaged in the mathematics lesson. The analysis considered each participant as unique to their style in interacting with the students and their manner of handling activities in the classroom. The analysis drew relevant data from field notes taken during classroom observations. This data then contributed to a sample lesson which included number sense and showed the actions and discourse between the teacher and students. This lesson illustrated the behaviour of the participant in action focused on what number sense concept was being developed, how students developed number sense and how the teacher facilitated that development.

A comparison was drawn between the participants' stated beliefs about number sense and it's teaching and learning, as expressed in the interviews, and the participants' actions in the classroom observations as recorded in the field notes. This comparison examined consistencies and/or inconsistencies between the teacher beliefs as described in the interviews and actual actions on the part of the participants with their students during the classroom observations.

Chapter 6

Findings

The findings are presented based on the following research questions previously set out in Chapter 1: (i) What beliefs do the teachers have about number sense, how students acquire or develop number sense and how this development of number sense can or should be facilitated in the mathematics classroom? (ii) What are the relationships between the teachers' beliefs about facilitating the development of number sense and their actions in the classroom when teaching classes they consider to involve facilitating the development of number sense?

6.1 Teachers' Beliefs about Number Sense, Development of Number Sense and Facilitating Number Sense Development

This section sets forth the beliefs of the three participating teachers based on the description of their thinking and classroom behaviours. For each participant, the beliefs are presented in three categories: number sense, the development of number sense and the facilitation of the development of number sense. The participant's beliefs are summarized in tables 6.1a (Karen), 6.1b (Louise), and 6.1c (Mary). Each table is accompanied by an elaboration of the teachers' thinking. The notations (such as M-6) refer to the page number of the participants' interviews.

6.1a Karen's Case

A. Number Sense

Karen's beliefs about number sense include a number of characteristics that have to do with making sense of the world in terms of numbers. Her concept of number sense appears to, Karen's Beliefs about Number Sense and its Pedagogy

A. Number Sense is:

- a way to organize our thinking
- a way to view the world- make sense of it, interact with it and be part of it
- the awareness of what number is in relation to objects, in relation to each other, in relation to the world
- a language of thinking where we can communicate about how many and how much

B. Development of Number Sense involves students:

- representing
- articulating
- engaging
- working with tangible things
- playing

C. Facilitating Development of Number Sense involves the teacher:

- providing opportunities for students to:
  - build concrete, pictorial and mental representations
  - talk about what they are thinking and doing
  - interact with each other
  - play
- building on students' past experiences
- questioning students about their thinking
- observing what students are doing
- discussing how to solve problems or work with patterns

Table 6.1a

be a prominent part of her life and thinking. Karen summarized her comments saying,

> It's [number sense] in everything we do and it's a way in which we organize our thinking so mathematics and number sense is one of those ways to view the world, to make sense of it, to interact with the world and be part of it. [K-4]

Karen also associated number sense with awareness of numbers, particularly in terms of relationships.

> It's kind of the awareness [of] what numbers are in relation to objects, in relation to each other, in relation to the world. [K-1]

She continued, stating that number sense provides a basis of communicating your thinking about quantity in everyday life, a "language of thinking" as she illustrated with the following account:.

> There's a certain number of plants sitting on my counter. . . . I have to have a plate under each one as I water them. . . . I have my experience, the plants and the pots and the everyday life but then we get to a place where if I talk about it in terms of number it's a close thing where you and I can communicate about how many and how much and what do we need here so in a sense it's a language of thinking. [K-4]

Belief about number sense for Karen seems to be about understanding numbers, relationships between and among numbers and application of numbers in life.

B. Development of Number Sense

Karen's beliefs of how students acquire or develop number sense focuses on the type of behaviours students need to engage in to allow them to build their number sense. These behaviours consist of representing, articulating, engaging, and working with tangible things in both formal learning and play situations.

For Karen, representing involves "How they [students] picture stuff in their heads and how they can show it to me on paper [K-3]". Articulating involves students "sharing with somebody else [K-5]". Engaging students means they actively participate in problem solving activities that are interesting to them, such as, problems related to their life experiences. Working with tangible things involves students using objects as a basis to create and recognize patterns or number relationships, for example, "this doesn't change if I put it over here or if I put it under there. It's still one kind of object [K-3]". Finally, playing involves both in school and out of school activities as Karen explained:

> They do lots of number stuff in their play. . . when they play at centers, when they play at the house center and do the dishes. . . get the dolls out and have one plate for each doll. They do lots of number sense stuff when they're building with Lego or other building materials. [K-2]

Generally, Karen believed that the development of number sense in students' learning includes learning through representation of number activities, articulating what is happening, being engaged in meaningful problem solving activities, working with tangible objects and playing. Karen does not view these features of student learning as separate entities but more as intertwined aspects with what the students need to do.

C. Facilitating the Development of Number Sense

Karen's beliefs about how students' develop number sense suggest that students learning can or should be facilitated in the mathematics classroom by creating an environment to allow students access to number sense through activities and by creating opportunities for interaction with others. Thus, Karen's view of facilitating number sense development has students involved in learning situations that build number concepts including students working in activities that require them to apply these developing concepts.

Karen believes that providing opportunities for students to build concrete, pictorial and mental representations is important to allow them to develop number sense. She described situations in which she provided such opportunities for her students. For example, in a whole class activity, students had to represent the boys and girls in the class by drawing pictures of each member of the class. She explained,

> I get kids so they see it in the sense of understanding. . . sense of having a visual mental picture of what number is. . . I have to present them with some opportunities to make those pictures themselves. [K- 2]

> I try really hard to try to work on how they picture stuff in their head and how they can show it to me on paper . . . on white boards or blackboards . . . lots of it is trying to represent it with pictures. [K- 3]

> I can think of talking to them about outfits they're wearing. . . . I've done this with kids and how they are dressed and how many outfits can you get out of that kind of thing and then trying to draw pictures of that kind of stuff. [K-3]

> . . . sometimes it doesn't work very well when you are doing it in pictures so then we have to get physical things and demonstrate it in that kind of way. [K-7]

Karen's students were encouraged to discuss, explain and describe their thinking and their representations to each other. She explained,

> Kids have a good way of representing. . . . they have ways of their thinking that they can articulate with somebody else in that they can share with somebody else. . . that sharing in and of itself helps the growth of the group. [K-5]

> We put things together . . . and use language to describe how things get sorted together and put together in sets. [K- 3]

Karen also believes that providing opportunities for students to play and interact with each other is important to learning. She encourages such interaction with her students to clarify a situation or concept for learners.

> If kids weren't getting something or it didn't feel to me like they were getting something, I'd go back and try it in a slightly different way or I'd try and um.. single out some child who was getting it and get them involved in how they were getting it. You know get them involved in some kind conversation. [K-7]

Karen emphasized the importance of games in her teaching and learning.

> It's [number sense activities] a lot of games. We play cards a lot. . . . we played lots and lots and lots of card games and board games. [K-3]

Karen spoke of the value of combining play and interaction through the use of communal learning centers in her classroom. She believes that centers promote play with meaningful materials and created situations where students can interact, share their ideas and strategies and dialogue with peers about the process. She explained,

> The reason for doing these centers was because so much of their learning language was through their play with one another their interaction with others. They learned from these situations. . . . And always in those play situation there was math stuff to be had. They were making things and there was measuring to see if the roll was going to be big enough. I mean, not standard measure but you would try to see them trying to construct something out of a bunch of meat trays, toilet paper rolls . . . you would see they were trying to figure things out and that was number sense. And we did sewing stuff and sewing was a great thing and there was lots of, lots of math kind of things around those activities. Dress-up always had... dishes and tea parties and all that kind of stuff. [K-3/4]

Karen believes that observing what students actually are doing is important for gaining understanding of students' progress in developing number sense. She stated,

> A lot I do when I watch them playing a game is not just [to] watch the actual object of the game but also watch how they [the students] go about the game too because it tells you a lot. Tells you a lot about where they're at and what they've got and what they haven't got, you know. [K- 4]

Karen believes that questioning students about their thinking processes is important to further expand the depth and breadth of their learning.

