# THE UNIVERSITY OF CALGARY 

## SOME MIXED BOUNDARY VALUE PROBLEMS IN THE LINEAR THEORY OF ELASTICITY

by

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## THE UNIVERSITY OF CALGARY FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Some Mixed Boundary Value Problems In The Linear Theory Of Elasticity", submitted by Wenhua He in partial fulfillment of the requirements for the degree of Doctor of Philosophy.


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## ABSTRACT

The present thesis deals with some torsion and crack problems in the linear theory of elasticity. In chapter 1 , we have given a brief summary of the linear theory of the elasticity. Chapter 2 deals with three different problems of Reissner-Sagoci type for composite cylindrical regions. In chapter 3, we have discussed a torsion problem of two bonded nonhomogeneous elastic layers with a penny-shaped flaw at the interface. In chapter 4, we have solved a problem of Griffith crack at the interface of two dissimilar orthotropic elastic layers. In chapter 5, we have solved a penny-shaped interface crack problem between two dissimilar transversely isotropic layers. .The numerical values of the physical quantities have been obtained and displayed graphically.

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Approval page ..... ii
Abstract ..... iii
Acknowledgements ..... iv
List of Symbols ..... ix
List of Figures ..... x
Chaper 1 Basic Equations of Mathematical Theory of Elasticity ..... 001
1.1 Strain and Stress ..... 002
1.2 Generalized Hooke's Law ..... 011
1.3 Plane Strain ..... 017
1.4 Polar Cylindrical Coordinates ..... 018
1.5 Penny-shaped Crack Problem ..... 025
1.6 Torsion Problem ..... 026
Chapter 2 Problems of Reissner-Sagoci Type For Composite Cylindrical Regions ..... 028
2.1 Introduction ..... 028
2.2 Finite Composite Elastic Cylinder ..... 033
2.2.1 The Statement of The Problem ..... 033
2.2.2 Derivation of The Dual Integral Equations ..... 035
2.2.3 Reduction to Integral Equation of Fredholm Type ..... 038
2.2.4 Numerical Results and Conclusions ..... 043
2.3 Finite Elastic Cylinder Embedded In An Elastic Layer ..... 049
2.3.1 The Statement of the Problem ..... 049
2.3.2 Derivation of The Dual Integral Equations ..... 051
2.3.3 Reduction to Integral Equation of Fredholm Type ..... 053
2.3.4 Particular Cases ..... 055
2.3.5 Numerical Results and Conclusions ..... 057
2.4 Semi-infinite Composite Elastic Cylinder ..... 062
2.4.1 The Statement of The Problem ..... 062
2.4.2 Derivation of The Dual Integral Equations ..... 064
2.4.3 Reduction to Integral Equation of Fredholm Type ..... 067
2.4.5 Numerical Results and Conclusions ..... 069
Chapter 3 Torsion of Two Nonhomogeneous Elastic Layers With Penny-Shaped Flaw At The Interface ..... 071
3.1 Introduction ..... 071
3.2 Inclusion Problem with The Surface $z=h_{2}$ Stress-Free ..... 075
3.2.1 Statement of The Problem ..... 075
3.2.2 Analysis ..... 076
3.2.3 Solution for The Homogeneous Case and Numerical Results ..... 081
3.2.4 Solution for The Case $h_{2} \rightarrow \infty$ and Numerical Results ..... 092
3.3 Crack Problem with The Surface $z=h_{2}$ Stress-Free ..... 096
3.3.1 Statement of The Problem ..... 096
3.3.2 Analysis ..... 096
3.3.3 Solution for The Homogeneous Case and Numerical Results ..... 101
3.3.4 Solution for The Case $h_{2} \rightarrow \infty$ and Numerical Results ..... 112
3.4 Inclusion Problem with The Surface $z=h_{2}$ rigidly fixed ..... 116
3.4.1 Analysis ..... 116
3.4.2 Solution for The Homogeneous Case and Numerical Results ..... 117
3.5 Crack Problem With The Surface $z=h_{2}$ Rigidly Fixed ..... 128
3.5.1 Analysis ..... 128
3.5.2 Solution for The Homogeneous Case and Numerical Results ..... 129
3.6 Some Comments on The Solutions ..... 140
Chapter 4 Griffith Crack At The Interface of Two Orthotropic Elastic Layers ..... 141
4.1 Introduction ..... 141
4.2 Basic Equations and Their Solution ..... 143
4.3 Statement of The Problem and Derivation of The Singular Integral Equations ..... 145
4.4 Other Cases ..... 157
4.4.1 One Face Fixed and The Other Stress-Free ..... 157
4.4.2 Both Faces Fixed ..... 158
4.4.3 One Face Rigidly Restrained and The Other Fixed ..... 159
4.4.4 One Face Rigidly Restrained and The Other Stress- Free ..... 160
4.4.5 Both Faces Rigidly Restrained ..... 161
4.5 Numerical Results and Discussion ..... 162
Chapter 5 Penny-Shaped Interface Crack Between Two Dissimilar Transversely Isotropic Layers ..... 175
5.1 Introduction ..... 175
5.2 Basic Equations and Their Solution ..... 178
5.3 Statement of The Problem and Boundary Conditions ..... 180
5.4 Analysis ..... 182
5.5 Other Cases ..... 194
5.5.1 One Face Fixed and The Other Stress-Free ..... 194
5.5.2 Both Faces Fixed ..... 195
5.5.3 One Face Rigidly Restrained and The Other Fixed ..... 195
5.5.4 One face Rigidly Restrained and The Other Stress- Free ..... 197
5.5.5 Both faces rigidly Restrained ..... 197
5.6 Numerical Results and Discussion ..... 198
Recommendation for future work ..... 209
References ..... 210

## Nomenclature

$c_{\mathrm{ij}} \quad$ the elastic moduli for anisotropic material.
$\lambda, \mu \quad$ Lame constants for isotropic elastic material.
$u_{\mathrm{i}} \quad$ the displacement component along $i$-direction.
$\sigma_{\mathrm{ij}} \quad$. the stress tensor components.
$e_{\mathrm{ij}} \quad$ the strain tensor components.
$\delta_{\mathrm{ij}} \quad$ the Kronecker delta.
$J_{\nu}(x) \quad$ the Bessel functions of the first kind and of order $\nu$.
$Y_{\nu}(x) \quad$ the Bessel functions of the second kind and of order $\nu$.
$I_{\nu}(\mathrm{x}) \quad$ the modified Bessel functions of the first kind and of order $\nu$.
$K_{\nu}(x) \quad$ the modified Bessel functions of the second kind and of order $\nu$.
$\mathscr{F}_{\mathrm{s}} \quad$ the Fourier sine transform defined by the equation

$$
\mathscr{F}_{\mathrm{s}}[f(z) ; z \rightarrow \xi]=\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_{0}^{\infty} f(z) \sin (\xi z) d z
$$

$\mathscr{F}_{\mathrm{c}} \quad$ the Fourier cosine transform defined by the equation

$$
\mathscr{F}_{\mathrm{c}}[f(z) ; z \rightarrow \xi]=\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_{0}^{\infty} f(z) \cos (\xi z) d z
$$

$\mathscr{H}$ the Hankel transform defined by the equation

$$
\mathscr{H}[f(\xi) ; \xi \rightarrow r]=\int_{0}^{\infty} \xi f(\xi) J_{\nu}(\xi r) d \xi
$$

$\mathscr{S}_{1}$ the Abel transform of the first kind defined by the equation

$$
\mathscr{C}_{1}[f(t) ; t \rightarrow r]=\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_{0}^{r} f(t) /\left[r^{2-t^{2}}\right]^{\frac{1}{2}} d t
$$

$\mathscr{S}_{2}$ the Abel transform of the second kind defined by the equation

$$
\mathscr{C}_{2}[f(t) ; t \rightarrow r]=\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_{r}^{\infty} f(t) /\left[t^{2}-\mathrm{r}^{2}\right]^{\frac{1}{2}} d t
$$

$H(x) \quad$ the Heaviside function.
$\Gamma(x) \quad$ the Gamma fuction.
$P_{\mathrm{n}}{ }^{(\alpha, \beta)}(x)$ the Jacobi polynomials.

## LIST OF FIGURES

Fig.1.1.1 Displacement and strain. ..... 003
Fig.1.1.2 Stress vector ..... 006
Fig.1.1.3 Stress and stress tensor ..... 008
Fig.1.4.1 Displacement and stress in cylindrical polarcoordinates ......................................... 020

Fig.2.2.1 Torsion of an elastic cylinder bonded to a dissimilar elastic cylindrical shell034

Fig.2.2.2 Numerical values of the ratio of the torques $T / T_{0}$ against $d / c$ for $\bar{\mu}=0.5, a / d=1.1$ for various values of $b / c=0.2,0.3,0.4,0.5,1.0,2.0,10.0$045

Fig.2.2.3 Numerical values of the ratio of the torques $T / T_{0}$ against $d / c$ for $\bar{\mu}=0.5, a / d=1.5$ for various values of $b / c=0.2,0.3,0.4,0.5,1.0,2.0,10.0$046

Fig.2.2.4 Numerical values of the ratio of the torques $T / T_{0}$ against $d / c$ for $\bar{\mu}=2.0, a / d=1.1$ for various values of $b / c=0.2,0.3,0.4,0.5,1.0,2.0,10.0$047

Fig.2.2.5 Numerical values of the ratio of the torques $T / T_{0}$ against $d / c$ for $\bar{\mu}=2.0, a / d=1.5$ for various values of $b / c=0.2,0.3,0.4,0.5,1.0,2.0,10.0$ 048

Fig.2.3.1 Torsion of an elastic cylinder bonded to a dissimilar elastic layer

Fig.2.3.2 Values of $T / T_{0}$ displayed against $d / c$ for $b / c=0.2$, $0.3,0.4,0.5,1.0,2.0,10.0$ and $\bar{\mu}=0.0$

Fig.2.3.3 Values of $T / T_{0}$ displayed against $d / c$ for $b / c=0.2$, $0.3,0.4,0.5,1.0,2.0,10.0$ and $\bar{\mu}=0.5$

Fig.2.3.4 Values of $T / T_{0}$ displayed against $d / c$ for $b / c=0.2$,
$0.3,0.4,0.5,1.0,2.0,10.0$ and $\bar{\mu}=2.0$
Fig.2.3.5 Values of $T / T_{0}$ displayed against $d / c$ for $b / c=0.2$, $0.3,0.4,0.5,1.0,2.0,10.0$ and $\bar{\mu} \rightarrow \infty$

Fig.2.4.1 A problem of Reissner-Sagoci type for a semi-infinite composite elastic cylinder

Fig.2.4.2 Numerical values of ratio of the torques $T / T_{0}$ against $d / c$ for $\bar{\mu}=0.5,1.0,2.0 ; a / b=1.1,1.5$

Fig.3.1.1 Torsion of two elastic layers by a rigid shaft.
Figs.3.2.1-3.2.4 For Inclusion Problem, Numerical values of $\phi(1)$, $\Psi(1), M / M_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against. $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25$, $0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=0.5, \mathrm{~h}_{1} / \mathrm{a}=0.5 \ldots$

Figs.3.2.5-3.2.8 For Inclusion Problem, Numerical values of $\phi(1)$, $\Psi(1), M / M_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $b / a$ for $h_{2} / h_{1}=0.25$, $0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=1.0, \mathrm{~h}_{1} / \mathrm{a}=0.5 \ldots$

Figs.3.2.9-3.2.12 For Inclusion Problem, Numerical values of $\phi(1)$, $\Psi(1), M / M_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $b / a$ for $h_{2} / h_{1}=0.25$, $0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=2.0, h_{1} / \mathrm{a}=0.5 \ldots$

Figs.3.2.13-3.2.16 For Inclusion Problem, Numerical values of $\phi(1)$,
$\Psi(1), M / M_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $b / a$ for $h_{2} / h_{1}=0.25$,
$0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \dot{\mu_{1}}=0.5, \mathrm{~h}_{1} / \mathrm{a}=1.0 \ldots$
Figs.3.2.17-3.2.20 For Inclusion Problem, Numerical values of $\phi(1)$,
$\Psi(1), M / M_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $b / a$ for $h_{2} / h_{1}=0.25$, $0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=1.0, \mathrm{~h}_{1} / \mathrm{a}=1.0 \ldots$

Figs.3.2.21-3.2.24 For Inclusion Problem, Numerical values of $\phi(1)$, $\Psi(1), M / M_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $b / a$ for $h_{2} / h_{1}=0.25$, $0.5,1.0,2.0,5.0$ and $\vec{\mu}=\mu_{2} / \mu_{1}=2.0, \mathrm{~h}_{1} / \mathrm{a}=1.0 \ldots$

Figs.3.2.25-3.2.28 For Inclusion Problem, Numerical values of $\phi(1)$, $\Psi(1), M / M_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $b / a$ for $h_{2} / h_{1}=0.25$, $0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=0.5, \mathrm{~h}_{1} / \mathrm{a}=2.0 \ldots$

Figs.3.2.29-3.2.32 For Inclusion Problem, Numerical values of $\phi(1)$, $\Psi(1), M / M_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $b / a$ for $h_{2} / h_{1}=0.25$, $0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=1.0, \mathrm{~h}_{1} / \mathrm{a}=2.0 \ldots$

Figs.3.2.33-3.2.36 For Inclusion Problem, Numerical values of $\phi(1)$, $\Psi(1), M / M_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25$, $0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=2.0, \mathrm{~h}_{1} / \mathrm{a}=2.0 \ldots$

Figs.3.2.37-3.2.40 For Inclusion Problem, Numerical values of $\Phi(1)$, $\Psi(1), \mathrm{M} / \mathrm{M}_{0}, \beta=\epsilon_{1} / \epsilon_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{1} / \mathrm{a}=0.2,0.3$, $0.4,0.5,0.6,0.8,1.0,2.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=0.5 \ldots$.

Figs.3.2.41-3.2.44 For Inclusion Problem, Numerical values of $\Phi(1)$,
$\Psi(1), M / M_{0}, \beta=\epsilon_{1} / \epsilon_{0}$ against $b / a$ for $h_{1} / a=0.2,0.3$, $0.4,0.5,0.6,0.8,1.0,2.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=1.0 \ldots .$.

Figs.3.2.45-3.2.48 For Inclusion Problem, Numerical values of $\Phi(1)$,
$\Psi(1), M / M_{0}, \beta=\epsilon_{1} / \epsilon_{0}$ against $b / a$ for $h_{1} / a=0.2,0.3$,
$0.4,0.5,0.6,0.8,1.0,2.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=2.0 \ldots .$.
Figs.3.3.1-3.3.3 For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and $M / M_{0}$ against $b / a$ for $h_{2} / h_{1}=0.25,0.5$, $1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=0.5, \mathrm{~h}_{1} / \mathrm{a}=0.5 \ldots \ldots .$.

Figs.3.3.4-3.3.6 For Crack Problem, Numerical values of $\phi(1)$,
$\Psi(1)$ and $M / M_{0}$ against $b / a$ for $h_{2} / h_{1}=0.25,0.5$, $1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=1.0, \mathrm{~h}_{1} / \mathrm{a}=0.5$

Figs.3.3.7-3.3.9 For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and $M / M_{0}$ against $b / a$ for $h_{2} / h_{1}=0.25,0.5$, $1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=2.0, \mathrm{~h}_{1} / \mathrm{a}=0.5 \ldots \ldots .$.

Figs.3.3.10-3.3.12 For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and $M / M_{0}$ against $b / a$ for $h_{2} / h_{1}=0.25,0.5$, $1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=0.5, \mathrm{~h}_{1} / \mathrm{a}=1.0 \ldots \ldots .$. 106

Figs.3.3.13-3.3.15 For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and $M / M_{0}$ against $b / a$ for $h_{2} / h_{1}=0.25,0.5$, $1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=1.0, \mathrm{~h}_{1} / \mathrm{a}=1.0 \ldots \ldots .$.

Figs.3.3.16-3.3.18 For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and $M / M_{0}$ against $b / a$ for $h_{2} / h_{1}=0.25,0.5$, $1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=2.0, \mathrm{~h}_{1} / \mathrm{a}=1.0 \ldots \ldots .$.

Figs.3.3.19-3.3.21 For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and $M / M_{0}$ against $b / a$ for $h_{2} / h_{1}=0.25,0.5$, $1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=0.5, \mathrm{~h}_{1} / \mathrm{a}=2.0 \ldots \ldots .$.

Figs.3.3.22-3.3.24 For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and $\mathrm{M} / \mathrm{M}_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5$, 1.0, 2.0, 5.0 and $\bar{\mu}=\mu_{2} / \mu_{1}=1.0, \mathrm{~h}_{1} / \mathrm{a}=2.0 \ldots \ldots .$.

Figs.3.3.25-3.3.27 For Crack Problem, Numerical values of $\phi(1)$,
$\Psi(1)$ and $M / M_{0}$ against $b / a$ for $h_{2} / h_{1}=0.25,0.5$,
$1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=2.0, \mathrm{~h}_{1} / \mathrm{a}=2.0 \ldots \ldots .$.
Figs.3.3.28-3.3.30 For Crack Problem, Numerical values of $\Phi(1)$, $\Psi(1)$ and $M / M_{0}$ against $b / a$ for $h_{1} / a=0.2,0.3,0.4$, $0.5,0.6,0.8,1.0,2.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=0.5 \ldots \ldots . .$.

Figs.3.3.31-3.3.33 For Crack Problem, Numerical values of $\Phi(1)$,
$\Psi(1)$ and $M / M_{0}$ against $b / a$ for $h_{1} / a=0.2,0.3,0.4$; $0.5,0.6,0.8,1.0,2.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=1.0$

Figs.3.3.34-3.3.36 For Crack Problem, Numerical values of $\Phi(1)$,
$\Psi(1)$ and $M / M_{0}$ against $b / a$ for $h_{1} / a=0.2,0.3,0.4$,
$0.5,0.6,0.8,1.0,2.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=2.0$
Figs.3.4.1-3.4.4 For Inclusion Problem, Numerical values of $\phi(1)$, $\Psi(1), M / M_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $b / a$ for $h_{2} / h_{1}=0.25$, $0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=0.5, \mathrm{~h}_{1} / a=0.5 \ldots$

Figs.3.4.5-3.4.8 For Inclusion Problem, Numerical values of $\phi(1)$, $\Psi(1), M / M_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $b / a$ for $h_{2} / h_{1}=0.25$, $0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=1.0, \mathrm{~h}_{1} / \mathrm{a}=0.5 \ldots$

Figs.3.4.9-3.4.12 For Inclusion Problem, Numerical values of $\phi(1)$, $\Psi(1), M / M_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $b / a$ for $h_{2} / h_{1}=0.25$, $0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=2.0, \mathrm{~h}_{1} / a=0.5 \ldots$

Figs.3.4.13-3.4.16 For Inclusion Problem, Numerical values of $\phi(1)$, $\Psi(1), M / M_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $b / a$ for $h_{2} / h_{1}=0.25$, $0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=0.5, \mathrm{~h}_{1} / \mathrm{a}=1.0 \ldots$

Figs.3.4.17-3.4.20 For Inclusion Problem, Numerical values of $\phi(1)$,
$\Psi(1), \mathrm{M} / \mathrm{M}_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25$,
$0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=1.0, \mathrm{~h}_{1} / \mathrm{a}=1.0 \ldots$
Figs.3.4.21-3.4.24 For Inclusion Problem, Numerical values of $\phi(1)$,
$\Psi(1), M / M_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $b / a$ for $h_{2} / h_{1}=0.25$,
$0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=2.0, \mathrm{~h}_{1} / \mathrm{a}=1.0 \ldots$
Figs.3.4.25-3.4.28 For Inclusion Problem, Numerical values of $\phi(1)$,
$\Psi(1), M / M_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $b / a$ for $h_{2} / h_{1}=0.25$,
$0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=0.5, \mathrm{~h}_{1} / \mathrm{a}=2.0 \ldots$
Figs.3.4.29-3.4.32 For Inclusion Problem, Numerical values of $\phi(1)$,
$\Psi(1), M / M_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $b / a$ for $h_{2} / h_{1}=0.25$,
$0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=1.0, \mathrm{~h}_{1} / \mathrm{a}=2.0 \ldots$

Figs.3.4.33-3.4.36 For Inclusion Problem, Numerical values of $\phi(1)$, $\Psi(1), M / M_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25$, $0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=2.0, h_{1} / a=2.0 \ldots$
Figs.3.5.1-3.5.3 For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and $\mathrm{M} / \mathrm{M}_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5,1.0$, $2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=0.5, h_{1} / a=0.5$
Figs.3.5.4-3.5.6 For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and $M / M_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5,1.0$, $2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=1.0, \mathrm{~h}_{1} / \mathrm{a}=0.5$ 132

Figs.3.5.7-3.5.9 For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and $\mathrm{M} / \mathrm{M}_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5,1.0$, $2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=2.0, \mathrm{~h}_{1} / \mathrm{a}=0.5$ 133

Figs.3.5.10-3.5.12 For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and $M / M_{0}$ against $b / a$ for $h_{2} / h_{1}=0.25,0.5,1.0$, 2.0, 5.0 and $\bar{\mu}=\mu_{2} / \mu_{1}=0.5, h_{1} / a=1.0$ 134

Figs.3.5.13-3.5.15 For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and $M / M_{0}$ against $b / a$ for $h_{2} / h_{1}=0.25,0.5,1.0$, $2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=1.0, \mathrm{~h}_{1} / \mathrm{a}=1.0$ 135

Figs.3.5.16-3.5.18 For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and $M / M_{0}$ against $b / a$ for $h_{2} / h_{1}=0.25,0.5,1.0$, $2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=2.0, h_{1} / \mathrm{a}=1.0$
Figs.3.5.19-3.5.21 For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and $M / M_{0}$ against $b / a$ for $h_{2} / h_{1}=0.25,0.5,1.0$, $2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=0.5, h_{1} / a=2.0$

Figs.3.5.22-3.5.24 For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and $M / M_{0}$ against $b / a$ for $h_{2} / h_{1}=0.25,0.5,1.0$, $2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=1.0, h_{1} / \mathrm{a}=2.0$

Figs.3.5.25-3.5.27 For Crack Problem, Numerical values of $\phi(1)$, $\Psi(1)$ and $M / M_{0}$ against $b / a$ for $h_{2} / h_{1}=0.25,0.5,1.0$, $2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=2.0, \mathrm{~h}_{1} / \mathrm{a}=2.0$ 139

Fig.4.3.1 Griffith crack at the interface of two dissimilar orthotropic elastic layers 147

Fig.4.5.1 Numerical values of the stress intensity factor $K_{1}$ against $h / a$ ( 0.2 to 1.0 ) for a fixed crack length $a$ and equal layer thickness $\left(h_{1}=h_{2}=h\right)$

Fig.4.5.2 Numerical values of the stress intensity factor $K_{1}$ against $h / a$ (1.0 to 10.0 ) for a fixed crack length $a$ and equal layer thickness $\left(h_{1}=h_{2}=h\right)$

Fig.4.5.3 Numerical values of the stress intensity factor $K_{2}$ against $h / a$ (0.2 to 1.0) for a fixed crack length $a$ and layers of equal thickness ( $h_{1}=h_{2}=h$ )
Fig.4.5.4 Numerical values of the stress intensity factor $K_{2}$ against $h / a$ (1.0 to 10.0 ) for a fixed crack length $a$ and layers of equal thickness ( $h_{1}=h_{2}=h$ )167

Fig.4.5.5 Numerical values of the stress intensity factor $K_{1}$ against the crack length $a$ (1.0 to 10.0 ) for the layer thickness $h_{1}=h_{2} \rightarrow \infty$

Fig.4.5.6 Numerical values of the stress intensity factor $K_{2}$ against the crack length $a$ (1.0 to 10.0 ) for the layer thickness $h_{1}=h_{2} \rightarrow \infty$
Fig.4.5.7 Numerical values of the stress intensity factor $K_{1}$ against the crack length $a$ (1.0 to 10.0) for the layer thickness $h_{1}=h_{2}=20$

Fig.4.5.8 Numerical values of the stress intensity factor $K_{2}$ against the crack length $a$ ( 1.0 to 10.0 ) for the layer thickness $h_{1}=h_{2}=20$...........................................

Fig.4.5.9 Numerical values of the stress intensity factor $K_{1}$ against the crack length $a$ ( 1.0 to 10.0 ) for the layer thickness $h_{1}=h_{2}=10$

Fig.4.5.10 Numerical values of the stress intensity factor $K_{2}$ against the crack length $a$ ( 1.0 to 10.0 ) for the layer thickness $h_{1}=h_{2}=10$

Fig.5.3.1 Penny-shaped interface crack between two dissimilar transversely isotrpic layers

Fig.5.6.1 Numerical values of the stress intensity factor $K_{1}$ against $h / a$ ( 0.2 to 1.0 ) for a fixed crack length $a$ and layers of equal thickness ( $h_{1}=h_{2}=h$ )
Fig.5.6.2 Numerical values of the stress intensity factor $K_{1}$ against $h / a$ ( 1.0 to 10.0 ) for a fixed crack length $a$ and layers of equal thickness $\left(h_{1}=h_{2}=h\right)$
Fig.5.6.3 Numerical values of the stress intensity factor $K_{2}$ against $h / a$ ( 0.2 to 1.0 ) for a fixed crack length $a$ and layers of equal thickness ( $h_{1}=h_{2}=h$ )
Fig.5.6.4 Numerical values of the stress intensity factor $K_{2}$ against $h / a$ (1.0 to 10.0) for a fixed crack length $a$ and layers of equal thickness ( $h_{1}=h_{2}=h$ )

Fig.5.6.5 Numerical values of the stress intensity factor $K_{1}$ against the crack length $a$ ( 1.0 to 10.0 ) for the layer thickness $h_{1}=h_{2} \rightarrow \infty$

Fig.5.6.6 Numerical values of the stress intensity factor $K_{2}$ against the crack length $a$ (1.0 to 10.0 ) for the layer thickness $h_{1}=h_{2} \rightarrow \infty$ 205

Fig.5.6.7 Numerical values of the stress intensity factor $K_{1}$ against the crack length $a$ (1.0 to 10.0 ) for the layer thickness $h_{1}=h_{2}=20$ 206

Fig.5.6.8 Numerical values of the stress intensity factor $K_{2}$ against the crack length $a$ (1.0 to 10.0) for the layer thickness $h_{1}=h_{2}=20$

## CHAPTER 1

## BASIC EQUATIONS OF MATHEMATICAL THEORY OF ELASTICITY

The mathematical theory of elasticity has a long history. Hooke, Bernoulli, Navier, Cauchy and Green made a lot of contributions to the developments of the mathematical theory of elasticity. There are many excellent books which give introduction to the basic theory of elasticity. Sokolnikoff [1], Green and Zerna [2], Love [3] and Fung [3a] are good reference books for isotropic elasticity, while Lekhnitskii [4] and Hearman [5] are good reference books for anisotropic elasticity. In this chapter we will give an outline of the linear theory of elasticity and some basic formulae which are needed later.

In the study of the distribution of stresses and deformations in an elastic body, we regard an elastic body as a solid continuous medium. The configuration of a solid body is described by a region of a Euclidean point space whose geometrical points are identified with the position of the material particles of the body.

Let a system of coordinates $x_{1}, x_{2}, x_{3}$ be chosen so that a point $P$ of an elastic body at a certain instant of time has the coordinates $x_{i}(i=1,2,3)$. Under some physical actions the configuration of the solid body changes. Suppose at a later instant of time the point $P$ moves to $P^{*}$ with coordinates $y_{\mathrm{i}}(i=1,2,3)$ with respect to a new system of coordinates $y_{1}, y_{2}, y_{3}$. The change of the
configuration of the elastic body can be thought of a one-to-one mapping between two configurations. The mapping can be written as

$$
y_{\mathrm{i}}=\tilde{y}_{\mathrm{i}}\left(x_{1}, x_{2}, x_{3}\right), \quad(i=1,2,3) ;
$$

with the unique inverse mapping

$$
x_{\mathrm{i}}=\tilde{x}_{\mathrm{i}}\left(y_{1}, y_{2}, y_{3}\right), \quad(i=1,2,3)
$$

for every point of the elastic body. The change of configuration is assumed to be continuous and smooth. In fact, we are assuming that the functions $\tilde{x}_{i}$ and $\tilde{y}_{\mathrm{i}}(i=1,2,3)$ are twice continuously differentiable.

### 1.1 Strain and Stress

If the distance between particles of a body is changed under an action, the body is said to be deformed and otherwise the body is said to be undeformed. To study deformation of an elastic body, let us fix a cartesian coordinate system $O-x_{1} x_{2} x_{3}$. Suppose a point $P\left(\xi_{1}, \xi_{2}, \xi_{3}\right)$ is moved to the position $P^{*}\left(\xi_{1}{ }^{*}, \xi_{2}{ }^{*}, \xi_{3}{ }^{*}\right)$ under a physical action. The difference $u_{\mathrm{i}}(P)=\xi_{\mathrm{i}}{ }^{*}-\xi_{\mathrm{i}}(i=1,2,3)$ is called the displacement of point $P$ along the $x_{\mathrm{i}}$ direction. The displacement vector $\left\{u_{1}, u_{2}, u_{3}\right\}$ varies, in general, from point to point of the body and is twice continuously differentiable.

Now suppose that a neighborhood point of $P$, say $Q\left(\eta_{1}, \eta_{2}, \eta_{3}\right)$ is moved to $Q^{*}\left(\eta_{1}{ }^{*}, \eta_{2}{ }^{*}, \eta_{3}{ }^{*}\right)$ with the displacement components $u_{\mathrm{i}}(Q)=\eta_{\mathrm{i}}{ }^{*}-\eta_{\mathrm{i}},(i=1,2,3)$. Let the vector joining the points $P$ and $Q$ be $\neq$ with the components $\tau_{1}, \tau_{2}$ and $\tau_{3}$, and ${ }^{*}$ *be the vector joining the points $P^{*}$ and $Q^{*}$. From the Fig.1.1.1 we know that the vector $\delta \neq \neq{ }^{*}-\neq$ has the components

$$
\delta \tau_{\mathrm{i}}=\left(\eta_{\mathrm{i}}^{*}-\xi_{\mathrm{i}}^{*}\right)-\left(\eta_{\mathrm{i}}-\xi_{\mathrm{i}}\right)=\left(\eta_{\mathrm{i}}^{*}-\eta_{\mathrm{i}}\right)-\left(\xi_{\mathrm{i}}^{*}-\xi_{\mathrm{i}}\right)=u_{\mathrm{i}}(Q)-u_{\mathrm{i}}(P), \quad i=1,2,3
$$



Fig. 1.1.1

If we assume that the displacement components $u_{i}(i=1,2,3)$, and their partial derivatives are so small that their product can be neglected then we have

$$
\begin{align*}
& \delta \tau_{\mathrm{i}}=u_{i, j} \tau_{j} \\
& =\frac{1}{2}\left[\left(u_{i, j}+u_{j, i}\right)+\left(u_{i, j} u_{j, i}\right)\right] \tau_{j} \\
& =\left(e_{i, j}+w_{i, j}\right) \tau_{j} \tag{1.1.1}
\end{align*}
$$

where

$$
\begin{align*}
& e_{i j}=\frac{1}{2}\left(u_{i, j}+u_{j, i}\right),  \tag{1.1.2}\\
& w_{i j}=\frac{1}{2}\left(u_{i, j} u_{j, i}\right), \tag{1.1.3}
\end{align*}
$$

and the comma in the subscript means a partial differentiation, e.g $u_{i, j}=\frac{\partial u_{i}}{\partial x_{j}}$.
The symmetric coefficients $e_{i j}$ are called components of strain tensor at the point $P$, which characterize a pure deformation; while the skew-symmetric coefficients $w_{i j}$ correspond to a rigid body rotation.

We consider next the transformation of the components of the strain tensor under a rotation of axes of a Cartesian coordinate system. Let the two coordinates be connected by the following linear relations

$$
\begin{equation*}
x_{j}^{\prime}=\beta_{j i} x_{i}, \quad i, j=1,2,3 \tag{1.1.4}
\end{equation*}
$$

where $\beta_{i j}$ are the direction cosines of the $x_{i}^{\prime}$-axis with respect to the $x_{j}$ -axis. The matrix $T=\left(\beta_{j i}\right)$ is orthogonal and

$$
\frac{\partial x_{i}}{\partial x_{j}^{\prime}}=\beta_{j i}, \quad \frac{\partial x_{i}^{\prime}}{\partial x_{j}}=\beta_{i j}, \quad i, j=1,2,3
$$

Then the relations of the components of displacement vector with respect to the two coordinate systems are given by

$$
\begin{equation*}
u_{i}^{\prime}=\beta_{i j} u_{j}, \quad i, j=1,2,3 \tag{1.1.5}
\end{equation*}
$$

Substituting equations (1.1.4) and (1.1.5) into equations (1.1.2) we obtain

$$
\begin{align*}
e_{i j}^{\prime} & =\frac{1}{2}\left(u_{i, j}^{\prime}+u_{j, i}^{\prime}\right)=\frac{1}{2}\left[\frac{\partial}{\partial x_{j}^{\prime}}\left(\beta_{i k} u_{k}\right)+\frac{\partial}{\partial x_{i}^{\prime}}\left(\beta_{j m} u_{m}\right)\right] \\
& =\frac{1}{2}\left[\beta_{i k}\left(\frac{\partial x_{l}}{\partial x_{j}^{\prime}} \frac{\partial}{\partial x_{l}} u_{k}\right)+\beta_{j m}\left(\frac{\partial x_{l}}{\partial x_{i}^{\prime}} \frac{\partial}{\partial x_{l}} u_{m}\right)\right] \\
& =\frac{1}{2}\left[\beta_{i k} \beta_{j l} u_{k, l}+\beta_{j m} \beta_{i l} u_{m, l}\right] \\
& =\frac{1}{2}\left[\beta_{i k} \beta_{j l} u_{k, l}+\beta_{j l} \beta_{i k} u_{l, k}\right] \\
& =\frac{1}{2} \beta_{i k^{\prime}} \beta_{j l}\left(u_{k, l}+u_{l, k}\right) \\
& =\beta_{i k} \beta_{j l} e_{k l}, \tag{1.1.6}
\end{align*}
$$

this shows that $e_{i j}$ are really components of a second order tensor.
When deformation occurs there is a surface force acting from a portion of an elastic body upon the other portion of the body. Let us consider a surface element $\Delta S$ of the body, see Fig.1.1.2, which is located in the interior of the body. Drawing a unit normal vector $\vec{y}$ from a point on $\Delta S$, we can distinguish the two sides of $\Delta S$ according to the sense of $\vec{\nu}$. Suppose that the portion of the material lying on the positive side of the normal exerts a force $\Delta F$ on the other portion of the material. Obviously, the force $\Delta F$ is a function of the area $\Delta s$ of the surface element $\Delta S$ and varies when the normal $\vec{\nu}$ changes. As $\Delta s \rightarrow$ 0 , we get

$$
\vec{T}^{\nu}=\lim _{\Delta s \rightarrow 0} \frac{\Delta F}{\Delta s},
$$

where the subscript $\nu$ denotes the direction of the unit normal $\vec{\nu}$ of the surface element $\Delta S$. The vector $\vec{T}^{\nu}$ is called "stress vector" or "traction", which represents a force per unit area acting on the surface with normal $\vec{v}$.

The projection of $\vec{T}^{\nu}$ along the direction of coordinate axis $x_{i}$ is denoted by $T_{i}^{\nu}$. When $\vec{\nu}$ is a unit vector in the direction vector $x_{j}$, we write $T_{i}^{\nu}=\sigma_{j i}$,which are called the components of stress tensor. Particularly the


Fig. 1.1. 2
components $\sigma_{11}, \sigma_{22}$ and $\sigma_{33}$ are called "normal stresses" and $\sigma_{i j}(i \neq j)$ are called "shear stresses". To see that a stress vector in any direction can be written in terms of stress components, let us consider an infinitesimal tetrahedron formed by three surfaces parallel to the coordinate planes and one normal to the unit vector $\vec{\nu}$ with components $\nu_{i}(i=1,2,3)$ (see Fig.1.1.3). Suppose that the lengths of the sides of the tetrahedron along the $x_{i}$-direction are $d x_{i}, i=1,2,3$; respectively. Denote the area of the surface normal to $\vec{v}$ by $d s$ then the area of the surfaces normal to direction $x_{i}$ is given by $d s_{i}=\nu_{i} d s$, $i=1,2,3 ;$ respetively.

Let $\vec{T}^{\nu}$ be the stress vector acting on a surface element with the normal $\vec{\nu}$ which passes through the point of vertex of the tetrahedron and $h$ be the height of the vertex from the base of the tetrahedron. By assuming the continuity of the stress vector $\vec{T}^{\nu}$, the $i$-component of the force acting on the surface of the tetrahedron which is normal to $\vec{\nu}$ is $\left(T_{i}^{\nu}+\epsilon_{i}\right) d s$ with $\epsilon_{i} \rightarrow 0$, as $h \rightarrow 0$. And the $i$-components of the force acting on the surfaces of the tetrahedron which is normal to $j$-direction are $\left(-\sigma_{j i}+\epsilon_{\ddot{j}}\right) d s_{j} j=1,2,3$; with $\epsilon_{\ddot{j}}$ $\rightarrow 0$, as $h \rightarrow 0$. If the body force, per unit mass, is given by $\left\{F_{1}, F_{2}, F_{3}\right\}$ at the vertex, then the $i$-component of the body force acting on the tetrahedron is $\frac{\rho}{3}\left(F_{i}+\tilde{\epsilon}_{i}\right) h d s$ with $\tilde{\epsilon}_{i} \rightarrow 0$ as $h \rightarrow 0$, where $\rho$ is the density of the material. Hence the equilibrium of forces on the tetrahedron yield

$$
\left(T_{i}^{\nu}+\epsilon_{i}\right) d s+\left(-\sigma_{j i}+\epsilon_{\ddot{j}}\right) \nu_{j} d s+\frac{\rho}{3}\left(F_{i}+\tilde{\epsilon}_{i}\right) h d s=0 .
$$

Dividing by $d s$ and letting $h \rightarrow 0$ we get

$$
\begin{equation*}
T_{i}^{\nu}=\sigma_{\ddot{j}}{ }_{j} \tag{1.1.7}
\end{equation*}
$$

Hence, knowing the stress components $\sigma_{i j}$ at a point we can calculate the stress vector in any direction at the point.


Fig. 1.1.3

Now let $\Omega$ be a portion of an elastic body and $S$ be the boundary of $\Omega$. Each point of $S$ is subjected to a traction $\vec{T}^{\nu}$ with $\vec{\nu}$ being the unit normal to $S$ at the point considered. Each mass element of $\Omega$ is subjected to a body force ( per unit mass ) $\vec{F}$ with components $F_{i}, i=1,2,3$. For equilibrium, both the resultant force and the resultant moment acting on $\Omega$ must vanish, which leads to the following equations

$$
\begin{align*}
& \int_{\Omega} \rho F_{i} d V+\int_{S} T_{i}^{\nu} d S=0  \tag{1.1.8}\\
& \int_{\Omega} \rho \gamma_{i j k} F_{j} x_{k} d V+\int_{S} \gamma_{i j k} T_{j}^{\nu} x_{k} d S=0 \tag{1.1.9}
\end{align*}
$$

where $\gamma_{i j k}$ is defined as

$$
\gamma_{i j k}=\left\{\begin{aligned}
1 & \text { if } i j k \text { represents an even permutation of } 123 . \\
0 & \text { if any two of } i j k \text { indices are same. } \\
-1 & \text { if } i j k \text { represents an odd permutation. }
\end{aligned}\right.
$$

Substituting $T_{i}^{\nu}$ from equation (1.1.7) into equation (1.1.8) and using the divergence theorem we obtain

$$
\begin{equation*}
\int_{\Omega}\left(\rho F_{i}+\sigma_{j \ddot{j}, j}\right) d V=0 . \tag{1.1.10}
\end{equation*}
$$

The continuity of the integrand in the equation (1.1.10) and arbitrariness of the region $\Omega$ lead to the following equilibrium equation

$$
\begin{equation*}
\sigma_{j i, j}+\rho F_{i}=0 \tag{1.1.11}
\end{equation*}
$$

On the other hand, by divergence theorem we have

$$
\begin{align*}
\int_{S} \gamma_{i j k} T_{j}^{\nu} x_{k} d S & =\int_{S} \gamma_{i j k} \sigma_{l j} \nu_{l} x_{k} d S \\
& =\int_{\Omega}\left(\gamma_{i j k} \sigma_{l j} x_{k}\right)_{, l} d V \\
& =\int_{\Omega}\left(\gamma_{i j k} \sigma_{l j, l} x_{k}+\gamma_{i j k} \sigma_{l j} \delta_{k l}\right) d V \tag{1.1.12}
\end{align*}
$$

where $\delta_{k l}$ is the Kronecker delta.
Using the equation (1.1.11) and observing that $\delta_{k l} \sigma_{l j}=\sigma_{k j}$, equation (1.1.12) can be written as

$$
\begin{equation*}
\int_{S} \gamma_{i j k} T_{j}^{\nu} x_{k} d S=\int_{\Omega}\left(-\rho \gamma_{i j k} F_{j}^{x_{k}}+\gamma_{i j k} \sigma_{k j}\right) d V \tag{1.1.13}
\end{equation*}
$$

Substituting equation(1.1.13) into equation (1.1.9) we get

$$
\begin{equation*}
\int_{\Omega} \gamma_{i j k} \sigma_{k j} d V=0 \tag{1.1.14}
\end{equation*}
$$

Again the continuity of the integrand in the equation and the arbitrariness of the region $\Omega$ lead to $\gamma_{i j k} \sigma_{k j}=0$, which gives

$$
\begin{equation*}
\sigma_{i j}=\sigma_{j i}, \quad(i, j=1,2,3) \tag{1.1.15}
\end{equation*}
$$

hence the stress tensor is symmetric. Considering the equation (1.1.7) we know that the state of stress at any point of the body is determined entirely by six independent components of the symmetric stress tensor.

Let the surface elements $\Delta S$ and $\Delta S^{\prime}$, with unit normals $\vec{\nu}$ and $\vec{\nu}^{\prime}$, pass through a point $P$. By virtue of equations (1.1.7) and (1.1.15) we can show that the component of the stress vector $T^{\nu}$ (acting on $\Delta S$ ) in the direction of $\vec{\nu}^{\prime}$ is the same as the component of the stress vector $T^{\nu^{\prime}}$ (acting on $\Delta S^{\prime}$ ) in the direction of $\vec{\nu}$. In fact,

$$
\begin{equation*}
T^{\nu^{\prime}} \cdot \vec{\nu}=T_{i}^{\nu^{\prime}} \nu_{i}=\sigma_{\ddot{j}} \nu_{j}^{\prime} \nu_{i}=\left(\sigma_{i j} \nu_{i}\right) \nu_{j}^{\prime}=T^{\nu} \cdot \vec{\nu}^{\prime} \tag{1.1.16}
\end{equation*}
$$

It will be used to derive formulae of transformation of the components of the stress tensor $\sigma_{i j}$ to $\sigma_{i j}^{\prime}$ when the latter is referred to a new coordinate system $x_{i}^{\prime}$ obtained from the old one by a rotation of axes.

Let the two coordinates be connected by the linear relations defined by equations (1.1.4). Then the components of stress tensor with respect to the coordinate system $x_{i}^{\prime}$ are given by

$$
\begin{equation*}
\sigma_{k m}^{\prime}=T^{\nu^{\prime}} \cdot \vec{\nu} \tag{1.1.17}
\end{equation*}
$$

where $\vec{\nu}^{\prime}=\left\{\beta_{k 1}, \beta_{k 2}, \beta_{k 3}\right\}$ is the unit vector parallel to $x_{k}^{\prime}$-direction and $\vec{\nu}=\left\{\beta_{m 1}, \beta_{m 2}, \beta_{m 3}\right\}$ is the unit vector parallel to $x_{m}^{\prime}$-direction, referring to the old system. Using equation (1.1.7) we get

$$
\begin{equation*}
\sigma_{k m}^{\prime}=\sigma_{j i} \nu_{j}^{\prime} \nu_{i}=\sigma_{i j} \beta_{k j} \beta_{m i} \tag{1.1.18}
\end{equation*}
$$

indeed, this shows that $\sigma_{i j}$ is really a second order tensor.

### 1.2 Generalized Hooke's Law

If an elastic material is maintained at a fixed temperature and under a state of strain, the generalized Hooke's law states that the components of the stress are linearly related to the components of the strain at the given point. The generalized Hooke's law can be written in the following form

$$
\begin{equation*}
\sigma_{i j}=c_{i j k l} e_{k l}, \quad(i, j, k, l=1,2,3) \tag{1.2.1}
\end{equation*}
$$

where coefficients $c_{i j k l}$ are called elastic constants or moduli of the material. If coefficients $c_{i j k l}$ vary from point to point of the material, then the material is called non-homogeneous. If, however, the coefficients $c_{i j k l}$ are independent of the position of the point, the material is called elastically homogeneous. Since $e_{i j}$ and $\sigma_{i j}$ are symmetric there are, at most, 36 independent elastic coefficients.

Introducing the following notations

$$
\begin{array}{llr}
\sigma_{i i}=\sigma_{i}, & e_{i i}=e_{i}, & (i=1,2,3) \\
\sigma_{23}=\sigma_{4}, & \sigma_{31}=\sigma_{5}, & \sigma_{12}=\sigma_{6} \\
2 e_{23}=e_{4}, & 2 e_{31}=e_{5}, & 2 e_{12}=e_{6}
\end{array}
$$

the equation (1.2.1) becomes

$$
\begin{equation*}
\sigma_{i}=c_{i j} e_{j}, \quad(i, j=1,2, \cdots, 6) \tag{1.2.2}
\end{equation*}
$$

or

$$
\begin{align*}
\sigma_{\mathrm{xx}} & =c_{11} e_{\mathrm{xx}}+c_{12} e_{\mathrm{yy}}+c_{13} e_{\mathrm{zz}}+c_{14} \gamma_{\mathrm{yz}}+c_{15} \gamma_{\mathrm{xz}}+c_{16} \gamma_{\mathrm{xy}} \\
\sigma_{\mathrm{yy}} & =c_{21} e_{\mathrm{xx}}+c_{22} e_{\mathrm{yy}}+c_{23} e_{\mathrm{zz}}+c_{24} \gamma_{\mathrm{yz}}+c_{25} \gamma_{\mathrm{xz}}+c_{26} \gamma_{\mathrm{xy}} \\
\sigma_{\mathrm{zz}} & =c_{31} e_{\mathrm{xx}}+c_{32} e_{\mathrm{yy}}+c_{33} e_{\mathrm{zz}}+c_{34} \gamma_{\mathrm{yz}}+c_{35} \gamma_{\mathrm{xz}}+c_{36} \gamma_{\mathrm{xy}} \\
\sigma_{\mathrm{yz}} & =c_{41} e_{\mathrm{xx}}+c_{42} e_{\mathrm{yy}}+c_{43} e_{\mathrm{zz}}+c_{44} \gamma_{\mathrm{yz}}+\mathrm{c}_{45} \gamma_{\mathrm{xz}}+c_{46} \gamma_{\mathrm{xy}} \\
\sigma_{\mathrm{xz}} & =c_{51} e_{\mathrm{xx}}+c_{52} e_{\mathrm{yy}}+c_{53} e_{\mathrm{zz}}+c_{54} \gamma_{\mathrm{yz}}+c_{55} \gamma_{\mathrm{xz}}+c_{56} \gamma_{\mathrm{xy}} \\
\sigma_{\mathrm{xy}} & =c_{61} e_{\mathrm{xx}}+c_{62} e_{\mathrm{yy}}+c_{63} e_{z z}+c_{64} \gamma_{\mathrm{yz}}+\mathrm{c}_{65} \gamma_{\mathrm{xz}}+c_{66} \gamma_{\mathrm{xy}} \tag{1.2.3}
\end{align*}
$$

if we let $x_{1}=x, x_{2}=y$ and $x_{3}=z$ and $\gamma_{\mathrm{xy}}=2 e_{\mathrm{xy}}, \gamma_{\mathrm{xz}}=2 \mathrm{e}_{\mathrm{xz}}, \gamma_{\mathrm{yz}}=2 e_{\mathrm{yz}}$.
When an elastic body is under deformation, there is energy stored in the body. By the assumption that the deformation occur isothermally, we can assume that there is a strain energy density function $W$ which is a single valued function of $e_{1}, e_{2}, \cdots, e_{6}$. If a volume element in a state of stress is subjected to a virtual strain $\delta e_{\mathrm{i}}$, then the stress components $\sigma_{\mathrm{i}}$ yield the work $-\sigma_{\mathrm{i}} \delta e_{\mathrm{i}}$. Hence as $e_{\mathrm{i}}$ gets an increment $\delta e_{\mathrm{i}}, W\left(e_{1}, e_{2}, \cdots, e_{6}\right)$ gets an increment

$$
\begin{equation*}
\delta W=\sigma_{\mathrm{i}} \delta e_{\mathrm{i}} \tag{1.2.4}
\end{equation*}
$$

On the other hand

$$
\begin{equation*}
\delta W=\frac{\partial W}{\partial e_{\mathrm{i}}} \delta e_{\mathrm{i}} . \tag{1.2.5}
\end{equation*}
$$

Comparing (1.2.4) and (1.2.5) we have

$$
\begin{equation*}
\sigma_{\mathrm{i}}=\frac{\partial W}{\partial e_{\mathrm{i}}} . \tag{1.2.6}
\end{equation*}
$$

Since

$$
\frac{\partial^{2} W}{\partial e_{\mathrm{i}} \partial e_{\mathrm{j}}}=\frac{\partial^{2} W}{\partial e_{\mathrm{j}} \partial e_{\mathrm{i}}},
$$

we obtain

$$
\frac{\partial \sigma_{\mathrm{i}}}{\partial e_{\mathrm{j}}}=\frac{\partial \sigma_{\mathrm{j}}}{\partial e_{\mathrm{i}}},
$$

from equation (1.2.6), hence $c_{\mathrm{ij}}=c_{\mathrm{ji}}$ in equations (1.2.2). In other words, among the 36 coefficients $c_{i j}$ 's only 21 are independent.

