THE UNIVERSITY OF CALGARY

A MARKOVIAN DECISION UNDER COMPETITION

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE

DEGREE OF MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS AND STATISTICS

CALGARY, ALBERTA

AUGUST, 1988

ISBN 0-315-46638-3

THE UNIVERSITY OF CALGARY

FACULTY OF GRADUATE STUDIES

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ABSTRACT

This thesis extends the paper by Enns and Ferenstein (1988), a one stream problem, to a two stream problem.

The game consists of two players who observe two Poisson streams of offers. Each of the two players observing these offers wishes to select exactly one offer in the time interval [0,T]. The game is structured in such a way that the players have different priorities for different streams when it comes to making their first selection. The winner of the game is the person who has the larger offer by the end of time T.

The offers are independent and identically distributed random variables from some known continuous distribution. On their arrival, the offers are observed sequentially. Furthermore, at each observation, a decision as to whether to accept or reject the offer must be made. Once accepted, an offer cannot be discarded; once rejected, an offer cannot be recalled later in the game.

The optimal strategies and the winning probabilities for both the players have been derived for this priorized decision scheme. The moment generating function of the fraction of the time of the first offer acceptance has been obtained. Asymptotic results are also available for all cases (including that for the asymptotic mean and variance of the fraction of the time of the first offer acceptance).

ACKNOWLEDGEMENT

I would firstly like to convey my thanks to my beloved and wonderful parents, Mr. and Mrs. Kannoo Pillai, whose support (both moral and financial) has made it possible for me to advance this far.

My sincere thanks also goes to my supervisor, Professor E.G. Enns, who helped me realise one of my goals by making it possible for me to defend my thesis within this short period of time. I would also like to take this opportunity to thank Dr. E.Z. Ferenstein, who had patiently helped me with my thesis during my supervisor's absence.

I am also thankful to the Department of Mathematics and Statistics for providing me with financial support during this period of time. Finally, I would like to thank Maria Dourado who had efficiently and painstakingly typed this manuscript.

Dedicated to my beloved grandmother, Sri Mariamman

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CHAPTER I

INTRODUCTION

1.1 SEQUENTIAL ANALYSIS

Although quite a number of people had been working simultaneously in the field of sequential methods since the Second World War, it was Abraham Wald who was generally regarded as the pioneer in that field.

In 1944, when John von Neumann and Oskar Morgenstern published their much celebrated book, 'Theory of Games and Economic Behaviour', little was known about the relation between the theory of games and the statistical theory of Neyman and Pearson. It took the genius of Wald to see the connection between the theory of games and the statistical theory of Neyman and Pearson and to come to the conclusion that the statistical theory was a game consisting of two players, namely the Statistician and Nature. The words 'theory of games' here refer to at least two decision makers with conflicting (or partially conflicting) interests who try to simultaneously choose their actions from their respective action spaces so as to maximize their payoffs.

With his series of papers from 1945, Wald bridged the gap between the two fields. Darling (1974) recounted how Wald, in his paper in 1945, was the first to define and study the important notion of a stopping time of a sequence of random variables. Ferguson (1967) remarked that Wald's theory of statistical decisions generalized and simplified the Neyman-Pearson statistical theory by making it a special case of the decision theory problem.

What is a stopping time? According to Darling, a stopping time is roughly a rule that utilizes only the present data and the data anterior to it to determine if one has to stop observing a sequence of random variables. We will look at optimal stopping rules and times later in this chapter.

Wald's outstanding research resulted in the classic 'Statistical Decision Functions' in 1950. Ironically, it was also in 1950 when the unfortunate Wald met with a fatal plane accident.

Following Wald's death, a prodigious amount of research on the general sequential decision theory was done by many people. As Ferguson (1974) remarked, the only two notably outstanding contributions which stood out amongst all this avalanche of work were

a. 'Theory of Games and Statistical Decisions', a book published by David Blackwell and M.A. Girshick in 1954, in which sequential decision theory was popularized and vastly applied, and

b. LeCam's paper in 1955 in which Wald's theory was extended and put into a modern mathematical framework.

1.2 OPTIMAL STOPPING PROBLEMS

To cover all that has been done in Sequential Decision Theory (let alone Sequential Analysis) would require a whole encyclopedia. Since this paper falls into a category of sequential decision theory problems called optimal stopping problems, we will devote our time discussing optimal stopping problems.

What is an optimal stopping problem? Chow, Robbins and Siegmund (1971) defined an optimal stopping problem as summarized below:

> The random variables $y_1, y_2, \ldots,$ having a known joint distribution, are observed sequentially. The observation process, which must be stopped, yields a reward of x_n (where x_n is a known function of $y_1, \ldots, y_n$) when the observation is stopped at stage n. The problem is to find the stopping rules that maximize the average reward.

They further remarked that optimal stopping in general had rapidly developed as a part of probability theory with particular but not exclusive application to statistics. These optimal stopping problems are usually practical problems which are colourful in nature.

The secretary problem (also known as the beauty contest problem, the dowry problem and the marriage problem), is an example of an optimal stopping problem that has a long history in the theory of probability. This well-known stopping problem has a large literature that dates back to Cayley (1875). Since the time a version of the secretary problem appeared in the article by Gardner (1960), the problem has been extended and generalized in many directions that culminated in a field of its own. This field is now known as probability-optimization. A review of the secretary problem and its extensions is given by Freeman (1983). Another interesting account of the secretary problem, a historical one, is given by Ferguson (1988).

Before looking at the extent to which we have extended the secretary problem, we will first define the problem. To do this, we will borrow the definition from Ferguson (1988) and restate it. He defined the secretary problem in its simplest form to have the following features.

a. There is only one secretarial position available.

b. The number of applicants, N, is known.

c. The applicants are interviewed sequentially in a random order.

d. All applicants can be ranked from the best to the worst without any ties. Further, the decision to accept or reject an applicant must be based solely on the relative ranks of the applicants interviewed.

e. An applicant once rejected cannot be recalled later.

f. The employer is satisfied with nothing but the very best. The payoff is 1 if the best of the N applicants is chosen and 0 otherwise.

Enns and Ferenstein (1985) disguised the standard secretary problem as a two-person zero sum game that was couched in terms of horsebets. Since the game comprised two players, they were not strictly interested in getting the largest of the N observations. Assuming that the N observations were identically and independently distributed and that they all came from the same continuous distribution function F, Enns and Ferenstein solved the problem of finding the players' winning probabilities when F was known and alternatively unknown. In 1988, they extended this paper by adding the following ingredients:

a. The problem was now posed in a continuous time setting.

b. The number of observations that arrived by the end of time T was a random variable with range $\{1,2,3,...\}$.

c. If neither of the players ends up with an offer by the end of time T, then both players lose the game. This condition no longer makes the game a zero sum game.

With these assumptions, they obtained results similar to those in their paper in 1985.

The present paper extends their 1988 paper by setting the problem in a scenario where there are two streams of offers instead of one. The problem here, thus, differs from the standard secretary problem in the following ways:

a. the problem is posed in a continuous time setting,

b. the number of observations that arrive in the time interval $[0,T]$ is a random variable, and

c. since we are dealing with two players, we are only interested in getting the larger of the two observations.

With this brief history, we will now proceed with the problem proper.

1.3 STATEMENT OF THE PROBLEM

There are two Poisson streams of offers (Stream 1 and Stream 2) that arrive in two sequences with known rates λ_1 and λ_2, respectively. Two players, player 1 and player 2, observing these offers wish to each make exactly one selection from the offers that arrive in the time interval [0,T]. A reward of 1 is given to the player with the larger offer and a reward of 0 to the player who does not hold the larger offer or who has not made any selection by the end of time T.

All the offers arriving via stream i (for i = 1, 2) are first presented to player i. As each offer is presented, player i has to decide whether to accept or reject the offer. If player i accepts the offer, he stays out of the game and waits for either the other player to make a selection or for time T to elapse (whichever is earlier). With player i out of the game, the other player has to now wait for an offer that is larger than the one accepted by player i, if such an offer arrives by time T.

Alternatively, if player i rejects the offer, the other player is then presented with the same offer which he must now decide to either accept or reject. Should the other player accept the offer, player i would then have to wait for a larger offer to arrive by time T in order to win the game. But if the other player rejects the offer, the whole process is repeated with the next observation.

By nature of the game, there is a possibility that neither player will have accepted any offer in the time interval [0,T]. If this prevails, then both players receive rewards of 0.

A further condition on the offers is that once an offer is accepted, it cannot be discarded and once rejected, it cannot be recalled later in the game. Also, since both players want to maximize their average rewards and the payoff is either a 0 or 1, we will have to find the optimal strategies that maximize the winning probabilities such that maximum average rewards are yielded.

In Chapter II, we first define the players' winning probabilities and their respective admissable game strategies. We then find the expressions for these winning probabilities which are functions of the time $T-t$ where $T-t \in [0,T]$ and t is the residual time for any given set of admissable strategies. In Chapter III, optimal strategies that maximise the winning probabilities are obtained. In Chapter IV, under the constraint that the probability that player 1 wins is less than a half, the players' winning probabilities are obtained when the game strategies are optimal. Under the constraint that the probability that player 1 wins is at least a half, the players' winning probabilities have been obtained in Chapter V using the optimal game strategies. Chapter VI discusses the distribution of the time of the first offer acceptance. The moment generating function of the fraction of the time of the first offer acceptance has also been obtained in this chapter. We conclude by finding the asymptotic winning probabilities and obtaining the asymptotic expressions for the moment generating function, the mean and the variance of the fraction of the time of the first offer acceptance in Chapter VII. Results similar to those of Enns and Ferenstein (1988) have also been obtained when $\lambda_1 = 1$ and $\lambda_2 = 0$.

1.4 PREVIOUS WORK AND POSSIBLE EXTENSIONS

Elving (1967) considered the same model of observations where a player chooses his optimal stopping time in order to maximize his mean reward. This paper was discussed in detail by Chow, Robbins and Siegmund (1971). Elving's work was also extended by Stadje (1986) who considered the case of a player choosing k optimal stopping times (where $k \geq 1$).

Presman and Sonin (1972) considered the classical secretary problem in a discrete time setting with the assumption that the number of secretaries that arrived in the time interval $[0,T]$ was a random variable from a known distribution. Cowan and Zabczyk (1978) considered a similar stopping time problem in the continuous time setting where a player was presented with a sequence of offers that arrived at random via a Poisson process with a known rate. The problem there was for the player to maximize his probability of picking the best offer. Bruss (1987) considered the same problem with one important distinction, i.e. the offers arrived via a Poisson process with an unknown rate λ. He even considered the case when the rate of the Poisson process was an inhomogeneous intensity function $\lambda(t)$ which was either supposed to be known or known up to some multiplicative constant.

Irle (1980), Abdel-Hamid, Bather and Trustrum (1982), Petrucelli (1983) and others had also studied alternative approaches to best-choice problems with an unknown number of offers under the assumption that the distribution of the number of offers was known.

Enns and Ferenstein (1985), as mentioned earlier, considered a similar two-person game where the maximum number of offers was known and specified. An alternative approach to the same problem was considered by Enns, Ferenstein and Sheahan (1986). A generalization of this game, for the case of many players and general rewards (which depended on the players' decisions and sample information), was also considered by Enns and Ferenstein (1987).

