Syntonic Appropriation for Growth in Mathematical Understanding: An Argument For Curated Robotics Experiences

Krista Francis
University of Calgary

Steven Khan
University of Alberta

Francis, K. & Khan, S. (2021). Syntonic Appropriation for Growth in Mathematical Understanding: An Argument for Curated Robotics Experiences (pp. 143-169). In Danesi M., Dragana, M., & Costa, S. (Eds.) *Math (Education) in the Information Age.* Springer.

Syntonic Appropriation for Growth in Mathematical Understanding: An Argument For Curated Robotics Experiences

Introduction: "There are no numbers between 5 and 6"

We open with an anecdote from our initial co-writing meetings about students' growth in mathematical understanding. We chose this scene to provide an illustrative example of what we intend to highlight, viz. that curated robotics experiences offer opportunities for syntonic appropriation that contributes to learners' growth in mathematical understanding.

> Krista: When I work with children aged 9-11 years old (Grade 4 to 6) with robots I've noticed in their conversations they often refer to the robot as themselves ("I need...") or extensions of themselves ("We need…"). Adults, teachers and pre-service teachers, also often refer to the robot as themselves. For example when they say "we're *so* close" as they try to get the robot to travel 100 centimeters exactly or "we need to go back one" on similar tasks. I've also noticed how quickly children seem to learn decimal numbers when they do this task or similar tasks. When I first started working in one school (3 years ago) and asked what's between 2 and 3 or 4 and 5, it didn't matter which grades, 4, 5 and 6 (Division 2 or Upper Elementary) the children would confidently answer "there is nothing" between 2 and 3 or between 4 and 5 and that's despite having up to three years of learning about decimal numbers.

> Steven: Since I work mostly with elementary pre-service teachers, let's look at the K-8 Program of Studies (Alberta Education, 2016) (Curriculum document) and the Achievement Indicators (2016) (Supplementary document) to see where decimals first appear, as what happens in classrooms here in Canada I've found is very much driven by the Provincial Curriculum documents. Decimals appear in Grade 4 in the Number Strand (See Appendix 1) and the goal of comparing and ordering decimal numbers becomes a required curriculum outcome by the end of Grade 5. The specific model of using a number line is not mentioned in the Program of Studies but is in the accompanying Achievement Indicators document which is meant to provide some ideas to teachers but is not the 'curriculum' and teachers are not legally required to report on the ideas presented there.

> Krista: Grade 4 teachers might say that this (using the number line for decimals) is not their responsibility. But the number line is such a powerful tool though (Braithwaite & Siegler, 2018; Obersteiner et al., 2019). It elaborates and extends the ordering of whole numbers, especially for decimal fractions, and points towards the real number continuum later on that it is surprising to me that it is not even considered until Grade 5.

> Steven: Checking the Program of Studies, I see that the number line is introduced as an expectation in Grade 1 with respect to benchmarks of the whole numbers 0, 5, 10

Francis, K. & Khan, S. (2021). Syntonic Appropriation for Growth in Mathematical Understanding: An Argument for Curated Robotics Experiences (pp. 143-169). In Danesi M., Dragana, M., & Costa, S. (Eds.) *Math (Education) in the Information Age.* Springer.

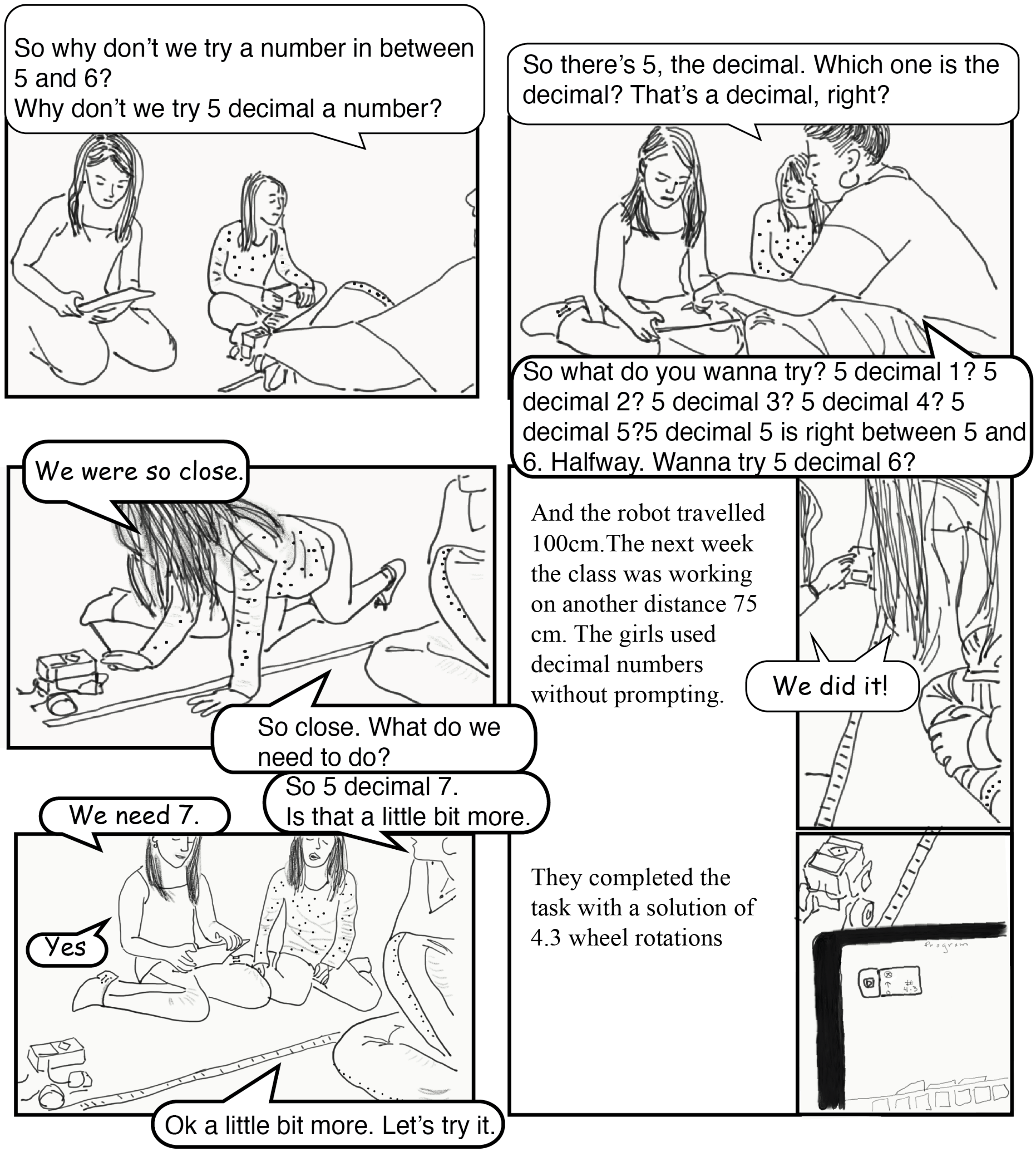
and 20 but is not with respect to fractions and decimals at least not in the stated expectations. It is however explicitly called on in the Achievement Indicators in Grade 4 (Name fractions between two given benchmarks on a numberline. Order a given set of fractions by placing them on a number line with given benchmarks) and Grade 5 (Position a given set of fractions with like and unlike denominators on a number line, and explain strategies used to determine the order; Order a given set of decimals by placing them on a number line that contains the benchmarks 0.0, 0.5 and 1.0.) and Grade 6 (Place a given set of fractions, including mixed numbers and improper fractions, on a number line, and explain strategies used to determine position.)

Krista: Well, I am going to keep this information in mind for future teaching and professional learning. Below (Figure 1) is an illustration of an interaction that two Grade 4 girls had with their teacher while programming their robot to travel exactly 100 cm. This was their first experience programming the robot to move and their first experience with decimal numbers. They had been working on this task for about half an hour before this interaction occurred. They had observed that 5-wheel rotations did not travel far enough and 6 was too far. Their challenge was understanding that there were numbers between 5 and 6.

Figure 1

How many wheel rotations for the robot to travel 100 cm?

Francis, K. & Khan, S. (2021). Syntonic Appropriation for Growth in Mathematical Understanding: An Argument for Curated Robotics Experiences (pp. 143-169). In Danesi M., Dragana, M., & Costa, S. (Eds.) *Math (Education) in the Information Age.* Springer.



Krista: The next week this pair of girls were using decimal numbers as they were trying to figure out how many wheel rotations to travel 73 cm. They solved the questions with an answer of 4.2. This was a variation of the context and the girls demonstrated a spatial sense of the number indicating it was closer to 4 than 5.

We framed our discussions about these incidents in terms of a growth in mathematical understanding following the model offered by Pirie and Kieren (1994a, 1994b). We believe that a well structured robotics inquiry (such as these described in this paper and others) allows students to discern critical features of a concept (Marton, 2014, 2018) through providing multiple instantiations of the concept (available through different embodied metaphors and enactions) and multiple opportunities to relate to its different aspects.

In this paper we argue that a well structured robotics inquiry can lead to what Pirie and Kieren (1994a; 1994b) called growth in mathematical understanding. In particular we offer that such structuring is a means to encourage processes of syntonic appropriation as introduced by Papert (1980). We start with the observation that some mathematical concepts are introduced to learners in ways that are disassociated from learners' bodies, experiences and/or culture(s). Consequently, learners struggle to apply/relate the mathematics concepts in novel circumstances - such as in a robotics environment. Learning in this case is superficial and fragmented though our intention is for such knowledge to become deep and connected.

The intentional design of instructional tasks can focus use of mathematical concepts for making the robots move more precisely in order to prompt growth in mathematical understanding around mathematical concepts such as the existence of rational numbers and their 'location' or 'magnitude' on the visuo-spatial representation of a number line. Our goal is to suggest and illustrate how the Pirie-Kieren model can be used to highlight/draw attention to some of the growth in mathematical understanding within a curated robotics learning experience. The growth in mathematical understanding we observe involves students (gradually) appropriating models and concepts in a syntonic way through curating their own experiences of learning from the opportunities provided by the teacher's previously curated robotics learning task(s). We use the metaphor of curating in this paper as we have found that there is value in shifting our thinking from the language of *accumulating* of experience (Khan, Francis & Davis, 2015) to the intentional and deliberate *curating* of such experience which involves keeping an imagined audience and their interactions in mind.

