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SOME MIXED BOUNDARY VALUE PROBLEMS IN THE LINEAR THEORY OF ELASTICITY

by

WENHUA HE

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Some Mixed Boundary Value Problems In The Linear Theory Of Elasticity", submitted by Wenhua He in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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Supervisor, R.S. Dhaliwal, Department of Mathematics and Statis tics

S.R. Majumdar, Department of Mathematics and Statistics

J.G. Rokne, Department of Computer Science

M.C. Singh, Department of Mechanical Engineering

External Examiner, M. Singh, Simon Fraser University

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<u>ABSTRACT</u>

The present thesis deals with some torsion and crack problems in the linear theory of elasticity. In chapter 1, we have given a brief summary of the linear theory of the elasticity. Chapter 2 deals with three different problems of Reissner-Sagoci type for composite cylindrical regions. In chapter 3, we have discussed a torsion problem of two bonded nonhomogeneous elastic layers with a penny-shaped flaw at the interface. In chapter 4, we have solved a problem of Griffith crack at the interface of two dissimilar orthotropic elastic layers. In chapter 5, we have solved a penny-shaped interface crack problem between two dissimilar transversely isotropic layers. The numerical values of the physical quantities have been obtained and displayed graphically.

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Nomenclature

c_{ij}	the elastic moduli for anisotropic material.
λ, μ	Lame constants for isotropic elastic material.
$u_{\mathbf{i}}$	the displacement component along <i>i</i> -direction.
$\sigma_{\mathbf{ij}}$	the stress tensor components.
e_{ij}	the strain tensor components.
$\delta_{ m ij}$	the Kronecker delta.
$J_{\nu}(x)$	the Bessel functions of the first kind and of order ν .
$Y_{\nu}(x)$	the Bessel functions of the second kind and of order ν .
$I_{\nu}(\mathbf{x})$	the modified Bessel functions of the first kind and of order ν .
$K_{\nu}(x)$	the modified Bessel functions of the second kind and of order ν .
${\mathcal F}_{\mathrm{s}}$	the Fourier sine transform defined by the equation
	$\mathscr{F}_{\mathrm{s}}[f(z) ; z \rightarrow \xi] = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_{0}^{\infty} f(z) \sin(\xi z) dz$
\mathcal{F}_{c}	the Fourier cosine transform defined by the equation
	$\mathscr{F}_{\mathrm{c}}[f(z) ; z \rightarrow \xi] = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_{0}^{\infty} f(z) \cos(\xi z) dz ,$
H,	the Hankel transform defined by the equation
	$\mathscr{H}_{\nu}[f(\xi) \ ; \ \xi \ ightarrow \ r] \ = \ \int_{0}^{\infty} \xi \ f(\xi) \ J_{\nu}(\xi r) d\xi \ .$
<i>S</i> ₁	the Abel transform of the first kind defined by the equation
	$\mathscr{I}_{1}[f(t) ; t \to r] = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_{0}^{r} f(t)/[r^{2}-t^{2}]^{\frac{1}{2}} dt$
Л ₂	the Abel transform of the second kind defined by the equation
	$\mathscr{I}_{2}[f(t) ; t \to r] = (\frac{2}{\pi})^{\frac{1}{2}} \int_{r}^{\infty} f(t)/[t^{2}-r^{2}]^{\frac{1}{2}} dt$
H(x)	the Heaviside function.
$\Gamma(x)$	the Gamma fuction.
$P_n^{(\alpha,\beta)}(x)$	the Jacobi polynomials.

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CHAPTER 1

BASIC EQUATIONS OF MATHEMATICAL THEORY OF ELASTICITY

The mathematical theory of elasticity has a long history. Hooke, Bernoulli, Navier, Cauchy and Green made a lot of contributions to the developments of the mathematical theory of elasticity. There are many excellent books which give introduction to the basic theory of elasticity. Sokolnikoff [1], Green and Zerna [2], Love [3] and Fung [3a] are good reference books for isotropic elasticity, while Lekhnitskii [4] and Hearman [5] are good reference books for anisotropic elasticity. In this chapter we will give an outline of the linear theory of elasticity and some basic formulae which are needed later.

In the study of the distribution of stresses and deformations in an elastic body, we regard an elastic body as a solid continuous medium. The configuration of a solid body is described by a region of a Euclidean point space whose geometrical points are identified with the position of the material particles of the body.

Let a system of coordinates x_1 , x_2 , x_3 be chosen so that a point P of an elastic body at a certain instant of time has the coordinates $x_i(i=1,2,3)$. Under some physical actions the configuration of the solid body changes. Suppose at a later instant of time the point P moves to P^* with coordinates $y_i(i=1,2,3)$ with respect to a new system of coordinates y_1 , y_2 , y_3 . The change of the

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configuration of the elastic body can be thought of a one-to-one mapping between two configurations. The mapping can be written as

$$y_i = \tilde{y}_i(x_1, x_2, x_3),$$
 (*i*=1,2,3);

with the unique inverse mapping

$$x_i = \tilde{x}_i(y_1, y_2, y_3),$$
 (*i*=1,2,3);

for every point of the elastic body. The change of configuration is assumed to be continuous and smooth. In fact, we are assuming that the functions \tilde{x}_i and \tilde{y}_i (*i*=1,2,3) are twice continuously differentiable.

1.1 Strain and Stress

If the distance between particles of a body is changed under an action, the body is said to be *deformed* and otherwise the body is said to be *undeformed*. To study deformation of an elastic body, let us fix a cartesian coordinate system $O-x_1x_2x_3$. Suppose a point $P(\xi_1,\xi_2,\xi_3)$ is moved to the position $P^*(\xi_1^*,\xi_2^*,\xi_3^*)$ under a physical action. The difference $u_i(P)=\xi_i^*-\xi_i$ (*i*=1,2,3) is called the *displacement* of point P along the x_i direction. The displacement vector $\{u_1,u_2,u_3\}$ varies, in general, from point to point of the body and is twice continuously differentiable.

Now suppose that a neighborhood point of P, say $Q(\eta_1, \eta_2, \eta_3)$ is moved to $Q^*(\eta_1^*, \eta_2^*, \eta_3^*)$ with the displacement components $u_i(Q) = \eta_i^* - \eta_i$, (i=1,2,3). Let the vector joining the points P and Q be \neq with the components τ_1, τ_2 and τ_3 , and \neq^* be the vector joining the points P^* and Q^* . From the Fig.1.1.1 we know that the vector $\delta \neq = \neq^* - \neq$ has the components

$$\delta \tau_{i} = (\eta_{i}^{*} - \xi_{i}^{*}) - (\eta_{i} - \xi_{i}) = (\eta_{i}^{*} - \eta_{i}) - (\xi_{i}^{*} - \xi_{i}) = u_{i}(Q) - u_{i}(P), \quad i = 1, 2, 3.$$



Fig. 1.1.1

If we assume that the displacement components u_i (i=1,2,3), and their partial derivatives are so small that their product can be neglected then we have

$$\begin{split} \delta \tau_{i} &= u_{i,j} \tau_{j} \\ &= \frac{1}{2} [(u_{i,j} + u_{j,i}) + (u_{i,j} - u_{j,i})] \tau_{j} \\ &= (e_{i,j} + w_{i,j}) \tau_{j} , \end{split} \tag{1.1.1}$$

where

$$e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}), \tag{1.1.2}$$

$$w_{ij} = \frac{1}{2} (u_{i,j} - u_{j,i}), \tag{1.1.3}$$

and the comma in the subscript means a partial differentiation, e.g $u_{i,j} = \frac{\partial u_i}{\partial x_i}$.

The symmetric coefficients e_{ij} are called components of *strain* tensor at the point *P*, which characterize a pure deformation; while the skew-symmetric coefficients w_{ij} correspond to a rigid body rotation.

We consider next the transformation of the components of the strain tensor under a rotation of axes of a Cartesian coordinate system. Let the two coordinates be connected by the following linear relations

$$x'_{j} = \beta_{ji} x_{i}$$
, $i, j = 1, 2, 3;$ (1.1.4)

where β_{ij} are the direction cosines of the x'_i -axis with respect to the x_j -axis. The matrix $T=(\beta_{ij})$ is orthogonal and

$$rac{\partial x_i}{\partial x'_j} = \beta_{ji} , \qquad rac{\partial x'_i}{\partial x_j} = \beta_{ij} , \qquad i,j = 1,2,3.$$

Then the relations of the components of displacement vector with respect to the two coordinate systems are given by

$$u'_i = \beta_{ij} u_j, \qquad i,j = 1,2,3.$$
 (1.1.5)

Substituting equations (1.1.4) and (1.1.5) into equations (1.1.2) we obtain

$$e_{ij}^{\prime} = \frac{1}{2} (u_{i,j}^{\prime} + u_{j,i}^{\prime}) = \frac{1}{2} [\frac{\partial}{\partial x_{j}^{\prime}} (\beta_{ik} \ u_{k}) + \frac{\partial}{\partial x_{i}^{\prime}} (\beta_{jm} \ u_{m})]$$

$$= \frac{1}{2} [\beta_{ik} \left(\frac{\partial x_{l}}{\partial x_{j}^{\prime}} \frac{\partial}{\partial x_{l}} u_{k} \right) + \beta_{jm} \left(\frac{\partial x_{l}}{\partial x_{i}^{\prime}} \frac{\partial}{\partial x_{l}} u_{m} \right)]$$

$$= \frac{1}{2} [\beta_{ik} \beta_{jl} \ u_{k,l} + \beta_{jm} \ \beta_{il} \ u_{m,l}]$$

$$= \frac{1}{2} [\beta_{ik} \beta_{jl} \ u_{k,l} + \beta_{jl} \ \beta_{ik} \ u_{l,k}]$$

$$= \frac{1}{2} \beta_{ik} \beta_{jl} \ (u_{k,l} + u_{l,k})$$

$$= \beta_{ik} \beta_{jl} \ e_{kl} , \qquad (1.1.6)$$

this shows that e_{ij} are really components of a second order tensor.

When deformation occurs there is a surface force acting from a portion of an elastic body upon the other portion of the body. Let us consider a surface element ΔS of the body, see Fig.1.1.2, which is located in the interior of the body. Drawing a unit normal vector $\vec{\nu}$ from a point on ΔS , we can distinguish the two sides of ΔS according to the sense of $\vec{\nu}$. Suppose that the portion of the material lying on the positive side of the normal exerts a force ΔF on the other portion of the material. Obviously, the force ΔF is a function of the area Δs of the surface element ΔS and varies when the normal $\vec{\nu}$ changes. As $\Delta s \rightarrow$ 0, we get

$$\vec{T}^{\nu} = \lim_{\Delta s \to 0} \frac{\Delta F}{\Delta s} ,$$

where the subscript ν denotes the direction of the unit normal $\vec{\nu}$ of the surface element ΔS . The vector \vec{T}^{ν} is called "stress vector" or "traction", which represents a force per unit area acting on the surface with normal $\vec{\nu}$.

The projection of \vec{T}^{ν} along the direction of coordinate axis x_i is denoted by T_i^{ν} . When $\vec{\nu}$ is a unit vector in the direction vector x_j , we write $T_i^{\nu} = \sigma_{ji}$, which are called the components of stress tensor. Particularly the



components σ_{11} , σ_{22} and σ_{33} are called "normal stresses" and σ_{ij} $(i \neq j)$ are called "shear stresses". To see that a stress vector in any direction can be written in terms of stress components, let us consider an infinitesimal tetrahedron formed by three surfaces parallel to the coordinate planes and one normal to the unit vector $\vec{\nu}$ with components ν_i (i=1,2,3) (see Fig.1.1.3). Suppose that the lengths of the sides of the tetrahedron along the x_i -direction are dx_i , i=1,2,3; respectively. Denote the area of the surface normal to $\vec{\nu}$ by ds then the area of the surfaces normal to direction x_i is given by $ds_i = \nu_i ds$, i=1,2,3; respectively.

Let \vec{T}^{ν} be the stress vector acting on a surface element with the normal \vec{r} which passes through the point of vertex of the tetrahedron and h be the height of the vertex from the base of the tetrahedron. By assuming the continuity of the stress vector \vec{T}^{ν} , the *i*-component of the force acting on the surface of the tetrahedron which is normal to \vec{r} is $(T_i^{\nu} + \epsilon_i)ds$ with $\epsilon_i \rightarrow 0$, as $h \rightarrow 0$. And the *i*-components of the force acting on the surfaces of the tetrahedron which is normal to *j*-direction are $(-\sigma_{ji} + \epsilon_{ji})ds_j$, j=1,2,3; with $\epsilon_{ji} \rightarrow 0$, as $h \rightarrow 0$. If the body force, per unit mass, is given by $\{F_1, F_2, F_3\}$ at the vertex, then the *i*-component of the body force acting on the tetrahedron is $\frac{\rho}{3}(F_i + \tilde{\epsilon}_i)hds$ with $\tilde{\epsilon}_i \rightarrow 0$ as $h \rightarrow 0$, where ρ is the density of the material. Hence the equilibrium of forces on the tetrahedron yield

$$(T_i^{\nu}+\epsilon_i)ds + (-\sigma_{ji}+\epsilon_{ji})\nu_j ds + \frac{\rho}{3}(F_i+\tilde{\epsilon}_i)hds = 0.$$

Dividing by ds and letting $h \rightarrow 0$ we get

$$T_{i}^{\nu} = \sigma_{ji}^{\nu} \nu_{j} . \tag{1.1.7}$$

Hence, knowing the stress components σ_{ij} at a point we can calculate the stress vector in any direction at the point.





Now let Ω be a portion of an elastic body and S be the boundary of Ω . Each point of S is subjected to a traction \vec{T}^{ν} with $\vec{\nu}$ being the unit normal to S at the point considered. Each mass element of Ω is subjected to a body force (per unit mass) \vec{F} with components F_i , i=1,2,3. For equilibrium, both the resultant force and the resultant moment acting on Ω must vanish, which leads to the following equations

$$\int_{\Omega} \rho F_i dV + \int_S T_i^{\nu} dS = 0, \qquad (1.1.8)$$

$$\int_{\Omega} \rho \gamma_{ijk} F_j x_k dV + \int_{S} \gamma_{ijk} T_j^{\nu} x_k dS = 0, \qquad (1.1.9)$$

where γ_{ijk} is defined as

$$\gamma_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ represents an even permutation of 123} \\ 0 & \text{if any two of } ijk \text{ indices are same.} \\ -1 & \text{if } ijk \text{ represents an odd permutation.} \end{cases}$$

Substituting T_i^{ν} from equation (1.1.7) into equation (1.1.8) and using the divergence theorem we obtain

$$\int_{\Omega} (\rho F_i + \sigma_{ji,j}) dV = 0.$$
 (1.1.10)

The continuity of the integrand in the equation (1.1.10) and arbitrariness of the region Ω lead to the following equilibrium equation

$$\sigma_{ji,j} + \rho F_i = 0. \tag{1.1.11}$$

On the other hand, by divergence theorem we have

$$\begin{split} \int_{S} \gamma_{ijk} \ T_{j}^{\nu} \ x_{k} dS &= \int_{S} \gamma_{ijk} \ \sigma_{lj} \ \nu_{l} \ x_{k} dS \\ &= \int_{\Omega} \ (\gamma_{ijk} \ \sigma_{lj} \ x_{k})_{,l} \ dV \\ &= \int_{\Omega} \ (\gamma_{ijk} \ \sigma_{lj,l} \ x_{k} + \ \gamma_{ijk} \ \sigma_{lj} \ \delta_{kl}) dV, \end{split} \tag{1.1.12}$$

where δ_{kl} is the Kronecker delta.

Using the equation (1.1.11) and observing that $\delta_{kl}\sigma_{lj}=\sigma_{kj}$, equation (1.1.12) can be written as

$$\int_{S} \gamma_{ijk} T^{\nu}_{j} x_k dS = \int_{\Omega} (-\rho \gamma_{ijk} F_j x_k + \gamma_{ijk} \sigma_{kj}) dV.$$
(1.1.13)

Substituting equation(1.1.13) into equation (1.1.9) we get

$$\int_{\Omega} \gamma_{ijk} \sigma_{kj} \, dV = 0. \tag{1.1.14}$$

Again the continuity of the integrand in the equation and the arbitrariness of the region Ω lead to $\gamma_{ijk}\sigma_{kj} = 0$, which gives

$$\sigma_{ij} = \sigma_{ji}$$
, $(i,j = 1,2,3);$ (1.1.15)

hence the stress tensor is symmetric. Considering the equation (1.1.7) we know that the state of stress at any point of the body is determined entirely by six independent components of the symmetric stress tensor.

Let the surface elements ΔS and $\Delta S'$, with unit normals $\vec{\nu}$ and $\vec{\nu}'$, pass through a point *P*. By virtue of equations (1.1.7) and (1.1.15) we can show that the component of the stress vector T^{ν} (acting on ΔS) in the direction of $\vec{\nu}'$ is the same as the component of the stress vector $T^{\nu'}$ (acting on $\Delta S'$) in the direction of $\vec{\nu}$. In fact,

$$T^{\nu'} \cdot \vec{\nu} = T^{\nu'}_{i} \nu_{i} = \sigma_{ji} \nu'_{j} \nu_{i} = (\sigma_{ij} \nu_{i}) \nu'_{j} = T^{\nu} \cdot \vec{\nu}'.$$
(1.1.16)

It will be used to derive formulae of transformation of the components of the stress tensor σ_{ij} to σ'_{ij} when the latter is referred to a new coordinate system x'_i obtained from the old one by a rotation of axes.

Let the two coordinates be connected by the linear relations defined by equations (1.1.4). Then the components of stress tensor with respect to the coordinate system x'_i are given by

$$\sigma'_{km} = T^{\nu'} \cdot \vec{\nu} , \qquad (1.1.17)$$

where $\nu' = \{\beta_{k1}, \beta_{k2}, \beta_{k3}\}$ is the unit vector parallel to x'_k -direction and $\nu = \{\beta_{m1}, \beta_{m2}, \beta_{m3}\}$ is the unit vector parallel to x'_m -direction, referring to the old system. Using equation (1.1.7) we get

$$\sigma'_{km} = \sigma_{ji} \nu'_j \nu_i = \sigma_{ij} \beta_{kj} \beta_{mi} , \qquad (1.1.18)$$

indeed, this shows that σ_{ij} is really a second order tensor.

1.2 Generalized Hooke's Law

If an elastic material is maintained at a fixed temperature and under a state of strain, the generalized Hooke's law states that the components of the stress are linearly related to the components of the strain at the given point. The generalized Hooke's law can be written in the following form

$$\sigma_{ij} = c_{ijkl} e_{kl} , \qquad (i,j,k,l = 1,2,3) \qquad (1.2.1)$$

where coefficients c_{ijkl} are called elastic constants or moduli of the material. If coefficients c_{ijkl} vary from point to point of the material, then the material is called non-homogeneous. If, however, the coefficients c_{ijkl} are independent of the position of the point, the material is called elastically homogeneous. Since e_{ij} and σ_{ij} are symmetric there are, at most, 36 independent elastic coefficients.

Introducing the following notations

$$\begin{split} \sigma_{ii} &= \sigma_i , \qquad e_{ii} = e_i , \qquad (i=1,2,3) \\ \sigma_{23} &= \sigma_4 , \qquad \sigma_{31} = \sigma_5 , \qquad \sigma_{12} = \sigma_6 , \\ 2e_{23} &= e_4 , \qquad 2e_{31} = e_5 , \qquad 2e_{12} = e_6 , \end{split}$$

the equation (1.2.1) becomes

$$\sigma_i = c_{ij} e_j, \qquad (i,j=1,2,\cdots,6).$$
 (1.2.2)

$$\begin{aligned} \sigma_{xx} &= c_{11}e_{xx} + c_{12}e_{yy} + c_{13}e_{zz} + c_{14}\gamma_{yz} + c_{15}\gamma_{xz} + c_{16}\gamma_{xy} ,\\ \sigma_{yy} &= c_{21}e_{xx} + c_{22}e_{yy} + c_{23}e_{zz} + c_{24}\gamma_{yz} + c_{25}\gamma_{xz} + c_{26}\gamma_{xy} ,\\ \sigma_{zz} &= c_{31}e_{xx} + c_{32}e_{yy} + c_{33}e_{zz} + c_{34}\gamma_{yz} + c_{35}\gamma_{xz} + c_{36}\gamma_{xy} ,\\ \sigma_{yz} &= c_{41}e_{xx} + c_{42}e_{yy} + c_{43}e_{zz} + c_{44}\gamma_{yz} + c_{45}\gamma_{xz} + c_{46}\gamma_{xy} ,\\ \sigma_{xz} &= c_{51}e_{xx} + c_{52}e_{yy} + c_{53}e_{zz} + c_{54}\gamma_{yz} + c_{55}\gamma_{xz} + c_{56}\gamma_{xy} ,\\ \sigma_{xy} &= c_{61}e_{xx} + c_{62}e_{yy} + c_{63}e_{zz} + c_{64}\gamma_{yz} + c_{65}\gamma_{xz} + c_{66}\gamma_{xy} , (1.2.3) \end{aligned}$$

if we let $x_1=x$, $x_2=y$ and $x_3=z$ and $\gamma_{xy} = 2 e_{xy}$, $\gamma_{xz} = 2 e_{xz}$, $\gamma_{yz} = 2 e_{yz}$.

When an elastic body is under deformation, there is energy stored in the body. By the assumption that the deformation occur isothermally, we can assume that there is a strain energy density function W which is a single valued function of e_1, e_2, \dots, e_6 . If a volume element in a state of stress is subjected to a virtual strain δe_i , then the stress components σ_i yield the work $-\sigma_i \delta e_i$. Hence as e_i gets an increment δe_i , $W(e_1, e_2, \dots, e_6)$ gets an increment

$$\delta W = \sigma_i \, \delta e_i \, . \tag{1.2.4}$$

On the other hand

$$\delta W = \frac{\partial W}{\partial e_{i}} \, \delta e_{i} \, . \tag{1.2.5}$$

Comparing (1.2.4) and (1.2.5) we have

$$\sigma_{i} = \frac{\partial W}{\partial e_{i}} . \tag{1.2.6}$$

Since

$$\frac{\partial^2 W}{\partial e_{\rm i} \partial e_{\rm j}} = \frac{\partial^2 W}{\partial e_{\rm j} \partial e_{\rm i}} ,$$

we obtain

$$rac{\partial \sigma_{\mathbf{i}}}{\partial e_{\mathbf{j}}} = rac{\partial \sigma_{\mathbf{j}}}{\partial e_{\mathbf{i}}}$$
 ,

from equation (1.2.6), hence $c_{ij} = c_{ji}$ in equations (1.2.2). In other words, among the 36 coefficients c_{ij} 's only 21 are independent.

If an elastic body is symmetric in a certain direction, the number of independent coefficients c_{ij} can be further reduced. First of all, let us consider a material which is elastically symmetric with respect to the x_1x_2 -plane. The symmetry means that the c_{ij} will remain the same under the transformation

$$x_1=\tilde{x}_1,$$
 $x_2=\tilde{x}_2,$ $x_3=-\tilde{x}_3.$

By the transformation of coordinates we get

$$\tilde{\sigma}_{i} = \sigma_{i}, \quad \tilde{e}_{i} = e_{i}, \quad (i = 1, 2, 3, 6)$$

 $\tilde{\sigma}_{4} = -\sigma_{4}, \quad \tilde{e}_{4} = -e_{4}, \quad \tilde{\sigma}_{5} = -\sigma_{5}, \quad \tilde{e}_{5} = -e_{5}.$
(1.2.7)

For i=1, the equation (1.2.2) yields

$$\tilde{\sigma}_{1} = c_{11}\tilde{e}_{1} + c_{12}\tilde{e}_{2} + c_{13}\tilde{e}_{3} + c_{14}\tilde{e}_{4} + c_{15}\tilde{e}_{5} + c_{16}\tilde{e}_{6} ,$$

$$\sigma_{1} = c_{11}e_{1} + c_{12}e_{2} + c_{13}e_{3} + c_{14}e_{4} + c_{15}e_{5} + c_{16}e_{6} . \qquad (1.2.8)$$

Substituting Equations (1.2.7) into the first equation of (1.2.8) and comparing with the second one we get

$$c_{14} = c_{15} = 0$$

Similarly by considering the equations for $\tilde{\sigma}_2, \dots, \tilde{\sigma}_6$ we obtain

$$c_{24} = c_{25} = c_{34} = c_{35} = c_{64} = 0$$
,
 $c_{41} = c_{42} = c_{43} = c_{46} = c_{51} = c_{52} = c_{53} = c_{56} = 0$.

Therefore the matrix of the coefficients of equation (1.2.2) for a material with the elastic symmetry with respect to x_1x_2 -plane can be written as

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{12} & c_{22} & c_{23} & 0 & 0 & c_{26} \\ c_{13} & c_{23} & c_{33} & 0 & 0 & c_{36} \\ 0 & 0 & 0 & c_{44} & c_{45} & 0 \\ 0 & 0 & 0 & c_{45} & c_{55} & 0 \\ c_{16} & c_{26} & c_{36} & 0 & 0 & c_{66} \end{bmatrix}.$$
 (1.2.9)

Materials like wood, for example, which have three mutually orthogonal planes of elastic symmetry are called to be <u>orthotropic</u>. In the study of orthotropic materials it is convenient for us to choose such axes of the coordinate system so that the coordinate planes coincide with the planes of the elastic symmetry. In such a case, besides the symmetry with respect to the x_1x_2 -plane, expressed by matrix (1.2.9), the coefficients c_{ij} must also be invariant under the transformation of coordinates defined by

 $x_1 = -\tilde{x}_1, \qquad x_2 = \tilde{x}_2, \qquad x_3 = \tilde{x}_3.$

Using the same method of as we used above, we find that more coefficients in

equations (1.2.2) vanish and the matrix of the c_{ij} takes the following form

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix},$$
(1.2.10)

there are only 9 independent coefficients.

In the case of an <u>isotropic</u> medium, whose elastic properties are independent of the orientation of the coordinate axes, the coefficients c_{ij} must keep the same when we introduce a new Cartesian coordinate system $O-\tilde{x}_1\tilde{x}_2\tilde{x}_3$ by rotating the $O-x_1x_2x_3$ through a right angle about the x_1 -axis. Hence we get

 $c_{12} = c_{13}$, $c_{33} = c_{22}$, $c_{66} = c_{55}$.

Similarly, by rotating the axes through a right angle about the x_3 -axis we get

 $c_{22} = c_{11}$, $c_{13} = c_{23}$, $c_{55} = c_{44}$.

Finally, let us consider the new coordinate system by rotating the $O-x_1x_2x_3$ through an angle of $\pi/4$ about the x_3 -axis, in this way we get

$$\tilde{\sigma}_6 = -\frac{1}{2} \sigma_1 + \frac{1}{2} \sigma_2 , \qquad \tilde{e}_6 = -e_1 + e_2 . \qquad (1.2.11)$$

From matrix (1.2.7) we have

$$\sigma_6 = c_{44}e_6 , \qquad \tilde{\sigma}_6 = c_{44}\tilde{e}_6 ,$$

and by using equations (1.2.11) we get

$$\frac{1}{2}(-\sigma_1 + \sigma_2) = c_{44}(-e_1 + e_2) . \qquad (1.2.12)$$

Considering matrix (1.2.10) again and observing that $c_{11}=c_{22}$, $c_{12}=c_{13}=c_{23}$ in this case, we have

 $\sigma_1 = c_{11}e_1 + c_{12}e_2 + c_{13}e_3 ,$ $\sigma_2 = c_{12}e_1 + c_{11}e_2 + c_{13}e_3 ,$

then

$$\frac{1}{2}(-\sigma_1 + \sigma_2) = \frac{1}{2}(c_{11} - c_{12})(e_2 - e_1). \qquad (1.2.13)$$

Equations (1.2.12) and (1.2.13) lead to

$$c_{44} = \frac{1}{2}(c_{11} - c_{12}).$$

Hence for an isotropic material the elastic coefficients matrix can be reduced to

$$\begin{bmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{12} & c_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(c_{11}-c_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(c_{11}-c_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(c_{11}-c_{12}) \end{bmatrix}, \quad (1.2.14)$$

there are only two independent coefficients c_{11} and c_{12} . For isotropic materials, we traditionally use λ (= c_{12}) and μ (= $\frac{1}{2}(c_{11}-c_{12})$) as two independent coefficients which are called Lame constants. In such a case the Hooke's law can be stated as following

$$\sigma_{ii} = \lambda \ \delta_{ii} \ \vartheta + 2 \ \mu \ e_{ii} \ , \tag{1.2.15}$$

where δ_{ij} is Kronecker delta and $\vartheta = e_{11} + e_{22} + e_{33}$.

A body is said to be in the state of *plane deformation*, or *plane strain*, parallel the x_1x_2 -plane, if the displacement component u_3 vanishes and the components u_1 and u_2 are the functions of the coordinates x_1 and x_2 , but not of x_3 . That is

$$u_i = u_i(x_1, x_2), \quad i=1,2; \quad u_3 = 0.$$
 (1.3.1)

By equation (1.1.2) we find that the components of the strain tensor are

$$e_{13} = e_{23} = e_{33} = 0,$$

$$e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}), \qquad i,j = 1, 2 , \qquad (1.3.2)$$

which do not depend on x_3 .

Particularly, in the <u>plane orthotropic</u> case, which can be thought of a plane strain problem for a three dimensional orthotropic medium, from equations (1.2.2) and (1.3.2) we get the nonvanising components of the stress tensor

$$\sigma_{11} = c_{11}e_{11} + c_{12}e_{22} ,$$

$$\sigma_{22} = c_{21}e_{11} + c_{22}e_{22} ,$$

$$\sigma_{12} = c_{66}\gamma_{12} ,$$
(1.3.3)

and

 $\sigma_{33} = c_{13}e_{11} + c_{23}e_{22} ,$

but from the first two equations of (1.3.3) we know that σ_{33} is entirely determined by σ_{11} and σ_{22} , and is independent of coordinate x_3 . Hence it is clear that the deformations and stresses of an orthotropic plane strain problem are completely determined by e_{ij} and u_i (i,j=1,2).
We consider next the equilibrium equations. First of all, since σ_{ij} do not depend on x_3 we conclude from equation (1.1.11) that the components F_1 and F_2 of the body force must be independent of the coordinate x_3 and $F_3=0$. Hence the equilibrium equations can be written as

$$\sigma_{ij,j} = -\rho F_i(x_1, x_2), \qquad i, j=1,2 \quad . \tag{1.3.4}$$

1.4 Polar Cylindrical Coordinates

Polar cylindrical coordinate system is often introduced in theory of elasticity when the boundary conditions can be simplified by such a frame of reference. It is appropriate to resolve the components of stress and strain in the direction of the coordinates and denote them by corresponding subscripts.

When we have a Cartesian coordinate system O-xyz, the components of displacement vector in the x, y and z directions are denoted by u_x , u_y and u_z respectively. We use e_{xx} , e_{yy} and e_{zz} to denote the normal strain components while e_{xy} , e_{xz} and e_{yz} to denote the shear strain components. Similarly we use σ_{xx} , σ_{yy} and σ_{zz} to denote the normal stress components while σ_{xy} , σ_{xz} and σ_{yz} to denote the shear stress components. Referring to a polar cylindrical coordinate system the components of displacement vector along the directions r, θ and z are denoted by u_r , u_{θ} and u_z respectively. We will use e_{rr} , $e_{\theta\theta}$ and e_{zz} to denote the normal strain components while $e_{r\theta}$, e_{rz} and $e_{\theta z}$ to denote the shear strain components. We also use σ_{rr} , $\sigma_{\theta\theta}$ and σ_{zz} to denote the normal stress components while $\sigma_{r\theta}$, σ_{rz} and $\sigma_{\theta z}$ to denote the shear stress components. The relations between the polar cylindrical coordinates r, θ , z and the Cartesian coordinates x, y, z are

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z.$$
 (1.4.1)

It follows that derivatives with respect to x, y and z in Cartesian equations can be written in terms of derivatives with respect to r, θ and z by using the following relations

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x} \frac{\partial}{\partial \theta} = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta},$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y} \frac{\partial}{\partial \theta} = \sin \theta \frac{\partial}{\partial r} - \frac{\cos \theta}{r} \frac{\partial}{\partial \theta},$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z}.$$
(1.4.2)

To relate the components between two systems let us select a local Cartesian frame of reference $x_1y_1z_1$ at the point (r,θ,z) , with the origin located at the point (r,θ,z) , the x_1 -axis in the direction of increasing r, the y_1 -axis in the direction of increasing θ , and the z_1 -axis parallel to z-axis (Fig. 1.4.1). Then, in conventional notation, $e_{x_1x_1}$, $e_{x_1y_1}$, \cdots , are well defined. By identifying r, θ , z with x_1 , y_1 , z_1 we have

$$\sigma_{rr} = \sigma_{x_1x_1}, \qquad \sigma_{r\theta} = \sigma_{x_1y_1}, \qquad \sigma_{\theta\theta} = \sigma_{y_1y_1}, \cdots,$$
$$e_{rr} = e_{x_1x_1}, \qquad e_{r\theta} = e_{x_1y_1}, \qquad e_{\theta\theta} = e_{y_1y_1}, \cdots,$$

etc. The matrix of the direction cosines of the axes x_1 , y_1 , z_1 relative to x, y, z is

$$(\beta_{ij}) = \begin{bmatrix} \cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1 \end{bmatrix}.$$
(1.4.3)





Hence, from equation(1.1.18) and the matrix (1.4.3) we get

$$\sigma_{xx} = \sigma_{rr} \cos^2 \theta + \sigma_{\theta\theta} \sin^2 \theta - \sigma_{r\theta} \sin 2\theta,$$

$$\sigma_{yy} = \sigma_{rr} \sin^2 \theta + \sigma_{\theta\theta} \cos^2 \theta + \sigma_{r\theta} \sin 2\theta,$$

$$\sigma_{zz} = \sigma_{zz},$$

$$\sigma_{xy} = (\sigma_{rr} - \sigma_{\theta\theta}) \sin \theta \cos \theta + \sigma_{r\theta} (\cos^2 \theta - \sin^2 \theta)$$

$$\sigma_{xz} = \sigma_{rz} \cos \theta - \sigma_{\theta z} \sin \theta,$$

$$\sigma_{yz} = \sigma_{rz} \sin \theta + \sigma_{\theta z} \cos \theta.$$
(1.4.4)

Similarly, from equation (1.1.6) and matrix (1.4.3) we obtain

$$e_{rr} = e_{xx} \cos^{2}\theta + e_{yy} \sin^{2}\theta + e_{xy} \sin 2\theta,$$

$$e_{\theta\theta} = e_{xx} \sin^{2}\theta + e_{yy} \cos^{2}\theta - e_{xy} \sin 2\theta,$$

$$e_{zz} = e_{zz},$$

$$e_{r\theta} = (e_{yy} - \sigma_{xx}) \sin \theta \cos \theta + e_{xy}(\cos^{2}\theta - \sin^{2}\theta)$$

$$e_{rz} = e_{xz} \cos \theta + e_{yz} \sin \theta,$$

$$e_{\theta z} = -e_{xz} \sin \theta + e_{yz} \cos \theta.$$
(1.4.5)

From the Fig. (1.4.1) the relations of displacement components between polar cylindrical coordinates and Cartesian coordinates for a displacement vector are given by

$$u_{\mathbf{x}} = u_{\mathbf{r}} \cos \theta - u_{\theta} \sin \theta,$$

$$u_{\mathbf{y}} = u_{\mathbf{r}} \sin \theta + u_{\theta} \cos \theta,$$

$$u_{\mathbf{z}} = u_{\mathbf{z}}.$$
(1.4.6)

Equations (1.1.2) now become

$$e_{xx} = \frac{\partial u_x}{\partial x} , \qquad e_{yy} = \frac{\partial u_y}{\partial y} , \qquad e_{zz} = \frac{\partial u_z}{\partial z} ,$$

$$e_{xy} = \frac{1}{2} \left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} \right) , \qquad e_{xz} = \frac{1}{2} \left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} \right) ,$$

$$e_{yz} = \frac{1}{2} \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \right) . \qquad (1.4.7)$$

Finally, substituting equations (1.4.7) into equations (1.4.5) and using equation (1.4.6) and equations (1.4.2) we obtain the strain-displacement relations in polar cylindrical coordinates

$$e_{rr} = \frac{\partial u_{r}}{\partial r} , \qquad e_{\theta\theta} = \frac{1}{r} \frac{\partial u_{\theta}}{\partial \theta} + \frac{u_{r}}{r} , \qquad e_{zz} = \frac{\partial u_{z}}{\partial z} ,$$

$$e_{r\theta} = \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_{r}}{\partial \theta} + \frac{\partial u_{\theta}}{\partial r} - \frac{u_{\theta}}{r} \right) ,$$

$$e_{rz} = \frac{1}{2} \left(\frac{\partial u_{r}}{\partial z} + \frac{\partial u_{z}}{\partial r} \right) ,$$

$$e_{\theta z} = \frac{1}{2} \left(\frac{\partial u_{\theta}}{\partial z} + \frac{1}{r} \frac{\partial u_{z}}{\partial \theta} \right) . \qquad (1.4.8)$$

To obtain the equilibrium equations for polar cylindrical coordinates we first resolve the body force per unit volume at the point (r, θ, z) into components F_r , F_{θ} , F_z along the r-, θ - and z-directions, then we have

$$F_{x} = F_{r} \cos \theta - F_{\theta} \sin \theta,$$

$$F_{y} = F_{r} \sin \theta + F_{\theta} \cos \theta,$$

$$F_{z} = F_{z}.$$
(1.4.9)

The equilibrium equations (1.1.11) for Cartesian coordinates x, y, z can be written as

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + \rho F_x = 0 ,$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + \rho F_y = 0 ,$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho F_z = 0 .$$
(1.4.10)

Substituting the equations (1.4.9) and (1.4.4) into the equations (1.4.10) and using the equations (1.4.2) to transform the derivatives, we obtain

$$\begin{pmatrix} \sigma_{\rm rr} \\ \partial r \end{pmatrix}^{\prime} + \frac{1}{r} \frac{\partial \sigma_{\rm r\theta}}{\partial \theta} + \frac{\partial \sigma_{\rm rz}}{\partial z} + \frac{\sigma_{\rm rr} - \sigma_{\theta\theta}}{r} \end{pmatrix} \cos \theta - \left(\frac{\partial \sigma_{\rm r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2}{r} \sigma_{\rm r\theta} \right) \sin \theta$$

 + $\rho F_{\rm r} \cos \theta - \rho F_{\theta} \sin \theta = 0 ,$ (1.4.11)

from the first equation of (1.4.10). The equation (1.4.11) must hold for all values of θ and letting $\theta = 0$ and $\theta = \pi/2$ respectively we get

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \rho F_r = 0 ,$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2}{r} \sigma_{r\theta} + \rho F_{\theta} = 0 , \qquad (1.4.12)$$

since the choice of the x-direction is arbitrary the equations (1.4.12) must be valid for all values of θ . Similarly, from the third equation of (1.4.10) we get

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{2}{r} \sigma_{rz} + \rho F_z = 0 . \qquad (1.4.13)$$

The generalized Hooke's law for the most general case in polar cylindrical coordinate system is stated as follows

$$\begin{aligned} \sigma_{\rm rr} &= c_{11}e_{\rm rr} + c_{12}e_{\theta\theta} + c_{13}e_{zz} + c_{14}\gamma_{\theta z} + c_{15}\gamma_{rz} + c_{16}\gamma_{r\theta} ,\\ \sigma_{\theta\theta} &= c_{21}e_{\rm rr} + c_{22}e_{\theta\theta} + c_{23}e_{zz} + c_{24}\gamma_{\theta z} + c_{25}\gamma_{rz} + c_{26}\gamma_{r\theta} ,\\ \sigma_{zz} &= c_{31}e_{\rm rr} + c_{32}e_{\theta\theta} + c_{33}e_{zz} + c_{34}\gamma_{\theta z} + c_{35}\gamma_{rz} + c_{36}\gamma_{r\theta} ,\\ \sigma_{\theta z} &= c_{41}e_{\rm rr} + c_{42}e_{\theta\theta} + c_{43}e_{zz} + c_{44}\gamma_{\theta z} + c_{45}\gamma_{rz} + c_{46}\gamma_{r\theta} ,\\ \sigma_{rz} &= c_{51}e_{\rm rr} + c_{52}e_{\theta\theta} + c_{53}e_{zz} + c_{54}\gamma_{\theta z} + c_{55}\gamma_{rz} + c_{56}\gamma_{r\theta} ,\\ \sigma_{r\theta} &= c_{61}e_{\rm rr} + c_{62}e_{\theta\theta} + c_{63}e_{zz} + c_{64}\gamma_{\theta z} + c_{65}\gamma_{rz} + c_{66}\gamma_{r\theta} , \end{aligned}$$
(1.4.14)

where $c_{ij}=c_{ji}$, and $\gamma_{\theta z} = 2 e_{\theta z}$, $\gamma_{rz} = 2 e_{rz}$, $\gamma_{r\theta} = 2 e_{r\theta}$.