If an elastic body is symmetric in a certain direction, the number of independent coefficients $c_{\mathrm{ij}}$ can be further reduced. First of all, let us consider a material which is elastically symmetric with respect to the $x_{1} x_{2}$-plane. The symmetry means that the $c_{i \mathrm{j}}$ will remain the same under the transformation

$$
x_{1}=\tilde{x}_{1}, \quad x_{2}=\tilde{x}_{2}, \quad x_{3}=-\tilde{x}_{3} .
$$

By the transformation of coordinates we get

$$
\begin{align*}
& \tilde{\sigma}_{\mathrm{i}}=\sigma_{\mathrm{i}}, \quad \tilde{e}_{\mathrm{i}}=e_{\mathrm{i}}, \quad(i=1,2,3,6) \\
& \tilde{\sigma}_{4}=-\sigma_{4}, \quad \tilde{e}_{4}=-e_{4}, \quad \tilde{\sigma}_{5}=-\sigma_{5}, \quad \tilde{e}_{5}=-e_{5} . \tag{1.2.7}
\end{align*}
$$

For $i=1$, the equation (1.2.2) yields

$$
\begin{align*}
& \tilde{\sigma}_{1}=c_{11} \tilde{e}_{1}+c_{12} \tilde{e}_{2}+c_{13} \tilde{e}_{3}+c_{14} \tilde{e}_{4}+c_{15} \tilde{e}_{5}+c_{16} \tilde{e}_{6}, \\
& \sigma_{1}=c_{11} e_{1}+c_{12} e_{2}+c_{13} e_{3}+c_{14} e_{4}+c_{15} e_{5}+c_{18} e_{6} . \tag{1.2.8}
\end{align*}
$$

Substituting Equations (1.2.7) into the first equation of (1.2.8) and comparing with the second one we get

$$
c_{14}=c_{15}=0
$$

Similarly by considering the equations for $\tilde{\sigma}_{2}, \cdots, \tilde{\sigma}_{6}$ we obtain

$$
\begin{aligned}
& c_{24}=c_{25}=c_{34}=c_{35}=c_{64}=0 \\
& c_{41}=c_{42}=c_{43}=c_{46}=c_{51}=c_{52}=c_{53}=c_{56}=0 .
\end{aligned}
$$

Therefore the matrix of the coefficients of equation (1.2.2) for a material with the elastic symmetry with respect to $x_{1} x_{2}$-plane can be written as

$$
\left[\begin{array}{cccccc}
c_{11} & c_{12} & c_{13} & 0 & 0 & c_{18}  \tag{1.2.9}\\
c_{12} & c_{22} & c_{23} & 0 & 0 & c_{28} \\
c_{13} & c_{23} & c_{33} & 0 & 0 & c_{38} \\
0 & 0 & 0 & c_{44} & c_{45} & 0 \\
0 & 0 & 0 & c_{45} & c_{55} & 0 \\
c_{16} & c_{26} & c_{36} & 0 & 0 & c_{66}
\end{array}\right] .
$$

Materials like wood, for example, which have three mutually orthogonal planes of elastic symmetry are called to be orthotropic. In the study of orthotropic materials it is convenient for us to choose such axes of the coordinate system so that the coordinate planes coincide with the planes of the elastic symmetry. In such a case, besides the symmetry with respect to the $x_{1} x_{2}$-plane, expressed by matrix (1.2.9), the coefficients $c_{\mathrm{ij}}$ must also be invariant under the transformation of coordinates defined by

$$
x_{1}=-\tilde{x}_{1}, \quad x_{2}=\tilde{x}_{2}, \quad x_{3}=\tilde{x}_{3} .
$$

Using the same method of as we used above, we find that more coefficients in
equations (1.2.2) vanish and the matrix of the $c_{\mathrm{ij}}$ takes the following form

$$
\left[\begin{array}{cccccc}
c_{11} & c_{12} & c_{13} & 0 & 0 & 0  \tag{1.2.10}\\
c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\
c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & c_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & c_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & c_{86}
\end{array}\right],
$$

there are only 9 independent coefficients.
In the case of an isotropic medium, whose elastic properties are independent of the orientation of the coordinate axes, the coefficients $c_{\mathrm{ij}}$ must keep the same when we introduce a new Cartesian coordinate system $O-\tilde{x}_{1} \tilde{x}_{2} \tilde{x}_{3}$ by rotating the $O-x_{1} x_{2} x_{3}$ through a right angle about the $x_{1}$-axis. Hence we get

$$
c_{12}=c_{13}, \quad c_{33}=c_{22}, \quad c_{66}=c_{55}
$$

Similarly, by rotating the axes through a right angle about the $x_{3}$-axis we get

$$
c_{22}=c_{11}, \quad c_{13}=c_{23}, \quad c_{55}=c_{44}
$$

Finally, let us consider the new coordinate system by rotating the $O-x_{1} x_{2} x_{3}$ through an angle of $\pi / 4$ about the $x_{3}$-axis, in this way we get

$$
\begin{equation*}
\tilde{\sigma}_{6}=-\frac{1}{2} \sigma_{1}+\frac{1}{2} \sigma_{2}, \quad \tilde{e}_{6}=-e_{1}+e_{2} \tag{1.2.11}
\end{equation*}
$$

From matrix (1.2.7) we have

$$
\sigma_{6}=c_{44} e_{6}, \quad \tilde{\sigma}_{6}=c_{44} \tilde{e}_{6}
$$

and by using equations (1.2.11) we get

$$
\begin{equation*}
\frac{1}{2}\left(-\sigma_{1}+\sigma_{2}\right)=c_{44}\left(-e_{1}+e_{2}\right) \tag{1.2.12}
\end{equation*}
$$

Considering matrix (1.2.10) again and observing that $c_{11}=c_{22}, c_{12}=c_{13}=c_{23}$ in this case, we have

$$
\begin{aligned}
& \sigma_{1}=c_{11} e_{1}+c_{12} e_{2}+c_{13} e_{3}, \\
& \sigma_{2}=c_{12} e_{1}+c_{11} e_{2}+c_{13} e_{3}
\end{aligned}
$$

then

$$
\begin{equation*}
\frac{1}{2}\left(-\sigma_{1}+\sigma_{2}\right)=\frac{1}{2}\left(c_{11}-c_{12}\right)\left(e_{2}-e_{1}\right) . \tag{1.2.13}
\end{equation*}
$$

Equations (1.2.12) and (1.2.13) lead to

$$
c_{44}=\frac{1}{2}\left(c_{11}-c_{12}\right)
$$

Hence for an isotropic material the elastic coefficients matrix can be reduced to

$$
\left[\begin{array}{cccccc}
c_{11} & c_{12} & c_{12} & 0 & 0 & 0  \tag{1.2.14}\\
c_{12} & c_{11} & c_{12} & 0 & 0 & 0 \\
c_{12} & c_{12} & c_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2}\left(c_{11}-c_{12}\right) & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2}\left(c_{11}-c_{12}\right) & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2}\left(c_{11}-c_{12}\right)
\end{array}\right]
$$

there are only two independent coefficients $c_{11}$ and $c_{12}$. For isotropic materials, we traditionally use $\lambda\left(=c_{12}\right)$ and $\mu\left(=\frac{1}{2}\left(c_{11}-c_{12}\right)\right)$ as two independent coefficients which are called Lame constants. In such a case the Hooke's law can be stated as following

$$
\begin{equation*}
\sigma_{\mathrm{ij}}=\lambda \delta_{\mathrm{ij}} \vartheta+2 \mu e_{\mathrm{ij}}, \tag{1.2.15}
\end{equation*}
$$

where $\delta_{\mathrm{ij}}$ is Kronecker delta and $\vartheta=e_{11}+e_{22}+e_{33}$.

### 1.3 Plane Strain

A body is said to be in the state of plane deformation, or plane strain, parallel the $x_{1} x_{2}$-plane, if the displacement component $u_{3}$ vanishes and the components $u_{1}$ and $u_{2}$ are the functions of the coordinates $x_{1}$ and $x_{2}$, but not of $x_{3}$. That is

$$
\begin{equation*}
u_{\mathrm{i}}=u_{\mathrm{i}}\left(x_{1}, x_{2}\right), \quad i=1,2 ; \quad u_{3}=0 \tag{1.3.1}
\end{equation*}
$$

By equation (1.1.2) we find that the components of the strain tensor are

$$
\begin{align*}
& e_{13}=e_{23}=e_{33}=0, \\
& e_{i j}=\frac{1}{2}\left(u_{i, j}+u_{j, i}\right), \quad i, j=1,2 \tag{1.3.2}
\end{align*}
$$

which do not depend on $x_{3}$.
Particularly, in the plane orthotropic case, which can be thought of a plane strain problem for a three dimensional orthotropic medium, from equations (1.2.2) and (1.3.2) we get the nonvanising components of the stress tensor

$$
\begin{align*}
& \sigma_{11}=c_{11} e_{11}+c_{12} e_{22} \\
& \sigma_{22}=c_{21} e_{11}+c_{22} e_{22} \\
& \sigma_{12}=c_{66} \gamma_{12} \tag{1.3.3}
\end{align*}
$$

and

$$
\sigma_{33}=c_{13} e_{11}+c_{23} e_{22}
$$

but from the first two equations of (1.3.3) we know that $\sigma_{33}$ is entirely determined by $\sigma_{11}$ and $\sigma_{22}$, and is independent of coordinate $x_{3}$. Hence it is clear that the deformations and stresses of an orthotropic plane strain problem are completely determined by $e_{\mathrm{ij}}$ and $u_{\mathrm{i}}(i, j=1,2)$.

We consider next the equilibrium equations. First of all, since $\sigma_{i j}$ do not depend on $x_{3}$ we conclude from equation (1.1.11) that the components $F_{1}$ and $F_{2}$ of the body force must be independent of the coordinate $x_{3}$ and $F_{3}=0$. Hence the equilibrium equations can be written as

$$
\begin{equation*}
\sigma_{i j, j}=-\rho F_{i}\left(x_{1}, x_{2}\right), \quad i, j=1,2 \tag{1.3.4}
\end{equation*}
$$

### 1.4 Polar Cylindrical Coordinates

Polar cylindrical coordinate system is often introduced in theory of elasticity when the boundary conditions can be simplified by such a frame of reference. It is appropriate to resolve the components of stress and strain in the direction of the coordinates and denote them by corresponding subscripts.

When we have a Cartesian coordinate system $0-x y z$, the components of displacement vector in the $x, y$ and $z$ directions are denoted by $u_{x}, u_{y}$ and $u_{z}$ respectively. We use $e_{\mathrm{xx}}, e_{\mathrm{yy}}$ and $e_{\mathrm{zz}}$ to denote the normal strain components while $e_{\mathrm{xy}}, e_{\mathrm{xz}}$ and $e_{\mathrm{yz}}$ to denote the shear strain components. Similarly we use $\sigma_{\mathrm{xx}}, \sigma_{\mathrm{yy}}$ and $\sigma_{\mathrm{zz}}$ to denote the normal stress components while $\sigma_{\mathrm{xy}}, \sigma_{\mathrm{xz}}$ and $\sigma_{\mathrm{yz}}$ to denote the shear stress components. Referring to a polar cylindrical coordinate system the components of displacement vector along the directions $r, \theta$ and $z$ are denoted by $u_{r}, u_{\theta}$ and $u_{z}$ respevtively. We will use $e_{\Gamma r}, e_{\theta \theta}$ and $e_{z z}$ to denote the normal strain components while $e_{\Gamma \theta}, e_{\mathrm{rz}}$ and $e_{\theta \mathrm{z}}$ to denote the shear strain components. We also use $\sigma_{\mathrm{rr}}, \sigma_{\theta \theta}$ and $\sigma_{\mathrm{zz}}$ to denote the normal stress components while $\sigma_{\mathrm{r} \theta}, \sigma_{\mathrm{IZ}}$ and $\sigma_{\theta \mathrm{Z}}$ to denote the shear stress components. The relations between the polar cylindrical coordinates $r, \theta, z$ and the Cartesian coordinates $x, y, z$ are

$$
\begin{equation*}
x=r \cos \theta, \quad y=r \sin \theta, \quad z=z \tag{1.4.1}
\end{equation*}
$$

It follows that derivatives with respect to $x, y$ and $z$ in Cartesian equations can be written in terms of derivatives with respect to $r, \theta$ and $z$ by using the following relations

$$
\begin{align*}
& \frac{\partial}{\partial x}=\frac{\partial r}{\partial x} \frac{\partial}{\partial r}+\frac{\partial \theta}{\partial x} \frac{\partial}{\partial \theta}=\cos \theta \frac{\partial}{\partial r}-\frac{\sin \theta}{r} \frac{\partial}{\partial \theta}, \\
& \frac{\partial}{\partial y}=\frac{\partial r}{\partial y} \frac{\partial}{\partial r}+\frac{\partial \theta}{\partial y} \frac{\partial}{\partial \theta}=\sin \theta \frac{\partial}{\partial r}-\frac{\cos \theta}{r} \frac{\partial}{\partial \theta}, \\
& \frac{\partial}{\partial z}=\frac{\partial}{\partial z} . \tag{1.4.2}
\end{align*}
$$

To relate the components between two systems let us select a local Cartesian frame of reference $x_{1} y_{1} z_{1}$ at the point $(r, \theta, z)$, with the origin located at the point $(r, \theta, z)$, the $x_{1}$-axis in the direction of increasing $r$, the $y_{1}$-axis in the direction of increasing $\theta$, and the $z_{1}$-axis parallel to $z$-axis (Fig. 1.4.1). Then, in conventional notation, $e_{x_{1} x_{1}}, e_{x_{1} y_{1}}, \cdots$, are well defined. By identifying $r, \theta, z$ with $x_{1}, y_{1}, z_{1}$ we have

$$
\begin{array}{ll}
\sigma_{\mathrm{rr}}=\sigma_{x_{1} x_{1}}, & \sigma_{\mathrm{r} \theta}=\sigma_{x_{1} y_{1}}, \\
e_{\mathrm{rr}}=e_{x_{1} x_{1}}, & e_{\mathrm{r} \mathrm{\theta}}=e_{x_{1} y_{1}}, \quad e_{\theta \theta}=e_{y_{1} y_{1} y_{1}, \cdots}, \cdots
\end{array}
$$

etc. The matrix of the direction cosines of the axes $x_{1}, y_{1}, z_{1}$ relative to $x, y$, $z$ is

$$
\left(\beta_{i j}\right)=\left[\begin{array}{ccc}
\cos \theta & \sin \theta & 0  \tag{1.4.3}\\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{array}\right]
$$



Fig. 1.4.1

Hence, from equation(1.1.18) and the matrix (1.4.3) we get

$$
\begin{align*}
& \sigma_{\mathrm{xx}}=\sigma_{\mathrm{rr}} \cos ^{2} \theta+\sigma_{\theta \theta} \sin ^{2} \theta-\sigma_{\mathrm{r} \theta} \sin 2 \theta, \\
& \sigma_{\mathrm{yy}}=\sigma_{\mathrm{rr}} \sin ^{2} \theta+\sigma_{\theta \theta} \cos ^{2} \theta+\sigma_{\mathrm{r} \theta} \sin 2 \theta, \\
& \sigma_{\mathrm{zz}}=\sigma_{\mathrm{zz}} \\
& \sigma_{\mathrm{xy}}=\left(\sigma_{\mathrm{rr}}-\sigma_{\theta \theta}\right) \sin \theta \cos \theta+\sigma_{\mathrm{r} \theta}\left(\cos ^{2} \theta-\sin ^{2} \theta\right) \\
& \sigma_{\mathrm{xz}}=\sigma_{\mathrm{rz}} \cos \theta-\sigma_{\theta \mathrm{z}} \sin \theta, \\
& \sigma_{\mathrm{yz}}=\sigma_{\mathrm{rz}} \sin \theta+\sigma_{\theta \mathrm{z}} \cos \theta . \tag{1.4.4}
\end{align*}
$$

Similarly, from equation (1.1.6) and matrix (1.4.3) we obtain

$$
\begin{align*}
& e_{\mathrm{rr}}=e_{\mathrm{xx}} \cos ^{2} \theta+e_{\mathrm{yy}} \sin ^{2} \theta+e_{\mathrm{xy}} \sin 2 \theta, \\
& e_{\theta \theta}=e_{\mathrm{xx}} \sin ^{2} \theta+e_{\mathrm{yy}} \cos ^{2} \theta-e_{\mathrm{xy}} \sin 2 \theta, \\
& e_{\mathrm{zz}}=e_{\mathrm{zz}}, \\
& e_{\mathrm{r} \theta}=\left(e_{\mathrm{yy}}-\sigma_{\mathrm{xx}}\right) \sin \theta \cos \theta+e_{\mathrm{xy}}\left(\cos ^{2} \theta-\sin ^{2} \theta\right) \\
& e_{\mathrm{rz}}=e_{\mathrm{xz}} \cos \theta+e_{\mathrm{yz}} \sin \theta, \\
& e_{\theta \mathrm{z}}=-e_{\mathrm{xz}} \sin \theta+e_{\mathrm{yz}} \cos \theta . \tag{1.4.5}
\end{align*}
$$

From the Fig. (1.4.1) the relations of displacement components between polar cylindrical coordinates and Cartesian coordinates for a displacement vector are given by

$$
\begin{align*}
& u_{\mathrm{x}}=u_{\mathrm{r}} \cos \theta-u_{\theta} \sin \theta \\
& u_{\mathrm{y}}=u_{\mathrm{r}} \sin \theta+u_{\theta} \cos \theta \\
& u_{\mathrm{z}}=u_{\mathrm{z}} \tag{1.4.6}
\end{align*}
$$

Equations (1.1.2) now become

$$
\begin{align*}
& e_{\mathrm{xx}}=\frac{\partial u_{\mathrm{x}}}{\partial x}, \quad e_{\mathrm{yy}}=\frac{\partial u_{\mathrm{y}}}{\partial y}, \quad e_{\mathrm{zz}}=\frac{\partial u_{\mathrm{z}}}{\partial z} \\
& e_{\mathrm{xy}}=\frac{1}{2}\left(\frac{\partial u_{\mathrm{y}}}{\partial x}+\frac{\partial u_{\mathrm{x}}}{\partial y}\right), \quad e_{\mathrm{xz}}=\frac{1}{2}\left(\frac{\partial u_{\mathrm{z}}}{\partial x}+\frac{\partial u_{\mathrm{x}}}{\partial z}\right) \\
& e_{\mathrm{yz}}=\frac{1}{2}\left(\frac{\partial u_{\mathrm{y}}}{\partial z}+\frac{\partial u_{z}}{\partial y}\right) \tag{1.4.7}
\end{align*}
$$

Finally, substituting equations (1.4.7) into equations (1.4.5) and using equation (1.4.6) and equations (1.4.2) we obtain the strain-displacement relations in polar cylindrical coordinates

$$
\begin{align*}
& e_{\mathrm{rr}}=\frac{\partial u_{r}}{\partial r}, \quad e_{\theta \theta}=\frac{1}{r} \frac{\partial u_{\theta}}{\partial \theta}+\frac{u_{r}}{r}, \quad e_{\mathrm{zz}}=\frac{\partial u_{\mathrm{z}}}{\partial z}, \\
& e_{\mathrm{r} \theta}=\frac{1}{2}\left(\frac{1}{r} \frac{\partial u_{\mathrm{r}}}{\partial \theta}+\frac{\partial u_{\theta}}{\partial r}-\frac{u_{\theta}}{r}\right), \\
& e_{\mathrm{rz}}=\frac{1}{2}\left(\frac{\partial u_{\mathrm{r}}}{\partial z}+\frac{\partial u_{\mathrm{z}}}{\partial r}\right) \\
& e_{\theta \mathrm{z}}=\frac{1}{2}\left(\frac{\partial u_{\theta}}{\partial z}+\frac{1}{r} \frac{\partial u_{\mathrm{z}}}{\partial \theta}\right) . \tag{1.4.8}
\end{align*}
$$

To obtain the equilibrium equations for polar cylindrical coordinates we first resolve the body force per unit volume at the point ( $r, \theta, z$ ) into components $F_{\mathrm{r}}, F_{\theta}, F_{\mathrm{z}}$ along the $r$-, $\theta$ - and $z$-directions, then we have

$$
\begin{align*}
& F_{\mathrm{x}}=F_{\mathrm{r}} \cos \theta-F_{\theta} \sin \theta \\
& F_{\mathrm{y}}=F_{\mathrm{r}} \sin \theta+F_{\theta} \cos \theta \\
& F_{\mathrm{z}}=F_{\mathrm{z}} \tag{1.4.9}
\end{align*}
$$

The equilibrium equations (1.1.11) for Cartesian coordinates $x, y, z$ can be written as

$$
\begin{align*}
& \frac{\partial \sigma_{\mathrm{xx}}}{\partial x}+\frac{\partial \sigma_{\mathrm{xy}}}{\partial y}+\frac{\partial \sigma_{\mathrm{xz}}}{\partial z}+\rho F_{\mathrm{x}}=0 \\
& \frac{\partial \sigma_{\mathrm{xy}}}{\partial x}+\frac{\partial \sigma_{\mathrm{yy}}}{\partial y}+\frac{\partial \sigma_{\mathrm{yz}}}{\partial z}+\rho F_{\mathrm{y}}=0 \\
& \frac{\partial \sigma_{\mathrm{xz}}}{\partial x}+\frac{\partial \sigma_{\mathrm{yz}}}{\partial y}+\frac{\partial \sigma_{\mathrm{zz}}}{\partial z}+\rho F_{\mathrm{z}}=0 \tag{1.4.10}
\end{align*}
$$

Substituting the equations (1.4.9) and (1.4.4) into the equations (1.4.10) and using the equations (1.4.2) to transform the derivatives, we obtain

$$
\begin{align*}
& \left(\frac{\sigma_{\mathrm{rr}}}{\partial r}+\frac{1}{r} \frac{\partial \sigma_{\mathrm{r}}}{\partial \theta}+\frac{\partial \sigma_{\mathrm{rz}}}{\partial z}+\frac{\sigma_{\mathrm{rr}}-\sigma_{\theta \theta}}{r}\right) \cos \theta \\
& -\left(\frac{\partial \sigma_{\mathrm{r} \theta}}{\partial r}+\frac{1}{r} \frac{\partial \sigma_{\theta \theta}}{\partial \theta}+\frac{\partial \sigma_{\theta \mathrm{z}}}{\partial z}+\frac{2}{r} \sigma_{\mathrm{r} \theta}\right) \sin \theta \\
& +\rho F_{\mathrm{r}} \cos \theta-\rho F_{\theta} \sin \theta=0, \tag{1.4.11}
\end{align*}
$$

from the first equation of (1.4.10). The equation (1.4.11) must hold for all values of $\theta$ and letting $\theta=0$ and $\theta=\pi / 2$ respectively we get

$$
\begin{align*}
& \frac{\partial \sigma_{\mathrm{rr}}}{\partial r}+\frac{1}{r} \frac{\partial \sigma_{\mathrm{r} \theta}}{\partial \theta}+\frac{\partial \sigma_{\mathrm{rz}}}{\partial z}+\frac{\sigma_{\mathrm{rr}}-\sigma_{\theta \theta}}{r}+\rho F_{\mathrm{r}}=0 \\
& \frac{\partial \sigma_{\mathrm{r} \theta}}{\partial r}+\frac{1}{r} \frac{\partial \sigma_{\theta \theta}}{\partial \theta}+\frac{\partial \sigma_{\theta z}}{\partial z}+\frac{2}{r} \sigma_{\mathrm{r} \theta}+\rho F_{\theta}=0 \tag{1.4.12}
\end{align*}
$$

since the choice of the $x$-direction is arbitrary the equations (1.4.12) must be valid for all values of $\theta$. Similarly, from the third equation of (1.4.10) we get

$$
\begin{equation*}
\frac{\partial \sigma_{\mathrm{rz}}}{\partial r}+\frac{1}{r} \frac{\partial \sigma_{\theta \mathrm{z}}}{\partial \theta}+\frac{\partial \sigma_{\mathrm{zz}}}{\partial z}+\frac{2}{r} \sigma_{\mathrm{rz}}+\rho F_{\mathrm{z}}=0 \tag{1.4.13}
\end{equation*}
$$

The generalized Hooke's law for the most general case in polar cylindrical coordinate system is stated as follows

$$
\begin{align*}
\sigma_{\mathrm{rr}} & =c_{11} e_{\mathrm{rr}}+c_{12} e_{\theta \theta}+c_{13} e_{\mathrm{zz}}+c_{14} \gamma_{\theta \mathrm{z}}+c_{15} \gamma_{\mathrm{rz}}+c_{16} \gamma_{\mathrm{r} \theta}, \\
\sigma_{\theta \theta} & =c_{21} e_{\mathrm{rr}}+c_{22} e_{\theta \theta}+c_{23} e_{\mathrm{zz}}+c_{24} \gamma_{\theta z}+c_{25} \gamma_{\mathrm{rz}}+c_{26} \gamma_{\mathrm{r} \theta}, \\
\sigma_{\mathrm{zz}} & =c_{31} e_{\mathrm{rr}}+c_{32} e_{\theta \theta}+c_{33} e_{\mathrm{zz}}+c_{34} \gamma_{\theta \mathrm{z}}+c_{35} \gamma_{\mathrm{rz}}+c_{36} \gamma_{\mathrm{r} \theta}, \\
\sigma_{\theta \mathrm{z}} & =c_{41} e_{\mathrm{rr}}+c_{42} e_{\theta \theta}+c_{43} e_{\mathrm{zz}}+c_{44} \gamma_{\theta \mathrm{z}}+c_{45} \gamma_{\mathrm{rz}}+c_{46} \gamma_{\mathrm{r} \theta}, \\
\sigma_{\mathrm{rz}} & =c_{51} e_{\mathrm{rr}}+c_{52} e_{\theta \theta}+c_{53} e_{\mathrm{zz}}+c_{54} \gamma_{\theta z}+c_{55} \gamma_{\mathrm{rz}}+c_{56} \gamma_{\mathrm{r} \theta}, \\
\sigma_{\mathrm{r} \theta} & =c_{61} e_{\mathrm{rr}}+c_{62} e_{\theta \theta}+c_{83} e_{\mathrm{zz}}+c_{64} \gamma_{\theta \mathrm{z}}+c_{85} \gamma_{\mathrm{rz}}+c_{68} \gamma_{\mathrm{r} \theta}, \tag{1.4.14}
\end{align*}
$$

where $c_{\mathrm{ij}}=c_{\mathrm{ji}}$, and $\gamma_{\theta \mathrm{z}}=2 e_{\theta \mathrm{z}}, \gamma_{\mathrm{rz}}=2 e_{\mathrm{rz}}, \gamma_{\mathrm{r} \theta}=2 e_{\mathrm{r} \theta}$.
An anisotropic medium is said cylindrical if a certain straight line $l$, the axis of anisotropy, is associated with the medium such that all directions intersecting the line $l$ at right angle are equivalent; correspondingly, all directions parallel to $l$-axis which pass through distinct points and all directions orthogonal to the first two directions are equivalent. For such kind of medium, it is more convenient to use the cylindrical polar coordinate system by taking the axis of the anisotropy as the $z$-axis of the cylindrical polar coordinates system ( $r, \theta, z$ ).

A material is called transversely isotropic if all directions in the planes, which are orthogonal to the axis of the anisotropy are equivalent, in other words, the isotropy occurs in $r \theta$-plane when the $z$-axis coincides with the axis of the anisotropy in cylindrical polar coordinate system.

For a transversely isotropic medium, letting the $z$-axis coincide with the axis of the anisotropy, and using the same method as we did for Cartesian coordinate system we can reduce the equation (1.4.14) to the following form

$$
\begin{align*}
\sigma_{\mathrm{rr}} & =c_{11} e_{\mathrm{rr}}+c_{12} e_{\theta \theta}+c_{13} e_{\mathrm{zz}} \\
\sigma_{\theta \theta} & =c_{12} e_{\mathrm{rr}}+c_{11} e_{\theta \theta}+c_{13} e_{\mathrm{zz}} \\
\sigma_{\mathrm{zz}} & =c_{13} e_{\mathrm{rr}}+c_{13} e_{\theta \theta}+c_{33} e_{\mathrm{zz}} \\
\sigma_{\theta \mathrm{z}} & =c_{44} \gamma_{\theta \mathrm{z}} \\
\sigma_{\mathrm{rz}} & =c_{44} \gamma_{\mathrm{rz}} \\
\sigma_{\mathrm{r} \theta} & =\frac{1}{2}\left(c_{11}-c_{12}\right) \gamma_{\mathrm{r} \theta} \tag{1.4.15}
\end{align*}
$$

by transformation of variables according to the symmetry of the medium.
In the case of isotropy, we have $c_{33}=c_{11}, c_{13}=c_{12}$ and $c_{44}=\frac{1}{2}\left(c_{11}-c_{12}\right)$. Again there are only two independent coefficients, i.e. Lame constants $\lambda=c_{12}$ and $\mu=\frac{1}{2}\left(c_{11}-c_{12}\right)$.

### 1.5 Penny-shaped crack problem

In Chapter 5, we will consider a penny-shaped crack problem for transversely isotropic medium, in that problem the displacement components are functions of variables $r$ and $z$ only, in fact, we have $u_{\mathrm{r}}=u_{\mathrm{r}}(r, z), u_{\theta}=0$ and $u_{z}=u_{z}(r, z)$. Using the strain-displacement relations (1.4.8) we find that $e_{r \theta}=e_{\theta z}=0$ and

$$
\begin{align*}
& e_{\mathrm{rr}}=\frac{\partial u_{\mathrm{r}}}{\partial r}, \quad e_{\theta \theta}=\frac{u_{\mathrm{r}}}{r}, \quad e_{\mathrm{zz}}=\frac{\partial u_{\mathrm{z}}}{\partial z} \\
& e_{\mathrm{rz}}=\frac{1}{2}\left(\frac{\partial u_{\mathrm{r}}}{\partial z}+\frac{\partial u_{\mathrm{z}}}{\partial r}\right) \tag{1.5.1}
\end{align*}
$$

Then for a homogeneous, transversely isotropic medium we get the following stress-displacement relations

$$
\begin{align*}
& \sigma_{\mathrm{rr}}=c_{11} \frac{\partial u_{\mathrm{r}}}{\partial r}+c_{12} \frac{u_{\mathrm{r}}}{r}+c_{13} \frac{\partial u_{\mathrm{z}}}{\partial z} \\
& \sigma_{\theta \theta}=c_{12} \frac{\partial u_{\mathrm{r}}}{\partial r}+c_{11} \frac{u_{\mathrm{r}}}{r}+c_{13} \frac{\partial u_{\mathrm{z}}}{\partial z} \\
& \sigma_{\mathrm{zz}}=c_{13}\left[\frac{\partial u_{\mathrm{r}}}{\partial r}+\frac{u_{\mathrm{r}}}{r}\right]+c_{33} \frac{\partial u_{\mathrm{z}}}{\partial z}, \\
& \sigma_{\mathrm{rz}}=c_{44}\left[\frac{\partial u_{\mathrm{r}}}{\partial z}+\frac{\partial u_{\mathrm{z}}}{\partial r}\right] \tag{1.5.2}
\end{align*}
$$

from equations (1.4.15). Substituting equations (1.5.2) into equations (1.4.12) and (1.4.13) we obtain the following equilibrium equations

$$
\begin{align*}
& c_{11}\left[\frac{\partial^{2} u_{\mathrm{r}}}{\partial r^{2}}+\frac{1}{\mathrm{r}} \frac{\partial u_{\mathrm{r}}}{\partial r}-\frac{u_{\mathrm{r}}}{r^{2}}\right]+c_{44} \frac{\partial^{2} u_{\mathrm{r}}}{\partial z^{2}}+\left(c_{13}+c_{44}\right) \frac{\partial^{2} u_{z}}{\partial r \partial z}=0, \\
& c_{44}\left[\frac{\partial^{2} u_{z}}{\partial r^{2}}+\frac{1}{\mathrm{r}} \frac{\partial u_{z}}{\partial r}\right]+c_{33} \frac{\partial^{2} u_{z}}{\partial z^{2}}+\left(c_{13}+c_{44}\right) \frac{\partial}{\partial z}\left[\frac{\partial u_{\mathrm{r}}}{\partial r}+\frac{u_{\mathrm{r}}}{r}\right]=0 . \tag{1.5.3}
\end{align*}
$$

### 1.6 Torsion problem

For an axisymmetric torsion problem the displacement components are given by $u_{\mathrm{r}}=0, u_{\mathrm{z}}=0$ and $u_{\theta}=u_{\theta}(r, z)$, which depends on $r$ and $z$ only. Hence from equations (1.4.8) we find that the non-zero strain components are the following

$$
\begin{align*}
& e_{\theta \theta}=\frac{1}{r} \frac{\partial u_{\theta}}{\partial \theta}, \\
& e_{\mathrm{r} \theta}=\frac{1}{2}\left(\frac{\partial u_{\theta}}{\partial r}-\frac{u_{\theta}}{r}\right), \\
& e_{\theta \mathrm{z}}=\frac{1}{2} \frac{\partial u_{\theta}}{\partial z} . \tag{1.6.1}
\end{align*}
$$

They depend only on $r$ and $z$. On the other hand, we know that stress components $\sigma_{\mathrm{rr}}=\sigma_{\theta \theta}=\sigma_{\mathrm{zz}}=\sigma_{\mathrm{rz}}=0$ and the non-zero stress components
are given the following

$$
\begin{align*}
\sigma_{\theta \mathrm{z}} & =c_{44} \gamma_{\theta \mathrm{z}} \\
\sigma_{\mathrm{r} \theta} & =\frac{1}{2}\left(c_{11}-c_{12}\right) \gamma_{\mathrm{r} \theta} \tag{1.6.2}
\end{align*}
$$

For a non-homogeneous and isotropic medium, in terms of displacement components, equations (1.6.2) yield

$$
\begin{align*}
& \sigma_{\theta \mathrm{z}}=\mu \frac{\partial u_{\theta}}{\partial z} \\
& \sigma_{\mathrm{r} \theta}=\mu\left(\frac{\partial u_{\theta}}{\partial r}-\frac{u_{\theta}}{r}\right) \tag{1.6.3}
\end{align*}
$$

where we have used the fact that $c_{44}=\frac{1}{2}\left(c_{11}-c_{12}\right)$ for isotropic medium and assumed that $\mu=\mu(z)=c_{44}$ depends on $z$ only. Since $\sigma_{\theta z}$ and $\sigma_{r_{\theta}}$ depend on $r$ and $z$ only, equation (14.13) and the first of equations (1.4.12) vanish and the second one of equations (1.4.12) becomes

$$
\begin{equation*}
\frac{\partial \sigma_{\mathrm{r} \theta}}{\partial r}+\frac{\partial \sigma_{\theta \mathrm{z}}}{\partial z}+\frac{2}{r} \sigma_{\mathrm{r} \theta}=0 \tag{1.6.4}
\end{equation*}
$$

in the absence of the body force.
Substituting equations (1.6.3) into equation (1.6.4) we get

$$
\begin{equation*}
\frac{\partial^{2} u_{\theta}}{\partial r^{2}}+\frac{1}{r} \frac{\partial u_{\theta}}{\partial z}-\frac{u_{\theta}}{r^{2}}+\frac{\partial^{2} u_{\theta}}{\partial z^{2}}+\frac{1}{\mu} \frac{\partial u_{\theta}}{\partial z} \frac{\partial \mu}{\partial z}=0 \tag{1.6.5}
\end{equation*}
$$

For a particular case, when $\mu=$ constant, i.e. homogeneous, isotropic case, equation (1.6.5) can be reduced to the following

$$
\begin{equation*}
\frac{\partial^{2} u_{\theta}}{\partial r^{2}}+\frac{1}{r} \frac{\partial u_{\theta}}{\partial z}-\frac{u_{\theta}}{r^{2}}+\frac{\partial^{2} u_{\theta}}{\partial z^{2}}=0 \tag{1.6.6}
\end{equation*}
$$

## CHAPTER 2

## PROBLEMS OF REISSNER-SAGOCI TYPE FOR COMPOSITE CYLINDRICAL REGIONS

### 2.1 Introduction

In 1937, E.Reissner [6] formulated several problems relating to torsional vibrations of an elastic half-space. He posed but didn't solve the following mixed boundary value problem

$$
\begin{array}{ll}
\vartheta(r, 0)=a r e^{i w t}, & 0<r<d, \\
\sigma_{\theta z}(r .0)=0, & r>d . \tag{2.1.1}
\end{array}
$$

Later Reissner and Sagoci [7] solved the static version of above problem by using oblate spherical coordinates. The problem posed by Reissner is called the Reissner-Sagoci problem (RS-problem in short ) now.

Sneddon [8] solved the RS-problem by a different method, by reducing it to a pair of dual integral equations using the Hankel transforms. In static case these integral equations could be solved by using various methods developed by Titchmarch [9], Busbridge [10] and Harding and Sneddon [11]. Bycroft [12] gave an approximate treatment to the dynamic RS-problem. Ufliand [13] set up the dual integral equations for the RS-problem for a circular disc on an elastic layer and reduced them to the solution of a Fredholm integral equation of the second kind.

Sneddon [14] returned to the RS-problem for a half space and obtained the solution by using his own elementary solution of dual integral equations. He also solved the RS-problem of determining the distribution of stress in a long circular cylinder of homogeneous isotropic material under the condition that the curved surface was fixed. Sneddon, Srivastava and Mathur [15] obtained a solution of the problem for a finite cylinder when the curved surface was stress free and the length of the cylinder was long compared with its radius.

Freeman and Keer [16] investigated a torsion problem of an elastic cylinder, which is attached to an elastic half space. The problem was reduced to the solution of dual integral equations and Dini-series. Rukhovets and Ufliand [17] presented a solution of RS-problem for an elastic half space with a circular inclusion. Gladwell [18] solved the RS-problem for an elastic layer of finite thickness, when the lower face is either stress free or rigidly clamped. Keer and Freeman [19] later extended their previous analysis to a finite elastic cylinder which is partially bonded to a semi-infinite elastic cylinder of the same radius which is embedded in an elastic half-space.

Luco [20] solved the problem of a rigid rod embedded in an elastic layer, the whole being perfectly bonded to a half-space of different material. Singh and Dhaliwal [21] investigated the RS-problem for an elastic layer under torsion by a pair of circular discs on opposite faces. And Dhaliwal, Singh and Sneddon [22] obtained a solution of the RS-problem for a semi-infinite elastic cylinder embedded in an elastic half-space. Low [23] investigated a RS-problem of an elastic half-space with a penny-shaped flaw in the form of an inclusion or a crack. Chebakov [24] considered a RS-problem for a finite cylinder with a torque applied to a rigid disc in the middle of the top face while the curved
surface and bottom face were fixed. Dhaliwal, Singh and Rokne [25] solved a torsion problem for a hemisphere embedded in an elastic half-space. Gladwell and Lemczyk [26] solved the static RS-problem for a finite cylinder.

The RS-problem in which a torque is applied over an annulus has also been considered by some researchers. In 1966 Boradachev and Boradacheva [27] investigated this problem by Hankel transforms and reduced it to the solution of triple integral equations. Arutinunian and Bobloian [28] investigated the problem in which a torque is applied on a circle $r<b$ on the surface of a half-space which has an inclusion occupying the cylinder $r<a(b<a)$. Shibuya et al.[29] considered the problem of an elastic layer under torsion by a pair of identical facing annular discs. Dhaliwal and Singh [30] investigated a problem of torsion, by an annular die, of an elastic layer bonded to an elastic half-space, the problem was reduced to the solution of a system of four Fredholm integral equations. Dhaliwal, Singh and Vrbik [31] considered the problem of a half-space with a cylindrical inclusion which was twisted by an annular die. Hasegawa [32] obtained an essential solution for a finite cylinder under torsion by a pair of identical annular stamps to its ends by using Green's function method.

The study of the static RS-problem for non-homogeneous material started in 1960's. In 1967 Protsenko [33] considered the torsion of a half-space with a shear modulus $\mu(z)=m z^{a}$, and later he [34] considered the half-space problem with a torque applied over an annular area. Kassir [35] solved the RS-problem for the half-space and semi-infinite cylinder by assuming the shear modulus of the material in the form of $\mu(z)=m z^{\lambda}$, and reduced it to the solution of a pair of dual integral equations. Kolybikhim [36] solved the above problem by assuming the shear modulus in the form of $\mu(r, z)=\mu_{0} r^{k} z^{\alpha}$. Chuaprasert and Kassir [37] considered a half-space and a
semi-infinite cylinder RS-problem by assuming $\mu(z)=\mu_{0}(1+z / c)^{\alpha}$. George [38] assumed $\mu(r)=\mu_{0} e^{-a r}$ for a semi-infinite cylinder RS-problem. In 1979, Dhaliwal and Singh [39] analyzed the RS-problem for an elastic layer with shear modulus $\mu(z)=\mu_{1}(z+b)^{\beta_{1}}$, which was bonded to an elastic half-space with shear modulus $\mu(z)=\mu_{2}(z+b)^{\beta_{2}}$. Selvadurai, Singh and Vrbik [40] considered the RS-problem for half-space by assuming shear modulus $\mu(z)=G_{1}+G_{2} e^{-\beta z}$. Dhaliwal [41] solved the RS-problem for a more general form of shear modulus $\mu(z)=\mu_{0}+\epsilon \mu_{1}(z)+\epsilon^{2} \mu_{2}(z)+\cdots$, where $\mu_{0}$ and $\epsilon \ll 1$ are positive real constants while $\mu_{\mathrm{i}}(z)$ are differentiable functions of $z$. Dhaliwal and Chehil [42] solved the RS-problem of non-homogeneous layer bonded to another non-homogeneous elastic layer with the shear modulus as $\mu_{\mathrm{i}}=\mu_{\mathrm{i}}\left(\mathrm{a}_{\mathrm{i}}+z\right)^{a_{\mathrm{i}}}, i=1,2$ for the two materials.

In this chapter we will consider the Reissner-Sagoci type problems for finite composite elastic cylinder (section 2.2), finite elastic cylinder embedded in an elastic layer (section 2.3) and semi-infinite composite elastic cylinder (section 2.4). The materials considered in this chapter are assumed to be elastic, homogeneous and isotropic.

By the use of integral transforms and the theory of dual integral equations, the problems are reduced to the solution of a Fredholm integral equation of the second kind. Numerical solution of the integral equation is obtained and the numerical values of the torque required to produce the given rotation are displayed graphically.

As discussed in Chapter 1, the displacement field of the medium under torsion considered in this chapter is given by $u_{\mathrm{r}}=u_{\mathrm{z}}=0$ and $u_{\theta}=u_{\theta}(r, z)$, depending on $r$ and $z$ only, where $(r, \theta, z)$ is the cylindrical polar coordinate system. And the corresponding non-zero stress components of the stress tensor
are given by equations (1.6.3). Consequently the equation of equilibrium is given by equation (1.6.5) when there are no body forces. To simplify, let us denote $u_{\mathrm{r}}, u_{\theta}, u_{z}$ by $u, v, w$ respectively, then we have the following basic equations for the problems under consideration in this chapter:

$$
\begin{align*}
& u=0, \quad v=v(r, z), \quad w=0 \\
& \sigma_{\theta \mathrm{z}}(r, z)=\mu \frac{\partial v}{\partial z}, \quad \sigma_{\mathrm{r} \theta}(r, z)=\mu\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right), \\
& \frac{\partial^{2} v}{\partial r^{2}}+\frac{1}{r} \frac{\partial v}{\partial r}-\frac{v}{r^{2}}+\frac{\partial^{2} v}{\partial z^{2}}=0 \tag{2.1.2}
\end{align*}
$$

Using the method of separation of variables, it is easy to show that the following are basic solutions of the last equation in (2.1.2) for $v(r, z)$ :
(I) $J_{1}(\xi r) \exp ( \pm \xi z)$,
(II) $\quad Y_{1}(\xi r) \exp ( \pm \xi z)$,
(III) $I_{1}(\xi r) \cos (\xi z)$ or $I_{1}(\xi r) \sin (\xi z)$,
(IV) $K_{1}(\xi r) \cos (\xi z)$ or $K_{1}(\xi r) \sin (\xi z)$,
(V) $r z, r, 1 / r, z / r$;
where $J_{\nu}, Y_{\nu}$ are Bessel functions of the first and second kind and of order $\nu$, and $I_{\nu}, K_{\nu}$ are modified Bessel functions of the first and second kind and of order $\nu$ respectively [94].

In the following three sections we will use three different combinations of the basic solutions stated above such that they will satisfy the boundary conditions of each of the three problems.

### 2.2 Finite composite elastic cylinder

### 2.2.1 The statement of the problem

In this section we consider the torsion of a finite elastic cylinder which is embedded in a finite elastic cylindrical shell with different shear modulus.

We assume that a finite elastic cylinder of radius $d$, height $b$ and shear modulus $\mu_{1}$ is embedded in a finite elastic cylindrical shell of outer radius $a$, height $b$, and shear modulus $\mu_{2}$ as shown in Fig.2.2.1. It is also assumed that the inner cylinder is perfectly bonded to the surrounding cylindrical shell and that a torque is applied to the inner cylinder, through a rigid disc of radius $c<d$, which is bonded to its flat surface. It is assumed that the bottom flat. surface of the composite cylinder is rigidly fixed and the curved outer surface of the composite cylinder is stress-free. In terms of cylindrical polar coordinates $(r, \theta, z)$, displacement field, the corresponding non-zero stress components and the equilibrium equation are given by (2.1.2). The basic solutions of the equilibrium equation are given by (2.1.3).

We further assume that the rigid disc bonded to the inner cylinder is turned through a small angle $\epsilon$. We therefore consider the problem of determining the stress and displacement field in the composite cylinder with the following boundary and continuity conditions:


Fig.2.2.1
Torsion of an elastic cylinder bonded to a dissimilar elastic cylindrical shell.

$$
\begin{array}{ll}
v(r, b)=0, & 0 \leq r<d,(2.2 .1) \\
\hat{v}(r, b)=0, & d<r<a,(2.2 .2) \\
\vartheta(r, 0)=\epsilon r, & 0<r<c,(2.2 .3) \\
\sigma_{\theta \mathrm{z}}(r, 0)=0, & c \leq r<d,(2.2 .4) \\
\hat{\sigma}_{\theta \mathrm{z}}(r, 0)=0, & d<r<a,(2.2 .5) \\
\hat{\sigma}_{\mathrm{r} \theta}(a, z)=0, & 0 \leq z \leq b,(2.2 .6) \\
\vartheta(d, z)=\hat{v}(d, z), & 0 \leq z \leq b, \quad(2.2 .7) \\
\sigma_{\mathrm{re}}(d, z)=\hat{\sigma}_{\mathrm{r} \theta}(d, z) & 0 \leq z \leq b,(2.2 .8)
\end{array}
$$

where $v, \sigma_{\theta z}$, and $\sigma_{\mathrm{r} \theta}$ are the non-zero displacement and stress components for the inner cylinder while $\hat{v}, \hat{\sigma}_{\theta z}$, and $\hat{\sigma}_{\mathrm{r} \theta}$ are their counterparts for the surrounding medium.

### 2.2.2. Derivation of the dual integral equations

Now we introduce a combination of the basic solutions given in section 2.1 for $v(r, z)$ and for $\hat{v}(r, z)$, by means of these combinations we are able to reduce the problem of solving the mixed boundary value problem stated in section 2.2.1 to that of solving a pair of dual integral equations.

For the inner cylinder, we assume that

$$
\begin{align*}
v(r, z)=a_{0} r(b-z) & +\mathscr{G}_{1}[\xi-1 A(\xi) \sinh [\xi(b-z)] ; \xi+r] \\
& +\sum_{\mathrm{n}=1}^{\infty} \xi_{\mathrm{n}}^{-1} B_{\mathrm{n}} \cos \left(\xi_{\mathrm{n}} z\right) I_{1}\left(\xi_{\mathrm{n}} r\right), \tag{2.2.9}
\end{align*}
$$

where $\mathscr{F}$, the Hankel operator, is defined by the equation

$$
\mathscr{H}_{\nu}[f(\xi) ; \xi \rightarrow r]=\int_{0}^{\infty} \xi f(\xi) J_{\nu}(\xi r) d \xi .
$$

For the surrounding medium we assume that

$$
\begin{equation*}
\hat{v}(r, z)=\sum_{\mathrm{n}=1}^{\infty} \xi_{\mathrm{n}}^{-1} C_{\mathrm{n}} \cos \left(\xi_{\mathrm{n}} z\right) I_{1}\left(\xi_{\mathrm{n}} r\right)+\sum_{\mathrm{n}=1}^{\infty} \xi_{\mathrm{n}}^{-1} D_{\mathrm{n}} \cos \left(\xi_{\mathrm{n}} z\right) K_{1}\left(\xi_{\mathrm{n}} r\right) \tag{2.2.10}
\end{equation*}
$$

where

$$
a_{0}=\epsilon / b,
$$

while $A, B_{\mathrm{n}}, C_{\mathrm{n}}, D_{\mathrm{n}}$ and $\xi_{\mathrm{n}}$ are arbitrary constants to be determined later by using the boundary and continuity conditions.