Arguing for Curating Tasks - Theoretical Framing

Knowing Is Doing

Our main theoretical commitment is to enactivism (see Khan, Francis & Davis, 2015). However, in this paper we also draw on theories of embodied cognition in the learning of mathematics (Lakoff and Nunez, 2000) and computational thinking (Buteau et al., 2016; Francis, Khan, & Davis, 2016; Grover & Pea, 2013; Wing, 2006).

Briefly, enactivism is (1) a theory of engagement (2) that is simultaneously attentive to the coupling of organisms and their environments, action as cognition, and sensorimotor coordination. (3) Attending to relevant phenomena of interest involves a methodological eclecticism (Di Jaegher & Di Paulo, 2013) that is concerned with inter-agent dynamics that include feedback from the system and the organism's responses. In our work students, teachers, researchers, tasks and technologies are dynamically coupled and provide feedback to each other.

Enactivist theories of human learning attend explicitly and deliberately to action, feedback, and discernment. They emphasize the bodily basis of meaning. As Brown and Coles (2011) note, ''[t]he enactive conception of knowledge is essentially performative'' (p. 861), i.e. knowing is doing. While constructivism can also be interpreted as performative, the focus is on the outcome of actions rather than the process of interactions as in enactivism. Enactivism is attentive to the many feedback structures in a greater-than-the-individual-learner system. It is the organism as a whole, together with its environment, which co-evolves in enactivism.

We work from the position of Varela, Thompson and Rosch (1991) that contends that the enactivist approach comprises two principles, viz. that ‘‘(1) perception consists in perceptually guided action and (2) cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided’’ (p. 173). That is to say, what an individual perceives is dependent on, but not determined by, the types of sensory stimuli that the individual’s body, its physical interface with the world encounters.

For example, with respect to perceptually guided action, Rushton (2008) states, ‘‘[t]o walk to a target you need to know where it is’’ (p. 36), i.e., our potential for action (walking or moving in terms of the robot and body syntonicity) and goal (destination) is dependent on the perception and selection of sensory information from the physical world. Work in developmental psychology (e.g., Keen et al., 2003), which analyses infants reaching for objects, exemplify both principles of the enactivist approach we are using.

Over time, repeated activity (action) establishes predictable (statistical) patterns of neural and neuromuscular activity that in turn influences the response to the stimuli sought or encountered. Humphreys et al. (2010) argue and present experimental evidence that, ‘‘our need to act upon the world not only imposes a general need for selection on our perceptual systems, but it directly mediates how selection operates. Attention is grounded in action’’ (p. 186). It is action then or rather potential for action that focuses attention on some features of the environment such that some aspects of the sensory landscape are perceived and others are not. It is these ongoing focusing of perceptions and sensations and evaluation of goal states (feedback) that guides action. This focusing of intention is an attribute of the individual learner as well as the learning designer-teacher-curator.

We distinguish between related enactivist and embodied perspectives in that in enactivism the external environment plays a significant role in understanding the dynamic unfolding of cognitive processes: what is in the environment is a resource for thinking, doing/knowing and being. While enactivism is attentive to ongoing co-constituted interaction among bodily action, cognition, and the environment, theories of embodied cognition focus on the relationship between cognition and *prior* action. In other words, cognition, in embodied models, is closely tied to prior sensorimotor experiences. Rather than predictions of learning a concept, enactivism is concerned with the learning *in* action since it is the potential for action in the world that focuses attention and drives learning. Embodied cognition is concerned with the learning *from* action. Embodied cognition can be regarded as a sort of consolidation of enactive action. In this work both of these theoretical frames are necessary in our attempt to make sense of learners’ growth in mathematical understanding in a robotics learning environment.

Hutto (2013) argued that enactivism, with its starting assumption that mental life can be understood as embodied activity, is a good candidate for ‘‘defining and demarcating [psychology’s] subject matter’’ (p. 174)—that is, in his terms, for ‘‘unifying psychology.’’ Traditional perspectives, he argued, delimit psychological explanations to ones that rely on inner representational states. He noted that enactivism, in its original formulation by Varela et al. (1991), attended explicitly to organisms’ varied engagements with contexts ‘‘not only of the biological kind but also of sociocultural varieties’’ (p. 177). The robotic moving task illustrated in Figure 1 could be interpreted as merely a manipulation of an inner representation but to our understanding from an enactivist perspective it is not at this point in time. Rather, they are in the process of constructing a mental number line (as an *object-to-think-with*) through action with the robot, the programming interface, the task specifications, and each other. By the last panel though where they have completed the task accurately, embodied perspectives help us to better understand how decimal numbers and a number line have been appropriated through metaphors of imagined robotic movement. We see value here in drawing readers' attention to this necessary shift in theoretical tools to better analyse and understand the growth in mathematical understanding of learners at two different points in developmental time. Bridging these two moments in time we use the Pirie-Kieren model which helps us to carefully notice and name subtle shifts in attention as evidenced by changes in language, gesture and performance (action).

Teaching is Presenting Appropriable Challenges

From Accumulating to Curating

In previous work we used the idea of learning from an enactivist perspective as an *accumulating* of sufficient and diverse experiences (Francis, Khan, Davis, 2016; Khan, Francis & Davis, 2015). Over time, in work with students and teachers the limits of this descriptor have become more apparent as we have critically appraised our own growth in understanding about how people (National Academies of Sciences, Engineering, and Medicine, 2018) and systems learn (Davis, Francis, & Friesen, 2019; Dehaene, 2020). Learners and teachers do appreciatively much more than 'accumulate' experiences, they attach affective (probabilistic) weights and meaning to these experiences. The metaphor of curating has emerged as a more apt descriptor than accumulating.

We take curating as a literacy practice. Looking to its linguistic origin we find both curation as a noun and a verb. As a noun, a curate refers to a person tasked with the care or cure of souls. We choose to read 'care' as deliberate and loving attention to the necessary aspects for the realization of well-being in an Other (including the self). We read 'cure' (of a soul) not in a medically restorative sense or the elimination of a disease, but rather as a learning how to live and be well in the world (eg. Seligman, 2011) with others. In more recent usage, the verb curate refers to the actions of selecting, organizing, and presenting something for an intended (or imagined) audience based on expert knowledge and values. According to art historian, Donald Preziosi (2019), curation,

> involves the *critical use* of parts of the material environment both for constructing and deconstructing the premises, promises, and potential consequences of what are conventionally understood as realities, or social, cultural, political, philosophical, or religious truths. It is a way of using things to think with and to reckon with—to struggle with and against—their possible consequences. It is an *epistemological technology*: a craft of thinking. As such, it is not innocent or innocuous…[It] entails the *conscious* juxtaposition and orchestration of what in various Western traditions were distinguished as "subjects" and "objects": what are conventionally differentiated as "agents," and as what is "acted upon." Curating not only precedes and is more fundamental than exhibitions, galleries, collections, and museums; but it is also not unique, nor exclusive, to any of those institutions and professions. In fact, it is not even an "it" at all but is, rather, a way of using things: potentially any things. In short, curating is a *creative performance* using the world to think about, and both affirm and transform, the world (p.11, italics added)

Curation here involves critical, craft and creative thinking with awareness or consciousness, i.e. it is not mere collection (or accumulation) and display, that is oriented towards an imagined audience and intended for learning. The perspective above shares many resonances with Papert's framing of bricolage (Papert, 1980). It is engaging in bricolage that provides the necessary and diverse occasions for syntonic appropriation.

As independent curator Glen Adamson argues, curation is about manipulating and trading in the attention economy, “You are drawing people’s attention to objects in a different and heightened way. The other big idea for me is that curation is really about attention. The medium you work in as a curator is attention, and we live in a so-called “attention economy,” in the sense that what people pay attention to is itself a form of value” (Acosta & Adamson, 2017). Attention is one pillar that is key to learning (Dehaene, 2020). Our goal is to suggest and illustrate how the Pirie-Kieren model can be used to highlight/draw attention to some of the growth in mathematical understanding within a curated robotics learning experience.

> Krista: When I first started doing this work my attention was on the engineering process (design), partly because I was taught/mentored in the design of robotics tasks but my son who was in engineering at the time. But as I started to recognize glimmers of the potential for mathematics learning, I began to curate and explore more mathematical tasks as opposed to design tasks. I found that as students gained some of the mathematical connections in programming robots, their skill at the programming design also improved, such as manoeuvring the robot precisely. Michael recently told me that if he would have learned how the move steering worked in Grade 9 (mathematically modeled) it would have put him years ahead in the robotics competitions in which he participated.

Rephrasing this in the language of our paper, a deeper more complex and focused syntonic appropriation and appreciation of the robot’s functioning might occur alongside (in synchrony with) a growth in mathematical understanding needed for personal goal achievement. Getting there however requires our intentional focusing of students’ attention through our curated task. Teachers, we think, have always worked the economics of attention in classrooms, schools and larger collectives, and in doing so have developed or utilised skills in curating. What we are trying to do is to draw attention to that challenging aspect (curating) of teachers’ work that is not quite captured in the idea of Mathematics-for-Teaching (M4T) (eg. Davis & Renert, 2014) or Mathematical Knowledge for Teaching (Ball et al., 2005) or Technological Pedagogical Content Knowledge (TPCK) (Koehler & Mishra, 2005) and to find ways to value it and develop it more intentionally as part of the work we do in our different professional networks with pre-service and in-service teachers and colleagues in different communities of practice.