An anisotropic medium is said cylindrical if a certain straight line l, the axis of anisotropy, is associated with the medium such that all directions intersecting the line l at right angle are equivalent; correspondingly, all directions parallel to l-axis which pass through distinct points and all directions orthogonal to the first two directions are equivalent. For such kind of medium, it is more convenient to use the cylindrical polar coordinate system by taking the axis of the anisotropy as the z-axis of the cylindrical polar coordinate polar coordinates system (r, θ, z) .

A material is called transversely isotropic if all directions in the planes, which are orthogonal to the axis of the anisotropy are equivalent, in other words, the isotropy occurs in $r\theta$ -plane when the z-axis coincides with the axis of the anisotropy in cylindrical polar coordinate system.

For a transversely isotropic medium, letting the z-axis coincide with the axis of the anisotropy, and using the same method as we did for Cartesian coordinate system we can reduce the equation (1.4.14) to the following form

$$\begin{aligned}
\sigma_{\rm rr} &= c_{11}e_{\rm rr} + c_{12}e_{\theta\theta} + c_{13}e_{zz} , \\
\sigma_{\theta\theta} &= c_{12}e_{\rm rr} + c_{11}e_{\theta\theta} + c_{13}e_{zz} , \\
\sigma_{zz} &= c_{13}e_{\rm rr} + c_{13}e_{\theta\theta} + c_{33}e_{zz} , \\
\sigma_{\thetaz} &= c_{44}\gamma_{\thetaz} , \\
\sigma_{\rm rz} &= c_{44}\gamma_{\rm rz} , \\
\sigma_{\rm r\theta} &= -\frac{1}{2} (c_{11}-c_{12})\gamma_{\rm r\theta} ,
\end{aligned}$$
(1.4.15)

by transformation of variables according to the symmetry of the medium.

In the case of isotropy, we have $c_{33}=c_{11}$, $c_{13}=c_{12}$ and $c_{44}=\frac{1}{2}(c_{11}-c_{12})$. Again there are only two independent coefficients, i.e. Lame constants $\lambda=c_{12}$ and $\mu=\frac{1}{2}(c_{11}-c_{12})$.

1.5 Penny-shaped crack problem

In Chapter 5, we will consider a penny-shaped crack problem for transversely isotropic medium, in that problem the displacement components are functions of variables r and z only, in fact, we have $u_r=u_r(r,z)$, $u_0=0$ and $u_z=u_z(r,z)$. Using the strain-displacement relations (1.4.8) we find that $e_{r0}=e_{0z}=0$ and

$$e_{\rm rr} = \frac{\partial u_{\rm r}}{\partial r} , \qquad e_{\theta\theta} = \frac{u_{\rm r}}{r} , \qquad e_{\rm zz} = \frac{\partial u_{\rm z}}{\partial z} ,$$
$$e_{\rm rz} = \frac{1}{2} \left(\frac{\partial u_{\rm r}}{\partial z} + \frac{\partial u_{\rm z}}{\partial r} \right) . \qquad (1.5.1)$$

Then for a homogeneous, transversely isotropic medium we get the following stress-displacement relations

$$\sigma_{\rm rr} = c_{11} \frac{\partial u_{\rm r}}{\partial r} + c_{12} \frac{u_{\rm r}}{r} + c_{13} \frac{\partial u_{\rm z}}{\partial z},$$

$$\sigma_{\theta\theta} = c_{12} \frac{\partial u_{\rm r}}{\partial r} + c_{11} \frac{u_{\rm r}}{r} + c_{13} \frac{\partial u_{\rm z}}{\partial z},$$

$$\sigma_{\rm zz} = c_{13} \left[\frac{\partial u_{\rm r}}{\partial r} + \frac{u_{\rm r}}{r} \right] + c_{33} \frac{\partial u_{\rm z}}{\partial z},$$

$$\sigma_{\rm rz} = c_{44} \left[\frac{\partial u_{\rm r}}{\partial z} + \frac{\partial u_{\rm z}}{\partial r} \right], \qquad (1.5.2)$$

from equations (1.4.15). Substituting equations (1.5.2) into equations (1.4.12) and (1.4.13) we obtain the following equilibrium equations

$$c_{11}\left[\frac{\partial^2 u_{r}}{\partial r^2} + \frac{1}{r} \frac{\partial u_{r}}{\partial r} - \frac{u_{r}}{r^2}\right] + c_{44}\frac{\partial^2 u_{r}}{\partial z^2} + (c_{13} + c_{44}) \frac{\partial^2 u_{z}}{\partial r \partial z} = 0 ,$$

$$c_{44}\left[\frac{\partial^2 u_{z}}{\partial r^2} + \frac{1}{r} \frac{\partial u_{z}}{\partial r}\right] + c_{33}\frac{\partial^2 u_{z}}{\partial z^2} + (c_{13} + c_{44}) \frac{\partial}{\partial z}\left[\frac{\partial u_{r}}{\partial r} + \frac{u_{r}}{r}\right] = 0 .$$
(1.5.3)

1.6 Torsion problem

For an axisymmetric torsion problem the displacement components are given by $u_r=0$, $u_z=0$ and $u_{\theta}=u_{\theta}(r,z)$, which depends on r and z only. Hence from equations (1.4.8) we find that the non-zero strain components are the following

$$e_{\theta\theta} = \frac{1}{r} \frac{\partial u_{\theta}}{\partial \theta} ,$$

$$e_{r\theta} = \frac{1}{2} \left(\frac{\partial u_{\theta}}{\partial r} - \frac{u_{\theta}}{r} \right) ,$$

$$e_{\theta z} = \frac{1}{2} \frac{\partial u_{\theta}}{\partial z} .$$
(1.6.1)

They depend only on r and z. On the other hand, we know that stress components $\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{zz} = \sigma_{rz} = 0$ and the non-zero stress components

$$\sigma_{\theta^{z}} = c_{44} \gamma_{\theta^{z}} ,$$

$$\sigma_{r_{\theta}} = \frac{1}{2} (c_{11} - c_{12}) \gamma_{r_{\theta}} .$$
 (1.6.2)

For a non-homogeneous and isotropic medium, in terms of displacement components, equations (1.6.2) yield

$$\sigma_{\theta z} = \mu \frac{\partial u_{\theta}}{\partial z} ,$$

$$\sigma_{r\theta} = \mu \left(\frac{\partial u_{\theta}}{\partial r} - \frac{u_{\theta}}{r} \right) ,$$
(1.6.3)

where we have used the fact that $c_{44} = \frac{1}{2} (c_{11}-c_{12})$ for isotropic medium and assumed that $\mu = \mu(z) = c_{44}$ depends on z only. Since $\sigma_{\theta z}$ and $\sigma_{r\theta}$ depend on r and z only, equation (1 4.13) and the first of equations (1.4.12) vanish and the second one of equations (1.4.12) becomes

$$\frac{\partial \sigma_{\mathbf{r}\theta}}{\partial r} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2}{r} \sigma_{\mathbf{r}\theta} = 0 , \qquad (1.6.4)$$

in the absence of the body force.

Substituting equations (1.6.3) into equation (1.6.4) we get

$$\frac{\partial^2 u_{\theta}}{\partial r^2} + \frac{1}{r} \frac{\partial u_{\theta}}{\partial z} - \frac{u_{\theta}}{r^2} + \frac{\partial^2 u_{\theta}}{\partial z^2} + \frac{1}{\mu} \frac{\partial u_{\theta}}{\partial z} \frac{\partial \mu}{\partial z} = 0 . \qquad (1.6.5)$$

For a particular case, when μ =constant, i.e. homogeneous, isotropic case, equation (1.6.5) can be reduced to the following

$$\frac{\partial^2 u_{\theta}}{\partial r^2} + \frac{1}{r} \frac{\partial u_{\theta}}{\partial z} - \frac{u_{\theta}}{r^2} + \frac{\partial^2 u_{\theta}}{\partial z^2} = 0 . \qquad (1.6.6)$$

CHAPTER 2

PROBLEMS OF REISSNER-SAGOCI TYPE FOR COMPOSITE CYLINDRICAL REGIONS

2.1 Introduction

In 1937, E.Reissner [6] formulated several problems relating to torsional vibrations of an elastic half-space. He posed but didn't solve the following mixed boundary value problem

$$v(r,0) = a \ r \ e^{iwt}$$
, $0 < r < d$,
 $\sigma_{\Theta r}(r,0) = 0$, $r > d$. (2.1.1)

Later Reissner and Sagoci [7] solved the static version of above problem by using oblate spherical coordinates. The problem posed by Reissner is called the Reissner-Sagoci problem (RS-problem in short) now.

Sneddon [8] solved the RS-problem by a different method, by reducing it to a pair of dual integral equations using the Hankel transforms. In static case these integral equations could be solved by using various methods developed by Titchmarch [9], Busbridge [10] and Harding and Sneddon [11]. Bycroft [12] gave an approximate treatment to the dynamic RS-problem. Ufliand [13] set up the dual integral equations for the RS-problem for a circular disc on an elastic layer and reduced them to the solution of a Fredholm integral equation of the second kind. Sneddon [14] returned to the RS-problem for a half space and obtained the solution by using his own elementary solution of dual integral equations. He also solved the RS-problem of determining the distribution of stress in a long circular cylinder of homogeneous isotropic material under the condition that the curved surface was fixed. Sneddon, Srivastava and Mathur [15] obtained a solution of the problem for a finite cylinder when the curved surface was stress free and the length of the cylinder was long compared with its radius.

Freeman and Keer [16] investigated a torsion problem of an elastic cylinder, which is attached to an elastic half space. The problem was reduced to the solution of dual integral equations and Dini-series. Rukhovets and Ufliand [17] presented a solution of RS-problem for an elastic half space with a circular inclusion. Gladwell [18] solved the RS-problem for an elastic layer of finite thickness, when the lower face is either stress free or rigidly clamped. Keer and Freeman [19] later extended their previous analysis to a finite elastic cylinder which is partially bonded to a semi-infinite elastic cylinder of the same radius which is embedded in an elastic half-space.

Luco [20] solved the problem of a rigid rod embedded in an elastic layer, the whole being perfectly bonded to a half-space of different material. Singh and Dhaliwal [21] investigated the RS-problem for an elastic layer under torsion by a pair of circular discs on opposite faces. And Dhaliwal, Singh and Sneddon [22] obtained a solution of the RS-problem for a semi-infinite elastic cylinder embedded in an elastic half-space. Low [23] investigated a RS-problem of an elastic half-space with a penny-shaped flaw in the form of an inclusion or a crack. Chebakov [24] considered a RS-problem for a finite cylinder with a torque applied to a rigid disc in the middle of the top face while the curved surface and bottom face were fixed. Dhaliwal, Singh and Rokne [25] solved a torsion problem for a hemisphere embedded in an elastic half-space. Gladwell and Lemczyk [26] solved the static RS-problem for a finite cylinder.

The RS-problem in which a torque is applied over an annulus has also been considered by some researchers. In 1966 Boradachev and Boradacheva [27] investigated this problem by Hankel transforms and reduced it to the solution of triple integral equations. Arutinunian and Bobloian [28] investigated the problem in which a torque is applied on a circle r < b on the surface of a half-space which has an inclusion occupying the cylinder r < a (b < a). Shibuya et al.[29] considered the problem of an elastic layer under torsion by a pair of identical facing annular discs. Dhaliwal and Singh [30] investigated a problem of torsion, by an annular die, of an elastic layer bonded to an elastic half-space, the problem was reduced to the solution of a system of four Fredholm integral equations. Dhaliwal, Singh and Vrbik [31] considered the problem of a half-space with a cylindrical inclusion which was twisted by an annular die. Hasegawa [32] obtained an essential solution for a finite cylinder under torsion by a pair of identical annular stamps to its ends by using Green's function method.

The study of the static RS-problem for non-homogeneous material started in 1960's. In 1967 Protsenko [33] considered the torsion of a half-space with a shear modulus $\mu(z) = m z^{\alpha}$, and later he [34] considered the half-space problem with a torque applied over an annular area. Kassir [35] solved the RS-problem for the half-space and semi-infinite cylinder by assuming the shear modulus of the material in the form of $\mu(z) = m z^{\lambda}$, and reduced it to the solution of a pair of dual integral equations. Kolybikhim [36] solved the above problem by assuming the shear modulus in the form of $\mu(r,z)=\mu_0 r^k z^{\alpha}$. Chuaprasert and Kassir [37] considered a half-space and a

semi-infinite cylinder RS-problem by assuming $\mu(z)=\mu_0(1+z/c)^a$. George [38] assumed $\mu(r)=\mu_0e^{-ar}$ for a semi-infinite cylinder RS-problem. In 1979, Dhaliwal and Singh [39] analyzed the RS-problem for an elastic layer with shear modulus $\mu(z)=\mu_1(z+b)^{\beta_1}$, which was bonded to an elastic half-space with shear modulus $\mu(z)=\mu_2(z+b)^{\beta_2}$. Selvadurai, Singh and Vrbik [40] considered the RS-problem for half-space by assuming shear modulus $\mu(z)=G_1+G_2e^{-\beta z}$. Dhaliwal [41] solved the RS-problem for a more general form of shear modulus $\mu(z)=\mu_0+\epsilon\mu_1(z)+\epsilon^2\mu_2(z)+\cdots$, where μ_0 and $\epsilon<<1$ are positive real constants while $\mu_i(z)$ are differentiable functions of z. Dhaliwal and Chehil [42] solved the RS-problem of non-homogeneous layer bonded to another non-homogeneous elastic layer with the shear modulus as $\mu_i=\mu_i(a_i+z)^{\alpha_i}$, i=1,2 for the two materials.

In this chapter we will consider the Reissner-Sagoci type problems for finite composite elastic cylinder (section 2.2), finite elastic cylinder embedded in an elastic layer (section 2.3) and semi-infinite composite elastic cylinder (section 2.4). The materials considered in this chapter are assumed to be elastic, homogeneous and isotropic.

By the use of integral transforms and the theory of dual integral equations, the problems are reduced to the solution of a Fredholm integral equation of the second kind. Numerical solution of the integral equation is obtained and the numerical values of the torque required to produce the given rotation are displayed graphically.

As discussed in Chapter 1, the displacement field of the medium under torsion considered in this chapter is given by $u_r=u_z=0$ and $u_{\theta}=u_{\theta}(r,z)$, depending on r and z only, where (r, θ, z) is the cylindrical polar coordinate system. And the corresponding non-zero stress components of the stress tensor are given by equations (1.6.3). Consequently the equation of equilibrium is given by equation (1.6.5) when there are no body forces. To simplify, let us denote $u_{\rm r}$, u_{θ} , $u_{\rm z}$ by u, v, w respectively, then we have the following basic equations for the problems under consideration in this chapter:

$$u = 0 , \quad v = v(r,z) , \quad w = 0 ,$$

$$\sigma_{\theta z}(r,z) = \mu \frac{\partial v}{\partial z} , \quad \sigma_{r\theta}(r,z) = \mu \left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) ,$$

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2} = 0 .$$
(2.1.2)

Using the method of separation of variables, it is easy to show that the following are basic solutions of the last equation in (2.1.2) for v(r,z):

(I)
$$J_1(\xi r) exp(\pm \xi z)$$
, (II) $Y_1(\xi r) exp(\pm \xi z)$,
(III) $I_1(\xi r) cos(\xi z)$ or $I_1(\xi r) sin(\xi z)$,
(IV) $K_1(\xi r) cos(\xi z)$ or $K_1(\xi r) sin(\xi z)$,
(V) rz , r , $1/r$, z/r ; (2.1.3)

where J_{ν} , Y_{ν} are Bessel functions of the first and second kind and of order ν , and I_{ν} , K_{ν} are modified Bessel functions of the first and second kind and of order ν respectively [94].

In the following three sections we will use three different combinations of the basic solutions stated above such that they will satisfy the boundary conditions of each of the three problems.

2.2.1 The statement of the problem

In this section we consider the torsion of a finite elastic cylinder which is embedded in a finite elastic cylindrical shell with different shear modulus.

We assume that a finite elastic cylinder of radius d, height b and shear modulus μ_1 is embedded in a finite elastic cylindrical shell of outer radius a, height b, and shear modulus μ_2 as shown in Fig.2.2.1. It is also assumed that the inner cylinder is perfectly bonded to the surrounding cylindrical shell and that a torque is applied to the inner cylinder, through a rigid disc of radius c < d, which is bonded to its flat surface. It is assumed that the bottom flat surface of the composite cylinder is rigidly fixed and the curved outer surface of the composite cylinder is stress-free. In terms of cylindrical polar coordinates (r, θ, z) , displacement field, the corresponding non-zero stress components and the equilibrium equation are given by (2.1.2). The basic solutions of the equilibrium equation are given by (2.1.3).

We further assume that the rigid disc bonded to the inner cylinder is turned through a small angle ϵ . We therefore consider the problem of determining the stress and displacement field in the composite cylinder with the following boundary and continuity conditions:





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where v, $\sigma_{\theta z}$, and $\sigma_{r\theta}$ are the non-zero displacement and stress components for the inner cylinder while \hat{v} , $\hat{\sigma}_{\theta z}$, and $\hat{\sigma}_{r\theta}$ are their counterparts for the surrounding medium.

2.2.2. Derivation of the dual integral equations

Now we introduce a combination of the basic solutions given in section 2.1 for v(r,z) and for $\hat{v}(r,z)$, by means of these combinations we are able to reduce the problem of solving the mixed boundary value problem stated in section 2.2.1 to that of solving a pair of dual integral equations.

For the inner cylinder , we assume that

$$v(r,z) = a_0 r(b-z) + \mathscr{H}_{I}[\xi^{-1}A(\xi)sinh[\xi(b-z)] ; \xi \to r] + \sum_{n=1}^{\infty} \xi_n^{-1}B_n cos(\xi_n z)I_1(\xi_n r) , \qquad (2.2.9)$$

where \mathcal{H}_{p} , the Hankel operator, is defined by the equation

$$\mathscr{H}_{\nu}[f(\xi) ; \xi \rightarrow r] = \int_0^\infty \xi f(\xi) J_{\nu}(\xi r) d\xi$$
.

For the surrounding medium we assume that

$$\hat{v}(r,z) = \sum_{n=1}^{\infty} \xi_n^{-1} C_n \cos(\xi_n z) I_1(\xi_n r) + \sum_{n=1}^{\infty} \xi_n^{-1} D_n \cos(\xi_n z) K_1(\xi_n r) , \qquad (2.2.10)$$

where

$$a_0 = \epsilon/b$$
,

while A, B_n , C_n , D_n and ξ_n are arbitrary constants to be determined later by using the boundary and continuity conditions.

Now from equations (2.1.2) we have

$$\sigma_{r\theta}(r,z) = -\mu_1 \mathscr{H}_2[A(\xi)sinh[\xi(b-z)]; \xi \to r] + \mu_1 \sum_{n=1}^{\infty} B_n cos(\xi_n z) I_2(\xi_n r), \qquad (2.2.11)$$

$$\hat{\sigma}_{r\theta}(r,z) = \mu_2 \sum_{n=1}^{\infty} C_n \cos(\xi_n z) I_2(\xi_n r) - \mu_2 \sum_{n=1}^{\infty} D_n \cos(\xi_n z) K_2(\xi_n r), \qquad (2.2.12)$$

$$\sigma_{\theta z}(r,z) = -\mu_1 \mathscr{H}[A(\xi) \cosh[\xi(b-z)] ; \xi \to r] - \mu_1 \sum_{n=1}^{\infty} B_n \sin(\xi_n z) I_1(\xi_n r) - a_0 \mu_1 r ,$$
(2.2.13)

$$\hat{\sigma}_{\theta z}(r,z) = -\mu_2 \sum_{n=1}^{\infty} C_n \sin(\xi_n z) I_1(\xi_n r) - \mu_2 \sum_{n=1}^{\infty} D_n \sin(\xi_n z) K_1(\xi_n r).$$
(2.2.14)

The conditions (2.2.1) and (2.2.2) will be satisfied if we take

$$\cos(\xi_n b) = 0 ,$$

which gives

$$\xi_n = (2n-1)\pi/2b$$
, $n = 1,2,3,....$ (2.2.15)

The condition (2.2.6) yields

$$\sum_{n=1}^{\infty} [C_n I_2(\xi_n a) - D_n K_2(\xi_n a)] \cos(\xi_n z) = 0 , \qquad (2.2.16)$$

.

and the conditions (2.2.7) and (2.2.8) give

$$a_{0} d(b-z) + \mathscr{H}_{I}[\xi^{-1}A(\xi)sinh[\xi(b-z)] ; \xi \to d]$$

= $\sum_{n=1}^{\infty} \xi_{n}^{-1}[C_{n}I_{I}(\xi_{n}d) - B_{n}I_{I}(\xi_{n}d) + D_{n}K_{I}(\xi_{n}d)]cos(\xi_{n}z) , \qquad (2.2.17)$

 $\quad \text{and} \quad$

.

$$\mathcal{H}_{2}[A(\xi)sinh[\xi(b-z)] ; \xi \rightarrow d]$$

$$= \sum_{n=1}^{\infty} [B_{n}I_{2}(\xi_{n}d) + \bar{\mu}D_{n}K_{2}(\xi_{n}d) - \bar{\mu}C_{n}I_{2}(\xi_{n}d)]cos(\xi_{n}z), \qquad (2.2.18)$$

where

٠

$$\bar{\mu} = \mu_2/\mu_1 \; .$$

Since $\{cos(\xi_n z)\}$ are orthogonal over the interval (0,b) and

$$\int_{0}^{b} sinh[\xi(b-z)]cos(\xi_{n}z)dz = \frac{\xi cosh(\xi b)}{(\xi^{2} + \xi_{n}^{2})} , \qquad (2.2.19)$$

$$\int_{0}^{b} (b-z)\cos(\xi_{n}z)dz = \frac{1}{\xi_{n}^{2}} , \qquad (2.2.20)$$

from equations (2.2.16) to (2.2.18) we obtain

$$C_n I_2(\xi_n a) = D_n K_2(\xi_n a)$$
, (2.2.21)

$$-B_{n}I_{1}(\xi_{n}d)+C_{n}I_{1}(\xi_{n}d)+D_{n}K_{1}(\xi_{m}d)=G_{1}(n) , \qquad (2.2.22)$$

$$B_{n}I_{2}(\xi_{n}d) - \bar{\mu}C_{n}I_{2}(\xi_{n}d) + \bar{\mu}D_{n}K_{2}(\xi_{n}d) = G_{2}(n) , \qquad (2.2.23)$$

where

.

$$G_{1}(n) = \frac{2\xi_{n}}{b} \left\{ \int_{0}^{\infty} \frac{\xi A(\xi) \cosh(\xi b) J_{1}(\xi d)}{\xi^{2} + \xi_{n}^{2}} d\xi + \frac{a_{0} d}{\xi_{n}^{2}} \right\},$$

$$G_{2}(n) = \frac{2}{b} \int_{0}^{\infty} \frac{\xi^{2} A(\xi) \cosh(\xi b) J_{2}(\xi d)}{\xi^{2} + \xi_{n}^{2}} d\xi . \qquad (2.2.24)$$

Eliminating C_n from equations (2.2.21) , (2.2.22) and (2.2.23) , we obtain

$$-B_n I_1(\xi_n d) + D_n Q_1(n) = G_1(n) , \qquad (2.2.25)$$

$$B_{n}I_{2}(\xi_{n}d) + \bar{\mu}D_{n}Q_{2}(n) = G_{2}(n) , \qquad (2.2.26)$$

hence

$$B_{\rm n} = [\bar{\mu}G_{\rm l}(n)Q_{\rm l}(n) - G_{\rm l}(n)Q_{\rm l}(n)]/\Delta(n) , \qquad (2.2.27)$$

where

$$\Delta(n) = -\overline{\mu}I_{1}(\xi_{n}d)Q_{2}(n) - I_{2}(\xi_{n}d)Q_{1}(n) ,$$

$$Q_{1}(n) = K_{1}(\xi_{n}d) + \frac{K_{2}(\xi_{n}a)}{I_{2}(\xi_{n}a)}I_{1}(\xi_{n}d) ,$$

$$Q_{2}(n) = K_{2}(\xi_{n}d) - \frac{K_{2}(\xi_{n}a)}{I_{2}(\xi_{n}a)}I_{2}(\xi_{n}d) .$$
(2.2.28)

From equation (2.2.14) we find that the condition (2.2.5) is identically satisfied and that the conditions (2.2.3) and (2.2.4) will be satisfied if $A(\xi)$ is the solution of the following dual integral equations

$$\mathcal{H}_{I}[\xi^{-1}A(\xi)\sinh(\xi b);\xi \rightarrow r] + \sum_{n=1}^{\infty} \xi_{n}^{-1}B_{n}I_{I}(\xi_{n}r) = 0 , \qquad r < c, \qquad (2.2.29)$$
$$\mathcal{H}_{I}[A(\xi)\cosh(\xi b);\xi \rightarrow r] = -a_{0}r , \qquad c < r < d. (2.2.30)$$

2.2.3 Reduction to integral equation of Fredholm type

To reduce the problem of solving the dual integral equations (2.2.29) and (2.2.30) to that of solving an integral equation of Fredhiom type of the second kind, we will make use of an integral representation for $A(\xi)$, which automatically satisfies equation (2.2.30).

It is known that if we take [26]

$$\phi_{0}(t) = -\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \begin{cases} t\left[\left(d^{2}-t^{2}\right)^{\frac{1}{2}}-\left(c^{2}-t^{2}\right)^{\frac{1}{2}}\right] & t < c, \\ t\left(d^{2}-t^{2}\right)^{\frac{1}{2}}, & c < t < d, (2.2.31) \\ 0, & t > d, \end{cases}$$

we have

$$\mathscr{F}_{s}[\phi_{0}(t) ; t \to \xi] = -\int_{c}^{d} r^{2} J_{1}(\xi r) dr , \qquad (2.2.32)$$

$$\mathscr{H}_{1}\{ \mathcal{F}_{s}[\phi_{0}(t) ; t \rightarrow \xi] ; \xi \rightarrow r\} = \begin{cases} 0, & 0 \leq r < c, \\ -r, & c < r < d, (2.2.33) \\ 0, & r > d, \end{cases}$$

where \mathcal{F}_s is Fourier sine transform defined by

$$\mathscr{F}_{\mathbf{s}}[f(z) ; z \to \xi] = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_{0}^{\infty} f(z) \sin(\xi z) dz ,$$

It is easy to show [43] that if we take

$$A(\xi) = \frac{a_0}{\cosh(\xi b)} \mathscr{F}_{s}[\phi(t) + \phi_0(t) ; t \to \xi] , \qquad (2.2.34)$$

where $\phi(t)$ is a new unknown fuction defined in $(0,\infty)$ such that $\phi(t)=0$ for t>c, then equation (2.2.30) will be satisfied automatically.

Substituting from equation (2.2.34) into equations (2.2.24) and using the integrals [44]

$$\int_{0}^{\infty} \frac{\xi^{2} \sin(\xi t) J_{2}(\xi d) d\xi}{\xi^{2} + \xi_{n}^{2}} = \xi_{n} \sinh(\xi_{n} t) K_{2}(\xi_{n} d) , \qquad t < d , \qquad (2.2.35)$$

$$\int_{0}^{\infty} \frac{\xi \sin(\xi t) J_{1}(\xi d) d\xi}{\xi^{2} + \xi_{n}^{2}} = \sinh(\xi_{n} t) K_{1}(\xi_{n} d) , \qquad t < d , \qquad (2.2.36)$$

we obtain

$$G_1(n) = g_{11}(n) + g_{12}(n) + g_{13}(n) , \qquad G_2(n) = g_{21}(n) + g_{22}(n) , \qquad (2.2.37)$$

where

$$g_{11}(n) = \frac{2a_0\xi_n}{b} (\frac{2}{\pi})^{\frac{1}{2}} \int_0^c \phi(t) \sinh(\xi_n t) K_1(\xi_n d) dt ,$$

$$g_{12}(n) = \frac{2a_0\xi_n}{b} (\frac{2}{\pi})^{\frac{1}{2}} \int_0^d \phi_0(t) \sinh(\xi_n t) K_1(\xi_n d) dt ,$$

$$g_{13}(n) = \frac{2a_0}{b\xi_n} \frac{d}{\xi_n} ,$$

$$g_{21}(n) = \frac{2a_0\xi_n}{b} (\frac{2}{\pi})^{\frac{1}{2}} \int_0^c \phi(t) \sinh(\xi_n t) K_2(\xi_n d) dt ,$$

$$g_{22}(n) = \frac{2a_0\xi_n}{b} (\frac{2}{\pi})^{\frac{1}{2}} \int_0^d \phi_0(t) \sinh(\xi_n t) K_2(\xi_n d) dt .$$
(2.2.38)

Operating on equation (2.2.29) by $x^{-1} \mathscr{N}_1^{-1}[r; x]$ and using the following results [45]:

$$x^{-1} \mathscr{K}_{1}^{-1} \{ r \mathscr{K}_{1}[\xi^{-1}F(\xi) ; \xi \to r] ; r \to x \} = \mathscr{F}_{s}[F(\xi) ; \xi \to x] , \qquad (2.2.39)$$

$$x^{-1} \mathscr{I}_{1}[r \ I_{1}(\xi_{n}r) \ ; \ r \to x] = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} sinh(\xi_{n}x) \ , \qquad (2.2.40)$$

we obtain

$$\mathscr{F}_{s}[A(\xi)sinh(\xi b);\xi \to x] + (\frac{2}{\pi})^{\frac{1}{2}} \sum_{n=1}^{\infty} \xi_{n}^{-1} B_{n} sinh(\xi_{n} x) = 0, \qquad 0 \leq x < c, \quad (2.2.41)$$

.

where \mathcal{M}_{1}^{1} is the inverse of \mathcal{M}_{1} , the Abel operator of the first kind, which is defined by the equation

$$\mathscr{I}_{1}[f(t) ; t \to r] = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_{0}^{r} f(t)[r^{2}-t^{2}]^{-\frac{1}{2}} dt$$

 $\quad \text{and} \quad$

$$\mathscr{I}_{1}^{-1}[f(r) ; t] = \frac{d}{dt} \mathscr{I}_{1}[rf(r) ; t]$$
.

Using expressions (2.2.34), (2.2.27) and (2.2.37), we get

$$\mathcal{F}_{s}[A(\xi)sinh(\xi b);\xi \rightarrow x] = a_{0}\phi(x) - \frac{4a_{0}}{\pi} \int_{0}^{c} \phi(t) dt \int_{0}^{\infty} (1 + e^{2\xi b})^{-1} sin(\xi t) sin(\xi x) d\xi + a_{0}\phi_{0}(x) - \frac{4a_{0}}{\pi} \int_{0}^{d} \phi_{0}(t) dt \int_{0}^{\infty} (1 + e^{2\xi b})^{-1} sin(\xi t) sin(\xi x) d\xi, \qquad (2.2.42)$$

and

$$(\frac{2}{\pi})^{\frac{1}{2}} \sum_{n=1}^{\infty} \xi_n^{-1} B_n sinh(\xi_n x)$$

= $a_0 \int_0^c \phi(t) S(x,t) dt + a_0 \int_0^d \phi_0(t) S(x,t) dt + a_0 R(x)$, (2.2.43)

where

...

$$S(x,t) = \frac{4}{\pi b_{n=1}} \sum_{n=1}^{\infty} \frac{\bar{\mu} K_1(\xi_n d) Q_2(n) - K_2(\xi_n d) Q_1(n)}{\Delta(n)} \sinh(\xi_n t) \sinh(\xi_n x), \qquad (2.2.44)$$

$$R(x) = \frac{2d\bar{\mu}}{b} (\frac{2}{\pi})^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{Q_2(n)sinh(\xi_n x)}{\Delta(n)\xi_n^2} . \qquad (2.2.45)$$

Finally equation (2.2.41) may be written as

$$\phi(x) - \int_0^c \phi(t) [M(x,t) - S(x,t)] dt = -f(x) , \qquad 0 \le x \le c, \quad (2.2.46)$$

where

$$f(x) = \phi_0(x) - \int_0^d \phi_0(t) [M(x,t) - S(x,t)] dt + R(x) , \qquad (2.2.47)$$

$$M(x,t) = \frac{4}{\pi} \int_0^\infty (1+e^{2\xi b})^{-1} \sin(\xi t) \sin(\xi x) d\xi \quad . \tag{2.2.48}$$

For a large n,

$$\frac{\overline{\mu}K_1(\xi_n d)Q_2(n)-K_2(\xi_n d)Q_1(n)}{\Delta(n)}\sinh(\xi_n t)\sinh(\xi_n x) = O(exp[-\xi_n(2d-x-t)]),$$

hence the convergence of the series in equation (2.2.44) is fast .

To obtain the torque T required to produce the prescribed rotation of the rigid disc bonded to the inner cylinder , as given by condition (2.2.3) , we have to calculate the value of σ_{θ_z} at z=0. Now from equations (2.2.13) and (2.2.34) , we get

$$\begin{split} \sigma_{\theta z}(r,0) &= -\mu_1 \mathscr{H}_{\mathbf{i}}[A(\xi)\cosh(\xi b) \ ;\xi \to r] \ - \ a_0\mu_1 r \\ &= -a_0\mu_1 \mathscr{H}_{\mathbf{i}}\{ \ \mathscr{F}_{\mathbf{s}}[\phi(t) \ ;t \to \xi] \ ;\xi \to r\} \ - \ a_0\mu_1 \mathscr{H}_{\mathbf{i}}\{ \ \mathscr{F}_{\mathbf{s}}[\phi_0(t) \ ;t \to \xi] \ ;\xi \to r\} \\ &- \ a_0\mu_1 r \ , \qquad \qquad r < c \ , \qquad (2.2.49) \end{split}$$

and we find that

$$\int_{0}^{c} r^{2} \mathscr{H}_{1} \{ \mathscr{F}_{s}[\phi_{0}(t) ; t \rightarrow \xi] ; \xi \rightarrow r \} dr = 0 , \qquad (2.2.50)$$

$$\mathscr{H}_{1}\{ \mathscr{F}_{\mathbf{s}}[\phi(t); t \to \xi]; \xi \to r\} = -\frac{d}{dr} \mathscr{H}_{0}\{\xi^{-1} \mathscr{F}_{\mathbf{s}}[\phi(t); t \to \xi]; \xi \to r\} .$$
(2.2.51)

The required torque is given by

$$T = -2\pi \int_{0}^{c} r^{2} \sigma_{\theta z}(r,0) dr . \qquad (2.2.52)$$

Using equations (2.2.49), (2.2.50) and (2.2.51) in (2.2.52) we obtain

$$T = \frac{\pi \mu_1 \epsilon c^4}{2b} + 4(2\pi)^{\frac{1}{2}} \frac{\mu_1 \epsilon}{b} \int_0^c t \phi(t) dt . \qquad (2.2.53)$$

It is worth mentioning that the solution for the corresponding semi-infinite composite cylinder problem cannot be derived from the present solution, since the S integral in equation (2.2.46) is divergent as $b\rightarrow\infty$. It may be noted that

$$\int_{0}^{c} \phi_{0}(t) S(x,t) dt$$

$$= \int_{0}^{c} \phi_{0}(t) \frac{4}{\pi b_{n=1}} \sum_{n=1}^{\infty} \frac{\overline{\mu} K_{1}(\xi_{n}d) Q_{2}(n) - K_{2}(\xi_{n}d) Q_{1}(n)}{\Delta(n)} \sinh(\xi_{n}t) \sinh(\xi_{n}x) dt. \quad (2.2.54)$$

Let us consider a particular case, when $\bar{\mu} = 0$, we have

$$\frac{\bar{\mu}K_{1}(\xi_{n}d)Q_{2}(n)-K_{2}(\xi_{n}d)Q_{1}(n)}{\Delta(n)} = \frac{K_{2}(\xi_{n}d)}{I_{2}(\xi_{n}d)}.$$
Since $\xi_{n} = (2n-1)\pi/(2b)$, the ξ_{n} are apart and formally

$$\lim_{b \to \infty} \frac{4}{\pi b_{n=1}^{\infty}} \frac{\bar{\mu}K_{1}(\xi_{n}d)Q_{2}(n)-K_{2}(\xi_{n}d)Q_{1}(n)}{\Delta(n)} \sinh(\xi_{n}t)\sinh(\xi_{n}x)$$

$$= \frac{4}{\pi^{2}} \int_{0}^{\infty} \frac{K_{2}(\xi_{n}d)}{I_{2}(\xi_{n}d)} \sinh(\xi_{n}t) \sinh(\xi_{n}x) d\xi. \qquad (2.2.55)$$

Sneddon [14] noted that the integral in (2.2.55) diverges as $O(\xi^{-2})$. In general, the integral in (2.2.54) is also divergent. Hence the solution for the case $b \rightarrow \infty$ will be presented separately.