Possible extensions of this paper could be obtained by making the following considerations.

a. If none of the players has accepted any offer in the time interval $[0,T]$, the last offer from stream i would be given to player i (for $i = 1, 2$) at the end of time T. This will, of course, guarantee a winner.

b. The rates of the Poisson processes are random variables with either a fully or partially known distribution which depend on the number of offers observed. Thus, we have to revise λ_i everytime an offer is observed from stream i.

CHAPTER II

GAME FORMULATION

2.1 INTRODUCTION

Two players, player 1 and player 2, must make a decision to either accept or reject an offer with no possibility of recall at each offer presented. The offers arrive in a sequence as a Poisson process in two different streams (Streams 1 and 2). We will assume that the offers in Stream i arrive at a rate λ_i.

Further, an offer arriving via stream i [for $i = 1,2$] is first presented to player i for a possible selection. If he selects the offer, then he stays out of the game and waits for either the other player to make a selection or time T to elapse [whichever is earlier]. Alternatively, if player i rejects the offer, the other player is then presented with the same offer for a possible selection. If this offer is accepted by the other player, he then stays out of the game and waits for either player i to make his selection or time T to elapse (whichever is earlier). But if this offer is rejected by the other player, the whole process is repeated with the next offer. The player with a larger offer at the end of the game will be declared a winner [in which case he gets a payoff of 1]. If neither of the players has made any selection by the end of time T, they both lose [in which case they both get payoffs of 0]. Also, we will assume that all these offers are independent and identically distributed (i.i.d.) random variables from a uniform distribution on the interval [0,1] since i.i.d. offers from any continuous distribution can be easily transformed to the $U[0,1]$ case without a loss of generality.

The strategy derived here is the optimal sequence of decisions that maximize a player's winning probability until an offer is first accepted by either player. Once an offer is accepted, the obvious strategy of the remaining player is to pick the first available offer greater than the one that has already been accepted by the other player, if such an offer arrives by time T.

2.2 DEFINITIONS AND DIFFERENTIAL EQUATIONS

Let $T-t$ be the time elapsed (i.e. t is the residual time).

$$p_i(T-t) = P[\text{player } i \text{ wins} \mid \text{first accepted offer is in time } (T-t,T)].$$

$$\phi_{ij}(x,T-t) = P[\text{player } i \text{ accepts offer } x \text{ at time } T-t \mid \text{offer is from stream } j \text{ and no offer has been previously selected}].$$

$$g(x,T-t) = P[\text{player making the first choice } x \text{ at time } T-t \text{ eventually wins}],$$

$$\lambda_1 = 1,$$

and $$\lambda_2 = \beta \in [0,1]$$

where $i,j = 1,2$.

By considering the possible events that occur in time $(T-t, T-t+\Delta(T-t))$, we have the following equations:

<u>For Player 1</u>:

$$\begin{aligned} p_1(T-t) &= [1-(1+\beta)\Delta(T-t)]\, p_1(T-t+\Delta(T-t)) \\ &+ \Delta(T-t)\left\{\int_0^1 [\phi_{11}g+(1-\phi_{11})\phi_{21}(1-g)+(1-\phi_{11})(1-\phi_{21})p_1(T-t+\Delta(t-t))]dx\right\} \\ &+ \beta\Delta(T-t)\left\{\int_0^1 [\phi_{22}(1-g)+(1-\phi_{22})\phi_{12}g+(1-\phi_{22})(1-\phi_{12})p_1(T-t+\Delta(T-t))]dx\right\} \\ &+ o(\Delta(T-t)). \end{aligned}$$

As $\Delta(T-t) \longrightarrow 0^+$, we have that

(2.2.1) $$\dot{p}_1(T-t) = (1+\beta)p_1$$

$$- \left\{\int_0^1 [\phi_{11}g+(1-\phi_{11})\phi_{21}(1-g)+(1-\phi_{11})(1-\phi_{21})p_1]dx\right\}$$

$$- \beta\left\{\int_0^1 [\phi_{22}(1-g)+(1-\phi_{22})\phi_{12}g+(1-\phi_{12})(1-\phi_{22})p_1]dx\right\},$$

where $\dot{p}(s) = \frac{dp}{ds}$ for all s.

<u>For Player 2</u>:

Similarly, for player 2 we have that

$$\text{(2.2.2)} \qquad \dot{p}_2(T-t) = (1+\beta)p_2$$

$$- \left\{\int_0^1 [\phi_{11}(1-g)+(1-\phi_{11})\phi_{21}g+(1-\phi_{11})(1-\phi_{21})p_2]dx\right\}$$

$$- \beta\left\{\int_0^1 [\phi_{22}g+(1-\phi_{22})\phi_{12}(1-g)+(1-\phi_{22})(1-\phi_{12})p_2]dx\right\},$$

where the arguments of p_i, ϕ_{ij} and g have been suppressed for the sake of clarity and brevity.

Since $g(x,T-t) = P$[player making the first choice x at time $T-t$ eventually wins], we may rewrite $g(x,T-t)$ as follows:

$$g(x,T-t) = \sum_{r=0}^{\infty} P[\text{all remaining offers are } \leq x|R=r]\ P[R=r]$$

where R is the number of remaining offers.

Thus, for Poisson arrivals we have that

$$g(x,T-t) = \sum_{r=0}^{\infty} x^r \frac{e^{-(1+\beta)t}(1+\beta)^r\ t^r}{r!}$$

$$= e^{-(1+\beta)t}\ e^{x(1+\beta)t}$$

or equivalently,

$$(2.2.3)\quad g(x,T-t) = e^{-(1+\beta)(1-x)t}.$$

Further, since player i has control of strategies ϕ_{i1} and ϕ_{i2}, he wishes to maximize his probability of winning, that is, to maximize $p_i(0)$ for $i = 1,2$. Therefore, letting $p_i(t;\ \phi_{11},\phi_{12},\phi_{21},\phi_{22}) = p_i(t)$ where the optimal strategies $\phi_{11},\phi_{12},\phi_{21}$ and ϕ_{22} constitute an equilibrium solution with respect to the mean rewards $p_1(0)$ and $p_2(0)$, we have that

$$p_1(0;\ \phi_{11},\phi_{12},\phi_{21},\phi_{22}) \geq p_1(0;\ \tilde{\phi}_{11},\tilde{\phi}_{12},\tilde{\phi}_{21},\tilde{\phi}_{22})$$

and

$$p_2(0;\ \phi_{11},\phi_{12},\phi_{21},\phi_{22}) \geq p_2(0;\ \tilde{\phi}_{11},\tilde{\phi}_{12},\tilde{\phi}_{21},\tilde{\phi}_{22})$$

where we have used the notation

$(\tilde{\phi}_{11},\tilde{\phi}_{12},\tilde{\phi}_{21},\tilde{\phi}_{22})$ to represent any game strategy with $0 \leq \tilde{\phi}_{ij}\ (x,\ T-t) \leq 1$ [for $i,j = 1,2$].

CHAPTER III

OPTIMAL STRATEGIES

3.1 INTRODUCTION

Since $p_i(T) = 0$ (for $i = 1,2$) and player i wishes to maximize $p_i(0)$, it is intuitively reasonable to claim that the optimal strategies must cause the function $p_i(T-t)$ to descend as steeply as possible. This means that $\dot{p}_i(T-t)$ must be as small as possible which also implies that the terms under the integrals in both (2.2.1) and (2.2.2) must be as large as possible. Because player 1 has control of ϕ_{11} and ϕ_{12}, and player 2 has control of ϕ_{21} and ϕ_{22}, we have the following strategies.

3.2 OPTIMAL STRATEGIES FOR STREAM 1

By (2.2.2), player 2 would wish to maximize

$$(1-\phi_{11})\, \phi_{21} g + (1-\phi_{11})(1-\phi_{21})\, p_2$$

which leads us to the optimal strategy

$$(3.2.1) \qquad \phi_{21}(x,T-t) = \begin{cases} 1 & \text{if} \quad g(x,T-t) \geq p_2(T-t) \\ 0 & \quad \text{otherwise.} \end{cases}$$

Since player 2 is not given the first opportunity to make a selection when the offers arrive via stream 1, he will only be presented with the offers that have been rejected by player 1. Thus, he will act conservatively by selecting an offer only if it gives him a winning probability at least as large as his optimal winning probability [i.e. $g(x, T-t) \geq p_2(T-t)$]. Hence, the optimal strategy given in (3.2.1) is intuitively reasonable.

By similar consideration of (2.2.1), player 1 would wish to maximize

$$\phi_{11}g + (1-\phi_{11})[\phi_{21}(1-g) + (1-\phi_{21})\ p_1]$$

from which we get the optimal strategy,

$$\phi_{11}(x,T-t) = \begin{cases} 1 & \text{if} \quad g \geq \phi_{21}(1-g) + (1-\phi_{21})\ p_1 \\ 0 & \text{otherwise.} \end{cases}$$

Splitting ϕ_{11} into 2 cases with respect to (3.2.1), we have that

i) If $g \geq p_2$, $\phi_{21} = 1$

$$\Rightarrow \phi_{11} = \begin{cases} 1 & \text{if} \quad g \geq 1-g. \\ 0 & \text{otherwise.} \end{cases}$$

ii) If $g < p_2$, $\phi_{21} = 0$

$$\Rightarrow \phi_{11} = \begin{cases} 1 & \text{if} \quad g \geq p_1 \\ 0 & \text{otherwise} \end{cases}$$

which is impossible, because $p_1 \geq p_2$ is implied by $\lambda_1 \geq \lambda_2$.

Thus, we may rewrite ϕ_{11} as

$$\phi_{11}(x,T-t) = \begin{cases} 1 & \text{if} \quad g(x,T-t) \geq \max[1-g(x,T-t),\ p_2(T-t)] \\ 0 & \text{otherwise.} \end{cases}$$

Further, since $p_2(T-t) < \frac{1}{2}$, $\phi_{11}(x,T-t)$ becomes

$$\phi_{11}(x,T-t) = \begin{cases} 1 & \text{if} \quad g(x,T-t) \geq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases} \tag{3.2.2}$$

Since player 1 is given the first opportunity to make a selection when the offers arrive via stream 1, he will tend to be choosy. Also, since $p_2 \leq p_1$, it is not surprising that player 1 will only select an offer which gives him a winning probability of at least $\frac{1}{2}$ [i.e. $g(x,\ T-t) \geq \frac{1}{2}$].

3.3 OPTIMAL STRATEGIES FOR STREAM 2

Similarly by (2.2.1), player 1 would wish to maximize

$$(1-\phi_{22})\phi_{12}g + (1-\phi_{12})(1-\phi_{22})\ p_1\ .$$

This would then give rise to the following optimal strategy:

$$\phi_{12}(x,T-t) = \begin{cases} 1 & \text{if} \quad g(x,T-t) \geq p_1(T-t) \\ 0 & \text{otherwise.} \end{cases} \tag{3.3.1}$$

Since player 1 is not given the first opportunity to make a selection when the offers arrive via stream 2, he will only be presented with the offers that have been rejected by player 2. Thus, he will act conservatively by selecting any offer which gives him a winning probability that is at least as large as his optimal winning probability [i.e. $g(x,\ T-t) \geq p_1(T-t)$]. Hence, the optimal strategy given in (3.3.1) is intuitively reasonable.