Now from equations (2.1.2) we have

$$
\begin{align*}
& \sigma_{\mathrm{r} \theta}(r, z)=-\mu_{1} \mathscr{H}_{2}[A(\xi) \sinh [\xi(b-z)] ; \xi \rightarrow r]+\mu_{1} \sum_{\mathrm{n}=1}^{\infty} B_{\mathrm{n}} \cos \left(\xi_{\mathrm{n}} z\right) I_{2}\left(\xi_{\mathrm{n}} r\right),  \tag{2.2.11}\\
& \hat{\sigma}_{\mathrm{r} \theta}(r, z)=\mu_{2} \sum_{\mathrm{n}=1}^{\infty} C_{\mathrm{n}} \cos \left(\xi_{\mathrm{n}} z\right) I_{2}\left(\xi_{\mathrm{n}} r\right)-\mu_{2} \sum_{\mathrm{n}=1}^{\infty} D_{\mathrm{n}} \cos \left(\xi_{\mathrm{n}} z\right) K_{2}\left(\xi_{\mathrm{n}} r\right),  \tag{2.2.12}\\
& \sigma_{\theta \mathrm{z}}(r, z)=-\mu_{1} \mathscr{\mathscr { H } _ { \mathrm { l } }}[A(\xi) \cosh [\xi(b-z)] ; \xi \rightarrow r]-\mu_{1} \sum_{\mathrm{n}=1}^{\infty} B_{\mathrm{n}} \sin \left(\xi_{\mathrm{n}} z\right) I_{1}\left(\xi_{\mathrm{n}} r\right)-a_{0} \mu_{1} r,
\end{align*}
$$

$$
\begin{equation*}
\hat{\sigma}_{\theta \mathrm{z}}(r, z)=-\mu_{2} \sum_{\mathrm{n}=1}^{\infty} C_{\mathrm{n}} \sin \left(\xi_{\mathrm{n}} z\right) I_{1}\left(\xi_{\mathrm{n}} r\right)-\mu_{2} \sum_{\mathrm{n}=1}^{\infty} D_{\mathrm{n}} \sin \left(\xi_{\mathrm{n}} z\right) K_{1}\left(\xi_{\mathrm{n}} r\right) \tag{2.2.13}
\end{equation*}
$$

The conditions (2.2.1) and (2.2.2) will be satisfied if we take

$$
\cos \left(\xi_{\mathrm{n}} b\right)=0
$$

which gives

$$
\begin{equation*}
\xi_{\mathrm{n}}=(2 n-1) \pi / 2 b, \quad n=1,2,3, \ldots \ldots \tag{2.2.15}
\end{equation*}
$$

The condition (2.2.6) yields

$$
\begin{equation*}
\sum_{\mathrm{n}=1}^{\infty}\left[C_{\mathrm{n}} I_{2}\left(\xi_{\mathrm{n}} a\right)-D_{\mathrm{n}} K_{2}\left(\xi_{\mathrm{n}} a\right)\right] \cos \left(\xi_{\mathrm{n}} z\right)=0 \tag{2.2.16}
\end{equation*}
$$

and the conditions (2.2.7) and (2.2.8) give

$$
\begin{align*}
& a_{0} d(b-z)+\mathscr{H}_{1}\left[\xi^{-1} A(\xi) \sinh [\xi(b-z)] ; \xi \rightarrow d\right] \\
& =\sum_{\mathrm{n}=1}^{\infty} \xi_{\mathrm{n}}^{-1}\left[C_{\mathrm{n}} I_{1}\left(\xi_{\mathrm{n}} d\right)-B_{\mathrm{n}} I_{1}\left(\xi_{\mathrm{n}} d\right)+D_{\mathrm{n}} K_{1}\left(\xi_{\mathrm{n}} d\right)\right] \cos \left(\xi_{\mathrm{n}} z\right), \tag{2.2.17}
\end{align*}
$$

and

$$
\begin{align*}
& \mathscr{C}_{2}[A(\xi) \sinh [\xi(b-z)] ; \xi \rightarrow d] \\
&= \sum_{\mathrm{n}=1}^{\infty}\left[B_{\mathrm{n}} I_{2}\left(\xi_{\mathrm{n}} d\right)+\bar{\mu} D_{\mathrm{n}} K_{2}\left(\xi_{\mathrm{n}} d\right)-\bar{\mu} C_{\mathrm{n}} I_{2}\left(\xi_{\mathrm{n}} d\right)\right] \cos \left(\xi_{\mathrm{n}} z\right), \tag{2.2.18}
\end{align*}
$$

where

$$
\bar{\mu}=\mu_{2} / \mu_{1} .
$$

Since $\left\{\cos \left(\xi_{\mathrm{n}} z\right)\right\}$ are orthogonal over the interval $(0, b)$ and

$$
\begin{align*}
& \int_{0}^{\mathrm{b}} \sinh [\xi(b-z)] \cos \left(\xi_{\mathrm{n}} z\right) d z=\frac{\xi \cosh (\xi b)}{\left(\xi^{2}+\xi_{\mathrm{n}}^{2}\right)}  \tag{2.2.19}\\
& \int_{0}^{\mathrm{b}}(b-z) \cos \left(\xi_{\mathrm{n}} z\right) d z=\frac{1}{\xi_{\mathrm{n}}^{2}} \tag{2.2.20}
\end{align*}
$$

from equations (2.2.16) to (2.2.18) we obtain

$$
\begin{align*}
& C_{\mathrm{n}} I_{2}\left(\xi_{\mathrm{n}} a\right)=D_{\mathrm{n}} K_{2}\left(\xi_{\mathrm{n}} a\right),  \tag{2.2.21}\\
& -B_{\mathrm{n}} I_{1}\left(\xi_{\mathrm{n}} d\right)+C_{\mathrm{n}} I_{1}\left(\xi_{\mathrm{n}} d\right)+D_{\mathrm{n}} K_{1}\left(\xi_{\mathrm{m}} d\right)=G_{1}(n),  \tag{2.2.22}\\
& B_{\mathrm{n}} I_{2}\left(\xi_{\mathrm{n}} d\right)-\bar{\mu} C_{\mathrm{n}} I_{2}\left(\xi_{\mathrm{n}} d\right)+\bar{\mu} D_{\mathrm{n}} K_{2}\left(\xi_{\mathrm{n}} d\right)=G_{2}(n), \tag{2.2.23}
\end{align*}
$$

where

$$
\begin{align*}
& G_{1}(n)=\frac{2 \xi_{\mathrm{n}}}{b}\left\{\int_{0}^{\infty} \frac{\xi A(\xi) \cosh (\xi b) J_{1}(\xi d)}{\xi^{2}+\xi_{\mathrm{n}}^{2}} d \xi+\frac{a_{0} d}{\xi_{\mathrm{n}}^{2}}\right\}, \\
& G_{2}(n)=\frac{2}{b} \int_{0}^{\infty} \frac{\xi^{2} A(\xi) \cosh (\xi b) J_{2}(\xi d)}{\xi^{2}+\xi_{\mathrm{n}}^{2}} d \xi . \tag{2.2.24}
\end{align*}
$$

Eliminating $C_{\mathrm{n}}$ from equations (2.2.21), (2.2.22) and (2.2.23), we obtain

$$
\begin{align*}
& -B_{\mathrm{n}} I_{1}\left(\xi_{\mathrm{n}} d\right)+D_{\mathrm{n}} Q_{1}(n)=G_{1}(n),  \tag{2.2.25}\\
& B_{\mathrm{n}} I_{2}\left(\xi_{\mathrm{n}} d\right)+\bar{\mu} D_{\mathrm{n}} Q_{2}(n)=G_{2}(n), \tag{2.2.26}
\end{align*}
$$

hence

$$
\begin{equation*}
B_{\mathrm{n}}=\left[\bar{\mu} G_{1}(n) Q_{2}(n)-G_{2}(n) Q_{1}(n)\right] / \Delta(n), \tag{2.2.27}
\end{equation*}
$$

where

$$
\begin{align*}
& \Delta(n)=-\bar{\mu}_{1}\left(\xi_{\mathrm{n}} d\right) Q_{2}(n)-I_{2}\left(\xi_{\mathrm{n}} d\right) Q_{1}(n), \\
& Q_{1}(n)=K_{1}\left(\xi_{\mathrm{n}} d\right)+\frac{K_{2}\left(\xi_{\mathrm{n}} a\right)}{I_{2}\left(\xi_{\mathrm{n}} a\right)} I_{1}\left(\xi_{\mathrm{n}} d\right), \\
& Q_{2}(n)=K_{2}\left(\xi_{\mathrm{n}} d\right)-\frac{K_{2}\left(\xi_{\mathrm{n}} a\right)}{I_{2}\left(\xi_{\mathrm{n}} a\right)} I_{2}\left(\xi_{\mathrm{n}} d\right) \tag{2.2.28}
\end{align*}
$$

From equation (2.2.14) we find that the condition (2.2.5) is identically satisfied and that the conditions (2.2.3) and (2.2.4) will be satisfied if $A(\xi)$ is the solution of the following dual integral equations

$$
\begin{array}{ll}
\mathscr{G}[\xi-1 A(\xi) \sinh (\xi b) ; \xi \rightarrow r]+\sum_{\mathrm{n}=1}^{\infty} \xi_{\mathrm{n}}^{-1} B_{\mathrm{n}} I_{\mathrm{l}}\left(\xi_{\mathrm{n}} r\right)=0, & r<c, \\
\mathscr{\mathscr { G } [}[A(\xi) \cosh (\xi b) ; \xi \rightarrow r]=-a_{0} r, & c<r<d .(2.2 .20)
\end{array}
$$

### 2.2.3 Reduction to integral equation of Fredholm type

To reduce the problem of solving the dual integral equations (2.2.29) and (2.2.30) to that of solving an integral equation of Fredhiom type of the second kind, we will make use of an integral representation for $A(\xi)$, which automatically satisfies equation (2.2.30).

It is known that if we take [26]

$$
\phi_{0}(t)=-\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left\{\begin{array}{lc}
t\left[\left(d^{2}-t^{2}\right)^{\frac{1}{2}}-\left(c^{2}-t^{2}\right)^{\frac{1}{2}}\right] & t<c, \\
t\left(d^{2}-t^{2}\right)^{\frac{1}{2}}, & c<t<d,(2.2 .31) \\
0, & t>d,
\end{array}\right.
$$

we have

$$
\begin{align*}
& \mathscr{F}_{\mathrm{s}}\left[\phi_{0}(t) ; t \rightarrow \xi\right]=-\int_{\mathbf{c}}^{\mathrm{d}} r^{2} J_{1}(\xi r) d r,  \tag{2.2.32}\\
& \mathscr{H}_{1}\left\{\mathscr{F}_{\mathrm{s}}\left[\phi_{0}(t) ; t \rightarrow \xi\right] ; \xi \rightarrow r\right\}= \begin{cases}0, & 0 \leq r<c \\
-\mathrm{r}, & c<r<d \\
0, & r>d\end{cases} \tag{2.2.33}
\end{align*}
$$

where $\mathscr{F}_{\mathrm{s}}$ is Fourier sine transform defined by

$$
\mathscr{F}_{\mathrm{s}}[f(z) ; z \rightarrow \xi]=\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_{0}^{\infty} f(z) \sin (\xi z) d z
$$

It is easy to show [43] that if we take

$$
\begin{equation*}
A(\xi)=\frac{a_{0}}{\cosh (\xi b)} \mathscr{F}_{\mathrm{s}}\left[\phi(t)+\phi_{0}(t) ; t \rightarrow \xi\right] \tag{2.2.34}
\end{equation*}
$$

where $\phi(t)$ is a new unknown fuction defined in $(0, \infty)$ such that $\phi(t)=0$ for $t>c$, then equation (2.2.30) will be satisfied automatically.

Substituting from equation (2.2.34) into equations (2.2.24) and using the integrals [44]

$$
\begin{array}{ll}
\int_{0}^{\infty} \frac{\xi^{2} \sin (\xi t) J_{2}(\xi d) d \xi}{\xi^{2}+\xi_{\mathrm{n}}^{2}}=\xi_{\mathrm{n}} \sinh \left(\xi_{\mathrm{n}} t\right) K_{2}\left(\xi_{\mathrm{n}} d\right), & t<d \\
\int_{0}^{\infty} \frac{\xi \sin (\xi t) J_{1}(\xi d) d \xi}{\xi^{2}+\xi_{\mathrm{n}}^{2}}=\sinh \left(\xi_{\mathrm{n}} t\right) K_{1}\left(\xi_{\mathrm{n}} d\right), \quad t<d \tag{2.2.36}
\end{array}
$$

we obtain

$$
\begin{equation*}
G_{1}(n)=g_{11}(n)+g_{12}(n)+g_{13}(n), \quad G_{2}(n)=g_{21}(n)+g_{22}(n) \tag{2.2.37}
\end{equation*}
$$

where

$$
\begin{align*}
& g_{11}(n)=\frac{2 a_{0} \xi_{\mathrm{n}}}{b}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_{0}^{\mathrm{c}} \phi(t) \sinh \left(\xi_{\mathrm{n}} t\right) K_{1}\left(\xi_{\mathrm{n}} d\right) d t \\
& g_{12}(n)=\frac{2 a_{0} \xi_{\mathrm{n}}}{b}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_{0}^{\mathrm{d}} \phi_{0}(t) \sinh \left(\xi_{\mathrm{n}} t\right) K_{1}\left(\xi_{\mathrm{n}} d\right) d t \\
& g_{13}(n)=\frac{2 a_{0} d}{b \xi_{\mathrm{n}}} \\
& g_{21}(n)=\frac{2 a_{0} \xi_{\mathrm{n}}}{b}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_{0}^{\mathrm{c}} \phi(t) \sinh \left(\xi_{\mathrm{n}} t\right) K_{2}\left(\xi_{\mathrm{n}} d\right) d t \\
& g_{22}(n)=\frac{2 a_{0} \xi_{\mathrm{n}}}{b}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_{0}^{\mathrm{d}} \phi_{0}(t) \sinh \left(\xi_{\mathrm{n}} t\right) K_{2}\left(\xi_{\mathrm{n}} d\right) d t \tag{2.2.38}
\end{align*}
$$

Operating on equation (2.2.29) by $x^{-1} \mathscr{6}_{1}^{-1}[r ; x]$ and using the following results [45]:

$$
\begin{align*}
& x^{-1} \mathscr{G}_{1}^{-1}\left\{r \mathscr{H}_{1}\left[\xi^{-1} F(\xi) ; \xi \rightarrow r\right] ; r-x\right\}=\mathscr{F}_{\mathrm{S}}[F(\xi) ; \xi \rightarrow x]  \tag{2.2.39}\\
& x^{-1} \mathscr{L}_{1}^{-1}\left[r I_{1}\left(\xi_{\mathrm{n}} r\right) ; r \rightarrow x\right]=\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sinh \left(\xi_{\mathrm{n}} x\right) \tag{2.2.40}
\end{align*}
$$

we obtain

$$
\begin{equation*}
\mathscr{F}_{\mathrm{s}}[A(\xi) \sinh (\xi b) ; \xi \rightarrow x]+\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sum_{\mathrm{n}=1}^{\infty} \xi_{\mathrm{n}}^{-1} B_{\mathrm{n}} \sinh \left(\xi_{\mathrm{n}} x\right)=0, \quad 0 \leq x<c \tag{2.2.41}
\end{equation*}
$$

where $\mathscr{\mathscr { S }}_{1}^{-1}$ is the inverse of $\mathscr{\mathscr { b }}_{1}$, the Abel operator of the first kind, which is defined by the equation

$$
\mathscr{C}_{1}[f(t) ; t \rightarrow r]=\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_{0}^{r} f(t)\left[r^{2}-t^{2}\right]^{-\frac{1}{2}} d t
$$

and

$$
\mathscr{S}_{1}^{-1}[f(r) ; t]=\frac{d}{d t} \mathscr{C}_{1}[r f(r) ; t]
$$

Using expressions (2.2.34), (2.2.27) and (2.2.37), we get

$$
\begin{align*}
\mathscr{F}_{\mathrm{s}} & {[A(\xi) \sinh (\xi b) ; \xi \rightarrow x]=a_{0} \phi(x)-\frac{4 a_{0}}{\pi} \int_{0}^{\mathrm{c}} \phi(t) d t \int_{0}^{\infty}\left(1+e^{2 \xi b}\right)^{-1} \sin (\xi t) \sin (\xi x) d \xi } \\
& +a_{0} \phi_{0}(x)-\frac{4 a_{0}}{\pi} \int_{0}^{\mathrm{d}} \phi_{0}(t) d t \int_{0}^{\infty}\left(1+e^{2 \xi b}\right)^{-1} \sin (\xi t) \sin (\xi x) d \xi \tag{2.2.42}
\end{align*}
$$

and

$$
\begin{align*}
& \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sum_{\mathrm{n}=1}^{\infty} \xi_{\mathrm{n}}^{-1} B_{\mathrm{n}} \sinh \left(\xi_{\mathrm{n}} x\right) \\
& =a_{0} \int_{0}^{\mathrm{c}} \phi(t) S(x, t) d t+a_{0} \int_{0}^{\mathrm{d}} \phi_{0}(t) S(x, t) d t+a_{0} R(x), \tag{2.2.43}
\end{align*}
$$

where

$$
\begin{align*}
& S(x, t))=\frac{4}{\pi b_{\mathrm{n}}} \sum_{=1}^{\infty} \bar{\mu} K_{1}\left(\xi_{\mathrm{n}} d\right) Q_{2}(n)-K_{2}\left(\xi_{\mathrm{n}} d\right) Q_{1}(n)  \tag{2.2.44}\\
& \Delta(n)  \tag{2.2.45}\\
& \sinh \left(\xi_{\mathrm{n}} t\right) \sinh \left(\xi_{\mathrm{n}} x\right) \\
& R(x)=\frac{2 d \bar{\mu}}{b}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sum_{\mathrm{n}=1}^{\infty} \frac{Q_{2}(n) \sinh \left(\xi_{\mathrm{n}} x\right)}{\Delta(n) \xi_{\mathrm{n}}^{2}}
\end{align*}
$$

Finally equation (2.2.41) may be written as

$$
\begin{equation*}
\phi(x)-\int_{0}^{c} \phi(t)[M(x, t)-S(x, t)] d t=-f(x), \quad 0 \leq x \leq c \tag{2.2.46}
\end{equation*}
$$

where

$$
\begin{align*}
& f(x)=\phi_{0}(x)-\int_{0}^{\mathrm{d}} \phi_{0}(t)[M(x, t)-S(x, t)] d t+R(x)  \tag{2.2.47}\\
& M(x, t)=\frac{4}{\pi} \int_{0}^{\infty}\left(1+e^{2 \xi b}\right)^{-1} \sin (\xi t) \sin (\xi x) d \xi \tag{2.2.48}
\end{align*}
$$

For a large $n$,

$$
\frac{\bar{\mu} K_{1}\left(\xi_{\mathrm{n}} d\right) Q_{2}(n)-K_{2}\left(\xi_{\mathrm{n}} d\right) Q_{1}(n)}{\Delta(n)} \sinh \left(\xi_{\mathrm{n}} t\right) \sinh \left(\xi_{\mathrm{n}} x\right)=O\left(\exp \left[-\xi_{\mathrm{n}}(2 d-x-t)\right]\right),
$$

hence the convergence of the series in equation (2.2.44) is fast .

To obtain the torque $T$ required to produce the prescribed rotation of the rigid disc bonded to the inner cylinder, as given by condition (2.2.3), we have to calculate the value of $\sigma_{\theta z}$ at $z=0$. Now from equations (2.2.13) and (2.2.34), we get

$$
\begin{align*}
\sigma_{\theta z}(r, 0)= & -\mu_{1} \mathscr{H}_{[ }[A(\xi) \cosh (\xi b) ; \xi \rightarrow r]-a_{0} \mu_{1} r \\
= & -a_{0} \mu_{1} \mathscr{H g}_{1}\left\{\mathscr{F}_{\mathrm{s}}[\phi(t) ; t \rightarrow \xi] ; \xi \rightarrow r\right\}-a_{0} \mu_{1} \mathscr{G}_{1}\left\{\mathscr{F}_{\mathrm{s}}\left[\phi_{0}(t) ; t \rightarrow \xi\right] ; \xi \rightarrow r\right\} \\
& -a_{0} \mu_{1} r, \quad r<c, \quad \tag{2.2.49}
\end{align*}
$$

and we find that

$$
\begin{align*}
& \int_{0}^{c} r^{2} \mathscr{H}_{1}\left\{\mathscr{F}_{s}\left[\phi_{0}(t) ; t \rightarrow \xi\right] ; \xi \rightarrow r\right\} d r=0,  \tag{2.2.50}\\
& \mathscr{H}_{1}\left\{\mathscr{F}_{\mathrm{s}}[\phi(t) ; t \rightarrow \xi] ; \xi \rightarrow r\right\}=-\frac{d}{d r} \mathscr{H}_{0}\left\{\xi^{-1} \mathscr{F}_{\mathrm{s}}[\phi(t) ; t \rightarrow \xi] ; \xi \rightarrow r\right\} . \tag{2.2.51}
\end{align*}
$$

The required torque is given by

$$
\begin{equation*}
T=-2 \pi \int_{0}^{\mathrm{c}} r^{2} \sigma_{\theta \mathrm{z}}(r, 0) d r . \tag{2.2.52}
\end{equation*}
$$

Using equations (2.2.49), (2.2.50) and (2.2.51) in (2.2.52) we obtain

$$
\begin{equation*}
T=\frac{\pi \mu_{1} \epsilon c^{4}}{2 b}+4(2 \pi)^{\frac{1}{2}} \frac{\mu_{1} \epsilon}{b} \int_{0}^{c} t \phi(t) d t \tag{2.2.53}
\end{equation*}
$$

It is worth mentioning that the solution for the corresponding semi-infinite composite cylinder problem cannot be derived from the present solution, since the $S$ integral in equation (2.2.46) is divergent as $b \rightarrow \infty$. It may be noted that

$$
\begin{align*}
& \int_{0}^{\mathrm{c}} \phi_{0}(t) S(x, t) d t \\
& =\int_{0}^{\mathrm{c}} \phi_{0}(t) \frac{4}{\pi b_{\mathrm{n}}} \sum_{1}^{\infty} \bar{\mu} K_{1}\left(\xi_{\mathrm{n}} d\right) Q_{2}(n)-K_{2}\left(\xi_{\mathrm{n}} d\right) Q_{1}(n)  \tag{2.2.54}\\
& \Delta(n) \\
& \sinh \left(\xi_{\mathrm{n}} t\right) \sinh \left(\xi_{\mathrm{n}} x\right) d t .
\end{align*}
$$

Let us consider a particular case, when $\bar{\mu}=0$, we have

$$
\frac{\bar{\mu} K_{1}\left(\xi_{\mathrm{n}} d\right) Q_{2}(n)-K_{2}\left(\xi_{\mathrm{n}} d\right) Q_{1}(n)}{\Delta(n)}=\frac{K_{2}\left(\xi_{\mathrm{n}} d\right)}{I_{2}\left(\xi_{\mathrm{n}} d\right)}
$$

Since $\xi_{\mathrm{n}}=(2 n-1) \pi /(2 b)$, the $\xi_{\mathrm{n}}$ are apart and formally

$$
\begin{align*}
& \lim _{b \rightarrow \infty} \frac{4}{\pi b_{\mathrm{n}}} \sum_{=1}^{\infty} \frac{\bar{\mu} K_{1}\left(\xi_{\mathrm{n}} d\right) Q_{2}(n)-K_{2}\left(\xi_{\mathrm{n}} d\right) Q_{1}(n)}{\Delta(n)} \sinh \left(\xi_{\mathrm{n}} t\right) \sinh \left(\xi_{\mathrm{n}} x\right) \\
& =\frac{4}{\pi^{2}} \int_{0}^{\infty} \frac{K_{2}(\xi d)}{I_{2}(\xi \mathrm{~d})} \sinh (\xi t) \sinh (\xi x) d \xi . \tag{2.2.55}
\end{align*}
$$

Sneddon [14] noted that the integral in (2.2.55) diverges as $O\left(\xi^{-2}\right)$. In general, the integral in (2.2.54) is also divergent. Hence the solution for the case $b \rightarrow \infty$ will be presented separately.

### 2.2.4 Numerical results and conclusions

Numerical values of $\phi(x)$ for $x=(0.0,0.10 .2, \ldots . .1 .0) c$ have been calculated from the integral equation (2.2.46) by reducing it to algebraic equations. And then the numerical values of the dimensionless ratio of torque $T / T_{0}$ have been calculated from equation (2.2.53), where $T_{0}=16 \mu_{1} \epsilon c^{3} / 3$ is the torque for the corresponding Reissner-Sagoci problem for the semi-infinite elastic space. The Simpson's rule is used to perform the numerical integrations and the Crout's factorisation method is used to solve the linear algebraic equations. In the numerical results the relative errors are controlled under 0.01 .

Numerical values of $T / T_{0}$ have been calculated for the following combination of values of

$$
\begin{aligned}
& b / c=0.2(0.1) 0.5,1.0,2.0,10.0 ; \quad d / c=1.0(0.2) 2.0,3.0(1.0) 10.0 \\
& a / d=1.1,1.5 ; \quad \bar{\mu}=0.5,2.0 ;
\end{aligned}
$$

and these have been displayed in Figs.2.2.2 to Fig.2.2.5. From the figures, we observe that for a fixed radius $c$ of the rigid disc the torque $T$ decreases as the height $b$ of the cylinder increases and the torque increases as the radius $d$ of the inner cylinder increases while the ratio $a / d$ of the radius of outer cylinder to the inner cylinder is kept the same. We also observe that the ratio $T / T_{0}$ approaches to 1 when $d / c$ and $b / c$ approach to infinity simultaneously. We notice that the values of $T / T_{0}$ have very negligible effect with the change in the values of $a / d$ from 1.1 to 1.5 or with the change in the values of $\bar{\mu}$ from 0.5 to 2.0 .


Fig.2.2.2
Numerical values of the ratio of the torques $T / T_{0}$ against $d / c$ for $\bar{\mu}=0.5, a / d=1.1$ for various values of $b / c=$ $0.2,0.3,0.4,0.5,1.0,2.0,10.0$.


Fig.2.2.3
Numerical values of the ratio of the torques $T / T_{0}$ against $d / c$ for $\bar{\mu}=0.5, a / d=1.5$ for various values of $b / c=0.2$, $0.3,0.4,0.5,1.0,2.0,10.0$.


Fig.2.2.4
Numerical values of the ratio of the torques $T / T_{0}$ against $d / c$ for $\bar{\mu}=2.0, a / d=1.1$ for various values of $b / c=0.2$, $0.3,0.4,0.5,1.0,2.0,10.0$.


Fig.2.2.5
Numerical values of the ratio of the torques $T / T_{0}$ against $d / c$ for $\bar{\mu}=2.0, a / d=1.5$ for various values of $b / c=0.2$, $0.3,0.4,0.5,1.0,2.0,10.0$.

### 2.3 Finite elastic cylinder embedded in an elastic layer

### 2.3.1 The statement of problem

The problem considered in this section is that of the torsion of a finite elastic cylinder which is embedded in an elastic layer of different shear modulus. We are assuming that an elastic cylinder of radius $d$ and shear modulus $\mu_{1}$ is embedded in an elastic layer whose shear shear moduis is $\mu_{2}$ as shown in Fig.2.3.1. It is also assumed that the cylinder is perfectly bonded to the surrounding elastic layer and a torque is applied to the cylinder, through a rigid disc of radius $c<d$, which is bonded to its top flat surface, and the flat bottom surface of the finite cylinder and the surrounding layer is fixed. In terms of cylindrical polar coordinates ( $r, \theta, z$ ) , displacement field is given by $u=w=0$ and $v=v(r, z)$, hence we have the basic equations (2.1.2) and the solutions (2.1.3) for $v(r, z)$.

We also assume that the rigid disc bonded to the cylinder is turned through a small angle $\epsilon$ and that. the height of the cylinder and the surrounding layer is $b$. We, therefore, consider the problem of determining the stress and displacement field in the cylinder and the surrounding layer with the following boundary and continuity conditions:


Fig.2.3.1
Torsion of an elastic cylinder bonded to a dissimilar elastic layer

$$
\begin{array}{ll}
v(r, 0)=\epsilon r, & 0 \leq r<c, \\
\sigma_{\theta z}(r, 0)=0, & c \leq r<d, \\
\hat{\sigma}_{\theta z}(r, 0)=0, & r>d, \\
v(r, b)=0, & r<d, \\
\hat{v}(r, b)=0, & r>d, \\
v(d, z)=\hat{v}(d, z), & 0 \leq z \leq b, \\
\sigma_{\mathrm{r} \mathrm{\theta}}(d, z)=\hat{\sigma}_{\mathrm{r} \theta}(d, z) & 0 \leq z \leq b, \\
\hat{v}(r, z) \rightarrow 0, \hat{\sigma}_{\theta \mathrm{z}}(r, z) \rightarrow 0, \hat{\sigma}_{\mathrm{r} \theta}(r, z) \rightarrow 0, & r \rightarrow \infty, \tag{2.3.7a}
\end{array}
$$

where $v, \sigma_{\theta z}$, and $\sigma_{\mathrm{r} \theta}$ are the non-zero displacement and stress components in the cylinder while $\hat{v}, \hat{\sigma}_{\theta z}$, and $\hat{\sigma}_{\mathrm{r} \theta}$ are their counterparts in the surrounding layer.

### 2.3.2. Derivation of the dual integral equations

Conditions (2.3.7a) at infinity are identically satisfied if in the combination of the basic solutions assumed for $\hat{v}(r, z)$ in section 2.2.2, we take, for the present problem, $C_{\mathrm{n}}=0$ for all $n$. By letting $C_{\mathrm{n}}=0$ for all $n$ we reach the following combinations.

For $0 \leq r<d$, we may assume the following representation for $v$

$$
\begin{equation*}
\vartheta(r, z)=a_{0} r(b-z)+\mathscr{F}_{1}\left[\xi^{-1} A(\xi) \sinh [\xi(b-z)] ; \xi-r\right]+\sum_{\mathrm{n}=1}^{\infty} \xi_{\mathrm{n}}^{-1} B_{\mathrm{n}} \cos \left(\xi_{\mathrm{n}} z\right) I_{1}\left(\xi_{\mathrm{n}} r\right), \tag{2.3.8}
\end{equation*}
$$

and for $r>d$ we may assume that

$$
\begin{equation*}
\hat{v}(r, z)=\sum_{\mathrm{n}=1}^{\infty} \xi_{\mathrm{n}}^{-1} D_{\mathrm{n}} \cos \left(\xi_{\mathrm{n}} z\right) K_{\mathrm{i}}\left(\xi_{\mathrm{n}} r\right), \tag{2.3.9}
\end{equation*}
$$

where $a_{0}=\epsilon / b$, while $A, B_{\mathrm{n}}, D_{\mathrm{n}}$ and $\xi_{\mathrm{n}}$ are to be determined later.

From equations (2.1.2) we have

$$
\begin{align*}
& \sigma_{\mathrm{r} \theta}(r, z)=-\mu_{1} \mathscr{O}_{2}[A(\xi) \sinh [\xi(b-z)] ; \xi \rightarrow r]+\mu_{1} \sum_{\mathrm{n}=1}^{\infty} B_{\mathrm{n}} \cos \left(\xi_{\mathrm{n}} z\right) I_{2}\left(\xi_{\mathrm{n}} r\right)  \tag{2.3.10}\\
& \hat{\sigma}_{\mathrm{r} \theta}(r, z)=-\mu_{2} \sum_{\mathrm{n}=1}^{\infty} D_{\mathrm{n}} \cos \left(\xi_{\mathrm{n}} z\right) K_{2}\left(\xi_{\mathrm{n}} r\right)  \tag{2.3.11}\\
& \sigma_{\theta \mathrm{z}}(r, z)=-\mu_{1} \mathscr{\mathscr { O }}[A(\xi) \cosh [\xi(b-z)] ; \xi \rightarrow r]-\mu_{\mu_{\mathrm{n}}} \sum_{n=1}^{\infty} B_{\mathrm{n}} \sin \left(\xi_{\mathrm{n}} z\right) I_{1}\left(\xi_{\mathrm{n}} r\right)-a_{0} \mu_{1} r  \tag{2.3.12}\\
& \hat{\sigma}_{\theta \mathrm{z}}(r, z)=-\mu_{\mathrm{n}} \sum_{\mathrm{n}=1}^{\infty} D_{\mathrm{n}} \sin \left(\xi_{\mathrm{n}} z\right) K_{1}\left(\xi_{\mathrm{n}} r\right) \tag{2.3.13}
\end{align*}
$$

The conditions (2.3.4) and (2.3.5) may be satisfied by taking

$$
\cos \left(\xi_{\mathrm{n}} b\right)=0
$$

which gives that

$$
\begin{equation*}
\xi_{\mathrm{n}}=(2 n-1) \pi / 2 b, \quad n=1,2,3, \ldots \ldots \tag{2.3.14}
\end{equation*}
$$

The condition (2.3.6) yields

$$
\begin{align*}
& a_{0} d(b-z)+\mathscr{O}_{1}\left[\xi^{-1} A(\xi) \sinh [\xi(b-z)] ; \xi \rightarrow d\right] \\
& =\sum_{\mathrm{n}=1}^{\infty} \xi_{\mathrm{n}}^{-1}\left[D_{\mathrm{n}} K_{1}\left(\xi_{\mathrm{n}} d\right)-B_{\mathrm{n}} I_{1}\left(\xi_{\mathrm{n}} d\right)\right] \cos \left(\xi_{\mathrm{n}} z\right), \tag{2.3.15}
\end{align*}
$$

and the condition (2.3.7) yields

$$
\begin{equation*}
\mathscr{\mathscr { E } _ { 2 }}[A(\xi) \sinh [\xi(b-z)] ; \xi \rightarrow d]=\sum_{\mathrm{n}=1}^{\infty}\left[B_{\mathrm{n}} I_{2}\left(\xi_{\mathrm{n}} d\right)+\bar{\mu} D_{\mathrm{n}} K_{2}\left(\xi_{\mathrm{n}} d\right)\right] \cos \left(\xi_{\mathrm{n}} z\right) \tag{2.3.16}
\end{equation*}
$$

where $\bar{\mu}=\mu_{2} / \mu_{1}$.
Since $\left\{\cos \left(\xi_{\mathrm{n}} z\right)\right\}_{\mathrm{n}=1,2,3} \ldots$ are orthogonal over the interval $(0, b)$ and

$$
\begin{align*}
& \int_{0}^{\mathrm{b}} \sinh [\xi(b-z)] \cos \left(\xi_{\mathrm{n}} z\right) d z=\frac{\xi \cosh (\xi b)}{\left(\xi^{2}+\xi^{2}\right)}  \tag{2.3.17}\\
& \int_{0}^{\mathrm{b}}(b-z) \cos \left(\xi_{\mathrm{n}} z\right) d z=\frac{1}{\xi_{\mathrm{n}}^{2}} \tag{2.3.18}
\end{align*}
$$

equations (2.3.15) and (2.3.16) lead to the following equations

$$
\begin{align*}
-B_{\mathrm{n}} I_{1}\left(\xi_{\mathrm{n}} d\right)+D_{\mathrm{n}} K_{1}\left(\xi_{\mathrm{n}} d\right) & =\frac{2 \xi_{\mathrm{n}}}{b}\left\{\int_{0}^{\infty} \frac{\xi A(\xi) \cosh (\xi b) J_{1}(\xi d)}{\xi^{2}+\xi_{\mathrm{n}}^{2}} d \xi+\frac{a_{0} d}{\xi_{\mathrm{n}}^{2}}\right\} \\
& =G_{1}(n),  \tag{2.3.19}\\
B_{\mathrm{n}} I_{2}\left(\xi_{\mathrm{n}} d\right)+\bar{\mu} D_{\mathrm{n}} K_{2}\left(\xi_{\mathrm{n}} d\right) & =\frac{2}{b} \int_{0}^{\infty} \frac{\xi^{2} A(\xi) \cosh (\xi b) J_{2}(\xi d)}{\xi^{2}+\xi_{\mathrm{n}}^{2}} d \xi=G_{2}(n) \tag{2.3.20}
\end{align*}
$$

Solving equations (2.3.19) and (2.3.20) for $B_{\mathrm{n}}$ we obtain

$$
\begin{equation*}
B_{\mathrm{n}}=\left[-G_{2}(n) K_{1}\left(\xi_{\mathrm{n}} d\right)+\bar{\mu} G_{1}(n) K_{2}\left(\xi_{\mathrm{n}} d\right)\right] / \Delta(n) \tag{2.3.21}
\end{equation*}
$$

where

$$
\begin{equation*}
\Delta(n)=-I_{2}\left(\xi_{\mathrm{n}} d\right) K_{1}\left(\xi_{\mathrm{n}} d\right)-\bar{\mu} I_{1}\left(\xi_{\mathrm{n}} d\right) K_{2}\left(\xi_{\mathrm{n}} d\right) \tag{2.3.22}
\end{equation*}
$$

From equation (2.3.13) we find that the condition (2.3.3) is identically satisfied and boundary conditions (2.3.1) and (2.3.2) will be satisfied if $A(\xi)$ is the solution of the following dual integral equations

$$
\begin{array}{lc}
\mathscr{B}_{1}\left[\xi^{-1} A(\xi) \sinh (\xi b) ; \xi-r\right]+\sum_{\mathrm{n}=1}^{\infty} \xi_{\mathrm{n}}^{-1} B_{\mathrm{n}} I_{1}\left(\xi_{\mathrm{n}} r\right)=0, & r<c \\
\mathscr{H}_{1}[A(\xi) \cosh (\xi b) ; \xi \rightarrow r]=-a_{0} r, & c \leq r<d \tag{2.3.24}
\end{array}
$$

### 2.3.3 Reduction to integral equation of Fredholm type

Equations (2.3.23) and (2.3.24) have the same appearances as the equations (2.2.29) and (2.2.30) only with different expression of $B_{\mathrm{n}}$. So we can use the same arguments as we did in section 2.2.3 to reduce the problem of solving the equations (2.3.23) and (2.3.24) to that of solving an integral equation of

Fredholm type of the second kind by means of an integral representation for $A(\xi)$.

Let $\phi_{0}(t)$ be the function defined in equation (2.2.31) and

$$
\begin{equation*}
A(\xi)=\frac{a_{0}}{\cosh (\xi b)} \mathscr{F}_{\mathrm{s}}\left[\phi(t)+\phi_{0}(t) ; t \rightarrow \xi\right] \tag{2.3.25}
\end{equation*}
$$

where $\phi(t)$ is a new unknown fuction defined in $(0, \infty)$ such that $\phi(t)=0$ for $t>c$. Defining $G_{1}(n)$ and $G_{2}(\mathrm{n})$ by the equations (2.2.37) and $g_{11}, g_{12}, g_{13}, g_{21,}$, $g_{22}$ by equations (2.2.38) we obtain that

$$
\begin{align*}
B_{\mathrm{n}}= & -\left\{\left[g_{21}(n) K_{1}\left(\xi_{\mathrm{n}} d\right)-\bar{\mu} g_{11}(n) K_{2}\left(\xi_{\mathrm{n}} d\right)\right]+\left[g_{22}(n) K_{1}\left(\xi_{\mathrm{n}} d\right)-\bar{\mu} g_{12}(n) K_{2}\left(\xi_{\mathrm{n}} d\right)\right]\right. \\
& \left.-\bar{\mu} g_{13}(n) K_{2}\left(\xi_{\mathrm{n}} d\right)\right\} / \Delta(n) \tag{2.3.26}
\end{align*}
$$

Operating on equation (2.3.23) by $x^{-1} \mathscr{L}_{1}^{-1}[r ; x]$ we get equations (2.2.41), (2.2.42) and

$$
\begin{align*}
& \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sum_{\mathrm{n}=1}^{\infty} \xi_{\mathrm{n}}^{-1} B_{\mathrm{n}} \sinh \left(\xi_{\mathrm{n}} x\right)=a_{0} \int_{0}^{\mathrm{c}} \phi(t)\left[\sum_{\mathrm{n}=1}^{\infty} Q(n) \sinh \left(\xi_{\mathrm{n}} t\right) \sinh \left(\xi_{\mathrm{n}} x\right)\right] d t \\
& \quad+a_{0} \int_{0}^{\mathrm{d}} \phi_{0}(t)\left[\sum_{\mathrm{n}=1}^{\infty} Q(n) \sinh \left(\xi_{\mathrm{n}} t\right) \sinh \left(\xi_{\mathrm{n}} x\right)\right] d t+a_{0} R(x) \tag{2.3.27}
\end{align*}
$$

where

$$
\begin{align*}
& Q(n)=\frac{4}{\pi b \Delta(n)}(\bar{\mu}-1) K_{1}\left(\xi_{\mathrm{n}} d\right) K_{2}\left(\xi_{\mathrm{n}} d\right) \\
& R(x)=\frac{2 d \bar{\mu}}{b}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sum_{\mathrm{n}=1}^{\infty} \frac{K_{2}\left(\xi_{\mathrm{n}} d\right) \sinh \left(\xi_{\mathrm{n}} x\right)}{\Delta(n) \xi_{\mathrm{n}}^{2}} \tag{2.3.28}
\end{align*}
$$

Substituting from equations (2.2.42) and (2.3.27) into equation(2.2.41) we find that $\phi(x)$ must satisfy the following integral equation

$$
\begin{equation*}
\phi(x)-\int_{0}^{c} \phi(t) M(x, t) d t+\int_{0}^{c} \phi(t) N(x, t) d t=-f(x), \quad 0 \leq x \leq c \tag{2.3.29}
\end{equation*}
$$

where

$$
\begin{align*}
& f(x)=\phi_{0}(x)-\int_{0}^{\mathrm{d}} \phi_{0}(t) M(x, t) d t+\int_{0}^{\mathrm{d}} \phi_{0}(t) N(x, t) d t+R(x)  \tag{2.3.30}\\
& M(x, t)=\frac{4}{\pi} \int_{0}^{\infty}\left(1+e^{2 \xi b}\right)^{-1} \sin (\xi t) \sin (\xi x) d \xi  \tag{2.3.31}\\
& N(x, t)=\sum_{\mathrm{n}=1}^{\infty} Q(n) \sinh \left(\xi_{\mathrm{n}} t\right) \sinh \left(\xi_{\mathrm{n}} x\right) . \tag{2.3.32}
\end{align*}
$$

For a large $\dot{n}$,

$$
Q(n) \sinh \left(\xi_{\mathrm{n}} t\right) \sinh \left(\xi_{\mathrm{n}} x\right)=O\left(\exp \left[-\xi_{\mathrm{n}}(2 d-x-t)\right]\right)
$$

and hence the convergence of the series in equation (2.3.32) is fast .
Following the same procedures as we did in section 2.2.3, we find that the torque $T$ necessary to produce the prescribed rotation of the rigid disc bonded to the cylinder is given by

$$
\begin{equation*}
T=\frac{\pi \mu_{1} \epsilon c^{4}}{2 b}+4(2 \pi)^{\frac{1}{2}} \frac{\mu_{1} \epsilon}{b} \int_{0}^{\mathrm{c}} t \phi(t) d t \tag{2.3.33}
\end{equation*}
$$

It is worth mentioning here that the integral equation (2.3.29) can be derived from (2.2.46) in section 2.2, by letting $a$, the radius of the outer elastic cylindrical shell, tend to infinity. However, when $d$, the radius of the cylinder, tends to infinity, the expression for displacement $v$ in (2.2.9) is unacceptable.

### 2.3.4 Particular cases

Case (a) $\mu_{2} \rightarrow \infty$
Letting $\mu_{2} \rightarrow \infty$ in the results of previous section, we get the results for
the case in which the elastic cylinder is embedded in a rigid layer as a limiting case. Since

$$
\frac{\bar{\mu}-1}{\Delta(n)} \rightarrow \frac{-1}{I_{1}\left(\xi_{\mathrm{n}} d\right) K_{2}\left(\xi_{\mathrm{n}} d\right)} \quad, \quad \text { as } \mu_{2} \rightarrow \infty
$$

we obtain the expression for the kernel

$$
N(x, t)=\frac{4}{\pi b} \sum_{\mathrm{n}=1}^{\infty} \frac{K_{1}\left(\xi_{\mathrm{n}} d\right)}{I_{1}\left(\xi_{\mathrm{n}} d\right)} \sinh \left(\xi_{\mathrm{n}} t\right) \sinh \left(\xi_{\mathrm{n}} x\right),
$$

and

$$
R(x)=\frac{2 d}{b}\left(\frac{2}{x}\right)^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{\sinh \left(\xi_{\mathrm{n}} x\right)}{I_{1}\left(\xi_{\mathrm{n}} d\right) \xi_{\mathrm{n}}^{2}},
$$

for $\mu_{2} \rightarrow \infty$. With these modifications the solution for this case is siven by equations (2.3.29) and (2.3.30).

Case (b) $\mu_{2} \rightarrow 0$.
In this case, if we let $\mu_{2} \rightarrow 0$, we get the solution for the case in which the elastic cylinder is free of stress on its curved surface. And

$$
\frac{\bar{\mu}-1}{\Delta(n)} \rightarrow \frac{1}{I_{2}\left(\xi_{\mathrm{n}} d\right) K_{1}\left(\xi_{\mathrm{n}} d\right)} \quad, \quad \text { as } \mu_{2} \rightarrow 0
$$

hence the kernel $N(x, t)$ becomes

$$
N(x, t)=\frac{4}{\pi b} \sum_{\mathrm{n}=1}^{\infty} \frac{K_{2}\left(\xi_{\mathrm{n}} d\right)}{I_{2}\left(\xi_{\mathrm{n}} d\right)} \sinh \left(\xi_{\mathrm{n}} t\right) \sinh \left(\xi_{\mathrm{n}} x\right)
$$

and $R(x)=0$, as $\mu_{2} \rightarrow 0$ and these results are in agreement with Gladwell and Lemczyk [26] .

### 2.3.5 Numerical results and conclusions

Numerical values of $\phi(x)$ for $x=(0.0,0.10 .2, \ldots . .1 .0) c$ have been calculated from the integral equation (2.3.29) by reducing it to algebraic equations. And then the numerical values of the dimensionless ratio of torque $T / T_{0}$ have been calculated from equation (2.3.33), where $T_{0}=16 \mu_{1} \epsilon c^{3} / 3$ is the torque for the corresponding Reissner-Sagoci problem for the semi-infinite space. The Simpson's rule is used to perform the numerical integrations and the Crout's factorisation method is used to solve the linear algebraic equations. In the numerical results the relative errors are controlled under 0.01 .

Numerical values of $T / T_{0}$ have been calculated for the following values of $b / c, d / c$ and $\bar{\mu}=\mu_{2} / \mu_{1}$ :

$$
\begin{aligned}
& b / c=0.2(0.1) 0.5,1.0,2.0,10.0 ; \quad d / c=1.0(0.2) 2.0,3.0(1.0) 10.0 ; \\
& \bar{\mu}=0.0,0.5,2.0 ; \text { and } \bar{\mu} \rightarrow \infty
\end{aligned}
$$

and these have been displayed in Fig.2.3.2 to Fig.2.3.5.
From the figures, we observe that the torque $T$ decreases as the height $b$ of the cylinder increases and the torque increases as the radius $d$ of the cylinder increases. We also observe that the ratio $T / T_{0}$ approaches 1 when $d / c$ and $b / c$ approach to infinity at the same time. We notice that the change in the values of $T / T_{0}$ when $b$ (the height of the elastic cylinder and layer) increases from 2 to 10 goes on decreasing as $\bar{\mu}$ increases from 0 to .


Fig.2.3.2
Values of $T / T_{0}$ displayed against $d / c$ for $b / c=0.2,0.3$, $0.4,0.5,1.0,2.0,10.0$ and $\bar{\mu}=0.0$.


Fig.2.3.3
Values of $T / T_{0}$ displayed against $d / c$ for $b / c=0.2,0.3$, $0.4,0.5,1.0,2.0,10.0$ and $\bar{\mu}=0.5$.


Fig.2.3.4
Values of $T / T_{0}$ displayed against $d / c$ for $b / c=0.2,0.3$, $0.4,0.5,1.0,2.0,10.0$ and $\bar{\mu}=2.0$.


Fig.2.3.5
Values of $T / T_{0}$ displayed against $d / c$ for $b / c=0.2,0.3$, $0.4,0.5,1.0,2.0,10.0$ and $\bar{\mu} \rightarrow \infty$.

### 2.4 Semi-infinite composite elastic cylinder

### 2.4.1 The statement of the problem

In this section we consider the torsion problem of a semi-infinite elastic cylinder which is embedded in a semi-infinite elastic cylindrical shell with different shear modulus.

We assume that a semi-infinite elastic cylinder of radius $d$ and shear modulus $\mu_{1}$ is embedded in a semi-infinite elastic cylindrical shell of outer radius $a$ and shear moduis $\mu_{2}$ as shown in Fig.2.4.1. It is assumed that curved outer surface of the semi-infinite composite elastic cylinder is stress-free. It is also assumed that the inner cylinder is perfectly bonded to the surrounding elastic medium and that a torque is applied to the inner cylinder, through a rigid disc with radius $c<d$, which is bonded to its top flat surface. In terms of cylindrical polar coordinates $(r, \theta, z)$, displacement field is given by $u=w=0$ and $v=v(r, z)$, hence we have the basic equations (2.1.2) and the solutions (2.1.3) for $v(r, z)$.