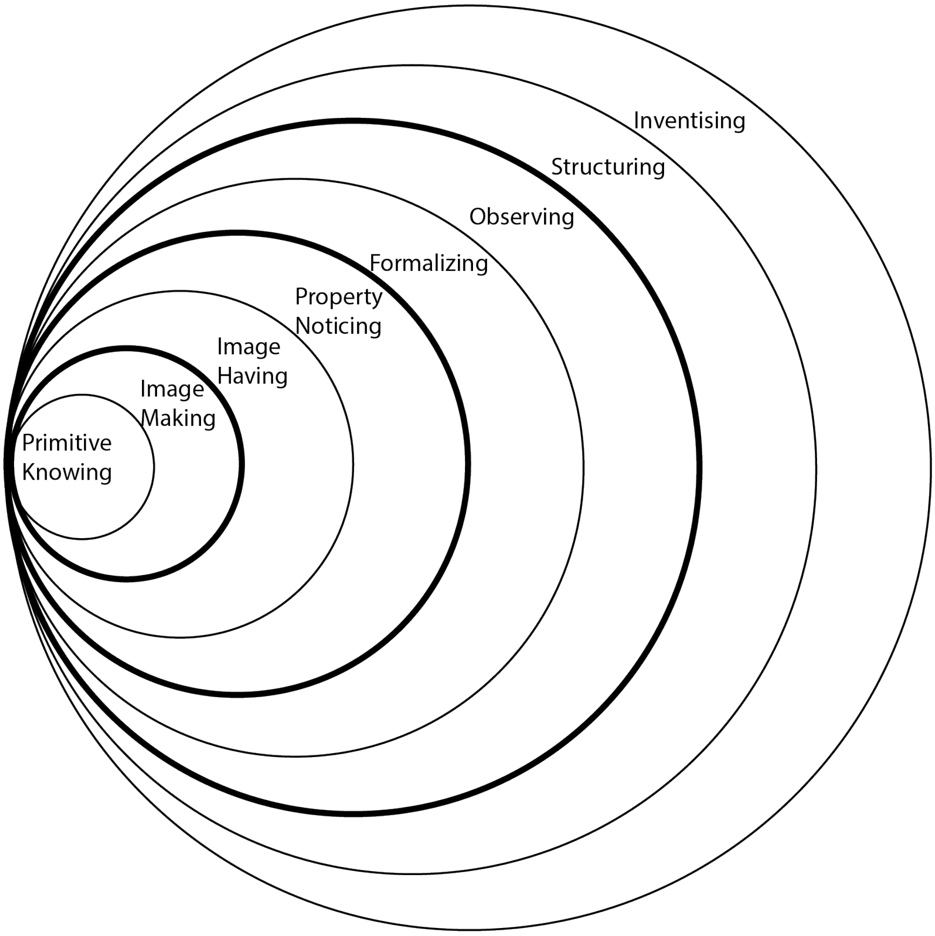
> Steven: Teachers and pre-service teachers have taught me the value of the emotional dimension for learners in tasks, I hope that I manage to shift their understanding that while learners might be engaged because of the emotional investment in the task, their attention as a teacher has to be on what is mathematically significant.]

In our next section, we introduce some elements of the Pirie-Kieren model of growth in Mathematical Understanding which we use as an analytic frame.

The Pirie-Kieren Model of Growth In Mathematical Understanding

There are a variety of framings of mathematical understanding (eg. Hiebert & Carpenter, 1992; Sfard, 1991; Sierpinska, 1994; Simon, 2006; Skemp, 1976). George (2017) offers a historical analysis of the concept in mathematics education. An enactivist framing of understanding however grounds it in terms of the dynamics of action (or co-action) and potential actions in a world, i.e. to say individual understandings are not static or 'fixed' but contextually and temporally dependent, grounded in experience and interpretations of experience, that may be challenging to articulate. This view of understanding is a non-linear or complex one (Davis & Simmt, 2003). As such, we draw on Pirie and Kieren's (1994a; 1994b) model of growth of mathematical understanding through non-linear back-and-forth movements of the following modes: *primitive knowing, image making, image having, property noticing, formalising, observing, structuring* and *inventising* (see Figure 2 below).

Figure 2
Illustration of Pirie and Kieran model of growth in mathematical understanding



We use the Pirie-Kieren model as an analytic tool to illustrate children's growth in understanding of mathematical concepts and computational thinking concepts (see Namukasa, 2019 for an example of the relationship). Coming to understanding starts with primitive knowing. "Primitive here does not imply low level mathematics, but is rather the starting place for growth of any particular mathematical understanding" (Pirie &Kieren, 1994a, p. 170). This is what a learner

Francis, K. & Khan, S. (2021). Syntonic Appropriation for Growth in Mathematical Understanding: An Argument for Curated Robotics Experiences (pp. 143-169). In Danesi M., Dragana, M., & Costa, S. (Eds.) *Math (Education) in the Information Age.* Springer.

brings with them at the beginning of a new task sequence. According to Pirie and Kieren (1994a), students' *primitive knowing* can be assumed, i.e., the skills they have initially. In the case of these students working with Lego EV3 robots such *primitive knowing* includes knowing related to mathematics such as spatial reasoning, proportional thinking, number; programming; technical procedural knowing such as how to connect the robots to the iPad and download and run programs; and knowledge about classroom routines, norms and procedures including how to work in small groups.

Image making is when a student records and reflects on *primitive knowing* (eg. creating an object through drawing or manipulatives). In the example above (Figure 1) the girls are beginning to think about (make an image) of what is between the numbers 5 and 6 (prompted by their engagement with the robot as well as their teacher and the task).

Image having is when a student no longer requires acting on the object. Pirie & Kieren (1994a) note that, "[a]t the mode of *image having* a person can use a mental construct about a topic without having to do the particular activities which brought it about" (p. 170). Between the second and the third frame of Figure 1 above, the students are starting to recognise how the decimal numbers are related incrementally, i.e. they are beginning to develop a spatial sense of (decimal) numbers on a number line.

At the fourth level of understanding, *property noticing*, one can manipulate or combine aspects of one's own images to "construct context specific properties"(Pirie & Kieren, 1994a, p. 170). In our example, the fourth frame when students say "we need 7", we take this as indicating a movement from *image-making* to i*mage having* in that students are no longer thinking directly 'with' the robot but are able to use their mental image of decimals on a number line. Mathematically, that is as far as we perceived the students' movement in the model. Having also worked with Marton's (2014; 2018) Variation Theory of Learning we see the various modes in the PK model as dynamic networks of critical discernments of student thinking or understanding. This is particularly evident with the *property noticing* mode.

However from a programming standpoint, in Frame 2 of Figure 1, the students very quickly moved from *primitive knowing* of how to program to move whole wheel rotations, through *image making, image having*, *property noticing* to *formalising* how to program decimals into the move steering block. *Formalising*, is when one can abstract a "method or common quality from the previous image" (Pirie & Kieren, 1994a, p. 171). We have demonstrated previously (Francis, Khan & Davis, 2016) that the block programming environment makes it easier to move into this formalising mode.

Deeper levels of the Pirie & Kieren model (1994a) are not observed in our analysis, but for reference they are mentioned briefly. *Observing* occurs when one reflects on one's *formalising* of images and proposes theorems. "*Structuring* occurs when one attempts to think about one's formal observations as a theory. This means that the person is aware of how a collection of theorems is interrelated and calls for justification or verification of statements through logical or meta-mathematical argument" (p. 171). With *inventising*, one has such a strong understanding that they are able to ask new questions which might grow into an entirely new concept. In our example in Figure 1, the grade 4 girls do not yet have sufficient number or sufficient diversity in the example space of curated experiences in mathematics or robotics or programming to move into these levels/modes of understanding. That is to say, they are in the process of developing and growing their understandings in each of these areas.

Syntonic Appropriation and Tools for Conviviality

As we mentioned earlier, the teacher and students in Figure 1 above referred to the robot in the first-person plural as ‘we’. This use of the word ‘we’ is an indication of the intimacy, familiarity and conviviality they have with the robot. As a similar example, Merleau-Ponty (1978) noted the phenomenon of such intimacy, familiarity and conviviality with technology in the example of driving a car. Drivers are intimately acquainted with how the car turns by moving the steering wheel, or how pressing the gas pedal changes the speed, or how pressing the brake arrests movement. Drivers are also intimately aware of the dimensions of the car so that when they are parking, they do not bump into other cars. There is a sense of knowing and intention such that the driver moves the car with almost the same spatial precision as their own body.

> “We said earlier that it is the body which “understands” in the acquisition of habituality. This way of putting it will appear absurd, if understanding is subsuming a sense datum under an idea, and if the body is an object. But the phenomenon of habituality is just what prompts us to revise our notion of “understand” and our notion of the body. To understand is to experience harmony between what we aim at and what is given, between the intention and the performance – and the body is our anchorage in the world” (pp 144).

The experience of harmony is part of what Papert intends by the term ‘syntonic appropriation’ and Illich by ‘convivial’. Tool use is not an end in itself nor is it the motivation for action. Doing something, creating something and the aesthetic dimensions of experience are the ends and motives.

> ... the ultimate theoretical task in advancing, for example, the learning of mathematics, is not producing a range of so-and-so-centric kinds of mathematical knowledge but rather finding ways of thinking about mathematical knowledge that will allow each individual to make what in *Mindstorms* I call a syntonic appropriation (Papert, 1980, page no).

In *The Children’s Machine,* Papert (1993) connects bricolage as a methodology for intellectual activity in the context of tinkering, building with Lego, and working in computer environments (programming in Logo and controlling robot turtles) with Illich’s (1973) concept of tools for conviviality, seeing the latter as analogous to his concept of syntonic appropriation. Papert writes,

> [t]he basic tenets of bricolage as a methodology for intellectual activity are: Use what you've got, improvise, make do. And for the true bricoleur the tools in the bag will have been selected over a long time by a process determined by more than pragmatic utility. These mental tools will be as well worn and comfortable as the physical tools of the traveling tinkerer; they will give a sense of the familiar, of being at ease with oneself; they will be what Illich calls "convivial" and I called "syntonic" in Mindstorms (Papert, 1993, p.144).

Papert also notes that there are different forms of syntonicity in the learning of mathematics, viz. ego-syntonic, body-syntonic and cultural syntonic which must all filter through a context for learning mathematics in which the aesthetic is foregrounded.

For Illich (1973) the term convivial - “with life” - is intentionally and deliberately chosen to, “designate a modern society of responsibly limited tools” (p.6) that,

> ...designate[s] the opposite of industrial productivity… [but rather] autonomous and creative intercourse among persons, and the intercourse of persons with their environment...individual freedom realized in personal interdependence...the freedom to make things among which they can live, to give shape to them according to their own tastes, and to put them to use in caring for and about others (p.17).

