THE UNIVERSITY OF CALGARY

EVOLUTION OF PERTURBATIONS OF SOME NON-LINEAR DISPERSIVE WAVE TRAINS

by

CHIRAKKAL VARIER EASWARAN

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE

DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS AND STATISTICS

CALGARY, ALBERTA

JUNE, 1987

ISBN 0-315-38004-7

THE UNIVERSITY OF CALGARY

FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled, "Evolution of Perturbations of Some Nonlinear Dispersive Wavetrains" submitted by Chirakkal V. Easwaran in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

S.R. Majumdar, supervisor,
Dept. of Maths. & Stats.

R.S. Dhaliwal, Maths. & Stats.

P. Lancaster, Maths. & Stats.

K.L. Chowdhury, Mechanical Engg.

J.F. Stanislav, Chemical Engg.

W. E. Olmstead

W.E. Olmstead, external examiner,
(California Inst. of Technology)
Northwestern University

DATE: 87.06.25

ABSTRACT

EVOLUTION OF PERTURBATIONS OF SOME NON-LINEAR DISPERSIVE WAVETRAINS

In Part I we study the effect of perturbations on two non-linear dispersive wave systems. First the Benjamin-Feir instability of capillary-gravity surface waves on a liquid layer of arbitrary but uniform depth h is considered. Explicit conditions in terms of two dimensionless parameters kh and $\tau k^2/g$, where k is the wave number, τ the surface tension coefficient per unit density and g the acceleration of gravity, are derived for the possible growth of sidebands. The automated computer algebra system MACSYMA is used to facilitate the analysis which involves heavy algebra. We then give a lagrangian for capillary-gravity waves and investigate the stability of the waves to slow modulations using the averaged lagrangian method of Whitham. The result of this analysis is compared with the Fourier mode analysis of the sidebands. Secondly we study the evolution of slowly varying solitary wave solutions of the perturbed Renormalised Long Wave equation. It is shown that the tail of the solitary wave decays exponentially unlike in the Korteweg de Vries case, where the tail is oscillatory.

In Part II we consider some general aspects of the flow of a class of non-Newtonian fluids called micropolar fluids which models rheologically complex liquids such as suspensions and polymers. Under fairly general conditions of smoothness and boundedness on the flow variables we prove that the coupled nonlinear partial differential equations governing the flow of such fluids admit at most one solution in two general cases: (i) Flow in a bounded region, which could be time dependent; (ii) Flow past a finite solid body in an unbounded region. This is accomplished using the method of energy integrals. We also show how explicit fundamental solutions could be constructed for the linearised unsteady equations of motion. This leads to an integral representation of flow variables, which at least yields to an asymptotic analysis in terms of small parameters in specific situations. In the final chapter we consider the flow of micropolar fluids in meandering channels, a model which could be of relevance in biological modelling.

Acknowledgements

It is a pleasure to thank Prof. S.R.Majumdar for his advice and encouragement during the preparation of this thesis. Thanks are also due to Dr. H.Ramkissoon and Prof. W.E.Olmstead for helpful discussions.

Moral support from many friends and colleagues, specially Dr. S.Kokal, Dr. J.F.Stanislav and Dr. P.Sankaran has been invaluable during a difficult period.

I wish to thank Pat Dalgetty for an excellent job of typing a difficult manuscript and cheerfully putting up with constant rewritings and corrections.

I also wish to thank the University of Calgary for financial support.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

NOMENCLATURE

PART I

a	Amplitude of the primary wave
g	Acceleration of gravity
h	Constant liquid depth
H	kh
k, k_i	wavenumber
L	Lagrangian
ϵ_i	Amplitude of sidebands
ρ	$\tau k^2/g$
$\varphi, \varphi_i, \tilde{\varphi}_i$	Potential functions
$\eta, \eta_i, \tilde{\eta}_i$	Surface profiles
ς, ς_i	Phase functions
ω, ω_i	Frequency
τ	Surface tension per unit density

NOMENCLATURE

PART II

a_{klm}	Microdeformation rate tensor
b_{klm}	Microdeformation rate tensor
d_{klm}	Deformation rate tensor
$f_i, \underline{f}$	External force
i_{kl}	Microinertia moments
j	Inertial spin coefficient
$l_i, \underline{l}$	External couple
m_{ij}	Couple stress tensor
p	Pressure
t_{ij}	Stress tensor
$u_i, \underline{u}$	Velocity
α, β, γ	Viscosity coefficients
κ, μ, λ	Viscosity coefficients
λ_{klm}	The first stress moments
ρ	Density
$\dot{\sigma}_{lm}$	Inertial spin
ν_{kl}	Gyration tensor
$\nu_i, \underline{\nu}$	Microrotation

PART I. EVOLUTION OF PERTURBATIONS OF SOME NON-LINEAR DISPERSIVE WAVE TRAINS

The equations of mathematical physics modelling natural phenomena are at best approximations to reality. This is partly because we isolate the phenomenon under study from its surroundings when building models. In nature interferences from various sources are inevitably present which we usually ignore as "noise". Thus it is interesting and necessary to enquire how our exact models behave under perturbations.

In the first part of this thesis we study the effect of perturbations on two non-linear dispersive wave systems. The first is the system of complete surface wave equations, including the effect of gravity and surface tension. The second system is a model for long non-linear dispersive waves known as the Renormalised Long Wave (RLW) equation.