We assume that the rigid disc bonded to the inner cylinder is turned through an angle $\epsilon$ and that the curved outer surface of the semi-infinite composite elastic cylinder is stress-free.. We therefore consider the problem of determining the stress and displacement field in the semi-infinite composite elastic cylinder with the following boundary and continuity conditions:


Fig.2.4.1
A problem of Reissner-Sagoci type for a semi-infinite composite elastic cylinder.

$$
\begin{array}{ll}
v(r, 0)=\epsilon r, & 0 \leq r<c,(2.4 .1) \\
\sigma_{\theta \mathrm{z}}(r, 0)=0, & c<r<d,(2.4 .2) \\
\hat{\sigma}_{\theta \mathrm{z}}(r, 0)=0, & d<r<a,(2.4 .3) \\
\hat{\sigma}_{\mathrm{r} \theta}(a, z)=0, & z \geq 0, \\
v(d, z)=\hat{v}(d, z), & z \geq 0, \\
\sigma_{\mathrm{r} \theta}(d, z)=\hat{\sigma}_{\mathrm{r} \theta}(d, z) & z \geq d ;
\end{array}
$$

where $v, \sigma_{\theta z}$, and $\sigma_{r \theta}$ are the non-zero displacement and stress components in the inner semi-infinite cylinder while $\hat{v}, \hat{\sigma}_{\theta z}$, and $\hat{\sigma}_{\mathrm{r} \theta}$ are their counterparts in the surrounding medium.

### 2.4.2 Derivation of the dual integral equations

First of all, we select a combination of the basic solutions for $v(r, z)$ and $\hat{v}(r, z)$ from the basic solutions listed in equation (2.1.3). By means of these solutions we are able to reduce the problem of solving the mixed boundary value problem stated in section 2.4 .1 to that of solving a pair of dual integral equations.

For the inner semi-infinite cylinder, we assume that

$$
\begin{equation*}
v(r, z)=\mathscr{H}_{1}\left[\xi^{-1} A(\xi) e^{-\xi z ;} \quad \xi \rightarrow r\right]+\mathscr{F}\left[\xi^{-1} B(\xi) I_{1}(\xi r) ; \quad \xi \rightarrow z\right], r<d, \tag{2.4.7}
\end{equation*}
$$

where $\mathscr{F}_{\mathrm{C}}$ is Fourier cosine transform defined by

$$
\mathscr{F}_{\mathrm{c}}[f(z) ; z \rightarrow \xi]=\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_{0}^{\infty} f(z) \cos (\xi z) d z
$$

and $\quad \mathscr{F}_{\mathrm{c}}^{-1}=\mathscr{F}_{\mathrm{c}}$, where $\mathscr{F}_{\mathrm{c}}^{-1}$ is the inverse of $\mathscr{F}_{\mathrm{c}}$.

For the surrounding medium we assume that

$$
\hat{v}(r, z)=\mathscr{F}\left[\xi-1\left\{C(\xi) I_{1}(\xi r)+D(\xi) K_{1}(\xi r)\right\} ; \xi \rightarrow z\right], \quad d<r<a,(2.4 .8)
$$

where $A(\xi), B(\xi), C(\xi)$ and $D(\xi)$ are to be determined later.
Now from equations (2.1.2) we have

$$
\begin{align*}
& \sigma_{\mathrm{r} \theta}(r, z)=-\mu_{1} \mathscr{H}_{2}\left[A(\xi) e^{-\xi z} ; \xi \rightarrow r\right]+\mu_{1} \mathscr{\mathscr { C }}\left[B(\xi) I_{2}(\xi r) ; \xi \rightarrow z\right],  \tag{2.4.9}\\
& \hat{\sigma}_{\mathrm{r} \theta}(r, z)=\mu_{2} \mathscr{F}\left[C(\xi) I_{2}(\xi r)-D(\xi) K_{2}(\xi r) ; \xi \rightarrow z\right],  \tag{2.4.10}\\
& \sigma_{\theta \mathrm{z}}(r, z)=-\mu_{1} \mathscr{H}_{1}\left[A(\xi) e^{-\xi z} ; \xi \rightarrow r\right]-\mu_{1} \mathscr{F}\left[B(\xi) I_{1}(\xi r) ; \xi \rightarrow z\right],  \tag{2.4.11}\\
& \hat{\sigma}_{\theta \mathrm{z}}(r, z)=-\mu_{2} \mathscr{F}\left[C(\xi) I_{1}(\xi r)+D(\xi) K_{1}(\xi r) ; \xi \rightarrow z\right] . \tag{2.4.12}
\end{align*}
$$

The conditions (2.4.5) and (2.4.6) give

$$
\begin{align*}
\mathscr{F}_{\mathrm{c}} & {\left[\xi^{-1}\left\{C(\xi) I_{1}(d \xi)+D(\xi) K_{1}(d \xi)-B(\xi) I_{1}(d \xi)\right\} ; \xi \rightarrow z\right] } \\
& =\mathscr{H}_{1}\left[\xi^{-1} A(\xi) e^{-\xi z} ; \xi \rightarrow d\right]  \tag{2.4.13}\\
\mathscr{F}_{\mathrm{c}} & {\left[B(\xi) I_{2}(d \xi)-\bar{\mu}\left\{C(\xi) I_{2}(d \xi)-D(\xi) K_{2}(d \xi)\right\} ; \xi \rightarrow z\right] } \\
& =\mathscr{H}_{2}\left[A(\xi) e^{-\xi z} ; \xi \rightarrow d\right] \tag{2.4.14}
\end{align*}
$$

where $\bar{\mu}=\mu_{2} / \mu_{1}$.
Taking the inverse Fourier cosine transform of equations (2.4.13) and (2.4.14) and making use of the following result

$$
\begin{equation*}
\mathscr{F}\left[e^{-\xi z} ; z-\zeta\right]=\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \xi\left(\xi^{2}+\zeta^{2}\right)^{-1}, \quad \xi>0 \tag{2.4.15}
\end{equation*}
$$

we obtain

$$
\begin{align*}
& C(\xi) I_{1}(d \xi)+D(\xi) K_{1}(d \xi)-B(\xi) I_{1}(d \xi)=G_{1}(\xi)  \tag{2.4.16}\\
& B(\xi) I_{2}(d \xi)-\bar{\mu}\left[C(\xi) I_{2}(d \xi)-D(\xi) K_{2}(d \xi)=G_{2}(\xi)\right. \tag{2.4.17}
\end{align*}
$$

where

$$
\begin{align*}
& G_{1}(\xi)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \xi \int_{0}^{\infty} \frac{\zeta A(\zeta) J_{1}(d \zeta)}{\xi^{2}+\zeta^{2}} d \zeta  \tag{2.4.18}\\
& G_{2}(\xi)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_{0}^{\infty} \frac{\zeta^{2} A(\zeta) J_{2}(d \zeta)}{\xi^{2}+\zeta^{2}} d \zeta \tag{2.4.19}
\end{align*}
$$

The condition (2.4.4) yields

$$
\begin{equation*}
C(\xi)=\frac{K_{2}(a \xi)}{I_{2}(a \xi)} D(\xi) . \tag{2.4.20}
\end{equation*}
$$

Eliminating $C(\xi)$ from equations (2.4.16) and (2.4.17) we obtain

$$
\begin{align*}
& -B(\xi) I_{1}(d \xi)+D(\xi) Q_{1}(\xi)=G_{1}(\xi)  \tag{2.4.21}\\
& B(\xi) I_{2}(d \xi)+\bar{\mu} D(\xi) Q_{2}(\xi)=G_{2}(\xi) \tag{2.4.22}
\end{align*}
$$

hence

$$
\begin{equation*}
B(\xi)=\left[\bar{\mu} G_{1}(\xi) Q_{2}(\xi)-G_{2}(\xi) Q_{1}(\xi)\right] / \Delta(\xi) \tag{2.4.23}
\end{equation*}
$$

where

$$
\begin{align*}
& \Delta(\xi)=-\bar{\mu} I_{1}(d \xi) Q_{2}(\xi)-I_{2}(d \xi) Q_{1}(\xi) \\
& Q_{1}(\xi)=K_{1}(d \xi)+\frac{K_{2}(a \xi)}{I_{2}(a \xi)} I_{1}(d \xi) \\
& Q_{2}(\xi)=K_{2}(d \xi)-\frac{K_{2}(a \xi)}{I_{2}(a \xi)} I_{2}(d \xi) \tag{2.4.24}
\end{align*}
$$

From equation (2.4.12) we find that the condition (2.4.3) is identically satisfied and that the conditions (2.4.1) and (2.4.2) will be satisfied if $A(\xi)$ is the solution of the following dual integral equations

$$
\begin{array}{ll}
\mathscr{G _ { 1 } [ \xi ^ { - 1 } A ( \xi ) ; \xi \rightarrow r ] + ( \frac { 2 } { x } ) ^ { \frac { 1 } { 2 } } \int _ { 0 } ^ { \infty } \xi ^ { - 1 } B ( \xi ) I _ { 1 } ( \xi r ) d \xi = \epsilon r ,} & 0 \leq r<c,(2.4 .25) \\
\mathscr{\mathscr { G } _ { 1 }}[A(\xi) ; \xi \rightarrow r]=0 . & c<r<d .(2.4 .26)
\end{array}
$$

### 2.4.3 Reduction to integral equation of Fredholm type

To reduce the problem of solving the dual integral equations (2.4.25) and (2.4.26) to that of solving an integral equation of Fredhiom type of the second kind, let us take

$$
\begin{equation*}
A(\xi)=\left(\frac{2}{x}\right)^{\frac{1}{2}} \epsilon \int_{0}^{\mathrm{c}} \phi(t) \sin (\xi t) d t \tag{2.4.27}
\end{equation*}
$$

where $\phi(t)$ is a new unknown function defined in $(0, \infty)$ such that $\phi(t)=0$ for $t>c$. The representation (2.4.27) satisfies the equation (2.4.26) identically .

By using the following result [43]

$$
\begin{equation*}
\mathscr{O}_{1}\left[\xi^{-1} \sin (\xi t) ; \xi \rightarrow r\right]=\frac{t H(r-t)}{r\left(r^{2}-t^{2}\right)^{\frac{1}{2}}} \tag{2.4.28}
\end{equation*}
$$

where $H(x)$ is the Heaviside function, then from equation (2.4.27) we obtain

$$
\begin{equation*}
\mathscr{H} \mathscr{B}_{1}\left[\xi^{-1} A(\xi) ; \xi \rightarrow r\right]=\epsilon r^{-1} \mathscr{L}_{1}[t \phi(t) ; r] . \tag{2.4.29}
\end{equation*}
$$

Substituting from equation (2.4.27) into equations (2.4.18) and (2.4.19) and using the integrals (2.2.35) and (2.2.36) we find that

$$
\begin{align*}
& G_{1}(\xi)=\frac{2 \epsilon}{\pi} \xi K_{1}(d \xi) \int_{0}^{\mathrm{c}} \phi(t) \sinh (\xi t) d t  \tag{2.4.30}\\
& G_{2}(\xi)=\frac{2 \epsilon}{\pi} \xi K_{2}(d \xi) \int_{0}^{\mathrm{c}} \phi(t) \sinh (\xi t) d t \tag{2.4.31}
\end{align*}
$$

By using equations (2.4.23) , (2.4.30) and (2.4.31) we obtain

$$
\begin{align*}
& \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_{0}^{\infty} \xi^{-1} B(\xi) I_{1}(\xi r) d \xi \\
& =\epsilon\left(\frac{2}{\pi}\right)^{\frac{3}{2}} \int_{0}^{\mathbf{c}} \phi(t) d t \int_{0}^{\infty} \frac{\mu K_{1}(d \xi) Q_{2}(\xi)-K_{2}(d \xi) Q_{1}(\xi)}{\Delta(\xi)} I_{1}(\xi r) \sinh (\xi t) d \xi \tag{2.4.32}
\end{align*}
$$

Operating equation (2.4.25) by $x^{-1}: \mathfrak{6}_{1}[r ; x]$, and using the results (2.2.39) and (2.2.40) we obtain

$$
\begin{equation*}
\phi(x)+\int_{0}^{c} \phi(t) S(x, t) d t=f(x), \quad 0 \leq x \leq c, \tag{2.4.33}
\end{equation*}
$$

where

$$
\begin{align*}
& S(x, t)=\frac{4}{\pi^{2}} \int_{0}^{\infty-} \frac{\bar{\mu} K_{1}(d \xi) Q_{2}(\xi)-K_{2}(d \xi) Q_{1}(\xi)}{\Delta(\xi)} \sinh (\xi x) \sinh (\xi t) d \xi, \\
& \left.f(x)=x^{-\frac{1}{d}} \frac{d}{d x} \mathscr{C}_{1}\left[r^{3} ; r-x\right)\right]=2\left(\frac{2}{\pi}\right)^{\frac{1}{2}} x . \tag{2.4.34}
\end{align*}
$$

To calculate the torque $T$ necessary to produce the prescribed rotation of the disc rigidly bonded to the inner semi-infinite cylinder , as given by condition (2.4.1), we have to calculate the value of $\sigma_{\theta \mathrm{z}}$ at $z=0$. Now from equations (2.4.11) and (2.4.27), we get

$$
\begin{align*}
\sigma_{\theta z}(r, 0) & =-\mu_{1} \mathscr{\mathscr { C }}[A(\xi) ; \xi \xi r] \\
& =\mu_{1} \frac{d}{d r} \mathscr{\mathscr { F }}[\xi-1 / A(\xi) ; \xi \rightarrow r] \\
& =\mu_{1} \epsilon\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{d}{d r} \int_{\mathrm{r}}^{\mathrm{c}} \phi(t)\left(t^{2}-r^{2}\right)^{-\frac{1}{2}} d t, \quad r<c . \tag{2.4.35}
\end{align*}
$$

The required torque $T$ is given by

$$
\begin{equation*}
T=-2 \pi \int_{0}^{\mathbf{c}} r^{2} \sigma_{\theta z}(r, 0) d r \tag{2.4.36}
\end{equation*}
$$

Using equation (2.4.35) we obtain

$$
\begin{equation*}
T=4 \mu_{1} \epsilon(2 \pi)^{\frac{1}{2}} \int_{0}^{\mathrm{c}} t \phi(t) d t . \tag{2.4.37}
\end{equation*}
$$

In a particular case, when the outer semi-infinite cylindrical shell is of large radius compared to the radius of the inner semi-infinite cylinder, we let the radius $a$ tend to infinity. As $a \rightarrow \infty, Q_{1}(\xi) \rightarrow K_{1}(d \xi)$ and $Q_{2}(\xi) \rightarrow K_{2}(d \xi)$,
hence

$$
\begin{equation*}
S(x, t)=\frac{4}{x^{2}}(\bar{\mu}-1) \int_{0}^{\infty} \frac{K_{1}(d \xi) K_{2}(d \xi)}{\Delta(\xi)} \sinh (\xi t) \sinh (\xi x) d \xi, \tag{2.4.38}
\end{equation*}
$$

which is in agreement with [22].

### 2.4.5 Numerical results and conclusions

Numerical values of $\phi(x)$ for $x=(0.0,0.10 .2, \ldots . .1 .0) c$ have been calculated from the integral equation (2.4.33) by reducing it to algebraic equations. And then the numerical values of the dimensionless ratio of torque $T / T_{0}$ have been calculated from equation (2.4.36), where $T_{0}=16 \mu_{1} \epsilon c^{3} / 3$ is the torque for the corresponding Reissner-Sagoci problem for the semi-infinite elastic space. The Simpson's rule is used to perform the numerical integrations and the Crout's factorisation method is used to solve the linear algebraic equations. In the numerical results the relative errors are controlled under 0.01 .

Numerical values of $T / T_{0}$ have been calculated for the following combination of values of

$$
a / d=1.1,1.5 ; \quad d / c=1.0(1.0) 10.0 ; \quad \vec{\mu}=0.5,1.0,2.0 ;
$$

and these have been displayed in Fig.2.4.2. From the figure we observe that for a fixed radius $c$ of the rigid disc the torque $T$ increases as the radius $b$ of the inner semi-infinite cylinder increases while the ratio $a / d$ of the radius of outer semi-infinite cylindrical shell to the inner semi-infinite cylinder is kept the same. We also observe that the ratio $T / T_{0}$ approaches 1 when the radius of the inner semi-infinite cylinder approach infinity.


Fig.2.4.2
Numerical values of ratio of the torques $T / T_{0}$ against $d / c$ for $\bar{\mu}=0.5,1.0,2.0 ; a / b=1.1,1.5$.

## CHAPTER 3

# TORSION OF TWO NONHOMOGENEOUS ELASTIC LAYERS WITH PENNY-SHAPED FLAW AT THE INTERFACE 

### 3.1 Introduction

In this chapter we investigate a torsion problem of two non-homogeneous isotropic elastic layers with a penny-shaped flaw at the interface of the layers. It is assumed that the flaw is in the form of an inclusion or a crack, and the rigidity of each of the two materials is a function of the variable $z$ in cylindrical polar coordinate system in the form $\mu(z)=\mu \exp (\alpha z)$, where $\mu$ and $\alpha$ are real constants. And it is also assumed that a rigid circular shaft is bonded to the free surface of the first layer .just above the circular flaw, and the circular shaft is rotated through a small angle by applying a twisting moment of torque $T$ and the rest of the surface $z=-h_{1}$ is kept stress-free. The lower surface $z=h_{2}$ of the second layer is either stress-free or rigidly fixed (see Fig.3.1.1). Four different cases are considered and the results for the corresponding problems of a layer and a half-space with a flaw at the interface are derived. The problem is reduced to solving a system of simultaneous Fredholm integral equations which have been solved numerically. Numerical values of the physical quantities have been displayed graphically.


Fig.3.1.1
Torsion of two elastic layers by a rigid shaft.

Under the assumption of axial symmetry of the problem, we know that the displacement components $u_{r}$ and $u_{z}$ vanish and $u_{\theta}$ depends on $r$ and $z$ only. To simplify, let us denote $u_{\mathrm{r}}, u_{\theta}$ and $u_{z}$ by $u, v$ and $w$ respectively. Then we have displacement components $u=w=0$ and $v=v(r, z)$. The stress-displacement relations are given by equations (1.6.3) and the equation of equilibrium is given by equation (1.6.5). So we have the following basic equations under consideration

$$
\begin{align*}
& u=w=0, \quad v=v(r, z), \\
& \sigma_{\theta^{z}}(r, z)=\mu(z) \frac{\partial v}{\partial z}, \quad \sigma_{\mathrm{r} \theta}(r, z)=\mu(z)\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right), \\
& \frac{\partial^{2} v}{\partial r^{2}}+\frac{1}{r} \frac{\partial v}{\partial r}-\frac{v}{r^{2}}+\frac{\partial^{2} v}{\partial z^{2}}+\frac{1}{\mu} \frac{\partial v}{\partial z} \frac{\partial \mu}{\partial z}=0 . \tag{3.1.1}
\end{align*}
$$

where $\mu(z)$ is the shear modulus of the medium and $v(r, z), \sigma_{\theta z}$ and $\sigma_{r z}$ are respectively the non-zero displacement and stress components in the polar cylindrical coordinates ( $r, \theta, z$ ).

We further assume that the two isotropic, non-homogeneous elastic layers, which occupy the regions $R_{1}\left(-h_{1} \leq z<0\right)$ and $R_{2}\left(0 \leq z \leq h_{2}\right)$, have the moduli of rigidity in the form of

$$
\mu(z)=\left\{\begin{array}{l}
\mu_{1} \mathrm{e}^{\alpha_{1} z}=\mu_{1}(z) ; \text { in } R_{1}\left(-h_{1}<z<0\right)  \tag{3.1.2}\\
\mu_{2} \mathrm{e}^{\alpha_{2} z}=\mu_{2}(z) ; \text { in } R_{2}\left(0<z<h_{2}\right)
\end{array}\right.
$$

are perfectly bonded except that there is a flaw (inclusion or crack) in the region $0 \leq r \leq b, z=0$. It is also assumed that a rigid circular shaft of radius $a$ units is bonded to the free surface of the first layer at $z=-h_{1}$ and it is rotated through a small angle $\epsilon_{0}$ by applying a twisting moment $T$ to the shaft and the rest of the surface $z=-h_{1}$ is kept stress-free. And the other surface $z=h_{2}$ is assumed either stress-free or rigidly fixed. By using the Hankel transform on the third equation of (3.1.1) we obtain the following
general solution $v$ for layer $R_{1}$ and $\hat{v}$ for layer $R_{2}$ respectively :

$$
\begin{align*}
& v(r, z)=\int_{0}^{\infty} \xi\left[A(\xi) \cosh \left(\delta_{1} z\right)+B(\xi) \sinh \left(\delta_{1} z\right)\right] e^{-\frac{1}{2} \alpha_{1} z} J_{1}(r \xi) d \xi \\
& -h_{1}<z<0  \tag{3.1.3}\\
& \hat{v}(r, z)=\int_{0}^{\infty} \xi\left[C(\xi) \cosh \left(\delta_{2} z\right)+D(\xi) \sinh \left(\delta_{2} z\right)\right] e^{-\frac{1}{2} \alpha_{2} z} J_{1}(r \xi) d \xi \\
& 0<z<h_{2} \tag{3.1.4}
\end{align*}
$$

where

$$
\delta_{\mathrm{i}}=\left(\xi^{2}+\frac{\alpha_{\mathrm{i}}^{2}}{4}\right)^{\frac{1}{2}}, \quad i=1,2
$$

and $A(\xi), B(\xi), C(\xi)$ and $D(\xi)$ are unknown functions to be determined such that the integrals in equation (3.1.3) and (3.1.4) are convergent and the boundary conditions are satisfied.

Substitution of equations (3.1.2) and (3.1.3) into the second equation of (3.1.1) yields

$$
\begin{array}{rlr}
\sigma_{\theta \mathrm{z}}= & \mu_{1}(z) \int_{0}^{\infty} \xi\left\{\left[\delta_{1} A(\xi) \sinh \left(\delta_{1} z\right)+\delta_{1} B(\xi) \cosh \left(\delta_{1} z\right)\right]-\frac{\alpha_{1}}{2}\left[A(\xi) \cosh \left(\delta_{1} z\right)\right.\right. \\
& \left.\left.+B(\xi) \sinh \left(\delta_{1} z\right)\right]\right\} e^{-\frac{1}{2} \alpha_{1} z} J_{1}(r \xi) d \xi, & -h_{1}<z<0, \\
\hat{\sigma}_{\theta \mathrm{z}}= & \mu_{2}(z) \int_{0}^{\infty} \xi\left\{\left[\delta_{2} C(\xi) \sinh \left(\delta_{2} z\right)+\delta_{2} D(\xi) \cosh \left(\delta_{2} z\right)\right]\right. & -\frac{\alpha_{2}}{2}\left[C(\xi) \cosh \left(\delta_{2} z\right)\right. \\
& \left.\left.+D(\xi) \sinh \left(\delta_{2} z\right)\right]\right\} e^{-\frac{1}{2} \alpha_{2} z} J_{1}(r \xi) d \xi, & 0<z<h_{2}, \tag{3.1.6}
\end{array}
$$

where $\sigma_{\theta z}$ and $\hat{\sigma}_{\theta z}$ denote the stress components in layer $R_{1}$ and $R_{2}$ respectively.

The common boundary and continuity conditions for the problem (excluding the conditions on flaw and the surface $z=h_{2}$ ) are the following

$$
\begin{array}{ll}
v\left(r,-h_{1}\right)=\epsilon_{0} r, & 0 \leq r<a, \\
\sigma_{\theta z}\left(r,-h_{1}\right)=0, & a<r<\infty, \\
v(r, 0)=\hat{v}\left(r, 0^{+}\right), & b \leq r<\infty, \\
\sigma_{\theta z}(r, 0)=\hat{\sigma}_{\theta z}\left(r, 0^{+}\right), & b \leq r<\infty .
\end{array}
$$

3.2 Inclusion problem with the surface $z=h_{2}$ stress-free.

### 3.2.1 Statement of the problem

In this section we will consider the problem stated in section 3.1 when the flaw is an inclusion and the surface $z=h_{2}$ is stress-free. In addition to the common conditions (3.1.7) to (3.1.10) we have

$$
\begin{array}{ll}
v(r, 0)=\hat{v}\left(r, 0^{+}\right)=\epsilon_{1} r ; & 0 \leq r \leq b \\
\hat{\sigma}_{\theta z}\left(r, h_{2}\right)=0, & 0 \leq r<\infty \tag{3.2.2}
\end{array}
$$

where we have assumed that as a result of the application of twisting moment $T$ on the shaft, the rigid inclusion on the interface will rotate through some unknown angle $\epsilon_{1}$.
3.2.2 Analysis

In this case the continuity condition (3.1.9) may be replaced by

$$
\begin{equation*}
v(r, 0)=\hat{v}\left(r, 0^{+}\right), \quad 0 \leq r<\infty \tag{3.2.3}
\end{equation*}
$$

by using condition (3.2.1), which then will be satisfied if we take

$$
\begin{equation*}
C(\xi)=A(\xi) \tag{3.2.4}
\end{equation*}
$$

Using the boundary condition (3.2.2) along with equation (3.1.6), we get

$$
\begin{equation*}
D(\xi)=E(\xi) C(\xi) \tag{3.2.5}
\end{equation*}
$$

where

$$
\begin{equation*}
E(\xi)=\frac{\left[\frac{\alpha_{2}}{2}-\delta_{2}\right]+\left[\frac{\alpha_{2}}{2}+\delta_{2}\right] \exp \left(-2 \delta_{2} h_{2}\right)}{-\left[\frac{\alpha_{2}}{2}-\delta_{2}\right]+\left[\frac{\alpha_{2}}{2}+\delta_{2}\right] \exp \left(-2 \delta_{2} h_{2}\right)} \tag{3.2.6}
\end{equation*}
$$

Applying conditions (3.1.7), (3.1.8), (3.1.10) and (3.2.1), the equations (3.1.3) to (3.1.6) give the following results:

$$
\begin{align*}
& \int_{0}^{\infty} \xi\left[A(\xi) \cdot \cosh \left(\delta_{1} h_{1}\right)-B(\xi) \sinh \left(\delta_{1} h_{1}\right)\right] J_{1}(r \xi) d \xi=\epsilon_{0} r \exp \left(-\frac{\alpha_{1}}{2} h_{1}\right) ; \\
& \int_{0}^{\infty} \xi\left\{\left[\frac{\alpha_{1}}{2} \cosh \left(\delta_{1} h_{1}\right)+\delta_{1} \sinh \left(\delta_{1} h_{1}\right)\right] A(\xi)-\left[\frac{\alpha_{1}}{2} \sinh \left(\delta_{1} h_{1}\right)+\right.\right.  \tag{3.2.7}\\
& \left.\left.\delta_{1} \cosh \left(\delta_{1} h_{1}\right) B(\xi)\right]\right\} J_{1}(r \xi) d \xi=0, \\
&  \tag{3.2.8}\\
& \int_{0}^{\infty} \xi A(\xi) J_{1}(r \xi) d \xi=\epsilon_{1} r,  \tag{3.2.9}\\
& \int_{0}^{\infty} \xi\left[\rho A(\xi)+\delta_{1} B(\xi)\right] J_{1}(r \xi) d \xi=0, \tag{3.2.10}
\end{align*}
$$

where

$$
\begin{equation*}
\bar{\mu}=\mu_{2} / \mu_{1}, \quad \rho=\bar{\mu}\left(\frac{\alpha_{2}}{2}-\delta_{2} E\right)-\frac{\alpha_{1}}{2} \tag{3.2.10a}
\end{equation*}
$$

Let us introduce two functions

$$
\begin{align*}
& P(\xi)=\left[\frac{\alpha_{1}}{2} \cosh \left(\delta_{1} h_{1}\right)+\delta_{1} \sinh \left(\delta_{1} h_{1}\right)\right] A(\xi)-\left[\frac{\alpha_{1}}{2} \sinh \left(\delta_{1} h_{1}\right)+\delta_{1} \cosh \left(\delta_{1} h_{1}\right)\right] B(\xi) \\
& Q(\xi)=\rho A(\xi)+\delta_{1} B(\xi) \tag{3.2.11}
\end{align*}
$$

Observing that [43]

$$
\begin{equation*}
\int_{0}^{\infty} \xi \sin (\xi t) J_{1}(r \xi) d \xi=-r\left(t^{2}-r^{2}\right)^{\frac{-3}{2}} H(t-r) \tag{3.2.12}
\end{equation*}
$$

we find that equations (3.2.8) and (3.2.10) will be identically satisfied if we take

$$
\begin{align*}
& P(\xi)=\int_{0}^{a} \phi(t) \sin (\xi t) d t \\
& Q(\xi)=\int_{0}^{b} \psi(t) \sin (\xi t) d t \tag{3.2.13}
\end{align*}
$$

where $\phi$ and $\psi$ are two new unknown functions.
Using the expressions (3.2.11) we find that

$$
\begin{align*}
& \xi\left[A(\xi) \cosh \left(\delta_{1} h_{1}\right)-B(\xi) \sinh \left(\delta_{1} h_{1}\right)\right]=\left[1+M_{1}(\xi)\right] P(\xi)+N_{1}(\xi) Q(\xi) \\
& \xi A(\xi)=M_{2}(\xi) P(\xi)+\left[\lambda+N_{2}(\xi)\right] Q(\xi) \tag{3.2.14}
\end{align*}
$$

where

$$
\begin{align*}
& \lambda=\frac{1}{1+\bar{\mu}} \\
& M_{1}(\xi)=\xi \frac{a_{1}(\xi)+a_{2}(\xi) \exp \left(-2 \delta_{1} h_{1}\right)}{a_{3}(\xi)+a_{4}(\xi) \exp \left(-2 \delta_{1} h_{1}\right)}-1 \\
& N_{1}(\xi)=M_{2}(\xi)=\xi \frac{2 \delta_{1} \exp \left(-\delta_{1} h_{1}\right)}{a_{3}(\xi)+a_{4}(\xi) \exp \left(-2 \delta_{1} h_{1}\right)}, \\
& N_{2}(\xi)=\xi \frac{a_{5}(\xi)+a_{6}(\xi) \exp \left(-2 \delta_{1} h_{1}\right)}{a_{3}(\xi)+a_{4}(\xi) \exp \left(-2 \delta_{1} h_{1}\right)}-\lambda \tag{3.2.15}
\end{align*}
$$

with

$$
\begin{array}{lr}
a_{1}(\xi)=\delta_{1}+\rho, & a_{2}(\xi)=\delta_{1}-\rho \\
a_{3}(\xi)=a_{1}(\xi) a_{5}(\xi), & a_{4}(\xi)=-a_{2}(\xi) a_{6}(\xi) \\
a_{5}(\xi)=\delta_{1}+\frac{1}{2} \alpha_{1}, & a_{6}(\xi)=\delta_{1}-\frac{1}{2} \alpha_{1} \tag{3.2.16}
\end{array}
$$

Now substituting from equations (3.2.14) into equation (3.2.7), using equations(3.2.13) and the following integral representations

$$
\begin{align*}
& J_{1}(\xi r)=\frac{2}{\pi r} \int_{0}^{r} \frac{x \sin (\xi x) d x}{\sqrt{r^{2}-x^{2}}}  \tag{3.2.17}\\
& \int_{0}^{\infty} J_{1}(r \xi) \sin (\xi t) d \xi=\frac{t H(r-t)}{r \sqrt{r^{2}-t^{2}}} \tag{3.2.18}
\end{align*}
$$

we obtain an Able integral equation

$$
\begin{array}{ll}
\int_{0}^{r} \frac{x}{\sqrt{r^{2}-x^{2}}}\left\{\phi(x)+\int_{0}^{a} \phi(t) L_{1}(x, t) d t+\int_{0}^{b} \psi(\mathrm{t}) K_{1}(x, t) d t\right\} d x \\
=\epsilon_{0} r^{2} \exp \left(-\frac{1}{2} \alpha_{1} h_{1}\right), & 0 \leq x \leq a \tag{3.2.19}
\end{array}
$$

which when inverted gives

$$
\begin{array}{r}
\phi(x)+\int_{0}^{a} \phi(t) L_{1}(x, t) d t+\int_{0}^{b} \psi(t) K_{1}(x, t) d t=\frac{4}{\pi} \epsilon_{0} \exp \left(-\frac{1}{2} \alpha_{1} h_{1}\right) x \\
0 \leq x \leq a \tag{3.2.20}
\end{array}
$$

Similarly, equation (3.2.9) gives

$$
\begin{align*}
\lambda \psi(x)+\int_{0}^{a} \phi(t) L_{2}(x, t) d t+\int_{0}^{b} \psi(t) K_{2}(x, t) d t=\frac{4}{\pi} \epsilon_{1} x & \\
& 0 \leq x \leq b \tag{3.2.21}
\end{align*}
$$

where

$$
\begin{align*}
& L_{\mathrm{i}}(x, t)=\frac{2}{\pi} \int_{0}^{\infty} M_{\mathrm{i}}(\xi) \sin (\xi x) \sin (\xi t) d \xi \\
& K_{\mathrm{i}}(x, t)=\frac{2}{\pi} \int_{0}^{a} N_{\mathrm{i}}(\xi) \sin (\xi x) \sin (\xi t) d \xi, \quad i=1,2 \tag{3.2.22}
\end{align*}
$$

From equations (3.1.5) and (3.2.11) and observing $J_{0}^{\prime}(x)=-J_{1}(x)$, we obtain

$$
\begin{equation*}
\sigma_{\theta z}\left(r,-h_{1}\right)=\mu_{1} \exp \left(-\frac{1}{2} \alpha_{1} h_{1}\right) \frac{d}{d r} \int_{0}^{a} \phi(t) d t \int_{0}^{\infty} J_{0}(\xi r) \sin (\xi t) d \xi \tag{3.2.23}
\end{equation*}
$$

Now using the fact that the inner integral on the right side of the equation (3.2.23) is zero for $t<r$ and $\left(t^{2}-r^{2}\right)^{\frac{-1}{2}}$ for $t>r$, we obtain

$$
\begin{equation*}
\sigma_{\theta \mathrm{z}}\left(r,-h_{1}\right)=\mu_{1} \exp \left(-\frac{1}{2} \alpha_{1} h_{1}\right) \frac{d}{d r} \int_{0}^{a} \frac{\phi(t) d t}{\sqrt{t^{2}-r^{2}}}, \quad 0 \leq r \leq a \tag{3.2.24}
\end{equation*}
$$

The moment $T$ required to produce the required rotation $\epsilon_{0}$ of the rigid shaft is given by

$$
\begin{equation*}
T=-2 \pi \int_{0}^{a} r^{2} \sigma_{\theta z}\left(r,-h_{1}\right) d r \tag{3.2.25}
\end{equation*}
$$

Substituting from equation (3.2.24) into (3.2.25), we find that

$$
\begin{equation*}
T=4 \pi \mu_{1} \exp \left(-\frac{1}{2} \alpha_{1} h_{1}\right) \int_{0}^{a} t \phi(t) d t \tag{3.2.26}
\end{equation*}
$$

In a similar way, the proviso on the vanishing of the moment applied to the inclusion leads to

$$
\begin{equation*}
\int_{0}^{b} t \psi(t) d t=0 \tag{3.2.27}
\end{equation*}
$$

Integrating by parts and performing the indicated differentiation in the
equation (3.2.24), we obtain

$$
\begin{array}{r}
\sigma_{\theta \mathrm{z}}\left(r,-h_{1}\right)=-\mu_{1}\left[\frac{a \phi(a)}{r \sqrt{a^{2}-r^{2}}}-\frac{1}{r} \int_{r}^{a} \frac{t \phi^{\prime}(t)}{\sqrt{t^{2}-r^{2}}} d t\right] \exp \left(-\frac{1}{2} \alpha_{1} h_{1}\right) \\
0 \leq \mathrm{r} \leq \mathrm{a} . \tag{3.2.28}
\end{array}
$$

The stress $\sigma_{\theta z}\left(r,-h_{1}\right)$ has a square root singularity at $r=a$ and the constant $\phi(\mathrm{a})$ is the measure of the strength of the singularity at the rim of the rigid shaft. In a similar way, we can show that

$$
\begin{align*}
& \sigma_{\theta \mathrm{z}}(r, 0)=-\mu_{1}\left[\frac{b \psi(b)}{r \sqrt{b^{2}-r^{2}}}-\frac{1}{r} \int_{r}^{b} \frac{t \psi^{\prime}(t)}{\sqrt{t^{2}-r^{2}}} d t\right]+\Omega(r) \\
& 0 \leq r<b \tag{3.2.29}
\end{align*}
$$

where $\Omega(\mathrm{r})$ is bounded, hence $\sigma_{\theta z}$ has square root singularity at the edge of the inclusion, and $\psi(\mathrm{b})$ is the measure of the strength of the singularity.

For numerical solution it is convenient to write the integral equations in dimensionless form. We, therefore, set

$$
\begin{array}{lr}
\phi(\eta)=\frac{\pi}{4 a \epsilon_{0}} \exp \left(\frac{1}{2} \alpha_{1} h_{1}\right) \phi(a \eta), & \Psi(\eta)=\frac{\pi}{4 b \epsilon_{0}} \exp \left(\frac{1}{2} \alpha_{1} h_{1}\right) \psi(b \eta), \\
L_{1}^{*}(\eta, \tau)=a L_{1}(a \eta, a \tau), & L_{2}^{*}(\eta, \tau)=\frac{a^{2}}{b} \mathrm{~L}_{2}(b \eta, a \tau), \\
K_{1}^{*}(\eta, \tau)=\frac{b^{2}}{a} K_{1}(a \eta, b \tau), & K_{2}^{*}(\eta, \tau)=b \mathrm{~K}_{2}(b \eta, b \tau) \\
T_{0}=\frac{16}{3} \cdot \mu_{1} \epsilon_{0} a^{3} . & \tag{3.2.30}
\end{array}
$$

and

$$
\begin{equation*}
\beta=\frac{\epsilon_{1}}{\epsilon_{0}} . \tag{3.2.31}
\end{equation*}
$$

Then the equations (3.2.20), (3.2.21), (3.2.26) and (3.2.27) can be written in
the following forms

$$
\begin{array}{ll}
\phi(\eta)+\int_{0}^{1} \phi(\tau) L_{1}^{*}(\eta, \tau) d \tau+\int_{0}^{1} \Psi(\tau) K_{1}^{*}(\eta, \tau) \mathrm{d} \tau=\eta, & 0<\eta<1 \\
\lambda \Psi(\eta)+\int_{0}^{1} \phi(\tau) L_{2}^{*}(\eta, \tau) d \tau+\int_{0}^{1} \Psi(\tau) K_{2}^{*}(\eta, \tau) d \tau=\beta \exp \left(\frac{1}{2} \alpha_{1} h_{1}\right) \eta \\
& 0<\eta<1 \\
3 \exp \left(\alpha_{1} h_{1}\right) \int_{0}^{1} \tau \phi(\tau) \mathrm{d} \tau=\frac{T}{T_{0}} & \\
\int_{0}^{1} \tau \Psi(\tau) \mathrm{d} \tau=0 \tag{3.2.35}
\end{array}
$$

for the determination of $\phi(\eta), \Psi(\eta), \beta$ and $T$. It is easy to see $\phi(1)$ and $\Psi(1)$ are the measure of the strength of stress singularities at the rim of the shaft and the edge of the inclusion respectively.

### 3.2.3 Solution for the homogeneous case and numerical results

When $\alpha_{1}=\alpha_{2}=0$, the problem considered above becomes a torsion problem of two homogeneous elastic layers with a penny-shaped inclusion at the interface of the layers which have the shear moduli $\mu_{1}$ and $\mu_{2}$ respectively. In this case, $E(\xi), M_{\mathrm{i}}$ and $N_{\mathrm{i}}, i=1,2$; have the following forms

$$
\begin{equation*}
E(\xi)=-\tanh \left(\xi h_{2}\right) \tag{3.2.36}
\end{equation*}
$$

and

$$
\begin{align*}
& M_{1}(\xi)=\frac{2 \gamma}{\exp \left(2 \xi h_{1}\right)-\gamma^{\prime}} \\
& N_{1}(\xi)=M_{2}(\xi)=\frac{2 \exp \left(\xi h_{1}\right)}{(1-\bar{\mu} E)\left(\exp \left(2 \xi h_{1}\right)-\gamma\right)} \\
& N_{2}(\xi)=\frac{1+\exp \left(2 \xi h_{1}\right)}{(1-\bar{\mu} E) \exp \left(2 \xi h_{1}\right)-(1+\bar{\mu} E)}-\frac{1}{1+\bar{\mu}}, \\
& \gamma=\frac{1+\bar{\mu} E}{1-\bar{\mu} E} \tag{3.2.37}
\end{align*}
$$

Numerical solution for this particular case has been obtained by solving the simultaneous Fredholm integral equations (3.2.32), (3.2.33) and (3.2.35) for $\phi(\eta), \Psi(\eta)$ and $\beta$, in which kernels $L_{\mathrm{i}}^{*}(x, t)$ and $K_{\mathrm{i}}^{*}(x, t) i=1,2$ are dependent on functions $M_{\mathrm{i}}(\xi), N_{\mathrm{i}}(\xi), i=1,2$ given by equations (3.2.37). To do this, we partition the interval $[0,1]$ into 20 equal subintervals and approximate the integral equations by a system of linear algebraic equations in $\phi\left(\eta_{\mathrm{i}}\right), \Psi\left(\eta_{\mathrm{i}}\right)$ ( with $\eta_{\mathrm{i}}=0.0(0.05) 1.0$ ) and $\beta$ for their determination. Then the values of $T / T_{0}$ are calculated by numerical integration of (3.2.34). The quadrature method has been employed to perform the numerical integrations of the kernels involved in the integral equations and the relative error is controlled under 0.01. The same method is also used in sections $3.3,3.4$ and 3.5 .

The numerical values for this problem have been calculated for $b / a=0.0$ (0.1) 1.0, 2.0, 3,0, 4,0; $h_{1} / a=0.5,1.0,2.0 ; h_{2} / h_{1}=0.25,0.5,1.0,2.0,5.0$ and $\mu_{2} / \mu_{1}=\bar{\mu}=0.5,1.0,2.0$. The numerical values of $T / T_{0}, \phi(1), \Psi(1)$ and $\beta$ have been displayed against $b / a$ for various values of $h_{2} / h_{1}$ for a combination of values of $\bar{\mu}=0.5,1.0,2.0$ and $h_{1} / a=0.5,1.0,2.0$ in Fig.3.2.1—Fig.3.2.36.


Figs.3.2.1-3.2.4
For Inclusion Problem, Numerical values of $\phi(1), \Psi(1)$, $\mathrm{M} / \mathrm{M}_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5$, $1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=0.5, h_{1} / a=0.5$.


Figs.3.2.5-3.2.8
For Inclusion Problem, Numerical values of $\phi(1), \Psi(1)$, $\mathrm{M} / \mathrm{M}_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $b / a$ for $h_{2} / h_{1}=0.25,0.5$, $1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=1.0, \mathrm{~h}_{1} / a=0.5$.


For Inclusion Problem, Numerical values of $\phi(1), \Psi(1)$, $\mathrm{M} / \mathrm{M}_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5$, $1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=2.0, \mathrm{~h}_{1} / \mathrm{a}=0.5$.


Figs.3.2.13-3.2.16
For Inclusion Problem, Numerical values of $\phi(1), \Psi(1)$, $\mathrm{M} / \mathrm{M}_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5$, $1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=0.5, h_{1} / a=1.0$.

S-free muz/mul-1.0.h1/o-1.0


Figs.3.2.17-3.2.20
For Inclusion Problem, Numerical values of $\phi(1), \Psi(1)$, $M / M_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $b / a$ for $h_{2} / h_{1}=0.25,0.5$, 1.0, 2.0, 5.0 and $\bar{\mu}_{1}=\mu_{2} / \mu_{1}=1.0, \mathrm{~h}_{1} / \mathrm{a}=1.0$.

S-free mu2/mul-2.0 hl/o-1.0


Figs.3.2.21-3.2.24
For Inclusion Problem, Numerical values of $\phi(1), \Psi(1)$, $\mathrm{M} / \mathrm{M}_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $b / a$ for $h_{2} / h_{1}=0.25,0.5$, $1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=2.0, \mathrm{~h}_{1} / \mathrm{a}=1.0$.


Figs.3.2.25-3.2.28
For Inclusion Problem, Numerical values of $\phi(1), \Psi(1)$, $\mathrm{M} / \mathrm{M}_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5$, 1.0, 2.0, 5.0 and $\bar{\mu}=\mu_{2} / \mu_{1}=0.5, \mathrm{~h}_{1} / \mathrm{a}=2.0$.


Figs.3.2.29-3.2.32
For Inclusion Problem, Numerical values of $\phi(1), \Psi(1)$, $\mathrm{M} / \mathrm{M}_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5$, $1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=1.0, h_{1} / a=2.0$.


Figs.3.2.33-3.2.36
For Inclusion Problem, Numerical values of $\phi(1), \Psi(1)$, $\mathrm{M} / \mathrm{M}_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5$, 1.0, 2.0, 5.0 and $\bar{\mu}=\mu_{2} / \mu_{1}=2.0, \mathrm{~h}_{1} / \mathrm{a}=2.0$.

### 3.2.4 Solution for the case $h_{2} \rightarrow \infty$ and numerical results

When $h_{2} \rightarrow \infty$, the problem considered becomes the problem of torsion of a nonhomogeneous elastic layer bonded to a nonhomogeneous elastic half-space with penny-shaped inclusion at the interface. From equations (3.2.6) and (3.2.10a) we find that when $h_{2} \rightarrow \infty$

$$
E=-1, \quad \rho=\frac{1}{2}\left(\bar{\mu} \alpha_{2}-\alpha_{1}\right)+\bar{\mu} \delta_{2},
$$

and the solution for this case is given by the results of section 3.2.2. In a particular case of $\alpha_{1}=\alpha_{2}=0$, equations (3.2.37) give

$$
\begin{align*}
& M_{1}(\xi)=\frac{2 \gamma}{\exp \left(2 \xi h_{1}\right)-\gamma} \\
& N_{1}(\xi)=M_{2}(\xi)=\frac{2 \exp \left(\xi h_{1}\right)}{(1+\bar{\mu})\left(\exp \left(2 \xi h_{1}\right)-\gamma\right)^{\prime}}, \\
& N_{2}(\xi)=\frac{1+\exp \left(2 \xi h_{1}\right)}{(1+\bar{\mu}) \exp \left(2 \xi h_{1}\right)-(1-\bar{\mu})}-\frac{1}{1+\bar{\mu}}, \\
& \gamma=\frac{1-\bar{\mu}}{1+\bar{\mu}} \tag{3.2.38}
\end{align*}
$$

Numerical solution for this particular case has been obtained by solving the simultaneous Fredholm integral equations (3.2.32), (3.2.33) and (3.2.35) for $\phi(\eta), \Psi(\eta)$ and $\beta$, in which kernels $L_{\mathrm{i}}^{*}(x, t)$ and $K_{\mathrm{i}}^{*}(x, t)(i=1,2)$ are dependent on functions $M_{\mathrm{i}}(\xi), N_{\mathrm{i}}(\xi)(i=1,2)$ given by equations (3.2.38). Then the values of $T / T_{0}$ are calculated by numerical integration of (3.2.34).

The numerical values for this problem have been calculated for $b / a=0.0$ (0.1) $1.0,2.0,3,0,4,0 ; h_{1} / a=0.2(0.1) 0.6,0.8,1.0,2.0$, and $\mu_{2} / \mu_{1}=\bar{\mu}=0.5$, 1.0, 2.0. The numerical values of $T / T_{0}, \phi(1), \Psi(1)$ and $\beta$ have been displayed against $b / a$ for various values of $h_{1} / a$ for $\bar{\mu}=0.5,1.0,2.0$ in Fig.3.2.37-Fig.3.2.48.


Figs.3.2.37-3.2.40
For Inclusion Problem, Numerical values of $\Phi(1), \Psi(1)$, $N: I / M_{0}, \beta=\epsilon_{1} / \epsilon_{0}$ against $b / a$ for $h_{1} / a=0.2,0.3,0.4$, $0.5,0.6,0.8,1.0,2.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=0.5$.


Figs.3.2.41-3.2.44
For Inclusion Problem, Numerical values of $\Phi(1), \Psi(1)$, $\mathrm{M} / \mathrm{M}_{0}, \beta=\epsilon_{1} / \epsilon_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{1} / \mathrm{a}=0.2,0.3,0.4$, $0.5,0.6,0.8,1.0,2.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=1.0$.
mu2/mul-2.0


Figs.3.2.45-3.2.48
For Inclusion Problem, Numerical values of $\Phi(1), \Psi(1)$, $\mathrm{M} / \mathrm{M}_{0}, \beta=\epsilon_{1} / \epsilon_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{1} / \mathrm{a}=0.2,0.3,0.4$, $0.5,0.6,0.8,1.0,2.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=2.0$.

### 3.3 Crack problem with the surface $z=h_{2}$ stress-free

### 3.3.1 Statement of the problem

In this section we will consider the problem stated in section 3.1 when the flaw is in the form of a crack and the surface $z=h_{2}$ is stress-free. Hence we have the basic equations (3.1.1) and the general solutions (3.1.3) to (3.1.6). Besides the common conditions (3.1.7) to (3.1.10) there are the following two additional conditions:

$$
\begin{array}{ll}
\sigma_{\theta z}(r, 0)=\hat{\sigma}_{\theta z}\left(r, 0^{+}\right)=0, & 0 \leq r<b \\
\hat{\sigma}_{\theta z}\left(r, h_{2}\right)=0 . & 0 \leq r<\infty
\end{array}
$$

Using the condition (3.3.1), the continuity condition (3.1.10) may be replaced by

$$
\begin{equation*}
\sigma_{\theta \mathrm{z}}(r, 0)=\hat{\sigma}_{\theta \mathrm{z}}\left(r, 0^{+}\right), \quad 0 \leq r<\infty \tag{3.3.3}
\end{equation*}
$$

### 3.3.2 Analysis

Now the conditions (3.3.2) and (3.3.3) will be satisfied by the using (3.1.5) and (3.1.6), if we take

$$
\begin{equation*}
D(\xi)=E(\xi) C(\xi) \tag{3.3.4}
\end{equation*}
$$

and

$$
\begin{equation*}
C(\xi)=\frac{\alpha_{1} A(\xi)-2 \delta_{1} B(\xi)}{\bar{\mu}\left[\alpha_{2}-2 \delta_{2} E(\xi)\right]} \tag{3.3.5}
\end{equation*}
$$

where $E(\xi)$ is given by equation (3.2.6).