By (2.2.2), player 2 would wish to maximize

$$\phi_{22}g + (1-\phi_{22})\phi_{12}(1-g) + (1-\phi_{22})(1-\phi_{12})\ p_2\ .$$

This leads us to

$$\phi_{22} = \begin{cases} 1 & \text{if} \quad g \geq \phi_{12}(1-g) + (1-\phi_{12})\, p_2 \\ 0 & \text{otherwise.} \end{cases}$$

Splitting ϕ_{22} into 2 cases with respect to (3.3.1), we have that

i) If $g \geq p_1$, $\phi_{12} = 1$

$$\Rightarrow \phi_{22} = \begin{cases} 1 & \text{if} \quad g \geq 1-g \\ 0 & \text{otherwise.} \end{cases}$$

ii) If $g < p_1$, $\phi_{12} = 0$

$$\Rightarrow \phi_{22} = \begin{cases} 1 & \text{if} \quad g \geq p_2 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, by (i) and (ii), we have that

a) <u>If $p_1(T-t) < \frac{1}{2}$</u>

(3.3.2) $$\phi_{22}(x,T-t) = \begin{cases} 1 \text{ if } g(x,T-t) \geq \frac{1}{2} \text{ or } p_2(T-t) \leq g(x,T-t) < p_1(T-t) \\ 0 \quad \text{otherwise.} \end{cases}$$

b) <u>If $p_1(T-t) \geq \frac{1}{2}$</u>

(3.3.3) $$\phi_{22}(x,T-t) = \begin{cases} 1 & \text{if} \quad g(x,T-t) \geq p_2(T-t) \\ 0 & \text{otherwise.} \end{cases}$$

Although player 2 is given the first opportunity to make a selection when the offers arrive via stream 2, he will act conservatively because $p_2 \le p_1$. When $p_1(T-t) < \frac{1}{2}$, it is not surprising that player 2 accepts an offer of $g(x, T-t) \ge \frac{1}{2}$. However, the interesting part about player 2's optimal strategy is that he does not accept any offer when $p_1(T-t) \le g(x, T-t) < \frac{1}{2}$ but would instead accept an offer when $p_2(T-t) \le g(x, T-t) < p_1(T-t)$. This is reasonable because by not selecting an offer when $p_1(T-t) \le g(x, T-t) < \frac{1}{2}$, player 2 forces upon player 1 to select an offer that gives player 1 a winning probability of less than a $\frac{1}{2}$. Furthermore, since player 1 does not accept any offer when $g(x, T-t) < p_1(T-t)$, player 2 will use this to his advantage by selecting an offer even when $g(x, T-t) < p_1(T-t)$ as long as it is larger than his optimal winning probability [i.e. he selects an offer when $p_2(T-t) \le g(x, T-t) < p_1(T-t)$]. This explains (3.3.2).

Also, since the winning probabilities increase over time and $p_1 \ge p_2$, there may be a time when the winning probability of player 1 reaches a half. Once $p_1(T-t) = \frac{1}{2}$ and if neither of the players has made any selection by this time, player 2 will be more conservative and pick any offer which gives him a winning probability that is at least as large as his optimal winning probability [i.e. $g(x, T-t) \ge p_2(T-t)$]. This is sensible because player 1 will only accept an offer from stream 2 if $g(x, T-t) \ge p_1(T-t)$ and also, since $p_1(T-t) \ge \frac{1}{2}$, player 1 will actually be accepting an offer which gives him a winning probability that is at least as large as half. Hence, the optimal strategy in (3.3.3) is intuitively reasonable.

3.4 A FORMAL PROOF FOR THE OPTIMAL STRATEGIES

We will now present a formal proof to show that the strategies given by (3.2.1), (3.2.2), (3.3.1), (3.3.2) and (3.3.3) are optimal in the class of all the admissable strategies $(\tilde{\phi}_{11}, \tilde{\phi}_{12}, \tilde{\phi}_{21}, \tilde{\phi}_{22})$ where $\tilde{\phi}_{ij}(x, T-t)$ [for $i,j = 1,2$] are continuous functions of $T-t \in [0,T]$ for nearly all $x \in [0,1]$.

Using the facts that

$$\phi_{11}g + (1-\phi_{11})[\phi_{21}(1-g)+(1-\phi_{21})p_1] = \max[g, \phi_{21}(1-g)+(1-\phi_{21})p_1]$$

and

$$(1-\phi_{22})\ \phi_{12}g + (1-\phi_{12})(1-\phi_{22})p_1 = (1-\phi_{22})\ \max[g, p_1],$$

and the intuitive result $p_1 \geq p_2$ [because of the priorities and the fact $\lambda_2 \leq \lambda_1$], we will first prove that $\phi_{11}(x, T-t)$ and $\phi_{12}(x, T-t)$ are the optimal strategies for player 1.

Rewriting (2.2.1), we have that

$$\dot{p}_1(T-t) = f(T-t, p_1(T-t)) \text{ with } 0 \leq T-t \leq T \text{ and } p_1(T) = 0, \tag{3.4.1}$$

where we have used the representation

$$f(T-t,\ y) = (1+\beta)y - \int_0^1 \max[g,\ \phi_{21}(1-g) + (1-\phi_{21})y]\ dx$$

$$- \beta \int_0^1 \left\{\phi_{22}(1-g) + (1-\phi_{22}) \max[g,y]\right\} dx.$$

Letting $0 \le \tilde{\phi}_{ij}(x,\ T-t) \le 1$ [for $i,j = 1,2$] be continuous functions in $T-t \in [0,T]$ for nearly all $x \in [0,1]$ and $\tilde{p}_1(T-t)$ be the solution to (2.2.1) that corresponds to the game strategy $(\tilde{\phi}_{11},\ \tilde{\phi}_{12},\ \phi_{21},\ \phi_{22})$, we have that

$$\dot{\tilde{p}}_1(T-t) = (1+\beta)\tilde{p}_1 - \int_0^1 \left\{\tilde{\phi}_{11}g+(1-\tilde{\phi}_{11})[\phi_{21}(1-g)+(1-\phi_{21})\ \tilde{p}_1]\right\} dx$$

$$- \beta \int_0^1 \left\{\phi_{22}(1-g)+(1-\phi_{22})[\tilde{\phi}_{12}g+(1-\tilde{\phi}_{12})\ \tilde{p}_1]\right\} dx$$

$$\ge (1+\beta)\tilde{p}_1 - \int_0^1 \max[g,\ \phi_{21}(1-g) + (1-\phi_{21})\ \tilde{p}_1]\ dx$$

$$- \beta \int_0^1 \left\{\phi_{22}(1-g)+(1-\phi_{22}) \max[g,\ \tilde{p}_1]\right\} dx.$$

Thus, it follows that

$$(3.4.2) \qquad \dot{\tilde{p}}_1(T-t) \geq f(T-t, \tilde{p}_1(T-t)) \text{ with } 0 \leq T-t \leq T \text{ and } \tilde{p}_1(T) = 0.$$

Using the fact that f is a continuous function and comparing (3.4.1) with (3.4.2), we can use a variation of the differential inequality in Flett [page 97, ex.4] to conclude that $p_1(T-t) \geq \tilde{p}_1(T-t)$ where $0 \leq T-t \leq T$.

By similarly constructing $\tilde{p}_2(T-t)$ which is a solution of (2.2.2) that corresponds to the game strategy $(\phi_{11}, \phi_{12}, \tilde{\phi}_{21}, \tilde{\phi}_{22})$, we can show that

$$p_2(T-t) \geq \tilde{p}_2(T-t) \quad \text{for} \quad 0 \leq T-t \leq T.$$

Thus, we have that

$$p_i(T-t) \geq \tilde{p}_i(T-t) \quad \text{for} \quad i = 1,2 \quad \text{and} \quad 0 \leq T-t \leq T$$

from which it follows that

$$p_i(0) \geq \tilde{p}_i(0) \quad \text{for} \quad i = 1,2. \qquad \square$$

CHAPTER IV

THE WINNING PROBABILITIES WHEN $p_1(T-t) < \frac{1}{2}$ AND THE TIME WHEN $p_1(T-t) = \frac{1}{2}$

4.1 INTRODUCTION

Since we have a different optimal strategy when $p_1(T-t) < \frac{1}{2}$ as opposed to $p_1(T-t) \geq \frac{1}{2}$, the problem of finding the winning probabilities $p_i(0)$ [for $i = 1,2$] can be done by breaking the problem down into two parts.

In the first part (Chapter IV), we will use $p_i(T) = 0$ as the initial conditions to solve (2.2.1) and (2.2.2) in order to determine the residual time t^* such that $p_1(T-t^*) = 0.5$. In Chapter V, we will use the known values of t^* and $p_i(T-t^*)$ as initial conditions to solve (2.2.1) and (2.2.2) so as to determine $p_i(0)$ [for $i = 1,2$]. Here we have assumed that T is at least as large as t^* which implies that $p_1(0) \geq 0.5$.

4.2 THE TIME ZONES

Our goal in this chapter is to find a residual time t^* such that $p_1(T-t^*) = 0.5$. We will achieve this by solving (2.2.1) and (2.2.2) with the initial conditions $p_i(T) = 0$ [for $i = 1,2$] in 4 time zones which are defined by the optimal strategies (3.2.1), (3.2.2), (3.3.1) and (3.3.2). Before we define the times which constitute these time zones, we will first rewrite the optimal strategies (3.2.1), (3.2.2), (3.3.1) and (3.3.2).

With the combination of (2.2.3), (3.2.1) may be rewritten as follows:

$$(4.2.1) \qquad \phi_{21}(x,T-t) = \begin{cases} 1 & \text{if} \quad x \geq \max\,[0,\ 1+\alpha\ \ell n\ p_2(T-t)] \\ 0 & \text{otherwise,} \end{cases}$$

$$\text{where} \quad \alpha = \frac{1}{(1+\beta)t}\ .$$

(3.2.2), (3.3.1) and (3.3.2), via similar combination of (2.2.3), may be rewritten as follows:

$$(4.2.2) \qquad \phi_{11}(x,T-t) = \begin{cases} 1 & \text{if} \quad x \geq \max\,[0,\ 1-\alpha\ \ell n\ 2] \\ 0 & \text{otherwise,} \end{cases}$$

$$(4.2.3) \qquad \phi_{12}(x,T-t) = \begin{cases} 1 & \text{if} \quad x \geq \max\,[0,\ 1+\alpha\ \ell n\ p_1(T-t)] \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(4.2.4)\ \phi_{22}(x,T-t) = \begin{cases} 1 \text{ if } x \geq \max\,[0, 1-\alpha\ell n 2] \text{ or } x \in [1+\alpha\ell n\ p_2, 1+\alpha\ell n p_1) \\ 0 \quad \text{otherwise,} \end{cases}$$

$$\text{where} \quad p_1(T-t) < \tfrac{1}{2}.$$

With these, we can now define the times $T-t_1'$, $T-t_2'$ and $T-t_3'$, where $0 < T-t_1' < T-t_2' < T-t_3' < T$, as follows:

$$\phi_{11}(x,T-t) = \phi_{22}(x,T-t) = 1 \quad \text{for all } x \text{ if } T-t_3' \le T-t \le T$$

$$\phi_{12}(x,T-t) = 1 \quad \text{for all } x \text{ if } T-t_2' \le T-t \le T, \text{ and}$$

$$\phi_{21}(x,T-t) = 1 \quad \text{for all } x \text{ if } T-t_1' \le T-t \le T.$$

Writing down the expressions for the times $T-t_1'$, $T-t_2'$ and $T-t_3'$, we have that

(4.2.5) $$t_3' = \frac{\ell n2}{1+\beta},$$

(4.2.6) $$p_1(T-t_2') = e^{-(1+\beta)t_2'}, \text{ and}$$

(4.2.7) $$p_2(T-t_1') = e^{-(1+\beta)t_1'},$$

since $p_1(T-t)$ and $p_2(T-t)$ are both monotone.