In typical situations the perturbing effects could arise from an imperfect wavemaker in an experimental tank or due to inhomogenieties in the medium. There have been several studies addressed to the question of stability of surface waves in the presence of perturbations. In a classic work Benjamin and Feir (1967) and Benjamin (1967) have demonstrated the instability of small amplitude waves on a liquid surface in the presence of "noise". They have found that if the wave number and frequency of the perturbing "noise" lies within a certain range, the main wavetrain could suffer an unbounded amplification of amplitude, thus leading to instability. This result complements the work of Whitham

(1967) who investigated the stability of wave trains undergoing gradual modulation, as, for example, in a slowly varying medium. This latter analysis also indicates instability, agreeing with the results of Benjamin in their regions of overlap.

In Chapter 1 we attempt to extend the analysis of Benjamin and Whitham to take into account the effect of surface tension on the wave trains. First we carry out a linear perturbation analysis on the non-linear surface wave equations, taking into consideration capillary effects. This results in a stability diagram depending in a somewhat subtle manner on two dimensionless parameters $\tau k^2/g$ and kh, where τ is the surface tension coefficient, k is the wavenumber, g is the acceleration of gravity and h is the liquid depth. Next we give a lagrangian for capillary-gravity waves and use it to investigate the slow modulation of wave trains using Whitham's averaged lagrangian approach.

In Chapter 2 we study the evolution of solitary wave solution of the RLW equation

$$u_t + 6\,u\,u_x - u_{xxt} = 0$$

under the effect of a perturbation:

$$u_t + 6\,u\,u_x - u_{xxt} = \epsilon\,u.$$

There has been extensive work in recent years on the perturbed Korteweg de Vries (KdV) equation

$$u_t + 6\,u\,u_x + u_{xxx} = \epsilon\,u$$

(see Karpman (1979),Grimshaw (1979,1981)) using perturbations on the inverse scattering method and series expansion techniques. This is in line with the exhaustive studies done on the KdV equation which has certain desirable properties like possessing an infinite number of conservation laws and N-soliton solutions. However the alternative equation with u_{xxx} replaced by $-u_{xxt}$ is an equally valid model for long non-linear dispersive waves (see Benjamin et al. (1972)). It has not been studied as widely as the KdV equation because it does not possess the "nice" properties of the KdV equation mentioned above. We will thus investigate the qualitative differences in the evolution of the RLW equation under perturbations compared with the KdV equation. It will be shown that the tail of the solitary wave evolves differently in the two cases. In the analysis we utilise a matched asymptotic expansion technique similar to that of Smyth (1984).

CHAPTER I

INSTABILITY OF CAPILLARY-GRAVITY WAVES ON A UNIFORM LIQUID LAYER *

1.1 Introduction

It has been known for a long time that the complete set of non-linear water wave equations do admit waves of permanent form. For periodic waves on deep water Levi-Civita (1925) proved the convergence of the power series in wave amplitude (whose leading terms were obtained by Stokes (1847) as approximate solution to the non-linear problem) if the ratio of wave amplitude to wavelength is sufficiently small. Soon Struik (1926) extended the proof to waves on water of arbitrary depth. In a significant later advance on the subject Krasovskii (1960, 1961) proved the existence of permanent periodic waves subject only to the restriction that their maximum slope is less than the limiting value of 30 degrees. The striking fact that has come out in recent years is that this state of dynamic equilibrium of waves of unchanging form is in fact unstable to perturbations.

In an early analysis Whitham (1965) showed that the equations governing the slow modulations of water waves are of elliptic type and

---

* Contents of this chapter have been accepted for publication in *Wave Motion*. Also presented at the Tenth U.S. National Congress of Applied Mechanics held in Austin, Texas, June 1986.

hence unstable. Later using a Lagrangian discovered by Luke (1967) which generates the water wave equations, Whitham (1967) introduced the averaged Lagrangian technique for slow modulations which has since proved extremely useful in analysing the wave properties including the question of modulational stability. On the other hand in two classic papers Benjamin and Feir (1967) and Benjamin (1967) demonstrated, theoretically and experimentally, the instability of small amplitude surface waves to modulations in the form of sidebands. In his analysis Benjamin employs a perturbation scheme on the Fourier modes resulting from a Stokes-wave expansion consistent with the assumption of small amplitude waves. The results of Whitham and Benjamin are complementary in the sense that the first deals with very gradual but not necessarily small perturbations whereas the second deals with very small perturbations. The significant result that comes out of these investigations is that Stokes waves of wavenumber k on a water layer of uniform depth h are unstable if $kh > 1.363\cdots$ and stable otherwise.

These investigations however ignore the presence of such factors as surface tension, viscous dissipation and the effect of incumbent air pressure. The inclusion of any of these effects renders the calculations involved extremely difficult, if not intractable. Recently the effect of surface tension on the development of Benjamin-Feir type instability in deep water waves has been studied by Barakat (1984). The effect of surface tension is expected to play a more profound role in the stability of waves on water of arbitrary depth and our study addresses this question. We first use the Fourier mode analysis to investigate the sideband instability of capillary-gravity waves on a liquid layer of arbitrary uniform depth. Then we give a new Lagrangian from which the complete set of water wave equations with surface tension could be derived, and use this Lagrangian to investigate the stability of the waves to slow modulation. The differences in the results of these two types of analyses are then discussed.

1.2 Benjamin - Feir Analysis

Our analysis of the sideband instability follows closely that of Benjamin (1967) and we retain many of his notations for easy reference. The main difficulty in the analysis is the excessive algebra involved. Even the task of deriving Stokes-wave solutions for capillary-gravity waves up to second order terms in ka (where a is the amplitude, small but finite) is formidable. We have been able to utilise the capabilities of the algebraic manipulation system MACSYMA (1983) to handle the algebra in our investigation.