Then the conditions (3.1.7), (3.1.8), (3.1.9) and (3.3.1) lead to the following system of dual integral equations

$$
\begin{array}{cc}
\int_{0}^{\infty} \xi\left[A(\xi) \cosh \left(\delta_{1} h_{1}\right)-B(\xi) \sinh \left(\delta_{1} h_{1}\right)\right] J_{1}(r \xi) d \xi=\epsilon_{0} r \exp \left(-\frac{\alpha_{1}}{2} h_{1}\right), \\
\int_{0}^{\infty} \xi\left\{\left[\frac{\alpha_{1}}{2} \cosh \left(\delta_{1} h_{1}\right)+\delta_{1} \sinh \left(\delta_{1} h_{1}\right)\right] A(\xi)-\left[\frac{\alpha_{1}}{2} \sinh \left(\delta_{1} h_{1}\right)+\right.\right. \\
\left.\left.\delta_{1} \cosh \left(\delta_{1} h_{1}\right) B(\xi)\right]\right\} J_{1}(r \xi) d \xi=0, & a<r<a, \\
\int_{0}^{\infty} \xi\left\{\left[1-\frac{\alpha_{1}}{\bar{\mu}\left(\alpha_{2}-2 \delta_{2} E\right)}\right] A(\xi)+\frac{2 \delta_{1}}{\bar{\mu}\left(\alpha_{2}-2 \delta_{2} E\right)} B(\xi)\right\} J_{1}(r \xi) d \xi=0, \\
\int_{0}^{\infty} \xi\left[-\frac{\alpha_{1}}{2} A(\xi)+\delta_{1} B(\xi)\right] J_{1}(r \xi) d \xi=0, & b \leq r \leq \infty,
\end{array}
$$

where $\bar{\mu}=\mu_{2} / \mu_{1}$.
The above system of dual integral equations may be rewritten in the following form (Note: here and what follows such functions as $\phi, \psi, K_{\mathrm{i}}, L_{\mathrm{i}}, M_{\mathrm{i}}$ $, N_{\mathrm{i}}, \cdots$, etc. will be introduced, but it should be clear that they are (presumably) different in different cases):

$$
\begin{array}{ll}
\int_{0}^{\infty}\left\{\left[1+M_{1}(\xi)\right] P_{1}(\xi)+N_{1}(\xi) Q_{1}(\xi)\right\} J_{1}(r \xi) d \xi=\epsilon_{0} \exp \left(\frac{-1}{2} \alpha_{1} h_{1}\right) \\
& 0 \leq r<a \\
\int_{0}^{\infty} \xi P_{1}(\xi) J_{1}(r \xi) d \xi=0, & a<r<\infty \\
\int_{0}^{\infty} \xi\left\{M_{2}(\xi) P_{1}(\xi)+\left[\lambda_{1}+N_{2}(\xi)\right] Q_{1}(\xi)\right\} J_{1}(r \xi) d \xi=0, & 0 \leq r<b \\
\int_{0}^{\infty} Q_{1}(\xi) J_{1}(r \xi) d \xi=0, & b<r<\infty \tag{3.3.13}
\end{array}
$$

where

$$
\begin{gather*}
P_{1}(\xi)=\left[\frac{\alpha_{1}}{2} \cosh \left(\delta_{1} h_{1}\right)+\delta_{1} \sinh \left(\delta_{1} h_{1}\right)\right] A(\xi)-\left[\frac{\alpha_{1}}{2} \sinh \left(\delta_{1} h_{1}\right)+\delta_{1} \cosh \left(\delta_{1} h_{1}\right)\right] B(\xi)  \tag{3.3.14}\\
Q_{1}(\xi)=\xi\left[1-\frac{\alpha_{1}}{\bar{\mu}\left(\alpha_{2}-2 \delta_{2} E\right)}\right] A(\xi)+\xi \frac{2 \delta_{1}}{\bar{\mu}\left(\alpha_{2}-2 \delta_{2} E\right)} B(\xi) \tag{3.3.15}
\end{gather*}
$$

and

$$
\begin{align*}
& M_{1}(\xi)=\xi \frac{a_{1}(\xi)+a_{2}(\xi) \exp \left(-2 \delta_{1} h_{1}\right)}{a_{3}(\xi)+a_{4}(\xi) \exp \left(-2 \delta_{1} h_{1}\right)}-1 \\
& N_{1}(\xi)=-M_{2}(\xi)=\frac{\delta_{1} \bar{\mu}\left(\alpha_{2}-2 \delta_{2} E\right) \exp \left(-\delta_{1} h_{1}\right)}{a_{3}(\xi)+a_{4}(\xi) \exp \left(-2 \delta_{1} h_{1}\right)} \\
& N_{2}(\xi)=\xi \frac{\bar{\mu}\left(\alpha_{2}-2 \delta_{2} E\right)\left[1-\exp \left(-2 \delta_{1} h_{1}\right)\right]}{2\left[a_{3}(\xi)+a_{4}(\xi) \exp \left(-2 \delta_{1} h_{1}\right)\right.}-\lambda_{1} \\
& \lambda_{1}=\frac{\bar{\mu}}{1+\bar{\mu}} \tag{3.3.16}
\end{align*}
$$

and $a_{\mathrm{i}},{ }^{7} i=1,2,3,4$; are the same as given by equations (3.2.16).
Now if we take

$$
\begin{align*}
P_{1}(\xi) & =\int_{0}^{a} \phi(t) \sin (\xi t) d t  \tag{3.3.17}\\
Q_{1}(\xi) & =\int_{0}^{b} \psi(t)\left[\frac{\sin (\xi t)}{\xi t}-\cos (\xi t)\right] d t \tag{3.3.18}
\end{align*}
$$

the integral equations (3.3.11) and (3.3.13) will be identically satisfied, whereas if we insert the above expressions for $P_{1}(\xi)$ and $Q_{1}(\xi)$ into equation (3.3.10), we obtain the following integral equation:

$$
\begin{array}{r}
\phi(x)+\int_{0}^{a} \phi(t) L_{1}(x, t) d t+\int_{0}^{b} \psi(\mathrm{t}) K_{1}(x, t) d t=\frac{4}{\pi} \epsilon_{0} x \exp \left(-\frac{1}{2} \alpha_{1} h_{1}\right) \\
0 \leq x \leq a \tag{3.3.19}
\end{array}
$$

where

$$
\begin{align*}
& L_{1}(x, t)=\frac{2}{\pi} \int_{0}^{\infty} M_{1}(\xi) \sin (\xi t) \sin (\xi x) d \xi \\
& K_{1}(x, t)=\frac{2}{\pi} \int_{0}^{\infty} N_{1}(\xi)\left[\frac{\sin (\xi t)}{\xi t}-\cos (\xi t)\right] \sin (\xi x) d \xi \tag{3.3.20}
\end{align*}
$$

Now to satisfy equation (3.3.12), let us first rewrite equation (3.3.18) in the form

$$
\begin{equation*}
\xi Q_{1}(\xi)=\int_{0}^{b}\left[\frac{\psi(t)}{t}+\psi^{\prime}(t)\right] \sin (\xi t) d t-\psi(b) \sin (\xi b) \tag{3.3.21}
\end{equation*}
$$

and then substitute for $P_{1}(\xi)$ and $Q_{1}(\xi)$ from equations (3.3.17) and (3.3.21) into equation (3.3.12) and use equations (3.2.17) and (3.2.18) to obtain the integral equation

$$
\begin{equation*}
\int_{0}^{r} \frac{G(\eta) d \eta}{\sqrt{r^{2}-\eta^{2}}}=0, \quad 0 \leq r<b \tag{3.3.22}
\end{equation*}
$$

where

$$
\begin{align*}
G(\eta) & =\lambda_{1} \frac{d}{d \eta}[\eta \psi(\eta)]+\frac{2}{\pi} \int_{0}^{a} \phi(t) d t \int_{0}^{\infty} \eta \xi M_{2}(\xi) \sin (\xi t) \sin (\xi \eta) d \xi \\
& +\frac{2}{\pi} \int_{0}^{b} \psi(t) d t \int_{0}^{\infty} \eta \xi N_{2}(\xi)\left[\frac{\sin (\xi t)}{\xi t}-\cos (\xi t)\right] \sin (\xi \eta) d \xi \tag{3.3.23}
\end{align*}
$$

Clearly equation (3.3.22) will be satisfied if $G(\eta)=0$. So if we let $G(\eta)=$ 0 and integrate (3.3.23) with respect to $\eta$ from 0 to $x$ for $0<x<b$ and then divide by $x$, we obtain:

$$
\begin{equation*}
\lambda_{1} \psi(x)+\int_{0}^{a} \phi(t) L_{2}(x, t) d t+\int_{0}^{b} \psi(t) K_{2}(x, t) d t=0, \quad 0 \leq x \leq b \tag{3.3.24}
\end{equation*}
$$

where

$$
\begin{align*}
& L_{2}(x, t)=\frac{2}{\pi} \int_{0}^{\infty} M_{2}(\xi)\left[\frac{\sin (\xi x)}{\xi x}-\cos (\xi x)\right] \sin (\xi t) d \xi \\
& K_{2}(x, t)=\frac{2}{\pi} \int_{0}^{\infty} N_{2}(\xi)\left[\frac{\sin (\xi x)}{\xi x}-\cos (\xi x)\right]\left[\frac{\sin \xi t}{\xi t}-\cos (\xi t)\right] d \xi \tag{3.3.25}
\end{align*}
$$

The boundary value of shear stress $\sigma_{\theta \mathrm{z}}\left(r,-h_{1}\right)$ is still given by equation (3.2.28). The value of the shear stress $\sigma_{\theta z}(r, 0), r>b$ at the interface of the two materials is given by

$$
\begin{equation*}
\sigma_{\theta z}(r, 0)=\mu_{1} \int_{0}^{\infty} \xi\left[\delta_{1} B(\xi)-\frac{1}{2} \alpha_{1} A(\xi)\right] J_{1}(\xi r) d \xi \tag{3.3.26}
\end{equation*}
$$

And substituting for $A(\xi)$ and $B(\xi)$ from equation (3.3.14), (3.3.15) and using equations (3.3.17), (3.3.18), (3.3.21) and (3.2.18), we obtain

$$
\begin{equation*}
\sigma_{\theta z}(r, 0)=-\lambda_{1} \mu_{1} \frac{b \psi(b)}{r \sqrt{r^{2}-b^{2}}}+R(r), \quad r>b \tag{3.3.27}
\end{equation*}
$$

where

$$
\begin{align*}
R(r)= & \mu_{1} \int_{0}^{\infty} \xi\left[M_{2}(\xi) P_{1}(\xi)+N_{2}(\xi) Q_{1}(\xi)\right] J_{1}(\xi r) d \xi \\
& +\lambda_{1} \mu_{1} \int_{0}^{\infty} J_{1}(\xi r) d \xi \int_{0}^{b}\left[\frac{\psi(t)}{t}+\psi^{\prime}(t)\right] \sin (\xi t) d t \tag{3.3.28}
\end{align*}
$$

is bounded, while $\sigma_{\theta \mathrm{z}}\left(r,-h_{1}\right)$ and $T$, the torque required to rotate the rigid shaft through a small angle $\epsilon_{0}$, have the same expressions as given by equations (3.2.28) and (3.2.26). Hence $\phi(a)$ and $\psi(b)$ are the measure of the strength of the stress singularity at the rim of the shaft and at the edge of the crack respectively.

By using the same transformations as given in (3.2.30) to get dimensionless form, the equations (3.3.19), (3.3.24) and (3.2.26) can be rewritten in the following forms

$$
\begin{align*}
& \phi(\eta)+\int_{0}^{1} \phi(\tau) L_{1}^{*}(\eta, \tau) d \tau+\int_{0}^{1} \Psi(\tau) K_{1}^{*}(\eta, \tau) \mathrm{d} \tau=\eta, \quad 0<\eta<1  \tag{3.3.29}\\
& \lambda_{1} \Psi(\eta)+\int_{0}^{1} \phi(\tau) L_{2}^{*}(\eta, \tau) d \tau+\int_{0}^{1} \Psi(\tau) K_{2}^{*}(\eta, \tau) d \tau=0, \quad 0<\eta<1  \tag{3.3.30}\\
& 3 \exp \left(\alpha_{1} h_{1}\right) \int_{0}^{1} \tau \phi(\tau) \mathrm{d} \tau=\frac{T}{T_{0}} \tag{3.3.31}
\end{align*}
$$

for the determination of $\phi(\eta), \Psi(\eta)$ and $T$. Again, it is easy to see that $\phi(1)$ and $\Psi(1)$ are the measure of the strength of stress singularity at the rim of the shaft and the edge of the crack respectively.

### 3.3.3 Solution for the homogeneous case and numerical results

As in section 3.2.3, when $\alpha_{1}=\alpha_{2}=0$, the problem considered above becomes a torsion problem of two homogeneous elastic layers with a penny-shaped crack at the interface of the layers which have the shear moduli $\mu_{1}$ and $\mu_{2}$ respectively. In this case we have

$$
\begin{equation*}
E(\xi)=-\tanh \left(\xi h_{2}\right) \tag{3.3.32}
\end{equation*}
$$

and

$$
\begin{align*}
& M_{1}(\xi)=\frac{2(1+\bar{\mu} E)}{(1-\bar{\mu} E) \exp \left(2 \xi h_{1}\right)-(1+\bar{\mu} E)} \\
& N_{1}(\xi)=-M_{2}(\xi)=\frac{-2 \bar{\mu} \exp \left(2 \xi h_{1}\right)}{(1-\bar{\mu} E) \exp \left(2 \xi h_{1}\right)-(1+\bar{\mu} E)} \\
& N_{2}(\xi)=\frac{\bar{\mu} E\left(1-\exp \left(2 \xi h_{1}\right)\right)}{(1-\bar{\mu} E) \exp \left(2 \xi h_{1}\right)-(1+\bar{\mu} E)}-\frac{\bar{\mu}}{1+\bar{\mu}} \tag{3.3.33}
\end{align*}
$$

Numerical solution for this particular case has been obtained by solving
the simultaneous Fredholm integral equations (3.3.29) and (3.3.30) for $\phi(\eta)$ and $\Psi(\eta)$, in which kernels $L_{i}^{*}(x, t)$ and $K_{\dot{1}}^{*}(x, t) \quad(i=1,2)$ are dependent on functions $M_{i}(\xi), N_{\mathrm{i}}(\xi)(i=1,2)$ given by (3.3.33). Then the values of $T / T_{0}$ are calculated by performing the numerical integration in (3.3.31).

The numerical values for this problem have been calculated for $b / a=0.0$ (0.1) $1.0,2.0,3,0,4,0 ; h_{1} / a=0.5,1.0,2.0 ; h_{2} / h_{1}=0.25,0.5,1.0,2.0,5.0$ and $\mu_{2} / \mu_{1}=\bar{\mu}=0.5,1.0,2.0$. The numerical values of $T / T_{0}, \phi(1)$ and $\Psi(1)$ have been displayed against $b / a$ for various values of $h_{2} / h_{1}$ for a combination of values of $\bar{\mu}=0.5,1.0,2.0$ and $h_{1} / a=0.5,1.0,2.0$ in Fig.3.3.1—Fig.3.3.27.


Figs.3.3.1-3.3.3
For Crack Problem, Numerical values of $\phi(1), \Psi(1)$ and $\mathrm{M} / \mathrm{M}_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=0.5, \mathrm{~h}_{1} / \mathrm{a}=0.5$.


Figs.3.3.4-3.3.6
For Crack Problem, Numerical values of $\phi(1), \Psi(1)$ and $\mathrm{M} / \mathrm{M}_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=1.0, \mathrm{~h}_{1} / \mathrm{a}=0.5$.


Figs.3.3.7-3.3.9
For Crack Problem, Numerical values of $\phi(1), \Psi(1)$ and $\mathrm{M} / \mathrm{M}_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=2.0, \mathrm{~h}_{1} / \mathrm{a}=0.5$.


Figs.3.3.10-3.3.12
For Crack Problem, Numerical values of $\phi(1), \Psi(1)$ and $\mathrm{M} / \mathrm{M}_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=0.5, \mathrm{~h}_{1} / \mathrm{a}=1.0$.


Figs.3.3.13-3.3.15
For Crack Problem, Numerical values of $\phi(1), \Psi(1)$ and $\mathrm{M} / \mathrm{M}_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5,1.0,2.0,5.0$ and $\bar{\mu}=. \mu_{2} / \mu_{1}=1.0, \mathrm{~h}_{1} / \mathrm{a}=1.0$.


Figs.3.3.16-3.3.18
For Crack Problem, Numerical values of $\phi(1), \Psi(1)$ and $\mathrm{M} / \mathrm{M}_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=2.0, \mathrm{~h}_{1} / \mathrm{a}=1.0$.


For Crack Problem, Numerical values of $\phi(1), \Psi(1)$ and $\mathrm{M} / \mathrm{M}_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=0.5, \mathrm{~h}_{1} / \mathrm{a}=2.0$.


Figs.3.3.22-3.3.24
For Crack Problem, Numerical values of $\phi(1), \Psi(1)$ and $\mathrm{M} / \mathrm{M}_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=1.0, \mathrm{~h}_{1} / \mathrm{a}=2.0$.


Figs.3.3.25-3.3.27
For Crack Problem, Numerical values of $\phi(1), \Psi(1)$ and $\mathrm{M} / \mathrm{M}_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=2.0, \mathrm{~h}_{1} / \mathrm{a}=2.0$.

### 3.3.4 Solution for the case $h_{2} \rightarrow \infty$ and the numerical results

As in section 3.2.4, when $h_{2} \rightarrow \infty$, the problem considered in section 3.3.2 becomes the problem of torsion of a nonhomogeneous elastic layer bonded to a nonhomogeneous elastic half-space with penny-shaped crack at the interface. From equation (3.2.6) we find that $E \rightarrow-1$ when $h_{2} \rightarrow \infty$. The solution for this case is given by the results of section 3.3.2. by taking $E=-1$. Particularly, if $\alpha_{1}=\alpha_{2}=0$ we have

$$
\begin{align*}
& M_{1}(\xi)=-2(\bar{\mu}-1) S(\xi), \\
& N_{1}(\xi)=M_{2}(\xi)=2 \exp \left(\xi h_{1}\right) S(\xi), \\
& N_{2}(\xi)=\left[\exp \left(2 \xi h_{1}\right)-1\right] S(\xi)-\lambda, \tag{3.3.34}
\end{align*}
$$

where

$$
S(\xi)=\left[(\bar{\mu}-1)+(\bar{\mu}+1) \exp \left(2 \xi h_{1}\right)\right]^{-1}
$$

Numerical solution for this particular case has been obtained by solving the simultaneous Fredholm integral equations (3.3.29) and (3.3.30) for $\phi(\eta)$ and $\Psi(\eta)$, in which kernels $L_{i}^{*}(x, t)$ and $K_{i}^{*}(x, t) \quad(i=1,2)$ are dependent on functions $M_{\mathrm{i}}(\xi), N_{\mathrm{i}}(\xi),(i=1,2)$ given by equations (3.3.34) Then the values of $T / T_{0}$ are calculated by numerical integration of (3.3.31).

The numerical values for this problem have been calculated for $b / a=0.0$ (0.1) 1.0, 2.0, $3,0,4,0 ; h_{1} / a=0.2(0.1) 0.6,0.8,1.0,2.0$, and $\mu_{2} / \mu_{1}=\bar{\mu}=0.5$, 1.0, 2.0. The numerical values of $T / T_{0}, \phi(1)$ and $\Psi(1)$ have been displayed against $b / a$ for various values of $h_{1} / a$ for $\bar{\mu}=0.5,1.0,2.0$ in Fig.3.3.28.—Fig.3.3.36.
mu2/mul-0.5


Figs.3.3.28-3.3.30
For Crack Problem, Numerical values of $\Phi(1), \Psi(1)$ and $\mathrm{M} / \mathrm{M}_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{1} / \mathrm{a}=0.2,0.3,0.4,0.5,0.6,0.8$, $1.0,2.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=0.5$.


Figs.3.3.31-3.3.33
For Crack Problem, Numerical values of $\Phi(1), \Psi(1)$ and $\mathrm{M} / \mathrm{M}_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{1} / \mathrm{a}=0.2,0.3,0.4,0.5,0.6,0.8$, $1.0,2.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=1.0$.


Fig.3.3.34-3.3.36
For Crack Problem, Numerical values of $\Phi(1), \Psi(1)$ and $\mathrm{M} / \mathrm{M}_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{1} / \mathrm{a}=0.2,0.3,0.4,0.5,0.6,0.8$, 1.0, 2.0 and $\bar{\mu}=\mu_{2} / \mu_{1}=2.0$.

### 3.4 Inclusion problem with the surface $z=h_{2}$ rigidly fixed.

### 3.4.1 Analysis

In this section we will consider the problem stated in section 3.1, when the flaw is an inclusion and the surface $z=h_{2}$ is rigidly fixed. Hence, in addition to the common conditions (3.1.7) to (3.1.10) we have the following conditions

$$
\begin{array}{ll}
v\left(r, 0^{-}\right)=\hat{v}\left(r .0^{+}\right)=\epsilon_{1} r, & 0 \leq r \leq b \\
\hat{v}\left(r, h_{2}\right)=0, & 0 \leq r<\infty \tag{3.4.2}
\end{array}
$$

where we have also assumed that as a result of the application of torque $T$ on the shaft the rigid inclusion will rotate through some unknown small angle $\epsilon_{1}$. The combination of the conditions (3.1.9) and (3.4.1) yield

$$
\begin{equation*}
v\left(r, 0^{-}\right)=\hat{v}\left(r .0^{+}\right), \quad 0 \leq r \leq \infty . \tag{3.4.3}
\end{equation*}
$$

The conditions (3.4.2) and (3.4.3) will be satisfied by the general solutions (3.1.3) and (3.1.4) if we choose

$$
\begin{equation*}
D(\xi)=E_{1}(\xi) C(\xi) \tag{3.4.4}
\end{equation*}
$$

where

$$
\begin{equation*}
E_{1}(\xi)=-\operatorname{coth}\left(\delta_{2} h_{2}\right) \tag{3.4.5}
\end{equation*}
$$

If we replace $E$ in the equation $(3.2 .6)$ by $E_{1}$ (consequently all functions involving $E$ will be changed) then most of the discussion in section 3.2 is valid in this case. We indicate the difference in the following.

In this problem the boundary and continuity conditions (3.1.7), (3.1.8), (3.4.1) will lead to equations (3.2.7), (3.2.8) and (3.2.9) while the condition (3.1.10) gives

$$
\begin{equation*}
\int_{0}^{\infty} \xi\left\{\left[\mu\left(\frac{\alpha_{2}}{2}-\delta_{2} E_{1}\right)-\frac{\alpha_{1}}{2}\right] A(\xi)+\delta_{1} B(\xi)\right\} J_{1}(r \xi) d \xi=0, \quad b<r<\infty \tag{3.4.6}
\end{equation*}
$$

As we have done in section 3.2, the equations (3.2.7), (3.2.8), (3.2.9) and (3.4.6) can be reduced to solution of the system of integral equations (3.2.20) and (3.2.21) for the unknown functions $\phi(\mathrm{t})$ and $\psi(\mathrm{t})$; while expressions (3.2.15), (3.2.16) and (3.2.22) are valid if we replace $E$ in equation (3.2.10a) by $\mathrm{E}_{1}$, which is given by equation (3.4.5). The transformations (3.2.30) and (3.2.31) lead us to the equations (3.2.32), (3.2.33), (3.2.34) and (3.2.35) for the determination of $\phi(\eta), \Psi(\eta), \beta$ and $T$ for this problem.

### 3.4.2 Solution for the homogeneous case and numerical results

When $\alpha_{1}=\alpha_{2}=0$, the problem we considered becomes a torsion problem of two homogeneous elastic layers with a penny-shaped inclusion at the interface of the layers which have the shear moduli $\mu_{1}$ and $\mu_{2}$ respectively with the surface $z=h_{2}$ fixed. In this case we have

$$
\begin{equation*}
E_{1}(\xi)=-\operatorname{coth}\left(\xi h_{2}\right) \tag{3.4.7}
\end{equation*}
$$

and

$$
\begin{align*}
& M_{1}(\xi)=\frac{2 \gamma_{1}}{\exp \left(2 \xi h_{1}\right)-\gamma_{1}} \\
& N_{1}(\xi)=M_{2}(\xi)=\frac{2 \exp \left(\xi h_{1}\right)}{\left(1-\bar{\mu} E_{1}\right)\left(\exp \left(2 \xi h_{1}\right)-\gamma_{1}\right)} \\
& N_{2}(\xi)=\frac{1+\exp \left(2 \xi h_{1}\right)}{\left(1-\bar{\mu} E_{1}\right) \exp \left(2 \xi h_{1}\right)-\left(1+\bar{\mu} E_{1}\right)}-\frac{1}{1+\bar{\mu}} \\
& \gamma_{1}=\frac{1+\bar{\mu} E_{1}}{1-\bar{\mu} E_{1}} \tag{3.4.8}
\end{align*}
$$

Numerical solution for this particular case has been obtained by solving simultaneous Fredholm integral equations (3.2.32), (3.2.33) and (3.2.35) for $\phi(\eta), \Psi(\eta)$ and $\beta$, in which kernels $L_{i}^{*}(x, t)$ and $K_{\mathrm{i}}^{*}(x, t)(i=1,2)$ are dependent on functions $M_{\mathrm{i}}(\xi), N_{\mathrm{i}}(\xi)(i=1,2)$ given by equations (3.4.8). Then the values of $T / T_{0}$ are calculated by the numerical integration of (2.3.34).

The numerical values for this problem have been calculated for $b / a=0.0$ (0.1) $1.0,2.0,3,0,4,0 ; h_{1} / a=0.5,1.0,2.0 ; h_{2} / h_{1}=0.25,0.5,1.0,2.0,5.0$ and $\mu_{2} / \mu_{1}=\bar{\mu}=0.5,1.0,2.0$. The numerical values of $T / T_{0}, \phi(1), \Psi(1)$ and $\beta$ have been displayed against $b / a$ for various values of $h_{2} / h_{1}$ for a combination of values of $\bar{\mu}=0.5,1.0,2.0$ and $h_{1} / a=0.5,1.0,2.0$ in Fig.3.4.1—Fig.3.4.36.

D-rree mu2/mul-0.5 hl/o-0.5


Figs.3.4.1-3.4.4
For Inclusion Problem, Numerical values of $\phi(1), \Psi(1)$, $\mathrm{M} / \mathrm{M}_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5$, $1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=0.5, \mathrm{~h}_{1} / \mathrm{a}=0.5$.


Figs.3.4.5-3.4.8
For Inclusion Problem, Numerical values of $\phi(1), \Psi(1)$, $\mathrm{M} / \mathrm{MI}_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5$, $1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=1.0, \mathrm{~h}_{1} / \mathrm{a}=0.5$.


Figs.3.4.9-3.4.12
For Inclusion Problem, Numerical values of $\phi(1), \Psi(1)$, $\mathrm{M} / \mathrm{M}_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5$, $1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=2.0, \mathrm{~h}_{1} / \mathrm{a}=0.5$.


Figs.3.4.13-3.4.16
For Inclusion Problem, Numerical values of $\phi(1), \Psi(1)$, $\mathrm{M} / \mathrm{M}_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5$, $1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=0.5, \mathrm{~h}_{1} / \mathrm{a}=1.0$.


Figs.3.4.17-3.4.20
For Inclusion Problem, Numerical values of $\phi(1), \Psi(1)$, $M / M_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $b / a$ for $h_{2} / h_{1}=0.25,0.5$,
$1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=1.0, h_{1} / \mathrm{a}=1.0$.


For Inclusion Problem, Numerical values of $\phi(1), \Psi(1)$, $\mathrm{M} / \mathrm{M}_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\dot{\mathrm{h}}_{2} / \mathrm{h}_{1}=0.25,0.5$, $1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=2.0, \mathrm{~h}_{1} / \mathrm{a}=1.0$.


Figs.3.4.25-3.4.28
For Inclusion Problem, Numerical values of $\phi(1), \Psi(1)$, $\mathrm{M} / \mathrm{M}_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5$,
1.0, 2.0, 5.0 and $\bar{\mu}=\mu_{2} / \mu_{1}=0.5, \mathrm{~h}_{1} / \mathrm{a}=2.0$.


Figs.3.4.29-3.4.32
For Inclusion Problem, Numerical values of $\phi(1), \Psi(1)$, $\mathrm{M} / \mathrm{M}_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5$, $1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=1.0, \mathrm{~h}_{1} / \mathrm{a}=2.0$.


Figs.3.4.33-3.4.36
For Inclusion Problem, Numerical values of $\phi(1), \Psi(1)$, $\mathrm{M} / \mathrm{M}_{0}$ and $\beta=\epsilon_{1} / \epsilon_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5$, $1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=2.0, \mathrm{~h}_{1} / a=2.0$.

### 3.5 Crack Problem with the surface $z=h_{2}$ rigidly fixed

### 3.5.1 Analysis

In this section we will consider the problem stated in section 3.1 when the flaw is a crack and the surface $z=h_{2}$ is rigidly fixed. Hence we have the basic equations (3.1.1) and general solutions (3.1.3) to (3.1.6). Besides the common conditions (3.1.7) to (3.1.10) there are two additional conditions given by

$$
\begin{array}{ll}
\sigma_{\theta z}\left(r, 0^{-}\right)=\hat{\sigma}_{\theta z}\left(r, 0^{+}\right)=0, & 0 \leq r<b \\
\hat{v}\left(r, h_{2}\right)=0 . & 0 \leq r<\infty
\end{array}
$$

Using the condition (3.5.1), the continuity condition (3.1.10) may be replaced by

$$
\begin{equation*}
\sigma_{\theta z}\left(r, 0^{-}\right)=\hat{\sigma}_{\theta z}\left(r, 0^{+}\right), \quad 0 \leq r<\infty \tag{3.5.3}
\end{equation*}
$$

In this case the conditions (3.5.2) and (3.5.3) will be satisfied by the general solutions (3.1.5) and (3.1.6), if we take

$$
\begin{equation*}
D(\xi)=E_{1}(\xi) C(\xi) \tag{3.5.4}
\end{equation*}
$$

and

$$
\begin{equation*}
C(\xi)=\frac{\alpha_{1} A(\xi)-2 \delta_{1} B(\xi)}{\bar{\mu}\left[\alpha_{2}-2 \delta_{2} E_{1}(\xi)\right]} \tag{3.5.5}
\end{equation*}
$$

where $E_{1}(\xi)$ is given by equation (3.4.5).
If we replace $E$ in the equation (3.4.5) by $E_{1}$ (consequently all functions involving $E$ will be changed) then most of the discussion in section 3.3 is valid in this case. We indicate the difference in the following.

In this problem the boundary and continuity conditions (3.1.7), (3.1.8), (3 5.1) will lead to equations (3.3.6), (3.3.7) and (3.3.9) while the condition (3.1.9) gives

$$
\begin{array}{r}
\int_{0}^{\infty} \xi\left\{\left[1-\frac{\alpha_{1}}{\bar{\mu}\left(\alpha_{2}-2 \delta_{2} E_{1}\right)}\right] A(\xi)+\frac{2 \delta_{1}}{\bar{\mu}\left(\alpha_{2}-2 \delta_{2} E_{1}\right)} B(\xi)\right\} J_{1}(r \xi) d \xi=0 \\
0 \leq r \leq b \tag{3.5.6}
\end{array}
$$

As we have done in section 3.3 , the equations (3.3.7), (3.3.7), (3.3.9) and (3.5.6) can be reduced to the solution of the system of integral equations (3.3.19) and (3.3.24) for unknown functions $\phi(\mathrm{t})$ and $\psi(\mathrm{t})$; while expressions (3.3.16), (3.2.16) and (3.3.25) are valid if we replace $E$ in equation (3.2.10a) by $E_{1}$, which is given by equation (3.4.5), only to keep in mind that in every function involving $E$ in section $3.3, E$ should be replaced by $E_{1}$ for this problem.

The transformations (3.2.30) will lead us to the equations (3.3.29), (3.3.30) and (3.3.31) for the determination of $\phi(\eta), \Psi(\eta)$ and $T$ for this problem.

### 3.5.2 Solution for the homogeneous case and numerical results

When $\alpha_{1}=\alpha_{2}=0$, the problem we considered becomes a torsion problem of two homogeneous elastic layers with a penny-shaped crack at the interface of the layers which have the shear moduli $\mu_{1}$ and $\mu_{2}$ respectively with the surface $z=h_{2}$ fixed. In this case we have

$$
\begin{equation*}
E_{1}(\xi)=-\operatorname{coth}\left(\xi h_{2}\right), \tag{3.5.7}
\end{equation*}
$$

and

$$
\begin{align*}
& M_{1}(\xi)=\frac{2\left(1+\bar{\mu} E_{1}\right)}{\left(1-\bar{\mu} E_{1}\right) \exp \left(2 \xi h_{1}\right)-\left(1+\bar{\mu} E_{1}\right)}, \\
& N_{1}(\xi)=-M_{2}(\xi)=\frac{-2 \bar{\mu} E_{1} \exp \left(\xi h_{1}\right)}{\left(1-\bar{\mu} E_{1}\right) \exp \left(2 \xi h_{1}\right)-\left(1+\bar{\mu} E_{1}\right)}, \\
& N_{2}(\xi)=\frac{\bar{\mu} E_{1}\left(1-\exp \left(2 \xi h_{1}\right)\right)}{\left(1-\bar{\mu} E_{1}\right) \exp \left(2 \xi h_{1}\right)-\left(1+\bar{\mu} E_{1}\right)}-\frac{\bar{\mu}}{1+\bar{\mu}} . \tag{3.5.8}
\end{align*}
$$

Numerical solution for this particular case has been obtained by solving simultaneous Fredholm integral equations (3.3.29) and (3.3.30) for $\phi(\eta)$ and $\Psi(\eta)$, in which kernels $L_{\mathrm{i}}^{*}(x, t)$ and $K_{\mathrm{i}}^{*}(x, t) \quad(i=1,2)$ are dependent on functions $M_{\mathrm{i}}(\xi), N_{\mathrm{i}}(\xi)(i=1,2)$ given by equations (3.5.8). Then the values of $T / T_{0}$ are calculated by the numerical integration of (3.2.34).

The numerical values for this problem have been calculated for $b / a=0.0$ (0.1) $1.0,2.0,3,0,4,0 ; h_{1} / a=0.5,1.0,2.0 ; h_{2} / h_{1}=0.25,0.5,1.0,2.0,5.0$ and $\mu_{2} / \mu_{1}=\bar{\mu}=0.5,1.0,2.0$. The numerical values of $T / T_{0}, \phi(1)$ and $\Psi(1)$ have been displayed against $b / a$ for various values of $h_{2} / h_{1}$ for a combination of values of $\bar{\mu}=0.5,1.0,2.0$ and $h_{1} / a=0.5,1.0,2.0$ in Fig.3.5.1-Fig.3.5.27.


Figs.3.5.1-3.5.3
For Crack Problem, Numerical values of $\phi(1), \Psi(1)$ and $\mathrm{M} / \mathrm{M}_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=0.5, \mathrm{~h}_{1} / \mathrm{a}=0.5$.


Figs.3.5.4-3.5.6
For Crack Problem, Numerical values of $\phi(1), \Psi(1)$ and $\mathrm{M} / \mathrm{M}_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=1.0, h_{1} / a=0.5$.


Figs.3.5.7-3.5.9
For Crack Problem, Numerical values of $\phi(1), \Psi(1)$ and $\mathrm{M} / \mathrm{M}_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=2.0, h_{1} / a=0.5$.


Figs.3.5.10-3.5.12
For Crack Problem, Numerical values of $\phi(1), \Psi(1)$ and $\mathrm{M} / \mathrm{M}_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=0.5, \mathrm{~h}_{1} / \mathrm{a}=1.0$.


Figs.3.5.13-3.5.15
For Crack Problem, Numerical values of $\phi(1), \Psi(1)$ and $\mathrm{M} / \mathrm{M}_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=1.0, \mathrm{~h}_{1} / \mathrm{a}=1.0$.


Figs.3.5.16-3.5.18
For Crack Problem, Numerical values of $\phi(1), \Psi(1)$ and $\mathrm{M} / \mathrm{M}_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=2.0, \mathrm{~h}_{1} / \mathrm{a}=1.0$.


Figs.3.5.19-3.5.21
For Crack Problem, Numerical values of $\phi(1), \Psi(1)$ and $\mathrm{M} / \mathrm{M}_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=0.5, \mathrm{~h}_{1} / \mathrm{a}=2.0$.


Figs.3.5.22-3.5.24
For Crack Problem, Numerical values of $\phi(1), \Psi(1)$ and $\mathrm{M} / \mathrm{M}_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=1.0, \mathrm{~h}_{1} / \mathrm{a}=2.0$.


Figs.3.5.25-3.5.27
For Crack Problem, Numerical values of $\phi(1), \Psi(1)$ and $\mathrm{M} / \mathrm{M}_{0}$ against $\mathrm{b} / \mathrm{a}$ for $\mathrm{h}_{2} / \mathrm{h}_{1}=0.25,0.5,1.0,2.0,5.0$ and $\bar{\mu}=\mu_{2} / \mu_{1}=2.0, \mathrm{~h}_{1} / \mathrm{a}=2.0$.

### 3.6 Some comment on the solutions

For two layers case, we make the following observations from the graphs for the inclusion problem: the strength of the singularity at the rim of the rigid shaft and the moment required to rotate the shaft increase as $b / a$ increases and decrease as $h_{2} / h_{1}$ increases but near $1.0<b / a<2.0$ they approach 1 ; the strength of the singularity at the edge of the inclusion increases as $h_{2} / h_{1}$ decreases and reaches a maximum near $b / a=1$; the ratio $\epsilon_{1} / \epsilon_{0}=\beta$ decreases as $b / a$ or $h_{2} / h_{1}$ increases.

For the crack problem we notice the following: the strength of the singularity at the rim of the shaft and moment required to rotate the shaft decrease as $b / a$ or $h_{2} / h_{1}$ increases; the strength of the singularity at the edge of the crack behaves in the same manner as in the inclusion problem.

For layer and half-space case, we make the following observations from the graphs for the inclusion problem: the strength of the singularity at the rim of the rigid shaft and the moment required to rotate the shaft increase as $b / a$ increases and decrease as $h_{1} / a$ increases ; the singularity at the edge of the inclusion increases as $h_{1} / a$ decreases and reaches a maximum near $b / a=1$; the ratio $\epsilon_{1} / \epsilon_{0}=\beta$ decreases as $b / a$ or $h_{1} / a$ increases.

For the crack problem we notice the following: the singularity at the rim of the shaft and the moment required to rotate the shaft decrease as $b / a$ increases and increase as $h_{1} / a$ increases but near $b / a=1$ they increase as $h_{1} / a$ increases to 1 ; the singularity at the edge of the crack behaves in the same manner as in the inclusion problem.

## CHAPTER 4

# GRIFFITH CRACK AT THE INTERFACE OF TWO ORTHOTROPIC ELASTIC LAYERS 

### 4.1 Introduction

The study of Griffith crack problems in the mathematical theory of elasticity originated in the classical work of Griffith [46]. A crack occupying the line segment
$y=0, \quad-c \leq x \leq c$,
in the $x y$-plane is called a Griffith crack.
In 1946, Sneddon and Elliot [47] considered the problem of determining the distribution of stress in the neighborhood of a Griffith crack which is subjected to an internal pressure varying along the length of the crack. They reduced the problem to a half-plane mixed boundary value problem and solved it by using Fourier transform methods. Green and Zerna [2] reduced the Griffith crack problem to the Hilbert problem. Willmore [48] solved the problem of two collinear Griffith cracks in an isotropic material by means of elliptic functions when a uniform pressure acts normally on the crack surface. Tranter [49] considered the problem of a normally varying pressure on collinear Griffith cracks.

Koiter [50], and England and Green [51] considered the problem of determining the stress field caused by an infinite row of collinear Griffith cracks of equal length when each crack is subject to the same constant
pressure. Sneddon and Srivastav [52] considered the same problem by assuming a varying pressure on each crack.

Lowengrub [53] solved the Griffith crack problem in a strip with stress-free edges, when the crack is parallel to the edges of the strip. Sneddon and Srivastav [52] investigated the problem of Griffith crack in a strip in which crack is perpendicular to the edges of the strip. Lowengrub [54] considered the distribution of stress in the neighborhood of external crack in an elastic plane.

The problem of radial cracks originating at the boundary of an internal circular hole in an infinite elastic plane was solved by Bowie [55]. The problem of determining the distribution of stress in the vicinity of a star crack formed by the intersection of a number of Griffith cracks was solved by Westman [56].

Williams [57] considered the situation in which a Griffith crack is present at the interface of two isotropic semi-infinite planes of dissimilar materials, he found that the analytic solution of stresses has a peculiar behavior near the tip of the interface crack where the stresses undergo a rapid reversal of sign. The oscillatory character takes the form $r^{-\frac{1}{2}} \sin$ (or cos) of the argument $\epsilon \log (r / a)$ where $r$ is the radial distance from the crack border, $a$ is the crack size and $\epsilon$ is a bimaterial constant depending upon the elastic properties of the adjoining materials. This behavior was also studied by Sih and Rice [58], they formulated the the problem of stress state near the crack-tip and derived a formula for the stress intensity factor for the problem considered by Williams. Erdogan and Gupta [59, 60], analyzed the plane and antiplane problems of stress distribution of multi-layered composites with a flaw by reducing them to a system of singular integral equations. They developed a direct approach to find the approximate solutions of singular integral equations of the first
(second) kind by using Chebyshev (Jacobi) polynomials. Lowengrub and Sneddon [61] solved the problem of a Griffith crack at the interface of two bonded dissimilar elastic half-planes by Fourier transforms and reduced the problem to a set a dual integral equations, they used Muskhelishvili's method to solve the system of singular integral equations. Dhaliwal [62], Mohapatra and Parhi [63] , Satpathi and Parhi [64] and Parihar and Lalitha [65] considered the Griffith crack problem in an orthotropic medium. Recently Dhaliwal, Saxena and Rokne [66] considered the crack at the interface of an orthotropic elastic layer bonded to a dissimilar orthotropic elastic half-space.

In this chapter we will consider the problem of determining the state of stress near a Griffith crack located at the interface of two dissimilar orthotropic elastic layers. By means of Fourier transforms the problem is reduced to a system of singular integral equations. These equations are further reduced to a system of simultaneous algebraic equations by using Jacobi polynomials approximation. Numerical methods are employed to determine the stress intensity factors, which have been displayed graphically.

### 4.2. Basic equations and their solution

As discussed in chapter 1, under the assumptions of plane strain in an orthotropic medium when the cartesian coordinate axes are chosen to coincide with the principle axes, we know that the displacements $u_{\mathrm{x}}, u_{\mathrm{y}}$ depend on $x$ and $y$ only, while $u_{z}$ vanishes. To simplify, let us denote $u_{x}, u_{y}$ and $u_{z}$ by $u$, $v$ and $w$ respectively, then we have

$$
\begin{equation*}
u=u(x, y) \quad, \quad v=v(x, y), \quad w=0 \tag{4.2.1}
\end{equation*}
$$

and the stress-displacement relations are given by

$$
\begin{align*}
& \sigma_{\mathrm{xx}}=c_{11} \frac{\partial u}{\partial x}+c_{12} \frac{\partial v}{\partial y}, \\
& \sigma_{\mathrm{yy}}=c_{12} \frac{\partial u}{\partial x}+c_{22} \frac{\partial v}{\partial y}, \\
& \sigma_{\mathrm{xy}}=c_{66}\left[\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right], \tag{4.2.2}
\end{align*}
$$

where $c_{\mathrm{ij}}$ 's are the elastic moduli of the orthotropic medium. The equations of equilibrium, in the absence of body forces, may be expressed as follows

$$
\begin{align*}
& c_{11} \frac{\partial^{2} u}{\partial x^{2}}+c_{66} \frac{\partial^{2} u}{\partial y^{2}}+\left(c_{12}+c_{66}\right) \frac{\partial^{2} v}{\partial x \partial y}=0,  \tag{4.2.3a}\\
& c_{66} \frac{\partial^{2} v}{\partial x^{2}}+c_{22} \frac{\partial^{2} v}{\partial y^{2}}+\left(c_{12}+c_{66}\right) \frac{\partial^{2} u}{\partial x \partial y}=0 . \tag{4.2.3b}
\end{align*}
$$

When the displacement $u$ is anti-symmetric and $v$ is symmetric with respect to $y$-axis (this is the case when the crack is subjected to symmetric normal pressure and anti-symmetric shear with respect to $y$-axis considered in this chapter), by applying the Fourier sine transform with respect to $x$ to equation (4.2.3a) and the Fourier cosine transform with respect to $x$ to equation (4.2.3b) respectively we obtain

$$
\begin{align*}
& \left(c_{66} \frac{\partial^{2}}{\partial y^{2}}-c_{11} \xi^{2}\right) \bar{u}_{\mathrm{s}}-\xi\left(c_{12}+c_{66}\right) \frac{\partial}{\partial y} \bar{v}_{\mathrm{c}}=0 \\
& \xi\left(c_{12}+c_{66}\right) \frac{\partial}{\partial y} \bar{u}_{\mathrm{s}}+\left(c_{22} \frac{\partial^{2}}{\partial y^{2}}-c_{66} \xi^{2}\right) \bar{v}_{\mathrm{c}}=0 \tag{4.2.4}
\end{align*}
$$

where $\bar{u}_{\mathrm{s}}$ is Fourier sine transform of $u$ and $\bar{v}_{\mathrm{c}}$ is the Fourier cosine transform of $v$.