While Papert connects syntonic appropriation with Illich’s tools for conviviality he has not elaborated upon the connection. We find that there is a need to elaborate this connection more fully in our work as this we believe is part of where attention needs to be drawn to extend frameworks like M4T/MKT/TPCK. When Papert talks about syntonic appropriation he is referring to those felt processes by which an object or tool becomes “an object-to-think-with” or following Sfard’s (2008) commognitive theory, an object to discourse with. Papert’s discussion of syntonic appropriation exists in the realm of Deweyan educative experience, i.e. of individual learning.

Illich on the other hand is very much concerned with the role of technology in society and the press that technology imposes on everyday life through increases in industrial productivity and efficiency. Illich’s argument is the need for responsibly limiting tools such that the locus of control remains with the individual in serving a community, i.e. tool use is a way to bring to life the imagination of the tool user within an ethical space. The Logo programming language and its evolution in forms like Scratch or the EV3 Lego robotic visual programming language is one way in which tool use can be responsibly limited. We can also approach the idea of “responsibly limiting” through the concept of “enabling constraints” of complexity informed approaches (eg. Davis & Simmt, 2003). While the tools themselves provide some degree of responsibly limited active engagement and immediate feedback, the intention of a designer-teacher-curator can powerfully focus and direct learners’ attention and consolidation of understanding through meaningful task design and ongoing dialogue. In this way the designer-teacher-curator participates (without over-determining the pace, unfolding and trajectory) in the process of the learners’ appropriation of the tools as tools for conviviality.

Syntonic appropriation allows for (mental & physical) tools to become partners in intellectual and creative life, to become discursive tools that give life to individual learner’s ideas and communities of practitioners and which thereby contribute to growth in understanding. This freedom to select and make things - intellectual independence - the space of learning and growth is an ethical space for both Papert and Illich and one might add an empathetic space for creating a life-giving or convivial community.

The mathematical tool/object-to-think-with that we are intending students to make a syntonic appropriation with is the number line. We worked to design/curate tasks that embed the number line as an object-to-think-with within students’ initial appropriation of robots as objects-to-think-with. We intended that students might make a (more) syntonic appropriation with the number line in this context than with other presentations and previous experiences. That is to say, we intended for it to become a tool for conviviality in relation to learning mathematics. In the next section, we describe the tasks and our process that we designed/implemented/refined/curated for embedding the number line as an object-to-think-with.

Appropriable Challenges

A Curated Task for Learning Mathematics

In our previous work (see Francis, Khan, & Davis, 2016), we investigated how enactivism was a good framework for studying children's engagement in spatial reasoning while programming robots to move. In this paper, we are using our understanding of spatial reasoning to work more explicitly to develop understandings of the rational/decimal number line. We are putting more attention on the aspects of the task that enable this. We are curating our past experiences to direct students' attention and actions towards enabling their syntonic appropriation of the number line as an object-to-think-with (or tool for conviviality) as the specific domain over which their growth in mathematical understanding is observed.

In the following task, learners are invited to explore how the <move steering> programming block works to turn the robot. Figure 3 below is an example of the EV3 <move steering> programming block's steering set to 25. Our intention in this task is to intentionally vary the steering settings incrementally, and thereby draw students attention to specific observations about (1) how the wheels rotate, (2) how the robot travels in terms of the radius of the robot's turn and the circumference of its circular path and (3) what the steering means in terms of differential percentage through direct questioning/ dialogue/ prompting.

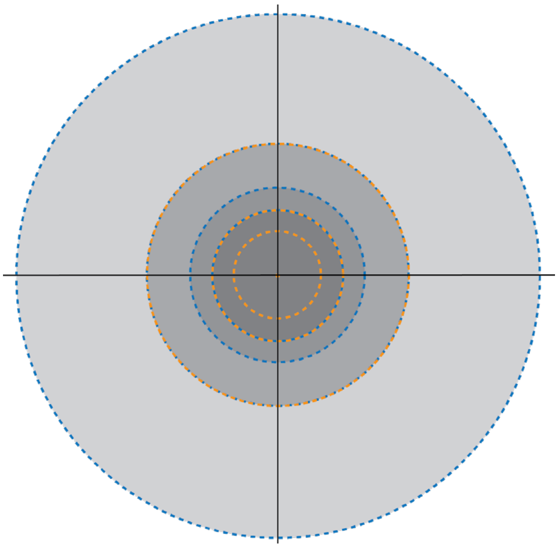
A recording sheet functions to create a shared focus of attention for discussion. Note that collaboration and dialogue are intended. designer-teacher-curator has intentionally designed the recording sheet in order to prompt certain awarenesses and questions[1]. In the first part of the recording sheet attention is drawn to how the wheels move and the direction of the robot's turn as the steering changes incrementally. For instance, when the steering is set to 25, then the right wheel rotates ½-wheel rotation forward and the left wheel rotates 1 rotation forward. The robot turns counterclockwise.

Next, in order to have students attend to features of the robot's circular path, a mat was designed (see Figure 3 below). Blue circles are the path of the outer wheel of the robot (at 25% steering the outer radius is 24cm, at 50% it is 12cm; 75% it is 8cm and 100% it is 6cm). The horizontal and vertical axes were included to give students access to the benchmark angles related to quarter, half and three-quarter turn and to allow for development of estimation strategies as well as serving as a marker for starting the robots off.

Figure 3
Steering mat for move steering task[2]

[1] The recording sheet is available under a Creative Commons CC BY NC ND license 4.0 at http://stem-education.ca/files/SteeringRecordingsheet-oldEV3_2020.pdf

[2] The steering mat is available at http://stem-education.ca/files/SteeringMat.pdf



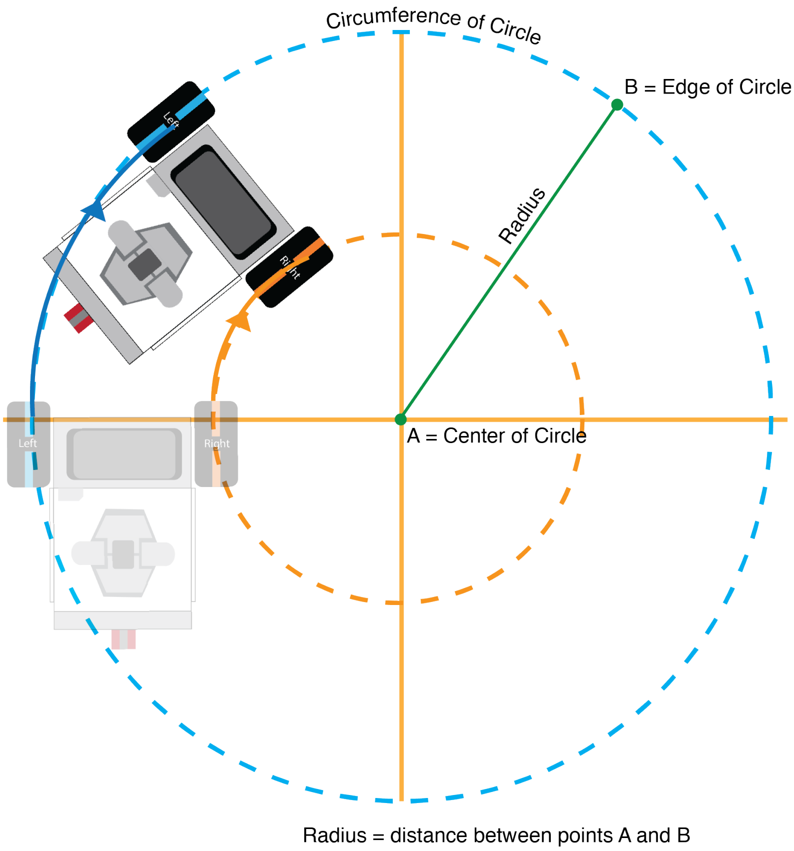
In order to easily follow the trace of the outer and inner wheels, the design of the mat allows students to discern that the robot has to be moved closer to the center of the circle as the steering increases. This reduces some of the cognitive load inherent in working with multiple aspects that vary simultaneously and allows a focusing on the critical learning intention of the task, viz. to learn how to turn the robot precisely.

Students are shown how to use the mat using the 25% steering - the robot is placed along one of the axes with the outer wheel on the largest blue circle (see Figure 4 below). The number of wheel rotations is varied in order to get the robot to follow one complete circle. Only the number of wheel rotations is being varied at this time. This allows them to discern the radius and circumference of the robot's circular path with 25% steering as a (decimal) multiple of the number of wheel rotations. Next students are guided to investigate the radius and circumference of the robot's circular path for 50%, 75%, 100%, -75%, -50% and -25% through the provided recording sheet.

Figure 4

A robot following a 24cm radius circle with steering is set to 25.

Steering Set at 25%



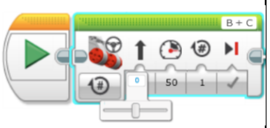
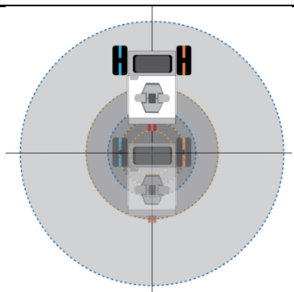
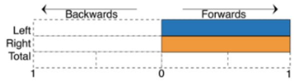
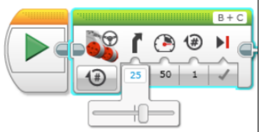
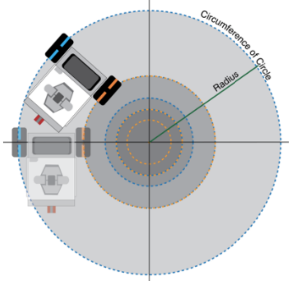
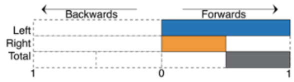
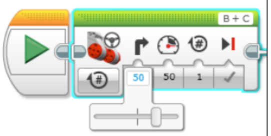
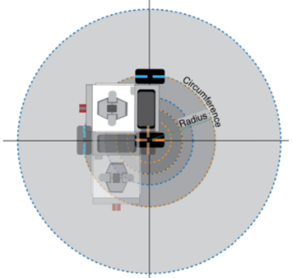
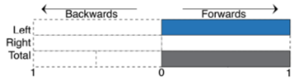
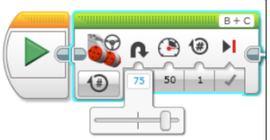
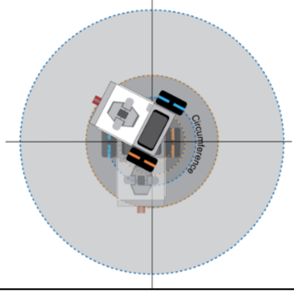
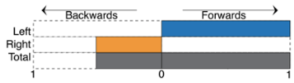
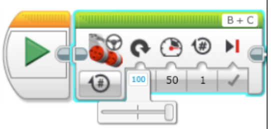
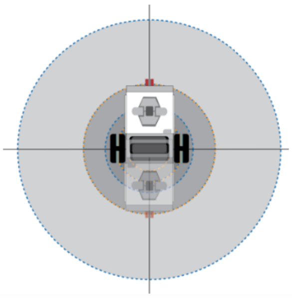
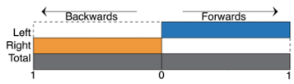
Students are required to find which circular path the robot travels with incremental variations of the steering. For instance, when the steering is set to 25, then the left outer wheel of the robot travels around the circle with a 24 cm radius. Next they are asked to determine how many wheel rotations it takes to complete one circle (the circumference). The extension of the second part is to convert the number of wheel rotations to cm.