First we briefly recall the Benjamin-Feir analysis. The primary wavetrain is assumed to have amplitude a, wavenumber k, and fundamental frequency ω. Let us call the argument of the fundamental mode ς, where $\varsigma = kx - \omega t$. Also present would be the higher harmonics with arguments $2\varsigma, 3\varsigma, \cdots$ each travelling with phase velocity ω/k but with decreasing amplitudes which can generally be assumed to be $O(k^n a^n)$ for the n^{th} harmonic. We now introduce two perturbing wavetrains with frequencies ω_1 and ω_2 and wavenumbers k_1 and k_2 which differ slightly from the frequency and wavenumber of the fundamental mode of the primary wavetrain:

$$k_1 = k(1+k'),\ k_2 = k(1-k')$$
$$\omega_1 = \omega(1+\omega'),\ \omega_2 = \omega(1-\omega') \qquad (1.1)$$

where ω' and k' are small compared to unity.
The amplitudes of the sidebands, ϵ_1 and ϵ_2, are assumed to be small compared to a, and slowly varying functions of time. The arguments of the sidebands are

$$\zeta_i = k_i x - \omega_i t - \gamma_i, \qquad i = 1,2 \qquad (1.2)$$

where γ_i are unknown slowly varying functions of time.
The sidebands and the various modes in the primary wavetrain interact nonlinearly giving rise to product terms. Consider for instance the following typical interactions between the sidebands and the second harmonics of the primary wavetrain:

$$(k^2a^2 \sin 2\zeta)(\epsilon_1 \cos\zeta_1) = 1/2\ k^2a^2\epsilon_1\{\sin(\zeta_2+\theta)+\sin(2\zeta+\zeta_1)\} \qquad (1.3)$$
$$(k^2a^2 \sin 2\zeta)(\epsilon_2 \cos\zeta_2) = 1/2\ k^2a^2\epsilon_2\{\sin(\zeta_1+\theta)+\sin(2\zeta+\zeta_2)\}. \qquad (1.4)$$

Ignoring the second term on the right of (1.3) and (1.4) we observe that if θ tends to a constant, then the pair of interactions becomes mutually resonant, with each sideband mode suffering a synchronous forcing effect proportional to the amplitude of the other. In this way the sidebands could grow in time, with the resultant distortion of the primary wavetrain. The main task of the Fourier mode analysis is to show that the condition that θ tends to a constant in time is possible.

1.3 Capillary-Gravity Stokes Waves

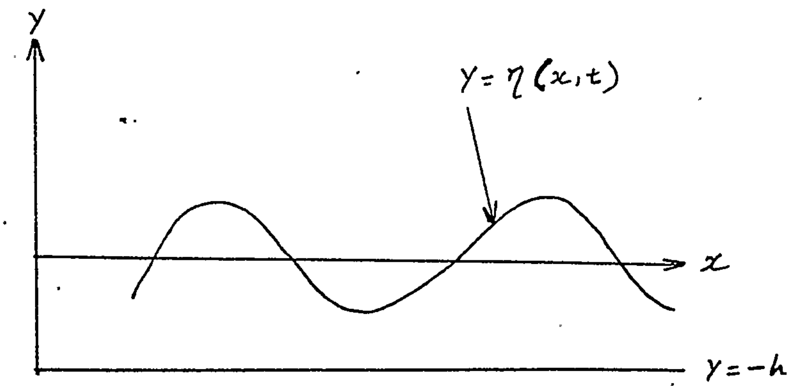

Figure 1.1 Coordinate system for surface waves

In this section we derive the Stokes-wave solution for periodic capillary-gravity waves in water of arbitrary uniform depth h. As shown in Figure 1.1, we take the x-axis horizontally on the water surface, the y-axis vertically up, with the origin at the undisturbed water surface. We denote the surface by

$$y = \eta(x,t). \tag{1.5}$$

Considering water as irrotational and inviscid, the velocity potential $\Phi(x,y,t)$ satisfies (suffix denotes partial derivative)

$$\Phi_{xx} + \Phi_{yy} = 0, \qquad t > 0, \tag{1.6}$$

throughout the liquid, subject to the no-slip condition at the bottom

$$\phi_y = 0, \; y = -h. \tag{1.7}$$

On the free surface, the kinematic condition is

$$\eta_t + \eta_x\phi_x - \phi_y = 0, \quad \text{on} \quad y = \eta(x,t) \tag{1.8}$$

and the constant-pressure condition gives

$$\phi_t + \frac{1}{2}(\phi_x^2+\phi_y^2) - \tau\eta_{xx}(1+\eta_x^2)^{-3/2} + g\eta = 0 \quad \text{on} \quad y = \eta(x,t). \tag{1.9}$$

In (1.9) τ denotes surface tension per unit density. See Whitham (1974) for a derivation of these equations.

We now look for small amplitude periodic wavetrain solutions for (1.5)-(1.9) in the form $\eta = \eta(x-ct)$, $\phi = \phi(x-ct,y)$, where c is a constant phase velocity. To facilitate the analysis we introduce the nondimensional groups

$$\rho = \tau k^2/g, \quad H = kh. \tag{1.10}$$

Then by successive approximation the following Stokes-wave solution is found up to $O(k^2a^2)$ terms:

$$\eta = ka^2\Delta + a\cos\varsigma + ka^2P\cos 2\varsigma \tag{1.11}$$

$$\phi = \frac{\omega a \cosh(ky+H)}{k \sinh H}\sin\varsigma + \omega a^2 Q\cos(2ky+2H)\sin 2\varsigma \tag{1.12}$$

where

$$\varsigma = kx - \omega t, \tag{1.13}$$

$$\Delta = - \frac{1}{2} (1+\rho)\operatorname{csch}(2H) \tag{1.14}$$

$$P = \frac{(1 - 2 \sinh^2 H)\sinh(2H) + \sinh(4H)}{4 \sinh^2 H \{2\omega_0^2 \cosh(2H) - gk(1+4\rho)\sinh(2H)\}} \omega_0^2 \tag{1.15}$$