Let $c_{\mathrm{ijk}}(k=1,2)$ be the elastic moduli for the layer $R_{\mathrm{k}}(k=1,2)$, where $R_{1}$ is the layer one ( $0 \leq y \leq h_{2}$ ) and $R_{2}$ is the layer two ( $-h_{1} \leq y \leq 0$ ). Solving equations (4.2.4) for $\bar{u}_{s}$ and $\bar{v}_{\mathrm{c}}$, then taking the inverse Fourier transforms we obtain the the following solution:

$$
\begin{align*}
u_{\mathrm{j}}(x, y)=\mathscr{F}_{\mathrm{s}}\left[\left(A_{\mathrm{j}} e^{\beta_{\mathrm{j}} \xi \mathrm{y}}+B_{\mathrm{j}} e^{-\beta_{\mathrm{j}} \xi \mathrm{y}}+C_{\mathrm{j}} e^{\delta_{\mathrm{j}} \xi \mathrm{y}}+D_{\mathrm{j}} e^{-\delta_{\mathrm{j}} \xi \mathrm{y}}\right) ; \xi \rightarrow x\right], \text { in } R_{\mathrm{j}}  \tag{4.2.5}\\
v_{\mathrm{j}}(x, y)=\mathscr{F}_{\mathrm{c}}\left[\left(A_{\mathrm{j}} a_{\mathrm{j}} e^{\beta_{\mathrm{j}} \xi \mathrm{y}}-B_{\mathrm{j}} \alpha_{\mathrm{j}} e^{-\beta_{\mathrm{j}} \xi \mathrm{y}}+C_{\mathrm{j}} \gamma_{\mathrm{j}} e^{\delta_{\mathrm{j}} \xi \mathrm{y}}-D_{\mathrm{j}} \gamma_{\mathrm{j}} e^{-\delta_{\mathrm{j}} \xi \mathrm{y}}\right) ; \xi \rightarrow x\right] \\
\text { in } R_{\mathrm{j}} \tag{4.2.6}
\end{align*}
$$

where $\mathscr{F}_{\mathbf{s}}, \mathscr{F}_{\mathrm{c}}$ are the Fourier sine and cosine transforms respectively, while $u_{\mathrm{j}}$ and $v_{\mathrm{j}}(j=1,2)$ are the displacements for the layer $R_{\mathrm{j}} ; A_{\mathrm{j}}, B_{\mathrm{j}}, C_{\mathrm{j}}$ and $D_{\mathrm{j}}$ ( $j=1,2$ ) are unknown functions of $\xi$, and

$$
\begin{align*}
& a_{\mathrm{j}}=\frac{c_{66 \mathrm{j}} \beta_{\mathrm{j}}-c_{11 \mathrm{j}} \beta_{\mathrm{j}}^{-1}}{c_{12 \mathrm{j}}+c_{66 \mathrm{j}}}, \quad \gamma_{\mathrm{j}}=\frac{c_{66 \mathrm{j}} \delta_{\mathrm{j}}-c_{11 \mathrm{j}} \delta_{\mathrm{j}}^{-1}}{c_{12 \mathrm{j}}+c_{66 \mathrm{j}}}, \\
& \left(\beta_{\mathrm{j}}, \delta_{\mathrm{j}}\right)=\left\{\frac{\eta_{\mathrm{j}} \pm \sqrt{\eta_{\mathrm{j}}^{2}-4 \cdot c_{11 \mathrm{j}} \cdot c_{22 \mathrm{j}} \cdot c_{66 \mathrm{j}}^{2}}}{2 \cdot c_{22 \mathrm{j}} \cdot c_{66 \mathrm{j}}}\right\}^{\frac{1}{2}}, \\
& \eta_{\mathrm{j}}=c_{11 \mathrm{j}} \cdot c_{33 \mathrm{j}}-2 \cdot c_{13 \mathrm{j}} \cdot c_{44 \mathrm{j}}-c_{13 \mathrm{j}}^{2} . \tag{4.2.7}
\end{align*}
$$

Substituting from equations (4.2.5) and (4.2.6) into (4.2.2), we obtain

$$
\begin{align*}
& \sigma_{\mathrm{yyj}}(x, y)=\mathscr{F}_{\mathrm{c}}\left[\xi\left(a_{\mathrm{j}} A_{\mathrm{j}} e^{\beta_{\mathrm{j}} \xi \mathrm{y}}+a_{\mathrm{j}} B_{\mathrm{j}} e^{-\beta_{\mathrm{j}} \xi \mathrm{y}}+b_{\mathrm{j}} C_{\mathrm{j}} e^{\delta_{\mathrm{j}} \xi y_{y}}+b_{\mathrm{j}} D_{\mathrm{j}} e^{-\delta_{\mathrm{j}} \xi \mathrm{y}}\right) ; \xi \rightarrow x\right] \text {, in } R_{\mathrm{j}}, \\
& \sigma_{\mathrm{xyj}}(x, y)=\mathscr{F}_{\mathrm{s}}\left[\xi\left(e_{\mathrm{j}} A_{\mathrm{j}} e^{\beta_{\mathrm{j}} \mathrm{\xi y}}-e_{\mathrm{j}} B_{\mathrm{j}} e^{-\beta_{\mathrm{j}} \xi \mathrm{y}}+d_{\mathrm{j}} C_{\mathrm{j}} e^{\delta_{\mathrm{j}} \xi \mathrm{y}}-d_{\mathrm{j}} D_{\mathrm{j}} e^{-\delta_{\mathrm{j}} \xi \mathrm{y}}\right) ; \xi \rightarrow x\right] \text {, in } R_{\mathrm{j}} \text {, } \tag{4.2.8}
\end{align*}
$$

where $\sigma_{\mathrm{yyj}}$ and $\sigma_{\mathrm{xyj}}$ are stress components for the region $R_{\mathrm{j}}(j=1,2)$ and

$$
\begin{align*}
& a_{\mathrm{j}}=c_{21 \mathrm{j}}+a_{\mathrm{j}} \beta_{\mathrm{j}} c_{22 \mathrm{j}}, \quad b_{\mathrm{j}}=c_{21 \mathrm{j}}+\gamma_{\mathrm{j}} \delta_{\mathrm{j}} c_{22 \mathrm{j}}, \\
& d_{\mathrm{j}}=c_{66 \mathrm{j}}\left(\delta_{\mathrm{j}}-\gamma_{\mathrm{j}}\right), \quad e_{\mathrm{j}}=c_{66 \mathrm{j}}\left(\beta_{\mathrm{j}}-a_{\mathrm{j}}\right), \quad j=1,2 . \tag{4.2.9}
\end{align*}
$$

4.3. Statment of the problem and derivation of the singular integral equations

We assume that two dissimilar orthotropic elastic layers, which occupy the regions $R_{1}\left(0 \leq y \leq h_{2}\right)$ and $R_{2}\left(-h_{1} \leq y \leq 0\right)$ respectively, are perfectly
bonded except that there is a crack in the interval $-a \leq x \leq a, y=0$ (see Fig 4.3.1). It is also assumed that the boundaries $y=-h_{1}$ and $y=h_{2}$ are stress-free, and the surfaces of the crack are subjected to symmetrical normal pressure and anti-symmetrical shear with respect to $y$-axis. Hence the problem of determining the stress and displacement field is subjected to the following boundary and continuity conditions:

$$
\begin{array}{lll}
\sigma_{\mathrm{yy} 1}\left(x, 0^{+}\right)=\sigma_{\mathrm{yy} 2}\left(x, 0^{-}\right)=p_{1}(x), & p_{1}(-x)=p_{1}(x) ; & |x|<a \\
\sigma_{\mathrm{xy} 1}\left(x, 0^{+}\right)=\sigma_{\mathrm{xy} 2}\left(x, 0^{-}\right)=p_{2}(x), & p_{2}(-x)=-p_{2}(x) ; & |x|<a \\
u_{1}\left(x, 0^{+}\right)=u_{2}\left(x, 0^{-}\right), \quad v_{1}\left(x, 0^{+}\right)=v_{2}\left(x, 0^{-}\right), & |x|>a \\
\sigma_{\mathrm{yy} 1}\left(x, 0^{+}\right)=\sigma_{\mathrm{yy} 2}\left(x, 0^{-}\right), \quad \sigma_{\mathrm{xy} 1}\left(x, 0^{+}\right)=\sigma_{\mathrm{xy} 2}\left(x, 0^{-}\right), & |x|>a \tag{4.3.4}
\end{array}
$$

and

$$
\begin{array}{ll}
\sigma_{\mathrm{yy} 1}\left(x, h_{2}\right)=0, & \sigma_{\mathrm{xy} 1}\left(x, h_{2}\right)=0, \\
\sigma_{\mathrm{yy} 2}\left(x,-h_{1}\right)=0, & \sigma_{\mathrm{xy} 2}\left(x,-h_{1}\right)=0,
\end{array}
$$

With the help of conditions (4.3.1) and (4.3.2), the conditions (4.3.4) may be replaced by

$$
\sigma_{\mathrm{yy} 1}\left(x, 0^{+}\right)=\sigma_{\mathrm{yy} 2}\left(x, 0^{-}\right), \quad \sigma_{\mathrm{xy} 1}\left(x, 0^{+}\right)=\sigma_{\mathrm{xy} 2}\left(x, 0^{-}\right), \quad \quad-\infty<x<\infty .(4.3 .4 \mathrm{a})
$$

The conditions (4.3.4a) yield

$$
\begin{align*}
& a_{1}\left(A_{1}+B_{1}\right)+b_{1}\left(C_{1}+D_{1}\right)=a_{2}\left(A_{2}+B_{2}\right)+b_{2}\left(C_{2}+D_{2}\right)  \tag{4.3.7}\\
& e_{1}\left(A_{1}-B_{1}\right)+d_{1}\left(C_{1}-D_{1}\right)=e_{2}\left(A_{2}-B_{2}\right)+d_{2}\left(C_{2}-D_{2}\right) \tag{4.3.8}
\end{align*}
$$

While the boundary conditions (4.3.5) and (4.3.6) give

$$
\begin{align*}
& a_{1} A_{1} e^{-\beta_{1} \xi \mathrm{~h}_{2}}+a_{1} B_{1} e^{-\beta_{1} \xi \mathrm{~h}_{2}}+b_{1} C_{1} e^{\delta_{1} \xi \mathrm{~h}_{2}}+b_{1} D_{1} e^{-\delta_{1} \xi \mathrm{~h}_{2}}=0  \tag{4.3.9}\\
& e_{1} A_{1} e^{\beta_{1} \xi \mathrm{~h}_{2}}-e_{1} B_{1} e^{-\beta_{1} \xi \mathrm{~h}_{2}}+d_{1} C_{1} e^{\delta \mathrm{h}_{2}}-d_{1} D_{1} e^{-\delta_{1} \xi \mathrm{~h}_{2}}=0  \tag{4.3.10}\\
& a_{2} A_{2} e^{-\beta_{2} \xi \mathrm{~h}_{1}}+a_{2} B_{2} e^{\beta_{2} \xi \mathrm{~h}_{1}}+b_{2} C_{2} e^{-\delta_{2} \xi \mathrm{~h}_{1}}+b_{2} D_{2} e^{\delta_{2} \xi \mathrm{~h}_{1}}=0  \tag{4.3.11}\\
& e_{2} A_{2} e^{-\beta_{2} \xi \mathrm{~h}_{1}}-e_{2} B_{2} e^{\beta_{2} \xi \mathrm{~h}_{1}}+d_{2} C_{2} e^{-\delta_{2} \xi \mathrm{~h}_{1}}-d_{2} D_{2} e^{\delta_{2} \xi \mathrm{~h}_{1}}=0 \tag{4.3.12}
\end{align*}
$$



Fig.4.3.1
Griffith crack at the interface of two dissimilar orthotropic elastic layers.

Solving equations (4.3.7) to (4.3.12). we get

$$
\begin{align*}
& C_{1}(\xi)=l_{11} A_{1} e^{\left(\beta_{1}-\delta_{1}\right) \xi h_{2}}+l_{12} B_{1} e^{-\left(\beta_{1}+\delta_{1}\right) \xi h_{2}}, \\
& D_{1}(\xi)=l_{12} A_{1} e^{\left(\beta_{1}+\delta \delta_{1}\right) \xi h_{2}}+l_{11} B_{1} e^{\left(-\beta_{1}+\delta_{1}\right) \xi h_{2}}, \\
& C_{2}(\xi)=l_{21} A_{2} e^{\left(-\beta_{2}+\delta_{2}\right) \xi \mathrm{h}_{1}}+l_{22} B_{2} e^{\left(\beta_{2}+\delta_{2}\right) \xi \mathrm{h}_{1}}, \\
& D_{2}(\xi)=l_{22} A_{2} e^{-\left(\beta_{2}+\delta_{2}\right) \xi h_{1}}+l_{21} B_{2} e^{\left(\beta_{2}-\delta_{2}\right) \xi \mathrm{h}_{1}}, \tag{4.3.13}
\end{align*}
$$

and

$$
\begin{equation*}
A_{2}(\xi)=l_{31} A_{1}+l_{32} B_{1}, \quad B_{2}=l_{41} A_{1}+l_{42} B_{1} \tag{4.3.14}
\end{equation*}
$$

where

$$
\begin{array}{ll}
l_{1 \mathrm{j}}=-\left[\begin{array}{ll}
a_{1} d_{1}-(-1)^{\mathrm{j}} e_{1} b_{1}
\end{array}\right] /\left(2 b_{1} d_{1}\right), & l_{2 \mathrm{j}}=-\left[\begin{array}{ll}
\left.a_{2} d_{2}-(-1)^{\mathrm{j}}{ }_{2} b_{2}\right]
\end{array}\right] /\left(2 b_{2} d_{2}\right), \quad j=1,2 . \\
l_{31}=\left(\begin{array}{lll}
a_{11} & b_{22}-a_{22} & \left.b_{11}\right) / \Delta,
\end{array} l_{32}=\left(\begin{array}{ll}
a_{12} & b_{22}-a_{22}
\end{array} b_{12}\right) / \Delta,\right. \\
l_{41}=\left(\begin{array}{lll}
a_{21} & b_{11}-a_{11} & b_{21}
\end{array}\right) / \Delta, \quad l_{42}=\left(\begin{array}{lll}
a_{21} & b_{12}-a_{12} & b_{21}
\end{array}\right) / \Delta, \\
\Delta=a_{21} b_{22}-a_{22} b_{21}, \tag{4.3.15}
\end{array}
$$

and

$$
\begin{align*}
& a_{11}=a_{1}+b_{1} l_{11} e^{\left(\beta_{1}-\delta_{1}\right) \xi \mathrm{h}_{2}}+b_{1} l_{12} e^{-\left(\beta_{1}+\delta_{1}\right) \xi h_{2}}, \\
& a_{12}=a_{1}+b_{1} l_{12} e^{-\left(\beta_{1}+\delta_{1}\right) \xi \mathrm{h}_{2}}+b_{1} l_{11} e^{\left(-\beta_{1}+\delta_{1}\right) \xi \mathrm{h}_{2}}, \\
& a_{21}=a_{2}+b_{2} l_{21} e^{\left(-\beta_{2}+\delta_{2}\right) \xi \mathrm{h}_{1}}+b_{2} l_{22} e^{-\left(\beta_{2}+\delta \delta_{2}\right) \xi \mathrm{h}_{1}}, \\
& a_{22}=a_{2}+b_{2} l_{22} e^{\left(\beta_{2}+\delta \delta_{2}\right) \xi \mathrm{h}_{1}}+b_{2} l_{21} e^{\left(\beta_{2}-\delta_{2}\right) \xi \mathrm{h}_{1}}, \\
& b_{11}=e_{1}+d_{1} l_{11} e^{\left(\beta_{1}-\delta_{1}\right) \xi \mathrm{h}_{2}}-d_{1} l_{12} e^{\left(\beta_{1}+\delta_{1}\right) \xi \mathrm{h}_{2}}, \\
& b_{12}=-e_{1}+d_{1} l_{12} e^{-\left(\beta_{1}+\delta_{1}\right) \xi \mathrm{h}_{2}}-d_{1} l_{11} e^{\left(-\beta_{1}+\delta \delta_{1}\right) \xi \mathrm{h}_{2}}, \\
& b_{21}=e_{2}+d_{2} l_{21} e^{\left(-\beta_{2}+\delta_{2}\right) \xi \mathrm{h}_{1}}-d_{2} l_{22} e^{-\left(\beta_{2}+\delta \delta_{2}\right) \xi \mathrm{h}_{1}}, \\
& b_{22}=-e_{2}+d_{2} l_{22} e^{\left(\beta_{2}+\delta \delta_{2}\right) \xi \mathrm{h}_{1}}-d_{2} l_{21} e^{\left(\beta_{2}-\delta_{2}\right) \xi \mathrm{h}_{1}} . \tag{4.3.16}
\end{align*}
$$

Conditions (4.3.1) , (4.3.2) and (4.3.3) yield the following equations:

$$
\begin{align*}
& \mathscr{F}_{\mathrm{c}}\left[\xi\left(a_{1} A_{1}+a_{1} B_{1}+b_{1} \mathrm{C}_{1}+d_{1} D_{1}\right) ; \xi \rightarrow x\right]=p_{1}(x), . \quad 0<x<a ; \quad \text { (4.3.17) } \\
& \mathscr{F}_{\mathrm{S}}\left[\xi\left(e_{1} A_{1}-e_{1} B_{1}+d_{1} \mathrm{C}_{1}-d_{1} D_{1}\right) ; \xi \rightarrow x\right]=p_{2}(x), \quad 0<x<a ; \\
& \mathscr{F}_{\mathrm{s}}\left[\left(A_{1}+B_{1}+C_{1}+D_{1}-A_{2}-B_{2}-C_{2}-D_{2}\right) ; \xi-x\right]= \begin{cases}u_{1}\left(x, 0^{+}\right)-u_{2}\left(x, 0^{-}\right), & 0<x<a ; \\
0, & x>a ;\end{cases}  \tag{4.3.19}\\
& \mathscr{F}_{\mathrm{c}}\left[\left(\alpha_{1} A_{1}-\alpha_{1} B_{1}+\gamma_{1} C_{1}-\gamma_{1} D_{1}-\alpha_{2} A_{2}+\alpha_{2} B_{2}-\gamma_{2} C_{2}+\gamma_{2} D_{2}\right) ; \xi \rightarrow x\right] \\
& =\left\{\begin{array}{lr}
v_{1}\left(x, 0^{+}\right)-v_{2}\left(x, 0^{-}\right), & 0<x<a ; ~ \\
0, & x>a ; ~
\end{array}\right. \tag{4.3.20}
\end{align*}
$$

Differentiating (4.3.19) and (4.3.20) with respect to $x$ and setting

$$
\begin{align*}
& \xi\left(A_{1}+B_{1}+C_{1}+D_{1}-A_{2}-B_{2}-C_{2}-D_{2}\right)=\phi_{1}(\xi),  \tag{4.3.21}\\
& \xi\left(\alpha_{1} A_{1}-\alpha_{1} B_{1}+\gamma_{1} C_{1}-\gamma_{1} D_{1}-\alpha_{2} A_{2}+\alpha_{2} B_{2}-\gamma_{2} C_{2}+\gamma_{2} D_{2}\right)=\phi_{2}(\xi), \tag{4.3.22}
\end{align*}
$$

we obtain the following equations from equations (4.3.17) to (4.3.20)

$$
\begin{array}{ll}
\mathscr{F}_{\mathrm{c}}\left[\left(\lambda_{11} \phi_{1}+\lambda_{12} \phi_{2}\right) ; \xi \rightarrow x\right]=p_{1}(x), & 0<x<a \\
\mathscr{F}_{\mathrm{s}}\left[\left(\lambda_{21} \phi_{1}+\lambda_{22} \phi_{2}\right) ; \xi \rightarrow x\right]=p_{2}(x), & 0<x<a \\
\mathscr{F}_{\mathrm{c}}\left[\phi_{1}(\xi) ; \xi \rightarrow x\right]=\left\{\begin{array}{cc}
\frac{\partial}{\partial x}\left[u_{1}\left(x, 0^{+}\right)-\dot{u}_{2}\left(x, 0^{-}\right)\right], & x>a \\
0, & 0<x<a
\end{array}\right. \\
\mathscr{F}_{\mathrm{s}}\left[\phi_{2}(\xi) ; \xi \rightarrow x\right]=\left\{\begin{array}{cc}
\frac{\partial}{\partial x}\left[-v_{1}\left(x, 0^{+}\right)+v_{2}\left(x, 0^{-}\right)\right], & x>a \\
0, & 0
\end{array}\right. \tag{4.3.26}
\end{array}
$$

where $\lambda_{\mathrm{jk}}$ are functions of $\xi$ and are given by

$$
\begin{align*}
& \lambda_{11}=\left(a_{11} \gamma_{12}-a_{12} \gamma_{11}\right) / \Delta^{*}, \lambda_{12}=\left(a_{12} \gamma_{21}-a_{11} \gamma_{22}\right) / \Delta^{*} \\
& \lambda_{21}=\left(b_{11} \gamma_{12}-b_{12} \gamma_{11}\right) / \Delta^{*}, \lambda_{22}=\left(b_{12} \gamma_{21}-b_{11} \gamma_{22}\right) / \Delta^{*} \\
& \Delta^{*}=\gamma_{12} \gamma_{21}-\gamma_{11} \gamma_{22} \tag{4.3.27}
\end{align*}
$$

and $\gamma_{\mathrm{jk}}$ 's are given by the following equations

$$
\begin{align*}
& \gamma_{11}=a_{1}-\gamma_{1} l_{11} e^{\left(\beta_{1}-\delta_{1}\right) \xi \mathrm{h}_{2}}+\gamma_{1} l_{12} e^{\left(\beta_{1}+\delta_{1}\right) \xi \mathrm{h}_{2}-a_{2} l_{31}+a_{2} l_{41}-\gamma_{2} l_{51}+\gamma_{2} l_{61}}, \\
& \gamma_{12}=-a_{1}-\gamma_{1} l_{12} e^{-\left(\beta_{1}+\delta_{1}\right) \xi \mathrm{h}_{2}}+\gamma_{1} l_{11} e^{\left(-\beta_{1}+\delta_{1}\right) \xi \mathrm{h}_{2}}-a_{2} l_{32}+a_{2} l_{42}-\gamma_{2} l_{52}+\gamma_{2} l_{62}, \\
& \gamma_{21}=1-l_{11} e^{\left(\beta_{1}-\delta_{1}\right) \xi \mathrm{h}_{2}-l_{12} e^{\left(\beta_{1}+\delta_{1}\right) \xi \mathrm{h}_{2}-l_{31}-l_{41}-l_{51}-l_{61}},} \\
& \gamma_{22}=1-l_{12} e^{-\left(\beta_{1}+\delta_{1}\right) \xi \mathrm{h}_{2}-l_{11} e^{\left(-\beta_{1}+\delta_{1}\right) \xi \mathrm{h}_{2}-l_{32}-l_{42}-l_{52}-l_{62}},} \\
& l_{51}=-l_{21} l_{31} e^{\left(-\beta_{2}+\delta_{2}\right) \xi \mathrm{h}_{1}}-l_{22} l_{41} e^{\left(\beta_{2}+\delta \delta_{2}\right) \xi \mathrm{h}_{1}}, \\
& l_{52}=-l_{21} l_{32} e^{-\left(\beta_{2}+\delta_{2}\right) \xi \mathrm{h}_{1}}-l_{22} l_{42} e^{\left(\beta_{2}+\delta_{2}\right) \xi \mathrm{h}_{1}}, \\
& l_{61}=-l_{22} l_{31} e^{-\left(\beta_{2}+\delta_{2}\right) \xi \mathrm{h}_{1}}-l_{21} l_{41} e^{\left(\beta_{2}-\delta_{2}\right) \xi \mathrm{h}_{1}}, \\
& l_{62}=-l_{22} l_{32} e^{-\left(\beta_{2}+\delta_{2}\right) \xi \mathrm{h}_{1}}-l_{21} l_{42} e^{\left(\beta_{2}-\delta_{2}\right) \xi \mathrm{h}_{1}} . \tag{4.3.28}
\end{align*}
$$

It is easy to verify that when $\xi \rightarrow \infty, \lambda_{j k}(\xi)$ tends to constant, say $\lambda_{j k}(\infty)$, and

$$
\begin{equation*}
\lambda_{\mathrm{jk}}(\xi)-\lambda_{\mathrm{jk}}(\infty)=O\left(e^{-\left(\beta_{1}+\delta_{1}\right) \xi \mathrm{h}_{2}}\right) \tag{4.3.29}
\end{equation*}
$$

Now suppose that the dislocations at $\mathrm{y}=0$ are $f_{1}$ and $f_{2}$, i.e

$$
\begin{align*}
& \frac{\partial}{\partial x}\left[u_{1}\left(x, 0^{+}\right)-u_{2}\left(x, 0^{-}\right)\right]= \begin{cases}f_{1}(x), & |x|<a \\
0, & |x|>a\end{cases}  \tag{4.3.30}\\
& \frac{\partial}{\partial x}\left[-v_{1}\left(x, 0^{+}\right)+v_{2}\left(x, 0^{-}\right)\right]= \begin{cases}f_{2}(x), & |x|<a \\
0, & |x|>a\end{cases} \tag{4.3.31}
\end{align*}
$$

and

$$
\begin{equation*}
f_{1}(-x)=f_{1}(x), \quad f_{2}(-x)=-f_{2}(x), \quad|x|<a \tag{4.3.32}
\end{equation*}
$$

By taking $s=x / a$, equations (4.3.23) to (4.3.26) can be written in the following form

$$
\begin{array}{ll}
\mathscr{F}_{\mathrm{c}}\left[\left(\tilde{\lambda}_{11} \tilde{\phi}_{1}+\tilde{\lambda}_{12} \tilde{\phi}_{2}\right) ; \xi \rightarrow s\right]=\tilde{p}_{1}(s), & 0<s<1 \\
\mathscr{F}_{\mathrm{s}}\left[\left(\tilde{\lambda}_{21} \tilde{\phi}_{1}+\tilde{\lambda}_{22} \tilde{\phi}_{2}\right) ; \xi \rightarrow s\right]=\tilde{p}_{2}(s), & 0<s<1 \tag{4.3.34}
\end{array}
$$

$$
\begin{array}{lr}
\mathscr{F}_{\mathrm{C}}\left[\tilde{\phi}_{1} ; \xi \rightarrow s\right]=\left\{\begin{array}{cr}
\frac{\partial}{\partial s}\left[\tilde{u}_{1}\left(s, 0^{+}\right)-\tilde{u}_{2}\left(s, 0^{-}\right)\right], & 0<s<1 \\
0, & s>1
\end{array}\right. \\
\mathscr{F}_{\mathrm{s}}\left[\tilde{\phi}_{2} ; \xi \rightarrow s\right]=\left\{\begin{array}{cr}
\frac{\partial}{\partial s}\left[-\tilde{v}_{1}\left(s, 0^{+}\right)+\tilde{v}_{2}\left(s, 0^{-}\right)\right], & 0<s<1 \\
0, & s>1
\end{array}\right. \tag{4.3.36}
\end{array}
$$

where

$$
\begin{align*}
& \tilde{p}_{\mathrm{j}}(s)=a p_{\mathrm{j}}(a s), \quad \tilde{\lambda}_{\mathrm{jk}}(\xi)=\lambda_{\mathrm{jk}}(\xi / a), \quad \tilde{\phi}_{\mathrm{j}}(\xi)=\phi_{\mathrm{j}}(\xi / a), \\
& \tilde{u}_{\mathrm{j}}(s)=u_{\mathrm{j}}(a s), \quad \tilde{v}_{\mathrm{j}}(s)=v_{\mathrm{j}}(a s), \quad j, k=1,2 \tag{4.3.37}
\end{align*}
$$

Now equations (4.3.30) and (4.3.31) can be written in the following form

$$
\begin{align*}
& \frac{\partial}{\partial s}\left[\tilde{u}_{1}\left(s, 0^{+}\right)-\tilde{u}_{2}\left(s, 0^{-}\right)\right]= \begin{cases}\tilde{f}_{1}(s), & |s|<1 \\
0, & |s|>1\end{cases}  \tag{4.3.38}\\
& \frac{\partial}{\partial s}\left[\tilde{v}_{1}\left(s, 0^{+}\right)-\tilde{v}_{2}\left(s, 0^{-}\right)\right]= \begin{cases}-\tilde{f}_{2}(s), & |s|<1 \\
0, & |s|>1\end{cases} \tag{4.3.39}
\end{align*}
$$

where

$$
\begin{array}{ll}
\tilde{f}_{1}(-s)=\tilde{f}_{1}(s), \quad \tilde{f}_{2}(-s)=-\tilde{f}_{2}(s), & |s|<1 \\
\tilde{f}_{\mathrm{j}}(s)=a f_{\mathrm{j}}(a s), & j=1,2 \tag{4.3.40}
\end{array}
$$

The equations (4.3.35) and (4.3.36) are satisfied automatically and we have

$$
\begin{align*}
& \tilde{\phi}_{1}(\xi)=(2 / \pi)^{\frac{1}{2}} \int_{0}^{1} \tilde{f}_{1}(s) \cos (\xi s) d s  \tag{4.3.41}\\
& \tilde{\phi}_{2}(\xi)=(2 / \pi)^{\frac{1}{2}} \int_{0}^{1} \tilde{f}_{2}(s) \sin (\xi s) d s \tag{4.3.42}
\end{align*}
$$

Using integration by parts, from equation (4.3.41) we get

$$
\tilde{\phi}_{1}(\xi)=-(2 / \pi)^{\frac{1}{2}} \xi^{-1} \int_{0}^{1} \frac{d}{d s} \tilde{f}_{1}(s) \sin (\xi s) d s
$$

then using the result [44] that

$$
\mathscr{F}_{\mathrm{s}}\left[\xi^{-1} \sin (\xi \tau) . ; \quad \xi \rightarrow s\right]=\frac{1}{2}(2 / \pi)^{\frac{1}{2}} \log \left|\frac{s+\tau}{s-\tau}\right|
$$

we see that

$$
\mathscr{F}_{\mathrm{s}}\left[\tilde{\phi}_{1}(\xi) ; \xi \rightarrow s\right]=-\frac{1}{\pi} \int_{0}^{1} \tilde{f}_{1}(\tau)\left[\frac{1}{s+\tau}+\frac{1}{s-\tau}\right] d \tau
$$

Since $\tilde{f}_{1}(s)$ is an even function, we have

$$
\begin{equation*}
\mathscr{F}_{\mathrm{s}}\left[\tilde{\phi}_{1}(\xi) ; \xi \rightarrow s\right]=-\frac{1}{\pi} \int_{-1}^{1} \frac{\tilde{f}_{1}(t)}{t-s} d t \tag{4.3.43}
\end{equation*}
$$

Similarly, from equation (4.3.42) we get

$$
\begin{equation*}
\mathscr{F}_{\mathrm{c}}\left[\tilde{\phi}_{2}(\xi) ; \xi \rightarrow s\right]=\frac{1}{\pi} \int_{-1}^{1} \frac{\tilde{f}_{2}(t)}{t-s} d t \tag{4.3.44}
\end{equation*}
$$

For the continuity of the displacements on $y=0,|x|>a ; f_{1}$ and $f_{2}$ must satisfy the following conditions

$$
\begin{equation*}
\int_{-a}^{a} f_{\mathrm{j}}(x) d x=0, \text { or } \int_{-1}^{1} \tilde{f}_{\mathrm{j}}(s) d s=0, \quad j=1,2 \tag{4.3.45}
\end{equation*}
$$

(Note, for $j=2$, condition (4.3.45) holds obviously since $f_{2}(x)$ is an odd function) Substituting for $\tilde{\phi}_{1}, \tilde{\phi}_{2}$ from equations (4.3.41) and (4.3.42) into equations (4.3.33) and (4.3.34), we obtain

$$
\begin{align*}
\lambda_{11}(\infty) \tilde{f}_{1}(s) & +\frac{\lambda_{12}(\infty)}{\pi} \int_{-1}^{1} \frac{\tilde{f}_{2}(t)}{t-s} d t+\int_{-1}^{1} \tilde{f}_{1}(t) K_{11}(s, t) d t \\
& +\int_{-1}^{1} \tilde{f}_{2}(t) K_{12}(s, t) d t=\tilde{p}_{1}(s), \quad|s|<1  \tag{4.3.46}\\
\lambda_{22}(\infty) \tilde{f}_{2}(s) & -\frac{\lambda_{21}(\infty)}{\pi} \int_{-1}^{1} \frac{\tilde{f}_{1}(t)}{t-s} d t+\int_{-1}^{1} \tilde{f}_{1}(t) K_{21}(s, t) d t \\
& +\int_{-1}^{1} \tilde{f}_{2}(t) K_{22}(s, t) d t=\tilde{p}_{2}(s), \quad|s|<1 \tag{4.3.47}
\end{align*}
$$

where

$$
\begin{align*}
& K_{11}(s, t)=\frac{1}{\pi} \int_{0}^{\infty}\left[\tilde{\lambda}_{11}(\xi)-\lambda_{11}(\infty)\right] \cos (\xi t) \cos (\xi s) d \xi \\
& K_{12}(s, t)=\frac{1}{\pi} \int_{0}^{\infty}\left[\tilde{\lambda}_{12}(\xi)-\lambda_{12}(\infty)\right] \sin (\xi t) \cos (\xi s) d \xi \\
& K_{21}(s, t)=\frac{1}{\pi} \int_{0}^{\infty}\left[\tilde{\lambda}_{21}(\xi)-\lambda_{21}(\infty)\right] \cos (\xi t) \sin (\xi s) d \xi \\
& K_{22}(s, t)=\frac{1}{\pi} \int_{0}^{\infty}\left[\tilde{\lambda}_{22}(\xi)-\lambda_{22}(\infty)\right] \sin (\xi t) \sin (\xi s) d \xi \tag{4.3.48}
\end{align*}
$$

Equations (4.3.46) and (4.3.47) can be rewritten in the following forms:

$$
\begin{gather*}
\lambda \psi_{1}(s)+\frac{1}{\pi i} \int_{-1}^{1} \frac{\psi_{1}(t)}{t-s} d t+\int_{-1}^{1} \psi_{1}(t) M_{11}(s, t) d t+\int_{-1}^{1} \psi_{2}(t) M_{12}(s, t) d t=g_{1}(s), \\
|s|<1 ;  \tag{4.3.49}\\
\lambda \psi_{2}(s)-\frac{1}{\pi i \int_{-1}^{1}} \frac{\psi_{2}(t)}{t-s} d t+\int_{-1}^{1} \psi_{1}(t) M_{21}(s, t) d t+\int_{-1}^{1} \psi_{2}(t) M_{22}(s, t) d t=g_{2}(s), \\
|s|<1 \tag{4.3.50}
\end{gather*}
$$

where

$$
\begin{aligned}
& \lambda_{1}=\lambda_{11}(\infty) / \lambda_{12}(\infty), \quad \lambda_{2}=\lambda_{22}(\infty) / \lambda_{21}(\infty), \lambda=\sqrt{\lambda_{1} \lambda_{2}}, \\
& \psi_{1}(s)=\sqrt{\lambda_{1}} \tilde{f}_{1}(s)+\dot{i} \lambda_{2} \tilde{f}_{2}(s), \quad \psi_{2}(s)=\sqrt{\lambda_{1}} \tilde{f}_{1}(s)-i \sqrt{\lambda_{2}} \tilde{f}_{2}(s), \\
& g_{1}(s)=\sqrt{\lambda_{2}} \tilde{p}_{1}(s) / \lambda_{12}(\infty)+i \sqrt{\lambda_{1}} \tilde{p}_{2}(s) / \lambda_{21}(\infty), \\
& g_{2}(s)=\sqrt{\lambda_{2}} \tilde{p}_{1}(s) / \lambda_{12}(\infty)-i \sqrt{\lambda_{1}} \tilde{p}_{2}(s) / \lambda_{21}(\infty), \\
& M_{11}(s, t)=\frac{1}{2}\left\{\frac{\sqrt{\lambda_{1}}}{\lambda_{21}(\infty) \sqrt{\lambda_{2}}} K_{22}(s, t)+\frac{\sqrt{\lambda_{2}}}{\lambda_{12}(\infty) \sqrt{\lambda_{1}}} K_{11}(s, t)\right. \\
& \left.+i\left(\frac{K_{21}(s, t)}{\lambda_{21}(\infty)}-\frac{K_{12}(s, t)}{\lambda_{12}(\infty)}\right)\right\}, \\
& M_{12}(s, t)=\frac{1}{2}\left\{\frac{-\sqrt{\lambda_{1}}}{\lambda_{21}(\infty) \sqrt{\lambda_{2}}} K_{22}(s, t)+\frac{\sqrt{\lambda_{2}}}{\lambda_{12}(\infty) \sqrt{\lambda_{1}}} K_{11}(s, t)\right. \\
& \left.+i\left(\frac{K_{21}(s, t)}{\lambda_{21}(\infty)}+\frac{K_{12}(s, t)}{\lambda_{12}(\infty)}\right)\right\},
\end{aligned}
$$

$$
\begin{align*}
& M_{21}(s, t)= \frac{1}{2}\left\{\frac{-\sqrt{\lambda_{1}}}{\lambda_{21}(\infty) \sqrt{\lambda_{2}}} K_{22}(s, t)+\frac{\sqrt{\lambda_{2}}}{\lambda_{12}(\infty) \sqrt{\lambda_{1}}} K_{11}(s, t)\right. \\
&\left.-i\left(\frac{K_{12}(s, t)}{\lambda_{12}(\infty)}+\frac{K_{21}(s, t)}{\lambda_{21}(\infty)}\right)\right\}, \\
& M_{22}(s, t)=\frac{1}{2}\left\{\frac{\sqrt{\lambda_{1}}}{\lambda_{21}(\infty) \sqrt{\lambda_{2}}} K_{22}(s, t)+\frac{\sqrt{\lambda_{2}}}{\lambda_{12}(\infty) \sqrt{\lambda_{1}}} K_{11}(s, t)\right. \\
&\left.+i\left(\frac{K_{12}(s, t)}{\lambda_{12}(\infty)}-\frac{K_{21}(s, t)}{\lambda_{21}(\infty)}\right)\right\} . \tag{4.3.51}
\end{align*}
$$

The analytic solution of equations (4.3.49) and (4.3.50) has been extensively studied (see, for exemple, [67] and [68]) by using regularization method, which, in this case, however becomes cumbersome. Here we try to use an approximation method described by Erdogan [69] to find the stress intensity factors.

Since the kernels $M_{j k}(j, k=1,2)$ are bounded, we know, aside from a multiplication constant, the singular behavior of the functions $\psi_{1}$ and $\psi_{2}$ at the points $s= \pm 1$ is determined by the dominant part of the singular integral equations. The equations (4.3.49) and (4.3.50) will be solved under the assumption that $\psi_{1}$ and $\psi_{2}$ satisfy a Hölder condition on every closed part of the interval $(-1,1)$ not containing the ends.

The solution of the equations (4.3.49) and (4.3.50) may be assumed in the form of Jacobi polynomials $P_{\mathrm{n}}{ }^{\left(\sigma_{\mathrm{k}}, \tau_{\mathrm{k}}\right)}(s)$ [60] by .

$$
\begin{equation*}
\psi_{\mathrm{k}}=\sum_{\mathrm{n}=1}^{\infty} C_{\mathrm{kn}} W_{\mathrm{k}}(s) P_{\mathrm{n}}\left(\sigma_{\mathrm{k}}, \tau_{\mathrm{k}}\right)(s), \quad k=1,2 ; \tag{4.3.52}
\end{equation*}
$$

where

$$
\begin{array}{ll}
W_{\mathrm{k}}(s)=(1-s)^{\sigma_{\mathrm{k}}}(1+s)^{\tau_{\mathrm{k}}}, \quad \sigma_{\mathrm{k}}=-\frac{1}{2}+i \omega_{\mathrm{k}}, & \tau_{\mathrm{k}}=-\frac{1}{2}-i w_{\mathrm{k}}, \\
\omega_{\mathrm{k}}=(-1)^{\mathrm{k}+1} \omega, \quad \omega=\frac{1}{2 \pi} \ln \left[\frac{1+\lambda}{1-\lambda}\right], & k=1,2 ; \tag{4.3.53}
\end{array}
$$

and $C_{\mathrm{kn}}$ are unknown coefficents.

Observing that $P_{0}^{\left(\sigma_{\mathrm{k}}, \tau_{\mathrm{k}}\right)}(s)=1$ and the orthogonality relations of Jacobi polynomials

$$
\begin{align*}
& \int_{-1}^{1} W(t) P_{\mathrm{n}}{ }^{(\sigma, \tau)}(t) P_{\mathrm{m}}^{(\sigma, \tau)}(t) d t= \begin{cases}0 & n \neq m \\
\theta_{\mathrm{m}}^{(\sigma, \tau)}\end{cases} \\
& \theta_{\mathrm{m}}^{(\sigma, \tau)}=\frac{2^{\sigma+\tau+1} \mathrm{\Gamma}(m+\sigma+1) \Gamma(m+\tau+1)}{m!(2 m+\sigma+\tau+1) \Gamma(m+\sigma+\tau+1)} \tag{4.3.54}
\end{align*}
$$

we conclude that by choosing $C_{\mathrm{k} 0}=0, k=1,2$, the condition (4.3.45) will be satisfied automatically.

Using the following relation [70]

$$
\begin{align*}
& \frac{1}{\pi i \int_{-1}^{1}} W_{\mathrm{k}}(t) P_{\mathrm{n}}\left(\sigma_{\mathrm{k}}, \tau_{\mathrm{k}}\right)(t) \frac{d t}{t-s}+(-1)^{\mathrm{k}} \lambda W_{\mathrm{k}}(s) P_{\mathrm{n}}^{\left(\sigma_{\mathrm{k}}, \tau_{\mathrm{k}}\right)}(s) \\
& = \begin{cases}-\frac{\left(1-\lambda^{2}\right)^{\frac{1}{2}}}{2 i} P_{\mathrm{n}-1}^{\left(-\sigma_{\mathrm{k}},-\tau_{\mathrm{k}}\right)}(s), & |s|<1 \\
-\frac{\left(1-\lambda^{2}\right)^{\frac{1}{2}}}{2 i}\left[(s-1)^{\sigma_{\mathrm{k}}}(s+1)^{\tau_{\mathrm{k}}} P_{\mathrm{n}}\left(\sigma_{\mathrm{k}}, \tau_{\mathrm{k}}\right)\right. \\
\left.(x)+G_{\mathrm{kn}}^{\infty}(s)\right], & |s|>1\end{cases} \tag{4.3.55}
\end{align*}
$$

where $G_{\mathrm{kn}}^{\infty}(s)$ is the principal part of $W_{\mathrm{k}}(s) P_{\mathrm{n}}\left(\sigma_{\mathrm{k}}, \tau_{\mathrm{k}}\right)(s)$ at infinity, and substituting from equation (4.3.52) into equations (4.3.49) and (4.3.50) we obtain

$$
\begin{align*}
& \sum_{\mathrm{n}=1}^{\infty} C_{1 \mathrm{n}} \frac{\left(1-\lambda^{2}\right)^{\frac{1}{2}}}{2 i} P_{\mathrm{n}-1}^{\left(-\sigma_{1},-\tau_{1}\right)}(s)+\sum_{\mathrm{n}=1}^{\infty}\left[C_{1 \mathrm{n}} L_{11 \mathrm{n}}(s, t)+C_{2 \mathrm{n}} L_{12 \mathrm{n}}(s, t)\right]=g_{1}(s) \\
&|s|<1  \tag{4.3.56}\\
& \sum_{\mathrm{n}=1}^{\infty} C_{2 \mathrm{n}} \frac{\left(1-\lambda^{2}\right)^{\frac{1}{2}}}{2 i} P_{\mathrm{n}-1}^{\left(-\sigma_{2},-\tau_{2}\right)}(s)-\sum_{\mathrm{n}=1}^{\infty}\left[C_{\mathrm{ln}} L_{21 \mathrm{n}}(s, t)+C_{2 \mathrm{n}} L_{22 \mathrm{n}}(s, t)\right]=-g_{2}(s) \\
&|s|<1 \tag{4.3.57}
\end{align*}
$$

where

$$
\begin{equation*}
L_{\mathrm{kmn}}(s)=\int_{-1}^{1} M_{\mathrm{km}}(s, t) W_{\mathrm{m}}(t) P_{\mathrm{n}}\left(\sigma_{\mathrm{m}}, \tau_{\mathrm{m}}\right)(t) d t, \quad k, m=1,2 ; \mathrm{n}=1,2,3 \ldots \tag{4.3.58}
\end{equation*}
$$

Multiplying equations (4.3.56) and (4.3.57) by $W_{1}^{-1}(s) P_{j}^{\left(-\sigma_{1},-\tau_{1}\right)}(s)$ and $W_{2}^{-1}(s) P \int_{3}^{\left(-\sigma_{2},-\tau_{2}\right)}(s)$ respectively, integrating from -1 to 1 , and using the orthogonality relations (4.3.54) we obtain the following algebraic equations for the determination of $C_{\mathrm{km}}$

$$
\begin{align*}
& \frac{\left(1-\lambda^{2}\right)^{\frac{1}{2}}}{2 i} C_{1(\mathrm{j}+1)} \theta_{\mathrm{j}}^{\left(-\sigma_{1},-\tau_{1}\right)}+\sum_{\mathrm{n}=1}^{\infty}\left(L_{1 \mathrm{inj}}^{*} C_{1 \mathrm{n}}+L_{12 \mathrm{nj}}^{*} C_{2 \mathrm{n}}\right)=q_{1 \mathrm{j}},  \tag{4.3.59}\\
& \frac{\left(1-\lambda^{2}\right)^{\frac{1}{2}}}{2 i} C_{2(\mathrm{j}+1)} \theta_{\mathrm{j}}^{\left(-\sigma_{2},-\tau_{2}\right)}-\sum_{\mathrm{n}=1}^{\infty}\left(L_{21 \mathrm{nj}}^{*} C_{1 \mathrm{n}}+L_{22 \mathrm{nj}}^{*} C_{2 \mathrm{n}}\right)=q_{2 \mathrm{j}} \tag{4.3.60}
\end{align*}
$$

where

$$
\begin{align*}
& L_{\mathrm{kmnj}}^{*}=\int_{-1}^{1} L_{\mathrm{kmn}}(s) W_{\mathrm{k}}^{-1}(s) P_{\mathrm{j}}^{\left(-\sigma_{\mathrm{k}},-\tau_{\mathrm{k}}\right)}(s) d s, \\
& q_{\mathrm{kj}}=\int_{-1}^{1} g_{\mathrm{k}}(s) W_{\mathrm{k}}^{-1}(s) P_{\mathrm{j}}^{\left(-\sigma_{\mathrm{k}},-\tau_{\mathrm{k}}\right)}(s) d s, k, m=1,2 ; n, j=1,2,3, \ldots \tag{4.3.61}
\end{align*}
$$

After solving linear equations (4.3.59) and (4.3.60) for the unknowns $C_{\mathrm{kn}}$ , $k=1,2, n, j=1,2,3 \ldots N ;$ we can calculate the stress intensity factors for the crack. The stress intensity factors $K_{1}$ and $K_{2}$ may be calculated as follows:

$$
\begin{align*}
& \frac{\sqrt{\lambda_{2}}}{\lambda_{12}(\infty)} K_{1}+(-1)^{\mathrm{k}+1} i^{\frac{\sqrt{\lambda_{1}}}{\lambda_{21}(\infty)}} K_{2} \\
& =\lim _{x \rightarrow a^{+}}(x-a)^{\sigma_{\mathrm{k}}}(x+a)^{\tau_{\mathrm{k}}}\left[\frac{\sqrt{\lambda_{2}}}{\lambda_{12}(\infty)} \sigma_{\mathrm{yy1}}\left(x, 0^{+}\right)+(-1)^{\mathrm{k}+1} i^{\frac{\sqrt{\lambda_{1}}}{\lambda_{21}(\infty)}} \sigma_{\mathrm{xy1}}\left(x, 0^{+}\right)\right] \tag{4.3.62}
\end{align*}
$$

By making the substitution $s=x / a$, writing $\sigma_{y y 1}\left(x, 0^{+}\right)$and $\sigma_{\mathrm{xyl}}\left(x, 0^{+}\right)$in terms of $\psi_{1}$ and $\psi_{2}$, and using the equations (4.3.52) and (4.3.53) we obtain

$$
\begin{align*}
& \frac{\sqrt{\lambda_{2}}}{\lambda_{12}(\infty)^{2}} K_{1}+(-1)^{\mathrm{k}+1} i_{i} \frac{\sqrt{\lambda_{1}}}{\lambda_{21}(\infty)} K_{2} \\
& =\lim _{s \rightarrow 1^{+}}(s-1)^{-\sigma_{\mathrm{k}}}(s+1)^{-\tau_{\mathrm{k}}(-1)^{\mathrm{k}+1}} \\
& \pi i  \tag{4.3.63}\\
& \int_{-1}^{1} \sum_{\mathrm{n}=1}^{\infty} C_{\mathrm{kn}} P_{\mathrm{n}}^{\left(\sigma_{\mathrm{k}}, \tau_{\mathrm{k}}\right)}(t) \frac{W_{\mathrm{k}}(t)}{t-s} d t \\
& =i(-1)^{k+1} \sum_{\mathrm{n}=1}^{\infty} C_{\mathrm{kn}} P_{\mathrm{n}}^{\left(\sigma_{\mathrm{k}}, \tau_{\mathrm{k}}\right)}(1)
\end{align*}
$$

where we have used the fact that $G_{\mathrm{kn}}^{\infty}(s)$, the principal part of $W_{\mathrm{k}}(s) P_{\mathrm{n}}^{\left(\sigma_{\mathrm{k}}, \tau_{\mathrm{k}}\right)}(s)$, is bounded.

### 4.4. Other cases

In section 4.3, we have considered the crack problem when the two boundaries of the layers $y=-h_{1}$ and $y=h_{2}$ are stress-free. We will study below some other possible boundary conditions.

### 4.4.1 One face fixed and the other stress-free.

In this case, we are assuming that the boundary conditions (4.3.6) in section 4.3 are replaced by

$$
\begin{equation*}
u_{2}\left(x,-h_{1}\right)=0 \quad, \quad v_{2}\left(x,-h_{1}\right)=0 \tag{4.4.1}
\end{equation*}
$$

while other conditions are kept the same. Hence the equations (4.3.11) and (4.3.12) should be replaced by

$$
\begin{align*}
& A_{2} e^{-\beta_{2} \xi \mathrm{~h}_{1}}+B_{2} e^{\beta_{2} \xi \mathrm{~h}_{1}}+C_{2} e^{-\delta_{2} \xi \mathrm{~h}_{1}}+D_{2} e^{\delta_{2} \xi \mathrm{~h}_{1}}=0,  \tag{4.4.2}\\
& A_{2} \alpha_{2} e^{-\beta_{2} \xi \mathrm{~h}_{1}}-B_{2} \alpha_{2} e^{\beta_{2} \xi \mathrm{~h}_{1}}+C_{2} \gamma_{2} e^{-\delta_{2} \xi \mathrm{~h}_{1}}-D_{2} \gamma_{2} e^{\delta_{2} \xi \mathrm{~h}_{1}}=0 \tag{4.4.3}
\end{align*}
$$

Solving equations (4.3.9), (4.3.10), (4.4.2) and (4.4.3) we get the same expressions (4.3.13) for $C_{\mathrm{i}}$ and $D_{\mathbf{i}}(i=1,2)$ with following different values for the coefficients $l_{j k}(j, k=1,2)$

$$
\begin{align*}
& l_{11}=-\left(a_{1} d_{1}+e_{1} b_{1}\right) /\left(2 b_{1} d_{1}\right), \quad l_{12}=-\left(a_{1} d_{1}-e_{1} b_{1}\right) /\left(2 b_{1} d_{1}\right), \\
& l_{21}=-\left(\gamma_{2}+\alpha_{2}\right) /\left(2 \gamma_{2}\right), \quad l_{22}=-\left(\gamma_{2}-\alpha_{2}\right) /\left(2 \gamma_{2}\right) . \tag{4.4.4}
\end{align*}
$$

Hence the solution for this case is given by the results of section 4.3 when $l_{\mathrm{jk}}$ $(j, k=1,2)$ in equations (4.3.15) are replaced by their values in equations (4.4.4).