> Children make dots to represent and however they represent what they figure out. . . . I ask questions of those dots, of whatever we're talking about or tell us what they represent to you. [Can you] tell us what those represent to you and how you do it? Can you explain your thinking here? [K- 5]

Karen believes that building on students' past experiences is important in helping them to develop number sense. She views such experiences as enhancing what students find interesting in the problems they are given to solve.

> Kids come with problems...they come with their own... once they get onto this notion that this is how we do things... It [problem solving activity] develops if they [students] are engaged and interested. [K-7]

Karen spoke about her emphasis on the problem solving process in her teaching.

> We spend a lot of time on things to do with them knowing how to go about solving some kind of problems . . . using what they've figured out about number in a way they talk to someone else. [K-5]

Finally Karen believes it is important to discuss with students their strategies and processes in solving problems or in their working with patterns. She described her discussion with students about patterns in numbers.

> . . .we talk lots about pattern in language so then the natural thing is to go and talk about it in number as well. . . the idea that counting by two's is a pattern, that looking at a number chart there's a pattern. . . . tell me about the patterns in this number chart, in a hundred chart. They talk about the patterns of all the twenties begin with a 2 and when you look at a number chart, that all the 2's, you know, have ten in a row or that ones that end in 2 are in a row and that kind of stuff. Talk about patterns . . . multiples of stuff . . . 3 groups of 3 and how that becomes a pattern. . . . talk about patterns in ten's and ones . . . patterns in . . . tens, ones and hundreds. And the kids will just say sometimes "It's a pattern. Look, it's a pattern". And you get talking about patterns when you try and make 10's out of stuff. . . How you can make a pattern by finding the tens and adding those tens, I mean just kind of everywhere. [K- 6]

To summarize, Karen believes that number sense is a way of thinking and organizing thought. Number sense is developed through active engagement, representation and working with tangible objects. The teacher facilitates the development of number sense by engaging the students in such activities as games, problem solving, creating pictures and talking about those pictures.

6.1b Louise's Case

A. Number Sense

Louise's beliefs about number sense include a number of characteristics to do with using numbers as a means of helping to create order in everyday life and objects in the real world.

Such use of numbers has to do with counting objects, assigning numbers to objects, dealing with "touchable things" that can be counted, manipulating numbers to derive different questions and answers and seeing numbers as flexible rather than rigid. Louise's thinking reflects these characteristics of her beliefs about number sense.

> I apply numbers to everything now. Whether it's I have 4 things to do before I do something else, and then I count them off. I've got 5 errands to run before I'm finished. I assign numbers to everything. . . . It just makes things orderly. I'm a very orderly person so it's. . . everything is numerically correct. [L-6]

> [What I] suppose I think of is touchable things, amounts of things, things that can be counted and how you can move things around in your world with some kind of order. I think mathematics and number sense has a lot to do with the way our everyday life is put together. Without that kind of order we would be very disorderly. [L- 1]

> I guess what I see is how many different ways you can manipulate numbers to come up with totally different answers or.... with a number, the incredible number of questions that you can derive that would have a particular answer? Like, I've become more divergent in my idea about number and what it is. It isn't sort of rigid, like locked in little boxes. Like numbers can take you places. [L- 6]

Thus, Louise appears to view number sense as a means of ordering and structuring her world using numbers in applications to everyday life.

B. Development of Number Sense

Louise's beliefs about how students acquire or develop number sense includes a number of strategies students can employ to allow them to build their own number sense. Such strategies include noticing and discovering, talking and sharing, interpreting, visualizing and building on concepts. For Louise, students need to "discover on their own" [L-1]. For example,

> They need to notice something that had to do with number. [For example] license plates they bring to school, you know, patterns in the calendar that they notice . . . pages of things. [L-1]

Louise's Beliefs about Number Sense and its Pedagogy

A. Number Sense is:

- applying numbers
- a way to order everyday life and things in the world
- counting things
- a way to manipulate numbers
- not rigid

B. Development of Number Sense involves students:

- noticing and discovering
- talking and sharing
- interpreting
- visualizing
- building a concept

C. Facilitating Development of Number Sense involves the teacher:

- providing a variety of activities including:
    - moving objects
    - drawing diagrams
    - visualizing
    - solving problems
- questioning students
- using different teaching approaches

Table 6.1b

Louise also believes it is important for students to talk about and share their thinking process as part of their development of number sense. She stated,

> [Students] talk about how they arrived at what they . . . found out. By then they are helping share that number sense with their peers. [L- 2]

For Louise, students should interpret and visualize numerical situations or problems to be solved in their process of developing number sense. For example,

> they have to interpret from number instead of having a picture, the symbolic form of a problem, they can transfer that into an amount or a picture in their head. [L- 8]

> [They] start off with a problem and . . . something to handle so they can visualize what the problem is. [L- 2]

Louise believes that the development of number sense is a hierarchical process and that students begin with simple concepts and acquire increasingly more complex concepts. She explained,

> You start off with one small concept then take that into another activity where they just add on and they keep adding on more and more complex things and more and more ideas. [L-1]

To summarize, Louise believed that the development of number sense involves students participating in activities that allow discovering, noticing, talking, sharing, interpreting and visualizing. For her, the concepts being developed begin with simple ideas that grow to more complex ideas.

C. Facilitating the Development of Number Sense

Louise's beliefs about how students' development of number sense can or should be facilitated in the mathematics classroom involve students in activities that build their concepts of number and provide opportunity for students to apply them. Louise believes that providing students with a variety of activities including moving objects, drawing diagrams, visualizing situations and solving problems is important in enhancing the development of number sense. She described situations in which she provided such activities for her students. One noteworthy activity was the tub-type centers she developed. The centers were made up of tubs filled with differing sets of activities. She explained,

> I have 8 tubs in a round. . . I try to use a collection of different strands. They're not all practicing the same thing for eight rounds. . . . There will always be time, something to do with money. They've done... perhaps geometry or measurement... a collection of things. [L- 5]

The tub activities included many problem solving situations in which students needed to figure out the solution and to move objects or draw diagrams to demonstrate the solution. Louise explained,

> You know, I put a problem on the board. Who can figure this out? It's just that... it becomes a quest for them. Give me something to solve and they really enjoy it. It's a real triumph for them. [L- 8]

> That's another area, problem solving where it's a really difficult problem if you just read it or said it . . . The children really have to move objects or draw diagrams or complete some kind of a table. [L-7]

Louise believes instructional situations should assist students to visualize the task or make pictures in their heads. She explained how she engaged her students in such visualization activities,

> . . . practicing of visualization. . . . close your eyes. . . . to try and make a picture or make a movie of 3 blue squares. Like some children can't visualize and they almost have to draw it. They have to see it on paper. They can't do it in their heads. So... you allow them to do that. But just to practice seeing something and make one of them disappear, you know, add, put two more in, you know to get the picture in their head to change. [L-8]

Louise discussed with her students and directed them to visualize activities as though they were photos or movies.

> [The students] show how important visualization is, not only in math but writing and reading . . . I really started looking at my own movies. And that's really... you can photos in your head or movies in your head. And both are to be, could be very valuable. But the movement.... the whole movie idea in

> problem solving firstly with operations is.... you know.... even dividing up your candies. Now, can you see yourself putting 2 in Susie's hand and while you're going to put 2 in Freddie's hand and can you see John's kind of anxious to get his 2. He's grabbing them. The whole idea of making an everyday situation. They can see that. Now, you can see that in your head, do it at your table. Here's the 2 for John and here's the 2 for....they sort of make their picture come true in front of them. [L-9]

In addition to activities, Louise believes that questioning students and using a variety of teaching approaches are important for facilitating student development of number sense. Louise used questioning to encourage students to think, to speak about, and to discuss their thoughts and their meanings. Louise explained how she teaches.

> I do a lot of questioning and comparing. [I] try to get the kids to talk as much as they can. So I know where they're headed. [L-3]

Louise explained the use of a variety of teaching approaches to assist students to understand the concept being developed. If students experienced difficulty, she incorporated alternate media to assist them to make sense of the situation.