Lastly, in the final portion of the task, students are asked to pictorially model the first part of the task (the number of rotations for each wheel with incremental steering changes) using fraction bars (with the 'whole' being two fraction bars representing one complete wheel rotation forward and one complete rotation backwards. Then they are asked to convert the fractions to a percentage (which represents the steering on the <move steering> block). Figure 5, summarizes the details that the designer-teacher-curator wanted to call attention to.

Figure 5

Summary of how the robot turns with incremental changes to the steering

Francis, K. & Khan, S. (2021). Syntonic Appropriation for Growth in Mathematical Understanding: An Argument for Curated Robotics Experiences (pp. 143-169). In Danesi M., Dragana, M., & Costa, S. (Eds.) *Math (Education) in the Information Age.* Springer.

	Steering is set to 0. The wheel rotation is set to 1. Both wheels rotate forward 1-wheel rotation. The robot moves straight. Differential is 0%.		
	Steering is set to 25. The left wheel rotates forward 1 rotation. The right wheel rotates forward $\frac{1}{2}$ rotations. Differential is $\frac{1}{4}$ or 25%.		
	Steering is set to 50. The left wheel rotates forward 1 rotation. The right wheel does not rotate. The robot pivots on the right wheel. Differential is $\frac{1}{2}$ or 50%.		
	Steering is set to 75. The left wheel rotates forward 1 rotation. The right wheel rotates backwards $\frac{1}{2}$ rotation. The robot turns tightly to the right. Differential is $\frac{3}{4}$ or 75%.		
	Steering set to 100. Left wheel rotates forward 1-wheel rotation. Right wheel rotates backwards 1-wheel rotation. The robot pirouettes right. Differential is $\frac{2}{2}$ or 100%.		

A printable version is available under a Creative Commons CC BY NC ND license 4.0 (https://creativecommons.org/licenses/by-nc-nd/4.0/) at http://stem-education.ca/files/SteeringExplanationSummary.pdf

In Figure 5 above the steering parameter is being varied incrementally. It is the only critical aspect that is varied (everything else is held constant). This allows learners to discern the function of the steering block in terms of its gross effects on the robot's movement. However, to do this intentionally, the task has been designed and refined, or curated so that attention is explicitly drawn to the *number* (or *fraction*) of wheel rotations and the *direction* of the wheel rotation, and the *direction* that the robots turn through the provided recording sheet. Each of these is a dimension of variation that is opened up. Varying this one thing in a structured (incremental) way allows learners to notice/attend to changes in the robot's behaviour at a number of distinct levels - each of these is a new dimension of variation (critical aspect for further study). Learners can think in terms of wheel rotations only, or in terms of the direction of wheel rotations only or the direction of robot turning only, however, the next part of the task requires working simultaneously with all of these to develop an understanding of how the robot moves around a circle. In this part of the task students are working with mathematical ideas of percentages (though they are not necessarily aware of this at this point in the task), direction of motion, and rotation.

In the intentional design of this task we asked what growth in mathematical understanding is possible, i.e. for an individual learner what is it possible to learn? In terms of *primitive knowings* they know how to program the <move steering> block and download to their robot, they know magnitude of numbers, language for direction of movement (eg. clockwise and anti-clockwise or forward and backward), how to move the robot straight precisely (they have done tasks to do that, eg. how many wheel rotations to travel 100 cm), decimal numbers (involved in measurement), how to measure accurately with rulers.

The larger goal of the move steering task is to use mathematical modelling to understand the black box of what the steering means in terms of how the robot moves/turns. Within the larger task, attention is directed to enactively experience the number line as an *object-to-think-with* through the concepts of circumference and radius. In the next section, we describe a student's engagement in the move steering task.

Knowing is Appropriating Modes of Doing

Examples illustrating theory

We remind readers that our argument is that a well-structured robotics inquiry (such as one like we described above, where the teacher provides the initial questions and overview of what needs to be done, and the students work independently to formulate and analyze findings – with support and guidance from the teacher) - can lead to what Pirie and Kieren (1994a, 1994b) called growth in mathematical understanding (whose modes were exemplified in our anecdote in Figure 1) by encouraging processes of syntonic appropriation of specific mathematical *objects-to-think-with* (eg. the number line).

In this section we intend to illustrate how the Pirie-Kieren model can be used to highlight/draw attention to some of the growth in mathematical understanding within a curated robotics learning experience (Figure 6 below).

In previous observations of children programming their robots to trace a polygon (Francis & Poscente, 2017) we noticed that the children did not appear to move beyond the *image having* category of mathematical understanding. As Pirie and Kieren (1994b) describe, the *image having* mode is characterised by a strong dependence on metaphor and working with metaphor thus, "mathematics *is* the image that they have and their working with that image" (p.40, italics in original). In the shift in understanding to property noticing "similie comes into play - "is" becomes "is like" (p.40). In the context of the polygon task, a shift to *property noticing* would be when students notice the similarities between the triangle and other polygons. For instance a triangle program consists of a straight-turn 3 times. Noticing that a square "is like" a triangle because it is also a collection of straight-turn but it is 4 times instead of 3 times.

In this paper we created illustrations to call attention to exchanges between students, teachers and the technology. These illustrations (Figures 1, 6, and 7) are excerpts of videos that were obtained during weekly robotics classes in a local school (Weekly robotics classes were held and video recorded weekly throughout the year for the past 4 years). These particular exchanges were chosen after an exhaustive interpretive video interaction analysis (Knoblauch et al., 2013). We began an initial overview of relevant video selection. This initial overview required reviewing field notes for finding and selecting video for analysis. Next, the selected videos were reviewed and the selection was refined based on the quality of the images, sound, actions and interactions.

Transcripts of the video do not illustrate the actions of the participants. For that reason, sequenced still image freeze frames were extracted from the video. Sketches of the images were made to improve the comprehensibility of the verbalized text form and make it easier for the reader to understand the situation (Knoblauch et al., 2013). Consistent with McLeod (1990), the sketches were placed in a juxtaposed sequential comic strip format to convey interactions. This format is similar to studies by Plowman and Stephan (2008), and Heath et al. (2010). The speech bubbles represent the dialogue; the commentary represents the metaphors of number used. The removal of background information allows us to keep attention focused on speech, gesture, and actions only and removes distracting elements such as carpet, tables and chairs. We are not saying that the classroom context is not important and we are aware of the loss of other information such as around ethnicity and valid concerns about representation in research images, however, in this study and paper these are not our focus for analysis in looking at growth in mathematical understanding.

Developing modes of doing

The following Figure 6 below is an illustration of an exchange between two Grade 4 students, Luke and Kara, as they attempted to trace a pentagon. The exchange occurred quite early in the year and the students had familiarity with making their robot move straight and turn. In this exchange, Luke is attempting to rectify an issue with the robot's movements. The program works for the first two straight-turn increments of the pentagon's path. They had success making the first two straight-turn segments of the polygon. But the robot veers off path for the third straight-turn. Luke has attempted to correct the last two blocks of the assembled code three times. Figure 6 begins with his fourth attempt.