2.2.4 Numerical results and conclusions

Numerical values of $\phi(x)$ for $x = (0.0, 0.1 \ 0.2, \dots 1.0)c$ have been calculated from the integral equation (2.2.46) by reducing it to algebraic equations. And then the numerical values of the dimensionless ratio of torque T/T_0 have been calculated from equation (2.2.53), where $T_0=16\mu_1\epsilon c^3/3$ is the torque for the corresponding Reissner-Sagoci problem for the semi-infinite elastic space. The Simpson's rule is used to perform the numerical integrations and the Crout's factorisation method is used to solve the linear algebraic equations. In the numerical results the relative errors are controlled under 0.01.

Numerical values of T/T_0 have been calculated for the following combination of values of

b/c = 0.2(0.1)0.5, 1.0, 2.0, 10.0; d/c = 1.0(0.2)2.0, 3.0(1.0)10.0; a/d = 1.1, 1.5; $\bar{\mu} = 0.5, 2.0$; and these have been displayed in Figs.2.2.2 to Fig.2.2.5. From the figures, we observe that for a fixed radius c of the rigid disc the torque T decreases as the height b of the cylinder increases and the torque increases as the radius d of the inner cylinder increases while the ratio a/d of the radius of outer cylinder to the inner cylinder is kept the same. We also observe that the ratio T/T_0 approaches to 1 when d/c and b/c approach to infinity simultaneously. We notice that the values of T/T_0 have very negligible effect with the change in the values of a/d from 1.1 to 1.5 or with the change in the values of $\bar{\mu}$ from 0.5 to 2.0.





Numerical values of the ratio of the torques T/T_0 against d/c for $\bar{\mu} = 0.5$, a/d=1.1 for various values of b/c = 0.2, 0.3, 0.4, 0.5, 1.0, 2.0, 10.0.

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Fig.2.2.3

Numerical values of the ratio of the torques T/T_0 against d/c for $\bar{\mu}=0.5$, a/d=1.5 for various values of b/c = 0.2, 0.3, 0.4, 0.5, 1.0, 2.0, 10.0.





Numerical values of the ratio of the torques T/T_0 against d/c for $\bar{\mu}=2.0$, a/d=1.1 for various values of b/c = 0.2, 0.3, 0.4, 0.5, 1.0, 2.0, 10.0.





Numerical values of the ratio of the torques T/T_0 against d/c for $\bar{\mu}=2.0$, a/d=1.5 for various values of b/c = 0.2, 0.3, 0.4, 0.5, 1.0, 2.0, 10.0.

2.3 Finite elastic cylinder embedded in an elastic layer

2.3.1 The statement of problem

The problem considered in this section is that of the torsion of a finite elastic cylinder which is embedded in an elastic layer of different shear modulus. We are assuming that an elastic cylinder of radius d and shear modulus μ_1 is embedded in an elastic layer whose shear shear modulis is μ_2 as shown in Fig.2.3.1. It is also assumed that the cylinder is perfectly bonded to the surrounding elastic layer and a torque is applied to the cylinder, through a rigid disc of radius c < d, which is bonded to its top flat surface, and the flat bottom surface of the finite cylinder and the surrounding layer is fixed. In terms of cylindrical polar coordinates (r, θ, z) , displacement field is given by u=w=0 and v=v(r,z), hence we have the basic equations (2.1.2) and the solutions (2.1.3) for v(r,z).

We also assume that the rigid disc bonded to the cylinder is turned through a small angle ϵ and that the height of the cylinder and the surrounding layer is *b*. We, therefore, consider the problem of determining the stress and displacement field in the cylinder and the surrounding layer with the following boundary and continuity conditions:





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$v(r,0) = \epsilon r$,	$0 \leq r < c,$	(2.3.1)
$\sigma_{\theta z}(r,0) = 0 , \qquad \qquad$	$c \leq r < d$,	(2.3.2)
$\hat{\sigma}_{ heta z}(r,0) = 0$,	r > d,	(2.3.3)
v(r,b) = 0 ,	r < d,	(2.3.4)
$\hat{v}(r,b) = 0$,	r > d,	(2.3.5)
$v(d,z) = \hat{v}(d,z)$,	$0 \leq z \leq b,$	(2.3.6)
$\sigma_{\mathrm{r}\theta}(d,z) = \hat{\sigma}_{\mathrm{r}\theta}(d,z)$	$0 \leq z \leq b,$	(2.3.7)
$\hat{v}(r,z) \rightarrow 0$, $\hat{\sigma}_{\theta z}(r,z) \rightarrow 0$, $\hat{\sigma}_{r\theta}(r,z) \rightarrow 0$,	$r \rightarrow \infty$,	(2.3.7a)

where v, $\sigma_{\theta z}$, and $\sigma_{r\theta}$ are the non-zero displacement and stress components in the cylinder while \hat{v} , $\hat{\sigma}_{\theta z}$, and $\hat{\sigma}_{r\theta}$ are their counterparts in the surrounding layer.

2.3.2. Derivation of the dual integral equations

Conditions (2.3.7a) at infinity are identically satisfied if in the combination of the basic solutions assumed for $\hat{v}(r,z)$ in section 2.2.2, we take, for the present problem, $C_n=0$ for all *n*. By letting $C_n=0$ for all *n* we reach the following combinations.

For $0 \leq r < d$, we may assume the following representation for v

$$v(r,z) = a_0 r(b-z) + \mathcal{H}_1[\xi^{-1}A(\xi)sinh[\xi(b-z)]; \xi \to r] + \sum_{n=1}^{\infty} \xi_n^{-1}B_n cos(\xi_n z) I_1(\xi_n r) ,$$
(2.3.8)

and for r > d we may assume that

$$\hat{v}(r,z) = \sum_{n=1}^{\infty} \xi_n^{-1} D_n \cos(\xi_n z) K_1(\xi_n r) , \qquad (2.3.9)$$

where $a_0 = \epsilon/b$, while A, B_n , D_n and ξ_n are to be determined later.

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From equations (2.1.2) we have

$$\sigma_{r\theta}(r,z) = -\mu_1 \mathscr{H}_2[A(\xi) \sinh[\xi(b-z)]; \xi \to r] + \mu_1 \sum_{n=1}^{\infty} B_n \cos(\xi_n z) I_2(\xi_n r), \qquad (2.3.10)$$

$$\hat{\sigma}_{r\theta}(r,z) = -\mu_2 \sum_{n=1}^{\infty} D_n \cos(\xi_n z) K_2(\xi_n r) , \qquad (2.3.11)$$

$$\sigma_{\theta z}(r,z) = -\mu_1 \mathscr{H}[A(\xi) \cosh[\xi(b-z)] ; \xi \to r] -\mu_1 \sum_{n=1}^{\infty} B_n \sin(\xi_n z) I_1(\xi_n r) - a_0 \mu_1 r ,$$
(2.3.12)

$$\hat{\sigma}_{\theta z}(r,z) = -\mu_2 \sum_{n=1}^{\infty} D_n sin(\xi_n z) K_1(\xi_n r) \quad .$$
(2.3.13)

The conditions (2.3.4) and (2.3.5) may be satisfied by taking

$$cos(\xi_n b) = 0$$
,

which gives that

$$\xi_n = (2n-1)\pi/2b$$
, $n = 1, 2, 3, \dots$ (2.3.14)

The condition (2.3.6) yields

$$a_{0}d(b-z) + \mathscr{H}_{I}[\xi^{-1}A(\xi)sinh[\xi(b-z)] ; \xi \rightarrow d]$$

= $\sum_{n=1}^{\infty} \xi_{n}^{-1}[D_{n}K_{I}(\xi_{n}d) - B_{n}I_{I}(\xi_{n}d)]cos(\xi_{n}z) ,$ (2.3.15)

and the condition (2.3.7) yields

$$\mathscr{H}_{2}[A(\xi)sinh[\xi(b-z)];\xi \rightarrow d] = \sum_{n=1}^{\infty} [B_{n}I_{2}(\xi_{n}d) + \bar{\mu}D_{n}K_{2}(\xi_{n}d)]cos(\xi_{n}z), \qquad (2.3.16)$$

where $\bar{\mu} = \mu_2/\mu_1$.

Since $\{cos(\xi_n z)\}_{n=1,2,3}$... are orthogonal over the interval (0,b) and

$$\int_{0}^{b} sinh[\xi(b-z)]cos(\xi_{n}z)dz = \frac{\xi cosh(\xi b)}{(\xi^{2} + \xi^{2})} , \qquad (2.3.17)$$

$$\int_{0}^{b} (b-z) \cos(\xi_{n} z) dz = \frac{1}{\xi_{n}^{2}} , \qquad (2.3.18)$$

.

equations (2.3.15) and (2.3.16) lead to the following equations

$$-B_{n}I_{1}(\xi_{n}d)+D_{n}K_{1}(\xi_{n}d) = \frac{2\xi_{n}}{b} \left\{ \int_{0}^{\infty} \frac{\xi A(\xi)\cosh(\xi b)J_{1}(\xi d)}{\xi^{2}+\xi_{n}^{2}} d\xi + \frac{a_{0}d}{\xi_{n}^{2}} \right\}$$

$$= G_{i}(n)$$
, (2.3.19)

$$B_{n}I_{2}(\xi_{n}d) + \bar{\mu}D_{n}K_{2}(\xi_{n}d) = \frac{2}{b} \int_{0}^{\infty} \frac{\xi^{2}A(\xi)\cosh(\xi b)J_{2}(\xi d)}{\xi^{2} + \xi_{n}^{2}} d\xi = G_{2}(n). \quad (2.3.20)$$

Solving equations (2.3.19) and (2.3.20) for B_n we obtain

$$B_{n} = \left[-G_{2}(n)K_{1}(\xi_{n}d) + \bar{\mu}G_{1}(n)K_{2}(\xi_{n}d)\right]/\Delta(n) , \qquad (2.3.21)$$

where

$$\Delta(n) = -I_2(\xi_n d) K_1(\xi_n d) - \bar{\mu} I_1(\xi_n d) K_2(\xi_n d) . \qquad (2.3.22)$$

From equation (2.3.13) we find that the condition (2.3.3) is identically satisfied and boundary conditions (2.3.1) and (2.3.2) will be satisfied if $A(\xi)$ is the solution of the following dual integral equations

$$\mathscr{H}_{1}[\xi^{-1}A(\xi)\sinh(\xi b);\xi \to r] + \sum_{n=1}^{\infty} \xi_{n}^{-1}B_{n}I_{i}(\xi_{n}r) = 0 , \qquad r < c, \qquad (2.3.23)$$

$$\mathscr{H}[A(\xi)\cosh(\xi b);\xi \rightarrow r] = -a_0 r$$
, $c \leq r < d.$ (2.3.24)

2.3.3 Reduction to integral equation of Fredholm type

Equations (2.3.23) and (2.3.24) have the same appearances as the equations (2.2.29) and (2.2.30) only with different expression of B_n . So we can use the same arguments as we did in section 2.2.3 to reduce the problem of solving the equations (2.3.23) and (2.3.24) to that of solving an integral equation of
Fredholm type of the second kind by means of an integral representation for $A(\xi)$.

Let $\phi_0(t)$ be the function defined in equation (2.2.31) and

$$A(\xi) = \frac{a_0}{\cosh(\xi b)} \mathscr{F}_{s}[\phi(t) + \phi_0(t) ; t \to \xi] , \qquad (2.3.25)$$

where $\phi(t)$ is a new unknown fuction defined in $(0,\infty)$ such that $\phi(t)=0$ for t>c. Defining $G_1(n)$ and $G_2(n)$ by the equations (2.2.37) and g_{11} , g_{12} , g_{13} , g_{21} , g_{22} by equations (2.2.38) we obtain that

$$B_{n} = -\{[g_{21}(n)K_{1}(\xi_{n}d) - \bar{\mu}g_{11}(n)K_{2}(\xi_{n}d)] + [g_{22}(n)K_{1}(\xi_{n}d) - \bar{\mu}g_{12}(n)K_{2}(\xi_{n}d)] - \bar{\mu}g_{13}(n)K_{2}(\xi_{n}d)\}/\Delta(n) . \qquad (2.3.26)$$

Operating on equation (2.3.23) by $x^{-1} \mathscr{I}_{1}^{1}[r ; x]$ we get equations (2.2.41), (2.2.42) and

$$(\frac{2}{\pi})^{\frac{1}{2}} \sum_{n=1}^{\infty} \xi_n^{-1} B_n \sinh(\xi_n x) = a_0 \int_0^c \phi(t) [\sum_{n=1}^{\infty} Q(n) \sinh(\xi_n t) \sinh(\xi_n x)] dt + a_0 \int_0^d \phi_0(t) [\sum_{n=1}^{\infty} Q(n) \sinh(\xi_n t) \sinh(\xi_n x)] dt + a_0 R(x), \qquad (2.3.27)$$

where

$$Q(n) = \frac{4}{\pi b \Delta(n)} (\bar{\mu} - 1) K_1(\xi_n d) K_2(\xi_n d) ,$$

$$R(x) = \frac{2d\bar{\mu}}{b} (\frac{2}{\pi})^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{K_2(\xi_n d) \sinh(\xi_n x)}{\Delta(n) \xi_n^2} .$$
(2.3.28)

Substituting from equations (2.2.42) and (2.3.27) into equation(2.2.41) we find that $\phi(x)$ must satisfy the following integral equation

$$\phi(x) - \int_{0}^{c} \phi(t) M(x,t) dt + \int_{0}^{c} \phi(t) N(x,t) dt = -f(x), \qquad 0 \le x \le c, \quad (2.3.29)$$

where

$$f(x) = \phi_0(x) - \int_0^d \phi_0(t) M(x,t) dt + \int_0^d \phi_0(t) N(x,t) dt + R(x) , \qquad (2.3.30)$$

$$M(x,t) = \frac{4}{\pi} \int_0^\infty (1+e^{2\xi b})^{-1} \sin(\xi t) \sin(\xi x) d\xi , \qquad (2.3.31)$$

$$N(x,t) = \sum_{n=1}^{\infty} Q(n) \sinh(\xi_n t) \sinh(\xi_n x) . \qquad (2.3.32)$$

For a large n,

$$Q(n)sinh(\xi_n t)sinh(\xi_n x) = O(exp[-\xi_n(2d-x-t)]) ,$$

and hence the convergence of the series in equation (2.3.32) is fast.

Following the same procedures as we did in section 2.2.3, we find that the torque T necessary to produce the prescribed rotation of the rigid disc bonded to the cylinder is given by

$$T = \frac{\pi \mu_1 \epsilon c^4}{2b} + 4(2\pi)^{\frac{1}{2}} \frac{\mu_1 \epsilon}{b} \int_0^c t \phi(t) dt . \qquad (2.3.33)$$

It is worth mentioning here that the integral equation (2.3.29) can be derived from (2.2.46) in section 2.2, by letting *a*, the radius of the outer elastic cylindrical shell, tend to infinity. However, when *d*, the radius of the cylinder, tends to infinity, the expression for displacement v in (2.2.9) is unacceptable.

2.3.4 Particular cases

Case (a) $\mu_2 \rightarrow \infty$

Letting $\mu_2 \rightarrow \infty$ in the results of previous section , we get the results for

the case in which the elastic cylinder is embedded in a rigid layer as a limiting case. Since

$$\frac{\overline{\mu}-1}{\Delta(n)} \to \frac{-1}{I_1(\xi_n d)K_2(\xi_n d)} , \qquad \text{as } \mu_2 \to \infty ,$$

we obtain the expression for the kernel

$$N(x,t) = \frac{4}{\pi b} \sum_{n=1}^{\infty} \frac{K_1(\xi_n d)}{I_1(\xi_n d)} \sinh(\xi_n t) \sinh(\xi_n x) ,$$

and

$$R(x) = \frac{2d}{b} (\frac{2}{\pi})^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{\sinh(\xi_n x)}{I_1(\xi_n d)\xi_n^2} ,$$

for $\mu_2 \rightarrow \infty$. With these modifications the solution for this case is given by equations (2.3.29) and (2.3.30).

Case (b) $\mu_2 \rightarrow 0$.

In this case, if we let $\mu_2 \rightarrow 0$, we get the solution for the case in which the elastic cylinder is free of stress on its curved surface. And

$$\frac{\overline{\mu}-1}{\Delta(n)} \rightarrow \frac{1}{I_2(\xi_n d)K_1(\xi_n d)} , \qquad \text{as } \mu_2 \rightarrow 0 ,$$

hence the kernel N(x,t) becomes

$$N(x,t) = \frac{4}{\pi b} \sum_{n=1}^{\infty} \frac{K_2(\xi_n d)}{I_2(\xi_n d)} \sinh(\xi_n t) \sinh(\xi_n x) ,$$

and R(x)=0, as $\mu_2 \rightarrow 0$ and these results are in agreement with Gladwell and Lemczyk [26].

2.3.5 Numerical results and conclusions

Numerical values of $\phi(x)$ for $x = (0.0, 0.1 \ 0.2, \dots 1.0)c$ have been calculated from the integral equation (2.3.29) by reducing it to algebraic equations. And then the numerical values of the dimensionless ratio of torque T/T_0 have been calculated from equation (2.3.33), where $T_0=16\mu_1\epsilon c^3/3$ is the torque for the corresponding Reissner-Sagoci problem for the semi-infinite space. The Simpson's rule is used to perform the numerical integrations and the Crout's factorisation method is used to solve the linear algebraic equations. In the numerical results the relative errors are controlled under 0.01.

Numerical values of T/T_0 have been calculated for the following values of b/c, d/c and $\bar{\mu}=\mu_2/\mu_1$:

$$b/c = 0.2 \ (0.1)0.5, 1.0, 2.0, 10.0 ; \ d/c = 1.0(0.2)2.0, 3.0(1.0)10.0 ;$$

 $\bar{\mu} = 0.0, 0.5, 2.0; \text{ and } \bar{\mu} \to \infty$,

and these have been displayed in Fig.2.3.2 to Fig.2.3.5.

From the figures, we observe that the torque T decreases as the height b of the cylinder increases and the torque increases as the radius d of the cylinder increases. We also observe that the ratio T/T_0 approaches 1 when d/c and b/c approach to infinity at the same time. We notice that the change in the values of T/T_0 when b (the height of the elastic cylinder and layer) increases from 2 to 10 goes on decreasing as $\overline{\mu}$ increases from 0 to ∞ .



Values of T/T_0 displayed against d/c for b/c = 0.2, 0.3, 0.4, 0.5, 1.0, 2.0, 10.0 and $\bar{\mu}=0.0$.







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Values of T/T_0 displayed against d/c for b/c = 0.2, 0.3, 0.4, 0.5, 1.0, 2.0, 10.0 and $\bar{\mu}=2.0$.



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Fig.2.3.5

Values of T/T_0 displayed against d/c for b/c = 0.2, 0.3, 0.4, 0.5, 1.0, 2.0, 10.0 and $\overline{\mu} \rightarrow \infty$.

2.4 Semi-infinite composite elastic cylinder

2.4.1 The statement of the problem

In this section we consider the torsion problem of a semi-infinite elastic cylinder which is embedded in a semi-infinite elastic cylindrical shell with different shear modulus.

We assume that a semi-infinite elastic cylinder of radius d and shear modulus μ_1 is embedded in a semi-infinite elastic cylindrical shell of outer radius a and shear moduis μ_2 as shown in Fig.2.4.1. It is assumed that curved outer surface of the semi-infinite composite elastic cylinder is stress-free. It is also assumed that the inner cylinder is perfectly bonded to the surrounding elastic medium and that a torque is applied to the inner cylinder , through a rigid disc with radius c < d, which is bonded to its top flat surface. In terms of cylindrical polar coordinates (r, θ, z) , displacement field is given by u=w=0and v=v(r,z), hence we have the basic equations (2.1.2) and the solutions (2.1.3) for v(r,z).

We assume that the rigid disc bonded to the inner cylinder is turned through an angle ϵ and that the curved outer surface of the semi-infinite composite elastic cylinder is stress-free. We therefore consider the problem of determining the stress and displacement field in the semi-infinite composite elastic cylinder with the following boundary and continuity conditions:





A problem of Reissner-Sagoci type for a semi-infinite composite elastic cylinder.

$v(r,0) = \epsilon r ,$	$0 \leq r < c,$	(2.4.1)
$\sigma_{\theta_Z}(r,0) = 0 ,$	$c < r < d_{r}$, (2.4.2)
$\hat{\sigma}_{ heta_{\mathrm{Z}}}(r,0) = 0$,	d < r < a	, (2.4.3)
$\hat{\sigma}_{\mathrm{r}\theta}(a,z) = 0$,	$z \ge 0$,	(2.4.4)
$v(d,z) = \hat{v}(d,z)$,	$z \ge 0$,	(2.4.5)
$\sigma_{r\theta}(d,z) = \hat{\sigma}_{r\theta}(d,z)$	$z \geq d$;	(2.4.6)

where v, $\sigma_{\theta z}$, and $\sigma_{r\theta}$ are the non-zero displacement and stress components in the inner semi-infinite cylinder while \hat{v} , $\hat{\sigma}_{\theta z}$, and $\hat{\sigma}_{r\theta}$ are their counterparts in the surrounding medium.

2.4.2 Derivation of the dual integral equations

First of all, we select a combination of the basic solutions for v(r,z) and v(r,z) from the basic solutions listed in equation (2.1.3). By means of these solutions we are able to reduce the problem of solving the mixed boundary value problem stated in section 2.4.1 to that of solving a pair of dual integral equations.

For the inner semi-infinite cylinder, we assume that

$$v(r,z) = \mathscr{H}_{1}[\xi^{-1}A(\xi)e^{-\xi z}; \xi \to r] + \mathscr{H}_{C}[\xi^{-1}B(\xi)I_{1}(\xi r); \xi \to z], r < d,$$
(2.4.7)

where \mathscr{T}_{c} is Fourier cosine transform defined by

$$\mathscr{F}_{\mathrm{c}}[f(z) ; z \to \xi] = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_{0}^{\infty} f(z) \cos(\xi z) dz ,$$

and $\mathscr{F}_{c}^{-1} = \mathscr{F}_{c}$, where \mathscr{F}_{c}^{-1} is the inverse of \mathscr{F}_{c} .

For the surrounding medium we assume that

$$\hat{v}(r,z) = \mathscr{F}_{C}[\xi^{-1}\{C(\xi)I_{1}(\xi r) + D(\xi)K_{1}(\xi r)\}; \xi \to z], \qquad d < r < a, (2.4.8)$$

where $A(\xi)$, $B(\xi)$, $C(\xi)$ and $D(\xi)$ are to be determined later.

Now from equations (2.1.2) we have

$$\sigma_{\mathbf{r}\theta}(\mathbf{r},\mathbf{z}) = -\mu_1 \mathscr{H}_2[A(\xi)e^{-\xi \mathbf{z}}; \xi \to \mathbf{r}] + \mu_1 \mathscr{H}_2[B(\xi)I_2(\xi \mathbf{r}); \xi \to \mathbf{z}] , \qquad (2.4.9)$$

$$\hat{\sigma}_{r\theta}(r,z) = \mu_2 \quad \mathscr{F}_{c}[C(\xi)I_2(\xi r) - D(\xi)K_2(\xi r) ; \xi \rightarrow z],$$
(2.4.10)

$$\sigma_{\theta z}(r,z) = -\mu_1 \mathscr{H}[A(\xi)e^{-\xi z} ; \xi \rightarrow r] - \mu_1 \mathscr{H}[B(\xi)I_1(\xi r) ; \xi \rightarrow z] , \qquad (2.4.11)$$

$$\hat{\sigma}_{\theta z}(r,z) = -\mu_2 \quad \mathscr{F}_{s}[C(\xi)I_1(\xi r) + D(\xi)K_1(\xi r) ; \xi \to z].$$
(2.4.12)

The conditions (2.4.5) and (2.4.6) give

$$\mathcal{F}_{C}[\xi^{-1}\{C(\xi)I_{1}(d\xi)+D(\xi)K_{1}(d\xi)-B(\xi)I_{1}(d\xi)\}; \ \xi \rightarrow z]$$

$$=\mathcal{H}_{1}[\xi^{-1}A(\xi)e^{-\xi z} \ ; \ \xi \rightarrow d] \ , \qquad (2.4.13)$$

$$\mathcal{F}_{C}[B(\xi)I_{2}(d\xi)-\bar{\mu}\{C(\xi)I_{2}(d\xi)-D(\xi)K_{2}(d\xi)\}; \ \xi \rightarrow z]$$

$$=\mathcal{H}_{2}[A(\xi)e^{-\xi z} \ ; \ \xi \rightarrow d] \ , \qquad (2.4.14)$$

where $\bar{\mu} = \mu_2/\mu_1$.

Taking the inverse Fourier cosine transform of equations (2.4.13) and (2.4.14) and making use of the following result

$$\mathscr{F}_{\mathbf{C}}[e^{-\xi z} ; z \to \zeta] = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \xi(\xi^2 + \zeta^2)^{-1} , \qquad \xi > 0 , \qquad (2.4.15)$$

we obtain

$$C(\xi)I_{1}(d\xi) + D(\xi)K_{1}(d\xi) - B(\xi)I_{1}(d\xi) = G_{1}(\xi) , \qquad (2.4.16)$$

$$B(\xi)I_2(d\xi) - \bar{\mu}[C(\xi)I_2(d\xi) - D(\xi)K_2(d\xi) = G_2(\xi) , \qquad (2.4.17)$$

where

$$G_{1}(\xi) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \xi \int_{0}^{\infty} \frac{\zeta A(\zeta) J_{1}(d\zeta)}{\xi^{2} + \zeta^{2}} d\zeta , \qquad (2.4.18)$$

$$G_2(\xi) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{\zeta^2 A(\zeta) J_2(d\zeta)}{\xi^2 + \zeta^2} d\zeta . \qquad (2.4.19)$$

The condition (2.4.4) yields

$$C(\xi) = \frac{K_2(a\xi)}{I_2(a\xi)} D(\xi) \quad . \tag{2.4.20}$$

Eliminating $C(\xi)$ from equations (2.4.16) and (2.4.17) we obtain

$$-B(\xi)I_1(d\xi) + D(\xi)Q_1(\xi) = G_1(\xi) , \qquad (2.4.21)$$

$$B(\xi)I_2(d\xi) + \bar{\mu}D(\xi)Q_2(\xi) = G_2(\xi) , \qquad (2.4.22)$$

hence

•

•

$$B(\xi) = [\bar{\mu}G_1(\xi)Q_2(\xi) - G_2(\xi)Q_1(\xi)]/\Delta(\xi) , \qquad (2.4.23)$$

where

$$\begin{split} \Delta(\xi) &= -\bar{\mu}I_1(d\xi)Q_2(\xi) - I_2(d\xi)Q_1(\xi) , \\ Q_1(\xi) &= K_1(d\xi) + \frac{K_2(a\xi)}{I_2(a\xi)}I_1(d\xi) , \\ Q_2(\xi) &= K_2(d\xi) - \frac{K_2(a\xi)}{I_2(a\xi)}I_2(d\xi) . \end{split}$$
(2.4.24)

From equation (2.4.12) we find that the condition (2.4.3) is identically satisfied and that the conditions (2.4.1) and (2.4.2) will be satisfied if $A(\xi)$ is the solution of the following dual integral equations

$$\mathcal{H}_{I}[\xi^{-1}A(\xi) ; \xi \rightarrow r] + \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_{0}^{\infty} \xi^{-1}B(\xi)I_{I}(\xi r)d\xi = \epsilon r , \qquad 0 \le r < c, (2.4.25)$$
$$\mathcal{H}_{I}[A(\xi) ; \xi \rightarrow r] = 0 . \qquad c < r < d. (2.4.26)$$

2.4.3 Reduction to integral equation of Fredholm type

To reduce the problem of solving the dual integral equations (2.4.25) and (2.4.26) to that of solving an integral equation of Fredhiom type of the second kind, let us take

$$A(\xi) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \epsilon \int_{0}^{c} \phi(t) \sin(\xi t) dt , \qquad (2.4.27)$$

where $\phi(t)$ is a new unknown function defined in $(0,\infty)$ such that $\phi(t)=0$ for t>c. The representation (2.4.27) satisfies the equation (2.4.26) identically.

By using the following result [43]

$$\mathscr{H}_{1}[\xi^{-1}sin(\xi t) ; \xi \to r] = \frac{tH(r-t)}{r(r^{2}-t^{2})^{\frac{1}{2}}} , \qquad (2.4.28)$$

where H(x) is the Heaviside function, then from equation (2.4.27) we obtain

$$\mathscr{H}[\xi^{-1}A(\xi);\xi \to r] = \epsilon r^{-1} \mathscr{I}_1[t\phi(t);r] . \qquad (2.4.29)$$

Substituting from equation (2.4.27) into equations (2.4.18) and (2.4.19) and using the integrals (2.2.35) and (2.2.36) we find that

$$G_1(\xi) = \frac{2\epsilon}{\pi} \xi K_1(d\xi) \int_0^c \phi(t) \sinh(\xi t) dt , \qquad (2.4.30)$$

$$G_2(\xi) = \frac{2\epsilon}{\pi} \xi K_2(d\xi) \int_0^c \phi(t) \sinh(\xi t) dt . \qquad (2.4.31)$$

By using equations (2.4.23), (2.4.30) and (2.4.31) we obtain

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_{0}^{\infty} \xi^{-1} B(\xi) I_{1}(\xi r) d\xi$$

= $\epsilon \left(\frac{2}{\pi}\right)^{\frac{3}{2}} \int_{0}^{c} \phi(t) dt \int_{0}^{\infty} \frac{\mu K_{1}(d\xi) Q_{2}(\xi) - K_{2}(d\xi) Q_{1}(\xi)}{\Delta(\xi)} I_{1}(\xi r) \sinh(\xi t) d\xi.$ (2.4.32)

Operating equation (2.4.25) by $x^{-1} \mathscr{N}_1[r;x]$, and using the results (2.2.39) and (2.2.40) we obtain

$$\phi(x) + \int_{0}^{c} \phi(t) S(x,t) dt = f(x) , \qquad 0 \le x \le c, \quad (2.4.33)$$

where

$$S(x,t) = \frac{4}{\pi^2} \int_0^{\infty} \frac{\mu K_1(d\xi) Q_2(\xi) - K_2(d\xi) Q_1(\xi)}{\Delta(\xi)} \sinh(\xi x) \sinh(\xi t) d\xi ,$$

$$f(x) = x \cdot \frac{1}{dx} \mathscr{I}_1[r^3; r \to x] = 2(\frac{2}{\pi})^{\frac{1}{2}} x . \qquad (2.4.34)$$

To calculate the torque T necessary to produce the prescribed rotation of the disc rigidly bonded to the inner semi-infinite cylinder , as given by condition (2.4.1) , we have to calculate the value of $\sigma_{\theta z}$ at z=0. Now from equations (2.4.11) and (2.4.27) , we get

$$\begin{aligned} \sigma_{\theta z}(r,0) &= -\mu_1 \mathscr{H}[A(\xi) \ ; \xi \to r] \\ &= \mu_1 \frac{d}{dr} \mathscr{H}_0[\xi^{-1}A(\xi) \ ; \xi \to r] \\ &= \mu_1 \epsilon (\frac{2}{\pi})^{\frac{1}{2}} \frac{d}{dr} \int_r^c \phi(t) (t^2 - r^2)^{-\frac{1}{2}} dt , \qquad r < c . \quad (2.4.35) \end{aligned}$$

The required torque T is given by

$$T = -2\pi \int_{0}^{c} r^{2} \sigma_{\theta z}(r,0) dr . \qquad (2.4.36)$$

Using equation (2.4.35) we obtain

$$T = 4\mu_1 \epsilon (2\pi)^{\frac{1}{2}} \int_0^c t \phi(t) dt . \qquad (2.4.37)$$

In a particular case, when the outer semi-infinite cylindrical shell is of large radius compared to the radius of the inner semi-infinite cylinder, we let the radius *a* tend to infinity. As $a \to \infty$, $Q_1(\xi) \to K_1(d\xi)$ and $Q_2(\xi) \to K_2(d\xi)$, hence

$$S(x,t) = \frac{4}{\pi^2} (\bar{\mu} - 1) \int_0^\infty \frac{K_1(d\xi) K_2(d\xi)}{\Delta(\xi)} \sinh(\xi t) \sinh(\xi x) d\xi \quad , \tag{2.4.38}$$

which is in agreement with [22].

2.4.5 Numerical results and conclusions

Numerical values of $\phi(x)$ for $x = (0.0, 0.1, 0.2, \dots, 1.0)c$ have been calculated from the integral equation (2.4.33) by reducing it to algebraic equations. And then the numerical values of the dimensionless ratio of torque T/T_0 have been calculated from equation (2.4.36), where $T_0=16\mu_1\epsilon c^3/3$ is the torque for the corresponding Reissner-Sagoci problem for the semi-infinite elastic space. The Simpson's rule is used to perform the numerical integrations and the Crout's factorisation method is used to solve the linear algebraic equations. In the numerical results the relative errors are controlled under 0.01.

Numerical values of T/T_0 have been calculated for the following combination of values of

a/d = 1.1, 1.5; d/c = 1.0(1.0)10.0; $\bar{\mu} = 0.5, 1.0, 2.0$;

and these have been displayed in Fig.2.4.2. From the figure we observe that for a fixed radius c of the rigid disc the torque T increases as the radius b of the inner semi-infinite cylinder increases while the ratio a/d of the radius of outer semi-infinite cylindrical shell to the inner semi-infinite cylinder is kept the same . We also observe that the ratio T/T_0 approaches 1 when the radius of the inner semi-infinite cylinder approach infinity.



Numerical values of ratio of the torques T/T_0 against d/c for $\overline{\mu} = 0.5$, 1.0, 2.0; a/b=1.1,1.5.

CHAPTER 3

TORSION OF TWO NONHOMOGENEOUS ELASTIC LAYERS WITH PENNY-SHAPED FLAW AT THE INTERFACE

3.1 Introduction

In this chapter we investigate a torsion problem of two non-homogeneous isotropic elastic layers with a penny-shaped flaw at the interface of the layers. It is assumed that the flaw is in the form of an inclusion or a crack, and the rigidity of each of the two materials is a function of the variable z in cylindrical polar coordinate system in the form $\mu(z) = \mu \exp(\alpha z)$, where μ and α are real constants. And it is also assumed that a rigid circular shaft is bonded to the free surface of the first layer just above the circular flaw, and the circular shaft is rotated through a small angle by applying a twisting moment of torque T and the rest of the surface $z = -h_1$ is kept stress-free. The lower surface $z = h_2$ of the second layer is either stress-free or rigidly fixed (see Fig.3.1.1). Four different cases are considered and the results for the corresponding problems of a layer and a half-space with a flaw at the interface are derived. The problem is reduced to solving a system of simultaneous Fredholm integral equations which have been solved numerically. Numerical values of the physical quantities have been displayed graphically.



Fig.3.1.1 Torsion of two elastic layers by a rigid shaft.

Under the assumption of axial symmetry of the problem, we know that the displacement components u_r and u_z vanish and u_{θ} depends on r and zonly. To simplify, let us denote u_r , u_{θ} and u_z by u, v and w respectively. Then we have displacement components u=w=0 and v=v(r,z). The stress-displacement relations are given by equations (1.6.3) and the equation of equilibrium is given by equation (1.6.5). So we have the following basic equations under consideration

$$u=w=0, \qquad v=v(r,z),$$

$$\sigma_{\theta z}(r,z) = \mu(z) \frac{\partial v}{\partial z}, \qquad \sigma_{r\theta}(r,z) = \mu(z) \left(\frac{\partial v}{\partial r} - \frac{v}{r}\right),$$

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2} + \frac{1}{\mu} \frac{\partial v}{\partial z} \frac{\partial \mu}{\partial z} = 0. \qquad (3.1.1)$$

where $\mu(z)$ is the shear modulus of the medium and v(r,z), $\sigma_{\theta z}$ and σ_{rz} are respectively the non-zero displacement and stress components in the polar cylindrical coordinates (r, θ, z) .