We will make a further assumption that T is large enough so that $T-t_1'$, $T-t_2'$ and $T-t_3'$ will all be well defined positive quantities.

4.3 THE WINNING PROBABILITIES FOR THE TIME ZONES

Since we have a constraint for $p_1(T-t)$ [i.e we want to find a t^* such that $p_1(T-t^*) = 0.5$], it will be easier for us to solve (2.2.1) first. We will start by solving (2.2.1) in the time zone $T-t_3' \le T-t \le T$. Using the fact that $p_1(T-t)$ is a continuous solution, we will then work backwards to determine t^*, such that $p_1(T-t^*) = 0.5$. Once t^* has been established, we can proceed to determine $p_2(T-t^*)$ which is just a by-product.

Case i: For the time zone $T-t_3' \le T-t \le T$, with the aid of (4.2.1), (4.2.2), (4.2.3) and (4.2.4), one can rewrite (2.2.1) as

$$\dot{p}_1(T-t) = (1+\beta)p_1 - \beta - \alpha(1-\beta) + \alpha(1-\beta)\ e^{-\frac{1}{\alpha}} .$$

Using the initial condition $p_1(T) = 0$, the above linear differential equation leads us to the following solution:

$$(4.3.1) \qquad p_1(T-t) = \frac{\beta}{1+\beta}\left[1-e^{-(1+\beta)t}\right] + \frac{1-\beta}{1+\beta}\ e^{-(1+\beta)t}\ J(T-t,T),$$

where we define

$$(4.3.2) \qquad J(a,b) = \int_a^b \frac{e^{(1+\beta)t}-1}{t}\ d(T-t),$$

where $0 \le a \le b \le T$.

<u>Case ii</u>: For the time zone $T-t_2' \le T-t \le T-t_3'$, combining (4.2.1), (4.2.2), (4.2.3) and (4.2.4) with (2.2.1), one can rewrite (2.2.1) as follows:

$$\dot{p}_1(T-t) = (1+\beta)p_1 + \alpha(\ell n\ 2)(1-\beta) - 1 - \alpha\, e^{-\frac{1}{\alpha}}(1-\beta).$$

Using the fact that $p_1(T-t)$ is a continuous solution and equating the solution of the above ordinary differential equation with (4.3.1) at time $T-t_3'$, we have that

$$(4.3.3)\quad p_1(T-t) = \frac{1}{1+\beta} - \frac{(\ell n2)(1-\beta)\ e^{-(1+\beta)t}}{1+\beta}\ J(T-t,\ T-t_3')$$

$$+ \frac{e^{-(1+\beta)t}\ (1-\beta)(1-\ell n2)}{1+\beta}\left[(\ell n\ t) - \ell n\left[\frac{\ell n2}{1+\beta}\right]\right]$$

$$+ \frac{e^{-(1+\beta)t}}{1+\beta}\left[\beta - 2 + (1-\beta)\ J(T-t_3',T)\right].$$

<u>Case iii</u>: For the time zone $T-t_1' \le T-t \le T-t_2'$, one can similarly rewrite (2.2.1) as

$$(4.3.4)\quad \dot{p}_1(T-t) = -\ (1+\beta)(1-p_1) + \alpha\left[(\ell n2)(1-\beta)-(1+\beta)\ e^{-\frac{1}{\alpha}}+\beta(2p_1-\ell n\ p_1)\right].$$

Because (4.3.4) is a non-linear differential equation, we will deal with (4.3.4) later and solve it numerically.

<u>Case iv</u>: For the time zone $0 \le T-t \le T-t_1'$, one can rewrite (2.2.1) as

(4.3.5) $\dot{p}_1(T-t) = \alpha\Big[(1+\beta)((\ln p_2)-p_2-p_1\ln p_2)+\beta(2p_1-\ln p_1)+(\ln 2)(1-\beta)\Big].$

Since (4.3.5) is another non-linear differential equation, we will deal with it later and solve it numerically as well. Furthermore, since $\dot{p}_1(T-t)$ depends on both p_1 and p_2, to solve (4.3.5) numerically [using the Runge-Kutta method], we will need an expression for $\dot{p}_2(T-t)$. Thus, combining (4.2.1), (4.2.2), (4.2.3) and (4.2.4) with (2.2.2), one can rewrite (2.2.2) for the same time zone [$0 \le T-t \le T-t_1'$] as follows:

(4.3.6) $\dot{p}_2(T-t) = \alpha\Big[(1+\beta)\ p_2(1-\ln p_2)-(\ln 2)(1-\beta)+\beta(-2p_1+\ln p_1)\Big].$

Before we proceed to find t^*, we will first find an alternative expression for $p_1(T-t_1')$. This can be done via $p(T-t)$, where we define

$$\begin{aligned} p(T-t) &= p_1(T-t) + p_2(T-t) \\ &= P[\text{someone wins} \mid \text{first offer is accepted} \\ &\qquad \text{in time } (T-t,T)] \\ &\le 1 \text{ [as both players may lose the game].} \end{aligned}$$

Thus, by (2.2.1) and (2.2.2), we have that

$$\dot{p}(T-t) = (1+\beta)\, p - \left\{\int_0^1 [\phi_{11}+(1-\phi_{11})\phi_{21} + (1-\phi_{11})(1-\phi_{21})\ p]\ dx\right\} - \beta\left\{\int_0^1 [\phi_{22}+(1-\phi_{22})\phi_{12} + (1-\phi_{12})(1-\phi_{22})\ p]\ dx\right\}. \tag{4.3.7}$$

For the time zone $T-t_1' \leq T-t \leq T$, with the aid of (4.2.1), (4.2.2), (4.2.3) and (4.2.4), we can rewrite (4.3.7) as follows:

$$\dot{p}(T-t) = (1+\beta)p - (1+\beta).$$

Using the initial condition $p(T) = 0$, the above differential equation leads us to the following solution:

$$p(T-t) = 1 - e^{-(1+\beta)t}. \tag{4.3.8}$$

At time $T-t_1'$, (4.3.8) becomes

$$p(T-t_1') = 1 - e^{-(1+\beta)t_1'}$$

or equivalently, the probability that someone wins at time

$$T-t_1', \text{ is } \; 1 - e^{-(1+\beta)t_1'}.$$

At this stage, it is interesting to note that the probability of no one winning at time $T-t_1'$ is given by player 2's winning probability.

Further, by (4.2.7) we have that

$$p_2(T-t_1') = e^{-(1+\beta)t_1'}.$$

We now have an alternative expression for $p_1(T-t_1')$, namely,

$$p_1(T-t_1') = 1 - 2\,e^{-(1+\beta)t_1'}. \tag{4.3.9}$$

4.4 DETERMINATION OF THE RESIDUAL TIME t^* SUCH THAT $p_1(T-t^*) = 0.5$

With the necessary equations being established, we can now proceed to find t^* such that $p_1(T-t^*) = 0.5$. We will start by proving the claim $T-t^* < T-t_2' < T-t_3' < T$.

PROOF:

Since $T-t_2' < T-t_3' < T$ [by definition], it suffices to just show that $p_1(T-t_2') < 0.5$, as this would mean that $T-t^* < T-t_2'$.

By our definition, we have that

$$T-t_2' < T-t_3'$$

$$\text{or} \quad e^{-(1+\beta)t_2'} < e^{-(1+\beta)t_3'}.$$

Equivalently, with the aid of (4.2.5) and (4.2.6),

$$p_1(T-t_2') < e^{-(1+\beta)\left[\frac{\ell n2}{1+\beta}\right]} = \frac{1}{2}.$$

Thus, we have that $p_1(T-t_2') < \frac{1}{2}$, which implies that

$$T-t^* < T-t_2' < T-t_3' < T \qquad (\text{Q.E.D.}). \qquad \square$$

Now that we know $T-t^* < T-t_2'$, it remains for us to check whether $T-t^* < T-t_1'$. To be able to check this, we first have to solve for t_1' and then check if $p_1(T-t_1') < 0.5$ [in which case $0 < T-t^* < T-t_1'$] or if $p_1(T-t_1') > 0.5$ [in which case $T-t_1' < T-t^* < T-t_2'$].

To determine the value of t_1', we first need to know t_2' and $p_1(T-t_2')$ as these will be used as initial conditions when solving (4.3.4) for t_1'. Hence, we first have to determine t_2', then $p_1(T-t_2')$, followed by t_1', and finally $p_1(T-t_1')$. At time $T-t_2'$, (4.3.3) becomes

$$(4.4.1) \qquad p_1(T-t_2') = \frac{1}{1+\beta} - \frac{(\ell n2)(1-\beta)\ e^{-(1+\beta)t_2'}}{1+\beta}\ J(T-t_2',\ T-t_3')$$

$$+ \frac{e^{-(1+\beta)t_2'}(1-\beta)(1-\ell n2)}{1+\beta}\left[(\ell n\ t_2') - \ell n\left[\frac{\ell n2}{1+\beta}\right]\right]$$

$$+ \frac{e^{-(1+\beta)t_2'}}{1+\beta}\left[\beta - 2 + (1-\beta)\ J(T-t_3',T)\right].$$

Equating (4.2.6) with (4.4.1), it follows that

$$(4.4.2) \quad e^{-(1+\beta)t_2'} = \frac{1}{1+\beta} - \frac{(\ell n2)(1-\beta)\ e^{-(1+\beta)t_2'}}{1+\beta}\ J(T-t_2',\ T-t_3')$$

$$+ \frac{e^{-(1+\beta)t_2'}(1-\beta)(1-\ell n2)}{1+\beta}\left[(\ell n\ t_2') - \ell n\left[\frac{\ell n2}{1+\beta}\right]\right]$$

$$+ \frac{e^{-(1+\beta)t_2'}}{1+\beta}\left[\beta - 2 + (1-\beta)\ J(T-t_3',T)\right].$$

Before we solve (4.4.2) for t_2', we first have to evaluate the integral (4.3.2). By (4.3.2), we have that

$$J(a,b) = \int_a^b \frac{e^{(1+\beta)t}-1}{t}\ d(T-t)$$

$$= \int_a^b \frac{e^{(1+\beta)(T-u)}-1}{T-u}\ du$$

$$= \sum_{r=1}^{\infty} \frac{(1+\beta)^r (T-a)^r}{r.r!} - \sum_{r=1}^{\infty} \frac{(1+\beta)^r (T-b)^r}{r.r!} .$$

It thus follows that

(4.4.3) $$J(a,b) = F(T-a) - F(T-b),$$

where we define $F(u) = \sum_{r=1}^{\infty} \frac{(1+\beta)^r u^r}{r.r!}$.