Figure 6
Luke and Kara tracing a pentagon

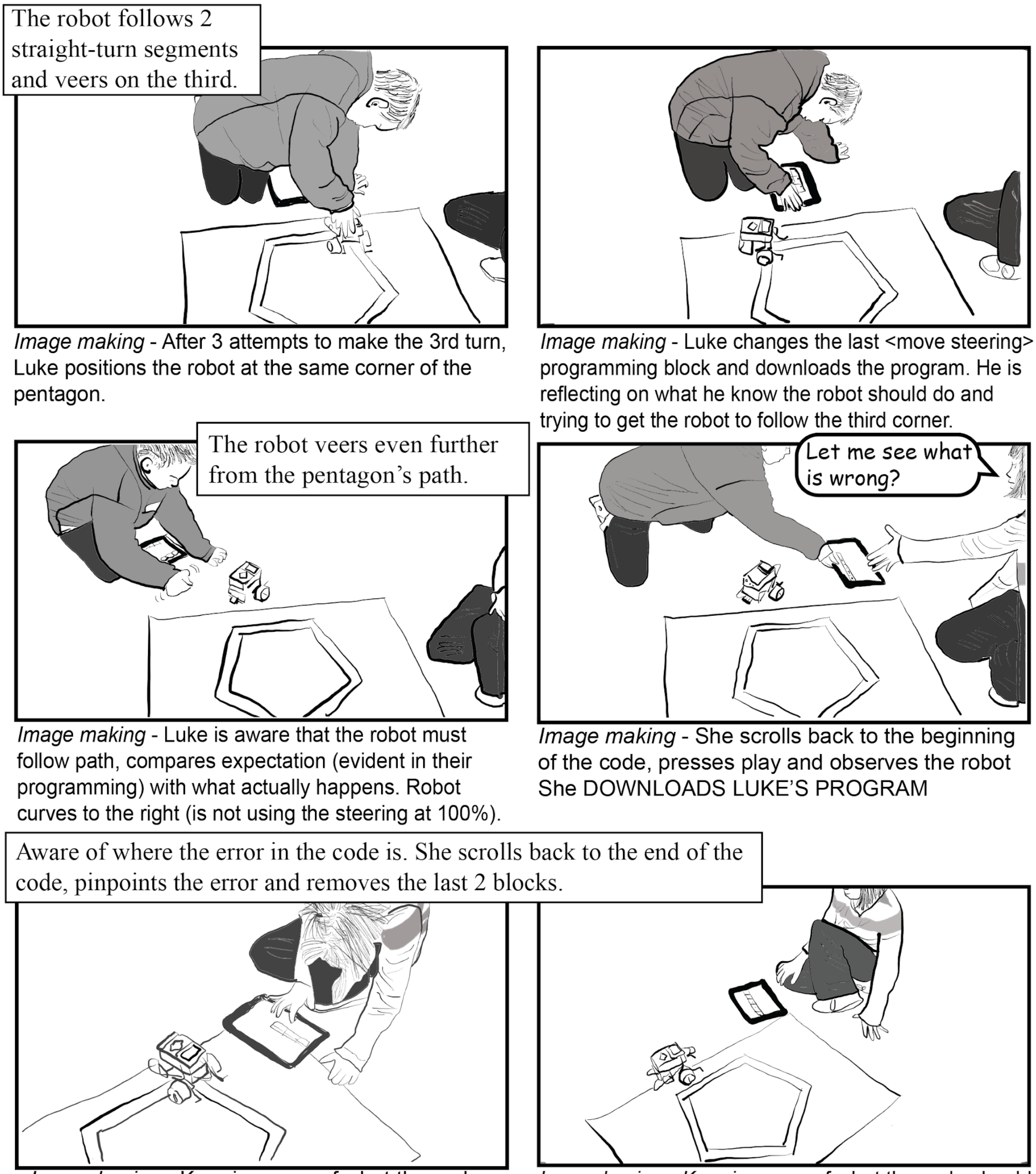


Image making - After 3 attempts to make the 3rd turn, Luke positions the robot at the same corner of the pentagon.

Image making - Luke changes the last <move steering> programming block and downloads the program. He is reflecting on what he know the robot should do and trying to get the robot to follow the third corner.

Image making - Luke is aware that the robot must follow path, compares expectation (evident in their programming) with what actually happens. Robot curves to the right (is not using the steering at 100%).

Image making - She scrolls back to the beginning of the code, presses play and observes the robot She DOWNLOADS LUKE'S PROGRAM

Image having - Kara is aware of what the code should be. Next, she adds 1 new move steering block. She changes the steering settings. First, she sets it to 100%, then she sets it close to 50%. This will have the robot pivot on one wheel.

Image having - Kara is aware of what the code should be. Then she adds another move steering block. She changes the steering to 100. The robot makes the appropriate turn.

See video here https://vimeo.com/343271775. As they tested and retested ideas, they engaged with multiple spatial reasoning elements simultaneously while moving back and forth between *image making* and *image having*. We did not observe the children move into *formalising*, nor were they able to program their robots's turns consistently. This pattern continued throughout

Francis, K. & Khan, S. (2021). Syntonic Appropriation for Growth in Mathematical Understanding: An Argument for Curated Robotics Experiences (pp. 143-169). In Danesi M., Dragana, M., & Costa, S. (Eds.) *Math (Education) in the Information Age.* Springer.

the year on many other robotics tasks. Each corner or distance for the robot to travel encountered in future tasks was approached with a guess and check process. Not to negate the importance of working in the space of *image making* and *image having*, we wondered if robotics tasks and inquiries could be structured in manners that could elicit deeper mathematical understanding that translates into more precise robotics movements and turns. Hence we developed the move steering task (described previously).

Developing more powerful modes of doing

In the graphic illustration below (Figure 7), Greg, aged 11, is working on a curated robotics task that is intended to help explain how the <move steering> programming block works. In this part of the task, the Greg is identifying which circle the outer wheel of the robot travels along, the radius of that circle in cm, the circumference of that circle in wheel rotations, and the direction the robot travels for <move steering> set at 100%, 75%, 50%, 25%, 0, -25%, -50%, -75%, and -100%. Figure 7 begins as Greg is attempting to determine the circumference of the outer circle that the robot follows when the steering is set to 25%.

Figure 7

Greg syntonically appropriating the unmarked number line the move steering task

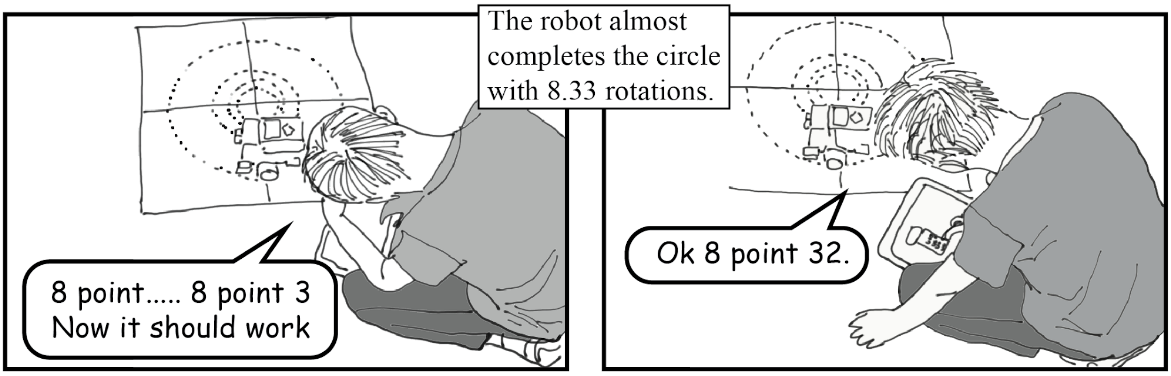


Image making - relating the number of wheel rotations to the circumference of the robots circlular path.

Image having - using the physical actions of the wheel's rotation to describe the measurement of motion.

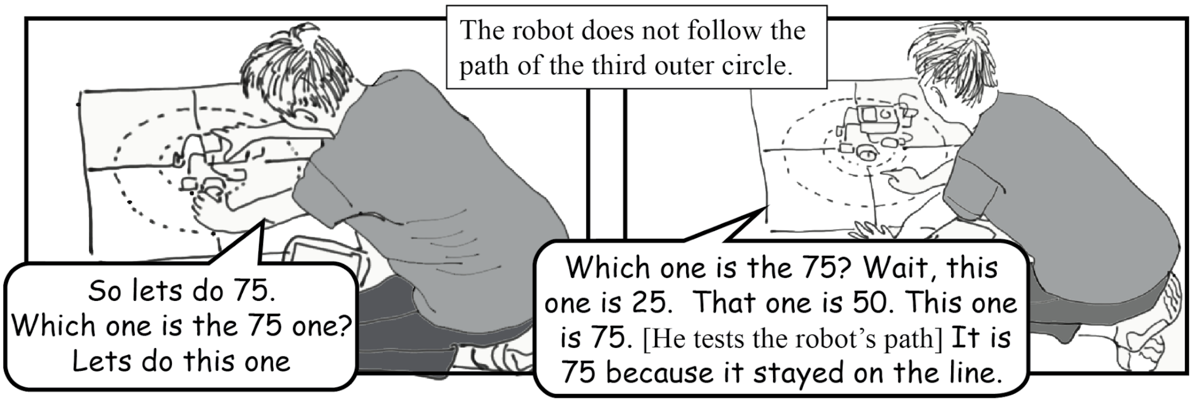
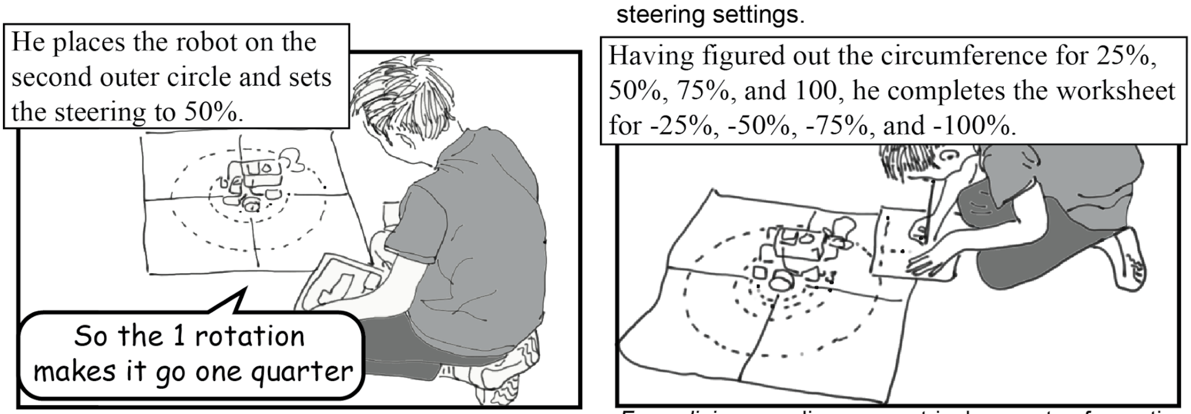


Image making - the boy is trying to use what he learned from the 25% steering turn for the 75% circle.

Property noticing - recognizes the corresponding decreasing radius of the circles with decreasing steering settings.



Formalising - applies symmetrical property of negative steering to complete worksheet.

As Pirie and Kieren (1994b) note *formalising* is characterised by, "a sense that one's mathematical methods work for all" relevant examples. Children who are formalising do not need the physical actions and images which brought them to the point of formalising*"* (p.43).

Greg knows which circle the radius traces for all the positive steering settings. From the previous week's tasks, Greg learned that negative steering is symmetrical to the positive steering, but the robot turns in a different direction. He applied this previous learning to complete the rest of the recording sheet [*formalising*] without needing to test each setting.