> You just keep using different media and try to approach it from different angles until it makes sense. So if they don't get it, it's obvious that you're not giving it to them in a way that they can. . . . [You] can test them by going back to some of the other ways if you finally hit on something that works then go back to the original ways you tried to explain it and hopefully then they'll see one of those first ways. [L-6]

To conclude, Louise believes facilitating the development of number sense requires engaging students in a variety of centers and activities. Students have opportunities to experience a variety of different approaches to make sense of numbers through visualization, through manipulation of objects and from drawing diagrams.

6.1c Mary's Case

A. Number Sense

Mary's beliefs about number sense include two related themes. The first views number sense in terms of having an understanding of or a feel for numbers. She explained,

> Number sense to me, means what children understand of a number. It may be a quantity, it may be a place in a row, it may... It's how this is the number 3. How they would feel about the number 3, how many fingers they have, maybe their sister is 3 years old, um... It's a quantity, also a place in a line of things, and so on. That's what number sense is to me. [M-1]

The second theme is the connection of number sense to estimation. Mary associates number sense with the ability to estimate, particularly in terms of determining reasonableness of a result to arithmetic computation including sum or product.

> Number sense would involve being able to estimate properly, being able to realize an answer is way out of line or in line. Being able to see that when you add... I guess number sense to me would mean something you would understand, you would have an idea in the ballpark where things should be. Or really, an estimating kind of thing but also that one on one. A number would be the end result of counting something. That would be number sense as well, I guess, and knowing that would be a feasible answer. That's what I think number sense is. [M-1]

B. Development of Number Sense

Mary's beliefs about how students acquire or develop number sense focuses on the types of strategies and processes students should exhibit that enhance their ability in building their number sense. These strategies include counting, working with real objects, estimating, talking about problems and problem solving. For Mary, students need to be "counting the objects rather than the amount" [M-4] and "counting backwards" [M-5]. They need to work with real objects and. "move things from one side to the other so they don't get mixed up in their counting . . ." [M-3]. They need to be "involved in activities that are of interest to them. . . . [since] children best learn through experiences." [M-4]

Mary's Beliefs about Number Sense and its Pedagogy

A. Number Sense is:

- having an understanding or feeling about a number
- being able to estimate properly
- being able to determine the reasonableness of an amount

B. Development of Number Sense involves students:

- counting
- working with things
- estimating
- talking
- problem solving

C. Facilitating Development of Number Sense involves the teacher:

- showing students
- providing activities involving:
  - grouping
  - estimating
  - counting
  - problem solving
  - working with concrete materials
- engaging students in talk
- asking students questions

Table 6.1c

Mary stated her beliefs that students need to estimate, for example,

> being able to think that this is a good ballpark figure, I think that it must be around this answer. It's the estimating again. . . also knowing what a number is. Knowing that one to one correspondence that they hardly have to think about it anymore. Some children are able to estimate really well when you show them a number of things on an overhead or in a jar. They're able to estimate much more closely than others. They'll know that they're over twenty or under twenty. They'll know that it's a large group so it must be over a certain amount. They're really estimators . . . they also have a sense of one more or one less or two more or two less. [M- 5]

Mary believes it is important for students to talk about learning activities and their experiences.

> I'm just thinking of a problem that I gave my children that dealt with money and a piggy bank. . . . The majority of them had come out to the answer and it may even have been through talking to their neighbour, that they came up with the answer. I allow a lot of talk in the classroom. Talk about what you're doing. Then I would have them all come back together as a large group and I would say to them, "Okay, how did you figure it out?" {I would ask them to] explain in their own words how they figured it out and show the children how they figured it out. [M-7]

Finally, Mary believes that students need to engage in problem solving to allow them to "figure out things" or "figure out how many there are just by listening to a story and do it in their head." [M-5]

C. Facilitating the Development of Number Sense

Mary's beliefs about how students' development of number sense can or should be facilitated in the mathematics classroom focus on showing concepts to students, providing a variety of learning activities, engaging students in talk about the concepts and activities, and questioning students about their understanding.

Mary believes students need specific concepts demonstrated and explained. She described this belief as follows.

> Place value . . . that's part of number sense . . . they have to know that the 2 in 123 is a twenty not a 2. I . . . show them that's not really 2 but twenty. [M-3]

[The students] . . . see them in groups of 2's, in groups of 3's . . . Could you see how many there were? I . . . show them how to group. [M-7]

I think with larger numbers showing how to group them so they can count them easily, putting them into groups of 5 or groups of 10 works so well in our number system would help them become more sensitive to what a number might be if it were a larger one. Just giving them practice grouping things might work. [M-5]

Mary believes that providing activities for concept development involving grouping, estimating, counting, problem solving, and working with concrete materials is important to enhance student development of number sense. Mary's statements during her interview showed her thinking and her description of her teaching style and disclosed these aspects of her beliefs.

I think that for grouping and configuration of numbers would be a real help to them. To have children look at a group of objects and see smaller groups of them maybe groups of two or maybe groups of three, five. That would be a way to help them out. Also estimating. [M- 5]

Lots of [students] don't know 25, 50, 75, 100. They can't remember the 75 so it's um. . . . you have to keep working with them on that, to count those things out and remember. That's one area where I have to get from 50 to 75. I have a very difficult time in getting children to figure that number is something they need to remember when they're counting quarters out in money. [M-8]

Throwing a bunch of things in a jar and saying. " How many do you think there are?" Having them guessing, then counting them out, then putting more in and then asking, "How many do you think there are now? Could there be possibly more or less? Do I have a greater number in the jar now than I did before?" [M-5]

. . .

I have activities that would have them call upon their ideas of number. For instance, I quite often use paper clips on an overhead projector, where I will throw several... this is the way I would plan a lesson. Okay, I'm going to take the overhead projector, I'm going to throw some paper clips on. Going to have them count that number of paper clips and... or tell me how ... or flash it on, flash it off. Tell me how many there are. [M-7]

You have to have kids physically count 10, bundle 10 up, move them over and start making a new group of ones to make another bundle of 10, move them over. I think place value is a very difficult thing for children to grasp. [M- 5]

That's what I do for number sense as well . . . have them do work with lots of moving things around on their desks, problem solving situations. [M- 7]

Mary believes during learning activities students need to be encouraged to engage in talk with each other and explain their understanding of what they are doing.

> I allow a lot of talk in the classroom. Talk about what you're doing. [M- 8]

> I have them explain in their own words how they figured it out and show the children how they figured it out. [M- 12]

> We do calendar activities everyday which focuses a lot on number sense. . . We use both ordinal and cardinal numbers, we talk about the days of the month. Then we talk about the number of days we've attended school which is a different number from the number of days that are in the calendar of the month. We talk about ... we put popsicle sticks into a little ones house for every day that they are at school and when they reach ten they know they can no longer stay in that house, they have to move to the tens house. They bundle up the popsicle sticks and go into the tens house and so on. We also talk about the days of the week being 7 days of the week, we have gone through Sunday and Monday, so how many days are left in the week. How many days have we gone through in a week? And then I have them do number centers like "Tell Me What In a Number Sentence." [M-6]

Mary believes allowing students to have and make choices about the way they work is positive and conducive to learning. Such openness allows students the opportunity to see how others work differently from themselves and on varied activities.

> So allowing the children to . . . be very free with the way, any way that they choose that works for them, is a good way for them to work. would be something that's a positive thing to do so that they think that they have... so that they realize that their number sense is right even though it may not be like someone else's. [M-8]

Finally, Mary believes that questioning students during their efforts at problem solving or as they work through activities or share their work is important in enhancing their development of number sense. She described situations in which she engaged her students in such questioning.

> I would then go to them and say, "Can you think of another way you could keep track of the ones you've counted?" and walk around. . . . I would ask each group and then I go back again and say, "Okay, did anyone find an easier way to do it?" [M-7]

> I would do a lesson as well as [by] presenting a problem, then having the children going away and working on it. Many different ways. Then having come back and share that activity. And then discussing and saying "Which way do you think would work best for you?" Maybe, "Would you like to do it this way?" or "Would you rather do it the way Sammy did?" [M-8, 9]

In summary, Mary believes facilitating the development of number sense requires showing students concepts, engaging students in talk, and asking students questions. The activities Mary provided involved grouping, estimating, counting, problem solving and working with concrete materials.