### 4.4.2 Both faces fixed

In this case, we are assuming that the boundary conditions (4.3.6) and (4.3.5) in section 4.3 are replaced respectively by conditions (4.4.1) and the following two conditions

$$
\begin{equation*}
u_{1}\left(x, h_{2}\right)=0 \quad, \quad v_{1}\left(x, h_{2}\right)=0 \tag{4.4.5}
\end{equation*}
$$

while other conditions are kept the same. So besides the replacement of equations (4.3.11) and (4.3.12) by equations (4.4.2) and (4.4.3), we should replace equations (4.3.9) and (4.3.10) by the following two equations

$$
\begin{align*}
& A_{1} e^{\beta_{1} \xi \mathrm{~h}_{2}}+B_{1} e^{-\beta_{1} \xi \mathrm{~h}_{2}}+C_{1} e^{\delta_{1} \xi \mathrm{~h}_{2}}+D_{1} e^{-\delta_{1} \xi \mathrm{~h}_{2}}=0,  \tag{4.4.6}\\
& A_{1} a_{1} e^{\beta_{1} \xi \mathrm{~h}_{2}}-B_{1} a_{1} e^{-\beta_{1} \xi \mathrm{~h}_{2}}+C_{1} \gamma_{1} e^{\delta_{1} \xi \mathrm{~h}_{2}}-D_{1} \gamma_{1} e^{-\delta_{1} \xi \mathrm{~h}_{2}}=0 . \tag{4.4.7}
\end{align*}
$$

Solving equations (4.4.2), (4.4.3), (4.4.6) and (4.4.7) we find that $C_{\mathrm{j}}$ and $D_{\mathrm{j}}$ $(j=1,2)$ are given by equations (4.3.13) as before but with the following values
for the coefficients $l_{\mathrm{jk}}(j, k=1,2)$ :

$$
\begin{array}{ll}
l_{11}=-\left(\gamma_{1}+a_{1}\right) /\left(2 \gamma_{1}\right), & l_{12}=-\left(\gamma_{1}-a_{1}\right) /\left(2 \gamma_{1}\right), \\
l_{21}=-\left(\gamma_{2}+a_{2}\right) /\left(2 \gamma_{2}\right), & l_{22}=-\left(\gamma_{2}-a_{2}\right) /\left(2 \gamma_{2}\right) . \tag{4.4.8}
\end{array}
$$

Again the solution for this case is given by the results of section 4.3 when $l_{\mathrm{jk}}$ ( $j, k=1,2$ ) in equations (4.3.15) are replaced by their values in equations (4.4.8).

### 4.4.3 One face rigidly restrained and the other fixed

In this case, we are assuming that the boundary conditions (4.3.5) and (4.3.6) of section 4.3 are replaced by the following boundary conditions

$$
\begin{array}{ll}
v_{1}\left(x, h_{2}\right)=0, & \sigma_{\mathrm{xy1}}\left(x, h_{2}\right)=0 ; \\
u_{2}\left(x,-h_{1}\right)=0, & v_{2}\left(x,-h_{1}\right)=0 \tag{4.4.10}
\end{array}
$$

while other conditions are kept the same. So the equations (4.3.9), and (4.3.12) in section 4.3 will be replaced by (4.4.7), (4.4.2) and (4.4.3). Solving equations (4.3.7), (4.3.8), (4.4.7), (4.3.10), (4.4.2) and (4.4.3) we can express $B_{1}, D_{1}, A_{2}, B_{2}, C_{2}$ and $D_{2}$ in terms of $A_{1}$ and $C_{1}$ by

$$
\begin{align*}
& B_{1}=A_{1} e^{2 \beta_{1} \xi \mathrm{~h}_{2}}, \quad D_{1}=C_{1} e^{2 \delta_{1} \xi \mathrm{~h}_{2}}, \\
& C_{2}=l_{21} A_{2} e^{\left(-\beta_{2}+\delta_{2}\right) \xi \mathrm{h}_{1}}+l_{22} B_{2} e^{\left(\beta_{2}+\delta_{2}\right) \xi h_{1}}, \\
& D_{2}=l_{22} A_{2} e^{-\left(\beta_{2}+\delta_{2}\right) \xi \mathrm{h}_{1}}+l_{21} B_{2} e^{\left(\beta_{2}-\delta_{2}\right) \xi \mathrm{h}_{1}}, \\
& A_{2}=l_{31} A_{1}+l_{32} C_{1}, \quad B_{2}=l_{41} A_{1}+l_{42} C_{1}, \tag{4.4.11}
\end{align*}
$$

where $l_{21}$ and $l_{22}$ are given by (4.4.4) but $l_{j k}(j=3,4 ; k=1,2$.) are given by (4.3.15) while the expression for $a_{\mathrm{jk}}$ and $b_{\mathrm{jk}}(j, k=1,2)$ are given by the
following:

$$
\begin{array}{lc}
a_{11}=a_{1}\left(1+e^{2 \beta_{1} \xi \mathrm{~h}_{2}}\right), & a_{12}=b_{1}\left(1+e^{2 \delta_{1} \xi \mathrm{~h}_{2}}\right), \\
a_{21}=a_{2}+b_{2} l_{21} e^{\left(-\beta_{2}+\delta_{2}\right) \xi \mathrm{h}_{1}}, & a_{22}=a_{2}+b_{2} l_{22} e^{\left(\beta_{2}+\delta_{2}\right) \xi \mathrm{h}_{1}}, \\
b_{11}=e_{1}\left(1-e^{2 \beta_{1} \xi \mathrm{~h}_{2}}\right), & b_{12}=d_{1}\left(1-e^{2 \delta_{1} \xi \mathrm{~h}_{2}}\right), \\
b_{21}=e_{2}+d_{2} l_{21} e^{\left(-\beta_{2}+\delta \delta_{2}\right) \xi \mathrm{h}_{1}}, & b_{22}=-e_{2}-d_{2} l_{22} e^{\left(\beta_{2}+\delta_{2}\right) \xi \mathrm{h}_{1}} . \tag{4.4.12}
\end{array}
$$

Consequently the expression for $\gamma_{j k}(j, k=1,2)$ for this case are given by

$$
\begin{align*}
& \gamma_{11}=\alpha_{1}-\alpha_{1} e^{2 \beta_{1} \xi \mathrm{~h}_{2}}-\alpha_{2} l_{31}+\alpha_{2} l_{41}-\gamma_{2} l_{51}+\gamma_{2} l_{61} \\
& \gamma_{12}=\gamma_{1}-\gamma_{1} e^{2 \delta_{1} \xi \mathrm{~h}_{2}}-\alpha_{2} l_{32}+\alpha_{2} l_{42}-\gamma_{2} l_{52}+\gamma_{2} l_{62} \\
& \gamma_{21}=1+e^{2 \beta_{1} \xi \mathrm{~h}_{2}-l_{31}-l_{41}-l_{51}-l_{61}} \\
& \gamma_{22}=1+e^{2 \delta_{1} \xi \mathrm{~h}_{2}-l_{32}-l_{42}-l_{52}-l_{62}} \tag{4.4.13}
\end{align*}
$$

where $l_{j k}(j=5,6 ; k=1,2)$ are the same as given in (4.3.28). The solution for this case is given by the results of section 4.3 after we have made above raplacements.

### 4.4.4 One face rigidly restrained and the other stress-free

In this case, the boundary conditions (4.3.5) are replaced by (4.4.9) and all the other boundary conditions remain the same as in section 4.3. So in equations (4.3.7) to (4.3.12) we only have to replace equation (4.3.9) by (4.4.7). Now we find that for this case $l_{j k}(j=2,3,4 ; k=1,2)$ are all given by (4.3.15) and $a_{j k}, b_{j k}(j, k=1,2)$ are given by (4.4.12) and the solution for this case is given by the results of section 4.3 after we have made above modifications.

### 4.4.5 Both faces rigidly restrained

In this case, we are assuming that the boundary conditions (4.3.5) and (4.3.6) are respectively replaced by conditions (4.4.9) and

$$
\begin{equation*}
v_{2}\left(x,-h_{1}\right)=0, \quad \sigma_{\mathrm{xy} 2}\left(x,-h_{1}\right)=0 \tag{4.4.14}
\end{equation*}
$$

while all other conditions remain the same as in section 4.3. The equations (4.3.9) and (4.3.11) will be replaced by equations (4.4.7) and (4.4.3) while all other equations remain the same. Solving equations (4.3.7), (4.3.8), (4.4.8), (4.3.10), (4.4.3) and (4.3.12) we find that

$$
\begin{align*}
& B_{1}=A_{1} e^{2 \beta \beta_{1} \xi h_{2}}, \quad D_{1}=C_{1} e^{2 \delta_{1} \xi \mathrm{~h}_{2}} \\
& B_{2}=A_{2} e^{-2 \beta \beta_{2} \xi \mathrm{~h}_{1}}, \quad D_{2}=C_{2} e^{-2 \delta_{2} \xi \mathrm{~h}_{1}} \\
& A_{2}=l_{31} A_{1}+l_{32} C_{1}, \quad C_{2}=l_{41} A_{1}+l_{42} C_{1} \tag{4.4.15}
\end{align*}
$$

where $l_{j k}\left(j=3,4 ; k=1,2\right.$.) are given by (4.3.15) but the expression for $a_{j k}$ and $b_{\mathrm{jk}}(j, k=1,2)$ are given by the following

$$
\begin{array}{ll}
a_{11}=a_{1}\left(1+e^{2 \beta_{1} \xi \mathrm{~h}_{2}}\right), & a_{12}=b_{1}\left(1+e^{2 \delta_{1} \xi \mathrm{~h}_{2}}\right), \\
a_{21}=a_{2}\left(1+e^{-2 \beta_{2} \xi \mathrm{~h}_{1}}\right), & a_{22}=b_{2}\left(1+e^{-2 \delta_{2} \xi \mathrm{~h}_{1}}\right), \\
b_{11}=e_{1}\left(1-e^{2 \beta \beta_{1} \xi \mathrm{~h}_{2}}\right), & b_{12}=d_{1}\left(1-e^{2 \delta_{1} \xi \mathrm{~h}_{2}}\right) \\
b_{21}=e_{2}\left(1-e^{-2 \beta \beta_{2} \xi \mathrm{~h}_{1}}\right), & b_{22}=d_{2}\left(1-e^{-2 \delta_{2} \xi \mathrm{~h}_{1}}\right) \tag{4.4.16}
\end{array}
$$

Consequently for this case $\gamma_{\mathrm{jk}}(j, k=1,2)$ are given by

$$
\begin{align*}
& \gamma_{11}=\alpha_{1}\left(1-e^{2 \beta_{1} \xi \mathrm{~h}_{2}}\right)-\alpha_{2} l_{31}\left(1-e^{-2 \beta_{2} \xi \mathrm{~h}_{1}}\right)-\gamma_{2} l_{41}\left(1-e^{-2 \delta_{2} \xi \mathrm{~h}_{1}}\right), \\
& \gamma_{12}=\gamma_{1}\left(1-e^{2 \delta_{1} \xi \mathrm{~h}_{2}}\right)-\alpha_{2} l_{32}\left(1-e^{-2 \delta_{2} \xi \mathrm{~h}_{1}}\right)-\gamma_{2} l_{42}\left(1-e^{-2 \delta_{2} \xi \mathrm{~h}_{1}}\right), \\
& \gamma_{21}=1+e^{2 \beta_{1} \xi \mathrm{~h}_{2}}-l_{31}\left(1+e^{-2 \beta_{2} \xi \mathrm{~h}_{1}}\right)-l_{41}\left(1+e^{-2 \delta_{2} \xi \mathrm{~h}_{1}}\right), \\
& \gamma_{22}=1+e^{2 \delta_{1} \xi \mathrm{~h}_{2}}-l_{32}\left(1+e^{-2 \delta_{2} \xi \mathrm{~h}_{1}}\right)-l_{42}\left(1+e^{-2 \delta_{2} \xi \mathrm{~h}_{1}}\right) . \tag{4.4.17}
\end{align*}
$$

Then the solution for this case is given by the results of section 4.3.

### 4.5. Numerical results and discussion

To evaluate the stress intensity factors, we truncate the infinite system of simultaneous algebraic equations (4.3.59) and (4.3.60) at $n=10$ and the Crout's factorisation method is used to solve these equations. And the Gaussian quadrature formula is used to perform the numerical integrations involved in the solution. The relative error is controlled under 0.01 . Numerical results for the stress intensity factors $K_{1}$ and $K_{2}$ are obtained for the case when the crack is subjected to a constant pressure $p_{1}(x)=p_{0}$ and $p_{2}(x)=0$, the thickness of the layers is the same (i.e, $h_{1}=h_{2}=h$ ) and the surfaces $y=-h$ and $y=h$ are stress-free. For the two orthotropic elastic materials considered here , the elastic moduli are the following: $\left(10^{11}\right.$ dynes $\left./ \mathrm{cm}^{2}\right)[5]$

|  | $c_{11}$ | $c_{12}$ | $c_{22}$ | $c_{66}$ |  |
| :--- | :--- | :--- | :--- | :--- | :--- |
| Beechwood | 0.170 | 0.150 | 1.580 | 0.103 | (layer 1) |
| $\alpha$-Uranium | 21.47 | 4.05 | 19.86 | 7.43 | (layer 2) |

Numerical values of the stress intensity factors have been calculated for following four particular cases:

Case 1. Let the length of the crack $a=1.0$ and let $h_{1} / a=h_{2} / a=h / a=$ $0.2(0.2) 1.0,2.0(2.0) 10.0$; the numerical values of the stress intensity factor $K_{1}$ against $h / a$ are displayed in Fig.4.5.1 and Fig.4.5.2, and the numerical values of the stress intensity factor $K_{2}$ against $h / a$ are displayed in Fig.4.5.3 and Fig.4.5.4.

Case 2. Let $h_{1}=h_{2} \rightarrow \infty$, and the length of the crack $a=1.0(1.0) 10.0$; the numerical values of the stress intensity factor $K_{1}$ against $a$ are displayed in Fig.4.5.5, and the numerical values of the stress intensity factor $K_{2}$ against $a$ are displayed in Fig.4.5.6.

Case 3. Let $h_{1}=h_{2}=20.0$ and the length of the crack $a=1.0(1.0) 10.0$; the numerical values of the stress intensity factor $K_{1}$ against $a$ are displayed in Fig.4.5.7, and the numerical valued of the stress intensity factor $K_{2}$ against $a$ are displayed in Fig.4.5.8.

Case 4. Let $h_{1}=h_{2}=10.0$ and the length of the crack $a=1.0(1.0) 10.0$; the numerical values of the stress intensity factor $K_{1}$ against $a$ are displayed in Fig.4.5.9, and the numerical values of stress intensity factor $K_{2}$ against $a$ are displayed in Fig.4.5.10.


Fig.4.5.1
Numerical values of the stress intensity factor $K_{1}$ against $h / a$ ( 0.2 to 1.0 ) for a fixed crack length $a$ and equal layer thickness ( $h_{1}=h_{2}=h$ ).


Fig.4.5.2
Numerical values of the stress intensity factor $K_{1}$ against $h / a$ (1.0 to 10.0 ) for a fixed crack length $a$ and equal layer thickness ( $h_{1}=h_{2}=h$ ).


Fig.4.5.3
Numerical values of the stress intensity factor $K_{2}$ against $h / a$ (0.2 to 1.0) for a fixed crack length $a$ and equal layer thickness ( $h_{1}=h_{2}=h$ ).


Fig.4.5.4
Numerical values of the stress intensity factor $K_{2}$ against $h / a$ (1.0 to 10.0) for a fixed crack length $a$ and equal layer thickness ( $h_{1}=h_{2}=h$ ).


Fig.4.5.5
Numerical values of the stress intensity factor $K_{1}$ against the crack length $a$ (1.0 to 10.0) for the layer thickness $h_{1}=h_{2} \rightarrow \infty$.


Fig.4.5.6
Numerical values of the stress intensity factor $K_{2}$ against the crack length $a(1.0$ to 10.0 ) for the layer thickness $h_{1}=h_{2} \rightarrow \infty$.


Fig.4.5.7
Numerical values of the stress intensity factor $K_{1}$ against the crack length $a$ (1.0 to 10.0 ) for the layer thickness $h_{1}=h_{2}=20$.


Fig.4.5.8
Numerical values of the stress intensity factor $K_{2}$ against the crack length $a$ (1.0 to 10.0 ) for the layer thickness $h_{1}=h_{2}=20$.


Fig.4.5.9
Numerical values of the stress intensity factor $K_{1}$ against the crack length $a$ ( 1.0 to 10.0 ) for the layer thickness $h_{1}=h_{2}=10$.


Fig.4.5.10
Numerical values of the stress intensity factor $K_{2}$ against the crack length $a$ ( 1.0 to 10.0 ) for the layer thickness $h_{1}=h_{2}=10$.

We make the following observations from the graphs : when the length of the crack $a$ is fixed at $a=1.0$, the stress intensity factor $K_{2}$ increases and $K_{1}$ decreases as the thickness of the layers increases from 0.2 to 1.0 while the stress intensity factor $K_{2}$ decreases and $K_{1}$ increases as the thickness of the layers increases from 1.0 to 10.0 ; and if the thickness of the layers is fixed (at $10.0,20.0$ or $\rightarrow \infty$ ) the stress intensity factor $K_{1}$ decreases and $K_{2}$ increases as the length of the crack increases.

## CHAPTER 5

# PENNY-SHAPED INTERFACE CRACK BETWEEN TWO DISSIMILAR TRANSVERSELY ISOTROPIC LAYERS 

### 5.1 Introduction

The study of internal penny-shaped cracks is of practical importance in stress analysis, since it represents an idealization of the shape of internal flaws that are inherent in many engineering materials. The formulation of this class of boundary value problems can be expressed most conveniently in terms of the cylindrical polar coordinates $(r, \theta, z)$. A crack lying in the $r \theta$-plane and occupying the region

$$
r \leq c, \quad z=0
$$

is called a penny-shaped crack.
In 1946 Sack [71] considered a penny-shaped crack in a three dimensional elastic space, he treated it as a limiting case of an ellipsoidal crack. It was Sneddon [72] who successfully introduced the application of Hankel transforms to solve a penny-shaped crack problem for an elastic solid when the surface of crack was under constant pressure. Green [73] solved the same problem by potential function methods. Collins [74] considered the case in which the surface of the crack was subjected to a variable pressure. Using Hankel transforms Muki [75] solved the problem of a penny-shaped crack under shear, and Sneddon [43] solved the problem of penny-shaped crack under torsion.

Some researchers have considered the dynamical problems concerning penny-shaped cracks. Craggs [76] and Atkinson [77] considered the expanding penny-shaped crack problem; the response of a penny-shaped crack to a loading in the form of a plane harmonic dilatational wave propagating along the axis of the crack was discussed by Mal [78]; the response to an incident plane harmonic shear wave polarized in a plane normal to the plane of the crack and propagating along the axis of the crack was considered by Mal [79].

Olesiak and Sneddon [80] discussed the distribution of thermal stresses in the vicinity of a penny-shaped crack by assuming that the thermal conditions on the upper surface of the crack were identical with those on the lower surface of the crack.

The distribution of stress in the vicinity of a penny-shaped crack in an elastic plate of finite thickness but infinite radius was discussed by Lowengrub [81], where the crack was taken to lie in the central plane of the plate with its surfaces parallel to those of the plate. Later Sneddon and Tait [82] and Sneddon and Welch [83] investigated the distribution of stress in a long circular cylinder $0 \leq r \leq a,-\infty<z<\infty$, containing a penny-shaped crack lying in the plane $z=0$ and the cylinder being under tension.

Many engineering structures are made by bonding together two or more materials with different elastic properties. The dissimilar material system is required to act as a single unit such that the loads are transmitted from one material to the next through the interfaces. The interface in bonded dissimilar materials often contains some flaws, such as cracks or hard inclusions, that may be induced during the process of joining the materials. These flaws, generally, form the nucleus of fracture initiation and propagation in the medium. The presence of flaws or cracks at the interface could cause high elevation of local stresses and lead to failure if the crack reaches a critical
size.
Mossakovkii and Rybka [84] introduced a way of formulating the axially symmetric penny-shaped crack problem, when the crack is located at the interface of two dissimilar isotropic materials. Willis [85] considered the problem of obtaining the stress intensity factor for a penny-shaped crack between two dissimilar materials. Erdogan [86] solved the interface penny-shaped crack problem by reducing it to a singular integral equation. Erdogan and Arin [87] considered a penny-shaped crack between an elastic layer and a half-space.

Elliott [88, 89] considered a penny-shaped crack in a transversely isotropic elastic solid. Kassir and Sih [90] investigated an elliptical crack problem in a transversely isotropic elastic solid, and Parhi and Atsumi [91] discussed the distribution of stress in a transversely isotropic elastic cylinder containing a penny-shaped crack. Recently, Saxena and Dhaliwal [92] considered a penny-shaped crack problem at the interface of two transversely isotropic half-spaces.

In this chapter we consider the penny-shaped interface crack between two dissimilar transversely isotropic elastic layers. By means of Hankel transforms and Fourier transforms the problem is reduced to the solution of a system of singular integral equations. These equations are further reduced to a system of simultaneous algebraic equations by using Jacobi polynomials approximation. Numerical methods are employed to determine the stress intensity factors, which have been displayed graphically.

### 5.2. Basic equations and their solution

As discussed in chapter 1 , under the assumptions of axially-symmetric deformations in cylindrical polar coordinates ( $r, \theta, z$ ), when the $z$-axis is the axis of anisotropy of transversely isotropic medium, the displacement components are defined by

$$
\begin{equation*}
u_{\mathrm{r}}=u_{\mathrm{r}}(r, z), \quad u_{\mathrm{z}}=u_{\mathrm{z}}(r, z), \quad u_{\theta}=0 \tag{5.2.1}
\end{equation*}
$$

along the $r-, z-$, and $\theta-$ directions respectively. The stress-displacement relations are given by

$$
\begin{align*}
& \sigma_{\mathrm{rr}}=c_{11} \frac{\partial u_{\mathrm{r}}}{\partial r}+c_{12} \frac{u_{\mathrm{r}}}{r}+c_{13} \frac{\partial u_{z}}{\partial z}, \\
& \sigma_{\theta \theta}=c_{12} \frac{\partial u_{\mathrm{r}}}{\partial r}+c_{11} \frac{u_{\mathrm{r}}}{r}+c_{13} \frac{\partial u_{z}}{\partial z}, \\
& \sigma_{z z}=c_{13}\left[\frac{\partial u_{\mathrm{r}}}{\partial r}+\frac{u_{\mathrm{r}}}{r}\right]+c_{33} \frac{\partial u_{z}}{\partial z}, \\
& \sigma_{\mathrm{rz}}=c_{44}\left[\frac{\partial u_{\mathrm{r}}}{\partial z}+\frac{\partial u_{z}}{\partial r}\right], \tag{5.2.2}
\end{align*}
$$

where $c_{\mathrm{ij}}$ 's are the elastic moduli of the transversely isotropic medium. In the absence of body forces, the equations of equilibrium may be written as follows

$$
\begin{align*}
& c_{11}\left[\frac{\partial^{2} u_{5}}{\partial r^{2}}+\frac{1}{r} \frac{\partial u_{r}}{\partial r}-\frac{u_{r}}{r^{2}}\right]+c_{44} \frac{\partial^{2} u_{r}}{\partial z^{2}}+\left(c_{13}+c_{44} \frac{\partial^{2} u_{z}}{\partial r \partial z}=0,\right.  \tag{5.2.3a}\\
& c_{44}\left[\frac{\partial^{2} u_{z}}{\partial r^{2}}+\frac{1}{r} \frac{\partial u_{z}}{\partial r}\right]+c_{33} \frac{\partial^{2} u_{z}}{\partial z^{2}}+\left(c_{13}+c_{44} \frac{\partial}{\partial z}\left[\frac{\partial u_{r}}{\partial r}+\frac{u_{r}}{r}\right]=0 .\right. \tag{5.2.3b}
\end{align*}
$$

Multiplying equations (5.2.3a) and (5.2.3b) by $r J_{1}(\xi r)$ and $r J_{0}(\xi r)$ respectively, then integrating with respect to $r$ from $0 \rightarrow \infty$, we get the following equations

$$
\begin{align*}
& -c_{11} \xi^{2} \tilde{u}_{\mathrm{r}}+c_{44} \frac{\partial^{2}}{\partial z^{2}} \tilde{u}_{\mathrm{r}}-\left(c_{13}+c_{44}\right) \xi \frac{\partial}{\partial z} \tilde{u}_{\mathrm{z}}=0 \\
& -c_{44} \xi^{2} \tilde{u}_{\mathrm{z}}+c_{33} \frac{\partial^{2}}{\partial z^{2}} \tilde{u}_{\mathrm{z}}+\left(c_{13}+c_{44}\right) \xi \frac{\partial}{\partial z} \tilde{u}_{\mathrm{r}}=0 \tag{5.2.4}
\end{align*}
$$

where $\tilde{u}_{r}$ and $\tilde{u}_{z}$ are the Hankel transforms of $u_{r}$ and $u_{z}$ of order 1 and 0 respectively.

Let $c_{\mathrm{ijk}}$ 's be the elastic moduli for the transversely isotropic medium in the region $R_{\mathrm{k}}(k=1,2)$ where $R_{1}$ is $0 \leq z \leq h_{2}$ and $R_{2}$ is $-h_{1} \leq z \leq 0$. Solving equations (5.2.4) for $\tilde{u}_{T}$ and $\tilde{u}_{\mathrm{z}}$ then taking the inverse Hankel transforms we obtain the following solution :

$$
\begin{align*}
u_{\mathrm{r}_{\mathrm{j}}}(r, z) & =\mathscr{H}_{\mathrm{b}}\left[\xi^{-1}\left\{A_{\mathrm{j}} e^{\beta_{\mathrm{j}} \xi \mathrm{z}}+B_{\mathrm{j}} e^{-\beta_{\mathrm{j}} \xi \mathrm{z}}+C_{\mathrm{j}} e^{\delta_{\mathrm{j}} \xi \mathrm{z}}+D_{\mathrm{j}} e^{-\delta_{\mathrm{j}} \xi \mathrm{z}}\right\} ; \xi \rightarrow r\right] \\
u_{z_{\mathrm{j}}}(r, z) & =\mathscr{H}_{0}\left[\xi^{-1}\left\{A_{\mathrm{j}} a_{\mathrm{j}} e^{\beta_{\mathrm{j}} \xi \mathrm{z}}-B_{\mathrm{j}} a_{\mathrm{j}} e^{-\beta_{\mathrm{j}} \xi \mathrm{z}}+C_{\mathrm{j}} \gamma_{\mathrm{j}} e^{\delta_{\mathrm{j}} \xi \mathrm{z}}-D_{\mathrm{j}} \gamma_{\mathrm{j}} e^{-\delta_{\mathrm{j}} \xi \mathrm{z}}\right\} ; \xi \rightarrow r\right] \tag{5.2.5}
\end{align*}
$$

where $u_{\mathrm{rj}}$ and $u_{\mathrm{zj}}$ denote the displacements for the region $R_{\mathrm{j}}(j=1,2)$,

$$
\begin{align*}
& a_{\mathrm{j}}=\frac{c_{44 \mathrm{j}} \beta_{\mathrm{j}}-c_{11 \mathrm{j}} \beta_{\mathrm{j}}^{-1}}{c_{13 \mathrm{j}}+c_{44 \mathrm{j}}}, \quad \gamma_{\mathrm{j}}=\frac{c_{44 \mathrm{j}} \delta_{\mathrm{j}}-c_{11 \mathrm{j}} \delta_{\mathrm{j}}{ }^{1}}{c_{13 \mathrm{j}}+c_{44 \mathrm{j}}}, \\
& \left(\beta_{\mathrm{j}}, \delta_{\mathrm{j}}\right)=\left\{\frac{\eta_{\mathrm{j}} \pm \sqrt{\eta_{\mathrm{j}}^{2}-4 \cdot c_{11 \mathrm{j}} \cdot c_{33 \mathrm{j}} \cdot c_{44 \mathrm{j}}^{2}}}{2 \cdot c_{33 \mathrm{j}} \cdot c_{44 \mathrm{j}}}\right\}^{\frac{1}{2}}, \\
& \eta_{\mathrm{j}}=c_{11 \mathrm{j} \cdot} \cdot c_{33 \mathrm{j}}-2 \cdot c_{13 \mathrm{j}} \cdot c_{44 \mathrm{j}}-c_{13 \mathrm{j}}^{2}, \tag{5.2.6}
\end{align*}
$$

and $A_{\mathrm{j}}, B_{\mathrm{j}}, C_{\mathrm{j}}$ and $D_{\mathrm{j}}(j=1,2)$ are unknown functions of $\xi$ to be determined by the boundary and continuity conditions.

Substituting from equation (5.2.5) into (5.2.2), we have

$$
\begin{align*}
\sigma_{z z \mathrm{j}}= & \mathscr{\mathscr { R } _ { 0 } [ \{ ( c _ { 1 3 \mathrm { j } } + c _ { 3 3 \mathrm { j } } \alpha _ { \mathrm { j } } \beta _ { \mathrm { j } } ) ( A _ { \mathrm { j } } e ^ { \beta _ { \mathrm { j } } \xi z } + B _ { \mathrm { j } } e ^ { - \beta _ { \mathrm { j } } \xi \mathrm { z } } )} \\
& \left.\left.+\left(c_{13 \mathrm{j}}+\gamma_{\mathrm{j}} \delta_{\mathrm{j}} c_{33 \mathrm{j}}\right)\left(C_{\mathrm{j}} e^{\delta_{\mathrm{j}} \xi \mathrm{z}}+D_{\mathrm{j}} e^{-\delta_{\mathrm{j}} \xi \mathrm{z}}\right)\right\} ; \xi \rightarrow r\right] \tag{5.2.7a}
\end{align*}
$$

$$
\begin{align*}
\sigma_{\mathrm{rzj}}= & c_{44 \mathrm{j}} \mathscr{H}_{\mathrm{B}}\left[\left\{\left(\beta_{\mathrm{j}}-a_{\mathrm{j}}\right)\left(A_{\mathrm{j}} e^{\beta_{\mathrm{j}} \xi \mathrm{z}}-B_{\mathrm{j}} e^{-\beta_{\mathrm{j}} \xi \mathrm{z}}\right)\right.\right. \\
& \left.\left.+\left(\delta_{\mathrm{i}}-\gamma_{\mathrm{i}}\right)\left(C_{\mathrm{i}} e^{\delta_{\mathrm{i}} \xi \mathrm{z}}-D_{\mathrm{i}} e^{-\delta_{\mathrm{i}} \xi \mathrm{z}}\right)\right\} ; \xi \rightarrow r\right] \tag{5.2.7b}
\end{align*}
$$

where $\sigma_{z z j}$ and $\sigma_{\mathrm{rzj}}$ denote stresses for the region $R_{\mathrm{j}}(j=1,2)$.

### 5.3. Statment of the problem and boundary conditions

We assume that two dissimilar transversely isotropic elastic layers, which occupy the regions $R_{1}\left(0 \leq z \leq h_{2}\right)$ and $R_{2}\left(-h_{1} \leq z \leq 0\right)$ respectively, are perfectly bonded except that there is a crack in the region $0 \leq r \leq a, z=0$. In the first case we also assume that the surfaces $z=-\dot{h}_{1}$ and $z=h_{2}$ are stress-free.(see Fig.5.3.1) It is assumed that the surfaces of crack are subjected to prescribed normal and shear stresses $p_{1}(r)$ and $p_{2}(r)$ respectively. Hence the problem of determining the stress and displacement field is subjected to the following boundary and continuity conditions:

$$
\begin{array}{ll}
\sigma_{\mathrm{zzj}}(r, 0)=p_{1}(r), \quad \sigma_{\mathrm{rzj}}(r, 0)=p_{2}(r), \quad j=1,2 ; & 0 \leq r \leq a, \\
u_{\mathrm{r} 1}\left(r, 0^{+}\right)=u_{\mathrm{r} 2}\left(r, 0^{-}\right), \quad u_{\mathrm{z} 1}\left(r, 0^{+}\right)=u_{\mathrm{z} 2}\left(r, 0^{-}\right), & r \geq a \\
\sigma_{\mathrm{zz} 1}\left(r, 0^{+}\right)=\sigma_{\mathrm{zz} 2}\left(r, 0^{-}\right), \quad \sigma_{\mathrm{rz} 1}\left(r, 0^{+}\right)=\sigma_{\mathrm{rz} 2}\left(r, 0^{-}\right), & r \geq a \\
\sigma_{\mathrm{zz} 1}\left(r, h_{2}\right)=0, \quad \sigma_{\mathrm{rz} 1}\left(r, h_{2}\right)=0, & r \geq 0, \\
\sigma_{z z 2}\left(r,-h_{1}\right)=0, \quad \sigma_{\mathrm{rz2}}\left(r,-h_{1}\right)=0, & r \geq 0 \tag{5.3.5}
\end{array}
$$

With the help of conditions (5.3.1), the conditions (5.3.3) may be replaced by the following:

$$
\begin{equation*}
\sigma_{\mathrm{zz} 1}\left(r, 0^{+}\right)=\sigma_{\mathrm{zz} 2}\left(r, 0^{-}\right), \quad \sigma_{\mathrm{rz} 1}\left(r, 0^{+}\right)=\sigma_{\mathrm{rz} 2}\left(r, 0^{-}\right), \quad r \geq 0 \tag{5.3.6}
\end{equation*}
$$



Fig.5.3.1
Penny-shaped interface crack between two dissimilar transversely isotropic layers.

### 5.4. Analysis

If we denote

$$
\begin{align*}
& a_{\mathrm{j}}=c_{13 \mathrm{j}}+a_{\mathrm{j}} \beta_{\mathrm{j}} c_{33 \mathrm{j}}, \quad b_{\mathrm{j}}=c_{44 \mathrm{j}}\left(\beta_{\mathrm{j}}-a_{\mathrm{j}}\right), \\
& d_{\mathrm{j}}=c_{13 \mathrm{j}}+\delta_{\mathrm{j}} \gamma_{\mathrm{j}} c_{33 \mathrm{j}}, \quad e_{\mathrm{j}}=c_{44 \mathrm{j}}\left(\delta_{\mathrm{j}}-\gamma_{\mathrm{j}}\right), \quad j=1,2 \tag{5.4.1}
\end{align*}
$$

then applying conditions (5.3.6), the equations (5.2.7) give

$$
\begin{align*}
& a_{1}\left(A_{1}+B_{1}\right)+d_{1}\left(C_{1}+D_{1}\right)=a_{2}\left(A_{2}+B_{2}\right)+d_{2}\left(C_{2}+D_{2}\right)  \tag{5.4.2a}\\
& b_{1}\left(A_{1}-B_{1}\right)+e_{1}\left(C_{1}-D_{1}\right)=b_{2}\left(A_{2}-B_{2}\right)+e_{2}\left(C_{2}-D_{2}\right) \tag{5.4.2b}
\end{align*}
$$

and the conditions (5.3.4) and (5.3.5) yield

$$
\begin{align*}
& a_{1} A_{1} e^{\beta_{1} \xi h_{2}}+a_{1} B_{1} e^{-\beta_{1} \xi h_{2}}+d_{1} C_{1} e^{\delta \xi h_{2}}+d_{1} D_{1} e^{-\delta_{1} \xi h_{2}}=0  \tag{5.4.3a}\\
& b_{1} A_{1} e^{\beta_{1} \xi h_{2}}-b_{1} B_{1} e^{-\beta_{1} \xi h_{2}}+e_{1} C_{1} e^{\delta \xi h_{2}}-e_{1} D_{1} e^{-\delta_{1} \xi h_{2}}=0  \tag{5.4.3b}\\
& a_{2} A_{2} e^{-\beta_{2} \xi h_{1}}+a_{2} B_{2} e^{\beta_{2} \xi h_{1}}+d_{2} C_{2} e^{-\delta_{2} \xi \mathrm{~h}_{1}}+d_{2} D_{2} e^{\delta \xi_{2} h_{1}}=0  \tag{5.4.3c}\\
& b_{2} A_{2} e^{-\beta_{2} \xi \mathrm{~h}_{1}}-b_{2} B_{2} e^{\beta_{2} \xi \mathrm{~h}_{1}}+e_{2} C_{2} e^{-\delta_{2} \xi \mathrm{~h}_{1}}-e_{2} D_{2} e^{\delta_{2} \xi \mathrm{~h}_{1}}=0 \tag{5.4.3d}
\end{align*}
$$

Solving the equations (5.4.2) and (5.4.3) we obtain

$$
\begin{align*}
& C_{1}=-l_{11} A_{1} e^{\left(\beta_{1}-\delta_{1}\right) \xi \mathrm{h}_{2}}-l_{12} B_{1} e^{-\left(\beta_{1}+\delta_{1}\right) \xi \mathrm{h}_{2}} \\
& D_{1}=-l_{12} A_{1} e^{\left(\beta_{1}+\delta_{1}\right) \xi \mathrm{h}_{2}}-l_{11} B_{1} e^{\left(-\beta_{1}+\delta_{1}\right) \xi \mathrm{h}_{2}} \\
& C_{2}=-l_{21} A_{2} e^{\left(-\beta_{2}+\delta_{2}\right) \xi \mathrm{h}_{1}}-l_{22} B_{2} e^{\left(\beta_{2}+\delta_{2}\right) \xi \mathrm{h}_{1}} \\
& D_{2}=-l_{22} A_{2} e^{-\left(\beta_{2}+\delta_{2}\right) \xi \mathrm{h}_{1}}-l_{21} B_{2} e^{\left(\beta_{2}-\delta_{2}\right) \xi \mathrm{h}_{1}} \\
& A_{2}=l_{31} A_{1}+l_{32} B_{1}, \quad B_{2}=l_{41} A_{1}+l_{42} B_{1} \tag{5.4.4}
\end{align*}
$$

where

$$
\begin{align*}
& l_{1 \mathrm{j}}=\left[a_{1} e_{1}-(-1)^{\mathrm{j}} b_{1} d_{1}\right] /\left(2 d_{1} e_{1}\right), \quad l_{2 \mathrm{j}}=\left[a_{2} e_{2}-(-1)^{\mathrm{j}} b_{2} d_{2}\right] /\left(2 d_{2} e_{2}\right) ; j=1,2 \\
& l_{31}=\left(a_{11} b_{22}-a_{22} b_{11}\right) / \Delta, \quad l_{32}=\left(a_{12} b_{22}-a_{22} b_{12}\right) / \Delta \\
& l_{41}=\left(a_{21} b_{11}-a_{11} b_{21}\right) / \Delta, \quad l_{42}=\left(a_{21} b_{12}-a_{12} b_{21}\right) / \Delta \\
& a_{11}=a_{1}-d_{1} l_{11} e^{\left(\beta_{1}-\delta_{1}\right) \xi \mathrm{h}_{2}}-d_{1} l_{12} e^{-\left(\beta_{1}+\delta_{1}\right) \xi \mathrm{h}_{2}}, \\
& a_{12}=a_{1}-d_{1} l_{12} e^{-\left(\beta_{1}+\delta_{1}\right) \xi \mathrm{h}_{2}}-d_{1} l_{11} e^{\left(-\beta_{1}+\delta_{1}\right) \xi h_{2}}, \\
& a_{21}=a_{2}-d_{2} l_{21} e^{\left(-\beta_{2}+\delta_{2}\right) \xi h_{1}}-d_{2} l_{22} e^{-\left(\beta_{2}+\delta_{2}\right) \xi \mathrm{h}_{1}}, \\
& a_{22}=a_{2}-d_{2} l_{22} e^{\left(\beta_{2}+\delta_{2}\right) \xi h_{1}}-d_{2} l_{21} e^{\left(\beta_{2}-\delta_{2}\right) \xi \mathrm{h}_{1}}, \\
& b_{11}=b_{1}-e_{1} l_{11} e^{\left(\beta_{1}-\delta_{1}\right) \xi \mathrm{h}_{2}}+e_{1} l_{12} e^{\left(\beta_{1}+\delta_{1}\right) \xi \mathrm{h}_{2}}, \\
& b_{12}=-b_{1}-e_{1} l_{12} e^{-\left(\beta_{1}+\delta_{1}\right) \xi \mathrm{h}_{2}}+e_{1} l_{11} e^{\left(-\beta_{1}+\delta_{1}\right) \xi \mathrm{h}_{2}}, \\
& b_{21}=b_{2}-e_{2} l_{21} e^{\left(-\beta_{2}+\delta_{2}\right) \xi \mathrm{h}_{1}}+e_{2} l_{22} e^{-\left(\beta_{2}+\delta_{2}\right) \xi \mathrm{h}_{1}}, \\
& b_{22}=-b_{2}-e_{2} l_{22} e^{\left(\beta_{2}+\delta_{2}\right) \xi \mathrm{h}_{1}}+e_{2} l_{21} e^{\left(\beta_{2}-, \delta_{2}\right) \xi \mathrm{h}_{1}}, \\
& \Delta=a_{21} b_{22}-a_{22} b_{21} \tag{5.4.5}
\end{align*}
$$

The conditions (5.3.1) and (5.3.2) lead to the following integral equations:

$$
\begin{align*}
& \mathscr{\mathscr { O }}\left[\left(a_{1} A_{1}+a_{1} B_{1}+d_{1} \mathrm{C}_{1}+d_{1} D_{1}\right) ; \xi \rightarrow r\right]=p_{1}(r) ; \quad r<a,  \tag{5.4.6}\\
& \mathscr{\mathscr { H } _ { 1 }}\left[\left(b_{1} A_{1}-b_{1} B_{1}+e_{1} \mathrm{C}_{1}-e_{1} D_{1}\right) ; \xi \rightarrow r\right]=p_{2}(r) ; \quad r<a,  \tag{5.4.7}\\
& \mathscr{\mathscr { H } _ { 1 }}\left[\xi^{-1}\left(A_{1}+B_{1}+C_{1}+D_{1}-A_{2}-B_{2}-C_{2}-D_{2}\right) ; \xi \rightarrow r\right]=0 ;  \tag{5.4.8}\\
& \mathscr{\mathscr { B }}\left[\xi ^ { - 1 } \left(a_{1} A_{1}-a_{1} \mathrm{~B}_{1}+\gamma_{1} C_{1}-\gamma_{1} D_{1}\right.\right. \\
& \quad r>a,(5.4 .8)  \tag{5.4.9}\\
& \left.\left.\quad-a_{2} A_{2}+a_{2} B_{2}-\gamma_{2} C_{2}+\gamma_{2} D_{2}\right) ; \xi \rightarrow r\right]=0 ; \quad r>a .
\end{align*}
$$

If we introduce two new unknown functions $\phi_{1}(\xi)$ and $\phi_{2}(\xi)$ by the relations

$$
\begin{align*}
& A_{1}+B_{1}+C_{1}+D_{1}-A_{2}-B_{2}-C_{2}-D_{2}=\phi_{1}(\xi), \\
& a_{1} A_{1}-a_{1} B_{1}+\gamma_{1} C_{1}-\gamma_{1} D_{1}-\alpha_{2} A_{2}+a_{2} B_{2}-\gamma_{2} C_{2}+\gamma_{2} D_{2}=\phi_{2}(\xi), \tag{5.4.10}
\end{align*}
$$

then the equations (5.4.6) to (5.4.9) may be rewritten in the form

$$
\begin{array}{ll}
\mathscr{H}\left[\left(\lambda_{11} \phi_{1}+\lambda_{12} \phi_{2}\right) ; \xi \rightarrow r\right]=p_{1}(r), & \mathrm{r}<a, \\
\mathscr{H}\left[\left(\lambda_{21} \phi_{1}+\lambda_{22} \phi_{2}\right) ; \xi \rightarrow r\right]=p_{2}(r), & \mathrm{r}<a, \\
\mathscr{H}\left[\xi^{-1} \phi_{1}(\xi) ; \xi \rightarrow r\right]=0, & \mathrm{r}>a, \\
\mathscr{H}\left[\xi^{-1} \phi_{2}(\xi) ; \xi \rightarrow r\right]=0, . & \mathrm{r}>a, \tag{5.4.14}
\end{array}
$$

where $\lambda_{\mathrm{jk}}$ are functions of $\xi$ and they are given by

$$
\begin{align*}
& \lambda_{11}=\left(a_{11} \gamma_{11}-a_{12} \gamma_{11}\right) / \Delta^{*}, \lambda_{12}=\left(a_{12} \gamma_{21}-a_{11} \gamma_{22}\right) / \Delta^{*}, \\
& \lambda_{21}=\left(b_{11} \gamma_{12}-b_{22} \gamma_{11}\right) / \Delta^{*}, \lambda_{22}=\left(b_{12} \gamma_{21}-b_{11} \gamma_{22}\right) / \Delta^{*}, \\
& \Delta^{*}=\gamma_{12} \gamma_{21}-\gamma_{11} \gamma_{22} . \tag{5.4.15}
\end{align*}
$$

and $\gamma_{j k}$ 's are defined as follows

$$
\begin{align*}
& \gamma_{11}=a_{1}-\gamma_{1} l_{11} e^{\left(\beta_{1}-\delta_{1}\right) \xi \mathrm{h}_{2}}+\gamma_{1} l_{12} e^{\left(\beta_{1}+\delta_{1}\right) \xi \mathrm{h}_{2}-a_{2} l_{31}+\alpha_{2} l_{41}-\gamma_{2} l_{51}+\gamma_{2} l_{61},} \\
& \gamma_{12}=-\alpha_{1}-\gamma_{1} l_{12} e^{-\left(\beta_{1}+\delta_{1}\right) \xi \mathrm{h}_{2}}+\gamma_{1} l_{11} e^{\left(-\beta_{1}+\delta_{1}\right) \xi \mathrm{h}_{2}}-\alpha_{2} l_{32}+a_{2} l_{22}-\gamma_{2} l_{52}+\gamma_{2} l_{62}, \\
& \gamma_{21}=1-l_{11} e^{\left(\beta_{1}-\delta_{1}\right) \xi \mathrm{h}_{2}-l_{12} e^{\left(\beta_{1}+\delta_{1}\right) \xi \mathrm{h}_{2}-l_{31}-l_{41}-l_{51}-l_{61}},} \\
& \gamma_{22}=1-l_{12} e^{-\left(\beta_{1}+\delta_{1}\right) \xi \mathrm{h}_{2}}-l_{11} e^{\left(-\beta_{1}+\delta_{1}\right) \xi \mathrm{h}_{2}-l_{32}-l_{42}-l_{52}-l_{62},} \\
& l_{51}=-l_{21} l_{31} e^{\left(-\beta_{2}+\delta_{2}\right) \xi \mathrm{h}_{1}-l_{22} l_{41}}{ }^{\left(\beta_{2}+\delta_{2}\right) \xi \mathrm{h}_{1}}, \\
& l_{52}=-l_{21} l_{32} e^{\left(-\beta_{2}+\delta_{2}\right) \xi h_{1}-l_{22} l_{42}}{ }^{\left(\beta_{2}+\delta_{2}\right) \xi h_{1}}, \\
& l_{61}=-l_{22} l_{31} e^{-\left(\beta_{2}+\delta_{2}\right) \xi \mathrm{h}_{1}}-l_{21} l_{41}\left(\beta_{2}-\delta_{2}\right) \xi \mathrm{h}_{1} \\
& l_{62}=-l_{22} l_{32} e^{-\left(\beta_{2}+\delta_{2}\right) \xi \mathrm{h}_{1}-l_{21} l_{42}}{ }^{\left(\beta_{2}-\delta_{2}\right) \xi \mathrm{h}_{1}} . \tag{5.4.16}
\end{align*}
$$

It can be seen that as $\xi \rightarrow \infty, \lambda_{j k}$ tends to constant, say $\lambda_{j k}(\infty)$, and

$$
\begin{equation*}
\lambda_{\mathrm{jk}}(\xi)-\lambda_{\mathrm{jk}}(\infty)=O\left(e^{-\left(\beta_{1}+\delta_{1}\right) \xi \mathrm{h}_{2}}\right) \tag{5.4.17}
\end{equation*}
$$

To solve the integral equations (5.4.11) to (5.4.14), let us recall the following results [45]

$$
\begin{align*}
& \mathscr{O}_{1}\left[r \mathscr{H}_{0}\{F(\xi) ; r\} ; y\right]=\mathscr{F}_{\mathrm{s}}[F(\xi) ; y]  \tag{5.4.18}\\
& \mathscr{C}_{2}\left[D_{\mathrm{r}} r \mathscr{H}_{1}\left\{\xi^{-1} F(\xi) ; r\right\} ; y\right]=\mathscr{F}_{\mathrm{c}}[F(\xi) ; y]  \tag{5.4.19}\\
& \mathscr{C}_{1}\left[\mathscr{H}_{1}\{F(\xi) ; r\} ; y\right]=y^{-1}\left\{\mathscr{F}_{\mathrm{c}}[F(\xi) ; 0]-\mathscr{F}_{\mathrm{c}}[F(\xi) ; y]\right\}  \tag{5.4.20}\\
& \mathscr{C}_{2}\left[D_{\mathrm{r}} \mathscr{H}_{0}\left\{\xi^{-1} F(\xi) ; r\right\} ; y\right]=-y^{-1} \mathscr{F}_{\mathrm{s}}[F(\xi) ; y] \tag{5.4.21}
\end{align*}
$$

where $D_{\mathrm{r}}$ denotes $\partial / \partial r$ and $\mathscr{S}_{1}, \mathscr{L}_{2}$ are Abel's operators $; \mathscr{F}_{\mathrm{s}}$ and $\mathscr{F}_{\mathrm{c}}$ are Fourier's operators defined by

$$
\begin{align*}
& \mathscr{G}_{1}[F(r) ; y]=(2 / \pi)^{\frac{1}{2}} \int_{0}^{\mathrm{y}} \frac{1}{\sqrt{y^{2}-r^{2}}} F(r) d r \\
& \mathscr{G}_{2}[F(r) ; y]=(2 / \pi)^{\frac{1}{2}} \int_{\mathrm{y}}^{\infty} \frac{1}{\sqrt{r^{2}-y^{2}}} F(r) d r \\
& \mathscr{F}_{\mathrm{s}}[F(r) ; y]=(2 / \pi)^{\frac{1}{2}} \int_{0}^{\infty} F(r) \sin (r y) d r \\
& \mathscr{F}_{\mathrm{c}}[F(r) ; y]=(2 / \pi)^{\frac{1}{2}} \int_{0}^{\infty} F(r) \cos (r y) d r \tag{5.4.22}
\end{align*}
$$