We have shown in the examples above that a well-structured (curated) robotics task can lead to growth in mathematical understanding. In this final section we still need to show that this occurs by encouraging processes of syntonic appropriation of specific mathematical *objects-to-think-with* (eg. the number line). In the many years of working with young children and robotics we have observed a syntonic appropriation of the robot in achieving the overall challenge goals (eg. get as close to the wall as fast as possible). However, what we now see as being enabled by the carefully curated robotics task is a more narrow but very powerful syntonic appropriation of specific, relevant mathematical *objects-to-think-with* such as the number line. This we believe is close to Papert's (1980) intent and description with learning environments such as Logo and Illich's (1973) view of technology as a tool for conviviality. Syntonic appropriation is not a one-off event or experience, it occurs at differential temporal paces and at different conceptual and affective grain-sizes for individual learners.

In the second frame of Figure 7 above, Greg holds the robot on the mat and physically moves the robot while closely observing how far the wheel rotates to get to the exact start place again. At this point the learner, the robot and the mat – specifically the outer circumference which represents an unmarked number line – are structurally coupled (in the enactivist sense) as one learning system. Greg is physically and concretely measuring the circumference with the wheel's rotations, similar to how a measuring wheel measures field sizes and so is coming to awareness of the need and existence of relevant sequential smaller units of decimal measure in relation to horizontal distance traveled. It is at such points of careful attention and focus that we believe processes of syntonic appropriation related to growth in mathematical understanding are at work.

In the follow up task in the subsequent week, every group was able to turn their robot precisely to complete a new challenge (to see videos of a turn, click here https://vimeo.com/415696291 and here https://vimeo.com/415697745). They knew it was a 100% steering turn. From our previous experience we have seen that other groups typically have approached turning on this task (and others) through a more guess-and-check approach and would take significantly longer to arrive at this understanding, some even after as much as a year of working with robots. Note, we are not claiming that these learners know everything about circumference and radius, but they have developed from very similar initial *primitive knowings* to more than an *image having* level of understanding about these concepts. We are claiming however that through the curated learning environment and enactive experiences learners are at the point of beginning to notice properties and formalise measurement with decimal numbers and are developing an enactive relationship with the rational number line (not the real continuum at this time).

Conclusion

The structure of a task matters especially for developing mathematical understanding. Through the examples in the paper we have shown our process of designing and curating a robotics learning task with the specific intention of directing learners' attention to the underlying number line as an *object-to-think-with*. The task affords students an opportunity to work with multiple instantiations of number - number as a count, number as a measure, number as distance moved/rolled and the existence of numbers between whole numbers or decimal/rational numbers – and grow their mathematical understanding of number. The structuring of the task, we have argued, encourages processes of syntonic appropriation such that learners have a personal and embodied meaning of the concept and the associated *object-to-think-with*.

When structurally coupled with the robot and the task learners are cognitively and affectively inserting themselves INTO (not merely onto) the number line in an enactive way, i.e. there is a meaning to being 'between' two points as a result not of discrete hops (as in whole number counting) but of continuous motion. When number lines are typically encountered in early elementary (if at all) they are used to model addition, subtraction and multiplication. These all involve discrete 'hops' or 'jumps' forward and backwards on the number line from one well-defined point to another. However, hopping, jumping, even stepping is only a small portion of the repertoire of movements made by human bodies, the majority of which are experienced (even if not consciously aware) as continuous movements. The specific task affordances allow students to put themselves into the robot in a way that allows them to 'experience' a continuous movement and so come to appreciate that there are a multitude of numbers between 2 and 3 in a meaningful way. It is in this way that the prior syntonic appropriation of the robot (building and earlier tasks) allows for the deeper syntonic appropriation of a powerful mathematical model - that of the (rational) number line and which leads to the deepening of multiple mathematical understandings (including that of number etc.) enabling both robot and number line to become a tool for conviviality in relation to learning mathematics. We acknowledge that there are many other aspects of mathematics that are likely being developed during and through this learning experience, however, we have chosen to focus on those aspects that we intentionally designed for, viz. appropriating the number line as an *object-to-think-with*.

Implications for classroom practice

Curated robotics experiences can provide opportunities for children to learn these concepts not from an external 'objective' perspective (disassociated way) but from a dynamic embodied and enactive perspective in ways that are meaningful. These curated experiences through providing a sufficiency of structure enable individual and collective processes of syntonic appropriation and sense-making that serves to enable growth in mathematical understanding.

One of the implications we see can be framed as teaching is presenting appropriable challenges in contrast to presenting content or experiences alone. We note from our own experiences that we, as teachers, also experience growth in our own pedagogical understanding of the teaching of mathematics. Earlier versions of robotics tasks we have used were not as appropriable for discerning the underlying *object-to-think-with* of the number line. Students were doing similar things BUT their attention was not being directed. In this instance the intentional design of the mats and recording sheet together with the task focuses students' attention on how the Move Steering works to turn the robot precisely. Turning the robot precisely, or as intended, is associated with positive (perhaps joyful) affect. To be clear, we do not believe that the learners have yet accumulated/curated sufficient and diverse sets of experiences as yet, and complementary mathematics learning is still needed to help students to formalise their understandings.

Teachers also have a (legal) responsibility to the curriculum which entails finding and developing appropriable tasks that address mathematical concepts in ways that enable syntonic appropriation through responsibly limiting (and gradually and deliberately increasing) the set of conceptual tools or *enactive-objects-to-think-wit*h (in contrast to merely *mental-objects-to-think-with* which represent a terminal goal of learning from action). Our explicit goal was to have the learners make a syntonic appropriation of the number line as an *object-to-think-with* to serve learners future growth in mathematical understanding. This future growth involves connecting their learning of multiple mathematical concepts beyond number. A teacher's role is to connect this knowing of the *enactive-object-to-think* with with other mathematical concepts. In our move steering task for example the circumference of a circle was first experienced as distance traveled. In the extension to this task (not presented in this paper), these experiences are shifted to more formalized understandings of radius and circumference with the algebraic expression of c=2πr (where c=circumference, r=radius) as an analysis of data collected.

Ideally a teacher would be able to continue to direct students' attention and action in using the *object-to-think-with* with other mathematical topics and concepts. This is where the designer-teacher-curator's role emerges as one that exceeds that of each of the individual categories (designer or teacher or curator) as each set of skills and dispositions is necessary but insufficient on its own. This requires, we think, a collaboration perhaps amongst task designers, classroom teachers and teacher-researchers who knit their curated nets of knowledge and experiences together. We see this as important as much as possible to develop resources and learning experiences that help focus or sharpen attention on the intended object or aspect of learning (Dehaene, 2020; Marton, 2014).

Implications for theory/research

Modes of Knowing: Connecting Enactivism & Embodied Cognition

One contribution we see this paper as making is explicitly identifying the periods of time during analysis for which enactivist frameworks and embodied cognition frameworks are useful in making sense of student learning. Earlier we noted that, with reference to Figure 1, enactivism is useful for understanding learning within the environment while an embodied cognition approach would foreground the relationships between prior engagements and outcome. Thus, we reiterate that enactivism is concerned with the learning *in* action since it is the potential for action in the world that focuses attention and drives learning while embodied cognition is concerned with the learning *from* action and thus a later consolidation of enactive action. We encourage others to see if this particular juxtaposition and blending of theoretical frames is useful for advancing understanding of student learning across developmental time both *in* action and *from* action.

Modes of Doing: Syntonic Appropriation

Our second contribution is to work explicitly with Papert's ideas around syntonic appropriation which we see as a missing element in frameworks like MKT/M4T/TPCK that acknowledge but do not deeply engage with the affective domain in teaching and learning mathematics. Specifically, we have introduced the idea of 'curating' experience as a useful metaphor in relation to the type of knowing that characterises the types of intentional learning spaces and opportunities that were designed and used in the task. Following Papert we investigated and reinforced his link with Illich's ideas of technology as a tool for conviviality through (initially) offering responsibly limited usage that thereby enables focused attention and growth in mathematical understanding.

Modes of Understanding: Pirie-Kieren model

We have incorporated technology into the Pirie-Kieren model of growth in mathematical understanding. Technology can provide enactivist experiences of concepts that support and strengthen those inner modes (*image making, image having*, and *property noticing*). Our work supports their view of growth in mathematical understanding as a non-linear process. The Pirie-Kieren model we believe is mostly about interpreting learning. What we have done is demonstrate how technology (structurally coupled with humans and a carefully curated task) can be used to *influence* learning. In learning to use the number line as an *object-to-think-with* for fractional (decimal) numbers and the idea of a number as a measure/distance, the use of robotics technology was critical.

In the current learning environment in schools with multiple competing learning initiatives and increasingly constrained teacher time, robotics platforms like the EV3, a well curated task and close collaboration with teachers allowed for addressing these multiple learning presses (STEM, CT, Multiliteracies, etc.). As mathematics learning evolves to increasingly include and depend on technology, designer-teacher-curators will require a complex and complementary set of interdisciplinary skills to both interpret learning in situ and design occasions to meaningfully influence learning.

Francis, K. & Khan, S. (2021). Syntonic Appropriation for Growth in Mathematical Understanding: An Argument for Curated Robotics Experiences (pp. 143-169). In Danesi M., Dragana, M., & Costa, S. (Eds.) *Math (Education) in the Information Age.* Springer.

References

Alberta Education. (2016). Mathematics Kindergarten to Grade 9 programs of study. In *Program of Studies* (updated). Retrieved from https://education.alberta.ca/media/3115252/2016_k_to_9_math_pos.pdf

Alberta Education (2016). Alberta K-9 Mathematics Achievement Indicators. Retrieved from https://open.alberta.ca/publications/9781460127292

Acosta, C. & Adamson, G. (2017, July 4). Glen Adamson: Curating attention. *True Living of Art and Design*

Magazine Retrieved from https://tlmagazine.com/glenn-adamson-curating-attention/?print=true

Ball, D. L., Hill, H. C., & Bass, H. (2005). Knowing mathematics for teaching: Who know mathematics well enough to third teach and how can we decide? *American Educator*, *Fall*, 14–21, 43–46.

Braithwaite, D. W., & Siegler, R. S. (2018). Developmental changes in the whole number bias. *Developmental Science*, *21*(2), 1–13. https://doi.org/10.1111/desc.12541

Brown, L., & Coles, A. (2011). Developing expertise: how enactivism reframes mathematics teacher development. *ZDM—The International Journal on Mathematics Education, 43*(6–7), 861–873. doi:10.1007/s11858-011-0343-4.