We further assume that the two isotropic, non-homogeneous elastic layers, which occupy the regions $R_1(-h_1 \leq z < 0)$ and $R_2(0 \leq z \leq h_2)$, have the moduli of rigidity in the form of

$$\mu(z) = \begin{cases} \mu_1 e^{\alpha_1 z} = \mu_1(z) ; \text{ in } R_1(-h_1 < z < 0), \\ \mu_2 e^{\alpha_2 z} = \mu_2(z) ; \text{ in } R_2(0 < z < h_2), \end{cases}$$
(3.1.2)

are perfectly bonded except that there is a flaw (inclusion or crack) in the region $0 \leq r \leq b$, z = 0. It is also assumed that a rigid circular shaft of radius *a* units is bonded to the free surface of the first layer at $z = -h_1$ and it is rotated through a small angle ϵ_0 by applying a twisting moment *T* to the shaft and the rest of the surface $z = -h_1$ is kept stress-free. And the other surface $z = h_2$ is assumed either stress-free or rigidly fixed. By using the Hankel transform on the third equation of (3.1.1) we obtain the following

general solution v for layer R_1 and \hat{v} for layer R_2 respectively :

$$v(r,z) = \int_{0}^{\infty} \xi [A(\xi) \cosh(\delta_{1}z) + B(\xi) \sinh(\delta_{1}z)] e^{-\frac{1}{2}\alpha_{1}z} J_{1}(r\xi) d\xi ,$$

$$-h_{1} < z < 0, (3.1.3)$$

$$\hat{v}(r,z) = \int_{0}^{\infty} \xi [C(\xi) \cosh(\delta_{2}z) + D(\xi) \sinh(\delta_{2}z)] e^{-\frac{1}{2}\alpha_{2}z} J_{1}(r\xi) d\xi ,$$

$$0 < z < h_{2}, (3.1.4)$$

where

$$\delta_{\rm i} = \left(\xi^2 + rac{lpha_{
m i}^2}{4}
ight)^{rac{1}{2}}, \qquad i=1,2;$$

and $A(\xi)$, $B(\xi)$, $C(\xi)$ and $D(\xi)$ are unknown functions to be determined such that the integrals in equation (3.1.3) and (3.1.4) are convergent and the boundary conditions are satisfied.

Substitution of equations (3.1.2) and (3.1.3) into the second equation of (3.1.1) yields

$$\begin{split} \sigma_{\theta z} &= \mu_1(z) \int_0^{\infty} \xi\{ [\delta_1 A(\xi) \sinh(\delta_1 z) + \delta_1 B(\xi) \cosh(\delta_1 z)] - \frac{\alpha_1}{2} [A(\xi) \cosh(\delta_1 z) \\ &+ B(\xi) \sinh(\delta_1 z)] \} e^{-\frac{1}{2}\alpha_1 z} J_1(r\xi)^{-} d\xi , \qquad -h_1 < z < 0, \ (3.1.5) \\ \hat{\sigma}_{\theta z} &= \mu_2(z) \int_0^{\infty} \xi\{ [\delta_2 C(\xi) \sinh(\delta_2 z) + \delta_2 D(\xi) \cosh(\delta_2 z)] - \frac{\alpha_2}{2} [C(\xi) \cosh(\delta_2 z) \\ &+ D(\xi) \sinh(\delta_2 z)] \} e^{-\frac{1}{2}\alpha_2 z} J_1(r\xi) d\xi , \qquad 0 < z < h_2, \ (3.1.6) \end{split}$$

where $\sigma_{\theta z}$ and $\hat{\sigma}_{\theta z}$ denote the stress components in layer R_1 and R_2 respectively.

The common boundary and continuity conditions for the problem (excluding the conditions on flaw and the surface $z=h_2$) are the following

$$\begin{aligned} v(r,-h_{1}) &= \epsilon_{0}r , & 0 \leq r < a, \quad (3.1.7) \\ \sigma_{\theta z}(r,-h_{1}) &= 0 , & a < r < \infty, \quad (3.1.8) \\ v(r,0^{-}) &= \hat{v}(r,0^{+}) , & b \leq r < \infty, \quad (3.1.9) \\ \sigma_{\theta z}(r,0^{-}) &= \hat{\sigma}_{\theta z}(r,0^{+}) , & b \leq r < \infty. \quad (3.1.10) \end{aligned}$$

3.2 Inclusion problem with the surface $z = h_2$ stress-free.

3.2.1 Statement of the problem

In this section we will consider the problem stated in section 3.1 when the flaw is an inclusion and the surface $z=h_2$ is stress-free. In addition to the common conditions (3.1.7) to (3.1.10) we have

$$v(r,0^{-}) = \hat{v}(r,0^{+}) = \epsilon_{1}r ; \qquad 0 \le r \le b, \qquad (3.2.1)$$

$$\hat{\sigma}_{\theta z}(r,h_{2}) = 0 , \qquad 0 \le r < \omega, \qquad (3.2.2)$$

where we have assumed that as a result of the application of twisting moment T on the shaft, the rigid inclusion on the interface will rotate through some unknown angle ϵ_1 .

In this case the continuity condition (3.1.9) may be replaced by

$$v(r,0^{-}) = \hat{v}(r,0^{+}), \qquad 0 \le r < \infty , \qquad (3.2.3)$$

by using condition (3.2.1), which then will be satisfied if we take

$$C(\xi) = A(\xi).$$
 (3.2.4)

Using the boundary condition (3.2.2) along with equation (3.1.6), we get

$$D(\xi) = E(\xi) C(\xi)$$
 (3.2.5)

where

$$E(\xi) = \frac{\left[\frac{\alpha_2}{2} - \delta_2\right] + \left[\frac{\alpha_2}{2} + \delta_2\right] exp(-2\delta_2h_2)}{-\left[\frac{\alpha_2}{2} - \delta_2\right] + \left[\frac{\alpha_2}{2} + \delta_2\right] exp(-2\delta_2h_2)}$$
(3.2.6)

Applying conditions (3.1.7), (3.1.8), (3.1.10) and (3.2.1), the equations (3.1.3) to (3.1.6) give the following results:

$$\int_{0}^{\infty} \xi[A(\xi) \cosh(\delta_{1}h_{1}) - B(\xi) \sinh(\delta_{1}h_{1})] J_{1}(r\xi) d\xi = \epsilon_{0} r \exp(-\frac{\alpha_{1}}{2} h_{1});$$

$$0 \leq r < a, \quad (3.2.7)$$

$$\int_{0}^{\infty} \xi\{\left[\frac{\alpha_{1}}{2} \cosh(\delta_{1}h_{1}) + \delta_{1} \sinh(\delta_{1}h_{1})\right] A(\xi) - \left[\frac{\alpha_{1}}{2} \sinh(\delta_{1}h_{1}) + \delta_{1} \cosh(\delta_{1}h_{1})B(\xi)\right]\}J_{1}(r\xi)d\xi = 0, \qquad a < r < \infty, \quad (3.2.8)$$
$$\int_{0}^{\infty} \xi A(\xi)J_{1}(r\xi)d\xi = \epsilon_{1}r \quad ; \qquad 0 \le r \le b, \quad (3.2.9)$$

$$\int_0^{\infty} \xi[\rho \ A(\xi) + \delta_1 B(\xi)] \ J_1(r\xi) d\xi = 0 \quad , \qquad b < r < \infty, \quad (3.2.10)$$

where

$$\overline{\mu} = \mu_2/\mu_1$$
, $\rho = \overline{\mu}(\frac{\alpha_2}{2} - \delta_2 E) - \frac{\alpha_1}{2}$. (3.2.10a)

Let us introduce two functions

$$P(\xi) = \left[\frac{\alpha_1}{2} \cosh(\delta_1 h_1) + \delta_1 \sinh(\delta_1 h_1)\right] A(\xi) - \left[\frac{\alpha_1}{2} \sinh(\delta_1 h_1) + \delta_1 \cosh(\delta_1 h_1)\right] B(\xi),$$

$$Q(\xi) = \rho A(\xi) + \delta_1 B(\xi).$$
(3.2.11)

Observing that [43]

$$\int_{0}^{\infty} \xi \sin(\xi t) J_{1}(r\xi) d\xi = -r (t^{2} - r^{2})^{-\frac{3}{2}} H(t - r), \qquad (3.2.12)$$

we find that equations (3.2.8) and (3.2.10) will be identically satisfied if we take

$$P(\xi) = \int_{0}^{a} \phi(t) \sin(\xi t) dt,$$

$$Q(\xi) = \int_{0}^{b} \psi(t) \sin(\xi t) dt,$$
(3.2.13)

where ϕ and ψ are two new unknown functions.

Using the expressions (3.2.11) we find that

$$\xi[A(\xi)\cosh(\delta_1h_1) - B(\xi)\sinh(\delta_1h_1)] = [1 + M_1(\xi)]P(\xi) + N_1(\xi)Q(\xi),$$

$$\xi A(\xi) = M_2(\xi)P(\xi) + [\lambda + N_2(\xi)]Q(\xi),$$
 (3.2.14)

where

.

$$\lambda = \frac{1}{1 + \overline{\mu}},$$

$$M_{1}(\xi) = \xi \frac{a_{1}(\xi) + a_{2}(\xi) \exp(-2\delta_{1}h_{1})}{a_{3}(\xi) + a_{4}(\xi) \exp(-2\delta_{1}h_{1})} - 1,$$

$$N_{1}(\xi) = M_{2}(\xi) = \xi \frac{2\delta_{1} \exp(-\delta_{1}h_{1})}{a_{3}(\xi) + a_{4}(\xi) \exp(-2\delta_{1}h_{1})},$$

$$N_{2}(\xi) = \xi \frac{a_{5}(\xi) + a_{6}(\xi)\exp(-2\delta_{1}h_{1})}{a_{3}(\xi) + a_{4}(\xi)\exp(-2\delta_{1}h_{1})} - \lambda,$$
(3.2.15)

with

.

$$a_{1}(\xi) = \delta_{1} + \rho , \qquad a_{2}(\xi) = \delta_{1} - \rho ,$$

$$a_{3}(\xi) = a_{1}(\xi)a_{5}(\xi), \qquad a_{4}(\xi) = -a_{2}(\xi)a_{6}(\xi),$$

$$a_{5}(\xi) = \delta_{1} + \frac{1}{2} \alpha_{1}, \qquad a_{6}(\xi) = \delta_{1} - \frac{1}{2} \alpha_{1}.$$
(3.2.16)

Now substituting from equations (3.2.14) into equation (3.2.7), using equations (3.2.13) and the following integral representations

$$J_1(\xi r) = \frac{2}{\pi r} \int_0^r \frac{x \sin(\xi x) \, dx}{\sqrt{r^2 - x^2}} \quad , \qquad (3.2.17)$$

$$\int_0^\infty J_1(r\xi) \sin(\xi t) d\xi = \frac{t \ H(r-t)}{r \sqrt{r^2 - t^2}} , \qquad (3.2.18)$$

we obtain an Able integral equation

$$\int_{0}^{r} \frac{x}{\sqrt{r^{2} - x^{2}}} \left\{ \phi(x) + \int_{0}^{a} \phi(t) L_{1}(x, t) dt + \int_{0}^{b} \psi(t) K_{1}(x, t) dt \right\} dx$$

= $\epsilon_{0} r^{2} exp(-\frac{1}{2} \alpha_{1} h_{1}), \qquad 0 \leq x \leq a, \quad (3.2.19)$

which when inverted gives

$$\phi(x) + \int_{0}^{a} \phi(t) L_{1}(x,t) dt + \int_{0}^{b} \psi(t) K_{1}(x,t) dt = \frac{4}{\pi} \epsilon_{0} \exp(-\frac{1}{2} \alpha_{1} h_{1}) x ,$$

$$0 \leq x \leq a. \quad (3.2.20)$$

Similarly, equation (3.2.9) gives

,

,

.

$$\lambda \psi(x) + \int_{0}^{a} \phi(t) L_{2}(x,t) dt + \int_{0}^{b} \psi(t) K_{2}(x,t) dt = \frac{4}{\pi} \epsilon_{1} x ,$$

$$0 \leq x \leq b, \qquad (3.2.21)$$

.

where

$$L_{i}(x,t) = \frac{2}{\pi} \int_{0}^{\infty} M_{i}(\xi) \sin(\xi x) \sin(\xi t) d\xi,$$

$$K_{i}(x,t) = \frac{2}{\pi} \int_{0}^{a} N_{i}(\xi) \sin(\xi x) \sin(\xi t) d\xi, \qquad i = 1,2. \qquad (3.2.22)$$

From equations (3.1.5) and (3.2.11) and observing $J'_0(x) = -J_1(x)$, we obtain

$$\sigma_{\theta z}(r,-h_1) = \mu_1 exp(-\frac{1}{2}\alpha_1 h_1) \frac{d}{dr} \int_0^a \phi(t) dt \int_0^\infty J_0(\xi r) sin(\xi t) d\xi. \qquad (3.2.23)$$

Now using the fact that the inner integral on the right side of the equation (3.2.23) is zero for t < r and $(t^2 - r^2)^{\frac{-1}{2}}$ for t > r, we obtain

$$\sigma_{\theta z}(r,-h_1) = \mu_1 exp(-\frac{1}{2}\alpha_1 h_1) \frac{d}{dr} \int_0^a \frac{\phi(t)dt}{\sqrt{t^2 - r^2}}, \qquad 0 \le r \le a. \quad (3.2.24)$$

The moment T required to produce the required rotation ϵ_0 of the rigid shaft is given by

$$T = -2\pi \int_{0}^{a} r^{2} \sigma_{\theta z}(r, -h_{1}) dr. \qquad (3.2.25)$$

Substituting from equation (3.2.24) into (3.2.25), we find that

$$T = 4\pi\mu_1 exp(-\frac{1}{2}\alpha_1 h_1) \int_0^a t\phi(t) dt. \qquad (3.2.26)$$

In a similar way, the proviso on the vanishing of the moment applied to the inclusion leads to

$$\int_{0}^{b} t\psi(t)dt = 0. (3.2.27)$$

Integrating by parts and performing the indicated differentiation in the

equation (3.2.24), we obtain

.

$$\sigma_{\theta z}(r,-h_1) = -\mu_1 \left[\frac{a\phi(a)}{r\sqrt{a^2-r^2}} - \frac{1}{r} \int_r^a \frac{t\phi'(t)}{\sqrt{t^2-r^2}} dt \right] exp(-\frac{1}{2}\alpha_1 h_1),$$

$$0 \le r \le a. \qquad (3.2.28)$$

The stress $\sigma_{\theta z}(r,-h_1)$ has a square root singularity at r = a and the constant $\phi(a)$ is the measure of the strength of the singularity at the rim of the rigid shaft. In a similar way, we can show that

$$\sigma_{\theta z}(r,0) = -\mu_1 \left[\frac{b\psi(b)}{r\sqrt{b^2 - r^2}} - \frac{1}{r} \int_r^b \frac{t\psi'(t)}{\sqrt{t^2 - r^2}} dt \right] + \mathfrak{l}(r),$$

$$0 \le r < b, \quad (3.2.29)$$

where $\Re(\mathbf{r})$ is bounded, hence $\sigma_{\theta z}$ has square root singularity at the edge of the inclusion, and $\psi(\mathbf{b})$ is the measure of the strength of the singularity.

For numerical solution it is convenient to write the integral equations in dimensionless form. We, therefore, set

$$\begin{split} \phi(\eta) &= \frac{\pi}{4a\epsilon_0} exp(\frac{1}{2} \alpha_1 h_1) \phi(a\eta), \qquad \Psi(\eta) = \frac{\pi}{4b\epsilon_0} exp(\frac{1}{2} \alpha_1 h_1) \psi(b\eta), \\ L_1^*(\eta, \tau) &= aL_1(a\eta, a\tau), \qquad L_2^*(\eta, \tau) = \frac{a^2}{b} L_2(b\eta, a\tau), \\ K_1^*(\eta, \tau) &= \frac{b^2}{a} K_1(a\eta, b\tau), \qquad K_2^*(\eta, \tau) = b K_2(b\eta, b\tau) \\ T_0 &= \frac{16}{3} \cdot \mu_1 \epsilon_0 a^3. \end{split}$$
(3.2.30)

and

$$\beta = \frac{\epsilon_1}{\epsilon_0}.\tag{3.2.31}$$

Then the equations (3.2.20), (3.2.21), (3.2.26) and (3.2.27) can be written in

the following forms

$$\begin{split} \phi(\eta) + \int_{0}^{1} \phi(\tau) L_{1}^{*}(\eta, \tau) d\tau &+ \int_{0}^{1} \Psi(\tau) K_{1}^{*}(\eta, \tau) d\tau = \eta, \quad 0 < \eta < 1, \quad (3.2.32) \\ \lambda \Psi(\eta) + \int_{0}^{1} \phi(\tau) L_{2}^{*}(\eta, \tau) d\tau &+ \int_{0}^{1} \Psi(\tau) K_{2}^{*}(\eta, \tau) d\tau = \beta \exp(\frac{1}{2}\alpha_{1}h_{1})\eta, \\ 0 < \eta < 1, \quad (3.2.33) \\ 3 \exp(\alpha_{1}h_{1}) \int_{0}^{1} \tau \phi(\tau) d\tau &= \frac{T}{T_{0}}, \end{split}$$

$$\int_{0}^{1} \tau \Psi(\tau) d\tau = 0$$
 (3.2.35)

for the determination of $\phi(\eta)$, $\Psi(\eta)$, β and T. It is easy to see $\phi(1)$ and $\Psi(1)$ are the measure of the strength of stress singularities at the rim of the shaft and the edge of the inclusion respectively.

3.2.3 Solution for the homogeneous case and numerical results

When $\alpha_1 = \alpha_2 = 0$, the problem considered above becomes a torsion problem of two homogeneous elastic layers with a penny-shaped inclusion at the interface of the layers which have the shear moduli μ_1 and μ_2 respectively. In this case, $E(\xi)$, M_i and N_i , i = 1,2; have the following forms

$$E(\xi) = -tanh(\xi h_2),$$
 (3.2.36)

and

$$M_{1}(\xi) = \frac{2\gamma}{exp(2\xi h_{1}) - \gamma},$$

$$N_{1}(\xi) = M_{2}(\xi) = \frac{2 exp(\xi h_{1})}{(1 - \overline{\mu}E)(exp(2\xi h_{1}) - \gamma)},$$

$$N_{2}(\xi) = \frac{1 + exp(2\xi h_{1})}{(1 - \overline{\mu}E)exp(2\xi h_{1}) - (1 + \overline{\mu}E)} - \frac{1}{1 + \overline{\mu}},$$

$$\gamma = \frac{1 + \overline{\mu}E}{1 - \overline{\mu}E}.$$
(3.2.37)

Numerical solution for this particular case has been obtained by solving the simultaneous Fredholm integral equations (3.2.32), (3.2.33) and (3.2.35) for $\phi(\eta)$, $\Psi(\eta)$ and β , in which kernels $L_i^*(x,t)$ and $K_i^*(x,t)$ i = 1,2 are dependent on functions $M_i(\xi)$, $N_i(\xi)$, i = 1,2 given by equations (3.2.37). To do this, we partition the interval [0,1] into 20 equal subintervals and approximate the integral equations by a system of linear algebraic equations in $\phi(\eta_i)$, $\Psi(\eta_i)$ (with $\eta_i = 0.0(0.05)1.0$) and β for their determination. Then the values of T/T_0 are calculated by numerical integration of (3.2.34). The quadrature method has been employed to perform the numerical integrations of the kernels involved in the integral equations and the relative error is controlled under 0.01. The same method is also used in sections 3.3, 3.4 and 3.5.

The numerical values for this problem have been calculated for b/a = 0.0(0.1) 1.0, 2.0, 3,0, 4,0; $h_1/a = 0.5$, 1.0, 2.0; $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\mu_2/\mu_1 = \overline{\mu} = 0.5$, 1.0, 2.0. The numerical values of T/T_0 , $\phi(1)$, $\Psi(1)$ and β have been displayed against b/a for various values of h_2/h_1 for a combination of values of $\overline{\mu} = 0.5$, 1.0, 2.0 and $h_1/a = 0.5$, 1.0, 2.0 in Fig.3.2.1—Fig.3.2.36.



Figs.3.2.1-3.2.4

For Inclusion Problem, Numerical values of $\phi(1)$, $\Psi(1)$, M/M₀ and $\beta = \epsilon_1/\epsilon_0$ against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu} = \mu_2/\mu_1 = 0.5$, $h_1/a = 0.5$.

S-free mu2/mu1-0.5 h1/o-0.5 .

S-free mu2/mu1-1.0 h1/0-0.5



For Inclusion Problem, Numerical values of $\phi(1)$, $\Psi(1)$, M/M_0 and $\beta = \epsilon_1/\epsilon_0$ against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu} = \mu_2/\mu_1 = 1.0$, $h_1/a = 0.5$.

S-free mu2/mu1-2.0 h1/0-0.5







Figs.3.2.13-3.2.16

For Inclusion Problem, Numerical values of $\phi(1)$, $\Psi(1)$, M/M₀ and $\beta = \epsilon_1/\epsilon_0$ against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu} = \mu_2/\mu_1 = 0.5$, $h_1/a = 1.0$.

S-free mu2/mu1-1.0 h1/o-1.0





For Inclusion Problem, Numerical values of $\phi(1)$, $\Psi(1)$, M/M₀ and $\beta = \epsilon_1/\epsilon_0$ against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu}_1 = \mu_2/\mu_1 = 1.0$, $h_1/a = 1.0$.





For Inclusion Problem, Numerical values of $\phi(1)$, $\Psi(1)$, M/M_0 and $\beta = \epsilon_1/\epsilon_0$ against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu} = \mu_2/\mu_1 = 2.0$, $h_1/a = 1.0$.





For Inclusion Problem, Numerical values of $\phi(1)$, $\Psi(1)$, M/M_0 and $\beta = \epsilon_1/\epsilon_0$ against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu} = \mu_2/\mu_1 = 0.5$, $h_1/a = 2.0$.








For Inclusion Problem, Numerical values of $\phi(1)$, $\Psi(1)$, M/M₀ and $\beta = \epsilon_1/\epsilon_0$ against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu} = \mu_2/\mu_1 = 2.0$, $h_1/a = 2.0$.

3.2.4 Solution for the case $h_2 \rightarrow \infty$ and numerical results

When $h_2 \rightarrow \infty$, the problem considered becomes the problem of torsion of a nonhomogeneous elastic layer bonded to a nonhomogeneous elastic half-space with penny-shaped inclusion at the interface. From equations (3.2.6) and (3.2.10a) we find that when $h_2 \rightarrow \infty$

$$E = -1, \qquad \rho = \frac{1}{2} (\overline{\mu}\alpha_2 - \alpha_1) + \overline{\mu}\delta_2,$$

and the solution for this case is given by the results of section 3.2.2. In a particular case of $\alpha_1 = \alpha_2 = 0$, equations (3.2.37) give

$$M_{1}(\xi) = \frac{2\gamma}{exp(2\xi h_{1}) - \gamma} ,$$

$$N_{1}(\xi) = M_{2}(\xi) = \frac{2exp(\xi h_{1})}{(1 + \overline{\mu}) (exp(2\xi h_{1}) - \gamma)},$$

$$N_{2}(\xi) = \frac{1 + exp(2\xi h_{1})}{(1 + \overline{\mu}) exp(2\xi h_{1}) - (1 - \overline{\mu})} - \frac{1}{1 + \overline{\mu}} ,$$

$$\gamma = \frac{1 - \overline{\mu}}{1 + \overline{\mu}} .$$
(3.2.38)

Numerical solution for this particular case has been obtained by solving the simultaneous Fredholm integral equations (3.2.32), (3.2.33) and (3.2.35) for $\phi(\eta)$, $\Psi(\eta)$ and β , in which kernels $L_i^*(x,t)$ and $K_i^*(x,t)$ (i = 1,2) are dependent on functions $M_i(\xi)$, $N_i(\xi)$ (i = 1,2) given by equations (3.2.38). Then the values of T/T_0 are calculated by numerical integration of (3.2.34).

The numerical values for this problem have been calculated for b/a = 0.0(0.1) 1.0, 2.0, 3,0, 4,0; $h_1/a = 0.2(0.1)0.6$, 0.8, 1.0, 2.0, and $\mu_2/\mu_1 = \overline{\mu} = 0.5$, 1.0, 2.0. The numerical values of T/T_0 , $\phi(1)$, $\Psi(1)$ and β have been displayed against b/a for various values of h_1/a for $\overline{\mu} = 0.5$, 1.0, 2.0 in Fig.3.2.37—Fig.3.2.48.







For Inclusion Problem, Numerical values of $\Phi(1)$, $\Psi(1)$, M/M_0 , $\beta = \epsilon_1/\epsilon_0$ against b/a for $h_1/a = 0.2$, 0.3, 0.4, 0.5, 0.6, 0.8, 1.0, 2.0 and $\overline{\mu} = \mu_2/\mu_1 = 0.5$.

mu2/mu1-1.0





For Inclusion Problem, Numerical values of $\Phi(1)$, $\Psi(1)$, M/M_0 , $\beta = \epsilon_1/\epsilon_0$ against b/a for $h_1/a = 0.2$, 0.3, 0.4, 0.5, 0.6, 0.8, 1.0, 2.0 and $\overline{\mu} = \mu_2/\mu_1 = 1.0$.

.

mu2/mu1-2.0



Figs.3.2.45-3.2.48

For Inclusion Problem, Numerical values of $\Phi(1)$, $\Psi(1)$, M/M_0 , $\beta = \epsilon_1/\epsilon_0$ against b/a for $h_1/a = 0.2, 0.3, 0.4, 0.5, 0.6, 0.8, 1.0, 2.0$ and $\overline{\mu} = \mu_2/\mu_1 = 2.0$.

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3.3 Crack problem with the surface $z=h_2$ stress-free

3.3.1 Statement of the problem

In this section we will consider the problem stated in section 3.1 when the flaw is in the form of a crack and the surface $z=h_2$ is stress-free. Hence we have the basic equations (3.1.1) and the general solutions (3.1.3) to (3.1.6). Besides the common conditions (3.1.7) to (3.1.10) there are the following two additional conditions:

$$\begin{aligned} \sigma_{\theta z}(r,0^{-}) &= \hat{\sigma}_{\theta z}(r,0^{+}) = 0, & 0 \leq r < b, \quad (3.3.1) \\ \hat{\sigma}_{\theta z}(r,h_2) &= 0. & 0 \leq r < \infty. \quad (3.3.2) \end{aligned}$$

Using the condition (3.3.1), the continuity condition (3.1.10) may be replaced by

$$\sigma_{\theta z}(r,0) = \hat{\sigma}_{\theta z}(r,0^{+}), \qquad 0 \leq r < \infty. \quad (3.3.3)$$

3.3.2 Analysis

Now the conditions (3.3.2) and (3.3.3) will be satisfied by the using (3.1.5) and (3.1.6), if we take

$$D(\xi) = E(\xi) C(\xi),$$
 (3.3.4)

and

$$C(\xi) = \frac{\alpha_1 A(\xi) - 2\,\delta_1 B(\xi)}{\overline{\mu} [\,\alpha_2 - 2\delta_2 E(\xi)]} , \qquad (3.3.5)$$

where $E(\xi)$ is given by equation (3.2.6).

Then the conditions (3.1.7), (3.1.8), (3.1.9) and (3.3.1) lead to the following system of dual integral equations

$$\int_{0}^{\infty} \xi[A(\xi)\cosh(\delta_{1}h_{1}) - B(\xi)\sinh(\delta_{1}h_{1})]J_{1}(r\xi)d\xi = \epsilon_{0}r \exp(-\frac{\alpha_{1}}{2}h_{1}),$$

$$0 \leq r < a, \quad (3.3.6)$$

$$\int_{0}^{\infty} \xi\{[\frac{\alpha_{1}}{2}\cosh(\delta_{1}h_{1}) + \delta_{1}\sinh(\delta_{1}h_{1})]A(\xi) - [\frac{\alpha_{1}}{2}\sinh(\delta_{1}h_{1}) + \delta_{1}\cosh(\delta_{1}h_{1})B(\xi)]\}J_{1}(r\xi)d\xi = 0, \quad a < r < \infty, \quad (3.3.7)$$

$$\int_{0}^{\infty} \xi\{\left[1 - \frac{\alpha_{1}}{\overline{\mu}(\alpha_{2}-2\delta_{2}E)}\right]A(\xi) + \frac{2\delta_{1}}{\overline{\mu}(\alpha_{2}-2\delta_{2}E)}B(\xi)\}J_{1}(r\xi)d\xi = 0, \quad b \leq r \leq \infty, \quad (3.3.8)$$

$$\int_{0}^{\infty} \xi[-\frac{\alpha_{1}}{2}A(\xi) + \delta_{1}B(\xi)]J_{1}(r\xi)d\xi = 0, \quad 0 < r < b. \quad (3.3.9)$$

where $\overline{\mu} = \mu_2/\mu_1$.

The above system of dual integral equations may be rewritten in the following form (Note: here and what follows such functions as ϕ , ψ , K_i , L_i , M_i , $N_i, \dots,$ etc. will be introduced, but it should be clear that they are (presumably) different in different cases):

$$\int_{0}^{\infty} \{ [1+M_{1}(\xi)] P_{1}(\xi) + N_{1}(\xi) Q_{1}(\xi) \} J_{1}(r\xi) d\xi = \epsilon_{0} \exp(\frac{1}{2}\alpha_{1}h_{1}),$$

$$0 \leq r < a, \quad (3.3.10)$$

$$\int_0^{\infty} \xi P_i(\xi) J_i(r\xi) d\xi = 0, \qquad a < r < \infty, \quad (3.3.11)$$

$$\int_{0}^{\infty} \xi \{ M_{2}(\xi) P_{1}(\xi) + [\lambda_{1} + N_{2}(\xi)] Q_{1}(\xi) \} J_{1}(r\xi) d\xi = 0, \quad 0 \leq r < b, \quad (3.3.12)$$

$$\int_{0}^{\infty} Q_{1}(\xi) J_{1}(r\xi) d\xi = 0, \qquad b < r < \infty, \quad (3.3.13)$$

where

$$P_{1}(\xi) = \left[\frac{\alpha_{1}}{2} \cosh(\delta_{1}h_{1}) + \delta_{1} \sinh(\delta_{1}h_{1})\right] A(\xi) - \left[\frac{\alpha_{1}}{2} \sinh(\delta_{1}h_{1}) + \delta_{1} \cosh(\delta_{1}h_{1})\right] B(\xi),$$
(3.3.14)

$$Q_{1}(\xi) = \xi \left[1 - \frac{\alpha_{1}}{\overline{\mu}(\alpha_{2} - 2\delta_{2}E)} \right] A(\xi) + \xi \frac{2\delta_{1}}{\overline{\mu}(\alpha_{2} - 2\delta_{2}E)} B(\xi), \qquad (3.3.15)$$

and

$$M_{1}(\xi) = \xi \frac{a_{1}(\xi) + a_{2}(\xi)}{a_{3}(\xi) + a_{4}(\xi)} \frac{exp(-2\delta_{1}h_{1})}{exp(-2\delta_{1}h_{1})} - 1,$$

$$N_{1}(\xi) = -M_{2}(\xi) = \frac{\delta_{1}\overline{\mu}(\alpha_{2} - 2\delta_{2}E)}{a_{3}(\xi) + a_{4}(\xi)} \frac{exp(-\delta_{1}h_{1})}{exp(-2\delta_{1}h_{1})},$$

$$N_{2}(\xi) = \xi \frac{\overline{\mu}(\alpha_{2} - 2\delta_{2}E)[1 - exp(-2\delta_{1}h_{1})]}{2[a_{3}(\xi) + a_{4}(\xi)} \frac{exp(-2\delta_{1}h_{1})}{exp(-2\delta_{1}h_{1})} - \lambda_{1},$$

$$\lambda_{1} = \frac{\overline{\mu}}{1 + \overline{\mu}},$$
(3.3.16)

and a_i , i = 1,2,3,4; are the same as given by equations (3.2.16).

Now if we take

$$P_{1}(\xi) = \int_{0}^{a} \phi(t) \sin(\xi t) dt, \qquad (3.3.17)$$

$$Q_{1}(\xi) = \int_{0}^{b} \psi(t) \left[\frac{\sin(\xi t)}{\xi t} - \cos(\xi t) \right] dt, \qquad (3.3.18)$$

the integral equations (3.3.11) and (3.3.13) will be identically satisfied, whereas if we insert the above expressions for $P_1(\xi)$ and $Q_1(\xi)$ into equation (3.3.10), we obtain the following integral equation:

$$\phi(x) + \int_{0}^{a} \phi(t) L_{1}(x,t) dt + \int_{0}^{b} \psi(t) K_{1}(x,t) dt = \frac{4}{\pi} \in_{0} x \ exp(-\frac{1}{2}\alpha_{1}h_{1}),$$

$$0 \le x \le a, \qquad (3.3.19)$$

where

$$L_{1}(x,t) = \frac{2}{\pi} \int_{0}^{\infty} M_{1}(\xi) \sin(\xi t) \sin(\xi x) d\xi,$$

$$K_{1}(x,t) = \frac{2}{\pi} \int_{0}^{\infty} N_{1}(\xi) [\frac{\sin(\xi t)}{\xi t} - \cos(\xi t)] \sin(\xi x) d\xi.$$
(3.3.20)

Now to satisfy equation (3.3.12), let us first rewrite equation (3.3.18) in the form

$$\xi Q_{1}(\xi) = \int_{0}^{b} \left[\frac{\psi(t)}{t} + \psi'(t)\right] \sin(\xi t) dt - \psi(b) \sin(\xi b), \qquad (3.3.21)$$

and then substitute for $P_1(\xi)$ and $Q_1(\xi)$ from equations (3.3.17) and (3.3.21) into equation (3.3.12) and use equations (3.2.17) and (3.2.18) to obtain the integral equation

$$\int_{0}^{r} \frac{G(\eta) d\eta}{\sqrt{r^{2} - \eta^{2}}} = 0, \qquad 0 \le r < b, \quad (3.3.22)$$

where

$$G(\eta) = \lambda_1 \frac{d}{d\eta} [\eta \ \psi(\eta)] + \frac{2}{\pi} \int_0^a \phi(t) dt \int_0^\infty \eta \ \xi \ M_2(\xi) sin(\xi t) sin(\xi \eta) d\xi + \frac{2}{\pi} \int_0^b \psi(t) dt \int_0^\infty \eta \ \xi \ N_2(\xi) [\frac{sin(\xi t)}{\xi t} - cos(\xi t)] sin(\xi \eta) d\xi.$$
(3.3.23)

Clearly equation (3.3.22) will be satisfied if $G(\eta) = 0$. So if we let $G(\eta) = 0$ and integrate (3.3.23) with respect to η from 0 to x for 0 < x < b and then divide by x, we obtain:

$$\lambda_1\psi(x) + \int_0^a \phi(t)L_2(x,t)dt + \int_0^b \psi(t)K_2(x,t)dt = 0, \qquad 0 \le x \le b, \qquad (3.3.24)$$

where

$$L_{2}(x,t) = \frac{2}{\pi} \int_{0}^{\infty} M_{2}(\xi) [\frac{\sin(\xi x)}{\xi x} - \cos(\xi x)] \sin(\xi t) d\xi,$$

$$K_{2}(x,t) = \frac{2}{\pi} \int_{0}^{\infty} N_{2}(\xi) [\frac{\sin(\xi x)}{\xi x} - \cos(\xi x)] [\frac{\sin\xi t}{\xi t} - \cos(\xi t)] d\xi.$$
(3.3.25)

The boundary value of shear stress $\sigma_{\theta z}(r,-h_1)$ is still given by equation (3.2.28). The value of the shear stress $\sigma_{\theta z}(r,0)$, r > b at the interface of the two materials is given by

$$\sigma_{\theta z}(r,0) = \mu_1 \int_0^\infty \xi[\delta_1 B(\xi) - \frac{1}{2} \alpha_1 \ A(\xi)] J_1(\xi r) d\xi. \qquad (3.3.26)$$

And substituting for $A(\xi)$ and $B(\xi)$ from equation (3.3.14), (3.3.15) and using equations (3.3.17), (3.3.18), (3.3.21) and (3.2.18), we obtain

$$\sigma_{\theta z}(r,0) = -\lambda_1 \mu_1 \frac{b\psi(b)}{r\sqrt{r^2 - b^2}} + R(r), \qquad r > b, \qquad (3.3.27)$$

where

$$R(r) = \mu_1 \int_0^\infty \xi [M_2(\xi) P_1(\xi) + N_2(\xi) Q_1(\xi)] J_1(\xi r) d\xi + \lambda_1 \mu_1 \int_0^\infty J_1(\xi r) d\xi \int_0^b [\frac{\psi(t)}{t} + \psi'(t)] sin(\xi t) dt, \qquad (3.3.28)$$

is bounded, while $\sigma_{\theta z}(r,-h_1)$ and T, the torque required to rotate the rigid shaft through a small angle ϵ_0 , have the same expressions as given by equations (3.2.28) and (3.2.26). Hence $\phi(a)$ and $\psi(b)$ are the measure of the strength of the stress singularity at the rim of the shaft and at the edge of the crack respectively.

By using the same transformations as given in (3.2.30) to get dimensionless form, the equations (3.3.19), (3.3.24) and (3.2.26) can be rewritten in the following forms

$$\phi(\eta) + \int_0^1 \phi(\tau) L_1^*(\eta, \tau) d\tau + \int_0^1 \Psi(\tau) K_1^*(\eta, \tau) d\tau = \eta , \quad 0 < \eta < 1, \quad (3.3.29)$$

$$\lambda_{1}\Psi(\eta) + \int_{0}^{1} \phi(\tau) L_{2}^{*}(\eta,\tau) d\tau + \int_{0}^{1} \Psi(\tau) K_{2}^{*}(\eta,\tau) d\tau = 0 , \quad 0 < \eta < 1, \quad (3.3.30)$$

3
$$exp(\alpha_1 h_1) \int_0^1 \tau \phi(\tau) d\tau = \frac{T}{T_0}$$
, (3.3.31)

for the determination of $\phi(\eta)$, $\Psi(\eta)$ and T. Again, it is easy to see that $\phi(1)$ and $\Psi(1)$ are the measure of the strength of stress singularity at the rim of the shaft and the edge of the crack respectively.

3.3.3 Solution for the homogeneous case and numerical results

As in section 3.2.3, when $\alpha_1 = \alpha_2 = 0$, the problem considered above becomes a torsion problem of two homogeneous elastic layers with a penny-shaped crack at the interface of the layers which have the shear moduli μ_1 and μ_2 respectively. In this case we have

$$E(\xi) = -tanh(\xi h_2),$$
 (3.3.32)

and

,

$$M_{1}(\xi) = \frac{2(1+\overline{\mu}E)}{(1-\overline{\mu}E)e\,x\,p\,(2\xi h_{1})-(1+\overline{\mu}E)},$$

$$N_{1}(\xi) = -M_{2}(\xi) = \frac{-2\overline{\mu}exp(2\xi h_{1})}{(1-\overline{\mu}E)\,exp(2\xi h_{1})-(1+\overline{\mu}E)},$$

$$N_{2}(\xi) = \frac{\overline{\mu}E(1-exp(2\xi h_{1}))}{(1-\overline{\mu}E)exp(2\xi h_{1})-(1+\overline{\mu}E)} - \frac{\overline{\mu}}{1+\overline{\mu}}.$$
(3.3.33)

Numerical solution for this particular case has been obtained by solving

the simultaneous Fredholm integral equations (3.3.29) and (3.3.30) for $\phi(\eta)$ and $\Psi(\eta)$, in which kernels $L_i^*(x,t)$ and $K_i^*(x,t)$ (i = 1,2) are dependent on functions $M_i(\xi)$, $N_i(\xi)$ (i = 1,2) given by (3.3.33). Then the values of T/T_0 are calculated by performing the numerical integration in (3.3.31).