These descriptions and explanations of the participant's beliefs about number sense, the students development of number sense and facilitating the development of number sense are gathered from the participants' interviews.

6.2 Teachers' Beliefs versus Classroom Actions

This section discusses the findings by comparing the teachers' beliefs reported above and the teaching practices based on classroom observations of the three teachers in their classrooms. The primary concern is on teachers' beliefs about facilitating students' development of number sense and the teachers' observed behaviour in teaching this concept. Each teacher is discussed individually, including a description of a sample lesson presented to show the nature of each teacher's practice. The lesson is followed by a discussion of their beliefs in action.

6.2a Karen's Case

6.2a (i) Sample Lesson

Karen led her class of eleven students into the class pushing a cart containing manipulatives, student scribblers and teacher's print materials. Students entered a classroom with a large circular table in the center of the room surrounded by 5 smaller rectangular tables placed along 3 walls. Blackboards lined two walls and a coat room was at the back of the room. Karen chatted with students about what they had been doing or the happenings in their lives. Students walked single file and took seats at the large table in no particular order. The researcher sat off to the side at another table to observe Karen and the students.

After the students were seated around the table, Karen stood by the table and the supply cart and handed out students' scribblers, whiteboards and coloured cubes. She was assisted by students nearby. Each student received his or her scribbler, a whiteboard, a marker and a set of coloured cubes.

Once students settled into their chairs with their materials Karen began, "Who can remember about multiplication and what we had done previously? Do you remember how to make arrays?" Some students were able to remember how to make arrays while some were not. She continued, "If you can't remember, look in your scribblers where you had it. That can help to remind you." Some looked and said, "Yeah, I get it! I remember." Arrays are one of the ways to show multiplication, so Karen asked the students to use blocks to make the array for 5x5. Student Sandra suggested they do 5x5, so students used that multiplication equation. Amanda completed the array but said, "I can do it 2 ways." She had built the array 5 rows of 5 with blocks and then she built 5 groups of 5 blocks. The whole group of students examined Amanda's work. Karen and other students asked Amanda what she meant by two ways. At Karen's direction students counted the array by 5's.

Amanda asked what they were going to build next. Karen said, "What we are going to do is go to a restaurant. I want you to build the restaurant. I'm going to give you more information about the restaurant. This is the table and we'll use these chairs, 1 table, 4 chairs. I'm going to ask you to get out 24 chairs. Build that restaurant. How many tables are there altogether at the restaurant?". Some students were unable to figure out how to use the blocks to arrange the tables and chairs. Karen demonstrated this by showing students how to designate one block as the table and place four other blocks around the table as chairs. Once the students understood how to place the blocks each built a restaurant. One student said, "Cool! Looks so real."

Students were confused as they put out the chairs with the tables. One student didn't have the same number of chairs around each tables and someone beside him tried to explain each table needed four chairs around it. Some students arranged the chairs around four sides of the table while others arranged two chairs on each of two sides of the table. Each student developed personal understanding of the instructions; discussion and disagreement began between various groups of students working at the table. Karen interrupted to explain the need for equal numbers of chairs around each table. Students rearranged their tables and chairs following Karen's additional instructions.

After students completed the task of placing their tables and chairs, Karen said, "Show me how you built your restaurant." Then she commented, "How could we write that. How would it look?" Students wrote on their whiteboard several equations including: 4+4+4+4+4+4=24 and 6x4=24. Karen stated, "Is it four 6's or six 4's? What did I ask you to do? I actually asked you to divide them into 4. How many tables?" Amanda replied, "Oh, it's six tables." Karen explained that one equation is addition and the other is multiplication. Someone in the class said, "Can we make our own? 30 chairs?" Karen circled the table saying, "Yes, get the tables out this time, 6 chairs at a table." The students worked placing 6 chairs around each table in various ways. Now the students were familiar with the task. Some wrote equations on their whiteboard including 6+6+6+6+6=30 and 6x5=30. Karen said, "Kathy, show the equation as multiplication." Kathy held up her whiteboard showing 5x6=30. "Kathy, would that be 6 groups of 5?" asked Karen. Kathy replied, "That would be a different kind of story, 5x6.

Karen had students follow a similar procedure and continued, "At the restaurant there are three tables with 4 chairs at each table. Build that restaurant. How many chairs are there altogether at the restaurant?" Some students counted the chairs and some students said, "three times four makes twelve." But David wrote 12÷ 4=3. Karen asked him to explain why he wrote 12÷ 4=3. David said, "You take 12 chairs divide around 4 tables." Paul replied, "3" [chairs at each table]. Karen then said, "Who has another story about 12 [chairs]? Anyone have another story about twelve?" The students generated different equations with 6 tables and 2 chairs, 12 tables and 1 chair, 2 tables and 6 chairs, 12 tables and 1 chair and Karen recorded the equations on her whiteboard:

3x4=12 6x2=12 1x12=12

Karen said, "You could use the blocks on the whiteboard to show the restaurant but then we can also build and draw. [Does anyone] want to see it on the whiteboard?". Karen drew three tables as squares and four chairs at each table as circles. She said, "Let's do 5 tables and 4 chairs at each table. How many chairs in all. This time you can build and draw. I want to see it on the whiteboard as well." The students built the tables and chairs followed by drawing of the tables and chairs. Some drew in the same spot as the blocks, removed the blocks and completed the drawing while others drew the tables and chairs on the whiteboard beside the blocks. Karen asked, "How many chairs in all?". Ed said, "What does it mean?" and Ashley said, "20". Keshia illustrated this by counting by 4's while the others watched. Sandra and Pete counted by 5's and someone asked, "Is it by 5's? How many chairs?". Karen corrected, "4's not 5's. Count with me." As she wrote the numbers, the children count 4, 8, 12, 16, 20. When the buzzer sounded to end the mathematics period Karen reminded students the lesson would continue next period and students would have the opportunity to build with tables and chairs again. One student gathered all the scribblers and returned them to the cart while other students helped to return the blocks to the container and on the cart. The students lined up at the door once all the materials were put away and changed classes.

6.2a (ii) Comparison of Karen's Beliefs and Practice

Karen's stated beliefs about the facilitation of number sense included the following five major categories: 1) providing opportunities for students to build concrete, pictorial, and mental representations; 2) talking about what students are thinking and doing, how students interact with each other and how students play during the activities; 3) building on students' past experiences; 4) questioning students about their thinking; observing what students are doing; and 5) discussing how to solve problems or work with patterns. These beliefs were observed in Karen's lessons not as a single identifiable action but as a series of teaching actions interwoven throughout the lessons. These beliefs were first noted in the interview and are described in more detail in examples taken from field notes of the classroom observations as Karen acted on her beliefs.

Karen's belief of the need to provide opportunities for students to build concrete, pictorial, and mental representations was pervasive in her practice. It was observed that Karen purposefully led students through steps to have them show representation of the problem about building the restaurant using blocks and drawing layouts. She began by having the students use blocks to represent the chair and table arrangements in the restaurant. Students seemed to understand the process of representation as they looked at the arrangements of their blocks and expressed comments of new insights. In the vernacular of the students expressed as, "Cool! Looks so real!" as students related their arrangement of blocks to an actual set of tables in a restaurant. After the students practiced using the blocks, Karen asked them to draw a map or a representation of the same tables and chairs laid out using the blocks. Karen guided students to remove the blocks and draw squares where the blocks had stood on the students' whiteboards. Judging from the conversation during the lesson, it appeared the students visualized the situation and made a mental picture of the restaurant without specific instructions from Karen.

Consistent with Karen's beliefs about students having opportunity to talk about their thinking, interacting with one another and playing, the researcher noted during the lesson the ease and frequency with which students engaged in such learning activities. It was not unusual for students to discuss quietly among themselves how they were going to "build their restaurant" and show someone sitting beside them the results of that building task. Embedded in student talk between peers were their own explanations of their thinking about how they planned and organized their arrangement of chairs and tables for each new task. Two students made differing arrangements of tables and chairs. However, after some negotiation and discussion they agreed that four chairs around three tables still resulted in the same number of chairs being set out. The students were uninhibited as they exchanged their ideas and thinking. Talking about their own thinking, moving the blocks and drawing the "map" of the restaurant were woven together as an activity. Students viewed these activities more as play than as an attempt to find solutions to a problems.