Combining (4.2.5) and (4.4.3) with (4.4.2), we have the following equation:

$$e^{(1+\beta)t_2'} - (\ell n2)(1-\beta)\left[F(t_2') - F\left[\frac{\ell n2}{1+\beta}\right]\right] + (1-\beta)(1-\ell n2)\left[\ell n\ t_2' - \ell n\left[\frac{\ell n2}{1+\beta}\right]\right]$$
$$+ (1-\beta)\ F\left[\frac{\ell n2}{1+\beta}\right] - 3 = 0.$$

Hence, for any given value of β, the above equation may be solved [using Newton's method recursively] for t_2'.

With β and the corresponding value of t_2' known, one can compute $p_1(T-t_2')$ using (4.2.6). With the computation of the initial condition t_2' and $p_1(T-t_2')$, we can now recursively solve (4.3.4) for t_1'.

To do this, we first apply the Runge-Kutta method [see Burden and Faires (1985)] to solve (4.3.4) at some arbitrary time $T-\hat{t}_1$ and obtain the solution of $p_1(T-\hat{t}_1)$. Then we will check whether this solution agrees with (4.3.9) when $T-t_1' = T-\hat{t}_1$. Alternating recursively between the Runge-Kutta method and a variation of the Bisection method would enable us to arrive at a $\tilde{t}$ such that $p_1(T-\tilde{t})$ [the solution of (4.3.4) at time $T-\tilde{t}$] agrees with (4.3.9) when $T-t_1' = T-\tilde{t}$. One can thus compute $t_1'[=\tilde{t}]$. Once t_1' has been computed, we can determine $p_1(T-t_1')$ and $p_2(T-t_1')$ via (4.3.9) and (4.2.7) respectively. [See Appendix A.]

To compute t^*, we proceed as follows:

(i) If $p_1(T-t_1') > 0.5$, we will solve (4.3.4) again [using (4.2.6) as the initial condition] numerically via the Runge-Kutta method at some arbitrary time $T-\hat{t}^*$ and check if $p_1(T-\hat{t}^*) = 0.5$. As before, alternating recursively between the Runge-Kutta method and a variation of the Bisection method, we will eventually arrive at a t^* such that $p_1(T-t^*) = 0.5$.

(ii) If $p_1(T-t_1') < 0.5$, we will use the known values of (4.3.9) and (4.2.7) as initial conditions to simultaneously solve (4.3.5) and (4.3.6). As before, we will alternate recursively between the Runge-Kutta method and a variation of the Bisection method to finally arrive at a t^* such that $p_1(T-t^*) = 0.5$.

(iii) The final but trivial alternative is when $p_1(T-t_1') = 0.5$. This simply means that $t^* = t_1'$. [See Appendix A.]

The following can be observed from Appendix A:

a. When $\beta \in [0, \ 0.1815)$, $t^* < t_1'$.

b. When $\beta = 0.1815$, $t^* = t_1'$.

c. When $\beta \in (0.1815, \ 1]$, $t^* > t_1'$.

With t^* and $p_1(T-t^*)$ [= 0.5] determined, we then find $p_2(T-t^*)$ as follows:

If $t^* \le t_1'$ [or equivalently $T-t_1' \le T-t^*$], then $p_2(T-t^*)$ is obtained by subtracting $p_1(T-t^*)$ from (4.3.8) at time $T-t^*$. Thus, we have that

$$p_2(T-t^*) = 0.5 - e^{-(1+\beta)t^*}. \tag{4.4.4}$$

Alternatively, if $t^* > t_1'$, then $p_2(T-t^*)$ is the solution of the non-linear differential equation (4.3.6) at time $T-t^*$ which is obtained as a by-product when solving (4.3.5) and (4.3.6) simultaneously for t^*. [See Appendix A.]

CHAPTER V

THE WINNING PROBABILITIES WHEN $p_1(T-t) \geq \frac{1}{2}$

5.1 INTRODUCTION

In this chapter, we will start with the known values of t^* and $p_i(T-t^*)$ [for $i = 1,2$] and then work backwards to determine the winning probabilities $p_i(0)$. Thus, we will be interested in (2.2.1) and (2.2.2) for the time interval $0 \leq T-t \leq T-t^*$ only.

5.2 THE TIME ZONES

To find the winning probabilities $p_i(0)$ [for $i = 1,2$], we will use t^*, $p_1(T-t^*)$ and $p_2(T-t^*)$ as the initial conditions and then solve (2.2.1) and (2.2.2) for the time interval $0 \leq T-t \leq T-t^*$. Since we have a different optimal strategy [with (3.3.3) replacing (3.3.2) because $p_1(T-t) \geq \frac{1}{2}$], the set of optimal strategies (3.2.1), (3.2.2), (3.3.1) and (3.3.3) will define 4 new time zones which are in no way related to the 4 time zones we had when $p_1(T-t) < \frac{1}{2}$. Before we define the times which constitute these 4 new time zones, we will rewrite the optimal strategies (3.2.1), (3.2.2), (3.3.1) and (3.3.3).

The optimal strategies (3.2.1), (3.2.2), (3.3.1) and (3.3.3) when combined with (2.2.3) will yield (5.2.1), (5.2.2), (5.2.3) and (5.2.4) respectively. Thus we have that

$$\text{(5.2.1)} \qquad \phi_{21}(x,\ T-t) = \begin{cases} 1 & \text{if} \quad x \geq \max[0,\ 1+\alpha\ \ell n\ p_2(T-t)] \\ 0 & \quad\quad \text{otherwise,} \end{cases}$$

$$\text{(5.2.2)} \qquad \phi_{11}(x,\ T-t) = \begin{cases} 1 & \text{if} \quad x \geq \max[0,\ 1-\alpha\ \ell n\ 2] \\ 0 & \quad\quad \text{otherwise,} \end{cases}$$

$$\text{(5.2.3)} \qquad \phi_{12}(x,\ T-t) = \begin{cases} 1 & \text{if} \quad x \geq \max[0,\ 1+\alpha\ \ell n\ p_1(T-t)] \\ 0 & \quad\quad \text{otherwise,} \end{cases}$$

and

$$\text{(5.2.4)} \qquad \phi_{22}(x,\ T-t) = \begin{cases} 1 & \text{if} \quad x \geq \max[0,\ 1+\alpha\ \ell n\ p_2(T-t)] \\ 0 & \quad\quad \text{otherwise,} \end{cases}$$

where $p_1(T-t) \geq \frac{1}{2}$.

Since these time zones have no relation with those when $p_1(T-t) < \frac{1}{2}$, we have decided to call the times which constitute these time zones by different names [namely $T-t_1$, $T-t_2$ and $T-t_3$], so as not to create confusion. With these, we can now define the times $T-t_1$, $T-t_2$ and $T-t_3$, where $0 < T-t_1 < T-t_2 < T-t_3 < T$, as follows:

$$\phi_{12}(x,T-t) = 1 \quad \text{for all} \quad x \quad \text{if} \quad T-t_3 \leq T-t \leq T,$$

$$\phi_{11}(x,T-t) = 1 \quad \text{for all} \quad x \quad \text{if} \quad T-t_2 \leq T-t \leq T,$$

and

$$\phi_{21}(x,T-t) = \phi_{22}(x,T-t) = 1 \quad \text{for all} \quad x \quad \text{if} \quad T-t_1 \leq T-t \leq T.$$

Writing down the expressions for the times $T-t_1$, $T-t_2$ and $T-t_3$, we have that

$$p_1(T-t_3) = e^{-(1+\beta)t_3}, \tag{5.2.5}$$

$$t_2 = \frac{\ln 2}{1+\beta}, \text{ and} \tag{5.2.6}$$

$$p_2(T-t_1) = e^{-(1+\beta)t_1}, \tag{5.2.7}$$

since $p_1(T-t)$ and $p_2(T-t)$ are both monotone.

We will assume, as before, that T is large enough so that $T-t_1$, $T-t_2$ and $T-t_3$ are all well defined positive quantities.

5.3 THE WINNING PROBABILITIES FOR THE TIME ZONES

Before we use the initial values t^*, $p_1(T-t^*)$ [= 0.5] and $p_2(T-t^*)$ to solve (2.2.1) and (2.2.2) for the time interval $0 \le T-t \le T - t^*$ so as to obtain $p_i(0)$ [for i = 1,2], we will first have to determine the zone in which the time $T-t^*$ [or equivalently, the residual time t^*] lies. In order to do this, we will use the boundary condition $p_1(T) = 0$ and solve (2.2.1) first in the time zone $T-t_3 \le T-t \le T$. Using the fact that $p_1(T-t)$ is a continuous solution, we will then work backwards to determine the zone which contains the time $T-t^*$.

<u>Case i</u>: For the time zone $T-t_2 \le T-t \le T$, with the aid of (5.2.1), (5.2.2), (5.2.3) and (5.2.4), one can rewrite (2.2.1) as

$$\dot{p}_1(T-t) = (1+\beta)p_1 - \beta - \alpha\,(1-\beta) + \alpha\,(1-\beta)\,e^{-\frac{1}{\alpha}}.$$

Using the initial condition $p_1(T) = 0$, the above differential equation leads us to the following solution:

$$(5.3.1) \qquad p_1(T-t) = \frac{\beta}{1+\beta}\left[1 - e^{-(1+\beta)t}\right] + \frac{1-\beta}{1+\beta}\,e^{-(1+\beta)t}\,J(T-t,T),$$

where $J(a,b)$ was defined in (4.3.2).

<u>Case ii</u>: For the time zone $T-t_1 \le T-t \le T-t_2$, combining (5.2.1), (5.2.2), (5.2.3) and (5.2.4) with (2.2.1), one can rewrite (2.2.1) as follows:

$$\dot{p}_1(T-t) = (1+\beta)p_1 - (1+\beta) + \alpha(\beta + \ln 2) - \alpha\,e^{-\frac{1}{\alpha}}\,(1+\beta).$$

Using the fact that $p_1(T-t)$ is a continuous solution and equating the solution of the above differential equation with (5.3.1) at time $T-t_2$, we have that

$$\begin{aligned}(5.3.2)\ p_1(T-t) = 1 - &\left[\frac{\beta+\ln 2}{1+\beta}\right] e^{-(1+\beta)t}\,J(T-t,\ T-t_2) + \\ &\left[\frac{1-\ln 2}{1+\beta}\right][\ln t - \ln t_2]\,e^{-(1+\beta)t} - \\ &e^{-(1+\beta)t}\left\{\frac{1}{1+\beta}\left[e^{(1+\beta)t_2} + \beta - (1-\beta)\,J(T-t_2,\ T)\right]\right\}.\end{aligned}$$

Case iii: For the time zone $0 \leq T\text{-}t \leq T\text{-}t_1$, one can similarly rewrite (2.2.1) as

$$(5.3.3) \qquad \dot{p}_1(T-t) = \alpha\Big[(1+\beta)((\ell n\ p_2) - p_1(\ell n\ p_2) - p_2) + (\beta+\ell n 2)\Big].$$

We will consider the above non-linear differential equation later in this chapter.