The numerical values for this problem have been calculated for b/a = 0.0(0.1) 1.0, 2.0, 3,0, 4,0; $h_1/a = 0.5$, 1.0, 2.0; $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\mu_2/\mu_1 = \overline{\mu} = 0.5$, 1.0, 2.0. The numerical values of T/T_0 , $\phi(1)$ and $\Psi(1)$ have been displayed against b/a for various values of h_2/h_1 for a combination of values of $\overline{\mu} = 0.5$, 1.0, 2.0 and $h_1/a = 0.5$, 1.0, 2.0 in Fig.3.3.1—Fig.3.3.27.



Figs.3.3.1-3.3.3

For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and M/M_0 against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu} = \mu_2/\mu_1 = 0.5$, $h_1/a = 0.5$.





For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and M/M_0 against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu} = \mu_2/\mu_1 = 1.0$, $h_1/a = 0.5$.



Figs.3.3.7-3.3.9

For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and M/M_0 against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu} = \mu_2/\mu_1 = 2.0$, $h_1/a = 0.5$.





For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and M/M_0 against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu} = \mu_2/\mu_1 = 0.5$, $h_1/a = 1.0$.





For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and M/M_0 against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu} = \mu_2/\mu_1 = 1.0$, $h_1/a = 1.0$.





For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and M/M_0 against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu} = \mu_2/\mu_1 = 2.0$, $h_1/a = 1.0$.



For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and M/M_0 against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu} = \mu_2/\mu_1 = 0.5$, $h_1/a = 2.0$.





For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and M/M_0 against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu} = \mu_2/\mu_1 = 1.0$, $h_1/a = 2.0$.





For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and M/M_0 against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu} = \mu_2/\mu_1 = 2.0$, $h_1/a = 2.0$.

3.3.4 Solution for the case $h_2 \rightarrow m$ and the numerical results

As in section 3.2.4, when $h_2 \rightarrow \infty$, the problem considered in section 3.3.2 becomes the problem of torsion of a nonhomogeneous elastic layer bonded to a nonhomogeneous elastic half-space with penny-shaped crack at the interface. From equation (3.2.6) we find that $E \rightarrow -1$ when $h_2 \rightarrow \infty$. The solution for this case is given by the results of section 3.3.2. by taking E = -1. Particularly, if $\alpha_1 = \alpha_2 = 0$ we have

$$M_{1}(\xi) = -2(\bar{\mu} - 1)S(\xi),$$

$$N_{1}(\xi) = M_{2}(\xi) = 2exp(\xi h_{1})S(\xi),$$

$$N_{2}(\xi) = [exp(2\xi h_{1})-1]S(\xi) - \lambda,$$
(3.3.34)

where

$$S(\xi) = [(\bar{\mu} - 1) + (\bar{\mu} + 1)exp(2\xi h_1)]^{-1}$$

Numerical solution for this particular case has been obtained by solving the simultaneous Fredholm integral equations (3.3.29) and (3.3.30) for $\phi(\eta)$ and $\Psi(\eta)$, in which kernels $L_i^*(x,t)$ and $K_i^*(x,t)$ (i = 1,2) are dependent on functions $M_i(\xi)$, $N_i(\xi)$, (i = 1,2) given by equations (3.3.34) Then the values of T/T_0 are calculated by numerical integration of (3.3.31).

The numerical values for this problem have been calculated for b/a = 0.0(0.1) 1.0, 2.0, 3,0, 4,0; $h_1/a = 0.2(0.1)0.6$, 0.8, 1.0, 2.0, and $\mu_2/\mu_1 = \overline{\mu} = 0.5$, 1.0, 2.0. The numerical values of T/T_0 , $\phi(1)$ and $\Psi(1)$ have been displayed against b/a for various values of h_1/a for $\overline{\mu} = 0.5$, 1.0, 2.0 in Fig.3.3.28.—Fig.3.3.36.



Figs.3.3.28-3.3.30

For Crack Problem, Numerical values of $\Phi(1)$, $\Psi(1)$ and M/M_0 against b/a for $h_1/a = 0.2, 0.3, 0.4, 0.5, 0.6, 0.8, 1.0, 2.0$ and $\overline{\mu} = \mu_2/\mu_1 = 0.5$.

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0.1

0^{.9}

phi (1)

0.6

0.5

0.0

Figs.3.3.31-3.3.33

For Crack Problem, Numerical values of $\Phi(1)$, $\Psi(1)$ and M/M_0 against b/a for $h_1/a = 0.2$, 0.3, 0.4, 0.5, 0.6, 0.8, 1.0, 2.0 and $\overline{\mu} = \mu_2/\mu_1 = 1.0$.

4.0

mu2/mu1-1.0



Fig.3.3.34-3.3.36

For Crack Problem, Numerical values of $\Phi(1)$, $\Psi(1)$ and M/M_0 against b/a for $h_1/a = 0.2, 0.3, 0.4, 0.5, 0.6, 0.8, 1.0, 2.0$ and $\overline{\mu} = \mu_2/\mu_1 = 2.0$.

3.4 Inclusion problem with the surface $z = h_2$ rigidly fixed.

3.4.1 Analysis

In this section we will consider the problem stated in section 3.1, when the flaw is an inclusion and the surface $z=h_2$ is rigidly fixed. Hence, in addition to the common conditions (3.1.7) to (3.1.10) we have the following conditions

$$v(r,0^{-}) = \hat{v}(r,0^{+}) = \epsilon_1 r$$
, $0 \leq r \leq b$, (3.4.1)

$$\hat{v}(r,h_2) = 0$$
, $0 \leq r < \infty$, (3.4.2)

where we have also assumed that as a result of the application of torque T on the shaft the rigid inclusion will rotate through some unknown small angle ϵ_1 . The combination of the conditions (3.1.9) and (3.4.1) yield

$$v(r,0^{-}) = \hat{v}(r,0^{+})$$
, $0 \leq r \leq \infty$. (3.4.3)

The conditions (3.4.2) and (3.4.3) will be satisfied by the general solutions (3.1.3) and (3.1.4) if we choose

$$D(\xi) = E_1(\xi)C(\xi),$$
 (3.4.4)

where

$$E_{1}(\xi) = - \cosh(\delta_{2}h_{2}). \qquad (3.4.5)$$

If we replace E in the equation (3.2.6) by E_1 (consequently all functions involving E will be changed) then most of the discussion in section 3.2 is valid in this case. We indicate the difference in the following.

In this problem the boundary and continuity conditions (3.1.7), (3.1.8), (3.4.1) will lead to equations (3.2.7), (3.2.8) and (3.2.9) while the condition (3.1.10) gives

$$\int_{0}^{\infty} \xi \{ [\overline{\mu}(\frac{\alpha_{2}}{2} - \delta_{2}E_{1}) - \frac{\alpha_{1}}{2}] A(\xi) + \delta_{1}B(\xi) \} J_{1}(r\xi) d\xi = 0, \quad b < r < \infty.$$
(3.4.6)

As we have done in section 3.2, the equations (3.2.7), (3.2.8), (3.2.9) and (3.4.6) can be reduced to solution of the system of integral equations (3.2.20) and (3.2.21) for the unknown functions $\phi(t)$ and $\psi(t)$; while expressions (3.2.15), (3.2.16) and (3.2.22) are valid if we replace E in equation (3.2.10*a*) by E_1 , which is given by equation (3.4.5). The transformations (3.2.30) and (3.2.31) lead us to the equations (3.2.32), (3.2.33), (3.2.34) and (3.2.35) for the determination of $\phi(\eta)$, $\Psi(\eta)$, β and T for this problem.

3.4.2 Solution for the homogeneous case and numerical results

When $\alpha_1 = \alpha_2 = 0$, the problem we considered becomes a torsion problem of two homogeneous elastic layers with a penny-shaped inclusion at the interface of the layers which have the shear moduli μ_1 and μ_2 respectively with the surface $z = h_2$ fixed. In this case we have

$$E_1(\xi) = -\cot(\xi h_2), \qquad (3.4.7)$$

and

$$M_{1}(\xi) = \frac{2\gamma_{1}}{exp(2\xi h_{1}) - \gamma_{1}},$$

$$N_{1}(\xi) = M_{2}(\xi) = \frac{2exp(\xi h_{1})}{(1 - \overline{\mu}E_{1})(exp(2\xi h_{1}) - \gamma_{1})},$$

$$N_{2}(\xi) = \frac{1 + exp(2\xi h_{1})}{(1 - \overline{\mu}E_{1})exp(2\xi h_{1}) - (1 + \overline{\mu}E_{1})} - \frac{1}{1 + \overline{\mu}},$$

$$\gamma_{1} = \frac{1 + \overline{\mu}E_{1}}{1 - \overline{\mu}E_{1}}.$$
(3.4.8)

Numerical solution for this particular case has been obtained by solving simultaneous Fredholm integral equations (3.2.32), (3.2.33) and (3.2.35) for $\phi(\eta)$, $\Psi(\eta)$ and β , in which kernels $L_i^*(x,t)$ and $K_i^*(x,t)$ (i = 1,2) are dependent on functions $M_i(\xi)$, $N_i(\xi)$ (i = 1,2) given by equations (3.4.8). Then the values of T/T_0 are calculated by the numerical integration of (2.3.34).

The numerical values for this problem have been calculated for b/a = 0.0(0.1) 1.0, 2.0, 3,0, 4,0; $h_1/a = 0.5$, 1.0, 2.0; $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\mu_2/\mu_1 = \overline{\mu} = 0.5$, 1.0, 2.0. The numerical values of T/T_0 , $\phi(1)$, $\Psi(1)$ and β have been displayed against b/a for various values of h_2/h_1 for a combination of values of $\overline{\mu} = 0.5$, 1.0, 2.0 and $h_1/a = 0.5$, 1.0, 2.0 in Fig.3.4.1—Fig.3.4.36.





For Inclusion Problem, Numerical values of $\phi(1)$, $\Psi(1)$, M/M₀ and $\beta = \epsilon_1/\epsilon_0$ against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu} = \mu_2/\mu_1 = 0.5$, $h_1/a = 0.5$.





For Inclusion Problem, Numerical values of $\phi(1)$, $\Psi(1)$, M/M_0 and $\beta = \epsilon_1/\epsilon_0$ against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu} = \mu_2/\mu_1 = 1.0$, $h_1/a = 0.5$.

0-free mu2/mu1-1.0 h1/0-0.5





For Inclusion Problem, Numerical values of $\phi(1)$, $\Psi(1)$, M/M_0 and $\beta = \epsilon_1/\epsilon_0$ against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu} = \mu_2/\mu_1 = 2.0$, $h_1/a = 0.5$.



Figs.3.4.13-3.4.16

For Inclusion Problem, Numerical values of $\phi(1)$, $\Psi(1)$, M/M_0 and $\beta = \epsilon_1/\epsilon_0$ against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu} = \mu_2/\mu_1 = 0.5$, $h_1/a = 1.0$.



Figs.3.4.17-3.4.20

For Inclusion Problem, Numerical values of $\phi(1)$, $\Psi(1)$, M/M_0 and $\beta = \epsilon_1/\epsilon_0$ against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu} = \mu_2/\mu_1 = 1.0$, $h_1/a = 1.0$.



For Inclusion Problem, Numerical values of $\phi(1)$, $\Psi(1)$, M/M_0 and $\beta = \epsilon_1/\epsilon_0$ against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu} = \mu_2/\mu_1 = 2.0$, $h_1/a = 1.0$.





For Inclusion Problem, Numerical values of $\phi(1)$, $\Psi(1)$, M/M_0 and $\beta = \epsilon_1/\epsilon_0$ against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu} = \mu_2/\mu_1 = 0.5$, $h_1/a = 2.0$.


Figs.3.4.29-3.4.32

For Inclusion Problem, Numerical values of $\phi(1)$, $\Psi(1)$, M/M_0 and $\beta = \epsilon_1/\epsilon_0$ against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu} = \mu_2/\mu_1 = 1.0$, $h_1/a = 2.0$.



Figs.3.4.33-3.4.36

For Inclusion Problem, Numerical values of $\phi(1)$, $\Psi(1)$, M/M_0 and $\beta = \epsilon_1/\epsilon_0$ against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu} = \mu_2/\mu_1 = 2.0$, $h_1/a = 2.0$.

0-free mu2/mu1-2.0 h1/o-2.0

3.5 Crack Problem with the surface $z=h_2$ rigidly fixed

3.5.1 Analysis

In this section we will consider the problem stated in section 3.1 when the flaw is a crack and the surface $z=h_2$ is rigidly fixed. Hence we have the basic equations (3.1.1) and general solutions (3.1.3) to (3.1.6). Besides the common conditions (3.1.7) to (3.1.10) there are two additional conditions given by

$$\sigma_{\theta z}(r,0^{-}) = \hat{\sigma}_{\theta z}(r,0^{+}) = 0, \qquad 0 \le r < b, \quad (3.5.1)$$
$$\hat{v}(r,h_2) = 0. \qquad 0 \le r < \infty. \quad (3.5.2)$$

Using the condition (3.5.1), the continuity condition (3.1.10) may be replaced by

$$\sigma_{\theta z}(r,0^{-}) = \hat{\sigma}_{\theta z}(r,0^{+}), \qquad 0 \leq r < \infty. \quad (3.5.3)$$

In this case the conditions (3.5.2) and (3.5.3) will be satisfied by the general solutions (3.1.5) and (3.1.6), if we take

$$D(\xi) = E_1(\xi) C(\xi), \qquad (3.5.4)$$

and

$$C(\xi) = \frac{\alpha_1 A(\xi) - 2\delta_1 B(\xi)}{\overline{\mu} [\alpha_2 - 2\delta_2 E_1(\xi)]}, \qquad (3.5.5)$$

where $E_{i}(\xi)$ is given by equation (3.4.5).

If we replace E in the equation (3.4.5) by E_1 (consequently all functions involving E will be changed) then most of the discussion in section 3.3 is valid in this case. We indicate the difference in the following.

In this problem the boundary and continuity conditions (3.1.7), (3.1.8), (3 5.1) will lead to equations (3.3.6), (3.3.7) and (3.3.9) while the condition (3.1.9) gives

$$\int_{0}^{\infty} \xi \{ \left[1 - \frac{\alpha_{1}}{\overline{\mu}(\alpha_{2} - 2\delta_{2}E_{1})} \right] A(\xi) + \frac{2\delta_{1}}{\overline{\mu}(\alpha_{2} - 2\delta_{2}E_{1})} B(\xi) \} J_{1}(r\xi) d\xi = 0 ,$$

$$0 \leq r \leq b, \qquad (3.5.6)$$

As we have done in section 3.3, the equations (3.3.7), (3.3.7), (3.3.9) and (3.5.6) can be reduced to the solution of the system of integral equations (3.3.19) and (3.3.24) for unknown functions $\phi(t)$ and $\psi(t)$; while expressions (3.3.16), (3.2.16) and (3.3.25) are valid if we replace E in equation (3.2.10*a*) by E_1 , which is given by equation (3.4.5), only to keep in mind that in every function involving E in section 3.3, E should be replaced by E_1 for this problem.

The transformations (3.2.30) will lead us to the equations (3.3.29), (3.3.30) and (3.3.31) for the determination of $\phi(\eta)$, $\Psi(\eta)$ and T for this problem.

3.5.2 Solution for the homogeneous case and numerical results

When $\alpha_1 = \alpha_2 = 0$, the problem we considered becomes a torsion problem of two homogeneous elastic layers with a penny-shaped crack at the interface of the layers which have the shear moduli μ_1 and μ_2 respectively with the surface

 $z = h_2$ fixed. In this case we have

$$E_1(\xi) = -coth(\xi h_2), \qquad (3.5.7)$$

and

$$M_{1}(\xi) = \frac{2(1+\overline{\mu}E_{1})}{(1-\overline{\mu}E_{1})\exp(2\xi h_{1})-(1+\overline{\mu}E_{1})},$$

$$N_{1}(\xi) = -M_{2}(\xi) = \frac{-2\overline{\mu}E_{1} \exp(\xi h_{1})}{(1-\overline{\mu}E_{1})\exp(2\xi h_{1})-(1+\overline{\mu}E_{1})},$$

$$N_{2}(\xi) = \frac{\overline{\mu}E_{1}(1-\exp(2\xi h_{1}))}{(1-\overline{\mu}E_{1})\exp(2\xi h_{1})-(1+\overline{\mu}E_{1})} - \frac{\overline{\mu}}{1+\overline{\mu}}.$$
(3.5.8)

Numerical solution for this particular case has been obtained by solving simultaneous Fredholm integral equations (3.3.29) and (3.3.30) for $\phi(\eta)$ and $\Psi(\eta)$, in which kernels $L_i^*(x,t)$ and $K_i^*(x,t)$ (i = 1,2) are dependent on functions $M_i(\xi)$, $N_i(\xi)$ (i = 1,2) given by equations (3.5.8). Then the values of T/T_0 are calculated by the numerical integration of (3.2.34).

The numerical values for this problem have been calculated for b/a = 0.0(0.1) 1.0, 2.0, 3,0, 4,0; $h_1/a = 0.5$, 1.0, 2.0; $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\mu_2/\mu_1 = \overline{\mu} = 0.5$, 1.0, 2.0. The numerical values of T/T_0 , $\phi(1)$ and $\Psi(1)$ have been displayed against b/a for various values of h_2/h_1 for a combination of values of $\overline{\mu} = 0.5$, 1.0, 2.0 and $h_1/a = 0.5$, 1.0, 2.0 in Fig.3.5.1—Fig.3.5.27.



Figs.3.5.1-3.5.3

For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and M/M_0 against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu} = \mu_2/\mu_1 = 0.5$, $h_1/a = 0.5$.

D-free mu2/mu1-1.0 h1/0-0.5



Figs.3.5.4-3.5.6

For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and M/M_0 against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu} = \mu_2/\mu_1 = 1.0$, $h_1/a = 0.5$.





For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and M/M_0 against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu} = \mu_2/\mu_1 = 2.0$, $h_1/a = 0.5$.





For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and M/M_0 against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu} = \mu_2/\mu_1 = 0.5$, $h_1/a = 1.0$.



Figs.3.5.13-3.5.15

For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and M/M_0 against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu} = \mu_2/\mu_1 = 1.0$, $h_1/a = 1.0$.

0.97.91 0.95 0.96 0.97 0.98 0.99 1.00 1.01 1.02 1.03 0.00.02 0.01 0.05 0.08 0.10 0.12 0.11 0.15 0.18 0.20 h2/h1=0.25 h2/hl≓5.0 0.0 1.0 2.0 b/a 3.0 3.0 4.0 0.0 1.0 2.0 4.0 b/a M/Mo 0.90.91 0.95 0.95 0.97 0.98 0.99 1.00 1.01 1.02 1.03 h2/hl=5.0

0-free mu2/mu1-2.0 h1/o-1.0



4.0

2.0

b/a

3.0

1.0

For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and M/M_0 against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu} = \mu_2/\mu_1 = 2.0$, $h_1/a = 1.0$.



Figs.3.5.19-3.5.21

For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and M/M_0 against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu} = \mu_2/\mu_1 = 0.5$, $h_1/a = 2.0$.





For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and M/M_0 against b/a for $h_2/h_1 = 0.25$, 0.5, 1.0, 2.0, 5.0 and $\overline{\mu} = \mu_2/\mu_1 = 1.0$, $h_1/a = 2.0$.

0-free mu2/mu1-2.0 h1/a-2.0 0.04

3.0

4.0



Figs.3.5.25-3.5.27

For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and M/M_0 against b/a for $h_2/h_1 = 0.25, 0.5, 1.0, 2.0, 5.0$ and $\overline{\mu} = \mu_2/\mu_1 = 2.0, h_1/a = 2.0.$

3.6 Some comment on the solutions

For two layers case, we make the following observations from the graphs for the inclusion problem: the strength of the singularity at the rim of the rigid shaft and the moment required to rotate the shaft increase as b/aincreases and decrease as h_2/h_1 increases but near 1.0 < b/a < 2.0 they approach 1; the strength of the singularity at the edge of the inclusion increases as h_2/h_1 decreases and reaches a maximum near b/a = 1; the ratio $\epsilon_1/\epsilon_0 = \beta$ decreases as b/a or h_2/h_1 increases.

For the crack problem we notice the following: the strength of the singularity at the rim of the shaft and moment required to rotate the shaft decrease as b/a or h_2/h_1 increases; the strength of the singularity at the edge of the crack behaves in the same manner as in the inclusion problem.

For layer and half-space case, we make the following observations from the graphs for the inclusion problem: the strength of the singularity at the rim of the rigid shaft and the moment required to rotate the shaft increase as b/aincreases and decrease as h_1/a increases; the singularity at the edge of the inclusion increases as h_1/a decreases and reaches a maximum near b/a = 1; the ratio $\epsilon_1/\epsilon_0 = \beta$ decreases as b/a or h_1/a increases.

For the crack problem we notice the following: the singularity at the rim of the shaft and the moment required to rotate the shaft decrease as b/aincreases and increase as h_1/a increases but near b/a = 1 they increase as h_1/a increases to 1; the singularity at the edge of the crack behaves in the same manner as in the inclusion problem.

CHAPTER 4

GRIFFITH CRACK AT THE INTERFACE OF TWO ORTHOTROPIC ELASTIC LAYERS

4.1 Introduction

The study of Griffith crack problems in the mathematical theory of elasticity originated in the classical work of Griffith [46]. A crack occupying the line segment

 $y = 0 , \qquad -c \leq x \leq c,$

in the xy-plane is called a Griffith crack.

In 1946, Sneddon and Elliot [47] considered the problem of determining the distribution of stress in the neighborhood of a Griffith crack which is subjected to an internal pressure varying along the length of the crack. They reduced the problem to a half-plane mixed boundary value problem and solved it by using Fourier transform methods. Green and Zerna [2] reduced the Griffith crack problem to the Hilbert problem. Willmore [48] solved the problem of two collinear Griffith cracks in an isotropic material by means of elliptic functions when a uniform pressure acts normally on the crack surface. Tranter [49] considered the problem of a normally varying pressure on collinear Griffith cracks.

Koiter [50], and England and Green [51] considered the problem of determining the stress field caused by an infinite row of collinear Griffith cracks of equal length when each crack is subject to the same constant

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pressure. Sneddon and Srivastav [52] considered the same problem by assuming a varying pressure on each crack.

Lowengrub [53] solved the Griffith crack problem in a strip with stress-free edges, when the crack is parallel to the edges of the strip. Sneddon and Srivastav [52] investigated the problem of Griffith crack in a strip in which crack is perpendicular to the edges of the strip. Lowengrub [54] considered the distribution of stress in the neighborhood of external crack in an elastic plane.

The problem of radial cracks originating at the boundary of an internal circular hole in an infinite elastic plane was solved by Bowie [55]. The problem of determining the distribution of stress in the vicinity of a star crack formed by the intersection of a number of Griffith cracks was solved by Westman [56].

Williams [57] considered the situation in which a Griffith crack is present at the interface of two isotropic semi-infinite planes of dissimilar materials, he found that the analytic solution of stresses has a peculiar behavior near the tip of the interface crack where the stresses undergo a rapid reversal of sign. The oscillatory character takes the form $r^{-\frac{1}{2}}sin$ (or cos) of the argument $\epsilon log(r/a)$ where r is the radial distance from the crack border, a is the crack size and ϵ is a bimaterial constant depending upon the elastic properties of the adjoining materials. This behavior was also studied by Sih and Rice [58], they formulated the the problem of stress state near the crack-tip and derived a formula for the stress intensity factor for the problem considered by Williams. Erdogan and Gupta [59, 60], analyzed the plane and antiplane problems of stress distribution of multi-layered composites with a flaw by reducing them to a system of singular integral equations. They developed a direct approach to find the approximate solutions of singular integral equations of the first (second) kind by using Chebyshev (Jacobi) polynomials. Lowengrub and Sneddon [61] solved the problem of a Griffith crack at the interface of two bonded dissimilar elastic half-planes by Fourier transforms and reduced the problem to a set a dual integral equations, they used Muskhelishvili's method to solve the system of singular integral equations. Dhaliwal [62], Mohapatra and Parhi [63], Satpathi and Parhi [64] and Parihar and Lalitha [65] considered the Griffith crack problem in an orthotropic medium. Recently Dhaliwal, Saxena and Rokne [66] considered the crack at the interface of an orthotropic elastic layer bonded to a dissimilar orthotropic elastic half-space.

In this chapter we will consider the problem of determining the state of stress near a Griffith crack located at the interface of two dissimilar orthotropic elastic layers. By means of Fourier transforms the problem is reduced to a system of singular integral equations. These equations are further reduced to a system of simultaneous algebraic equations by using Jacobi polynomials approximation. Numerical methods are employed to determine the stress intensity factors, which have been displayed graphically.

4.2. Basic equations and their solution

As discussed in chapter 1, under the assumptions of plane strain in an orthotropic medium when the cartesian coordinate axes are chosen to coincide with the principle axes, we know that the displacements u_x , u_y depend on x and y only, while u_z vanishes. To simplify, let us denote u_x , u_y and u_z by u, v and w respectively, then we have

u = u(x,y), v = v(x,y), w = 0, (4.2.1)

and the stress-displacement relations are given by

$$\sigma_{xx} = c_{11} \frac{\partial u}{\partial x} + c_{12} \frac{\partial v}{\partial y} ,$$

$$\sigma_{yy} = c_{12} \frac{\partial u}{\partial x} + c_{22} \frac{\partial v}{\partial y} ,$$

$$\sigma_{xy} = c_{66} \left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right] ,$$
(4.2.2)

where c_{ij} 's are the elastic moduli of the orthotropic medium. The equations of equilibrium, in the absence of body forces, may be expressed as follows

$$c_{11}\frac{\partial^2 u}{\partial x^2} + c_{66}\frac{\partial^2 u}{\partial y^2} + (c_{12} + c_{66}) \frac{\partial^2 v}{\partial x \partial y} = 0 , \qquad (4.2.3a)$$

$$c_{66}\frac{\partial^2 v}{\partial x^2} + c_{22}\frac{\partial^2 v}{\partial y^2} + (c_{12} + c_{66}) \frac{\partial^2 u}{\partial x \partial y} = 0 . \qquad (4.2.3b)$$

When the displacement u is anti-symmetric and v is symmetric with respect to y-axis (this is the case when the crack is subjected to symmetric normal pressure and anti-symmetric shear with respect to y-axis considered in this chapter), by applying the Fourier sine transform with respect to x to equation (4.2.3a) and the Fourier cosine transform with respect to x to equation (4.2.3b) respectively we obtain

$$(c_{66} \frac{\partial^2}{\partial y^2} - c_{11} \xi^2) \bar{u}_{\rm s} - \xi (c_{12} + c_{66}) \frac{\partial}{\partial y} \bar{v}_{\rm c} = 0,$$

$$\xi (c_{12} + c_{66}) \frac{\partial}{\partial y} \bar{u}_{\rm s} + (c_{22} \frac{\partial^2}{\partial y^2} - c_{66} \xi^2) \bar{v}_{\rm c} = 0,$$
 (4.2.4)

where \bar{u}_s is Fourier sine transform of u and \bar{v}_c is the Fourier cosine transform of v.

Let c_{ijk} (k=1,2) be the elastic moduli for the layer R_k (k=1,2), where R_1 is the layer one $(0 \le y \le h_2)$ and R_2 is the layer two $(-h_1 \le y \le 0)$. Solving equations (4.2.4) for \bar{u}_s and \bar{v}_c , then taking the inverse Fourier transforms we obtain the the following solution:

$$u_{j}(x,y) = \mathscr{F}_{s}[(A_{j}e^{\beta_{j}\xi y} + B_{j}e^{-\beta_{j}\xi y} + C_{j}e^{\delta_{j}\xi y} + D_{j}e^{-\delta_{j}\xi y}); \xi \to x], \text{ in } R_{j}.$$
(4.2.5)
$$v_{j}(x,y) = \mathscr{F}_{c}[(A_{j}a_{j}e^{\beta_{j}\xi y} - B_{j}a_{j}e^{-\beta_{j}\xi y} + C_{j}\gamma_{j}e^{\delta_{j}\xi y} - D_{j}\gamma_{j}e^{-\delta_{j}\xi y}); \xi \to x],$$
(4.2.6)
in $R_{j},$ (4.2.6)

where \mathscr{F}_s , \mathscr{F}_c are the Fourier sine and cosine transforms respectively, while u_j and v_j (j=1,2) are the displacements for the layer R_j ; A_j , B_j , C_j and D_j (j=1,2) are unknown functions of ξ , and

$$a_{j} = \frac{c_{66j} \beta_{j} - c_{11j} \beta_{j}^{-1}}{c_{12j} + c_{66j}} , \qquad \gamma_{j} = \frac{c_{66j} \delta_{j} - c_{11j} \delta_{j}^{-1}}{c_{12j} + c_{66j}} ,$$
$$(\beta_{j}, \delta_{j}) = \left\{ \frac{\eta_{j} \pm \sqrt{\eta_{j}^{2} - 4 \cdot c_{11j} \cdot c_{22j} \cdot c_{66j}^{2}}}{2 \cdot c_{22j} \cdot c_{66j}} \right\}^{\frac{1}{2}} ,$$
$$\eta_{j} = c_{11j} \cdot c_{33j} - 2 \cdot c_{13j} \cdot c_{44j} - c_{13j}^{2} . \qquad (4.2.7)$$

Substituting from equations (4.2.5) and (4.2.6) into (4.2.2), we obtain

$$\sigma_{\mathbf{y}\mathbf{y}\mathbf{j}}(x,y) = \mathscr{F}_{\mathbf{c}}[\xi(a_{\mathbf{j}}A_{\mathbf{j}}e^{\beta_{\mathbf{j}}\xi\mathbf{y}} + a_{\mathbf{j}}B_{\mathbf{j}}e^{-\beta_{\mathbf{j}}\xi\mathbf{y}} + b_{\mathbf{j}}C_{\mathbf{j}}e^{\delta_{\mathbf{j}}\xi\mathbf{y}} + b_{\mathbf{j}}D_{\mathbf{j}}e^{-\delta_{\mathbf{j}}\xi\mathbf{y}}); \xi \rightarrow x], \text{ in } R_{\mathbf{j}},$$

$$\sigma_{\mathbf{x}\mathbf{y}\mathbf{j}}(x,y) = \mathscr{F}_{\mathbf{s}}[\xi(e_{\mathbf{j}}A_{\mathbf{j}}e^{\beta_{\mathbf{j}}\xi\mathbf{y}} - e_{\mathbf{j}}B_{\mathbf{j}}e^{-\beta_{\mathbf{j}}\xi\mathbf{y}} + d_{\mathbf{j}}C_{\mathbf{j}}e^{\delta_{\mathbf{j}}\xi\mathbf{y}} - d_{\mathbf{j}}D_{\mathbf{j}}e^{-\delta_{\mathbf{j}}\xi\mathbf{y}}); \xi \rightarrow x], \text{ in } R_{\mathbf{j}},$$

$$(4.2.8)$$

where σ_{yyj} and σ_{xyj} are stress components for the region R_j (j=1,2) and

$$\begin{aligned} a_{j} &= c_{21j} + a_{j} \beta_{j} c_{22j} , \quad b_{j} &= c_{21j} + \gamma_{j} \delta_{j} c_{22j} , \\ d_{j} &= c_{66j} (\delta_{j} - \gamma_{j}) , \quad e_{j} &= c_{66j} (\beta_{j} - a_{j}) , \quad j = 1,2 . \end{aligned}$$
(4.2.9)

4.3. Statment of the problem and derivation of the singular integral equations

We assume that two dissimilar orthotropic elastic layers, which occupy the regions R_1 ($0 \le y \le h_2$) and R_2 ($-h_1 \le y \le 0$) respectively, are perfectly

bonded except that there is a crack in the interval $-a \leq x \leq a$, y=0 (see Fig 4.3.1). It is also assumed that the boundaries $y = -h_1$ and $y = h_2$ are stress-free, and the surfaces of the crack are subjected to symmetrical normal pressure and anti-symmetrical shear with respect to y-axis. Hence the problem of determining the stress and displacement field is subjected to the following boundary and continuity conditions:

$$\sigma_{yy1}(x,0^{+}) = \sigma_{yy2}(x,0^{-}) = p_1(x), \qquad p_1(-x) = p_1(x); \qquad |x| < a, \qquad (4.3.1)$$

$$\sigma_{xy1}(x,0^+) = \sigma_{xy2}(x,0^-) = p_2(x), \quad p_2(-x) = -p_2(x); \quad |x| < a, \quad (4.3.2)$$

$$u_1(x,0^+) = u_2(x,0^-), \quad v_1(x,0^+) = v_2(x,0^-), \quad |x| > a, \quad (4.3.3)$$

$$u_1(x,0^{-}) = u_2(x,0^{-}), \quad v_1(x,0^{-}) = v_2(x,0^{-}), \quad |x| > a, \quad (4.3.3)$$

$$\sigma_{yy1}(x,0^{+}) = \sigma_{yy2}(x,0^{-}), \quad \sigma_{xy1}(x,0^{+}) = \sigma_{xy2}(x,0^{-}), \qquad |x| > a, \qquad (4.3.4)$$

and

$$\begin{aligned} \sigma_{yy1}(x,h_2) &= 0 , & \sigma_{xy1}(x,h_2) &= 0 , & -\infty < x < \infty, (4.3.5) \\ \sigma_{yy2}(x,-h_1) &= 0 , & \sigma_{xy2}(x,-h_1) &= 0 , & -\infty < x < \infty. (4.3.6) \end{aligned}$$

With the help of conditions (4.3.1) and (4.3.2), the conditions (4.3.4) may be replaced by

$$\sigma_{yy1}(x,0^{+}) = \sigma_{yy2}(x,0^{-}), \quad \sigma_{xy1}(x,0^{+}) = \sigma_{xy2}(x,0^{-}), \quad -\infty < x < \infty. \quad (4.3.4a)$$

The conditions (4.3.4a) yield

$$a_1(A_1 + B_1) + b_1(C_1 + D_1) = a_2(A_2 + B_2) + b_2(C_2 + D_2)$$
, (4.3.7)

$$e_1(A_1 - B_1) + d_1(C_1 - D_1) = e_2(A_2 - B_2) + d_2(C_2 - D_2)$$
. (4.3.8)

While the boundary conditions (4.3.5) and (4.3.6) give

$$a_1 A_1 e^{\beta_1 \xi h_2} + a_1 B_1 e^{-\beta_1 \xi h_2} + b_1 C_1 e^{\delta_1 \xi h_2} + b_1 D_1 e^{-\delta_1 \xi h_2} = 0 , \qquad (4.3.9)$$

$$e_1 A_1 e^{\beta_1 \xi h_2} - e_1 B_1 e^{-\beta_1 \xi h_2} + d_1 C_1 e^{\delta_1 \xi h_2} - d_1 D_1 e^{-\delta_1 \xi h_2} = 0 , \qquad (4.3.10)$$

$$a_2 A_2 e^{-\beta_2 \xi h_1} + a_2 B_2 e^{\beta_2 \xi h_1} + b_2 C_2 e^{-\delta_2 \xi h_1} + b_2 D_2 e^{\delta_2 \xi h_1} = 0 , \qquad (4.3.11)$$

$$e_2 A_2 e^{-\beta_2 \xi h_1} - e_2 B_2 e^{\beta_2 \xi h_1} + d_2 C_2 e^{-\delta_2 \xi h_1} - d_2 D_2 e^{\delta_2 \xi h_1} = 0 . \qquad (4.3.12)$$





Griffith crack at the interface of two dissimilar orthotropic elastic layers.