Karen capitalized on situations in which students need to recall past information or skills to assist them to answer questions or to solve dilemmas, a practice that is consistent with her beliefs about building on students' past experiences. During the lesson observed she picked up on a conversation between a pair of students and joined in when Pete commented that he used his new watch with a timer to time the students moving into this class. It took 79 seconds. By asking the students how 79 seconds could be expressed as minutes, the students were presented with an opportunity to use past experiences. Karen drew on a teachable opportunity to see how students would respond to the impromptu problem. Amanda knew there are sixty seconds in a minute and showed the group that by counting on from sixty using her fingers she established that 79 seconds equaled one minute and nineteen seconds. Another incident illustrated a strategy Karen used to show students how to retrieve past understanding if they had difficulty remembering the specific mathematical information they needed to solve a problem. As the class began the restaurant activity, Karen asked students to recall how to show multiplication using the concept of arrays. Some students were confused and could not recall the concept of arrays and what it meant. Karen directed them to check their scribblers where the information was recorded in previous lessons. After looking in their scribblers the students recalled the concept of arrays and the lesson moved forward.

Throughout the lesson observed Karen posed questions for students including "How could we write that?" or "How would it look?" referring to the arrangement of tables and chairs to solve the problem. Posing questions appeared important to Karen to assist her to understand the students' thinking from their responses and their attempts to clarify understanding. During the observation David tried to explain three tables and six chairs as an equation. When David replied, "Six times three? Three times six?" in a questioning manner, Karen asked David to clarify his meaning. To probe for more information she asked him "Does that make sense?" and waited for him to think through and explain his meaning before she continued the lesson.

Throughout the lesson observed, Karen was sensitive to students, their use of the blocks, how they drew the arrangements and, in particular, the dialogue between and among students. This teaching behaviour is consistent with her belief that it is important for the teacher to observe what students are doing. Karen's approach to observations was more than simply watching students. Her observations were part of her on-going strategy to monitor the students' development of understanding so that she needed to enter into the activity when her monitoring showed it was required. For example, during another discussion about thirty chairs and combinations of six, Kathy responded with six times five. Karen queried Kathy about whether her display was six groups of five or five groups of six. Kathy was quick to tell Karen that five groups of six was another equation. In this way Karen watched Kathy build with blocks and waited for her explanation to check on Kathy's understanding of multiplication by groups.

Finally, Karen stated she believes in discussing how to solve problems. During the lesson Karen and her students considered the chair and table activity as a problem. Karen used the problem to allow students to build representations, to enter into discussion and dialogue and to explain their thinking. Karen watched students develop their ideas and share them.

While some of Karen's stated beliefs were not observed in the lesson explicitly further discussion during the lesson gave evidence that the students had prior experience with the concepts. For example, Karen did not use mental visualization specifically in the lessons. However, students seemed to have had some discussion about prior experiences in relation to visualization. This was noted when Karen and the students began the restaurant activity. Karen asked students to draw a map of the restaurant and a student asked to confirm that this was to be a map. Karen clarified her request by asking for only a drawing of the tables and chairs. Working with patterns was not the main focus of the activity but as students were reviewing the concept of multiplication the multiple counting and discussion by the students reflected an implicit understanding and experience with patterning.

6.2b Louise's Case

6.2b (i) Sample Lesson

The class began after students returned from recess, and hung up their coats on hooks along the wall to the right of the door. The classroom was a shared open area with a folding wall separating two classes. Desks were arranged in the middle of the room in groups of 4 with two students on each side facing one another. A blackboard and easel with chart paper stood to the left of the door beside a large meeting space. Storage bins and shelves stood beneath large windows facing into the classroom. Plastic tubs filled with manipulatives, games, activities and teacher-made booklets were labeled in the shelves.

Louise waited in the middle of the class and began the class with a warm-up that seemed to be familiar to the students. Louise said, "Get your number. Add 2. Add 1. Add 2 more. Subtract 2. Subtract first number. What's your answer?". The students randomly replied, some saying "I didn't get it." or "Yay, I got it!". Once all the students had settled into their seats, Louise asked them to move to the meeting space. Students took their places in no particular order on the carpeted floor The teacher stood next to the easel and blackboard with the tubs on the floor beside her.

Louise explained to students that a new round of centers was to begin. The tubs contained enough materials for more than one group to work using any particular tub. She showed the students the new recording booklet each students was to receive to accompany each center. The booklet included 8 tub activities with the objectives and instructions on what to do on each recording sheet in the booklet. The pages were for students to record their results as they worked on the tub center. Groups were arranged by the teacher for one center activity and changed when a new center began. The booklets for each group were placed in a bin in the storage area.

Louise took strips of orange and black cards with letters written in patterns across them. She held them up and showed them to the students. She also showed various coloured blocks in the tub and said, "This is the Spy Tub, Tub #7. In this tub you talk in special codes. You practice using the orange card but the black are special cards. Do I want to see anyone using the black to start? Of course not. There is a secret code. You've got to make a pattern with these blocks." Louise asked students how they might figure out the code. She said, "I'll give you a hint. What do you think these letters stand for? Morgan? What do you think?" Morgan replied, "G block, green. O block, orange." However, Louise presented a conundrum to the students when she said, "The problem is blue, brown and black start with the same letter. Does it matter which one you pick?". Morgan replied, "But there is only blue blocks in the tub." "Good observation", Louise responded. She continued saying, " You have to continue the pattern. You have to make the pattern to go longer. What will you have to count up to? Is something missing here? Yes, Gerry, it's in the book. Everyone needs to remember to fill in the blanks in your recording sheet."

Louise used similar explanations with students on how to use the materials and activities in the seven remaining tubs. When explanations for each tub were completed, Louise reminded each student to check the chart at the back of the easel if they did not remember which group they were in. The chart listed the titles for the eight tub activities. Each time a group was working on or had completed a particular tub their names were to be placed in the pocket beside that tub. While the students were still at the meeting spot, new booklets containing recording sheets for each tub were passed out to each student. Louise reminded students to make certain they wrote their names on the covers. The students seemed familiar with the routine. Once they got their books they set about getting pencils from their desks and finding their groups. Those who did not remember their group checked the posted chart. Often group members remembered who was in their group and called for those who were not sure. The groups talked and argued together trying to decide which tub they wanted to start with and understood they would have to complete each tub at some time. Once it was decided by the group which tub to begin, one member of the group got that tub.

The teacher circulated around the room as groups got organized. She helped those who were unsure about what they were to do and she helped others find a space to work. Five groups worked at desks and three groups were on the floor around the classroom. As groups settled into going through the tubs and finding the instructions, Louise moved to each group throughout the room questioning about what they were doing or giving suggestions or advice if students were experiencing difficulties getting started. She asked, "How's everything at tub 4? Have you done everything? Remember you need to take everything out of the tub." The group Louise approached was working on the "Make a Snake" tub. The students in the group used dice for a number and a + and - block for the operation. One member of the group rolled the dice to get a number and rolled the block to decide the operation to be used. Louise checked the student recording sheets as she conversed with the group. Ashley said, "We've done one. We have 3 and 6 and plus. 3 plus 6 equals 9." Louise replied, "I'm glad you know what to do. What numbers do you record? Nice neat numbers. I'm glad you write carefully." The children wrote the numbers according to the operation in the recording book following the shape of the snake. Louise approached the group with tub 2, Dinosaur Draw, and said, "What's in the bag?". Mark replied, "We're using green and blue dinosaurs. So far we have taken out 2 blue and 1 green." "What do you think comes next?", asked Louise. Each group member named a colour. One student put a hand in the bag and took out a blue dinosaur and students coloured in the blue dinosaur picture. Louise continued to visit each center until the buzzer sounded to end the math class. Students placed materials back into the tub and returned the tub and booklets to the storage areas. They returned to their desks and the teacher reminded them they would continue work on the centers next day.

6.2b (ii) Comparison of Louise's Beliefs and Practice

Louise's stated beliefs about facilitating number sense development included: 1) providing a variety of activities in which students manipulated objects, drew diagrams, visualized situations and/or solved problems; 2) questioning of students and 3) using a variety of teaching approaches. Louise demonstrated her beliefs about developing number sense through a novel program in which students worked in centers.