$$Q = \frac{(1 - 2 \sinh^2 H)\omega_0^2 + g(1+4\rho)\sinh(2H)}{4 \sinh^2 H \{2\omega_0^2 \cosh(2H) - gk(1+4\rho)\sinh(2H)\}} \tag{1.16}$$

and

$$\omega_0^2 = gk(1+\rho) \tanh H. \tag{1.17}$$

If the expansion is carried to the next term of order k^3a^3, one finds that the solution involves terms of the type $\varsigma \sin \varsigma$ which is unbounded in ς. Following Stokes, this secularity is suppressed by expanding ω in a power series

$$\omega = \omega_0(k) + k^2a^2\omega_2(k) + O(k^3a^3). \tag{1.18}$$

In this manner we find that the dispersion relation up to $O(k^2a^2)$ terms is

$$\omega^2 = \omega_0^2(1 + k^2a^2 D_2) \tag{1.19}$$

where

$$D_2 = 2Q \cosh^2 H + P \operatorname{csch}(2H) + 2\Delta \operatorname{csch}(2H) + 1. \qquad (1.20)$$

The perturbation solution breaks down when the denominators in (1.15) and (1.16) vanish; this occurs when

$$(1+4\rho)\sinh(2H) = 2(1+\rho)\tanh(H)\cosh(2H) \ . \qquad (1.21)$$

This breakdown is associated with second harmonic resonance and solutions valid at and near the critical wave numbers can be constructed using modified scales similar to the PLK method (see (Barakat and Houston(1968)). In this thesis we confine ourselves to nonsingular wave numbers where the solutions (1.11)-(1.12) are valid.

1.4 Perturbation Analysis

As is usual in similar situations, the perturbation scheme proceeds by introducing a small disturbance and investigating its asymptotic behaviour in time. Thus we set

$$\phi = \phi_1 + \epsilon\tilde{\phi} \qquad (1.22)$$

$$\eta = \eta_1 + \epsilon\tilde{\phi} \qquad (1.23)$$

where ϕ_1, η_1 are the main wave solutions given by (1.11) and (1.12), ϵ is a small parameter whose square can be ignored and $\tilde{\phi}$, $\tilde{\eta}$ are perturbations whose nature is to be determined.

To derive governing equations for $\tilde{\phi}$ and $\tilde{\eta}$ we substitute (1.22) and

(1.23) into (1.8) and (1.9) and reduce the resulting equations evaluated at $y = \eta_1$ to equations evaluated at $y = 0$ using a Taylor series about $y = 0$. This results in the following two equations:

$$\tilde{\eta}_t - \tilde{\phi}_y + \Big\{\eta_{1x}\tilde{\phi}_x - \eta_1\tilde{\phi}_{yy} + \eta_1\eta_{1x}\tilde{\phi}_{yx} - \frac{1}{2}\eta_1^2\tilde{\phi}_{yyy} + \tilde{\eta}(-\phi_{1yy} - \eta_1\phi_{1yyy} + \eta_{1x}\phi_{1yx}) + \tilde{\eta}_x(\phi_{1x} + \eta_{1x}\phi_{1yx})\Big\} = 0 \qquad (1.24)$$

$$g\tilde{\eta} - \tau\tilde{\eta}_{xx} + \tilde{\phi}_t + \Big\{\tilde{\phi}_x(\phi_{1x} + \eta_1\phi_{1yx}) + \tilde{\phi}_y(\phi_{1y} + \eta_1\phi_{1yy}) + \tilde{\phi}_{yx}(\eta_1\phi_1) + \tilde{\phi}_{yy}\eta_1\phi_{1y} + \eta_1\tilde{\phi}_{yt} + \frac{1}{2}\eta_1^2\tilde{\phi}_{yyt} + (\eta_{1x}\eta_{1yx} + \phi_{1y}\phi_{1yy} + \phi_{1yt} + \eta_1\phi_{1yyt})\tilde{\eta}\Big\} = 0 \qquad (1.25)$$

where all the quantities are evaluated at $y = 0$. Note that (1.24) and (1.25) are linear equations in $\tilde{\phi}$ and $\tilde{\eta}$. We now substitute the expressions (1.11) and (1.12) for η_1 and ϕ_1 into (1.24) and (1.25) and retain terms up to $O(a^2)$. In this way we obtain the following two linear equations governing the perturbations $\tilde{\phi}$ and $\tilde{\eta}$:

$$\tilde{\eta}_t-\tilde{\Phi}_y\Big|_{y=0} + a\left[-k\left\{\tilde{\Phi}_x+\tilde{\eta}\coth(H)\right\}\sin\varsigma + \left\{-\tilde{\Phi}_{yy}+\tilde{\eta}_x\coth(H)\right\}\cos\varsigma\right]_{y=0}$$

$$+ a^2\Bigg[\left\{k\omega\tilde{\eta}_x\left[Q\sinh(3H)\operatorname{csch}(H)-Q+\frac{1}{2}\right] - \frac{1}{4}\tilde{\Phi}_{yy} - Pk\tilde{\Phi}_{yy}\right\}\cos(2\varsigma)$$

$$+ \left\{k^2\omega\tilde{\eta}(2Q-2Q\sinh(3H)\operatorname{csch}(H)-1) - \frac{1}{2}k\tilde{\Phi}_{xy} - 2Pk^2\tilde{\Phi}_x\right\}\sin(2\varsigma)$$

$$+ \frac{1}{2}k\omega\tilde{\eta}_x - \frac{1}{4}\tilde{\Phi}_{yyy} - k\Delta\tilde{\Phi}_{yy}\Bigg]_{y=0} = 0 \qquad (1.26)$$