Solving equations (4.3.7) to (4.3.12) we get

$$C_{1}(\xi) = l_{11}A_{1}e^{(\beta_{1}-\delta_{1})\xih_{2}} + l_{12}B_{1}e^{-(\beta_{1}+\delta_{1})\xih_{2}},$$

$$D_{1}(\xi) = l_{12}A_{1}e^{(\beta_{1}+\delta_{1})\xih_{2}} + l_{11}B_{1}e^{(-\beta_{1}+\delta_{1})\xih_{2}},$$

$$C_{2}(\xi) = l_{21}A_{2}e^{(-\beta_{2}+\delta_{2})\xih_{1}} + l_{22}B_{2}e^{(\beta_{2}+\delta_{2})\xih_{1}},$$

$$D_{2}(\xi) = l_{22}A_{2}e^{-(\beta_{2}+\delta_{2})\xih_{1}} + l_{21}B_{2}e^{(\beta_{2}-\delta_{2})\xih_{1}},$$

(4.3.13)

and

$$A_2(\xi) = l_{31} A_1 + l_{32} B_1$$
, $B_2 = l_{41} A_1 + l_{42} B_1$, (4.3.14)

where

$$l_{1j} = -[a_1d_1 - (-1)^j e_1b_1]/(2b_1d_1) , \quad l_{2j} = -[a_2d_2 - (-1)^j e_2b_2]/(2b_2d_2) , \quad j = 1,2 .$$

$$l_{31} = (a_{11} \ b_{22} - a_{22} \ b_{11})/\Delta , \quad l_{32} = (a_{12} \ b_{22} - a_{22} \ b_{12})/\Delta ,$$

$$l_{41} = (a_{21} \ b_{11} - a_{11} \ b_{21})/\Delta , \quad l_{42} = (a_{21} \ b_{12} - a_{12} \ b_{21})/\Delta ,$$

$$\Delta = a_{21} \ b_{22} - a_{22} \ b_{21} , \qquad (4.3.15)$$

and

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$$a_{11} = a_{1} + b_{1}l_{11}e^{(\beta_{1}-\delta_{1})\xih_{2}} + b_{1}l_{12}e^{-(\beta_{1}+\delta_{1})\xih_{2}},$$

$$a_{12} = a_{1} + b_{1}l_{12}e^{-(\beta_{1}+\delta_{1})\xih_{2}} + b_{1}l_{11}e^{(-\beta_{1}+\delta_{1})\xih_{2}},$$

$$a_{21} = a_{2} + b_{2}l_{21}e^{(-\beta_{2}+\delta_{2})\xih_{1}} + b_{2}l_{22}e^{-(\beta_{2}+\delta_{2})\xih_{1}},$$

$$a_{22} = a_{2} + b_{2}l_{22}e^{(\beta_{2}+\delta_{2})\xih_{1}} + b_{2}l_{21}e^{(\beta_{2}-\delta_{2})\xih_{1}},$$

$$b_{11} = e_{1} + d_{1}l_{11}e^{(\beta_{1}-\delta_{1})\xih_{2}} - d_{1}l_{12}e^{(\beta_{1}+\delta_{1})\xih_{2}},$$

$$b_{12} = -e_{1} + d_{1}l_{12}e^{-(\beta_{1}+\delta_{1})\xih_{2}} - d_{1}l_{11}e^{(-\beta_{1}+\delta_{1})\xih_{2}},$$

$$b_{21} = e_{2} + d_{2}l_{21}e^{(-\beta_{2}+\delta_{2})\xih_{1}} - d_{2}l_{22}e^{-(\beta_{2}-\delta_{2})\xih_{1}},$$

$$b_{22} = -e_{2} + d_{2}l_{22}e^{(\beta_{2}+\delta_{2})\xih_{1}} - d_{2}l_{21}e^{(\beta_{2}-\delta_{2})\xih_{1}}.$$
(4.3.16)

Conditions (4.3.1), (4.3.2) and (4.3.3) yield the following equations:

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$$\begin{aligned} \mathscr{F}_{c}[\xi(a_{1}A_{1}+a_{1}B_{1}+b_{1}C_{1}+d_{1}D_{1}) ; \xi \rightarrow x] &= p_{1}(x) , & 0 < x < a; (4.3.17) \\ \mathscr{F}_{s}[\xi(e_{1}A_{1}-e_{1}B_{1}+d_{1}C_{1}-d_{1}D_{1}) ; \xi \rightarrow x] &= p_{2}(x) , & 0 < x < a; (4.3.18) \\ \mathscr{F}_{s}[(A_{1}+B_{1}+C_{1}+D_{1}-A_{2}-B_{2}-C_{2}-D_{2}) ; \xi \rightarrow x] &= \begin{cases} u_{1}(x,0^{+})-u_{2}(x,0^{-}), & 0 < x < a; \\ 0 , & x > a; \end{cases} \\ \end{aligned}$$

$$\begin{aligned} (4.3.19) \end{aligned}$$

$$\mathcal{F}_{c}[(a_{1}A_{1}-a_{1}B_{1}+\gamma_{1}C_{1}-\gamma_{1}D_{1}-a_{2}A_{2}+a_{2}B_{2}-\gamma_{2}C_{2}+\gamma_{2}D_{2});\xi \rightarrow x] = \begin{cases} v_{1}(x,0^{+})-v_{2}(x,0^{-}), & 0 < x < a; \\ 0, & x > a; \end{cases}$$
(4.3.20)

Differentiating (4.3.19) and (4.3.20) with respect to x and setting

$$\xi(A_1 + B_1 + C_1 + D_1 - A_2 - B_2 - C_2 - D_2) = \phi_1(\xi) \quad , \tag{4.3.21}$$

$$\xi(a_1A_1 - a_1B_1 + \gamma_1C_1 - \gamma_1D_1 - a_2A_2 + a_2B_2 - \gamma_2C_2 + \gamma_2D_2) = \phi_2(\xi) \quad , \tag{4.3.22}$$

we obtain the following equations from equations (4.3.17) to (4.3.20)

$$\begin{aligned} \mathscr{F}_{c}[(\lambda_{11}\phi_{1} + \lambda_{12}\phi_{2}) ; \xi \to x] &= p_{1}(x) , & 0 < x < a, (4.3.23) \\ \mathscr{F}_{s}[(\lambda_{21}\phi_{1} + \lambda_{22}\phi_{2}) ; \xi \to x] &= p_{2}(x) , & 0 < x < a, (4.3.24) \\ \mathscr{F}_{c}[\phi_{1}(\xi) ; \xi \to x] &= \begin{cases} \frac{\partial}{\partial x}[u_{1}(x,0^{+}) - u_{2}(x,0^{-})], & 0 < x < a, \\ 0, & x > a, \end{cases} & 0 < x < a, \\ x > a, \end{cases} & (4.3.25) \\ x > a, \end{cases} \\ \mathscr{F}_{s}[\phi_{2}(\xi) ; \xi \to x] &= \begin{cases} \frac{\partial}{\partial x}[-v_{1}(x,0^{+}) + v_{2}(x,0^{-})], & 0 < x < a, \\ 0, & x > a, \end{cases} & (4.3.26) \\ x > a, \end{cases} & (4.3.26) \end{aligned}$$

where λ_{jk} are functions of ξ and are given by

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$$\lambda_{11} = (a_{11}\gamma_{12} - a_{12}\gamma_{11})/\Delta^{*}, \ \lambda_{12} = (a_{12}\gamma_{21} - a_{11}\gamma_{22})/\Delta^{*},$$

$$\lambda_{21} = (b_{11}\gamma_{12} - b_{12}\gamma_{11})/\Delta^{*}, \ \lambda_{22} = (b_{12}\gamma_{21} - b_{11}\gamma_{22})/\Delta^{*},$$

$$\Delta^{*} = \gamma_{12} \gamma_{21} - \gamma_{11} \gamma_{22}, \qquad (4.3.27)$$

and γ_{jk} 's are given by the following equations

$$\gamma_{11} = a_{1} - \gamma_{1} l_{11} e^{(\beta_{1} - \delta_{1}) \xi h_{2}} + \gamma_{1} l_{12} e^{(\beta_{1} + \delta_{1}) \xi h_{2}} - a_{2} l_{31} + a_{2} l_{41} - \gamma_{2} l_{51} + \gamma_{2} l_{61} ,$$

$$\gamma_{12} = -a_{1} - \gamma_{1} l_{12} e^{-(\beta_{1} + \delta_{1}) \xi h_{2}} + \gamma_{1} l_{11} e^{(-\beta_{1} + \delta_{1}) \xi h_{2}} - a_{2} l_{32} + a_{2} l_{42} - \gamma_{2} l_{52} + \gamma_{2} l_{62} ,$$

$$\gamma_{21} = 1 - l_{11} e^{(\beta_{1} - \delta_{1}) \xi h_{2}} - l_{12} e^{(\beta_{1} + \delta_{1}) \xi h_{2}} - l_{31} - l_{41} - l_{51} - l_{61} ,$$

$$\gamma_{22} = 1 - l_{12} e^{-(\beta_{1} + \delta_{1}) \xi h_{2}} - l_{11} e^{(-\beta_{1} + \delta_{1}) \xi h_{2}} - l_{32} - l_{42} - l_{52} - l_{62} ,$$

$$l_{51} = -l_{21} l_{31} e^{(-\beta_{2} + \delta_{2}) \xi h_{1}} - l_{22} l_{41} e^{(\beta_{2} + \delta_{2}) \xi h_{1}} ,$$

$$l_{52} = -l_{21} l_{32} e^{-(\beta_{2} + \delta_{2}) \xi h_{1}} - l_{22} l_{42} e^{(\beta_{2} - \delta_{2}) \xi h_{1}} ,$$

$$l_{61} = -l_{22} l_{31} e^{-(\beta_{2} + \delta_{2}) \xi h_{1}} - l_{21} l_{41} e^{(\beta_{2} - \delta_{2}) \xi h_{1}} ,$$

$$l_{62} = -l_{22} l_{32} e^{-(\beta_{2} + \delta_{2}) \xi h_{1}} - l_{21} l_{42} e^{(\beta_{2} - \delta_{2}) \xi h_{1}} .$$

$$(4.3.28)$$

It is easy to verify that when
$$\xi \to \infty$$
, $\lambda_{jk}(\xi)$ tends to constant, say $\lambda_{jk}(\infty)$, and
 $\lambda_{jk}(\xi) - \lambda_{jk}(\infty) = O(e^{-(\beta_1 + \delta_1) \xi h_2})$. (4.3.29)

Now suppose that the dislocations at y=0 are f_1 and f_2 , i.e

$$\frac{\partial}{\partial x}[u_1(x,0^+)-u_2(x,0^-)] = \begin{cases} f_1(x), & |x| < a; \\ 0, & |x| > a; \end{cases}$$
(4.3.30)

$$\frac{\partial}{\partial x}[-v_1(x,0^+)+v_2(x,0^-)] = \begin{cases} f_2(x), & |x| < a; \\ 0, & |x| > a; \end{cases}$$
(4.3.31)

and

$$f_1(-x) = f_1(x)$$
, $f_2(-x) = -f_2(x)$, $|x| < a.$ (4.3.32)

By taking s=x/a, equations (4.3.23) to (4.3.26) can be written in the following form

$$\mathcal{F}_{c}[(\tilde{\lambda}_{11}\tilde{\phi}_{1} + \tilde{\lambda}_{12}\tilde{\phi}_{2}) ; \xi \to s] = \tilde{p}_{1}(s) , \qquad 0 < s < 1, \quad (4.3.33)$$
$$\mathcal{F}_{s}[(\tilde{\lambda}_{21}\tilde{\phi}_{1} + \tilde{\lambda}_{22}\tilde{\phi}_{2}) ; \xi \to s] = \tilde{p}_{2}(s) , \qquad 0 < s < 1, \quad (4.3.34)$$

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$$\mathcal{F}_{c}[\tilde{\phi}_{1} ; \xi \rightarrow s] = \begin{cases} \frac{\partial}{\partial s} [\tilde{u}_{1}(s,0^{+}) - \tilde{u}_{2}(s,0^{-})] , & 0 < s < 1; \\ 0 , & s > 1; \end{cases}$$
(4.3.35)
$$\mathcal{F}_{s}[\tilde{\phi}_{2} ; \xi \rightarrow s] = \begin{cases} \frac{\partial}{\partial s} [-\tilde{v}_{1}(s,0^{+}) + \tilde{v}_{2}(s,0^{-})] , & 0 < s < 1; \\ 0 , & s > 1; \end{cases}$$
(4.3.36)

where

$$\tilde{p}_{j}(s) = a p_{j}(as) , \quad \tilde{\lambda}_{jk}(\xi) = \lambda_{jk}(\xi/a) , \quad \tilde{\phi}_{j}(\xi) = \phi_{j}(\xi/a) ,$$

 $\tilde{u}_{j}(s) = u_{j}(as) , \quad \tilde{v}_{j}(s) = v_{j}(as) , \quad j,k=1,2 .$ (4.3.37)

Now equations (4.3.30) and (4.3.31) can be written in the following form

$$\frac{\partial}{\partial s} [\tilde{u}_{1}(s,0^{+}) - \tilde{u}_{2}(s,0^{-})] = \begin{cases} \tilde{f}_{1}(s), & |s| < 1; \\ 0, & |s| > 1; \end{cases}$$
(4.3.38)

$$\frac{\partial}{\partial s} [\tilde{v}_1(s,0^+) - \tilde{v}_2(s,0^-)] = \begin{cases} -\tilde{f}_2(s), & |s| < 1; \\ 0, & |s| > 1; \end{cases}$$
(4.3.39)

where

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$$\tilde{f}_{1}(-s) = \tilde{f}_{1}(s) , \quad \tilde{f}_{2}(-s) = -\tilde{f}_{2}(s) , \qquad |s| < 1 ,$$

 $\tilde{f}_{j}(s) = a f_{j}(as) , \qquad j = 1,2 . \qquad (4.3.40)$

The equations (4.3.35) and (4.3.36) are satisfied automatically and we have

$$\tilde{\phi}_{1}(\xi) = (2/\pi)^{\frac{1}{2}} \int_{0}^{1} \tilde{f}_{1}(s) \cos(\xi s) ds , \qquad (4.3.41)$$

$$\tilde{\phi}_2(\xi) = (2/\pi)^{\frac{1}{2}} \int_0^1 \tilde{f}_2(s) \sin(\xi s) \, ds \quad . \tag{4.3.42}$$

Using integration by parts, from equation (4.3.41) we get

$$\tilde{\phi}_{1}(\xi) = -(2/\pi)^{\frac{1}{2}} \xi^{-1} \int_{0}^{1} \frac{d}{ds} \tilde{f}_{1}(s) \sin(\xi s) ds,$$

then using the result [44] that

$$\mathscr{F}_{\mathbf{s}}[\xi^{-1}sin(\xi\tau); \xi \rightarrow s] = \frac{1}{2} \left(2/\pi\right)^{\frac{1}{2}} \log \left|\frac{s+\tau}{s-\tau}\right|,$$

we see that

$$\mathscr{F}_{s}[\tilde{\phi}_{1}(\xi) \; ; \; \xi \rightarrow s] \; = \; - \; \frac{1}{\pi} \; \int_{0}^{1} \tilde{f}_{1}(\tau) \Big[\frac{1}{s \; + \; \tau} \; + \; \frac{1}{s \; - \; \tau} \Big] \, d\tau.$$

Since $\tilde{f}_1(s)$ is an even function, we have

$$\mathscr{F}_{\mathbf{s}}[\tilde{\phi}_{1}(\xi) \ ; \ \xi \rightarrow s] = -\frac{1}{\pi} \int_{-1}^{1} \frac{\tilde{f}_{1}(t)}{t-s} \ dt. \qquad -1 \ < \ s \ < \ 1, \ (4.3.43)$$

Similarly, from equation (4.3.42) we get

$$\mathscr{F}_{\mathbf{c}}[\tilde{\phi}_{2}(\xi) \; ; \; \xi \rightarrow s] \; = \; \frac{1}{\pi} \; \int_{-1}^{1} \frac{\tilde{f}_{2}(t)}{t - s} \; dt. \qquad -1 \; < \; s \; < \; 1. \; (4.3.44)$$

For the continuity of the displacements on y=0 , |x|>a ; f_1 and f_2 must satisfy the following conditions

$$\int_{-a}^{a} f_{j}(x) dx = 0 , \text{ or } \int_{-1}^{1} \tilde{f}_{j}(s) ds = 0 , \qquad j = 1,2 . \quad (4.3.45)$$

(Note, for j=2, condition (4.3.45) holds obviously since $f_2(x)$ is an odd function) Substituting for $\tilde{\phi}_1$, $\tilde{\phi}_2$ from equations (4.3.41) and (4.3.42) into equations (4.3.33) and (4.3.34), we obtain

$$\lambda_{11}(\varpi) \tilde{f}_{1}(s) + \frac{\lambda_{12}(\varpi)}{\pi} \int_{-1}^{1} \frac{\tilde{f}_{2}(t)}{t-s} dt + \int_{-1}^{1} \tilde{f}_{1}(t) K_{11}(s,t) dt + \int_{-1}^{1} \tilde{f}_{2}(t) K_{12}(s,t) dt = \tilde{p}_{1}(s) , \qquad |s| < 1; \qquad (4.3.46)$$
$$\lambda_{22}(\varpi) \tilde{f}_{2}(s) - \frac{\lambda_{21}(\varpi)}{\pi} \int_{-1}^{1} \frac{\tilde{f}_{1}(t)}{t-s} dt + \int_{-1}^{1} \tilde{f}_{1}(t) K_{21}(s,t) dt + \int_{-1}^{1} \tilde{f}_{2}(t) K_{22}(s,t) dt = \tilde{p}_{2}(s) , \qquad |s| < 1; \qquad (4.3.47)$$

where

$$K_{11}(s,t) = \frac{1}{\pi} \int_{0}^{\infty} [\tilde{\lambda}_{11}(\xi) - \lambda_{11}(\omega)] \cos(\xi t) \cos(\xi s) d\xi ,$$

$$K_{12}(s,t) = \frac{1}{\pi} \int_{0}^{\infty} [\tilde{\lambda}_{12}(\xi) - \lambda_{12}(\omega)] \sin(\xi t) \cos(\xi s) d\xi ,$$

$$K_{21}(s,t) = \frac{1}{\pi} \int_{0}^{\infty} [\tilde{\lambda}_{21}(\xi) - \lambda_{21}(\omega)] \cos(\xi t) \sin(\xi s) d\xi ,$$

$$K_{22}(s,t) = \frac{1}{\pi} \int_{0}^{\infty} [\tilde{\lambda}_{22}(\xi) - \lambda_{22}(\omega)] \sin(\xi t) \sin(\xi s) d\xi .$$
(4.3.48)

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Equations (4.3.46) and (4.3.47) can be rewritten in the following forms:

$$\begin{split} \lambda\psi_{1}(s) + \frac{1}{\pi i J} \int_{-1}^{1} \frac{\psi_{1}(t)}{t-s} dt + \int_{-1}^{1} \psi_{1}(t) M_{11}(s,t) dt + \int_{-1}^{1} \psi_{2}(t) M_{12}(s,t) dt = g_{1}(s) , \\ |s| < 1; \qquad |s| < 1; \qquad (4.3.49) \\ \lambda\psi_{2}(s) - \frac{1}{\pi i J} \int_{-1}^{1} \frac{\psi_{2}(t)}{t-s} dt + \int_{-1}^{1} \psi_{1}(t) M_{21}(s,t) dt + \int_{-1}^{1} \psi_{2}(t) M_{22}(s,t) dt = g_{2}(s) , \\ |s| < 1; \qquad |s| < 1; \qquad (4.3.50) \end{split}$$

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where

$$\begin{split} \lambda_{1} &= \lambda_{11}(\varpi)/\lambda_{12}(\varpi) \ , \ \lambda_{2} &= \lambda_{22}(\varpi)/\lambda_{21}(\varpi) \ , \ \lambda &= \sqrt{\lambda_{1} \ \lambda_{2}} \ , \\ \psi_{1}(s) &= \sqrt{\lambda_{1}} \ \tilde{f}_{1}(s) \ + \ i\sqrt{\lambda_{2}} \ \tilde{f}_{2}(s) \ , \ \psi_{2}(s) &= \sqrt{\lambda_{1}} \ \tilde{f}_{1}(s) \ - \ i\sqrt{\lambda_{2}} \ \tilde{f}_{2}(s) \ , \\ g_{1}(s) &= \sqrt{\lambda_{2}} \ \tilde{p}_{1}(s)/\lambda_{12}(\varpi) \ + \ i\sqrt{\lambda_{1}} \ \tilde{p}_{2}(s)/\lambda_{21}(\varpi) \ , \\ g_{2}(s) &= \sqrt{\lambda_{2}} \ \tilde{p}_{1}(s)/\lambda_{12}(\varpi) \ - \ i\sqrt{\lambda_{1}} \ \tilde{p}_{2}(s)/\lambda_{21}(\varpi) \ , \\ M_{11}(s,t) &= \frac{1}{2} \ \left\{ \frac{\sqrt{\lambda_{1}}}{\lambda_{21}(\varpi)\sqrt{\lambda_{2}}} \ K_{22}(s,t) \ + \ \frac{\sqrt{\lambda_{2}}}{\lambda_{12}(\varpi)\sqrt{\lambda_{1}}} \ K_{11}(s,t) \\ &+ \ i \ \left(\frac{K_{21}(s,t)}{\lambda_{21}(\varpi)} \ - \ \frac{K_{12}(s,t)}{\lambda_{12}(\varpi)} \right) \right\} \ , \\ M_{12}(s,t) &= \frac{1}{2} \ \left\{ \frac{-\sqrt{\lambda_{1}}}{\lambda_{21}(\varpi)\sqrt{\lambda_{2}}} \ K_{22}(s,t) \ + \ \frac{\sqrt{\lambda_{2}}}{\lambda_{12}(\varpi)\sqrt{\lambda_{1}}} \ K_{11}(s,t) \\ &+ \ i \ \left(\frac{K_{21}(s,t)}{\lambda_{21}(\varpi)} \ + \ \frac{K_{12}(s,t)}{\lambda_{12}(\varpi)} \right) \right\} \ , \end{split}$$

$$M_{21}(s,t) = \frac{1}{2} \left\{ \frac{-\sqrt{\lambda_1}}{\lambda_{21}(\varpi)\sqrt{\lambda_2}} K_{22}(s,t) + \frac{\sqrt{\lambda_2}}{\lambda_{12}(\varpi)\sqrt{\lambda_1}} K_{11}(s,t) - i \left(\frac{K_{12}(s,t)}{\lambda_{12}(\varpi)} + \frac{K_{21}(s,t)}{\lambda_{21}(\varpi)}\right) \right\},$$

$$M_{22}(s,t) = \frac{1}{2} \left\{ \frac{\sqrt{\lambda_1}}{\lambda_{21}(\varpi)\sqrt{\lambda_2}} K_{22}(s,t) + \frac{\sqrt{\lambda_2}}{\lambda_{12}(\varpi)\sqrt{\lambda_1}} K_{11}(s,t) + i \left(\frac{K_{12}(s,t)}{\lambda_{12}(\varpi)} - \frac{K_{21}(s,t)}{\lambda_{21}(\varpi)}\right) \right\}.$$
(4.3.51)

The analytic solution of equations (4.3.49) and (4.3.50) has been extensively studied (see, for exemple, [67] and [68]) by using regularization method, which, in this case, however becomes cumbersome. Here we try to use an approximation method described by Erdogan [69] to find the stress intensity factors.

Since the kernels M_{jk} (j,k=1,2) are bounded, we know, aside from a multiplication constant, the singular behavior of the functions ψ_1 and ψ_2 at the points $s = \pm 1$ is determined by the dominant part of the singular integral equations. The equations (4.3.49) and (4.3.50) will be solved under the assumption that ψ_1 and ψ_2 satisfy a Hölder condition on every closed part of the interval (-1, 1) not containing the ends.

The solution of the equations (4.3.49) and (4.3.50) may be assumed in the form of Jacobi polynomials $P_n^{(\sigma_k, \tau_k)}(s)$ [60] by

$$\psi_{k} = \sum_{n=1}^{\infty} C_{kn} W_{k}(s) P_{n}(\sigma_{k}, \tau_{k})(s) , \qquad k=1,2 ; \qquad (4.3.52)$$

where

$$W_{k}(s) = (1-s)^{\sigma_{k}}(1+s)^{\tau_{k}}, \quad \sigma_{k} = -\frac{1}{2} + i\omega_{k}, \quad \tau_{k} = -\frac{1}{2} - i\omega_{k},$$

$$\omega_{k} = (-1)^{k+1}\omega, \quad \omega = \frac{1}{2\pi}\ln\left[\frac{1+\lambda}{1-\lambda}\right], \quad k = 1,2; \quad (4.3.53)$$

and C_{kn} are unknown coefficients.

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Observing that $P_0^{(\sigma_k,\tau_k)}(s)=1$ and the orthogonality relations of Jacobi polynomials

$$\int_{-1}^{1} W(t) P_{n}(\sigma,\tau)(t) P_{m}(\sigma,\tau)(t) dt = \begin{cases} 0 & n \neq m , \\ \theta_{m}(\sigma,\tau) & n = m , \end{cases}$$
$$\theta_{m}^{(\sigma,\tau)} = \frac{2^{\sigma+\tau+1} \Gamma(m+\sigma+1)\Gamma(m+\tau+1)}{m! (2m+\sigma+\tau+1)\Gamma(m+\sigma+\tau+1)}; \qquad (4.3.54)$$

we conclude that by choosing $C_{k0}=0$, k=1,2, the condition (4.3.45) will be satisfied automatically.

Using the following relation [70]

$$\frac{1}{\pi i \int_{-1}^{1} W_{k}(t) P_{n}^{(\sigma_{k},\tau_{k})}(t) \frac{dt}{t-s} + (-1)^{k} W_{k}(s) P_{n}^{(\sigma_{k},\tau_{k})}(s) \\
= \begin{cases} -\frac{(1-\lambda^{2})^{\frac{1}{2}}}{2i} P_{n-1}^{(-\sigma_{k},-\tau_{k})}(s) , & |s| < 1 ; \\ -\frac{(1-\lambda^{2})^{\frac{1}{2}}}{2i} [(s-1)^{\sigma_{k}}(s+1)^{\tau_{k}} P_{n}^{(\sigma_{k},\tau_{k})}(x) + G_{kn}^{0}(s)], & |s| > 1 ; \end{cases}$$
(4.3.55)

where $G_{kn}^{\infty}(s)$ is the principal part of $W_k(s)P_n^{(\sigma_k,\tau_k)}(s)$ at infinity, and substituting from equation (4.3.52) into equations (4.3.49) and (4.3.50) we obtain

$$\sum_{n=1}^{\infty} C_{1n} \frac{(1-\lambda^2)^{\frac{1}{2}}}{2i} P_{n-1}^{(-\sigma_1,-\tau_1)}(s) + \sum_{n=1}^{\infty} [C_{1n} L_{11n}(s,t) + C_{2n} L_{12n}(s,t)] = g_1(s) ,$$

$$|s| < 1; \qquad (4.3.56)$$

$$\sum_{n=1}^{\infty} C_{2n} \frac{(1-\lambda^2)^{\frac{1}{2}}}{2i} P_{n-1}^{(-\sigma_2,-\tau_2)}(s) - \sum_{n=1}^{\infty} [C_{1n}L_{21n}(s,t) + C_{2n}L_{22n}(s,t)] = -g_2(s) ,$$

$$|s| < 1; \qquad (4.3.57)$$

where

$$L_{\rm kmn}(s) = \int_{-1}^{1} M_{\rm km}(s,t) W_{\rm m}(t) P_{\rm n}(\sigma_{\rm m},\tau_{\rm m})(t) dt, \qquad k,m=1,2; \ n=1,2,3...$$
(4.3.58)

Multiplying equations (4.3.56) and (4.3.57) by $W_1^{-1}(s)P_j^{(-\sigma_1,-\tau_1)}(s)$ and $W_2^{-1}(s)P_j^{(-\sigma_2,-\tau_2)}(s)$ respectively, integrating from -1 to 1, and using the orthogonality relations (4.3.54) we obtain the following algebraic equations for the determination of $C_{\rm km}$

$$\frac{(1-\lambda^2)^{\frac{1}{2}}}{2i}C_{1(j+1)} \quad \theta_j^{(-\sigma_1,-\tau_1)} + \sum_{n=1}^{\infty} (L_{1\ln j}^*C_{1n} + L_{12nj}^*C_{2n}) = q_{1j} \quad , \qquad (4.3.59)^{\frac{1}{2}}$$

$$\frac{(1-\lambda^2)^{\frac{1}{2}}}{2i}C_{2(j+1)} \quad \theta_j^{(-\sigma_2,-\tau_2)} - \sum_{n=1}^{\infty} (L_{2\ln j}^* C_{\ln} + L_{22nj}^* C_{2n}) = q_{2j} \quad , \tag{4.3.60}$$

where

$$L_{\rm kmnj}^{*} = \int_{-1}^{1} L_{\rm kmn}(s) W_{\rm k}^{-1}(s) P_{\rm j}^{(-\sigma_{\rm k},-\tau_{\rm k})}(s) ds ,$$

$$q_{\rm kj} = \int_{-1}^{1} g_{\rm k}(s) W_{\rm k}^{-1}(s) P_{\rm j}^{(-\sigma_{\rm k},-\tau_{\rm k})}(s) ds , \ k,m=1,2; \ n,j=1,2,3,...$$
(4.3.61)

After solving linear equations (4.3.59) and (4.3.60) for the unknowns C_{kn} , k=1,2, n,j=1,2,3...N; we can calculate the stress intensity factors for the crack. The stress intensity factors K_1 and K_2 may be calculated as follows:

$$\frac{\sqrt{\lambda_2}}{\lambda_{12}(\varpi)}K_1 + (-1)^{\mathbf{k}+1}i\frac{\sqrt{\lambda_1}}{\lambda_{21}(\varpi)}K_2$$

= $\lim_{x \to a^+} (x-a)^{\sigma_{\mathbf{k}}}(x+a)^{\tau_{\mathbf{k}}}[\frac{\sqrt{\lambda_2}}{\lambda_{12}(\varpi)}\sigma_{\mathbf{yy1}}(x,0^+) + (-1)^{\mathbf{k}+1}i\frac{\sqrt{\lambda_1}}{\lambda_{21}(\varpi)}\sigma_{\mathbf{xy1}}(x,0^+)].$ (4.3.62)

By making the substitution s=x/a, writing $\sigma_{yy1}(x,0^+)$ and $\sigma_{xy1}(x,0^+)$ in terms of ψ_1 and ψ_2 , and using the equations (4.3.52) and (4.3.53) we obtain

$$\frac{\sqrt{\lambda_{2}}}{\lambda_{12}(\omega)}K_{1} + (-1)^{k+1}i\frac{\sqrt{\lambda_{1}}}{\lambda_{21}(\omega)}K_{2}$$

$$= \lim_{s \to 1^{+}} (s-1)^{-\sigma_{k}}(s+1)^{-\tau_{k}} \frac{(-1)^{k+1}}{\tau_{i}} \int_{-1}^{1} \sum_{n=1}^{\infty} C_{kn}P_{n}^{(\sigma_{k},\tau_{k})}(t) \frac{W_{k}(t)}{t-s} dt$$

$$= i(-1)^{k+1} \sum_{n=1}^{\infty} C_{kn}P_{n}^{(\sigma_{k},\tau_{k})}(1) , \qquad (4.3.63)$$

where we have used the fact that $G_{kn}^{\infty}(s)$, the principal part of $W_k(s)P_n^{(\sigma_k,\tau_k)}(s)$, is bounded.

4.4. Other cases

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In section 4.3, we have considered the crack problem when the two boundaries of the layers $y = -h_1$ and $y = h_2$ are stress-free. We will study below some other possible boundary conditions.

4.4.1 One face fixed and the other stress-free.

In this case, we are assuming that the boundary conditions (4.3.6) in section 4.3 are replaced by

$$u_2(x,-h_1) = 0$$
, $v_2(x,-h_1) = 0$, (4.4.1)

while other conditions are kept the same. Hence the equations (4.3.11) and (4.3.12) should be replaced by

$$A_2 e^{-\beta_2 \xi h_1} + B_2 e^{\beta_2 \xi h_1} + C_2 e^{-\delta_2 \xi h_1} + D_2 e^{\delta_2 \xi h_1} = 0 , \qquad (4.4.2)$$

$$A_{2}a_{2}e^{-\beta_{2}\xih_{1}} - B_{2}a_{2}e^{\beta_{2}\xih_{1}} + C_{2}\gamma_{2}e^{-\delta_{2}\xih_{1}} - D_{2}\gamma_{2}e^{\delta_{2}\xih_{1}} = 0 \quad .$$
 (4.4.3)

Solving equations (4.3.9), (4.3.10), (4.4.2) and (4.4.3) we get the same expressions (4.3.13) for C_i and D_i (*i*=1,2) with following different values for the coefficients l_{jk} (*j*,*k*=1,2)

$$l_{11} = -(a_1d_1 + e_1b_1)/(2b_1d_1) , \quad l_{12} = -(a_1d_1 - e_1b_1)/(2b_1d_1) ,$$

$$l_{21} = -(\gamma_2 + a_2)/(2\gamma_2) , \quad l_{22} = -(\gamma_2 - a_2)/(2\gamma_2) . \qquad (4.4.4)$$

Hence the solution for this case is given by the results of section 4.3 when l_{jk} (j,k=1,2) in equations (4.3.15) are replaced by their values in equations (4.4.4).

4.4.2 Both faces fixed

In this case, we are assuming that the boundary conditions (4.3.6) and (4.3.5) in section 4.3 are replaced respectively by conditions (4.4.1) and the following two conditions

$$u_1(x,h_2) = 0$$
, $v_1(x,h_2) = 0$, (4.4.5)

while other conditions are kept the same. So besides the replacement of equations (4.3.11) and (4.3.12) by equations (4.4.2) and (4.4.3), we should replace equations (4.3.9) and (4.3.10) by the following two equations

$$A_{1}e^{\beta_{1}\xi\mathbf{h}_{2}} + B_{1}e^{-\beta_{1}\xi\mathbf{h}_{2}} + C_{1}e^{\delta_{1}\xi\mathbf{h}_{2}} + D_{1}e^{-\delta_{1}\xi\mathbf{h}_{2}} = 0 , \qquad (4.4.6)$$

$$A_{1}a_{1}e^{\beta_{1}\xih_{2}}-B_{1}a_{1}e^{-\beta_{1}\xih_{2}}+C_{1}\gamma_{1}e^{\delta_{1}\xih_{2}}-D_{1}\gamma_{1}e^{-\delta_{1}\xih_{2}}=0.$$
(4.4.7)

Solving equations (4.4.2), (4.4.3), (4.4.6) and (4.4.7) we find that C_j and D_j (j=1,2) are given by equations (4.3.13) as before but with the following values

for the coefficients l_{jk} (j,k=1,2):

$$l_{11} = -(\gamma_1 + \alpha_1)/(2\gamma_1) , \quad l_{12} = -(\gamma_1 - \alpha_1)/(2\gamma_1) ,$$

$$l_{21} = -(\gamma_2 + \alpha_2)/(2\gamma_2) , \quad l_{22} = -(\gamma_2 - \alpha_2)/(2\gamma_2) . \qquad (4.4.8)$$

Again the solution for this case is given by the results of section 4.3 when l_{jk} (j,k=1,2,) in equations (4.3.15) are replaced by their values in equations (4.4.8).

4.4.3 One face rigidly restrained and the other fixed

In this case, we are assuming that the boundary conditions (4.3.5) and (4.3.6) of section 4.3 are replaced by the following boundary conditions

$$v_1(x,h_2) = 0$$
 , $\sigma_{xy1}(x,h_2) = 0$; (4.4.9)

$$u_2(x,-h_1) = 0$$
, $v_2(x,-h_1) = 0$, (4.4.10)

while other conditions are kept the same. So the equations (4.3.9), (4.3.11)and (4.3.12) in section 4.3 will be replaced by (4.4.7), (4.4.2) and (4.4.3). Solving equations (4.3.7), (4.3.8), (4.4.7), (4.3.10), (4.4.2) and (4.4.3) we can express B_1 , D_1 , A_2 , B_2 , C_2 and D_2 in terms of A_1 and C_1 by

$$B_{1} = A_{1}e^{2\beta_{1}\xih_{2}}, \quad D_{1} = C_{1}e^{2\delta_{1}\xih_{2}},$$

$$C_{2} = l_{21}A_{2}e^{(-\beta_{2}+\delta_{2})\xih_{1}} + l_{22}B_{2}e^{(\beta_{2}+\delta_{2})\xih_{1}},$$

$$D_{2} = l_{22}A_{2}e^{-(\beta_{2}+\delta_{2})\xih_{1}} + l_{21}B_{2}e^{(\beta_{2}-\delta_{2})\xih_{1}},$$

$$A_{2} = l_{31}A_{1} + l_{32}C_{1}, \quad B_{2} = l_{41}A_{1} + l_{42}C_{1},$$

$$(4.4.11)$$

where l_{21} and l_{22} are given by (4.4.4) but l_{jk} (j=3,4; k=1,2.) are given by (4.3.15) while the expression for a_{jk} and b_{jk} (j,k=1,2) are given by the

following:

$$a_{11} = a_{1}(1 + e^{2\beta_{1}\xih_{2}}) , \qquad a_{12} = b_{1}(1 + e^{2\delta_{1}\xih_{2}}) ,$$

$$a_{21} = a_{2} + b_{2}l_{21}e^{(-\beta_{2}+\delta_{2})\xih_{1}} , \qquad a_{22} = a_{2} + b_{2}l_{22}e^{(-\beta_{2}+\delta_{2})\xih_{1}} ,$$

$$b_{11} = e_{1}(1 - e^{2\beta_{1}\xih_{2}}) , \qquad b_{12} = d_{1}(1 - e^{2\delta_{1}\xih_{2}}) ,$$

$$b_{21} = e_{2} + d_{2}l_{21}e^{(-\beta_{2}+\delta_{2})\xih_{1}} , \qquad b_{22} = -e_{2} - d_{2}l_{22}e^{(-\beta_{2}+\delta_{2})\xih_{1}} . \qquad (4.4.12)$$

Consequently the expression for γ_{jk} (j,k=1,2) for this case are given by

$$\gamma_{11} = a_1 - a_1 e^{2\beta_1 \xi h_2} - a_2 l_{31} + a_2 l_{41} - \gamma_2 l_{51} + \gamma_2 l_{61} ,$$

$$\gamma_{12} = \gamma_1 - \gamma_1 e^{2\delta_1 \xi h_2} - a_2 l_{32} + a_2 l_{42} - \gamma_2 l_{52} + \gamma_2 l_{62} ,$$

$$\gamma_{21} = 1 + e^{2\beta_1 \xi h_2} - l_{31} - l_{41} - l_{51} - l_{61} ,$$

$$\gamma_{22} = 1 + e^{2\delta_1 \xi h_2} - l_{32} - l_{42} - l_{52} - l_{62} ,$$

(4.4.13)

where l_{jk} (j=5,6; k=1,2) are the same as given in (4.3.28). The solution for this case is given by the results of section 4.3 after we have made above raplacements.

4.4.4 One face rigidly restrained and the other stress-free

In this case, the boundary conditions (4.3.5) are replaced by (4.4.9) and all the other boundary conditions remain the same as in section 4.3. So in equations (4.3.7) to (4.3.12) we only have to replace equation (4.3.9) by (4.4.7). Now we find that for this case l_{jk} (j=2,3,4; k=1,2) are all given by (4.3.15) and a_{jk},b_{jk} (j,k=1,2) are given by (4.4.12) and the solution for this case is given by the results of section 4.3 after we have made above modifications.