Buteau, C., Muller, E., Marshall, N., Sacristán, A. I., & Mgombelo, J. (2016). Undergraduate Mathematics Students Appropriating Programming as a Tool for Modelling, Simulation, and Visualization: A Case Study. *Digital Experiences in Mathematics Education*, *2*(2), 142–166. https://doi.org/10.1007/s40751-016-0017-5.

Davis, B., Francis, K., & Friesen, S. (2019). *STEM education by design: opening horizons of possibilities*. New York: Routledge

Davis, B. & Renert, M. (2014). *The math teachers know: Profound understanding of emergent mathematics*. New York, NY: Routledge.

Davis, B., & Simmt, E. (2003). Understanding learning systems: Mathematics education and complexity science. *Journal for Research in Mathematics Education*, *2003*(2), 137-167.

Dehaene, S. (2020). *How we learn: Why brains learn better than any machine...for now.* Viking.

Di Jaegher, H., & Di Paulo, E. (2013). Enactivism is not interactionism. *Frontiers in Human Neuroscience, 6*(345), 1–2. doi:10.3389/fnhum.2012.00345.

Francis, K., Khan, S., & Davis, B. (2016). Enactivism, spatial reasoning and coding. *Digital Experiences in Mathematics Education*, *2*(1), 1–20. https://doi.org/10.1007/s40751-015-0010-4

Francis, K. & Poscente, M. (2017). Building number sense with Lego robots. *Teaching children mathematics, 23*(5), 310-312.

George, L. G. (2017). Children's learning of the partititive quotient fraction sub-construct and the elaboration of the don't need boundary feature of the Pirie-Kieren theory. University of Southampton, Faculty of Social, Human and Mathematical Sciences, Phd. Thesis, retrieved from https://eprints.soton.ac.uk/411237/1/Lois_George_Thesis_Final_submission_29_03_2017_.pdf on April 14, 2020.

Grover, S., & Pea, R. (2013). Computational thinking in K-12: a review of the state of the field. *Educational Researcher*, *42*(1), 38–43. https://doi.org/10.3102/0013189X12463051

Heath, C., Hindmarsh, J., & Luff, P. (2010). *Video in qualitative research* (p. 173). p. 173. London, UK: Sage Publications

Hiebert, J.E. and Carpenter, T.P. (1992) Learning and teaching with understanding. In Grouws, D.A. (ed.), *Handbook of research on mathematics teaching and learning*. New York: Mcmillan, 65-97

Humphreys, G. W., Yoon, E. Y., Kumar, S., Lestou, V., Kitadono, K., Roberts, K. L., et al. (2010). The interaction of attention and action: from seeing action to acting on perception. *British Journal of Psychology, 101*(2), 185–206. doi:10.1348/000712609X458927.

Hutto, D. D. (2013). Psychology unified: from folk psychology to radical enactivism. *Review of General Psychology*, *17*(2), 174–178. doi:10.1037/a0032930.

Illich, I. (1973). *Tools for conviviality* (R. N. Anshen, Ed.). Harper & Row.

Keen, R., Carrico, R. L., Sylvia, M. R., & Berthier, N. E. (2003). How infants use perceptual information to guide action. *Developmental Science, 6*(2), 221–231. doi:10.1111/1467-7687.00274.

Khan, S., Francis, K., & Davis, B. (2015). Accumulation of experience in a vast number of cases: enactivism as a fit framework for the study of spatial reasoning in mathematics education. *ZDM Mathematics Education*, *47*(2), 269–279. https://doi.org/10.1007/s11858-014-0623-x

Knoblauch, H., Tuma, R., & Schnettler, B. (2013). *Videography: Introduction to interpretive videoanalysis of*

social situations (K. Luft, Trans.). Frankfurt: Peter Lang.

Koehler, M. J., & Mishra, P. (2005). What happens when teachers design educational technology? The development of Technological Pedagogical Content Knowledge. *Journal of Educational Computing Research*, *32*(2), 131–152.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from : how the embodied mind brings mathematics into being*. New York: Basic Books.

Marton, F. (2014). *Necessary conditions of Learning*. New York, NY: Routledge.

Marton, F. (2018). Towards a pedagogical theory of learning. In K. Matsushita (Ed.), *Deep Active Learning* (pp. 59–77). https://doi.org/10.1007/978-981-10-5660-4_4

McLeod, S. (1990). *Understanding comics: The invisible art*. New York, NY: HarperPerennial.

Merleau-Ponty, M. (1978). *Phenomenology of perception*. London: Routledge.

Namukasa, I. (2019). Integrated curricular and computational thinking concepts. Math and Code Zine, 4(1) Retrieved from https://researchideas.ca/mc/integrated-concepts/

National Academies of Sciences, Engineering, and Medicine (2018). *How People Learn II: Learners, Contexts, and Cultures*. Washington, DC: The National Academies Press. https://doi.org/10.17226/24783.

Obersteiner, A., Dresler, T., Bieck, S. M., & Moeller, K. (2019). Understanding fractions: Integrating results from mathematics education, cognitive psychology, and neuroscience. In A. Norton & M. Alibali (Eds.), *Constructing number* (pp. 135–162). https://doi.org/10.1007/978-3-030-00491-0_7

Papert, S. (1980). Mindstorms: children, computers, and powerful ideas. Retrieved from http://worrydream.com/refs/Papert - Mindstorms 1st ed.pdf

Papert, S. (1993). *The children's machine: rethinking school in the age ot the computer*. New York, NY: Basic Books.

Pirie, S. & Kieren, T. (1994a). Growth in mathematical understanding: How can we characterise it and how can we represent it? *Educational Studies in Mathematics, 26*, 165-190.

Pirie, S. E. B. & Kieren, T. E. (1994b). Beyond metaphor: Formalising in mathematical understanding within Constructivist environments. For the Learning of Mathematics, 14(1), 39-43. https://flm-journal.org/Articles/83AE5FE212FCAC7DB193917FADF90.pdf

Plowman, L., & Stephen, C. (2008). The big picture? Video and the representation of interaction. *Educational Research Journal*, *34*(4), 541–565. https://doi.org/10.1080/01411920701609422

Preziosi, D. (2019). Curatorship as Bildungsroman: Or, from Hamlet to Hjelmslev. In M. Hansen, A. F. Henningsen, & A. Gregersen (Eds.). *Curational challenges: Interdisciplinary perspectives on contemporary curating, (pp.)*. London, UK: Routledge: Taylor and Francis Group. Retrieved from https://doi.org/10.4324/9781351174503

Rushton, S. K. (2008). Perceptually guided action: a step in the right direction. *Current Biology, 18*(1), R36–R37. doi:10.1016/j.cub.2007.10.064.

Seligman. M. P. (2011). *Flourish: A visionary new understanding of happiness and Well-being*. New York, NY: Simon & Schuster.

Sfard, A. (1991) On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. Educational Studies in Mathematics, 22 (1), 1-36.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses and mathematising.* Cambridge University Press.

Sierpinska, A. (1994). *Understanding in Mathematics. Studies in Mathematics Education Series: 2.* (P. Earnest, Ed.). London: Falmer Press.

Simon, M.A. (2006) Key developmental understandings in mathematics: A direction for investigating and establishing learning goals. *Mathematical Thinking and Learning, 8* (4), 359-371.

Skemp, R.R. (1976) Instrumental understanding and relational understanding. Mathematics *Teaching, 77,* 20-26.

Varela, F. J., Thompson, E., & Rosch, E. (1991). *The embodied mind : cognitive science and human experience*. Massachusetts: MIT Press.

Wing, J. M. (2006). Computational thinking. *Communications of the ACM*, *49*(3). Retrieved from https://www.cs.cmu.edu/~15110-s13/Wing06-ct.pdf

Appendix 1

Program of Studies	Achievement Indicators Document
9. Represent and describe decimals (tenths and hundredths), concretely, pictorially and symbolically.	• Write the decimal for a given concrete or pictorial representation of part of a set, part of a region or part of a unit of measure. • Represent a given decimal, using concrete materials or a pictorial representation. • Explain the meaning of each digit in a given decimal with all digits the same. • Represent a given decimal, using money values (dimes and pennies). • Record a given money value, using decimals. • Provide examples of everyday contexts in which tenths and hundredths are used. • Model,usingmanipulativesorpictures,thatagivententhcanbeexpressedasahundredth; e.g., 0.9 is equivalent to 0.90, or 9 dimes is equivalent to 90 pennies.
10. Relate decimals to fractions and fractions to decimals (to hundredths).	• Express, orally and in written form, a given fraction with a denominator of 10 or 100 as a decimal. • Read decimals as fractions; e.g., 0.5 is zero and five tenths. • Express, orally and in written form, a given decimal in fraction form. • Express a given pictorial or concrete representation as a fraction or decimal; e.g., 15 shaded squares on a hundredth grid can be expressed as 0.15 or 15/100 • Express, orally and in written form, the decimal equivalent for a given fraction; e.g., 50/100 expressed as 0.50.
Demonstrate an understanding of addition and subtraction of decimals (limited to hundredths) by: • using personal strategies to determine sums and	• Predict sums and differences of decimals, using estimation strategies. • Determine the sum or difference of two given decimal numbers, using a mental mathematics strategy, and explain the strategy. • Refine personal strategies to increase their efficiency. • Solve problems, including money problems, which involve addition and subtraction of decimals, limited

differences ● estimating sums and differences ● using mental mathematics strategies	to hundredths. ● Determine the approximate solution of a given problem not requiring an exact answer.
10. Compare and order decimals (to thousandths) by using: ● benchmarks ● place value ● equivalent decimals.	● Order a given set of decimals by placing them on a number line that contains the benchmarks 0.0, 0.5 and 1.0. ● Order a given set of decimals including only tenths, using place value. ● Order a given set of decimals including only hundredths, using place value. ● Order a given set of decimals including only thousandths, using place value. ● Explain what is the same and what is different about 0.2, 0.20 and 0.200. ● Order a given set of decimals including tenths, hundredths and thousandths, using equivalent decimals; e.g., 0.92, 0.7, 0.9, 0.876, 0.925 in order is: 0.700, 0.876, 0.900, 0.920, 0.925.