An alternative expression for $p_1(T-t_1)$ will now be obtained by considering

$$p(T-t) = p_1(T-t) + p_2(T-t) \leq 1.$$

This would now lead us to a differential equation $\dot{p}(T-t)$ which is identical to (4.3.7).

Using the initial condition $p(T) = 0$, for the time zone $T-t_1 \leq T-t \leq T$ we would have

$$(5.3.4) \qquad p(T-t) = 1 - e^{-(1+\beta)t}.$$

At time $T-t_1$, it follows that

$$(5.3.5) \qquad p_1(T-t_1) = 1 - 2e^{-(1+\beta)t_1}.$$

Now that we have established the necessary groundwork, we will proceed to find the zone in which $T-t^*$ lies.

5.4 DETERMINATION OF THE TIME ZONE WHICH CONTAINS THE TIME $T-t^*$

As before, we will start by proving the claim

$T-t^* < T-t_2 < T-t_3 < T.$

PROOF: Since $T-t_2 < T-t_3 < T$ [by definition], it suffices to just show that $T-t^* < T-t_2$ or equivalently $t^* > t_2$.

By (5.2.6), it follows that

$$\begin{aligned} t_2 &= \frac{\ell n\ 2}{1+\beta} \\ &\leq \ell n\ 2 \\ &< \inf_{\beta\in[0,1]} t^* = 1.0597. \qquad \text{[See Appendix A.]} \end{aligned}$$

Thus, we have that

$$T-t^* < T-t_2 < T-t_3 < T \qquad \text{(Q.E.D.).} \quad \square$$

Now that we have established the claim that $T-t^* < T-t_2$, it just remains for us to check if $T-t^* < T-t_1$. In order to do this, we have to find t_1 and then compare t_1 with t^* [which has been tabulated in Appendix A].

To find t_1, we just have to equate (5.3.2) at time $T-t_1$ with (5.3.5). This then leads us to the following:

$$P_1(T-t_1) = 1 - \left[\frac{\beta+\ln 2}{1+\beta}\right] e^{-(1+\beta)t_1} J(T-t_1, T-t_2) +$$

$$\left[\frac{1-\ln 2}{1+\beta}\right] [\ln t_1 - \ln t_2] e^{-(1+\beta)t_1} -$$

$$e^{-(1+\beta)t_1} \left\{\frac{1}{1+\beta} \left[e^{(1+\beta)t_2} + \beta - (1-\beta) J(T-t_2, T)\right]\right\}$$

$$= 1 - 2e^{-(1+\beta)t_1}.$$

Further simplification leads us to the following equation:

$$(5.4.1)\quad (\beta+\ln 2) \left[F(t_1) - F\left[\frac{\ln 2}{1+\beta}\right]\right] - (1-\ln 2) \left[\ln t_1 - \ln\left[\frac{\ln 2}{1+\beta}\right]\right] - \beta$$
$$- (1-\beta) F\left[\frac{\ln 2}{1+\beta}\right] = 0.$$

Thus, for any given value of β, (5.4.1) can be recursively solved for t_1 using the Newton's method. With the obtained value of t_1, one can easily check whether $t_1 < t^*$ [in which case $T-t_1 >$ $T-t^*$]. [See Appendix B].

The following can be easily observed from Appendix B:

a. When $\beta \in [0,0.1821)$, $t^* < t_1$.

b. When $\beta = 0.1821$, $t^* = t_1$.

c. When $\beta \in (0.1821, 1]$, $t^* > t_1$.

5.5 THE WINNING PROBABILITIES FOR THE TIME INTERVAL $0 \le T-t \le T-t^*$

Using the known values of t^*, $p_1(T-t^*)$ [= 0.5] and $p_2(T-t^*)$ as initial conditions, we will solve (2.2.1) and (2.2.2) in 2 time zones [namely $0 \le T-t \le T-t_1$ and $T-t_1 \le T-t \le T-t^*$] when $T-t_1 < T-t^*$ and in 1 time zone [namely $0 \le T-t \le T-t^*$] when $T-t^* < T-t_1$.

Thus, the problem can now be split into the following 2 cases:

CASE A: $0 \le \beta < 0.1821$, in which case $T-t_2 > T-t^* > T-t_1$

and

CASE B: $0.1821 \le \beta \le 1$, in which case $0 < T-t^* \le T-t_1$.

5.5.1 CASE A: $0 \le \beta < 0.1821$ [where $T-t_1 < T-t^* < T-t_2$]

Case A.1: For the time zone $T-t_1 \le T-t \le T-t^*$, (2.2.1) with the aid of (5.2.1), (5.2.2), (5.2.3) and (5.2.4) can be rewritten as

$$\dot{p}_1(T-t) = (1+\beta)p_1 - (1+\beta) + \alpha(\beta+\ln 2) - \alpha(1+\beta)\ e^{-\frac{1}{\alpha}}.$$

Using the initial condition $p_1(T-t^*) = 0.5$, the above differential equation has the following solution:

$$(5.5.1.1) \qquad p_1(T-t) = 1 - e^{-(1+\beta)t} \left\{ \left[\frac{\beta+\ln 2}{1+\beta}\right] J(T-t,\ T-t^*) - \left[\frac{1-\ln 2}{1+\beta}\right] (\ln t - \ln t^*) + \frac{1}{2}\, e^{(1+\beta)t^*} \right\}.$$

With the known value of t_1 [obtained by solving (5.4.1)] and t^*, one can solve (5.5.1.1) to obtain $p_1(T-t_1)$. [See Appendix B.]

Case A.2: For the time zone $0 \leq T-t \leq T-t_1$, (2.2.1) will be identical to (5.3.3),

i.e.

$$\dot{p}_1(T-t) = \alpha\ [(1+\beta)((\ell n\ p_2) - p_1(\ell n\ p_2) - p_2) + (\beta+\ell n2)].$$

Since the above differential equation depends on both p_1 and p_2, we will rewrite (2.2.2) as follows for the same time zone ($0 \leq T-t \leq T-t_1$) in order to obtain $\dot{p}_2(T-t)$:

$$\dot{p}_2(T-t) = \alpha[(1+\beta)\ p_2(1-\ell n\ p_2) - (\beta+\ell n2)]$$

or equivalently,

$$(5.5.1.2) \qquad \int_{p_2(T-t)}^{p_2(T-t_1)} \frac{d\ p_2}{\ell n\left[\left[\frac{e}{p_2}\right]^{p_2} e^{-\left[\frac{\ell n2+\beta}{1+\beta}\right]}\right]} = \ell nt - \ell nt_1.$$

Although we can solve (5.5.1.2) numerically, we will instead consider it asymptotically since we are interested in $p_i(0)$. [See Chapter VII]

$p_2(T-t_1)$ can be easily found via $p(T-t_1)$, by making the following consideration,

i.e.

$$p(T-t) = p_1(T-t) + p_2(T-t).$$

At time $T-t^*$, $p(T-t^*) = p_1(T-t^*) + p_2(T-t^*)$

$= m(\beta)$ [where $m(\beta)$ is a value that depends on β].

This value $m(\beta)$ can be computed via Appendix A where both $p_1(T-t^*)$ and $p_2(T-t^*)$ have been tabulated for different values of β.

Using the optimal strategies (5.2.1), (5.2.2), (5.2.3) and (5.2.4), we can rewrite (4.3.7) for the time zone $T-t_1 \le T-t \le T-t^*$ as follows:

$$\dot{p}(T-t) = (1+\beta)p - (1+\beta).$$

This ordinary differential equation has the following solution:

$$p(T-t) = 1 + [m(\beta) - 1]\, e^{(1+\beta)(t^*-t)}$$

where we have used the initial condition $p(T-t^*) = m(\beta)$.

Thus, $p_2(T-t_1)$ can be obtained by subtracting (5.5.1.1) from $p(T-t)$ and setting $t = t_1$. It is also obvious that as T increases, both the winning probabilities $p_1(T-t_1)$ and $p_2(T-t_1)$ will increase.

In the time zone $0 \leq T\text{-}t \leq T\text{-}t_1$, (4.3.7) becomes

$$\dot{p}(T-t) = \frac{(\ell n\ p_2)(1-p)}{t}$$

and the probability that someone wins may be expressed as

$$(5.5.1.3) \qquad \ell n[1-p(0)] = \ell n[1-p(T-t_1)] + \int_0^{T-t_1} \frac{\ell n\ p_2(T-t)}{t}\, d(T-t).$$

We will consider (5.5.1.3) further in Chapter VII.

5.5.2 <u>CASE B: $0.1821 \leq \beta \leq 1$ [where $0 < T\text{-}t^* \leq T\text{-}t_1$]</u>

For the time zone $0 \leq T\text{-}t \leq T\text{-}t^*$, (2.2.1) can be rewritten as (5.3.3), that is,

$$\dot{p}_1(T-t) = \alpha[(1+\beta)((\ell n\ p_2) - p_1(\ell n\ p_2) - p_2) + (\beta+\ell n2)].$$

Since this differential equation depends on both p_1 and p_2, we will again need an expression for $\dot{p}_2(T-t)$. Rewriting (2.2.2) for the time zone $0 \leq T\text{-}t \leq T\text{-}t^*$, we have that

$$\dot{p}_2(T-t) = \alpha[(1+\beta)\ p_2\ (1-\ell n\ p_2) - (\beta+\ell n2)]$$

or equivalently,

$$(5.5.2.1) \qquad \int_{p_2(T-t)}^{p_2(T-t^*)} \frac{d\,p_2}{\ln\left[\left[\frac{e}{p_2}\right]^{p_2} e^{-\left[\frac{\ln 2+\beta}{1+\beta}\right]}\right]} = \ln t - \ln t^*.$$

where we have used the known values of t^* and $p_2(T-t^*)$ as the initial condition.

As stated earlier, since we are interested in $p_i(0)$, we will consider (5.5.2.1) asymptotically in Chapter VII.

For this same time zone, we can rewrite (4.3.7) as follows:

$$\dot{p}(T-t) = \frac{(\ln p_2)(1-p)}{t}.$$

Thus, the probability that someone wins may be expressed as

$$(5.5.2.2) \quad \ln[1-p(0)] = \ln[1-p(T-t^*)] + \int_0^{T-t^*} \frac{\ln p_2(T-t)}{t}\, d(T-t).$$

We will again, as before, consider (5.5.2.2) in Chapter VII.

CHAPTER VI

THE TIME OF THE FIRST ACCEPTANCE

6.1 THE DISTRIBUTION OF $T-\tau$

Let $T-\tau$ be the time when an offer is first accepted by either player 1 or 2. Further, let $g(T-\tau)$ be the density of $T-\tau$. Thus, we will have that

$$g(T-\tau) = \sum_{n=1}^{\infty} g_n(T-\tau)$$

$$\text{where } [g_n(T-\tau)]\, \Delta(T-\tau) = P\begin{bmatrix} n^{\text{th}} \text{ offer arriving in time} \\ (T-\tau,\ T-\tau+\Delta(T-\tau)) \text{ is the first} \\ \text{to be accepted} \end{bmatrix}.$$

It is seen that when $0 \le T-\tau < T-t_1$, an offer is only accepted when $x \ge \xi(T-\tau) = 1 + \dfrac{\ell n\ p_2(T-\tau)}{(1+\beta)\tau}$. Alternatively, when $T-t_1 \le T-\tau \le T$, the first offer is readily accepted.