$$\tilde{\Phi}_t\Big|_{y=0} - \tau\tilde{\eta}_{xx}+g\tilde{\eta} + a\left[\omega\tilde{\Phi}_y\sin\varsigma+\left\{-\omega^2\tilde{\eta}+\omega\tilde{\Phi}_x\coth(H) + \tilde{\Phi}_{ty}\right\}\cos\varsigma\right]_{y=0}$$

$$+ a^2\Bigg[\Bigg\{k\omega\tilde{\Phi}_y\left[\frac{1}{2}Q\sinh(4H)\operatorname{csch}^2(H) - 2Q\coth(H) + \frac{1}{2}\coth(H)\right]$$

$$+ \frac{1}{2}\omega\tilde{\Phi}_{yy}\Bigg\}\sin(2\varsigma) + \Bigg\{Qk\omega^2\tilde{\eta}\left[\frac{7}{2}\coth(H) - \sinh(4H)\operatorname{csch}^2(H)\right]$$

$$+ k\omega\tilde{\Phi}_x\left[\frac{1}{2}Q\cosh(4H)\operatorname{csch}^2(H) + \frac{1}{2}Q\operatorname{csch}^2(H)(1-2\cosh(H)) + \frac{1}{2}\right]$$

$$+ \frac{1}{2}\omega\tilde{\Phi}_{xy}\coth(H) + \frac{1}{4}\tilde{\Phi}_{tyy} + kP\tilde{\Phi}_{ty}\Bigg\}\cos(2\varsigma) + \frac{1}{2}k\omega^2\tilde{\eta}\coth(H)$$

$$+ \frac{1}{2}k\omega\tilde{\Phi}_x + \frac{1}{2}\omega\tilde{\Phi}_{xy}\coth(H) + \frac{1}{4}\tilde{\Phi}_{tyy} + k\Delta\tilde{\Phi}_{ty}\Bigg]_{y=0} = 0\,. \qquad (1.27)$$

The next step in the analysis is the introduction of the right form of perturbing wavetrains. These wavetrains $\tilde{\eta}$ and $\tilde{\Phi}$ are assumed to consist of two sideband modes together with the product of their interaction with the main wavetrain. Following Benjamin we take

$$\tilde{\eta} = \tilde{\eta}_1 + \tilde{\eta}_2 .$$

where each η_i (i = 1,2) has the following form

$$\tilde{\eta}_i = \epsilon_i \cos \zeta_i + ka\epsilon_i \left\{ A_i \cos(\zeta+\zeta_i) + B_i \cos(\zeta-\zeta_i) \right\} + O(k^2a^2\epsilon_i) \tag{1.28}$$

where ζ_i is given by (1.2). Among the neglected terms of $O(k^2a^2\epsilon_i)$ in (1.28), terms with arguments $(2\zeta+\zeta_i)$ are non-resonant and can be neglected while terms with argument $2\zeta-\zeta_i$ are important, but can be merged in to the terms of the present expansion, as seen by (1.3) and (1.4). It is further assumed that ϵ_i, γ_i are slowly varying functions of time with

$$\dot{\epsilon}_i = O(\omega k^2a^2\epsilon_i), \quad \dot{\gamma}_i = O(\omega k^2a^2) . \tag{1.29}$$

It will also be necessary to assume that A_i, B_i are O(1).

The perturbing velocity potential is such that

$$\nabla^2\tilde{\Phi} = 0 \tag{1.30}$$

$$\tilde{\Phi}_y = 0, \quad y = -h . \tag{1.31}$$

We thus assume that

$$\tilde{\phi}_i = \frac{\cosh k_i(y+h)}{k_i \sinh(k_i h)} \left\{\epsilon_i(\omega_i L_i + \dot{\gamma}_i M_i)\sin \zeta_i + \dot{\epsilon}_i N_i \cos \zeta_i\right\}$$

$$+ \omega a\, \epsilon_i \left[C_i \frac{\cosh|(k+k_i)(y+h)|}{\sinh|(k+k_i)h|} \sin(\zeta+\zeta_i) + \right.$$

$$\left. D_i \frac{\cosh|(k-k_i)(y+h)|}{\sinh|(k-k_i)h|} \sin(\zeta-\zeta_i) \right], \quad (i = 1,2). \quad (1.32)$$

Here, too, the coefficients L_i, M_i, N_i, C_i, D_i, whose nature is unknown, will be taken to be O(1).

We now proceed to determine the equations governing the coefficients in $\tilde{\eta}_i$ and $\tilde{\phi}_i$. Towards this end equations (1.28) and (1.32) for $\tilde{\eta}_i$, $\tilde{\phi}_i$ are substituted in (1.26) and (1.27) and all terms are reduced to simple harmonic components. The process is quite laborious and the excellent abilities of MACSYMA were amply used. The resulting equations are supposed to hold over a continuous and unbounded range of x. Thus they must be satisfied by each set of components out of the boundary conditions, thus leading to independent equations for the coefficients A_i, B_i, etc.