4.4.5 Both faces rigidly restrained

In this case, we are assuming that the boundary conditions (4.3.5) and (4.3.6) are respectively replaced by conditions (4.4.9) and

$$v_2(x,-h_1)=0$$
, $\sigma_{xy2}(x,-h_1)=0$, (4.4.14)

while all other conditions remain the same as in section 4.3. The equations (4.3.9) and (4.3.11) will be replaced by equations (4.4.7) and (4.4.3) while all other equations remain the same. Solving equations (4.3.7), (4.3.8), (4.4.8), (4.3.10), (4.4.3) and (4.3.12) we find that

$$B_{1} = A_{1}e^{2\beta_{1}\xih_{2}}, \quad D_{1} = C_{1}e^{2\delta_{1}\xih_{2}},$$

$$B_{2} = A_{2}e^{-2\beta_{2}\xih_{1}}, \quad D_{2} = C_{2}e^{-2\delta_{2}\xih_{1}},$$

$$A_{2} = l_{31}A_{1} + l_{32}C_{1}, \quad C_{2} = l_{41}A_{1} + l_{42}C_{1}, \qquad (4.4.15)$$

where l_{jk} (j=3,4; k=1,2.) are given by (4.3.15) but the expression for a_{jk} and b_{jk} (j,k=1,2) are given by the following

$$a_{11} = a_1(1 + e^{2\beta_1 \xi h_2}) , \quad a_{12} = b_1(1 + e^{2\delta_1 \xi h_2}) ,$$

$$a_{21} = a_2(1 + e^{-2\beta_2 \xi h_1}) , \quad a_{22} = b_2(1 + e^{-2\delta_2 \xi h_1}) ,$$

$$b_{11} = e_1(1 - e^{2\beta_1 \xi h_2}) , \quad b_{12} = d_1(1 - e^{2\delta_1 \xi h_2}) ,$$

$$b_{21} = e_2(1 - e^{-2\beta_2 \xi h_1}) , \quad b_{22} = d_2(1 - e^{-2\delta_2 \xi h_1}) .$$
(4.4.16)
Consequently for this case γ_{jk} (*j*,*k*=1,2) are given by

$$\gamma_{11} = a_1 (1 - e^{2\beta_1 \xi h_2}) - a_2 l_{31} (1 - e^{-2\beta_2 \xi h_1}) - \gamma_2 l_{41} (1 - e^{-2\delta_2 \xi h_1}) ,$$

$$\gamma_{12} = \gamma_1 (1 - e^{2\delta_1 \xi h_2}) - a_2 l_{32} (1 - e^{-2\delta_2 \xi h_1}) - \gamma_2 l_{42} (1 - e^{-2\delta_2 \xi h_1}) ,$$

$$\gamma_{21} = 1 + e^{2\beta_1 \xi h_2} - l_{31} (1 + e^{-2\beta_2 \xi h_1}) - l_{41} (1 + e^{-2\delta_2 \xi h_1}) ,$$

$$\gamma_{22} = 1 + e^{2\delta_1 \xi h_2} - l_{32} (1 + e^{-2\delta_2 \xi h_1}) - l_{42} (1 + e^{-2\delta_2 \xi h_1}) .$$

(4.4.17)

Then the solution for this case is given by the results of section 4.3.

4.5. Numerical results and discussion

To evaluate the stress intensity factors, we truncate the infinite system of simultaneous algebraic equations (4.3.59) and (4.3.60) at n = 10 and the Crout's factorisation method is used to solve these equations. And the Gaussian quadrature formula is used to perform the numerical integrations involved in the solution. The relative error is controlled under 0.01. Numerical results for the stress intensity factors K_1 and K_2 are obtained for the case when the crack is subjected to a constant pressure $p_1(x) = p_0$ and $p_2(x) = 0$, the thickness of the layers is the same (i.e., $h_1 = h_2 = h$) and the surfaces y = -h and y = h are stress-free. For the two orthotropic elastic considered here , following: materials the elastic moduli are the $(10^{11} dynes/cm^2)[5]$

	<i>c</i> ₁₁	c_{12}	C ₂₂	C66	
Beechwood	0.170	0.150	1.580	0.103	(layer 1)
a–Uranium	21.47	4.05	19.86	7.43	(layer 2)

Numerical values of the stress intensity factors have been calculated for following four particular cases:

Case 1. Let the length of the crack a = 1.0 and let $h_1/a = h_2/a = h/a = 0.2(0.2)1.0, 2.0(2.0)10.0$; the numerical values of the stress intensity factor K_1 against h/a are displayed in Fig.4.5.1 and Fig.4.5.2, and the numerical values of the stress intensity factor K_2 against h/a are displayed in Fig.4.5.3 and Fig.4.5.4.

Case 2. Let $h_1 = h_2 \rightarrow \infty$, and the length of the crack a = 1.0(1.0)10.0; the numerical values of the stress intensity factor K_1 against a are displayed in Fig.4.5.5, and the numerical values of the stress intensity factor K_2 against a are displayed in Fig.4.5.6.

Case 3. Let $h_1=h_2=20.0$ and the length of the crack a = 1.0(1.0)10.0; the numerical values of the stress intensity factor K_1 against a are displayed in Fig.4.5.7, and the numerical valued of the stress intensity factor K_2 against a are displayed in Fig.4.5.8.

Case 4. Let $h_1=h_2=10.0$ and the length of the crack a = 1.0(1.0)10.0; the numerical values of the stress intensity factor K_1 against a are displayed in Fig.4.5.9, and the numerical values of stress intensity factor K_2 against a are displayed in Fig.4.5.10.



Fig.4.5.1

Numerical values of the stress intensity factor K_1 against h/a (0.2 to 1.0) for a fixed crack length a and equal layer thickness $(h_1=h_2=h)$.



Fig.4.5.2

Numerical values of the stress intensity factor K_1 against h/a (1.0 to 10.0) for a fixed crack length a and equal layer thickness $(h_1=h_2=h)$.



Numerical values of the stress intensity factor K_2 against h/a (0.2 to 1.0) for a fixed crack length a and equal layer thickness $(h_1=h_2=h)$.



Numerical values of the stress intensity factor K_2 against h/a (1.0 to 10.0) for a fixed crack length a and equal layer thickness $(h_1=h_2=h)$.



Numerical values of the stress intensity factor K_1 against the crack length a (1.0 to 10.0) for the layer thickness $h_1=h_2 \rightarrow \infty$.



Numerical values of the stress intensity factor K_2 against the crack length a (1.0 to 10.0) for the layer thickness $h_1=h_2 \rightarrow \infty$.



Numerical values of the stress intensity factor K_1 against the crack length a (1.0 to 10.0) for the layer thickness $h_1=h_2=20$.



Numerical values of the stress intensity factor K_2 against the crack length a (1.0 to 10.0) for the layer thickness $h_1=h_2=20$.



Numerical values of the stress intensity factor K_1 against the crack length a (1.0 to 10.0) for the layer thickness $h_1=h_2=10$.



Numerical values of the stress intensity factor K_2 against the crack length a (1.0 to 10.0) for the layer thickness $h_1=h_2=10$.

We make the following observations from the graphs : when the length of the crack *a* is fixed at a = 1.0, the stress intensity factor K_2 increases and K_1 decreases as the thickness of the layers increases from 0.2 to 1.0 while the stress intensity factor K_2 decreases and K_1 increases as the thickness of the layers increases from 1.0 to 10.0 ; and if the thickness of the layers is fixed (at 10.0, 20.0 or $\rightarrow \infty$) the stress intensity factor K_1 decreases and K_2 increases as the length of the crack increases.

CHAPTER 5

PENNY-SHAPED INTERFACE CRACK BETWEEN TWO DISSIMILAR TRANSVERSELY ISOTROPIC LAYERS

5.1 Introduction

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The study of internal penny-shaped cracks is of practical importance in stress analysis, since it represents an idealization of the shape of internal flaws that are inherent in many engineering materials. The formulation of this class of boundary value problems can be expressed most conveniently in terms of the cylindrical polar coordinates (r, θ, z) . A crack lying in the $r\theta$ -plane and occupying the region

 $r \leq c, \qquad z = 0;$

is called a penny-shaped crack.

In 1946 Sack [71] considered a penny-shaped crack in a three dimensional elastic space, he treated it as a limiting case of an ellipsoidal crack. It was Sneddon [72] who successfully introduced the application of Hankel transforms to solve a penny-shaped crack problem for an elastic solid when the surface of crack was under constant pressure. Green [73] solved the same problem by potential function methods. Collins [74] considered the case in which the surface of the crack was subjected to a variable pressure. Using Hankel transforms Muki [75] solved the problem of a penny-shaped crack under shear, and Sneddon [43] solved the problem of penny-shaped crack under torsion. Some researchers have considered the dynamical problems concerning penny-shaped cracks. Craggs [76] and Atkinson [77] considered the expanding penny-shaped crack problem; the response of a penny-shaped crack to a loading in the form of a plane harmonic dilatational wave propagating along the axis of the crack was discussed by Mal [78]; the response to an incident plane harmonic shear wave polarized in a plane normal to the plane of the crack and propagating along the axis of the crack was considered by Mal [79].

Olesiak and Sneddon [80] discussed the distribution of thermal stresses in the vicinity of a penny-shaped crack by assuming that the thermal conditions on the upper surface of the crack were identical with those on the lower surface of the crack.

The distribution of stress in the vicinity of a penny-shaped crack in an elastic plate of finite thickness but infinite radius was discussed by Lowengrub [81], where the crack was taken to lie in the central plane of the plate with its surfaces parallel to those of the plate. Later Sneddon and Tait [82] and Sneddon and Welch [83] investigated the distribution of stress in a long circular cylinder $0 \leq r \leq a, -\infty < z < \infty$, containing a penny-shaped crack lying in the plane z = 0 and the cylinder being under tension.

Many engineering structures are made by bonding together two or more materials with different elastic properties. The dissimilar material system is required to act as a single unit such that the loads are transmitted from one material to the next through the interfaces. The interface in bonded dissimilar materials often contains some flaws, such as cracks or hard inclusions, that may be induced during the process of joining the materials. These flaws, generally, form the nucleus of fracture initiation and propagation in the medium. The presence of flaws or cracks at the interface could cause high elevation of local stresses and lead to failure if the crack reaches a critical size.

Mossakovkii and Rybka [84] introduced a way of formulating the axially symmetric penny-shaped crack problem, when the crack is located at the interface of two dissimilar isotropic materials. Willis [85] considered the problem of obtaining the stress intensity factor for a penny-shaped crack between two dissimilar materials. Erdogan [86] solved the interface penny-shaped crack problem by reducing it to a singular integral equation. Erdogan and Arin [87] considered a penny-shaped crack between an elastic layer and a half-space.

Elliott [88, 89] considered a penny-shaped crack in a transversely isotropic elastic solid. Kassir and Sih [90] investigated an elliptical crack problem in a transversely isotropic elastic solid, and Parhi and Atsumi [91] discussed the distribution of stress in a transversely isotropic elastic cylinder containing a penny-shaped crack. Recently, Saxena and Dhaliwal [92] considered a penny-shaped crack problem at the interface of two transversely isotropic half-spaces.

In this chapter we consider the penny-shaped interface crack between two dissimilar transversely isotropic elastic layers. By means of Hankel transforms and Fourier transforms the problem is reduced to the solution of a system of singular integral equations. These equations are further reduced to a system of simultaneous algebraic equations by using Jacobi polynomials approximation. Numerical methods are employed to determine the stress intensity factors, which have been displayed graphically.

5.2. Basic equations and their solution

As discussed in chapter 1, under the assumptions of axially-symmetric deformations in cylindrical polar coordinates (r, θ, z) , when the z-axis is the axis of anisotropy of transversely isotropic medium, the displacement components are defined by

$$u_{\rm r} = u_{\rm r}(r,z)$$
 , $u_{\rm z} = u_{\rm z}(r,z)$, $u_{\theta} = 0$, (5.2.1)

along the r-, z-, and θ - directions respectively. The stress-displacement relations are given by

$$\begin{split} \sigma_{\rm rr} &= c_{11} \frac{\partial u_{\rm r}}{\partial r} + c_{12} \frac{u_{\rm r}}{r} + c_{13} \frac{\partial u_{\rm z}}{\partial z} ,\\ \sigma_{\theta\theta} &= c_{12} \frac{\partial u_{\rm r}}{\partial r} + c_{11} \frac{u_{\rm r}}{r} + c_{13} \frac{\partial u_{\rm z}}{\partial z} ,\\ \sigma_{\rm zz} &= c_{13} \left[\frac{\partial u_{\rm r}}{\partial r} + \frac{u_{\rm r}}{r} \right] + c_{33} \frac{\partial u_{\rm z}}{\partial z} ,\\ \sigma_{\rm rz} &= c_{44} \left[\frac{\partial u_{\rm r}}{\partial z} + \frac{\partial u_{\rm z}}{\partial r} \right] , \end{split}$$
(5.2.2)

where c_{ij} 's are the elastic moduli of the transversely isotropic medium. In the absence of body forces, the equations of equilibrium may be written as follows

$$c_{11}\left[\frac{\partial^2 u_{\Gamma}}{\partial r}\right] + \frac{1}{r} \frac{\partial u_{\Gamma}}{\partial r} - \frac{u_{\Gamma}}{r}\left] + c_{44}\frac{\partial^2 u_{\Gamma}}{\partial z^2} + (c_{13} + c_{44})\frac{\partial^2 u_{Z}}{\partial r \partial z} = 0, \quad (5.2.3a)$$

$$c_{44}\left[\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r} \frac{\partial u_z}{\partial r}\right] + c_{33}\frac{\partial^2 u_z}{\partial z^2} + (c_{13} + c_{44})\frac{\partial}{\partial z}\left[\frac{\partial u_r}{\partial r} + \frac{u_r}{r}\right] = 0.$$
(5.2.3b)

Multiplying equations (5.2.3a) and (5.2.3b) by $rJ_1(\xi r)$ and $rJ_0(\xi r)$ respectively, then integrating with respect to r from $0 \rightarrow \infty$, we get the following equations

$$-c_{11}\xi^{2} \tilde{u}_{r} + c_{44}\frac{\partial^{2}}{\partial z^{2}} \tilde{u}_{r} - (c_{13} + c_{44})\xi\frac{\partial}{\partial z} \tilde{u}_{z} = 0,$$

$$-c_{44}\xi^{2} \tilde{u}_{z} + c_{33}\frac{\partial^{2}}{\partial z^{2}} \tilde{u}_{z} + (c_{13} + c_{44})\xi\frac{\partial}{\partial z} \tilde{u}_{r} = 0,$$
 (5.2.4)

where \tilde{u}_r and \tilde{u}_z are the Hankel transforms of u_r and u_z of order 1 and 0 respectively.

Let c_{ijk} 's be the elastic moduli for the transversely isotropic medium in the region R_k (k=1,2) where R_1 is $0 \le z \le h_2$ and R_2 is $-h_1 \le z \le 0$. Solving equations (5.2.4) for \tilde{u}_r and \tilde{u}_z then taking the inverse Hankel transforms we obtain the following solution :

$$u_{rj}(r,z) = \mathscr{H}_{1}[\xi^{-1}\{A_{j}e^{\beta_{j}\xi_{z}} + B_{j}e^{-\beta_{j}\xi_{z}} + C_{j}e^{\delta_{j}\xi_{z}} + D_{j}e^{-\delta_{j}\xi_{z}}\}; \xi \to r] ,$$

$$u_{zj}(r,z) = \mathscr{H}_{0}[\xi^{-1}\{A_{j}a_{j}e^{\beta_{j}\xi_{z}} - B_{j}a_{j}e^{-\beta_{j}\xi_{z}} + C_{j}\gamma_{j}e^{\delta_{j}\xi_{z}} - D_{j}\gamma_{j}e^{-\delta_{j}\xi_{z}}\}; \xi \to r], \quad (5.2.5)$$

where u_{rj} and u_{zj} denote the displacements for the region R_j (j=1,2),

$$a_{j} = \frac{c_{44j} \beta_{j} - c_{11j} \beta_{j}^{-1}}{c_{13j} + c_{44j}} , \qquad \gamma_{j} = \frac{c_{44j} \delta_{j} - c_{11j} \delta_{j}^{-1}}{c_{13j} + c_{44j}} ,$$
$$(\beta_{j}, \delta_{j}) = \left\{ \frac{\eta_{j} \pm \sqrt{\eta_{j}^{2} - 4 \cdot c_{11j} \cdot c_{33j} \cdot c_{44j}^{2}}}{2 \cdot c_{33j} \cdot c_{44j}} \right\}^{\frac{1}{2}} , \qquad j=1,2 .$$
$$(5.2.6)$$

and A_j , B_j , C_j and D_j (j = 1,2) are unknown functions of ξ to be determined by the boundary and continuity conditions.

Substituting from equation (5.2.5) into (5.2.2), we have

$$\sigma_{zzj} = \mathscr{H}_{0}[\{(c_{13j} + c_{33j} \alpha_{j} \beta_{j})(A_{j} e^{\beta_{j}\xi z} + B_{j} e^{-\beta_{j}\xi z}) + (c_{13j} + \gamma_{j}\delta_{j}c_{33j})(C_{j}e^{\delta_{j}\xi z} + D_{j}e^{-\delta_{j}\xi z})\}; \xi \to r], \qquad (5.2.7a)$$

$$\sigma_{\rm rzj} = c_{44j} \mathscr{H}_{\rm I}[\{(\beta_{\rm j} - a_{\rm j})(A_{\rm j} \ e^{\beta_{\rm j}\xi_{\rm z}} - B_{\rm j} \ e^{-\beta_{\rm j}\xi_{\rm z}}) + (\delta_{\rm i} - \gamma_{\rm i})(C_{\rm i}e^{\delta_{\rm i}\xi_{\rm z}} - D_{\rm i}e^{-\delta_{\rm i}\xi_{\rm z}})\}; \xi \to r] , \qquad (5.2.7b)$$

where σ_{zzj} and σ_{rzj} denote stresses for the region R_j (j=1,2).

5.3. Statment of the problem and boundary conditions

We assume that two dissimilar transversely isotropic elastic layers, which occupy the regions R_1 ($0 \le z \le h_2$) and R_2 ($-h_1 \le z \le 0$) respectively, are perfectly bonded except that there is a crack in the region $0 \le r \le a$, z=0. In the first case we also assume that the surfaces $z = -h_1$ and $z = h_2$ are stress-free.(see Fig.5.3.1) It is assumed that the surfaces of crack are subjected to prescribed normal and shear stresses $p_1(r)$ and $p_2(r)$ respectively. Hence the problem of determining the stress and displacement field is subjected to the following boundary and continuity conditions:

$\sigma_{zzj}(r,0) = p_1(r), \sigma_{rzj}(r,0) = p_2(r), j=1,2;$	$0 \leq r \leq a$,	(5.3.1)
$u_{r1}(r,0^+) = u_{r2}(r,0^-), u_{z1}(r,0^+) = u_{z2}(r,0^-),$	$r \geq a$,	(5.3.2)
$\sigma_{zz1}(r,0^{+}) = \sigma_{zz2}(r,0^{-}), \sigma_{rz1}(r,0^{+}) = \sigma_{rz2}(r,0^{-}),$	$r \geq a$,	(5.3.3)
$\sigma_{zz1}(r,h_2) = 0, \qquad \sigma_{rz1}(r,h_2) = 0,$	$r \geq 0$,	(5.3.4)
$\sigma_{xx2}(r,-h_1) = 0$, $\sigma_{rx2}(r,-h_1) = 0$,	$r \geq 0.$	(5.3.5)

With the help of conditions (5.3.1), the conditions (5.3.3) may be replaced by the following:

$$\sigma_{zz1}(r,0^{+}) = \sigma_{zz2}(r,0^{-}), \quad \sigma_{rz1}(r,0^{+}) = \sigma_{rz2}(r,0^{-}), \qquad r \ge 0.$$
(5.3.6)





Penny-shaped interface crack between two dissimilar transversely isotropic layers.

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5.4. Analysis

If we denote

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$$a_{j} = c_{13j} + a_{j} \beta_{j} c_{33j} , \quad b_{j} = c_{44j} (\beta_{j} - a_{j}) ,$$

$$d_{j} = c_{13j} + \delta_{j} \gamma_{j} c_{33j} , \quad e_{j} = c_{44j} (\delta_{j} - \gamma_{j}), \quad j = 1,2 ; \qquad (5.4.1)$$

then applying conditions (5.3.6), the equations (5.2.7) give

$$a_1(A_1 + B_1) + d_1(C_1 + D_1) = a_2(A_2 + B_2) + d_2(C_2 + D_2)$$
, (5.4.2a)

$$b_1(A_1 - B_1) + e_1(C_1 - D_1) = b_2(A_2 - B_2) + e_2(C_2 - D_2)$$
; (5.4.2b)

and the conditions (5.3.4) and (5.3.5) yield

$$a_1A_1e^{\beta_1\xih_2} + a_1B_1e^{-\beta_1\xih_2} + d_1C_1e^{\delta_1\xih_2} + d_1D_1e^{-\delta_1\xih_2} = 0$$
, (5.4.3a)

$$b_1 A_1 e^{\beta_1 \xi h_2} - b_1 B_1 e^{-\beta_1 \xi h_2} + e_1 C_1 e^{\delta_1 \xi h_2} - e_1 D_1 e^{-\delta_1 \xi h_2} = 0 , \qquad (5.4.3b)$$

$$a_2 A_2 e^{-\beta_2 \xi h_1} + a_2 B_2 e^{\beta_2 \xi h_1} + d_2 C_2 e^{-\delta_2 \xi h_1} + d_2 D_2 e^{\delta_2 \xi h_1} = 0 , \qquad (5.4.3c)$$

$$b_2 A_2 e^{-\beta_2 \xi h_1} - b_2 B_2 e^{\beta_2 \xi h_1} + e_2 C_2 e^{-\delta_2 \xi h_1} - e_2 D_2 e^{\delta_2 \xi h_1} = 0 .$$
 (5.4.3d)

Solving the equations (5.4.2) and (5.4.3) we obtain

$$C_{1} = -l_{11}A_{1}e^{(\beta_{1}-\delta_{1})\xih_{2}} - l_{12}B_{1}e^{-(\beta_{1}+\delta_{1})\xih_{2}},$$

$$D_{1} = -l_{12}A_{1}e^{(\beta_{1}+\delta_{1})\xih_{2}} - l_{11}B_{1}e^{(-\beta_{1}+\delta_{1})\xih_{2}},$$

$$C_{2} = -l_{21}A_{2}e^{(-\beta_{2}+\delta_{2})\xih_{1}} - l_{22}B_{2}e^{(\beta_{2}+\delta_{2})\xih_{1}},$$

$$D_{2} = -l_{22}A_{2}e^{-(\beta_{2}+\delta_{2})\xih_{1}} - l_{21}B_{2}e^{(\beta_{2}-\delta_{2})\xih_{1}},$$

$$A_{2} = l_{31}A_{1} + l_{32}B_{1}, \quad B_{2} = l_{41}A_{1} + l_{42}B_{1},$$
(5.4.4)

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where

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$$l_{1j} = [a_{1}e_{1} - (-1)^{j}b_{1}d_{1}]/(2d_{1}e_{1}), \quad l_{2j} = [a_{2}e_{2} - (-1)^{j}b_{2}d_{2}]/(2d_{2}e_{2}); \; j=1,2 \; .$$

$$l_{31} = (a_{11}b_{22} - a_{22}b_{11})/\Delta , \quad l_{32} = (a_{12}b_{22} - a_{22}b_{12})/\Delta ,$$

$$l_{41} = (a_{21}b_{11} - a_{11}b_{21})/\Delta , \quad l_{42} = (a_{21}b_{12} - a_{12}b_{21})/\Delta ,$$

$$a_{11} = a_{1} - d_{1}l_{11}e^{(\beta_{1}^{-} \delta_{1})\xih_{2}} - d_{1}l_{12}e^{-(\beta_{1}^{+} \delta_{1})\xih_{2}} ,$$

$$a_{12} = a_{1} - d_{1}l_{12}e^{-(\beta_{1}^{+} \delta_{1})\xih_{2}} - d_{1}l_{11}e^{(-\beta_{1}^{+} \delta_{1})\xih_{2}} ,$$

$$a_{21} = a_{2} - d_{2}l_{21}e^{(-\beta_{2}^{+} \delta_{2})\xih_{1}} - d_{2}l_{22}e^{-(\beta_{2}^{+} \delta_{2})\xih_{1}} ,$$

$$a_{22} = a_{2} - d_{2}l_{22}e^{(\beta_{2}^{+} \delta_{2})\xih_{1}} - d_{2}l_{21}e^{(\beta_{1}^{-} \delta_{1})\xih_{2}} ,$$

$$b_{11} = b_{1} - e_{1}l_{11}e^{(\beta_{1}^{-} \delta_{1})\xih_{2}} + e_{1}l_{12}e^{(\beta_{1}^{+} \delta_{1})\xih_{2}} ,$$

$$b_{12} = -b_{1} - e_{1}l_{2}e^{-(\beta_{1}^{+} \delta_{1})\xih_{2}} + e_{1}l_{11}e^{(-\beta_{1}^{+} \delta_{1})\xih_{2}} ,$$

$$b_{21} = b_{2} - e_{2}l_{21}e^{(-\beta_{2}^{+} \delta_{2})\xih_{1}} + e_{2}l_{22}e^{-(\beta_{2}^{+} \delta_{2})\xih_{1}} ,$$

$$b_{22} = -b_{2} - e_{2}l_{22}e^{(\beta_{2}^{+} \delta_{2})\xih_{1}} + e_{2}l_{21}e^{(\beta_{2}^{-}, \delta_{2})\xih_{1}} ,$$

$$\Delta = a_{21} \; b_{22} - a_{22} \; b_{21} . \qquad (5.4.5)$$

The conditions (5.3.1) and (5.3.2) lead to the following integral equations:

$$\mathcal{H}_{0}[(a_{1}A_{1} + a_{1}B_{1} + d_{1}C_{1} + d_{1}D_{1}); \xi \to r] = p_{1}(r); \quad r < a, \quad (5.4.6)$$

$$\mathcal{H}_{1}[(b_{1}A_{1} - b_{1}B_{1} + e_{1}C_{1} - e_{1}D_{1}); \xi \to r] = p_{2}(r); \quad r < a, \quad (5.4.7)$$

$$\mathcal{H}_{1}[\xi^{-1}(A_{1} + B_{1} + C_{1} + D_{1} - A_{2} - B_{2} - C_{2} - D_{2}); \xi \to r] = 0; \quad r > a, \quad (5.4.8)$$

$$\mathcal{H}_{0}[\xi^{-1}(a_{1}A_{1} - a_{1}B_{1} + \gamma_{1}C_{1} - \gamma_{1}D_{1} - a_{2}A_{2} + a_{2}B_{2} - \gamma_{2}C_{2} + \gamma_{2}D_{2}); \xi \to r] = 0; \quad r > a. \quad (5.4.9)$$

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If we introduce two new unknown functions $\phi_1(\xi)$ and $\phi_2(\xi)$ by the relations

$$A_{1} + B_{1} + C_{1} + D_{1} - A_{2} - B_{2} - C_{2} - D_{2} = \phi_{1}(\xi) ,$$

$$a_{1}A_{1} - a_{1}B_{1} + \gamma_{1}C_{1} - \gamma_{1}D_{1} - a_{2}A_{2} + a_{2}B_{2} - \gamma_{2}C_{2} + \gamma_{2}D_{2} = \phi_{2}(\xi) , \qquad (5.4.10)$$

then the equations (5.4.6) to (5.4.9) may be rewritten in the form

$$\mathcal{H}_{0}[(\lambda_{11} \phi_{1} + \lambda_{12} \phi_{2}) ; \xi \to r] = p_{1}(r) , \qquad r < a , \qquad (5.4.11)$$

$$\mathscr{H}_{1}[(\lambda_{21} \phi_{1} + \lambda_{22} \phi_{2}) ; \xi \to r] = p_{2}(r) , \qquad r < a , \qquad (5.4.12)$$

$$\mathscr{H}_{1}[\xi^{-1} \phi_{1}(\xi) ; \xi \to r] = 0$$
, $r > a$, (5.4.13)

$$\mathcal{H}_0[\xi^{-1} \phi_2(\xi) ; \xi \to r] = 0 , \qquad r > a , \qquad (5.4.14)$$

where λ_{jk} are functions of ξ and they are given by

$$\lambda_{11} = (a_{11}\gamma_{11} - a_{12}\gamma_{11})/\Delta^{*}, \ \lambda_{12} = (a_{12}\gamma_{21} - a_{11}\gamma_{22})/\Delta^{*},$$

$$\lambda_{21} = (b_{11}\gamma_{12} - b_{22}\gamma_{11})/\Delta^{*}, \ \lambda_{22} = (b_{12}\gamma_{21} - b_{11}\gamma_{22})/\Delta^{*},$$

$$\Delta^{*} = \gamma_{12} \gamma_{21} - \gamma_{11} \gamma_{22}.$$
(5.4.15)

and γ_{jk} 's are defined as follows

$$\begin{aligned} \gamma_{11} &= a_1 - \gamma_1 l_{11} e^{(\beta_1^{-} \delta_1) \xi h_2} + \gamma_1 l_{12} e^{(\beta_1^{+} \delta_1) \xi h_2} - a_2 l_{31} + a_2 l_{41} - \gamma_2 l_{51} + \gamma_2 l_{61},} \\ \gamma_{12} &= -a_1 - \gamma_1 l_{12} e^{-(\beta_1^{+} \delta_1) \xi h_2} + \gamma_1 l_{11} e^{(-\beta_1^{+} \delta_1) \xi h_2} - a_2 l_{32} + a_2 l_{42} - \gamma_2 l_{52} + \gamma_2 l_{62},} \\ \gamma_{21} &= 1 - l_{11} e^{(\beta_1^{-} \delta_1) \xi h_2} - l_{12} e^{(\beta_1^{+} \delta_1) \xi h_2} - l_{31} - l_{41} - l_{51} - l_{61},} \\ \gamma_{22} &= 1 - l_{12} e^{-(\beta_1^{+} \delta_1) \xi h_2} - l_{11} e^{(-\beta_1^{+} \delta_1) \xi h_2} - l_{32} - l_{42} - l_{52} - l_{62},} \\ l_{51} &= -l_{21} l_{31} e^{(-\beta_2^{+} \delta_2) \xi h_1} - l_{22} l_{41}^{-(\beta_2^{+} \delta_2) \xi h_1},} \\ l_{52} &= -l_{21} l_{32} e^{(-\beta_2^{+} \delta_2) \xi h_1} - l_{22} l_{42}^{-(\beta_2^{-} \delta_2) \xi h_1},} \\ l_{61} &= -l_{22} l_{31} e^{-(\beta_2^{+} \delta_2) \xi h_1} - l_{21} l_{41}^{-(\beta_2^{-} \delta_2) \xi h_1}, \\ l_{62} &= -l_{22} l_{32} e^{-(\beta_2^{+} \delta_2) \xi h_1} - l_{21} l_{42}^{-(\beta_2^{-} \delta_2) \xi h_1}. \end{aligned}$$
(5.4.16)

It can be seen that as $\xi \to \infty$, λ_{jk} tends to constant, say $\lambda_{jk}(\varpi), \text{and}$

$$\lambda_{jk}(\xi) - \lambda_{jk}(\omega) = O(e^{-(\beta_1 + \delta_1) \xi h_2}) . \qquad (5.4.17)$$

To solve the integral equations (5.4.11) to (5.4.14), let us recall the following results [45]

$$\mathscr{I}_{1}[r\mathscr{H}_{0}\{F(\xi) ; r\} ; y] = \mathscr{F}_{s}[F(\xi) ; y] , \qquad (5.4.18)$$

$$\mathscr{I}_{2}[D_{r}r\mathscr{H}_{1}\{\xi^{-1}F(\xi) ; r\} ; y] = \mathscr{I}_{c}[F(\xi) ; y] , \qquad (5.4.19)$$

$$\mathscr{A}_{1}[\mathscr{H}_{1}\{F(\xi) ; r\} ; y] = y^{-1}\{\mathscr{F}_{c}[F(\xi) ; 0] - \mathscr{F}_{c}[F(\xi) ; y]\}, \qquad (5.4.20)$$

$$\mathscr{I}_{2}[D_{r}\mathscr{H}_{0}\{\xi^{-1}F(\xi) ; r\} ; y] = -y^{-1}\mathscr{I}_{s}[F(\xi) ; y] , \qquad (5.4.21)$$

where D_r denotes $\partial/\partial r$ and \mathcal{M}_1 , \mathcal{M}_2 are Abel's operators ; \mathcal{P}_s and \mathcal{P}_c are Fourier's operators defined by

$$\mathcal{M}_{1}[F(r) \; ; \; y \;] = (2/\pi)^{\frac{1}{2}} \int_{0}^{y} \frac{1}{\sqrt{y^{2} - r^{2}}} F(r) dr \; ,$$

$$\mathcal{M}_{2}[F(r) \; ; \; y \;] = (2/\pi)^{\frac{1}{2}} \int_{y}^{\infty} \frac{1}{\sqrt{r^{2} - y^{2}}} F(r) dr \; ,$$

$$\mathcal{H}_{s}[F(r) \; ; \; y \;] = (2/\pi)^{\frac{1}{2}} \int_{0}^{\infty} F(r) \; sin(ry \;) \; dr \; ,$$

$$\mathcal{H}_{c}[F(r) \; ; \; y \;] = (2/\pi)^{\frac{1}{2}} \int_{0}^{\infty} F(r) \; cos(ry \;) \; dr \; .$$
(5.4.22)

Applying the operators $\mathscr{K}_1 r$, \mathscr{K}_1 , $\mathscr{K}_2 D_r r$ and $\mathscr{K}_2 D_r$ to the equations (5.4.11), (5.4.12), (5.4.13) and (5.4.14) respectively, we obtain

$$\begin{aligned} \mathscr{F}_{s}[(\lambda_{11} \ \phi_{1} + \lambda_{12} \ \phi_{2}) \ ; \ y \] &= f_{1}(y) \ , & 0 \le y \le a, \ (5.4.23) \\ \mathscr{F}_{c}[(\lambda_{21} \ \phi_{1} + \lambda_{22} \ \phi_{2}) \ ; \ y \] &= f_{2}(y) \ , & 0 \le y \le a, \ (5.4.24) \\ \mathscr{F}_{c}[\phi_{1}(\xi) \ ; \ y \] &= 0 \ , & y > a \ ; \ (5.4.25) \\ \mathscr{F}_{s}[\phi_{2}(\xi) \ ; \ y \] &= 0 \ , & y > a \ ; \ (5.4.26) \end{aligned}$$

where

$$f_{1}(y) = \mathscr{K}_{1}[rp_{1}(r) ; y] , f_{2}(y) = y \mathscr{K}_{1}[p_{2}(r) ; y] + c ,$$

$$c = \mathscr{F}_{c}[(\lambda_{21} \phi_{1} + \lambda_{22} \phi_{2}) ; 0] . \qquad (5.4.27)$$

By taking y=ax, equations (5.4.23) to (5.4.26) can be written in the following form

$$\mathcal{F}_{s}[(\tilde{\lambda}_{11} \ \tilde{\phi}_{1} + \tilde{\lambda}_{12} \ \tilde{\phi}_{2}) \ ; \ x \] = \tilde{f}_{1}(x \) \ , \qquad 0 \le x \le 1, \qquad (5.4.28)$$

$$\mathcal{F}_{c}[(\tilde{\lambda}_{21} \ \tilde{\phi}_{1} + \tilde{\lambda}_{22} \ \tilde{\phi}_{2}) \ ; \ x \] = \tilde{f}_{2}(x) \ , \qquad 0 \le x \le 1, \qquad (5.4.29)$$

$$\mathcal{F}_{c}[\tilde{\phi}_{1} \ ; \ x \] = 0 \ , \qquad x > 1, \qquad (5.4.30)$$

$$\mathscr{F}_{s}[\phi_{2} ; x] = 0$$
, $x > 1$, (5.4.31)

where

$$\tilde{\lambda}_{jk}(\xi) = \lambda_{jk}(\xi/a) , \qquad \tilde{\phi}_{j}(\xi) = \phi_{j}(\xi/a) ,$$

$$\tilde{f}_{j}(x) = a f_{j}(ax) , \qquad j, k = 1, 2.$$
(5.4.32)

If we introduce two new unknown functions $\psi_1(t)$ and $\psi_2(t)$ such that

$$\begin{aligned} \mathscr{F}_{\mathbf{c}}[\tilde{\phi}_{1}(x) \ ; \ t] &= \begin{cases} & \psi_{1}(t), & 0 < t < 1; \\ & 0, & t > 1; \\ \\ \mathscr{F}_{\mathbf{s}}[\tilde{\phi}_{2}(x) \ ; \ t] &= \begin{cases} & \psi_{2}(t), & 0 < t < 1; \\ & 0, & t > 1; \\ \\ & 0, & t > 1; \end{cases} \end{aligned} \tag{5.4.33}$$

the equations (5.4.30) and (5.4.31) are identically satisfied. Let $\psi_e(t)$ and $\psi_o(t)$ be the even extension of $\psi_1(t)$ and odd extension of $\psi_2(t)$ on (-1, 1) respectively, then as we did in section 4.3, we have

$$\mathscr{F}_{s}[\tilde{\phi}_{1}(t) ; x] = -\frac{1}{\pi} \int_{-1}^{1} \frac{\psi_{e}(t)}{t-x} dt , \qquad (5.4.35)$$

$$\mathscr{F}_{\mathbf{C}}[\widetilde{\phi}_{2}(t) \; ; \; x] \; = \; \frac{1}{\pi} \; \int_{-1}^{1} \frac{\psi_{0}(t)}{t-x} dt. \tag{5.4.36}$$

Now if we denote $F_1(x)$ the odd extension of $\tilde{f}_1(x)$ and $F_2(x)$ the even extension of $\tilde{f}_2(x)$ on (-1, 1), the equations (5.4.28) and (5.4.29) become

$$\lambda_{12}(\varpi)\psi_{0}(x) - \frac{\lambda_{11}(\varpi)}{\pi} \int_{-1}^{1} \frac{\psi_{e}(t)}{t - x} dt + \int_{-1}^{1} \psi_{e}(t) K_{11}(x, t) dt + \int_{-1}^{1} \psi_{0}(t) K_{12}(x, t) dt = F_{1}(x) , \qquad |x| < 1 , \quad (5.4.37)$$
$$\lambda_{21}(\varpi)\psi_{e}(x) + \frac{\lambda_{22}(\varpi)}{\pi} \int_{-1}^{1} \frac{\psi_{0}(t)}{t - x} dt + \int_{-1}^{1} \psi_{e}(t) K_{21}(x, t) dt + \int_{-1}^{1} \psi_{0}(t) K_{22}(x, t) dt = F_{2}(x) , \qquad |x| < 1 , \quad (5.4.38)$$

where the kernels K_{jk} are given by

$$K_{11}(x,t) = \frac{1}{\pi} \int_{0}^{\infty} [\tilde{\lambda}_{11}(\xi) - \lambda_{11}(\infty)] \cos(\xi t) \sin(\xi x) d\xi ,$$

$$K_{12}(x,t) = \frac{1}{\pi} \int_{0}^{\infty} [\tilde{\lambda}_{12}(\xi) - \lambda_{12}(\infty)] \sin(\xi t) \sin(\xi x) d\xi ,$$

$$K_{21}(x,t) = \frac{1}{\pi} \int_{0}^{\infty} [\tilde{\lambda}_{21}(\xi) - \lambda_{21}(\infty)] \sin(\xi t) \cos(\xi x) d\xi ,$$

$$K_{22}(x,t) = \frac{1}{\pi} \int_{0}^{\infty} [\tilde{\lambda}_{22}(\xi) - \lambda_{22}(\infty)] \cos(\xi t) \cos(\xi x) d\xi .$$
(5.4.39)

Equations (5.4.37) and (5.4.38) may be rewritten as follows:

$$\lambda \zeta_{1}(x) + \frac{1}{\pi i J} \int_{-1}^{1} \frac{\zeta_{1}(t)}{t - x} dt + \int_{-1}^{1} \zeta_{1}(t) M_{11}(x, t) dt + \int_{-1}^{1} \zeta_{2}(t) M_{12}(x, t) dt = g_{1}(x), \quad (5.4.40)$$

$$\lambda \zeta_2(x) - \frac{1}{\pi i} \int_{-1}^{1} \frac{\zeta_2(t)}{t-x} dt + \int_{-1}^{1} \zeta_1(t) M_{21}(x,t) dt + \int_{-1}^{1} \zeta_2(t) M_{22}(x,t) dt = g_2(x), \quad (5.4.41)$$

where

$$\begin{split} \lambda_{1} &= \lambda_{21}(\varpi)/\lambda_{22}(\varpi) , \quad \lambda_{2} &= \lambda_{12}(\varpi)/\lambda_{11}(\varpi) , \quad \lambda &= \sqrt{\lambda_{1} \ \lambda_{2}} , \\ \zeta_{1}(x) &= \sqrt{\lambda_{1}}\psi_{e}(x) + i \sqrt{\lambda_{2}}\psi_{0}(x) , \quad \zeta_{2}(x) &= \sqrt{\lambda_{1}}\psi_{e}(x) - i \sqrt{\lambda_{2}}\psi_{0}(x) , \\ g_{1}(x) &= \sqrt{\lambda_{2}} \ F_{2}(x)/\lambda_{22}(\varpi) + i \sqrt{\lambda_{1}} \ F_{1}(x)/\lambda_{11}(\varpi) , \\ g_{2}(x) &= \sqrt{\lambda_{2}} \ F_{2}(x)/\lambda_{22}(\varpi) - i \sqrt{\lambda_{1}} \ F_{1}(x)/\lambda_{11}(\varpi) , \\ M_{11}(x,t) &= \frac{1}{2} \left\{ \frac{\sqrt{\lambda_{1}}}{\lambda_{11}(\varpi)\sqrt{\lambda_{2}}} \ K_{12}(x,t) + \frac{\sqrt{\lambda_{2}}}{\lambda_{22}(\varpi)\sqrt{\lambda_{1}}} \ K_{21}(x,t) \right. \\ &+ i \left(\frac{K_{11}(x,t)}{\lambda_{11}(\varpi)} - \frac{K_{22}(x,t)}{\lambda_{22}(\varpi)} \right) \right\} , \\ M_{12}(x,t) &= \frac{1}{2} \left\{ \frac{\sqrt{\lambda_{1}}}{\lambda_{11}(\varpi)\sqrt{\lambda_{2}}} \ K_{12}(x,t) + \frac{\sqrt{\lambda_{2}}}{\lambda_{22}(\varpi)\sqrt{\lambda_{1}}} \ K_{21}(x,t) \right. \\ &+ i \left(\frac{K_{11}(x,t)}{\lambda_{11}(\varpi)} + \frac{K_{22}(x,t)}{\lambda_{22}(\varpi)} \right) \right\} , \\ M_{21}(x,t) &= \frac{1}{2} \left\{ \frac{\sqrt{\lambda_{1}}}{\lambda_{11}(\varpi)\sqrt{\lambda_{2}}} \ K_{12}(x,t) + \frac{\sqrt{\lambda_{2}}}{\lambda_{22}(\varpi)\sqrt{\lambda_{1}}} \ K_{21}(x,t) \right. \\ &- i \left(\frac{K_{11}(x,t)}{\lambda_{11}(\varpi)} + \frac{K_{22}(x,t)}{\lambda_{22}(\varpi)} \right) \right\} , \\ M_{22}(x,t) &= \frac{1}{2} \left\{ \frac{\sqrt{\lambda_{1}}}{\lambda_{11}(\varpi)\sqrt{\lambda_{2}}} \ K_{12}(x,t) + \frac{\sqrt{\lambda_{2}}}{\lambda_{22}(\varpi)\sqrt{\lambda_{1}}} \ K_{21}(x,t) \right. \\ &+ i \left(\frac{K_{11}(x,t)}{\lambda_{11}(\varpi)} - \frac{K_{22}(x,t)}{\lambda_{22}(\varpi)} \right) \right\} . \end{split}$$
(5.4.42)

The analytic solution of equations (5.4.40) and (5.4.41) has been extensively studied (see, for exemple, [67] and [68]) by using regularization method, which, in this case, however becomes cumbersome. Here we try to use an approximation method described by Erdogan [69] to find the stress intensity factors.