The use of tub centers allowed students the opportunity of moving objects, drawing diagrams and solving problems just as she had described in her discussion. Louise began the round of tubs by explaining to students what objects and materials were in the tubs and what was expected from the students working in each center. She emphasized the objective of each tub. The students were organized into groups of three and four and each group of students worked on different concepts and used different materials.

In the lesson observed students had many opportunities to move objects. For example, in the Block Puzzle tub, the students used base ten blocks to build numbers by using tens and ones to cover a shape. The students then counted the tens and ones together to arrive at a number and recorded the number in their workbooks. At the Dinosaur Draw center small dinosaur models were used in a probability activity. The students put five blue and five green dinosaurs in a bag and pulled one out at a time. Before taking the dinosaurs from the bag the students were to predict which colour would most often be pulled from the bag. The picture of the dinosaur was coloured matching the colour of the dinosaur taken out of the bag. The dinosaur model was put back in the bag and ten tries completed one round. The format of many of the centers presented problem solving opportunities for the students in Louise's view. Other centers in this round of tubs used objects such as money, dice, tiles and blocks.

During observation students worked in the centers and recorded their findings from the activities in the center. For example, in tub 4, Make a Snake, the students rolled a die with numbers and a die with the addition and subtraction symbols. Each student had a turn to roll each dice, use the number and operation to and try to record the number on the snake path. The students were not drawing diagrams specifically in these centers but they were recording their results as they progressed through the centers. Although specific drawing of diagrams was not seen in the centers observed, it might be likely that the students had drawn diagrams in other centers and might have experiences not only drawing diagrams but also using symbols to record some parts of their activities.

Although Louise did not present an activity in which students demonstrated visualization, the discussion in the classroom demonstrated an implicit understanding by the students about visualization. For example, as Louise began the warm-ups she asked students to get a number. Students did not write down a number or get any objects but it was implied that they think of a number. Louise gave further instructions of operations applied to the number chosen. She asked students how they might get better at this warm-up activity. The class discussion led to students responding that they needed to think what was tricky and then how to get the answer. In other instances Louise asked students to put a picture in their head as she showed students the centers. For example, as she began the probability center, she asked the students to picture in their head what is in the bag and picture their hand going in the bag. Visualization was not the focus of the activity but was one technique that Louise used to help students work on the activity.

The activities presented in the centers involved solving problems in Louise's view. For example, Louise saw the Dinosaur Draw as a problem in which the students were to explore probability and make predictions. On completing the activity Louise felt students had gone on a quest, searched and came back with their solution. The centers could be a problem solving activity or an activity to practice a particular skill. For example, the Make a Snake tub was to practice addition and subtraction. Tub 7, Over and Over, allowed students to built patterns and appeared to be a problem activity. Students were uninhibited in working at problem-type centers and in their responses. They were at ease in joining discussions and conversations in problem situations which might indicate previous experience working in problem solving activities.

Louise's beliefs about questioning students was observed both in her instruction about the centers and during the student participation in using the centers. As explained previously, she began the class with warm-up questions as students entered the mathematics class. Her questioning continued throughout the class period as she outlined the centers for students. During the explanation of the Block Puzzle center her questions were framed as, "What do you do after you pick a number off the board? What's interesting about this tub? What if you get a wrong answer?" as she attempted to engage students in conversation. Louise circulated to the groups while students worked in groups. She asked specific questions of the groups as they were working. For example, before a student reached in the bag to take out a dinosaur, Louise asked, "Can you guess what colour the dinosaur will be that you take out?" or when another student was rolling the dice, she asked, "Can you predict what number you will get?" Further discussion ensued about what might be possibilities in response to Louise's questions. As students told Louise their predictions they appeared excited and anxious to find how close their predictions were to the actual number. Louise used questioning techniques to invoke students to think, talk and discuss with her and the others in the group.

Consistent with Louise's belief in using different teaching approaches, the use of the centers themselves illustrated an interesting and out-of-the-ordinary mode of instruction which allowed students opportunity to experience eight activities during each round of centers. For this round of centers, students worked on different concepts including operations of numbers, place value, probability and patterning. The students were often requested to make estimates or predictions as they worked and completed the activity to arrive at some conclusions. Not only did the activities allow Louise time to observe the students' actions, she made suggestions or assisted the students while they were working at the centers if difficulty was encountered. Louise did not use alternative approaches during the observations to help a student understand the lessons observed. Possibly as students became more familiar with the centers her assistance may have been necessary if they encountered difficulty.

6.3c Mary's Case

6.3c (i) Sample Lesson

The bell rang and the students came in from recess. Mary stood by the door greeting the children as they came in and talked to them about recess. They hung their coats up and without direction, went to storage bins and took out folders. The students sat in their groups on the floor or at desks without any particular plan, got materials and started working in their groups. The researcher sat at a table to the side of the class to observe Mary and her students.

The activities appeared to be a continuation of lessons from the previous day. Students appeared to understand the routine of their groups and continuing their work. The folders contained activities and manipulatives were available to go along with the activities in the storage area. The teacher did not use textbooks. The folder contained a set of activities with instructions. For example, one group of students was working with tangrams, a Chinese puzzle in which seven geometric shapes form a square. Student were using tangram pieces to fill a variety of outlined shapes. Each group of students worked on an activity together. For example, one group worked on geoboards while another group was working with nets to make a cube. The activities were from a variety of strands. Students independently rotated through the activities. There were ten activities to be completed in a round. Once an activity had been completed by a group, the teacher would have a discussion with the group before they would get another activity.

The students used a pocket chart to track which folders they had completed and which ones they needed to complete. The teacher used the pocket chart to track the students' progress. The large hanging chart had clear plastic pockets for placing 5x7 cards. The chart had slots for the activities and the group names. When a group completed the activities in a folder, they put their name by that folder. For example, if there were 10 folders to complete, each time students completed a folder they put their names by the folder. The teacher could look at the chart to see how many folders each group had completed. Students worked in groups of 3 or 4 on the activity folders. Students were familiar with the activities and knew the procedure.

The teacher circulated around the room to each group. She asked each group specific questions about their activities or if the group needed help. For example, a group of girls tried to make a cube with 4 different nets. Group members needed to decide which shape would form the cube. Mary asked how they were going to figure out how to make the cube. The group talked about drawing and cutting out a cube to see if they could draw the net. Mary asked if they could tell by looking at the nets and was it necessary to draw and cut a net out. Mary worked with each of the other groups. She did not ask questions such as "How are you doing?" She asked students specific questions about the processes and the problems. For example, when Mary talked with a group trying to fit tangram pieces into a shape, she asked if they had used all the pieces of the tangram to fill in the shape. Students hadn't completed the activity. Mary asked if they believed all the pieces would fit into the shape and if not, how did they intend to figure it out. Students spoke with ease in conversation with Mary. It was not necessary for them to put up their hand as they spoke with her. These center activities involved approximately 20 minutes of the lesson.

Mary moved to the meeting area and clapped a familiar pattern to gain the students attention. She asked them to return the folders and materials to the storage area. She invited them to come to the meeting area. Students followed Mary directions and gathered on the floor in the meeting area.

Mary instructed the children in developing the concept of large numbers, a new activity for them. She sat on a low chair beside an easel that brought her physically closer the students. Base ten materials were placed on the easel ledge and on the floor. She asked, "Can anyone help me count from 160 to 200?" and "What kinds of things can we do?" The group offered answers including, "We could count by 5's, by 10's". Students reviewed counting by 5's and 10's. Mary said, "If we don't have base ten materials, how do we count by 25's?" The students were slower at skip counting by 25's. Mary said, "If we started at 150, could you count by 25's?" Then the class counted by 25's to 200.

Students didn't put up their hands and groups of students responded when answers came to them. Mary responded to students' questions/comments. Students offered responses to Mary's questions. If they did not understand the question they asked for clarification The dialogue went back and forth between students and Mary. At times Mary asked someone to help her. She presented more than one example, such as the first activity using 160 and counting by 10's. She asked, "Richard, if I was counting from 230 by 10's what would be the next numbers?" Richard replied, "230. 240. 250." She asked the other students, "Does that make sense?" and "Do you think Richard is right?". Students responded that Richard's response made sense while others nodded their heads. For example, Susan was not quite sure, she seemed perplexed. Mary posed the question a second time and attempted to clarify the concept for Susan. She drew the flats (100's) while the class counted by 100's to 500.