Separating components with arguments $\zeta-\zeta_i$ and $\zeta+\zeta_i$ we obtain the following 8 linear equations for A_i, B_i, C_i and D_i (i = 1,2):

$$k(\omega+\omega_i)A_i-\omega(k+k_i)C_i = \frac{1}{4}(k+k_i)\operatorname{csch}(H)\operatorname{csch}(k_ih)\Big\{(\omega+\omega_i)$$

$$(\sinh((k+k_i)h)+\sinh((k-k_i)h))\Big\}, \quad i = 1,2 \tag{1.33}$$

$$k\left[g+\tau(k+k_i)^2\right]A_i-\omega(\omega+\omega_i)\coth((k+k_i)h)C_i = \frac{1}{4}\operatorname{csch}(H)\operatorname{csch}(k_ih)$$

$$\Big\{(\omega^2+\omega_i^2)\cosh((k+k_i)h)-(\omega+\omega_i)^2\cosh((k-k_i)h)\Big\}, \ (i = 1,2) \tag{1.34}$$

$$k(\omega-\omega_i)B_i-\omega(k-k_i)D_i = \frac{1}{4}\operatorname{csch}(H)\operatorname{csch}(k_ih)\Big\{(\omega+\omega_i)$$

$$\sinh((k+k_i)h)-(\omega-\omega_i)\sinh((k-k_i)h)\Big\}, \quad (i = 1,2) \tag{1.35}$$

$$k\left[g+\tau k(k-k_i)^2\right]B_i-\omega(\omega-\omega_i)\coth((k-k_i)h)D_i = \frac{1}{4}\operatorname{csch}(H)\operatorname{csch}(k_ih)$$

$$\Big\{(\omega-\omega_i)^2\cosh((k+k_i)h)-(\omega^2+\omega_i^2)\cosh((k-k_i)h)\Big\}, \ i = 1,2. \tag{1.36}$$

Before proceeding further it is necessary to consider the ratio of wave number and frequency perturbations, k' and ω' respectively. For this purpose we note that to a first approximation the sidebands may be expected to satisfy the linear dispersion relation at wave number k. Noting that k', ω' are assumed to be much smaller than k and ω, we have, to a first approximation,

$$\frac{\omega\omega'}{kk'} = \text{group velocity at wave number } k$$

$$= \frac{d}{dk}\left[(gk+\tau k^3)\tanh(kh)\right]$$

or
$$\frac{\omega'}{k'} = \frac{1}{2}\left\{\frac{1+3\rho}{1+\rho} + 2H\,\mathrm{csch}(2H)\right\} = \lambda, \text{ say.} \tag{1.37}$$

The expression (1.37) has been obtained by omitting $O(k^2a^2)$ terms.

The approximate solutions of (1.33)-(1.36) are now readily obtained, in the limit $\omega_i \to \omega$, $k_i \to k$:

$$A_i = 2P \tag{1.38}$$

$$B_i = -\left\{\frac{\lambda\coth(H) + \frac{1}{2}H\,\mathrm{csch}^2(H)}{H\coth(H) - \lambda^2}\right\} \tag{1.39}$$

$$C_i = 2Q\sinh(2H), \qquad i = 1,2 \tag{1.40}$$

$$D_{1,2} = \pm H\left\{\frac{\frac{1}{2}\lambda\,\mathrm{csch}^2(H) + \coth^2(H)}{H\coth(H) - \lambda^2}\right\}. \tag{1.41}$$

Next we separate components at wave numbers k_i from (1.26) and (1.27) with (1.28) and (1.32) substituted in. Taking approximations to $O(\omega k^2a^2\epsilon_i)$ and $O(\omega^2ka^2\epsilon_i)$ terms, we obtain the pair of equations (1.42) and (1.45):

$$\epsilon_i\left[\omega_i(1-L_i)+\dot{\gamma}_i(1-M_i)\right]\sin\varsigma_i + \dot{\epsilon}_i(1-N_i)\cos\varsigma_i$$

$$= \frac{1}{2}\,\omega k^2 a^2 \Big[\epsilon_i\ R \sin \varsigma_i + (\delta_{i1}\epsilon_2 + \delta_{i2}\epsilon_1) S \sin(\varsigma_i + \theta)\Big],\ i = 1,2 \qquad (1.42)$$

where

$$R = \frac{3}{2} + (2\Delta + A + B)\coth(H) + 2C \coth(2H) - |D|/H \qquad (1.43)$$

$$S = \frac{3}{4} + (P + B)\coth(H) + 2Q \cosh(2H) - |D|/H \qquad (1.44)$$

and A, B, C, D are given by (1.38)-(1.41) and $\theta = \gamma_1 + \gamma_2$

$$\epsilon_i \Big[\omega_i^{-1}(gk_i + \tau k_i^3)\tanh(k_i h) - \omega_i L_i - \dot{\gamma}_i (1 + M_i)\Big] \cos \varsigma_i$$

$$+ \dot{\epsilon}_i (1 + N_i) \sin(\varsigma_i) = \frac{1}{2}\,\omega k^2 a^2 \Big[\epsilon_i U \cos \varsigma_i$$

$$+ (\delta_{i1}\epsilon_2 + \delta_{i2}\epsilon_1) V \cos(\varsigma_i + \theta)\Big] \qquad (1.45)$$

where

$$U = -\frac{5}{2} + (2\Delta + A + B)\tanh(H) - 2C \operatorname{csch}(2H) + |D|/H \qquad (1.46)$$

$$V = -\frac{5}{4} + (P + B)\tanh(H) - 2Q + |D|/H\ . \qquad (1.47)$$

One observes that the forms of these equations are similar to those of Benjamin (1967) (except for a change of sign in front of $|D|$ which is probably a misprint in that paper) into which they reduce if surface tension is made zero.