Since the kernels M_{jk} 's are bounded, aside from a multiplication constant, the singular behavior of the functions ζ_j (j=1,2) at the points $x = \pm 1$ is determined by the dominant part of the singular integral equations. The equations (5.4.40) and (5.4.41) will be solved under the assumption that ζ_j (j=1,2) satisfy a Hölder condition on every closed part of the interval (-1, 1) not containing the ends.

The solution of the equations (5.4.40) and (5.4.41) may be assumed in the form of Jacobi polynomials $P_n^{(\sigma_k,\tau_k)}(x)$ [69] by

$$\zeta_{k} = \sum_{n=1}^{\infty} C_{kn} W_{k}(x) P_{n}^{(\sigma_{k}, \tau_{k})}(x) , \qquad (5.4.43)$$

where

$$W_{k}(x) = (1-x)^{\sigma_{k}} (1 + x)^{\tau_{k}} , \quad \sigma_{k} = i\omega_{k} , \quad \tau_{k} = -i\omega_{k} ,$$

$$\omega_{k} = (-1)^{k+1}\omega , \quad \omega = \frac{1}{2\pi} ln \left[\frac{1+\lambda}{1-\lambda} \right] , \quad k = 1, 2 ; \quad (5.4.44)$$

and C_{kn} are unknown coefficients.

Substituting from equation (5.4.43) into equations (5.4.40) and (5.4.41) we obtain

$$\sum_{n=1}^{\infty} C_{1n} \frac{(1-\lambda^2)^{\frac{1}{2}}}{2i} P_{n-1}^{(-\sigma_1,-\tau_1)}(x) + \int_{-1}^{1} [\sum_{n=1}^{\infty} (C_{1n}M_{11}(x,t) W_1(t) P_n^{(\sigma_1,\tau_1)}(t) + C_{2n}M_{12}(x,t) W_2(t) P_n^{(\sigma_2,\tau_2)}(t))] dt = g_1(x), \quad |x| < 1 , \quad (5.4.45)$$

$$\sum_{n=1}^{\infty} C_{2n} \frac{(1-\lambda^2)^{\frac{1}{2}}}{2i} P_{n-1}^{(-\sigma_2,-\tau_2)}(x) - \int_{-1}^{1} [\sum_{n=1}^{\infty} (C_{1n}M_{21}(x,t) W_1(t) P_n^{(\sigma_1,\tau_1)}(t) + C_{2n}M_{22}(x,t) W_2(t) P_n^{(\sigma_2,\tau_2)}(t))] dt = -g_2(x), \quad |x| < 1 , \quad (5.4.46)$$

where we have used the following result by Karpenko [70]:

$$\frac{1}{\pi i J_{-1}} W_{k}(t) P_{n}(\sigma_{k},\tau_{k})(t) \frac{dt}{t-x} + (-1)^{k} W_{k}(x) P_{n}(\sigma_{k},\tau_{k})(x)$$
$$= -(1-\lambda^{2})^{\frac{1}{2}} P_{n}(-\sigma_{k},-\tau_{k})(x) , \qquad |x| < 1 . \quad (5.4.47)$$

Multiplying equations (5.4.45) and (5.4.46) by $W_1^{-1}(x)P_j^{(-\sigma_1,-\tau_1)}(x)$ and $W_2^{-1}(x)P_j^{(-\sigma_2,-\tau_2)}(x)$ respectively, and using the orthogonality relations of Jacobi polynomials

$$\int_{-1}^{1} W(t) P_{n}(\sigma,\tau)(t) P_{m}(\sigma,\tau)(t) dt = \begin{cases} 0 & n \neq m , \\ \theta_{m}(\sigma,\tau) & n = m , \end{cases}$$

$$\theta_{m}^{(\sigma,\tau)} = \frac{2^{\sigma+\tau+1} \Gamma(m+\sigma+1)\Gamma(m+\tau+1)}{m! (2m+\sigma+\tau+1)\Gamma(m+\sigma+\tau+1)}, \qquad (5.4.48)$$

we get the following infinite system of simultaneous algebraic equations for the determination of $C_{\rm km}$.

$$-(1-\lambda^2)^{\frac{1}{2}}C_{\rm im} \ \theta_{\rm m}^{(-\sigma_1,-\tau_1)} + \sum_{n=1}^{\infty} (L_{1\,\rm inm}^* \ C_{\rm in} + L_{1\,\rm 2nm}^* \ C_{2n}) = G_{\rm im} \ , \quad (5.4.49)$$

$$-(1-\lambda^2)^{\frac{1}{2}}C_{2m} \ \theta_m^{(-\sigma_1,-\tau_1)} - \sum_{n=1}^{\infty} (L_{2\,1nm}^* \ C_{1n} + L_{2\,2nm}^* \ C_{2n}) = G_{2m} \ , \qquad (5.4.50)$$

where

$$L_{kjnm}^{*} = \int_{-1}^{1} L_{kjn}(x) W_{j}^{-1}(x) P_{m}^{(-\sigma_{j},-\tau_{j})}(x) dx ,$$

$$L_{kjn} = \int_{-1}^{1} M_{kj}(x) W_{j}(x) P_{n}^{(\sigma_{j},\tau_{j})}(x) dx ,$$

$$G_{km} = \int_{-1}^{1} g_{k}(x) W_{k}^{-1}(x) P_{m}^{(-\sigma_{k},-\tau_{k})}(x) dx ,$$

$$k, j = 1 , 2 ; \quad n,m = 1,2,3,4,..... \qquad (5.4.51)$$

After we find the coefficients C_{kn} by solving the system of linear algebraic equations (5.4.49) and (5.4.50), by taking $n, m = 1, 2, \dots, N$; we can determine the stress intensity factors. To do so, we make use of the following integrals [44]:

$$(\mathfrak{S}\mathfrak{L},\mathfrak{G}) = H(\mathfrak{Y})^{-1}($$

(53.4.3),
$$[\hat{\tau}_0(\xi y)]_1(\tau\xi)d\xi = yH(\tau^2-y)[\tau(\chi^2-y^2)]_1$$
, (5.4.53)

$$\int_{0}^{0} \cos(\xi y) J_{0}(r\xi) d\xi = H(r-y)/(r^{2}-y^{2})^{\frac{1}{2}}, \qquad (5.4.54)$$

$$(\delta \delta \cdot f \cdot \delta) = \frac{1}{2} \log(\xi y) J_1(\tau \xi y) = \frac{1}{2} \int_0^\infty \log(\xi y) J_1(\tau \xi y) dx = \frac{1}{2} \int_0^\infty \log(\xi y) J_1(\tau \xi y) dx = \frac{1}{2} \int_0^\infty \log(\xi y) J_1(\tau \xi y) dx = \frac{1}{2} \int_0^\infty \log(\xi y) J_1(\tau \xi y) dx = \frac{1}{2} \int_0^\infty \log(\xi y) J_1(\tau \xi y) dx = \frac{1}{2} \int_0^\infty \log(\xi y) J_1(\tau \xi y) dx = \frac{1}{2} \int_0^\infty \log(\xi y) J_1(\tau \xi y) dx = \frac{1}{2} \int_0^\infty \log(\xi y) J_1(\tau \xi y) dx = \frac{1}{2} \int_0^\infty \log(\xi y) J_1(\tau \xi y) dx = \frac{1}{2} \int_0^\infty \log(\xi y) J_1(\tau \xi y) dx = \frac{1}{2} \int_0^\infty \log(\xi y) J_1(\tau \xi y) dx = \frac{1}{2} \int_0^\infty \log(\xi y) dx = \frac{1}{2}$$

:swollof The stresses on z = 0 for r > a can be written in terms of ϕ_1 and ϕ_2 as

$$\begin{split} \sigma_{2ZI}(r,0^{*}) &= \mathscr{K}[\lambda_{11}(\xi)\phi_{1}(\xi) + \lambda_{12}(\xi)\phi_{2}(\xi);\xi - \lambda_{12}(\xi)\phi_{2}(\xi);\xi - \xi] \\ &= \mathscr{K}[(\lambda_{11}(\xi) - \lambda_{11}(\alpha))\phi_{1}(\xi);\eta + \mathscr{K}[\lambda_{11}(\alpha))\phi_{1}(\xi);\eta], \qquad r > \alpha, \quad (5.4.56) \\ &+ \mathscr{K}[(\lambda_{12}(\xi) - \lambda_{21}(\alpha))\phi_{1}(\xi);\eta + \mathscr{K}[\lambda_{21}(\alpha))\phi_{2}(\xi);\eta], \qquad r > \alpha, \quad (5.4.56) \\ &= \mathscr{K}[(\lambda_{21}(\xi) - \lambda_{21}(\alpha))\phi_{1}(\xi);\eta + \mathscr{K}[\lambda_{21}(\alpha))\phi_{1}(\xi);\eta], \qquad r > \alpha, \quad (5.4.57) \\ &+ \mathscr{K}[(\lambda_{22}(\xi) - \lambda_{22}(\alpha))\phi_{2}(\xi);\eta + \mathscr{K}[\lambda_{22}(\alpha))\phi_{2}(\xi);\eta], \qquad \eta > \alpha. \quad (5.4.57) \\ &+ \mathscr{K}[(\lambda_{22}(\xi) - \lambda_{22}(\alpha))\phi_{2}(\xi);\eta + \mathscr{K}[\lambda_{22}(\alpha))\phi_{2}(\xi);\eta], \qquad \eta > \alpha. \quad (5.4.57) \end{split}$$

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$$= \frac{\alpha^{2}}{\alpha^{2}} \left\{ \lambda_{11}(\mathfrak{m})\phi_{1}(\xi) \ ; \ \tau] = \lambda_{11}(\mathfrak{m}) \left\{ s \right\} \left\{ s \right\} \left\{ s \right\} \left\{ \phi_{1}(\xi) dt \right\} \left\{ \phi_{1}(\xi) dt \right\} \left\{ s \right\} \left\{ s$$

(83.4.5) .a < r $, 0 = [\tau ; (b)_{I}\phi(\omega)_{II}\lambda]$

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which by using equation (5.4.55) gives

$$\mathcal{H}_{0}[\lambda_{12}(\varpi)\phi_{2}(\xi) \ ; \ r] = \frac{\lambda_{12}(\varpi)}{a^{2}} \mathcal{H}_{0}[\tilde{\phi}_{2}(\xi) \ ; \ s]$$

$$= \frac{\lambda_{12}(\varpi)}{a^{2}} \frac{1}{s} \frac{\partial}{\partial s} \{ s \mathcal{H}_{1}[\xi^{-1}\tilde{\phi}_{2}(\xi) \ ; \ s] \}$$

$$= \frac{\lambda_{12}(\varpi)}{a^{2}} (2/\pi)^{\frac{1}{2}} \frac{1}{s} \frac{\partial}{\partial s} [s \int_{0}^{1} \psi_{2}(t) dt \int_{0}^{\infty} J_{1}(s\xi) sin(t\xi) d\xi], \qquad r > a, \quad (5.4.60)$$

which by using equation (5.4.53) gives

$$\mathcal{H}_{0}[\lambda_{12}(\mathbf{w})\phi_{2}(\xi) ; r] = \frac{\lambda_{12}(\mathbf{w})}{a^{2}} (2/\pi)^{\frac{1}{2}} \frac{1}{s} \frac{\partial}{\partial s} [\int_{0}^{1} \psi_{2}(t) t / (s^{2} - t^{2})^{\frac{1}{2}} dt, \qquad r > a. \qquad (5.4.61)$$

Similarly by means of equations (5.4.52) and (5.4.54) we get

$$\mathcal{H}_{1}[\lambda_{22}(m)\phi_{2}(\xi) ; r] = 0$$
, $r > a$, (5.4.62)

$$\mathcal{H}_{1}[\lambda_{21}(\omega)\phi_{1}(\xi) ; r] = \frac{\lambda_{21}(\omega)}{a^{2}} (2/\pi)^{\frac{1}{2}} \frac{1}{s} \frac{\partial}{\partial s} [\int_{0}^{1} \psi_{1}(t)/(s^{2}-t^{2})^{\frac{1}{2}} dt, \qquad r > a. \qquad (5.4.63)$$

Now

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$$\mathcal{H}_{0}[(\lambda_{11}(\xi) - \lambda_{11}(\omega))\phi_{1}(\xi) ; r]$$

$$= \frac{1}{a^{2}}(2/\pi)^{\frac{1}{2}} \int_{0}^{\infty} \xi[\tilde{\lambda}_{11}(\xi) - \lambda_{11}(\omega)] J_{0}(s\xi) d\xi \int_{0}^{1} \psi_{1}(t) \cos(\xi t) dt$$

$$= \frac{1}{a^{2}}(2/\pi)^{\frac{1}{2}} \int_{0}^{1} \psi_{1}(t) dt \int_{0}^{\infty} \xi[\tilde{\lambda}_{11}(\xi) - \lambda_{11}(\omega)] J_{0}(s\xi) d\xi$$

$$= O(1) , \qquad \text{as } r \to a^{+} , \quad (5.4.64)$$

and when $r \to a^+$, $\mathscr{K}_0[(\lambda_{11}(\xi) - \lambda_{11}(\omega))\phi_1(\xi); r]$ has no singularity. Similarly we have

$$\mathcal{H}_{0}[(\lambda_{12}(\xi) - \lambda_{12}(\omega))\phi_{2}(\xi) ; r] = O(1) , \qquad \text{as } r \to a^{+} ;$$

$$\mathcal{H}_{1}[(\lambda_{21}(\xi) - \lambda_{21}(\omega))\phi_{1}(\xi) ; r] = O(1) , \qquad \text{as } r \to a^{+} ;$$

$$\mathcal{H}_{1}[(\lambda_{22}(\xi) - \lambda_{22}(\omega))\phi_{2}(\xi) ; r] = O(1) , \qquad \text{as } r \to a^{+} . \qquad (5.4.65)$$

So as $r \rightarrow a^+$,

$$\sigma_{zz1}(r,0^{+})a^{2}(\pi/2)^{\frac{1}{2}}\sqrt{\lambda_{2}}/\lambda_{12}(\omega) = \lim_{s \to 1} \frac{\partial}{\partial s} \int_{0}^{1} \sqrt{\lambda_{2}} \psi_{2}(t) t / (s^{2} - t^{2})^{\frac{1}{2}} dt + O(1), \qquad (5.4.66)$$

$$\sigma_{\rm rz\,i}(r,0^{+})a^{2}(\pi/2)^{\frac{1}{2}}\sqrt{\lambda_{1}}/\lambda_{21}(\omega) = \lim_{s \to 1} \frac{\partial}{\partial s} \int_{0}^{1} \sqrt{\lambda_{1}}\psi_{1}(t)/(s^{2}-t^{2})^{\frac{1}{2}}dt + O(1).$$
(5.4.67)

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$$\frac{\partial}{\partial s} \int_{0}^{1} \sqrt{\lambda_{2}} \psi_{2}(t) t / (s^{2} - t^{2})^{\frac{1}{2}} dt = \frac{1}{2 i \partial s} \int_{0}^{1} [\zeta_{1}(t) - \zeta_{2}(t)] t / (s^{2} - t^{2})^{\frac{1}{2}} dt$$
$$= \frac{1}{2 i} \sum_{n=1k=1}^{\infty} \sum_{k=1}^{2} (-1)^{k+1} C_{kn} [\frac{\partial}{\partial s} \int_{0}^{1} t [\frac{1-t}{1+t}]^{i\omega_{k}} P_{n}^{(\sigma_{k}, \tau_{k})}(t) (s^{2} - t^{2})^{-\frac{1}{2}} dt], \qquad (5.4.68)$$

and

$$\frac{\partial}{\partial s} \int_{0}^{1} (\overline{\lambda_{1}} \psi_{1}(t) / (s^{2} - t^{2})^{\frac{1}{2}} dt = \frac{1\partial}{2\partial s} \int_{0}^{1} [\zeta_{1}(t) + \zeta_{2}(t)] / (s^{2} - t^{2})^{\frac{1}{2}} dt$$
$$= \frac{1}{2} \sum_{n=1}^{\infty} \sum_{k=1}^{2} (-1)^{k+1} C_{kn} [\frac{\partial}{\partial s} \int_{0}^{1} [\frac{1 - t}{1 + t}]^{i\omega_{k}} P_{n}^{(\sigma_{k}, \tau_{k})}(t) (s^{2} - t^{2})^{-\frac{1}{2}} dt].$$
(5.4.69)

If we define the stress intensity factors K_1 and K_2 by

$$\frac{\sqrt{\lambda_{2}}}{\lambda_{12}(\omega)}K_{1} + (-1)^{k+1}i\frac{\sqrt{\lambda_{1}}}{\lambda_{21}(\omega)}K_{2} = \lim_{r \to a^{+}} (\pi/2)^{\frac{1}{2}}[(r-a)^{\frac{1}{2}-i\omega_{k}} (r+a)^{\frac{1}{2}+i\omega_{k}}]$$
$$\cdot a^{2}[\frac{\sqrt{\lambda_{2}}}{\lambda_{12}(\omega)}\sigma_{zz1}(r,0^{+}) + (-1)^{k+1}i\frac{\sqrt{\lambda_{1}}}{\lambda_{21}(\omega)}\sigma_{rz1}(r,0^{+})] , k=1,2; \qquad (5.4.70)$$

then using the substitution s=r/a, equations (5.4.66) to (5.4.69) and following the method of Goldstein and Vainshelbarm [93] by taking

$$x=1-\frac{s-1}{s-t},$$

we get

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$$\frac{\sqrt{\lambda_{2}}}{\lambda_{12}(\varpi)}K_{1} + (-1)^{k+1}i\frac{\sqrt{\lambda_{1}}}{\lambda_{21}(\varpi)}K_{2} = (-1)^{k+1}ia[2(1-\lambda^{2})]^{\frac{1}{2}}\frac{\Gamma(1+i\omega_{k})}{\Gamma(\frac{1}{2}+i\omega_{k})}$$
$$\cdot \sum_{n=1}^{\infty} C_{kn}P_{n}(\sigma_{k},\tau_{k})(1) , \qquad k=1,2.$$
(5.4.71)

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In the previous two sections we have considered the case when the surfaces $z=-h_1$ and $z=h_2$ are stress-free. In this section we will consider other kinds of boundary conditions at the faces $z=-h_1$ and $z=h_2$ while the conditions at the interface z=0 are kept the same.

5.5.1 One face fixed and the other stress-free

In this case, we assume that all the conditions (5.3.1) to (5.3.4) remain the same and the conditions (5.3.5) are replaced by

$$u_{22}(r,-h_1) = 0$$
, $u_{r2}(r,-h_1) = 0$, $r \ge 0$. (5.5.1)

In this case condition (5.5.1) yield the following algebraic equations

$$A_2 e^{-\beta_2 \xi h_1} + B_2 e^{\beta_2 \xi h_1} + C_2 e^{-\delta_2 \xi h_1} + D_2 e^{\delta_2 \xi h_1} = 0 , \qquad (5.5.2)$$

$$A_{2}a_{2}e^{-\beta_{2}\xih_{1}} - B_{2}a_{2}e^{\beta_{2}\xih_{1}} + C_{2}\gamma_{2}e^{-\delta_{2}\xih_{1}} - D_{2}\gamma_{2}e^{\delta_{2}\xih_{1}} = 0 , \qquad (5.5.3)$$

which replace the algebraic equations (5.4.3a) and (5.4.3b) respectively. Comparing the two sets of equations, we find that if we take

$$a_2=1$$
, $d_2=1$, $b_2=a_2$, $e_2=\gamma_2$, (5.5.4)

in the results of section 5.4, we get the solution for this case.

5.5.2 Both faces fixed

In this case all the conditions (5.3.1) to (5.3.3) remain the same but the conditions (5.3.4) are replaced by the following:

$$u_{z1}(r,h_2) = 0$$
 , $u_{r1}(r,h_2) = 0$, $r \ge 0$; (5.5.5)

and (5.3.5) are replaced by (5.5.1).

In this case , equations (5.4.3) will be replaced by equations (5.5.2), (5.5.3) and the following equations

$$A_1 e^{\beta_1 \xi h_2} + B_1 e^{-\beta_1 \xi h_2} + C_1 e^{\delta_1 \xi h_2} + D_2 e^{-\delta_1 \xi h_2} = 0 , \qquad (5.5.6)$$

$$A_{1}a_{1}e^{\beta_{1}\xi h_{2}} - B_{1}a_{1}e^{-\beta_{1}\xi h_{2}} + C_{1}\gamma_{1}e^{-\delta_{1}\xi h_{2}} - D_{1}\gamma_{1}e^{-\delta_{1}\xi h_{2}} = 0 \quad .$$
 (5.5.7)

Comparing the two sets of equations, we find that if we take

$$a_i = d_i = 1$$
, $b_i = a_i$, $e_i = \gamma_i$, $(i=1,2)$, $(5.5.8)$

in the results of section 5.4, we get the solution for this case.

5.5.3 One face rigidly restrained and the other fixed

In this case, we assume that the boundary conditions (5.3.5) are replaced by (5.5.1) and conditions (5.3.4) are replaced by

$$u_{z1}(z,h_2) = 0$$
 , $\sigma_{rz1}(z,h_2) = 0$; (5.5.9)

and all the other conditions remain the same. So the equations (5.4.3) should be replaced by (5.5.7), (5.4.3b), (5.5.2) and (5.5.3). Solving equations (5.4.2), (5.5.7), (5.4.3b), (5.5.2) and (5.5.3) we can express B_1 , D_1 , A_2 , B_2 , C_2 and D_2 in terms of A_1 and C_1 by the following equations :

$$B_{1}=A_{1}e^{2\beta_{1}\xih_{2}}, \quad D_{1}=C_{1}e^{2\delta_{1}\xih_{2}},$$

$$C_{2}=-l_{21}A_{2}e^{(-\beta_{2}+\delta_{2})\xih_{1}}-l_{22}B_{2}e^{(\beta_{2}+\delta_{2})\xih_{1}},$$

$$D_{2}=-l_{22}A_{2}e^{-(\beta_{2}+\delta_{2})\xih_{1}}-l_{21}B_{2}e^{(\beta_{2}-\delta_{2})\xih_{1}},$$

$$A_{2}=l_{31}A_{1}+l_{32}C_{1}, \quad B_{2}=l_{41}A_{1}+l_{42}C_{1}, \qquad (5.5.10)$$

where l_{21} and l_{22} are given by

$$l_{21} = (\gamma_2 + a_2)/2\gamma_2$$
, $l_{22} = (\gamma_2 - a_2)/2\gamma_2$, (5.5.11)

and l_{jk} (j=3,4; k=1,2.) are still given by (5.4.5) while the expression for a_{jk} and b_{jk} (j,k=1,2) are given by the following:

$$a_{11} = a_{1}(1 + e^{2\beta_{1}\xih_{2}}) , \quad a_{12} = d_{1}(1 + e^{2\delta_{1}\xih_{2}}) ,$$

$$a_{21} = a_{2} - d_{2}l_{21}e^{(-\beta_{2}+\delta_{2})\xih_{1}} , \quad a_{22} = a_{2} + d_{2}l_{22}e^{(\beta_{2}+\delta_{2})\xih_{1}} ,$$

$$b_{11} = b_{1}(1 - e^{2\beta_{1}\xih_{2}}) , \quad b_{12} = b_{1}(1 - e^{2\delta_{1}\xih_{2}}) ,$$

$$b_{21} = b_{2} - e_{2}l_{21}e^{(-\beta_{2}+\delta_{2})\xih_{1}} , \quad b_{22} = -b_{2} + e_{2}l_{22}e^{(\beta_{2}+\delta_{2})\xih_{1}} .$$
(5.5.12)

Consequently the expressions for γ_{jk} (j,k=1,2) for this case are given by

$$\gamma_{11} = a_1 - a_1 e^{2\beta_1 \xi h_2} - a_2 l_{31} + a_2 l_{41} - \gamma_2 l_{51} + \gamma_2 l_{61} ,$$

$$\gamma_{12} = \gamma_1 - \gamma_1 e^{2\beta_1 \xi h_2} - a_2 l_{32} + a_2 l_{42} - \gamma_2 l_{52} + \gamma_2 l_{62} ,$$

$$\gamma_{21} = 1 + e^{2\beta_1 \xi h_2} - l_{31} - l_{41} - l_{51} - l_{61} ,$$

$$\gamma_{22} = 1 + e^{2\beta_1 \xi h_2} - l_{32} - l_{42} - l_{52} - l_{62} ,$$

(5.5.13)

where l_{jk} (j=5,6; k=1,2) are the same as given in (5.4.16). The solution for this case is given by the results of section 5.4 after we have made above raplacements.

5.5.4 One face rigidly restrained and the other stress-free

In this case, the boundary conditions (5.3.4) are replaced by (5.5.9) and all the other boundary conditions remain the same as in section 5.4. So in equations (5.4.2) and (5.4.3) we only have to replace equation (5.4.3a) by (5.5.7). Now we find that for this case l_{jk} (j=2,3,4; k=1,2) are all given by (5.4.5), a_{jk} and b_{jk} (j,k=1,2) are given by (5.5.12) and the solution for this case is given by the results of section 5.4 after we have made above modifications.

5.5.5 Both faces rigidly restrained

In this case, we are assuming that the boundary conditions (5.3.4) and (5.3.5) are replaced by conditions (5.5.9) and

$$u_{z2}(z,-h_1)=0$$
, $\sigma_{rz2}(z,-h_1)=0$. (5.5.14)

The equations (5.4.3a) and (5.4.3c) will be replaced by equations (5.5.7) and (5.5.3) while all other equations remain the same. Solving equations (5.4.2), (5.5.7), (5.4.3b), (5.5.3) and (5.4.3d) we can write

$$B_{1}=A_{1}e^{2\beta_{1}\xih_{2}}, \quad D_{1}=C_{1}e^{2\delta_{1}\xih_{2}},$$

$$B_{2}=A_{2}e^{-2\beta_{2}\xih_{1}}, \quad D_{2}=C_{2}e^{-2\delta_{2}\xih_{1}},$$

$$A_{2}=l_{31}A_{1}+l_{32}C_{1}, \quad C_{2}=l_{41}A_{1}+l_{42}C_{1},$$

(5.5.15)
where l_{jk} (j=3,4; k=1,2.) are given by (5.4.5) but the expression for a_{jk} and b_{jk} (j,k=1,2) should be replaced by the following

$$a_{11} = a_1(1 + e^{2\beta_1 \xi h_2}) , \quad a_{12} = d_1(1 + e^{2\delta_1 \xi h_2}) ,$$

$$a_{21} = a_2(1 + e^{-2\beta_2 \xi h_1}) , \quad a_{22} = d_2(1 + e^{-2\delta_2 \xi h_1}) ,$$

$$b_{11} = b_1(1 - e^{2\beta_1 \xi h_2}) , \quad b_{12} = e_1(1 - e^{2\delta_1 \xi h_2}) ,$$

$$b_{21} = b_2(1 - e^{-2\beta_2 \xi h_1}) , \quad b_{22} = e_2(1 - e^{-2\delta_2 \xi h_1}) .$$
(5.5.16)

Consequently for this case we should replace γ_{jk} (j,k=1,2) by

$$\gamma_{11} = a_1(1 - e^{2\beta_1 \xi h_2}) - a_2 l_{31}(1 - e^{-2\beta_2 \xi h_1}) - \gamma_2 l_{41}(1 - e^{-2\delta_2 \xi h_1}) ,$$

$$\gamma_{12} = \gamma_1(1 - e^{2\delta_1 \xi h_2}) - a_2 l_{32}(1 - e^{-2\delta_2 \xi h_1}) - \gamma_2 l_{42}(1 - e^{-2\delta_2 \xi h_1}) ,$$

$$\gamma_{21} = 1 + e^{2\beta_1 \xi h_2} - l_{31}(1 + e^{-2\beta_2 \xi h_1}) - l_{41}(1 + e^{-2\delta_2 \xi h_1}) ,$$

$$\gamma_{22} = 1 + e^{2\delta_1 \xi h_2} - l_{32}(1 + e^{-2\delta_2 \xi h_1}) - l_{42}(1 + e^{-2\delta_2 \xi h_1}) .$$

(5.5.17)

Then the solution for this case is given by the results of section 5.4.

5.6. Numerical results and discussion

To evaluate the stress intensity factors, we truncate the infinite system of simultaneous algebraic equations (5.4.49) and (5.4.50) at n = 10 and the Crout's factorisation method is used to solve these equations. And Gaussian quadrature formula is used to perform the numerical integrations involved in the solution. The relative error is controlled under 0.01. Numerical results for the stress intensity factors K_1 and K_2 are obtained for the case when the crack is subjected to a constant pressure $p_1(r)=p_0$ and $p_2(r)=0$, the thickness of the

layers is kept the same (i.e, $h_1 = h_2 = h$) and the surfaces z = -h and z = hare stress-free. For the two transversely isotropic materials considered here, the numerical values of the elastic moduli are taken as follows $(10^{11} dynes/cm^2)[5]$:

	c ₁₁	<i>c</i> ₁₂	C ₁₃	C33	C44	
Cadmium	11.00	4.04	3.83	4.69	1.56	(layer 1)
Beryl	26.94	9.61	6.61	23.63	6.53	(layer 2)

Numerical values of the stress intensity factors have been calculated for following three particular cases:

Case 1. The length of crack a = 1.0 and $h_1/a = h_2/a = h/a = 0.2(0.2), 1.0, 2.0(2.0)10.0$; the numerical values of the stress intensity factor K_1 against h/a are displayed in Fig.5.6.1 and Fig.5.6.2, and the numerical values of the stress intensity factor K_2 against h/a are displayed in Fig.5.6.3 and Fig.5.6.4.

Case 2. $h_1 = h_2 \rightarrow \infty$, and the length of the crack a = 1.0(1.0)10.0; the numerical values of the stress intensity factor K_1 against a are displayed in Fig.5.6.5, and the numerical values of the stress intensity factor K_2 against a are displayed in Fig.5.6.6.

Case 3. $h_1=h_2=20.0$ and the length of the crack a = 1.0(1.0)10.0; the numerical values of the stress intensity factor K_1 against a are displayed in Fig.5.6.7, and the numerical values of the stress intensity factor K_2 against a are displayed in Fig.5.6.8.



Numerical values of the stress intensity factor K_1 against h/a (0.2 to 1.0) for a fixed crack length a and equal layer thickness $(h_1=h_2=h)$.



Numerical values of the stress intensity factor K_1 against h/a (1.0 to 10.0) for a fixed crack length a and equal layer thickness $(h_1=h_2=h)$.

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Fig.5.6.3

Numerical values of the stress intensity factor K_2 against h/a (0.2 to 1.0) for a fixed crack length a and equal layer thickness $(h_1=h_2=h)$.



Numerical values of the stress intensity factor K_2 against h/a (1.0 to 10.0) for a fixed crack length a and equal layer thickness $(h_1=h_2=h)$.



Fig.5.6.5

Numerical values of the stress intensity factor K_1 against the crack length a (1.0 to 10.0) for the layer thickness $h_1=h_2 \rightarrow \infty$.



Fig.5.6.6

Numerical values of the stress intensity factor K_2 against the crack length a (1.0 to 10.0) for the layer thickness $h_1=h_2 \rightarrow \infty$.



Numerical values of the stress intensity factor K_1 against the crack length a (1.0 to 10.0) for the layer thickness $h_1=h_2=20$.





Numerical values of the stress intensity factor K_2 against the crack length a (1.0 to 10.0) for the layer thickness $h_1=h_2=20$.

We make the following observations from the graphs : when the length of the crack *a* is fixed at a=1.0, the stress intensity factor K_2 increases as the thickness of the layers increases; while the stress intensity factor K_1 decreases as the thickness of the layers increases from 0.1 to 1.0, but increases when the thickness of the layers increases from 1.0 to 10.0. If the thickness of the layers is fixed at 20.0 or $\rightarrow \infty$ the stress intensity factors K_1 and K_2 decrease as the length of the crack increases.

Recommendation for future work

As a continuation of the present work, the following problems may be considered in the future :

As an extension of the work of chapter 2 one could consider the Reissner-Sagoci problem of a finite elastic cylinder embedded in an infinite elastic layer and the whole is perfectly bonded to an elastic half-space.

The problem considered in chapter 3 may be extended to two dissimilar transversely isotropic layers with a penny-shaped flaw located at the interface, when a rigid shaft bonded to the elastic layer is rotated through a small angle.

The problem of chapter 4 may be extended to study the determination of stress intensity factors at the tips of a column of Griffith cracks, which are parallel to the x-axis and are equally spaced along the y-axis, located at the interfaces of the orthotropic multilayer composite materials.

Similarly, a problem related to chapter 5 is a problem of determination of stress intensity factors at the edges of of a column of penny-shaped cracks, which are parallel to the xy-plane and are equally spaced along the z-axis, located at the interfaces of the transversely isotropic multilayer composite material.

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