For the time interval $0 \le T-\tau < T-t_1$, one can show that

$$g_n(T-\tau) = (1+\beta)^n\ [1-\xi(T-\tau)]\ \frac{\left[\displaystyle\int_0^{T-\tau} \xi(T-y)\ d(T-y)\right]^{n-1}}{(n-1)!}\ \exp[-(1+\beta)(T-\tau)]$$

or equivalently,

$$(6.1.1)\qquad g(T-\tau) = (1+\beta)\ [1-\xi(T-\tau)]\ \exp\left[(1+\beta)\int_0^{T-\tau} [\xi(T-y)-1]\,d(T-y)\right].$$

Similarly, for the time interval $T-t_1 \le T-\tau \le T$, we have

$$g_n(T-\tau) = (1+\beta)^n \frac{\left[\int_0^{T-t_1} \xi(T-y)\ d(T-y)\right]^{n-1}}{(n-1)!} \exp[-(1+\beta)(T-\tau)].$$

or equivalently,

$$(6.1.2)\qquad g(T-\tau) = (1+\beta)\ e^{-(1+\beta)(T-\tau)}\ e^{(1+\beta)\int_0^{T-t_1} \xi(T-y)d(T-y)} .$$

6.2 THE MOMENTS OF $\frac{T-\tau}{T}$

To determine $E(T-\tau)$ and $Var\ (T-\tau)$, we could first find an expression for the moment generating function (m.g.f.) of the time of the first offer acceptance [i.e. $E(e^{z(T-\tau)})$]. But, since we are interested in these moments asymptotically, we will instead find an expression for the m.g.f. of the fraction of the time of the first offer acceptance.

Furthermore, since $g(T-\tau)$ is a defective density function, we will define the conditional m.g.f.

$$M\left[z, \frac{T-\tau}{T}\right] = E\left(e^{z\left[\frac{T-\tau}{T}\right]} \,\middle|\, \text{an offer is accepted}\right)$$

and also since

$$\begin{aligned} p(0) &= p(\text{someone wins}) \\ &= p(\text{an offer is accepted}), \end{aligned}$$

we can rewrite $M\left[z, \frac{T-\tau}{T}\right]$ with the aid of (6.1.1) and (6.1.2) as follows:

$$\begin{aligned} M\left[z, \frac{T-\tau}{T}\right] &= \int_0^T e^{z\left[\frac{T-\tau}{T}\right]} g(T-\tau)\, d(T-\tau) \\ &= \int_0^{T-t_1} e^{z\left[\frac{T-\tau}{T}\right]} (1+\beta)[1-\xi(T-\tau)] e^{(1+\beta)\int_0^{T-\tau} [\xi(T-y)-1]d(T-y)} d(T-\tau) \\ &+ \int_{T-t_1}^{T} e^{z\left[\frac{T-\tau}{T}\right]} (1+\beta)\, e^{-(1+\beta)(T-\tau)}\, e^{(1+\beta)\int_0^{T-t_1} \xi(T-y)d(T-y)} d(T-\tau). \end{aligned}$$

After some simplification, we have that

$$M\left[z, \frac{T-\tau}{T}\right] = 1 + \frac{z}{T}\int_0^{T-t_1} e^{z\left[\frac{T-\tau}{T}\right]} e^{(1+\beta)\int_0^{T-\tau}[\xi(T-y)-1]d(T-y)} d(T-\tau)$$
$$+ e^{(1+\beta)\int_0^{T-t_1}\xi(T-y)d(T-y)} \left\{-\frac{z}{z-(1+\beta)T} e^{\left[\frac{T-t_1}{T}\right][z-(1+\beta)T]} + \frac{(1+\beta)T}{z-(1+\beta)T} e^{[z-(1+\beta)T]}\right\}. \tag{6.2.1}$$

We will consider (6.2.1) asymptotically in the next chapter.

CHAPTER VII

ASYMPTOTIC RESULTS

In this chapter, we will derive results for $p_1(0)$, $p_2(0)$, $E[e^{z\left[\frac{T-\tau}{T}\right]}]$, $E\left[\frac{T-\tau}{T}\right]$ and $\mathrm{Var}\left[\frac{T-\tau}{T}\right]$ as $T \longrightarrow \infty$.

7.1 THE ASYMPTOTIC PROBABILITIES

Asymptotically, (5.5.1.2) and (5.5.2.1) yield the same result regardless of their initial conditions. We can similarly conclude that (5.5.1.3) and (5.5.2.2) will asymptotically yield the same result. Thus, it will suffice to just consider (5.5.1.2) and (5.5.1.3) [or alternatively (5.5.2.1) and (5.5.2.2)] asymptotically.

From (5.5.1.3), one can observe that as $T \longrightarrow \infty$,

$$\int_0^{T-t_1} \frac{\ell n\; p_2(T-t)}{t}\, d(T-t) \longrightarrow -\infty.$$

It thus follows from the left hand side of (5.5.1.3) that $\lim\limits_{T\to\infty} p(0) = 1$. This means that we definitely have a winner, but in an asymptotic sense. Thus, if $\lim\limits_{T\to\infty} p_2(0) = u$, it follows that $\lim\limits_{T\to\infty} p_1(0) = 1-u$.

Careful observation of (5.5.1.2) tells us that when $T-t = 0$ and $T \to \infty$, the right hand side of (5.5.1.2) diverges. This means that the denominator of the integrand in (5.5.1.2) must vanish. Thus, we have that

$$\left[\frac{e}{u}\right]^u e^{-\left[\frac{\ell n2+\beta}{1+\beta}\right]} = 1,$$

where we have used the notation $\lim_{T\to\infty} p_1(0) = 1-u$ and $\lim_{T\to\infty} p_2(0) = u$

or equivalently,

$$\left[\frac{u}{e}\right]^u e^{\frac{\beta+\ell n2}{1+\beta}} = 1 . \tag{7.1.1}$$

For a given value of β, we can solve (7.1.1) using Newton's method to obtain the value of u. Once u has been found, we can then compute the asymptotic probability of player 1 winning. [See Appendix B and C.] From Appendix B, we can also observe that when $\beta = 0$, our asymptotic probabilities agree with the results in Enns and Ferenstein (1988).

7.2 THE ASYMPTOTIC MOMENTS

First, we will show that the asymptotic expression for $M\left[z, \frac{T-\tau}{T}\right]$ is $1 + e^z \sum_{r=0}^{\infty} \frac{(-1)^r z^{r+1}}{r!(r+1-\ell nu)}$, following which we will show that the asymptotic values of $E\left[\frac{T-\tau}{T}\right]$ and $Var\left[\frac{T-\tau}{T}\right]$ are $(1 - \ell n\ u)^{-1}$ and $[2(1 - \ell n\ u)^{-1} (2 - \ell n\ u)^{-1} - (1 - \ell n\ u)^{-2}]$ respectively.

Rewriting (6.2.1), we have that

$$M\left[z, \frac{T-\tau}{T}\right] = L_1 + L_2 + 1,$$

where
$$L_1 = e^{(1+\beta)\int_0^{T-t_1} \xi(T-y)d(T-y)} \left\{- \frac{z}{z-(1+\beta)T}\, e^{\left[\frac{T-t_1}{T}\right][z-(1+\beta)T]} + \frac{(1+\beta)T}{z-(1+\beta)T}\, e^{[z-(1+\beta)T]}\right\}$$

and
$$L_2 = \frac{z}{T}\int_0^{T-t_1} e^{z\left[\frac{T-\tau}{T}\right]}\, e^{(1+\beta)\int_0^{T-\tau} [\xi(T-y)-1]d(T-y)}\, d(T-\tau).$$

Now, we will show that $\lim\limits_{T\to\infty} L_1 = 0.$

Due to the fact that we definitely have a winner in the asymptotic sense,

$$L_1 \le e^{(1+\beta)(T-t_1)+(\ell nu)\left\{\ell n\left[\frac{T}{T-(T-t_1)}\right]\right\}} \cdot \left\{-\frac{z}{z-(1+\beta)T}\, e^{\left[\frac{T-t_1}{T}\right][z-(1+\beta)T]}\right.$$

$$\left. + \frac{(1+\beta)T}{z-(1+\beta)T}\, e^{[z-(1+\beta)T]}\right\}$$

$$= u^{-\ell n\left[1-\frac{T-t_1}{T}\right]} \left\{-\frac{z}{z-(1+\beta)T}\, e^{z\left[\frac{T-t_1}{T}\right]} + \frac{(1+\beta)}{\frac{z}{T}-(1+\beta)}\, e^{[z-(1+\beta)t_1]}\right\}$$

$$\longrightarrow 0 \quad [\text{as} \quad T \longrightarrow \infty].$$

Rewriting L_2, we have that

$$L_2 = \frac{z}{T}\int_0^{T-t_1} e^{z\left[\frac{T-\tau}{T}\right]}\, e^{\int_0^{T-\tau}\left[\frac{\ell n\, p_2(T-y)}{y}\right] d(T-y)}\, d(T-\tau).$$

Letting $\ell n\, p_2(T-y) = \ell n\, u + c(T-y)$,

where $c(T-y) = \ell n\, p_2(T-y) - \ell n\, u$,

we have that

$$L_2 = J_1(T-\tau) - J_2(T-\tau),$$

$$\text{where} \quad J_1(T-\tau) = \frac{z}{T} \int_0^{T-t_1} e^{z\left[\frac{T-\tau}{T}\right]} \left[\frac{T}{T-(T-\tau)}\right]^{\ell nu} d(T-\tau)$$

$$\text{and} \quad J_2(T-\tau) = \frac{z}{T} \int_0^{T-t_1} e^{z\left[\frac{T-\tau}{T}\right]} \left[\frac{T}{T-(T-\tau)}\right]^{\ell nu} \left[1-e^{\int_0^{T-\tau} \frac{c(T-y)}{y}\, d(T-y)} \right] d(T-\tau).$$

With this, we will show that $\lim\limits_{T\to\infty} J_1(T-\tau) = e^z \sum\limits_{r=0}^{\infty} \frac{(-1)^r\, z^{r+1}}{r!(r+1-\ell nu)}$ and $\lim\limits_{T\to\infty} J_2(T-\tau) = 0$.