Applying the operators $\mathscr{大}_{1} r, \dot{\mathscr{G}}_{1}, \mathscr{\mathscr { C }}_{2} D_{\mathrm{r}} r$ and $\mathscr{\mathscr { C }}_{2} D_{\mathrm{r}}$ to the equations (5.4.11), (5.4.12), (5.4.13) and (5.4.14) respectively, we obtain

$$
\begin{array}{ll}
\mathscr{F}_{\mathrm{s}}\left[\left(\lambda_{11} \phi_{1}+\lambda_{12} \phi_{2}\right) ; y\right]=f_{1}(y), & 0 \leq y \leq a \\
\mathscr{F}_{\mathrm{c}}\left[\left(\lambda_{21} \phi_{1}+\lambda_{22} \phi_{2}\right) ; y\right]=f_{2}(y), & 0 \leq y \leq a \\
\mathscr{F}_{\mathrm{c}}\left[\phi_{1}(\xi) ; y\right]=0, & y>a ; \\
\mathscr{F}_{\mathrm{s}}\left[\phi_{2}(\xi) ; y\right]=0, & y>a ; \tag{5.4.26}
\end{array}
$$

where

$$
\begin{align*}
& f_{1}(y)=\mathscr{b}_{1}\left[r p_{1}(r) ; y\right] \quad, f_{2}(y)=y \mathscr{b}_{1}\left[p_{2}(r) ; y\right]+c \\
& c=\mathscr{F}_{c}\left[\left(\lambda_{21} \phi_{1}+\lambda_{22} \phi_{2}\right) ; 0\right] \tag{5.4.27}
\end{align*}
$$

By taking $y=a x$, equations (5.4.23) to (5.4.26) can be written in the following form

$$
\begin{array}{ll}
\mathscr{F}_{\mathrm{s}}\left[\left(\tilde{\lambda}_{11} \tilde{\phi}_{1}+\tilde{\lambda}_{12} \tilde{\phi}_{2}\right) ; x\right]=\tilde{f}_{1}(x), & 0 \leq x \leq 1 \\
\mathscr{F}_{\mathrm{c}}\left[\left(\tilde{\lambda}_{21} \tilde{\phi}_{1}+\tilde{\lambda}_{22} \tilde{\phi}_{2}\right) ; x\right]=\tilde{f}_{2}(x), & 0 \leq x \leq 1 \\
\mathscr{F}_{\mathrm{c}}\left[\tilde{\phi}_{1} ; x\right]=0, & x>1 \\
\mathscr{F}_{\mathrm{s}}\left[\tilde{\phi}_{2} ; x\right]=0, & x>1 \tag{5.4.31}
\end{array}
$$

where

$$
\begin{array}{ll}
\tilde{\lambda}_{\mathrm{jk}}(\xi)=\lambda_{\mathrm{jk}}(\xi / a), & \tilde{\phi}_{\mathrm{j}}(\xi)=\phi_{\mathrm{j}}(\xi / a) \\
\tilde{f}_{\mathrm{j}}(x)=a f_{\mathrm{j}}(a x), & j, k=1,2 \tag{5.4.32}
\end{array}
$$

If we introduce two new unknown functions $\psi_{1}(t)$ and $\psi_{2}(t)$ such that

$$
\begin{align*}
& \mathscr{F}_{\mathrm{c}}\left[\tilde{\phi}_{1}(x) ; t\right]=\left\{\begin{array}{lr}
\psi_{1}(t), & 0<t<1 \\
0, & t>1
\end{array}\right.  \tag{5.4.33}\\
& \mathscr{F}_{\mathrm{s}}\left[\tilde{\phi}_{2}(x) ; t\right]=\left\{\begin{array}{lr}
\psi_{2}(t), & 0<t<1 \\
0, & t>1
\end{array}\right. \tag{5.4.34}
\end{align*}
$$

the equations (5.4.30) and (5.4.31) are identically satisfied. Let $\psi \mathrm{e}(t)$ and $\psi_{0}(t)$ be the even extension of $\psi_{1}(t)$ and odd extension of $\psi_{2}(t)$ on $(-1,1)$ respectively, then as we did in section 4.3 , we have

$$
\begin{align*}
& \mathscr{F}_{\mathrm{s}}\left[\tilde{\phi}_{1}(t) ; x\right]=-\frac{1}{\pi} \int_{-1}^{1} \frac{\psi_{\mathrm{e}}(t)}{t-x} d t  \tag{5.4.35}\\
& \mathscr{F}_{\mathrm{c}}\left[\tilde{\phi}_{2}(t) ; x\right]=\frac{1}{\pi} \int_{-1}^{1} \frac{\psi_{0}(t)}{t-x} d t \tag{5.4.36}
\end{align*}
$$

Now if we denote $F_{1}(x)$ the odd extension of $\tilde{f}_{1}(x)$ and $F_{2}(x)$ the even extension of $\tilde{f}_{2}(x)$ on $(-1,1)$, the equations (5.4.28) and (5.4.29) become

$$
\begin{align*}
\lambda_{12}(\infty) \psi_{0}(x)- & \frac{\lambda_{11}(\infty)}{\pi} \int_{-1}^{1} \frac{\psi_{\mathrm{e}}(t)}{t-x} d t+\int_{-1}^{1} \psi_{\mathrm{e}}(t) K_{11}(x, t) d t \\
& +\int_{-1}^{1} \psi_{0}(t) K_{12}(x, t) d t=F_{1}(x), \quad|x|<1,  \tag{5.4.37}\\
\lambda_{21}(\infty) \psi_{\mathrm{e}}(x)+ & \frac{\lambda_{22}(\infty)}{\pi} \int_{-1}^{1} \frac{\psi_{0}(t)}{t-x} d t+\int_{-1}^{1} \psi_{\mathrm{e}}(t) K_{21}(x, t) d t \\
& +\int_{-1}^{1} \psi_{0}(t) K_{22}(x, t) d t=F_{2}(x), \quad|x|<1 \tag{5.4.38}
\end{align*}
$$

where the kernels $K_{j k}$ are given by

$$
\begin{align*}
& K_{11}(x, t)=\frac{1}{\pi} \int_{0}^{\infty}\left[\tilde{\lambda}_{11}(\xi)-\lambda_{11}(\infty)\right] \cos (\xi t) \sin (\xi x) d \xi \\
& K_{12}(x, t)=\frac{1}{\pi} \int_{0}^{\infty}\left[\tilde{\lambda}_{12}(\xi)-\lambda_{12}(\infty)\right] \sin (\xi t) \sin (\xi x) d \xi \\
& K_{21}(x, t)=\frac{1}{\pi} \int_{0}^{\infty}\left[\tilde{\lambda}_{21}(\xi)-\lambda_{21}(\infty)\right] \sin (\xi t) \cos (\xi x) d \xi \\
& K_{22}(x, t)=\frac{1}{\pi} \int_{0}^{\infty}\left[\tilde{\lambda}_{22}(\xi)-\lambda_{22}(\infty)\right] \cos (\xi t) \cos (\xi x) d \xi \tag{5.4.39}
\end{align*}
$$

Equations (5.4.37) and (5.4.38) may be rewritten as follows:

$$
\begin{align*}
& \lambda \zeta_{1}(x)+\frac{1}{\pi i} \int_{-1}^{1} \frac{\zeta_{1}(t)}{t-x} d t+\int_{-1}^{1} \zeta_{1}(t) M_{11}(x, t) d t+\int_{-1}^{1} \zeta_{2}(t) M_{12}(x, t) d t=g_{1}(\mathrm{x})  \tag{5.4.40}\\
& \lambda \zeta_{2}(x)-\frac{1}{x i} \int_{-1}^{1} \frac{\zeta_{2}(t)}{t-x} d t+\int_{-1}^{1} \zeta_{1}(t) M_{21}(x, t) d t+\int_{-1}^{1} \zeta_{2}(t) M_{22}(x, t) d t=g_{2}(\mathrm{x}) \tag{5.4.41}
\end{align*}
$$

where

$$
\begin{align*}
& \lambda_{1}=\lambda_{21}(\infty) / \lambda_{22}(\infty), \quad \lambda_{2}=\lambda_{12}(\infty) / \lambda_{11}(\infty), \lambda=\sqrt{\lambda_{1} \lambda_{2}}, \\
& \zeta_{1}(x)=\sqrt{\lambda_{1}} \psi_{\mathrm{e}}(x)+i \sqrt{\lambda_{2}} \psi_{0}(x), \quad \zeta_{2}(x)=\sqrt{\lambda_{1}} \psi_{\mathrm{e}}(x)-i \sqrt{\lambda_{2}} \psi_{0}(x), \\
& g_{1}(x)=\sqrt{\lambda_{2}} F_{2}(x) / \lambda_{22}(\infty)+i \sqrt{\lambda_{1}} F_{1}(x) / \lambda_{11}(\infty), \\
& g_{2}(x)=\sqrt{\lambda_{2}} F_{2}(x) / \lambda_{22}(\infty)-i \sqrt{\lambda_{1}} F_{1}(x) / \lambda_{11}(\infty), \\
& M_{11}(x, t)=\frac{1}{2}\left\{\frac{\sqrt{\lambda_{1}}}{\lambda_{11}(\infty) \sqrt{\lambda_{2}}} K_{12}(x, t)+\frac{\sqrt{\lambda_{2}}}{\lambda_{22}(\infty) \sqrt{\lambda_{1}}} K_{21}(x, t)\right. \\
& \left.+i\left(\frac{K_{11}(x, t)}{\lambda_{11}(\infty)}-\frac{K_{22}(x, t)}{\lambda_{22}(\infty)}\right)\right\}, \\
& M_{12}(x, t)=\frac{1}{2}\left\{\frac{\sqrt{\lambda_{1}}}{\lambda_{11}(\infty) \sqrt{\lambda_{2}}} K_{12}(x, t)+\frac{\sqrt{\lambda_{2}}}{\lambda_{22}(\infty) \sqrt{\lambda_{1}}} K_{21}(x, t)\right. \\
& +i\left(\frac{K_{11}(x, t)}{\lambda_{11}(\infty)}+\frac{K_{22}(x, t)}{\lambda_{22}(\infty)}\right\}, \\
& M_{21}(x, t)=\frac{1}{2}\left\{\frac{\sqrt{\lambda_{1}}}{\lambda_{11}(\infty) \sqrt{\lambda_{2}}} K_{12}(x, t)+\frac{\sqrt{\lambda_{2}}}{\lambda_{22}(\infty) \sqrt{\lambda_{1}}} K_{21}(x, t)\right. \\
& \left.-i\left(\frac{K_{11}(x, t)}{\lambda_{11}(\infty)}+\frac{K_{22}(x, t)}{\lambda_{22}(\infty)}\right)\right\}, \\
& M_{22}(x, t)=\frac{1}{2}\left\{\frac{\sqrt{\lambda_{1}}}{\lambda_{11}(\infty) \sqrt{\lambda_{2}}} K_{12}(x, t)+\frac{\sqrt{\lambda_{2}}}{\lambda_{22}(\infty) \sqrt{\lambda_{1}}} K_{21}(x, t)\right. \\
& \left.+i\left(\frac{K_{11}(x, t)}{\lambda_{11}(\infty)}-\frac{K_{22}(x, t)}{\lambda_{22}(\infty)}\right)\right\} . \tag{5.4.42}
\end{align*}
$$

The analytic solution of equations (5.4.40) and (5.4.41) has been extensively studied (see, for exemple, [67] and [68]) by using regularization method, which, in this case, however becomes cumbersome. Here we try to use an approximation method described by Erdogan [69] to find the stress intensity factors.

Since the kernels $M_{j k}$ 's are bounded, aside from a multiplication constant, the singular behavior of the functions $\zeta_{j}(j=1,2)$ at the points $x= \pm 1$ is
determined by the dominant part of the singular integral equations. The equations (5.4.40) and (5.4.41) will be solved under the assumption that $\zeta_{j}$ $(j=1,2)$ satisfy a Hölder condition on every closed part of the interval $(-1,1)$ not containing the ends.

The solution of the equations (5.4.40) and (5.4.41) may be assumed in the form of Jacobi polynomials $P_{\mathrm{n}}{ }^{\left(\sigma_{\mathrm{k}}, \tau_{\mathrm{k}}\right)}(x)$ [69] by

$$
\begin{equation*}
\zeta_{\mathrm{k}}=\sum_{\mathrm{n}=1}^{\infty} C_{\mathrm{kn}} W_{\mathrm{k}}(x) P_{\mathrm{n}}\left(\sigma_{\mathrm{k}}, \tau_{\mathrm{k}}\right)(x) \tag{5.4.43}
\end{equation*}
$$

where

$$
\begin{align*}
& W_{\mathrm{k}}(x)=(1-x)^{\sigma_{\mathrm{k}}}(1+x)^{\tau_{\mathrm{k}}}, \quad \sigma_{\mathrm{k}}=i \omega_{\mathrm{k}}, \quad \tau_{\mathrm{k}}=-i \omega_{\mathrm{k}}, \\
& \omega_{\mathrm{k}}=(-1)^{\mathrm{k}+1} \omega, \quad \omega=\frac{1}{2 \pi} \ln \left[\frac{1+\lambda}{1-\lambda}\right], \quad k=1,2 ; \tag{5.4.44}
\end{align*}
$$

and $C_{\mathrm{kn}}$ are unknown coefficents.
Substituting from equation (5.4.43) into equations (5.4.40) and (5.4.41) we obtain

$$
\begin{align*}
& \sum_{\mathrm{n}=1}^{\infty} C_{1 \mathrm{n}} \frac{\left(1-\lambda^{2}\right)^{\frac{1}{2}}}{2 i} P_{\mathrm{n}-1}^{\left(-\sigma_{1},-\tau_{1}\right)}(x)+\int_{-1}^{1}\left[\sum _ { \mathrm { n } = 1 } ^ { \infty } \left(C_{1 \mathrm{n}} M_{11}(x, t) W_{1}(t) P_{\mathrm{n}}\left(\sigma_{1}, \tau_{1}\right)(t)\right.\right. \\
&  \tag{5.4.45}\\
& \left.\left.\quad+C_{2 \mathrm{n}} M_{12}(x, t) W_{2}(t) P_{\mathrm{n}}\left(\sigma_{2}, \tau_{2}\right)(t)\right)\right] d t=g_{1}(x), \quad|x|<1, \\
& \sum_{\mathrm{n}=1}^{\infty} C_{2 \mathrm{n}} \frac{\left(1-\lambda^{2}\right)^{\frac{1}{2}}}{2 i} P_{\mathrm{n}-1}^{\left(-\sigma_{2},-\tau_{2}\right)}(x)-\int_{-1}^{1}\left[\sum _ { \mathrm { n } = 1 } ^ { \infty } \left(C_{1 \mathrm{n}} M_{21}(x, t) W_{1}(t) P_{\mathrm{n}}\left(\sigma_{1}, \tau_{1}\right)(t)\right.\right.  \tag{5.4.46}\\
& \\
& \left.\left.\quad+C_{2 \mathrm{n}} M_{22}(x, t) W_{2}(t) P_{\mathrm{n}}\left(\sigma_{2}, \tau_{2}\right)(t)\right)\right] d t=-g_{2}(x), \quad|x|<1
\end{align*}
$$

where we have used the following result by Karpenko [70]:

$$
\begin{align*}
& \frac{1}{\pi i} \int_{-1}^{1} W_{\mathrm{k}}(t) P_{\mathrm{n}}{ }^{\left(\sigma_{\mathrm{k}}, \tau_{\mathrm{k}}\right)}(t) \frac{d t}{t-x}+(-1)^{\mathrm{k}} \lambda W_{\mathrm{k}}(x) P_{\mathrm{n}}{ }^{\left(\sigma_{\mathrm{k}}, \tau_{\mathrm{k}}\right)}(x) \\
& =-\left(1-\lambda^{2}\right)^{\frac{1}{2}} P_{\mathrm{n}}\left(-\sigma_{\mathrm{k}},-\tau_{\mathrm{k}}\right)(x), \quad|x|<1 \tag{5.4.47}
\end{align*}
$$

Multiplying equations (5.4.45) and (5.4.46) by $W_{1}^{-1}(x) P_{j}^{\left(-\sigma_{1},-\tau_{1}\right)}(x)$ and $W_{2}^{-1}(x) P_{\mathrm{j}}^{\left(-\sigma_{2},-\tau_{2}\right)}(x)$ respectively, and using the orthogonality relations of Jacobi polynomials

$$
\begin{align*}
& \int_{-1}^{1} W(t) P_{\mathrm{n}}{ }^{(\sigma, \tau)}(t) P_{\mathrm{m}}(\sigma, \tau)(t) d t= \begin{cases}0 & n \neq m, \\
\theta_{\mathrm{m}}^{(\sigma, \tau)} & n=m,\end{cases} \\
& \theta_{\mathrm{m}}^{(\sigma, \tau)}=\frac{2^{\sigma+\tau+1} \Gamma(m+\sigma+1) \Gamma(m+\tau+1)}{m!(2 m+\sigma+\tau+1) \Gamma(m+\sigma+\tau+1)}, \tag{5.4.48}
\end{align*}
$$

we get the following infinite system of simultaneous algebraic equations for the determination of $C_{\mathrm{km}}$.

$$
\begin{align*}
& -\left(1-\lambda^{2}\right)^{\frac{1}{2}} C_{1 \mathrm{~m}} \theta_{\mathrm{m}}\left(-\sigma_{1},-\tau_{1}\right)+\sum_{\mathrm{n}=1}^{\infty}\left(L_{11 \mathrm{~nm}}^{*} C_{\mathrm{in}}+L_{12 \mathrm{~nm}}^{*} C_{2 \mathrm{n}}\right)=G_{1 \mathrm{~m}},  \tag{5.4.49}\\
& -\left(1-\lambda^{2}\right)^{\frac{1}{2}} C_{2 \mathrm{~m}} \theta_{\mathrm{m}}^{\left(-\sigma_{1},-\tau_{1}\right)}-\sum_{\mathrm{n}=1}^{\infty}\left(L_{2 \mathrm{~nm}}^{*} C_{1 \mathrm{n}}+L_{22 \mathrm{~nm}}^{*} C_{2 \mathrm{n}}\right)=G_{2 \mathrm{~m}}, \tag{5.4.50}
\end{align*}
$$

where

$$
\begin{align*}
& L_{\mathrm{kjnm}}^{*}=\int_{-1}^{1} L_{\mathrm{kjn}}(x) W_{\mathrm{j}}^{-1}(x) P_{\mathrm{m}}^{\left(-\sigma_{\mathrm{j}},-\tau_{\mathrm{j}}\right)}(x) d x \\
& L_{\mathrm{kjn}}=\int_{-1}^{1} M_{\mathrm{kj}}(x) W_{\mathrm{j}}(x) P_{\mathrm{n}}^{\left(\sigma_{\mathrm{j}}, \tau_{\mathrm{j}}\right)}(x) d x \\
& G_{\mathrm{km}}=\int_{-1}^{1} g_{\mathrm{k}}(x) W_{\mathrm{k}}^{-1}(x) P_{\mathrm{m}}^{\left(-\sigma_{\mathrm{k}},-\tau_{\mathrm{k}}\right)}(x) d x \\
& k, j=1,2 ; \quad n, m=1,2,3,4, \ldots \ldots \tag{5.4.51}
\end{align*}
$$

After we find the coefficients $C_{\mathrm{kn}}$ by solving the system of linear algebraic equations (5.4.49) and (5.4.50), by taking $n, m=1,2, \ldots . . . N$; we can determine the stress intensity factors. To do so, we make use of the following integrals [44]:


$\left(89^{\circ} \mp \mathfrak{G}\right) \quad{ }^{\prime} b<\iota$

${ }^{\prime}\left[\mu^{\prime}!(\xi)^{z} \phi((\infty) z \tau]\left[\mathscr{O}+\left[\mu!(\xi)^{z} \phi\left((\infty)^{z \tau} \gamma-(\xi)^{z \tau} \gamma\right)\right] \mathscr{O}+\right.\right.$
 $\left[\mu-\xi!(\xi)^{z} \phi(\xi)^{z z} \gamma+(\xi)^{I} \phi(\xi)^{I z} \gamma\right]_{\mathscr{O}} \mathscr{B}=\left(+0^{6} l\right)^{\mathrm{Iza}} \rho$
$\left(99^{\circ} \mp \cdot G\right) \quad$ ' $b<\downarrow$



:SMOIIOJ


$(\boxed{G} \cdot \mp \cdot 9)$

$$
{ }_{f}\left(\tau^{l-} \kappa\right) /(\mu-\kappa) H=\left\{p(\xi \mu)^{0} \Gamma(\kappa \xi) u \imath s_{\infty}^{0} \int\right.
$$

$$
\begin{aligned}
& {\left[s!(\xi){ }^{I} \phi\right] \log \frac{a^{n}}{(\infty)^{I T}}=}
\end{aligned}
$$

$$
\begin{align*}
& \mathscr{\mathscr { C } _ { 0 }}\left[\lambda_{12}(\infty) \phi_{2}(\xi) ; r\right]=\frac{\lambda_{12}(\infty)}{a^{2}} \mathscr{\mathscr { H }}\left[\tilde{\phi}_{2}(\xi) ; s\right] \\
& =\frac{\lambda_{12}(\infty)}{a^{2}} \frac{1}{s} \frac{\partial}{\partial s}\left\{s \mathscr{H}_{1}\left[\xi^{-1} \tilde{\phi}_{2}(\xi) ; s\right]\right\} \\
& =\frac{\lambda_{12}(\infty)}{a^{2}}(2 / \pi)^{\frac{1}{2}} \frac{1 \partial}{s \partial s}\left[s \int_{0}^{1} \phi_{2}(t) d t \int_{0}^{\infty} J_{1}(s \xi) \sin (t \xi) d \xi\right], \quad r>a \tag{5.4.60}
\end{align*}
$$

which by using equation (5.4.53) gives

$$
\begin{align*}
& \mathscr{H}_{0}\left[\lambda_{12}(\infty) \phi_{2}(\xi) ; r\right] \\
& =\frac{\lambda_{12}(\infty)}{a^{2}}(2 / \pi)^{\frac{1}{2}} \frac{1}{s} \frac{\partial}{\partial s}\left[\int_{0}^{1} \phi_{2}(t) t /\left(s^{2}-t^{2}\right)^{\frac{1}{2}} d t, \quad r>a .\right. \tag{5.4.61}
\end{align*}
$$

Similarly by means of equations (5.4.52) and (5.4.54) we get

$$
\begin{array}{ll}
\mathscr{H}_{1}\left[\lambda_{22}(\infty) \phi_{2}(\xi) ; r\right]=0, & r>a, \\
\mathscr{H}_{1}\left[\lambda_{21}(\infty) \phi_{1}(\xi) ; r\right] & \\
=\frac{\lambda_{21}(\infty)}{a^{2}}(2 / \pi)^{\frac{1}{2}} \frac{1}{s} \frac{\partial}{\partial s}\left[\int_{0}^{1} \phi_{1}(t) /\left(s^{2}-t^{2}\right)^{\frac{1}{2}} d t,\right. & r>a . \tag{5.4.63}
\end{array}
$$

Now

$$
\begin{align*}
& \mathscr{\mathscr { L }}\left[\left(\lambda_{11}(\xi)-\lambda_{11}(\infty)\right) \phi_{1}(\xi) ; r\right] \\
& =\frac{1}{a^{2}}(2 / \pi)^{\frac{1}{2}} \int_{0}^{\infty} \xi\left[\tilde{\lambda}_{11}(\xi)-\lambda_{11}(\infty)\right] J_{0}(s \xi) d \xi \int_{0}^{1} \phi_{1}(t) \cos (\xi t) d t \\
& =\frac{1}{a^{2}}(2 / \pi)^{\frac{1}{2}} \int_{0}^{1} \phi_{1}(t) d t \int_{0}^{\infty} \xi\left[\tilde{\lambda}_{11}(\xi)-\lambda_{11}(\infty)\right] J_{0}(s \xi) d \xi \\
& =O(1), \quad \text { as } r \rightarrow a^{+}, \tag{5.4.64}
\end{align*}
$$

and when $r \rightarrow a^{+}, \mathscr{H}_{0}\left[\left(\lambda_{11}(\xi)-\lambda_{11}(\infty)\right) \phi_{1}(\xi) ; r\right]$ has no singularity. Similarly we have

$$
\begin{align*}
& \mathscr{\mathscr { L } _ { 0 }}\left[\left(\lambda_{12}(\xi)-\lambda_{12}(\infty)\right) \phi_{2}(\xi) ; r\right]=O(1), \quad \text { as } r \rightarrow a^{+} ; \\
& \mathscr{H}_{1}\left[\left(\lambda_{21}(\xi)-\lambda_{21}(\infty)\right) \phi_{1}(\xi) ; r\right]=O(1), \quad \text { as } r \rightarrow a^{+} ; \\
& \mathscr{\mathscr { G }}_{1}\left[\left(\lambda_{22}(\xi)-\lambda_{22}(\omega)\right) \phi_{2}(\xi) ; r\right]=O(1), \quad \text { as } r \rightarrow a^{+} . \tag{5.4.65}
\end{align*}
$$

So as $r \rightarrow a^{+}$,
$\sigma_{\mathrm{zz} 1}\left(r, 0^{+}\right) a^{2}(\pi / 2)^{\frac{1}{2}} \sqrt{\lambda_{2}} / \lambda_{12}(\infty)=\lim _{s \rightarrow 1} \frac{\partial}{\partial \Delta s} \int_{0}^{1} \sqrt{\lambda_{2}} \psi_{2}(t) t /\left(s^{2}-t^{2}\right)^{\frac{1}{2}} d t+O(1)$,
$\sigma_{\mathrm{rz1}}\left(r, 0^{+}\right) a^{2}(\pi / 2)^{\frac{1}{2}} \sqrt{\lambda_{1}} / \lambda_{21}(\infty)=\lim _{s \rightarrow 1^{+}} \frac{\partial}{\partial s} \int_{0}^{1} \sqrt{\lambda_{1}} \phi_{1}(t) /\left(s^{2}-t^{2}\right)^{\frac{1}{2}} d t+O(1)$.
But

$$
\begin{align*}
& \frac{\partial}{\partial s} \int_{0}^{1} \sqrt{\lambda_{2} \psi_{2}}(t) t /\left(s^{2}-t^{2}\right)^{\frac{1}{2}} d t=\frac{1 \partial}{2 i \partial s} \int_{0}^{1}\left[\zeta_{1}(t)-\zeta_{2}(t)\right] t /\left(s^{2}-t^{2}\right)^{\frac{1}{2}} d t \\
& =\frac{1}{2} \sum_{\mathrm{n}=1 \mathrm{k}=1}^{\infty} \sum^{2}(-1)^{k+1} C_{\mathrm{kn}}\left[\frac{\partial}{\partial s} \int_{0}^{1} t\left[\frac{1-t}{1+t}\right]^{i \omega_{\mathrm{k}}} P_{\mathrm{n}}\left(\sigma_{\mathrm{k}},{ }_{\mathrm{T}} \mathrm{k}\right)(t)\left(s^{2}-t^{2}\right)^{-\frac{1}{2}} d t\right], \tag{5.4.68}
\end{align*}
$$

and

$$
\begin{align*}
& \frac{\partial}{\partial s} \int_{0}^{1} \sqrt{\lambda_{1}} \psi_{1}(t) /\left(s^{2}-t^{2}\right)^{\frac{1}{2}} d t=\frac{1 \partial}{2 \partial s} \int_{0}^{1}\left[\zeta_{1}(t)+\zeta_{2}(t)\right] /\left(s^{2}-t^{2}\right)^{\frac{1}{2}} d t \\
& =\frac{1}{2} \sum_{\mathrm{n}=1 \mathrm{k}=1}^{\infty} \sum_{2}^{2}(-1)^{k+1} C_{\mathrm{kn}}\left[\frac{\partial}{\partial s} \int_{0}^{1}\left[\frac{1-t}{1+t}\right]^{i \omega_{\mathrm{k}}} P_{\mathrm{n}}\left(\sigma_{\mathrm{k}}{ }^{{ }^{\top} \mathrm{k}}\right)\right.  \tag{5.4.69}\\
& \left.(t)\left(s^{2}-t^{2}\right)^{-\frac{1}{2}} d t\right] .
\end{align*}
$$

If we define the stress intensity factors $K_{1}$ and $K_{2}$ by

$$
\begin{align*}
& \frac{\sqrt{\lambda_{2}}}{\lambda_{12}(\infty)} K_{1}+(-1)^{\mathrm{k}+1} \frac{\sqrt{\lambda_{1}}}{\lambda_{21}(\infty)^{2}} K_{2}=\lim _{r \rightarrow a^{+}}(\pi / 2)^{\frac{1}{2}}\left[(r-a)^{\frac{1}{2}-i \omega_{\mathrm{k}}}(r+a)^{\frac{1}{2}+i \omega_{\mathrm{k}}}\right] \\
& \quad \cdot a^{2}\left[\frac{\sqrt{\lambda_{2}}}{\lambda_{12}(\infty)} \sigma_{\mathrm{zz1}}\left(r, 0^{+}\right)+(-1)^{\mathrm{k}+1} \frac{\sqrt{\lambda_{1}}}{\lambda_{21}(\infty)} \sigma_{\mathrm{rz1}}\left(r, 0^{+}\right)\right], k=1,2 \tag{5.4.70}
\end{align*}
$$

then using the substitution $s=r / a$, equations (5.4.66) to (5.4.69) and following the method of Goldstein and Vainshelbarm [93] by taking

$$
x=1-\frac{s-1}{s-t}
$$

we get

$$
\begin{array}{r}
\frac{\sqrt{\lambda_{2}}}{\lambda_{12}(\infty)} K_{1}+(-1)^{\mathrm{k}+1} \frac{\sqrt{\lambda_{1}}}{\lambda_{21}(\infty)} K_{2}=(-1)^{\mathrm{k}+1} i a\left[2\left(1-\lambda^{2}\right)\right]^{\frac{1}{2}} \frac{\Gamma\left(1+i \omega_{\mathrm{k}}\right)}{\Gamma\left(\frac{1}{2}+i \omega_{\mathrm{k}}\right)} \\
\cdot \sum_{\mathrm{n}=1}^{\infty} C_{\mathrm{kn}} P_{\mathrm{n}}{ }^{\left(\sigma_{\mathrm{k}}, \tau_{\mathrm{k}}\right)}(1), \quad k=1,2 . \tag{5.4.71}
\end{array}
$$

### 5.5. Other cases

In the previous two sections we have considered the case when the surfaces $z=-h_{1}$ and $z=h_{2}$ are stress-free. In this section we will consider other kinds of boundary conditions at the faces $z=-h_{1}$ and $z=h_{2}$ while the conditions at the interface $z=0$ are kept the same.

### 5.5.1 One face fixed and the other stress-free

In this case, we assume that all the conditions (5.3.1) to (5.3.4) remain the same and the conditions (5.3.5) are replaced by.

$$
\begin{equation*}
u_{z 2}\left(r,-h_{1}\right)=0, \quad u_{\mathrm{r} 2}\left(\mathrm{r},-h_{1}\right)=0, \quad \mathrm{r} \geq 0 \tag{5.5.1}
\end{equation*}
$$

In this case condition (5.5.1) yield the following algebraic equations

$$
\begin{align*}
& A_{2} e^{-\beta_{2} \xi h_{1}}+B_{2} e^{\beta_{2} \xi h_{1}}+C_{2} e^{-\delta_{2} \xi h_{1}}+D_{2} e^{\delta_{2} \xi h_{1}}=0  \tag{5.5.2}\\
& A_{2} \alpha_{2} e^{-\beta_{2} \xi h_{1}}-B_{2} \alpha_{2} e^{\beta_{2} \xi h_{1}}+C_{2} \gamma_{2} e^{-\delta_{2} \xi h_{1}}-D_{2} \gamma_{2} e^{\delta_{2} \xi h_{1}}=0 \tag{5.5.3}
\end{align*}
$$

which replace the algebraic equations (5.4.3a) and (5.4.3b) respectively. Comparing the two sets of equations, we find that if we take

$$
\begin{equation*}
a_{2}=1, \quad \mathrm{~d}_{2}=1, \quad b_{2}=a_{2}, \quad e_{2}=\gamma_{2}, \tag{5.5.4}
\end{equation*}
$$

in the results of section 5.4 , we get the solution for this case.

### 5.5.2 Both faces fixed

In this case all the conditions (5.3.1) to (5.3.3) remain the same but the conditions (5.3.4) are replaced by the following:

$$
\begin{equation*}
u_{\mathrm{z} 1}\left(r, h_{2}\right)=0 \quad, \quad u_{\mathrm{r} 1}\left(r, h_{2}\right)=0, \quad r \geq 0 \tag{5.5.5}
\end{equation*}
$$

and (5.3.5) are replaced by (5.5.1).
In this case , equations (5.4.3) will be replaced by equations (5.5.2), (5.5.3) and the following equations

$$
\begin{align*}
& A_{1} e^{\beta_{1} \xi \mathrm{~h}_{2}}+B_{1} e^{-\beta_{1} \xi \mathrm{~h}_{2}}+C_{1} e^{\delta_{1} \xi \mathrm{~h}_{2}}+D_{2} e^{-\delta_{1} \xi \mathrm{~h}_{2}}=0  \tag{5.5.6}\\
& A_{1} \alpha_{1} e^{\beta_{1} \xi \mathrm{~h}_{2}}-B_{1} \alpha_{1} e^{-\beta_{1} \xi \mathrm{~h}_{2}}+C_{1} \gamma_{1} e^{-\delta_{1} \xi \mathrm{~h}_{2}}-D_{1} \gamma_{1} e^{-\delta_{1} \xi \mathrm{~h}_{2}}=0 \tag{5.5.7}
\end{align*}
$$

Comparing the two sets of equations, we find that if we take

$$
\begin{equation*}
a_{\mathrm{i}}=d_{\mathrm{i}}=1, \quad b_{\mathrm{i}}=a_{\mathrm{i}}, \quad e_{\mathrm{i}}=\gamma_{\mathrm{i}}, \quad(i=1,2) \tag{5.5.8}
\end{equation*}
$$

in the results of section 5.4 , we get the solution for this case.

### 5.5.3 One face rigidly restrained and the other fixed

In this case, we assume that the boundary conditions (5.3.5) are replaced by (5.5.1) and conditions (5.3.4) are replaced by

$$
\begin{equation*}
u_{\mathrm{z} 1}\left(z, h_{2}\right)=0 \quad, \quad \sigma_{\mathrm{IZ} 1}\left(z, h_{2}\right)=0 ; \tag{5.5.9}
\end{equation*}
$$

and all the other conditions remain the same.So the equations (5.4.3) should be replaced by (5.5.7), (5.4.3b), (5.5.2) and (5.5.3). Solving equations (5.4.2), (5.5.7), (5.4.3b), (5.5.2) and (5.5.3) we can express $B_{1}, D_{1}, A_{2}, B_{2}, C_{2}$ and $D_{2}$
in terms of $A_{1}$ and $C_{1}$ by the following equations :

$$
\begin{align*}
& B_{1}=A_{1} e^{2 \beta_{1} \xi \mathrm{~h}_{2}}, \quad D_{1}=C_{1} e^{2 \delta_{1} \xi \mathrm{~h}_{2}} \\
& C_{2}=-l_{21} A_{2} e^{\left(-\beta_{2}+\delta_{2}\right) \xi \mathrm{h}_{1}-l_{22} B_{2} e^{\left(\beta_{2}+\delta_{2}\right) \xi \mathrm{h}_{1}}} \\
& D_{2}=-l_{22} A_{2} e^{-\left(\beta_{2}+\delta_{2}\right) \xi \mathrm{h}_{1}-l_{21} B_{2} e^{\left(\beta_{2}-\delta_{2}\right) \xi \mathrm{h}_{1}}} \\
& A_{2}=l_{31} A_{1}+l_{32} C_{1}, \quad B_{2}=l_{41} A_{1}+l_{42} C_{1} \tag{5.5.10}
\end{align*}
$$

where $l_{21}$ and $l_{22}$ are given by

$$
\begin{equation*}
l_{21}=\left(\gamma_{2}+a_{2}\right) / 2 \gamma_{2}, \quad l_{22}=\left(\gamma_{2}-a_{2}\right) / 2 \gamma_{2} \tag{5.5.11}
\end{equation*}
$$

and $l_{j k}\left(j=3,4 ; k=1,2\right.$.) are still given by (5.4.5) while the expression for $a_{j k}$ and $b_{\mathrm{jk}}\left(j_{, k} k=1,2\right)$ are given by the following:

$$
\begin{align*}
& a_{11}=a_{1}\left(1+e^{2 \beta_{1} \xi \mathrm{~h}_{2}}\right), \quad a_{12}=d_{1}\left(1+e^{2 \delta_{1} \xi \mathrm{~h}_{2}}\right) \\
& a_{21}=a_{2}-d_{2} l_{21} e^{\left(-\beta_{2}+\delta_{2}\right) \xi \mathrm{h}_{1}}, \quad a_{22}=a_{2}+d_{2} l_{22} e^{\left(\beta_{2}+\delta_{2}\right) \xi \mathrm{h}_{1}}, \\
& b_{11}=b_{1}\left(1-e^{2 \beta_{1} \xi \mathrm{~h}_{2}}\right), \quad b_{12}=b_{1}\left(1-e^{2 \delta_{1} \xi \mathrm{~h}_{2}}\right) \\
& b_{21}=b_{2}-e_{2} l_{21} e^{\left(-\beta_{2}+\delta_{2}\right) \xi \mathrm{h}_{1}}, \quad b_{22}=-b_{2}+e_{2} l_{22} e^{\left(\beta_{2}+\delta_{2}\right) \xi \mathrm{h}_{1}} \tag{5.5.12}
\end{align*}
$$

Consequently the expressions for $\gamma_{j k}(j, k=1,2)$ for this case are given by

$$
\begin{align*}
& \gamma_{11}=a_{1}-a_{1} e^{2 \beta_{1} \xi \mathrm{~h}_{2}}-\alpha_{2} l_{31}+a_{2} l_{41}-\gamma_{2} l_{51}+\gamma_{2} l_{61} \\
& \gamma_{12}=\gamma_{1}-\gamma_{1} e^{2 \delta_{1} \xi \mathrm{~h}_{2}}-\alpha_{2} l_{32}+\alpha_{2} l_{42}-\gamma_{2} l_{52}+\gamma_{2} l_{62} \\
& \gamma_{21}=1+e^{2 \beta_{1} \xi \mathrm{~h}_{2}-l_{31}-l_{41}-l_{51}-l_{61}} \\
& \gamma_{22}=1+e^{2 \delta_{1} \xi \mathrm{~h}_{2}-l_{32}-l_{42}-l_{52}-l_{62}} \tag{5.5.13}
\end{align*}
$$

where $l_{j k}(j=5,6 ; k=1,2)$ are the same as given in (5.4.16). The solution for this case is given by the results of section 5.4 after we have made above raplacements.

### 5.5.4 One face rigidly restrained and the other stress-free

In this case, the boundary conditions (5.3.4) are replaced by (5.5.9) and all the other boundary conditions remain the same as in section 5.4 . So in equations (5.4.2) and (5.4.3) we only have to replace equation (5.4.3a) by (5.5.7). Now we find that for this case $l_{\mathrm{jk}}(j=2,3,4 ; k=1,2)$ are all given by (5.4.5), $a_{j k}$ and $b_{j k}(j, k=1,2)$ are given by (5.5.12) and the solution for this case is given by the results of section 5.4 after we have made above modifications.

### 5.5.5 Both faces rigidly restrained

In this case, we are assuming that the boundary conditions (5.3.4) and (5.3.5) are replaced by conditions (5.5.9) and

$$
\begin{equation*}
u_{z 2}\left(z,-h_{1}\right)=0, \quad \sigma_{\mathrm{rz} 2}\left(z,-h_{1}\right)=0 \tag{5.5.14}
\end{equation*}
$$

The equations (5.4.3a) and (5.4.3c) will be replaced by equations (5.5.7) and (5.5.3) while all other equations remain the same. Solving equations (5.4.2), (5.5.7), (5.4.3b), (5.5.3) and (5.4.3d) we can write

$$
\begin{align*}
& B_{1}=A_{1} e^{2 \beta_{1} \xi \mathrm{~h}_{2}}, \quad D_{1}=C_{1} e^{2 \delta_{1} \xi \mathrm{~h}_{2}} \\
& B_{2}=A_{2} e^{-2 \beta_{2} \xi \mathrm{~h}_{1}}, \quad D_{2}=C_{2} e^{-2 \delta_{2} \xi \mathrm{~h}_{1}} \\
& A_{2}=l_{31} A_{1}+l_{32} C_{1}, \quad C_{2}=l_{41} A_{1}+l_{42} C_{1} \tag{5.5.15}
\end{align*}
$$

where $l_{j k}\left(j=3,4 ; k=1,2\right.$.) are given by (5.4.5) but the expression for $a_{j k}$ and $b_{\mathrm{jk}}(j, k=1,2)$ should be replaced by the following

$$
\begin{array}{ll}
a_{11}=a_{1}\left(1+e^{2 \beta_{1} \xi \mathrm{~h}_{2}}\right), & a_{12}=d_{1}\left(1+e^{2 \delta_{1} \xi \mathrm{~h}_{2}}\right), \\
a_{21}=a_{2}\left(1+e^{-2 \beta_{2} \xi \mathrm{~h}_{1}}\right), & a_{22}=d_{2}\left(1+e^{-2 \delta_{2} \xi \mathrm{~h}_{1}}\right), \\
b_{11}=b_{1}\left(1-e^{2 \beta_{1} \xi \mathrm{~h}_{2}}\right), & b_{12}=e_{1}\left(1-e^{2 \delta_{1} \xi \mathrm{~h}_{2}}\right), \\
b_{21}=b_{2}\left(1-e^{-2 \beta_{2} \xi \mathrm{~h}_{1}}\right), & b_{22}=e_{2}\left(1-e^{-2 \delta_{2} \xi \mathrm{~h}_{1}}\right), \tag{5.5.16}
\end{array}
$$

Consequently for this case we should replace $\gamma_{\mathrm{jk}}(j, k=1,2)$ by

$$
\begin{align*}
& \gamma_{11}=\alpha_{1}\left(1-e^{2 \beta_{1} \xi \mathrm{~h}_{2}}\right)-\alpha_{2} l_{31}\left(1-e^{-2 \beta_{2} \xi \mathrm{~h}_{1}}\right)-\gamma_{2} l_{41}\left(1-e^{-2 \delta_{2} \xi \mathrm{~h}_{1}}\right), \\
& \gamma_{12}=\gamma_{1}\left(1-e^{2 \delta_{1} \xi \mathrm{~h}_{2}}\right)-\alpha_{2} l_{32}\left(1-e^{-2 \delta_{2} \xi \mathrm{~h}_{1}}\right)-\gamma_{2} l_{42}\left(1-e^{-2 \delta_{2} \xi \mathrm{~h}_{1}}\right), \\
& \gamma_{21}=1+e^{2 \beta_{1} \xi \mathrm{~h}_{2}-l_{31}\left(1+e^{-2 \beta_{2} \xi \mathrm{~h}_{1}}\right)-l_{41}\left(1+e^{-2 \delta_{2} \xi \mathrm{~h}_{1}}\right)} \\
& \gamma_{22}=1+e^{2 \delta_{1} \xi \mathrm{~h}_{2}}-l_{32}\left(1+e^{-2 \delta_{2} \xi \mathrm{~h}_{1}}\right)-l_{42}\left(1+e^{-2 \delta_{2} \xi \mathrm{~h}_{1}}\right) \tag{5.5.17}
\end{align*}
$$

Then the solution for this case is given by the results of section 5.4.

### 5.6. Numerical results and discussion

To evaluate the stress intensity factors, we truncate the infinite system of simultaneous algebraic equations (5.4.49) and (5.4.50) at $\mathrm{n}=10$ and the Crout's factorisation method is used to solve these equations. And Gaussian quadrature formula is used to perform the numerical integrations involved in the solution. The relative error is controlled under 0.01 . Numerical results for the stress intensity factors $K_{1}$ and $K_{2}$ are obtained for the case when the crack is subjected to a constant pressure $p_{1}(r)=p_{0}$ and $p_{2}(r)=0$, the thickness of the
layers is kept the same (i.e, $h_{1}=h_{2}=h$ ) and the surfaces $z=-h$ and $z=h$ are stress-free. For the two transversely isotropic materials considered here, the numerical values of the elastic moduli are taken as follows $\left(10^{11}\right.$ dynes $\left./ \mathrm{cm}^{2}\right)[5]:$

|  | $c_{11}$ | $c_{12}$ | $c_{13}$ | $c_{33}$ |  |  |  |  | $c_{44}$ |  |
| :--- | :---: | :---: | :---: | :---: | :---: | ---: | :---: | :---: | :---: | :---: |
| Cadmium | 11.00 | 4.04 | 3.83 | 4.69 | 1.56 | (layer 1) |  |  |  |  |
| Beryl | 26.94 | 9.61 | 6.61 | 23.63 | 6.53 | (layer 2 ) |  |  |  |  |

Numerical values of the stress intensity factors have been calculated for following three particular cases:

Case 1. The length of crack $a=1.0$ and $h_{1} / a=h_{2} / a=h / a=$ $0.2(0.2), 1.0,2.0(2.0) 10.0$; the numerical values of the stress intensity factor $K_{1}$ against $h / a$ are displayed in Fig.5.6.1 and Fig.5.6.2, and the numerical values of the stress intensity factor $K_{2}$ against $h / a$ are displayed in Fig.5.6.3 and Fig.5.6.4.

Case 2. $h_{1}=h_{2} \rightarrow \infty$, and the length of the crack $a=1.0(1.0) 10.0$; the numerical values of the stress intensity factor $K_{1}$ against $a$ are displayed in Fig.5.6.5, and the numerical values of the stress intensity factor $K_{2}$ against $a$ are displayed in Fig.5.6.6.

Case 3. $h_{1}=h_{2}=20.0$ and the length of the crack $a=1.0(1.0) 10.0$; the numerical values of the stress intensity factor $K_{1}$ against $a$ are displayed in Fig.5.6.7, and the numerical values of the stress intensity factor $K_{2}$ against $a$ are displayed in Fig.5.6.8.


Fig.5.6.1
Numerical values of the stress intensity factor $K_{1}$ against $h / a$ (0.2 to 1.0) for a fixed crack length $a$ and equal layer thickness ( $h_{1}=h_{2}=h$ ).


Fig.5.6.2
Numerical values of the stress intensity factor $K_{1}$ against $h / a$ (1.0 to 10.0) for a fixed crack length $a$ and equal layer thickness ( $h_{1}=h_{2}=h$ ).


Fig.5.6.3
Numerical values of the stress intensity factor $K_{2}$ against $h / a$ ( 0.2 to 1.0 ) for a fixed crack length $a$ and equal layer thickness ( $h_{1}=h_{2}=h$ ).


Fig.5.6.4
Numerical values of the stress intensity factor $K_{2}$ against $h / a$ (1.0 to 10.0) for a fixed crack length $a$ and equal layer thickness $\left(h_{1}=h_{2}=h\right)$.


Fig.5.6.5
Numerical values of the stress intensity factor $K_{1}$ against the crack length $a$ ( 1.0 to 10.0 ) for the layer thickness $h_{1}=h_{2} \rightarrow \infty$.


Fig.5.6.6
Numerical values of the stress intensity factor $K_{2}$ against the crack length $a$ (1.0 to 10.0) for the layer thickness $h_{1}=h_{2} \rightarrow \infty$.


Fig.5.6.7
Numerical values of the stress intensity factor $K_{1}$ against the crack length $a(1.0$ to 10.0 ) for the layer thickness $h_{1}=h_{2}=20$.


Fig.5.6.8
Numerical values of the stress intensity factor $K_{2}$ against the crack length $a$ ( 1.0 to 10.0 ) for the layer thickness $h_{1}=h_{2}=20$.

We make the following observations from the graphs : when the length of the crack $a$ is fixed at $a=1.0$, the stress intensity factor $K_{2}$ increases as the thickness of the layers increases; while the stress intensity factor $K_{1}$ decreases as the thickness of the layers increases from 0.1 to 1.0 , but increases when the thickness of the layers increases from 1.0 to 10.0 . If the thickness of the layers is fixed at 20.0 or $\rightarrow \infty$ the stress intensity factors $K_{1}$ and $K_{2}$ decrease as the length of the crack increases.

## Recommendation for future work

As a continuation of the present work, the following problems may be considered in the future :

As an extension of the work of chapter 2 one could consider the Reissner-Sagoci problem of a finite elastic cylinder embedded in an infinite elastic layer and the whole is perfectly bonded to an elastic half-space.

The problem considered in chapter 3 may be extended to two dissimilar transversely isotropic layers with a penny-shaped flaw located at the interface, when a rigid shaft bonded to the elastic layer is rotated through a small angle.

The problem of chapter 4 may be extended to study the determination of stress intensity factors at the tips of a column of Griffith cracks, which are parallel to the $x$-axis and are equally spaced along the $y$-axis, located at the interfaces of the orthotropic multilayer composite materials.

Similarly, a problem related to chapter 5 is a problem of determination of stress intensity factors at the edges of of a column of penny-shaped cracks, which are parallel to the $x y$-plane and are equally spaced along the $z$-axis, located at the interfaces of the transversely isotropic multilayer composite material.

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