To continue, Mary stated, "I have a challenge for you. How many tens in one hundred?" Mary held up some longs(10's) and a flat (100) and said, "These are called base ten. Why are they called base ten? What does this make easier to do?" The students replied, "Easier to count." Mary drew a sample of the base ten blocks on the whiteboard. She repeated the question, "How many tens in one hundred?" and students counted with her by tens as she drew the longs on the whiteboard. She reminded the children that this way of counting is not new and she reminded them saying, "We've done this before". With Mary's help the group concluded there are ten tens in 100. They counted the longs Mary had drawn on the board. She used the materials to help the students count. Mary also used the term "Counting on" saying, "We don't have to start at the beginning but start from the number given and count on from there."

Next Mary said, "I want to give you a new challenge. How many tens in 160? How many do you think? Can you make a prediction?" The students responded with various guesses before Mary said, "In prediction, is anyone wrong?" The students replied, "No." At Mary's direction, students returned to their desks and one student collected a set of base ten materials to use in group activities. The teacher handed out paper for recording and reminded the students that they were trying to find out how many tens were in 160. Mary circulated among the groups seated at desks and observed them as they gathered the materials. Each group tried to figure out how to count out the longs. One group counted out and someone from the group said, "Do we have 150 or 160? 116 or 160?" Mary reminded them that the number was 160. After the groups worked to develop their ideas, Mary asked one group to show their responses to the challenge on the overhead projector in the classroom. Two boys and two girls from the group were ready and volunteered to share their answers. While the two boys watched, the two girls placed the tens on the overhead, counted by 10's to 160 and said, "There are 16 here. 16 tens." The rest of the class concurred. Mary posed another example and said, "I'm going to give you four minutes. How many tens are in 210? Don't record. Just build." Again, Mary circulated among the groups and checked their progress as the children counted out their tens. Within 4 minutes, Mary said, "Which table wants to share? Do you think you will have more than 10 tens or less than 10?". Mary asked, "May I have table two to share?" The students from that group went to the overhead and showed 210 using the base ten blocks in a manner similar to the first group. Mary applauded the work of the group saying, "Great work!" Mary asked, "How many tens in 280 stickers? Can you figure out a pattern?". Some students did not appear to see a pattern related to the number of tens while some responded saying, "28".

Mary directed the students to return to the meeting area. They seated themselves on the carpet and Mary said, "We have been building for a long time. If you don't have the base ten materials you can draw the numbers." She wrote the number 160 on the whiteboard and drew 16 longs underneath the number. She wrote the number 210 and recorded 21 longs underneath. "Anyone see what is happening?", she asked. As Mary erased her recordings she said, "I'm going to erase what is on the board. Just use your head. How many tens in this number?" Mary wrote 330 and students responded with the number 33. "Does this follow that pattern?" Mary asked. One student said, "It is easy. You look at the first number." Mary then writes 1020 and asked, "How many tens in this number?". Lindsay responded, "102." As the buzzer sounded to signal the end of the math period Mary said, "Maybe now we can count our pop can tabs (a large jar of collected tabs was at the back of the room)." At Mary's direction students put away the base ten materials and recording sheets in the storage bins. Students then prepared for the next class.

6.3c (ii) Comparison of Mary's Beliefs and Practice

Mary's stated beliefs about the facilitation of the development of number sense in learners included: 1) showing students concepts: 2) providing activities involving grouping, estimating, counting, problem solving and working with concrete materials; 3) engaging students in talk and 4) asking students questions.

Mary believes students develop number sense through explanation of specific concepts. Mary demonstrated this belief during the classroom observations when she instructed students in detail on the difference between amounts and values. She began by asking students the difference between the number of things and their value. To illustrate her point she asked a student to provide her with some blocks. Richard picked 7 longs. Mary asked the students the value of the blocks Richard selected. The class counted the blocks by tens to seventy but Mary said, "That's not what I had in my head. I have to give you more information. Seven blocks but three are tens." Richard readjusted his blocks and built three tens and four ones. Mary explained that thirty-four is the value of the blocks and seven is the number of or the amount of blocks. During another lesson Mary had the students estimate and count using base ten blocks. She wanted to know how many tens were in 160. She first asked the students to make a prediction, then had the students use the tens blocks and count them out. This lesson developed with further examples given and the students estimated how many tens were needed and counted them out.

Throughout the lesson Mary wove together the components of developing number sense. Counting, estimating, and grouping occurred within Mary's discussion about a concept or as an activity was progressing. For example, while explaining the activity of building large numbers, she asked students to count by tens and hundreds. When students were to find how many tens in 160, they counted the base ten blocks to 160 and the number of ten rods they had. The students also estimated throughout the activities. Before Mary began, she asked the students to estimate how many tens might be needed to make 160. The estimates were noted and on completion of an activity the students were asked to compare the answer to their estimates. The students did not specifically show grouping during the observations, however, the students showed that they had experience with grouping. This was observed in one instance when, as class was ending, a student noted that the large jar of pop tabs would need to be counted. Mary just asked the class to think of an easy way for the class to complete this task and asked students to consider a possible estimate of the number of pop tabs in the jar. Students responded with possibilities of grouping the tabs by fives, tens and twenty-five. It appeared the students were interested in continuing to count the pop tabs in a future class and would be ready to group them.

Mary believed in engaging students in talk to explain and share their work. Consistent with her belief, she demonstrated engagement of talk with students during large group instruction which involved the need for students to watch and listen. Mary began and said, "We have been building for a long time. If you don't have the base ten materials you can draw the numbers." She wrote the number 160 on the whiteboard and drew 16 longs underneath the number. She then wrote the number 210 and recorded similar longs underneath. "Anyone see what is happening?", she asked. Mary erased the recordings and said, "I'm going to erase what is on the board. Just use your head. How many tens in this number?". The students watched intently as she wrote 830. Students responded with various answers including seeing a pattern or crossing out the zero to get the answer.

At another time during observation Mary discussed measuring the students height and comparing that measurement to the length of a dinosaur. She engaged the students in talking about the comparisons and how measurements would be made. Some students suggested use of rulers while Scott thought a calculator would be useful. The interest rose as the students in the class tried to relate how to measure large objects and how a calculator might be used to measure a dinosaur. Following Mary's direction, the students were led through a discussion about estimation, comparison of using base ten blocks and rulers, and whether calculator was useful as a measurement tool. The students were interested and made unique contributions. Suggestions explored included some that were reasonable and some that were far-fetched.

Mary's questioning techniques were complicated and sought to explore what students were thinking. Students were asked to explain rather than offer only yes or no answers. Mary dispensed her questions throughout student discussions or while students participated in activities. Mary's questioning technique was demonstrated as she concluded a lesson on how many tens in a large number. She drew on the whiteboard a picture of the base ten blocks and the number, erased the pictures and repeated two more examples. Her only questions as the students watched the drawing and erasing was, "Anyone see what is happening?" The students were intent and watchful of Mary's actions ready to give their explanation. Another example during observation of a small group activity demonstrated how Mary tried to draw the students out specifically about what they were doing in a tangram activity. She approached the group as they were fitting tangram pieces into a shape. Mary asked if they thought all the pieces would go into the shape and how were they going to figure it out. The students told Mary that they had some problems because pieces were sticking out in some attempts but they were willing to continue. One student thought it would be useful to remember what pieces had been tried but the group found it difficult to use any particular strategy as they worked together.

CHAPTER 7

CONCLUSIONS

7.1 Discussion of Results

This study began as a result of my search for a way to consider a deeper level of meaning of teachers' practice and thinking about number sense. A review of the literature on teacher thinking indicated that teachers' beliefs are important factors in predicting teaching practice and professional development. The review provided the rationale for a study that focused specifically on teachers' beliefs as they are related to instructional practice in mathematics. In this study I chose number sense as a topic of mathematics that warranted exploration because number sense is at the heart of elementary school mathematics and is the foundation of later mathematics. My search led to investigating what number sense means to teachers and what guides their teaching of number sense.