By our definition of $J_1(T-\tau)$, we can show that

$$J_1(T-\tau) = \frac{z}{T} e^z \int_0^{T-t_1} e^{-z\left[1-\frac{T-\tau}{T}\right]} \left[1 - \frac{T-\tau}{T}\right]^{-\ell nu} d(T-\tau)$$

$$= \frac{z}{T} e^z \sum_{r=0}^{\infty} \int_0^{T-t_1} \frac{(-1)^r\, z^r \left[1-\frac{T-\tau}{r}\right]^{r-\ell nu}}{r!} d(T-\tau)$$

$$= z\, e^z \left\{ \sum_{r=0}^{\infty} \frac{(-1)^r\, z^r}{r!(r+1-\ell nu)} - \sum_{r=0}^{\infty} \frac{(-1)^r\, z^r \left[1-\frac{T-t_1}{T}\right]^{r+1-\ell nu}}{r!(r+1-\ell nu)} \right\}$$

$$= e^z \left\{ \sum_{r=0}^{\infty} \frac{(-1)^r z^{r+1}}{r!(r+1-\ell nu)} \right\} \quad \text{as} \quad T \longrightarrow \infty .$$

Now, we will show that $\lim_{T\to\infty} J_2(T-\tau) = 0$.

$$J_2(T-\tau) = \frac{z}{T}\int_0^{T-t_1} e^{z\left[\frac{T-\tau}{T}\right]}\left[\frac{T}{T-(T-\tau)}\right]^{\ell nu\left[1-e^{\int_0^{T-\tau}\frac{c(T-y)}{y}\,d(T-y)}\right]} d(T-\tau).$$

Letting $\delta = -\ell n\, u$, $J_2(T-\tau)$ may be rewritten as

$$J_2(T-\tau) = \frac{z}{T}\int_0^{T-t_1} e^{z\left[\frac{T-\tau}{T}\right]}\left[1-\frac{T-\tau}{T}\right]^{\delta\left[1-e^{\int_0^{T-\tau}\frac{c(T-y)}{y}\,d(T-y)}\right]} d(T-\tau),$$

where

$$0 \le J_2(T-\tau) \le \frac{z}{T}\int_0^{T-t_1} \lim_{\epsilon\to 0^+} \left\{ e^{z\left[\frac{T-\tau}{T}\right]}\left[1-\frac{T-\tau}{T}\right]^{\epsilon\left[1-e^{\int_0^{T-\tau}\frac{c(T-y)}{y}\,d(T-y)}\right]}\right\} d(T-\tau).$$

Letting $f(T-\tau) = \lim_{\epsilon\to 0^+}\left\{ e^{\frac{z}{T}(T-\tau)}\left[1-\frac{T-\tau}{T}\right]^{\epsilon\left[1-e^{\int_0^{T-\tau}\frac{c(T-y)}{y}\,d(T-y)}\right]}\right\}$,

we see that when $T \longrightarrow \infty$,

$$f(0) = 0 \quad \text{and} \quad f(T-t_1) = 0.$$

Thus, it now remains for us to show that $f(T-\tau)$ is either monotonically increasing or decreasing when $(T-\tau) \in [0, T-t_1]$. It can be seen that $f'(T-\tau) > 0$. Thus, we have that

$$\lim_{T\to\infty} \int_0^{T-t_1} f(T-\tau)\, d(T-\tau) = 0$$

or equivalently,

$$\lim_{T\to\infty} J_2(T-\tau) = 0.$$

With this, one can now write the asymptotic expression for the m.g.f. of the fraction of the time of the first offer acceptance as follows:

$$\text{(7.2.1)} \qquad \lim_{T\to\infty} M\left[z, \frac{T-\tau}{T}\right] = \lim_{T\to\infty} E\left[e^{z\left[\frac{T-\tau}{T}\right]}\right]$$

$$= e^{z} \sum_{r=0}^{\infty} \frac{(-1)^r z^{r+1}}{r!\,(r+1-\ell nu)} + 1 .$$

Using (7.2.1), we can compute $\lim_{T\to\infty} E\left[\frac{T-\tau}{T}\right]$ and $\lim_{T\to\infty} \mathrm{Var}\left[\frac{T-\tau}{T}\right]$ as follows:

$$\lim_{T\to\infty} M\left[z, \frac{T-\tau}{T}\right] = \frac{d}{dz}\left\{e^z \sum_{r=0}^{\infty} \frac{(-1)^r z^{r+1}}{r!(r+1-\ell n u)} + 1\right\}\Bigg|_{z=0}$$

$$= (1 - \ell n\ u)^{-1},$$ [which agrees with the result in Enns and Ferenstein (1988)].

Furthermore, since

$$\lim_{T\to\infty} \mathrm{Var}\left[\frac{T-\tau}{T}\right] = \frac{d^2}{dz^2}\left[\lim_{T\to\infty} M\left[z, \frac{T-\tau}{T}\right]\right]\Bigg|_{z=0} - [(1 - \ell n\ u)^{-1}]^2,$$

we can arrive at the result

$$\lim_{T\to\infty} \mathrm{Var}\left[\frac{T-\tau}{T}\right] = 2(1 - \ell n\ u)^{-1}\ (2 - \ell n\ u)^{-1} - (1 - \ell n\ u)^{-2}.$$

WINNING PROBABILITIES OF THE PLAYERS WHEN THE RESIDUAL TIMES ARE t'_2, t'_1 AND t^*

APPENDIX A

β	t'_2	$p_1(T-t'_2)$	t'_1	$p_1(T-t'_1)$	$p_2(T-t'_1)$	t^*	$p_1(T-t^*)$	$p_2(T-t^*)$
0.0000	0.8039	0.4476	1.5288	0.5664	0.2168	1.0597	0.5000	0.1534
0.0360	0.7928	0.4399	1.4439	0.5519	0.2241	1.0906	0.5000	0.1769
0.0720	0.7816	0.4326	1.3671	0.5381	0.2310	1.1175	0.5000	0.1982
0.1080	0.7706	0.4258	1.2974	0.5250	0.2375	1.1402	0.5000	0.2173
0.1440	0.7596	0.4194	1.2338	0.5124	0.2438	1.1585	0.5000	0.2343
0.1800	0.7488	0.4133	1.1757	0.5005	0.2498	1.1728	0.5000	0.2494
0.1815	0.7483	0.4131	1.1733	0.5000	0.2500	1.1733	0.5000	0.2500
0.1821	0.7482	0.4130	1.1725	0.4998	0.2501	1.1735	0.5000	0.2502
0.1900	0.7458	0.4117	1.1604	0.4973	0.2514	1.1763	0.5000	0.2533
0.3900	0.6886	0.3840	0.9164	0.4405	0.2798	1.3540	0.5000	0.3244
0.5900	0.6371	0.3631	0.7530	0.3959	0.3020	1.8094	0.5000	0.3870
0.7900	0.5915	0.3469	0.6377	0.3613	0.3194	2.1962	0.5000	0.4457
0.9900	0.5512	0.3339	0.5529	0.3345	0.3328	2.3054	0.5000	0.4976
1.0000	0.5493	0.3333	0.5493	0.3333	0.3333	∞	0.5000	0.5000

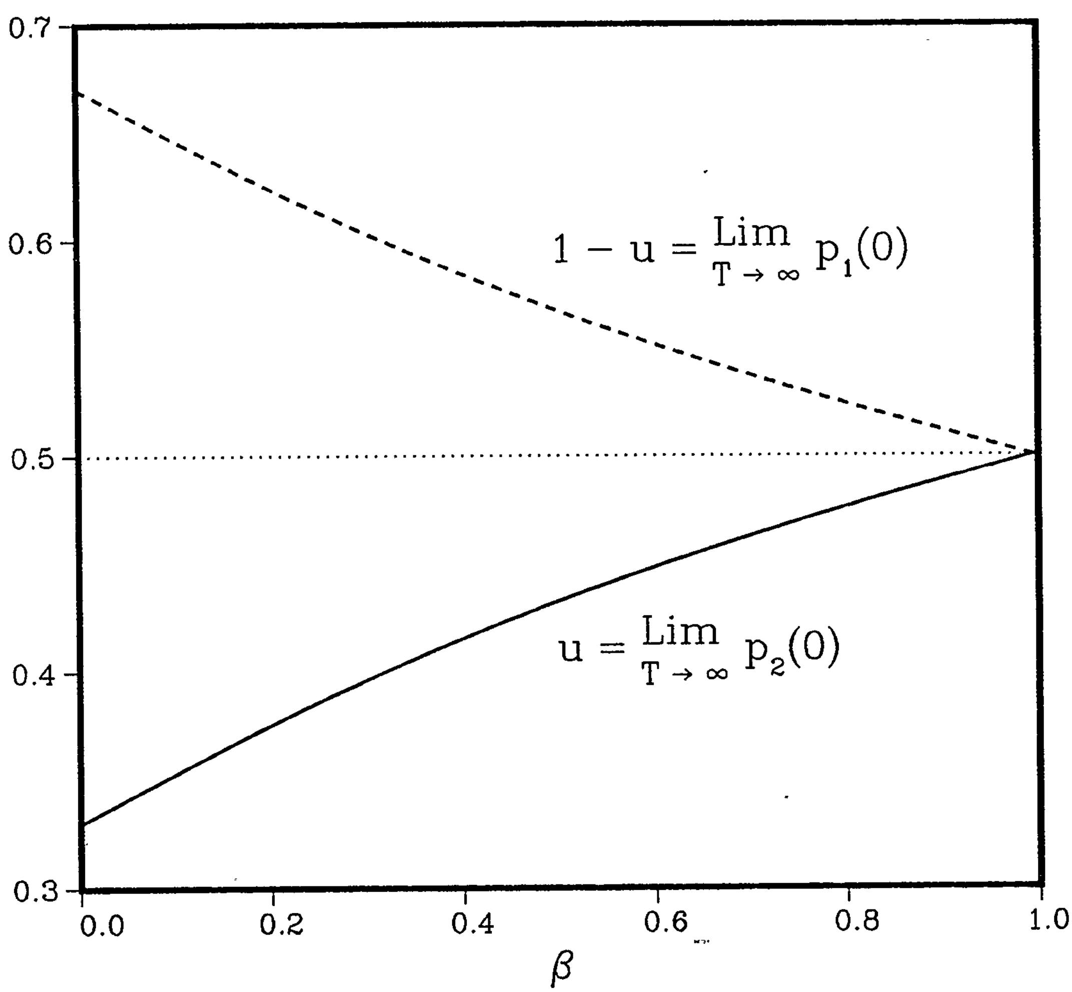
WINNING PROBABILITIES OF THE PLAYERS WHEN THE RESIDUAL TIMES ARE t_1 AND ∞

APPENDIX B

β	t^*	t_1	$p_1(T-t_1)$	$p_2(T-t_1)$	$1-u = \lim_{T\to\infty} p_1(0)$	$u = \lim_{T\to\infty} p_2(0)$
0.0000	1.0597	1.5300	0.5669	0.2165	0.6724	0.3276
0.0360	1.0906	1.4447	0.5523	0.2239	0.6628	0.3372
0.0720	1.1175	1.3678	0.5384	0.2309	0.6535	0.3465
0.1080	1.1402	1.2980	0.5252	0.2375	0.6446	0.3554
0.1440	1.1585	1.2346	0.5126	0.2438	0.6361	0.3639
0.1800	1.1728	1.1767	0.5007	0.2499	0.6279	0.3721
0.1815	1.1733	1.1743	0.5002	0.2501	0.6276	0.3724
0.1821	1.1735	1.1735	0.5000	0.2502	0.6274	0.3726
0.1900	1.1763	1.1615	–	–	0.6257	0.3743
0.3900	1.3540	0.9198	–	–	0.5858	0.4142
0.5900	1.8094	0.7582	–	–	0.5529	0.4471
0.7900	2.1962	0.6433	–	–	0.5250	0.4750
0.9900	2.3054	0.5579	–	–	0.5011	0.4989
1.0000	∞	0.5542	–	–	0.5000	0.5000

ASYMPTOTIC WINNING PROBABILITIES OF BOTH PLAYERS FOR DIFFERENT VALUES OF β

APPENDIX C



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