The research provided the following conclusions:

- the teachers beliefs about number sense were different.
- the teachers' beliefs and behaviours in the classroom about the development of number sense were similar despite differences in the teachers' beliefs about number sense
- the teachers' beliefs and behaviours in the classroom about the facilitation of the development of number sense were similar despite differences in teachers' beliefs about number sense.

Beliefs about number sense were unique and individual for each of the three teachers in the study and showed a pragmatic, an affective, or a philosophical view of number sense. For example, number sense from Louise's perspective appeared to be pragmatic. She viewed number sense as applying and manipulating numbers. Mary's view of number sense was somewhat pragmatic and affective. She viewed number sense as being able to estimate and determine the reasonableness of number but she also felt number sense was having a feel for number. Karen appeared to be philosophical and to view number sense as a way to organize thinking, a way to view the world, make sense of it, and interact with it. The three participants clearly articulated their unique understanding of number sense. When the participants worked in their mathematics classrooms, they connected these individualistic beliefs about number sense to their students' development of number sense. Recognition that teachers are individualistic in their beliefs is important. As Ernest (1989) suggests, teachers have mental contents or personal schemas they value.

Despite the differences in the participant's beliefs about number sense, they viewed the development of number sense similarly. All the participant's belief about the development of number sense had a strong student-centered focus supported by specific expectations and intentions. They valued talk and discussion, used specific questioning techniques and accepted classroom discourse as part of the development of number sense. Slight variations in classroom discourse were shown. For example, Karen described students articulating, Louise described students talking and sharing, and Mary described students being encouraged to talk and explain. For each participant, students were involved in activities that developed number sense. For example, the participants viewed use of objects in the activities as important. Karen and Mary were explicit in the use of tangible objects and Louise demonstrated her use of objects as described in her tub centers. While engaged in the activities the students used various techniques to develop number sense. For example, Karen assisted students in building understanding of numbers through representation. Louise assisted students to build a concept through noticing, discovering, interpreting and visualizing. Mary assisted students to develop number sense by counting, estimating, and problem solving. Although each participant expressed her belief about student development of number sense using different methods, the students were involved in similar ways using objects, engaged in classroom discourse, and involved in building concepts.

The participants' beliefs and their behaviour in the classroom relating to the facilitation of the development of number sense were similar despite differences in their beliefs about number sense. The participants provided students with activities or situations to develop number sense. For example, Karen provided opportunities for students to build mental, pictorial and concrete representations, talk about their thinking, interact and play with peers. Louise provided tub centers in which students had opportunities to move objects, draw diagrams, visualize and solve problems. Mary provided opportunities for students to group, estimate, count, solve problems and work with concrete materials. Each of the participants engaged students in their own form of classroom discourse. Karen believed in questioning students about their thinking and discussing how to solve problems. Louise believed in questioning techniques and using different teaching approaches. Mary believed in engaging students in talk and asking students questions. Facilitation of the development of number sense was similar from the participants' perspective. They provided similar activities or situations, assisted students to develop their understanding and engaged students in classroom discourse.

The research showed consistency between these teachers' beliefs about how development of number sense can or should be facilitated in the mathematics classroom and the observed teachers' actions in instructional situations in the classroom. For example, providing suitable situations and activities for students to develop number sense was viewed as important in both the teachers' beliefs and in their actions. Karen provided problem situations, Louise organized and provided tub centers, and Mary provided centers, large group instruction, and problem solving. They also included in these situations specific methods of number sense development. These methods were not the same but were similar in their intent. For example, these teachers provided classroom situations in which students had opportunities to articulate ideas, engage in talk, and respond to questions. The teacher's intention in each case was to have students explain or share their ideas or thinking. The teachers provided opportunity for students to use objects while involved in activities. Specific development techniques were used by the teachers and were both explained in their beliefs and observed in the classroom. These techniques included the use of pictorial and concrete representation by Karen, noticing and discovering by Louise and working with materials by Mary. In particular, beliefs about the development of number sense and the facilitation of that development appears to have provided structure for these teachers to organize their knowledge and direct their behaviour. Their beliefs seem to play an important role in directing their behaviour (Chapman, 2000).

The teachers' beliefs and actions about the facilitation of the development of number sense were closely related to suggestions from the literature. Greeno (1991) discussed working in a conceptual environment. Karen, Louise, and Mary provided activities in which concepts were developed. They used techniques Greeno suggested were useful such as finding patterns, apply understanding, solving routine problems, and using physical objects. They provided opportunities for students to work together to understand math concepts and procedures as suggested by Greeno. Specific techniques used by the teachers in the study reflected those suggested in the literature. For example, Mary felt it was important to prepare activities that present opportunities to explore estimation as is suggested by Sowder (1989). Louise used warm-ups as the students entered the classroom which is an activity described by Thornton/Tucker (1996). Each participant believed in facilitating the development of number sense through a conscious, coordinated effort to build connections and meaning through use of strategies such as promoting internal questioning and encouraging exploration, thinking, and discussion as suggested by Reys (1994). The question/discussion approach during classroom activities was discussed in the beliefs of the teachers and was observed in the classroom. Methods and techniques described and actioned by these teachers were not consistent with methods and techniques used in a traditional classroom. These teachers facilitated the development of number sense concepts such as number meaning, number relationships, the relative magnitude of number through specific situations and activities as outlined in the NCTM standards (1989, 2000) and the Alberta Education (1995) program of studies. It appears that these teachers "were undoubtedly teaching in ways that led their students to develop a sound, intuitive "feel" for numbers" (Sowder, 1994, p.4).

Although beliefs about the meaning of number sense were distinct and different for the teachers in each of these case studies, their beliefs and actions about students' development of number sense and the facilitation of that development were similar. The results of this study suggest that the teacher thinking about number sense development is consistent with ideas presented in the literature and that change from traditional classroom practice is possible.

7.2 Implications of Results

This study recognizes professional development of inservice teachers from an alternative perspective, valuing teachers' beliefs. Professional development activities could center on discussions and shared thinking about how teachers view and develop their work and what their beliefs are about their practices. The results of the study imply some ideas for such professional development activities as follows:

- Teachers involved in professional development might be able to compare how the case study teachers in this research viewed mathematical concepts to how they view mathematical concepts. This comparison might open up conversations about how their view of mathematics is framed. Discussion about or exposure to teachers' beliefs about topics such as number sense might help develop deeper understanding about other mathematical concepts or help lead to a search for additional information.

- Teachers might discuss their practice in reference to how they view the learner. Teachers in the study did not use the same methods; however, they had a responsibility for common outcomes of student understanding. Conversations and discussions of teachers' beliefs about students' development of mathematical concepts, their actions in the classroom and what that might look like could help teachers explore the student's role in the development of mathematical concepts.
- The case study teachers planned and presented activities and situations in their own personal ways. Other teachers might value viewing these plans and activities and be open to trying them. Good suggestions explaining additional situations and activities were also outlined in the literature and were closely related to those used by the teachers in the study. A collection of these activities could be presented, not as a recipe ,but as a way to add new ideas to one's current practice. Ideas, strategies, and techniques from these case study teachers might show different directions for teachers of mathematics to take.
- Teachers involved in professional development should be made aware of the concept of number sense so that they in turn can make their students aware to enhance a deeper understanding of mathematics.

7.3 Limitations of the Study

The opportunity to design a research project in which the emphasis was placed on human participants and their perceptions regarding the questions has the potential to raise a number of concerns with regard to the methodology. A few of these concerns are presented below to acknowledge the possibility for alternative approaches, or variations in strategy for any further research.

The participants were interested in their roles in the research of this topic. There are concerns in terms of the selection of the participants which was based on their being familiar to the researcher. Although an attempt was made to capture the participants' responses and actions independent of the researcher, it is recognized that this may not have been entirely possible. Selection of other participants might confirm the findings or other aspects of beliefs. Consideration might be given to participants including preservice, novice or apprentice teachers; participants teaching in upper elementary or secondary mathematics classrooms; or investigation of teachers of more than one gender.

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