Once again we require that the separate simple harmonic components satisfy (1.42) and (1.45) independently. Hence upon equating coefficients of cos ς_i and sin ς_i in these equations we obtain

$$\dot{\epsilon}_i(1-N_i) = \frac{1}{2}\,\omega k^2 a^2 \left\{(\delta_{i1}\epsilon_2+\delta_{i2}\epsilon_1)S \sin\theta\right\} \tag{1.48}$$

$$\dot{\epsilon}_i(1+N_i) = -\frac{1}{2}\,\omega k^2 a^2 \left\{(\delta_{i1}\epsilon_2+\delta_{i2}\epsilon_1)V \sin\theta\right\}\,. \tag{1.49}$$

Adding these two equations, we have

$$\dot{\epsilon}_i = \left[\frac{1}{2}\,\omega k^2 a^2 \;X \sin\theta\right]\left[\delta_{i1}\epsilon_2+\delta_{i2}\epsilon_1\right] \tag{1.50}$$

where

$$X = \frac{1}{2}\,(S-V) = 1 + (P+B)\operatorname{csch}(2H) + 2Q\cosh^2(H) - |D|/H\,. \tag{1.51}$$

To obtain equations governing $\dot{\theta}$ we separate sin ς_i components from (1.42) and cos ς_i components from (1.45) and get, respectively,

$$\epsilon_i\left[\omega_i(1-L_i)+\dot{\gamma}_i(1-M_i)\right]$$

$$= \frac{1}{2}\,\omega k^2 a^2 \left[\epsilon_i R + (\delta_{i1}\epsilon_2+\delta_{i2}\epsilon_1)S\cos\theta\right] \quad (i = 1,2) \tag{1.52}$$

$$\epsilon_i\left[\omega_i^{-1}(gk_i+\tau k_i^3)\tanh(k_i h)-\omega_i L_i-\dot{\gamma}_i(1+M_i)\right]$$

$$= \frac{1}{2}\,\omega k^2 a^2\left[\epsilon_i U+(\delta_{i1}\epsilon_2+\delta_{i2}\epsilon_1)V\cos\theta\right] \quad (i = 1,2). \tag{1.53}$$

Subtracting (1.53) from (1.52) results in

$$\dot{\gamma}_i = \frac{1}{2}\,\omega_i^{-1}\left[(gk_i+\tau k_i^3)\tanh(k_i h)-\omega_i^2\right] + \frac{1}{2}\,\omega k^2 a^2\left[\frac{1}{2}\,(R-U)\right.$$

$$\left. + \frac{\delta_{i1}\epsilon_2+\delta_{i2}\epsilon_1}{\delta_{i1}\epsilon_1+\delta_{i2}\epsilon_2}\,X\cos\theta\right]. \tag{1.54}$$

Adding the two equations contained in (1.54) gives

$$\dot{\theta} = \frac{1}{2}\left[\omega_1^{-1}\left\{(gk_1+\tau k_1^3)\tanh(k_1 h)-\omega_1^2\right\} + \omega_2^{-1}\left\{(gk_2+\tau k_2^3)\tanh(k_2 h)-\omega_2^2\right\}\right]$$

$$+ \frac{1}{2}\,\omega k^2 a^2\left[(R-U) + \frac{\epsilon_1^2+\epsilon_2^2}{\epsilon_1\epsilon_2}\,X\cos\theta\right]. \tag{1.55}$$

Let

$$f(k_i) = \left[(gk_i+\tau k_i^3)\tanh(k_i h)\right]^{1/2} \tag{1.56}$$

then recalling (1.1), we have, to a second approximation,

$$\begin{aligned} f^2(k_i)-\omega_i^2 &= f^2(k(1\pm k'))-\omega^2(1\pm\omega')^2 \\ &= f^2(k) + kk'(f^2)' + \frac{1}{2}\,(kk')^2(f^2)'' - \omega^2(1\pm\omega')^2 \\ &= f^2(k) \pm 2(kk')ff' + \frac{1}{2}\,\frac{(\omega\omega')^2}{(f')^2}\,(f^2)'' - \omega^2 \mp 2\omega\omega'^2 - (\omega\omega')^2 \end{aligned}$$

$$= f^2(k) - \omega^2 - (\omega\omega')^2\, Y(k) \tag{1.57}$$

where

$$Y = 1 - \frac{1}{2}\frac{(f^2)''}{(f')^2} \, . \tag{1.58}$$

The primes in (1.58) denote k-derivatives.

Using this expression and noting that

$$\frac{1}{2}\,(R-U)-D_2 = X, \tag{1.59}$$

we get from (1.55)

$$\dot{\theta} = \omega k^2 a^2 X\left\{1 + \frac{\epsilon_1^2+\epsilon_2^2}{2\epsilon_1\epsilon_2}\cos\theta\right\} - \omega\,\omega'^2 Y \, . \tag{1.60}$$

This expression is crucial to the whole stability issue. If only dispersive effects were present, we will have, if c_g is the group velocity at wave number k,

$$\frac{\partial\theta}{\partial t} + c_g\frac{\partial\theta}{\partial x} = (k'k)^2 f''(k) = \omega'^2\,\frac{f^2(k)f''(k)}{[f'(k)]^2} \, , \tag{1.61}$$

where $f(k)$ is the dispersion relation $\omega = f(k)$. Thus the property $\theta \to$ constant cannot be realised unless there are effects counteracting the dispersion. This is precisely the first term on the right of (1.60) which represents the nonlinear effects balancing the dispersive effects given by the second term.

1.4. Instability Criteria

Integrating (1.50) we find

$$\epsilon_i(t) = \epsilon_i(0)\cosh\left\{\frac{1}{2}\,\omega k^2a^2X\int_0^t \sin\theta\, dt\right\} +$$

$$+\left[\delta_{i1}\epsilon_2(0)+\epsilon_{i2}\epsilon_1(0)\right]\sinh\left\{\frac{1}{2}\,\omega k^2a^2\int_0^t \sin\theta\, dt\right\} \tag{1.62}$$

so that if θ = constant ($\neq 0,\pi$), as $t \to \infty$, $\epsilon_i(t) \to \infty$, so that the sideband instability can be achieved. To obtain precise criteria for instability we need to uncouple the differential equations for ϵ_1 and ϵ_2. This is achieved as follows:

Defining $T = \omega k^2a^2t$, we have from (1.60),

$$-\epsilon_1\epsilon_2\,\frac{d}{dT}\,(\cos\theta) = \left[X - \frac{\omega'^2 Y}{k^2a^2}\right]\epsilon_1\epsilon_2\,\sin\theta + \frac{1}{2}\left[\epsilon_1^2+\epsilon_2^2\right]X\,\sin\theta\,\cos\theta \tag{1.63}$$

and from (1.50),

$$\frac{d\epsilon_1^2}{dT} - \frac{d\epsilon_2^2}{dT} = X\,\epsilon_1\epsilon_2\,\sin\theta\ . \tag{1.64}$$

Using (1.63) in (1.64), we get

$$\frac{d}{dT}\left\{\epsilon_1\epsilon_2 \cos\theta + \left[1 - \frac{\omega'^2 Y}{k^2 a^2 X}\right]\epsilon_1^2\right\} = 0 \tag{1.65}$$

or

$$\epsilon_1\epsilon_2 \cos\theta + \alpha\epsilon_1^2 = \text{constant} = \sigma, \tag{1.66}$$

say, where

$$\alpha = 1 - \frac{\omega'^2 Y}{k^2 a^2 X}\,. \tag{1.67}$$

From (1.64), we have

$$\epsilon_1^2 - \epsilon_2^2 = \text{constant} = 2\sigma\alpha(1-\mu),\ \mu \text{ being a constant.} \tag{1.68}$$

From (1.64), upon using (1.66) and (1.68), we get

$$\left[\frac{d\epsilon_1^2}{dT}\right]^2 = X^2\left\{(1-\alpha^2)\epsilon_1^4 + 2\sigma\alpha\mu\,\epsilon_1^2 - \sigma^2\right\}\,. \tag{1.69}$$

The stability analysis based on the differential equation (1.69) for the sideband amplitude is similar to that in Benjamin (1967). For completeness we reproduce the main arguments.

Let

$$Q = (1-\alpha^2)\,\epsilon_1^4 + 2\alpha\sigma\mu\,\epsilon_1^2 - \sigma^2. \tag{1.70}$$

Since ϵ_1 is must be real, the solution ϵ_1^2 of (1.69) is restricted to the range of positive values over which $Q > 0$, and a positive root of Q represents an extremum of ϵ_1^2. The two roots of Q may be written as

$$A = -\frac{\alpha\mu\sigma}{1-\alpha^2}, \qquad B = \frac{\sigma(1-\alpha^2+\alpha^2\mu^2)^{1/2}}{|1-\alpha^2|}$$

We now have the following three cases.

i) <u>$-1 < \alpha < 1$, The case of instability</u>

In this case only one root, A+B, is positive and any value of ϵ_1^2 greater than this makes $Q > 0$, so that unbounded growth of ϵ_1^2 with T is possible.

Writing Q as

$$Q = (1-\alpha^2)\left\{ (\epsilon_1^2 - A)^2 - B^2 \right\} \tag{1.72}$$

we obtain from (1.69), upon integration,

$$(1-\alpha^2)^{1/2}\, T\, |X| = \int_{\epsilon_1^2(0)}^{\epsilon_1^2(T)} \left\{(\varphi - A)^2 - B\right\}^{-1/2} d\varphi$$

$$= \left[\cosh^{-1}\left\{\frac{\Psi-A}{B}\right\}\right]_{\epsilon_1^2(0)}^{\epsilon_1^2(T)} \tag{1.73}$$

If we denote the initial value of the expression on the right of (1.73) by $(1-\alpha^2)^{1/2}\ \tau\ |X|$, we have

$$\epsilon_1^2 = A + B \cosh\left\{(1-\alpha^2)^{1/2}\ |X|\ (T+\tau)\right\}. \tag{1.74}$$

Thus $\epsilon_1^2 \sim \exp(1/2\ (1-\alpha^2)^{1/2}\ |X|\ T)$ for large T and we have instability.

ii) <u>$\alpha = -1$, Marginal Instability</u>

In this case the right side of (1.69) becomes a linear function of ϵ_1^2, and the resulting equation is easily solved:

$$\epsilon_1^2 = -1/2\ \mu\sigma \left\{\sigma^2 + (T+\tau)^2\ X^2\right\}. \tag{1.75}$$

Thus $\epsilon_1^2 \sim T\ |X|$ for large times. This linear growth may be classed as an instability.

iii) <u>$\alpha < -1$, Stability</u>

Putting $(1-\alpha^2)^{1/2} = i\ (\alpha^2 - 1)^{1/2}$ in (1.74), we have, in this case,

$$\epsilon_1^2 = A + B \cos\left\{(\alpha^2-1)^{1/2}\ (T+\tau)\ |X|\right\}. \tag{1.76}$$

Thus ϵ_1^2 varies between $A \pm B$ and hence is bounded.

It should be noted that the present conclusions apply to ϵ_2 as well. Hence we have shown that there is instability if $-1 \leq \alpha < 1$. Going back to the original definition of α in (1.67), we find that the Stokes waves are stable to sideband perturbations if

$$X/Y < 0 \tag{1.77}$$

If $X/Y > 0$, then the sidebands can grow unbounded so long as the perturbation frequency ω' satisfies

$$0 < \omega'^2 \leq 2k^2a^2X/Y \tag{1.78}$$

This instability criterion is similar to that derived by Benjamin (1967) for gravity waves. In that case Y is always positive, and the stability depends on the sign of X. It was found that $X > 0$ for $kh > 1.363...$ and negative otherwise. Thus gravity waves are stable to sideband perturbations if $kh < 1.363....$

In the present case for capillary-gravity waves the stability criterion is not so simple because of the nature of X and Y. In figure 1.2 we have plotted the instability regions in terms of the two dimensionless parameters $\tau k^2/g$ and kh. A discussion of the results, in comparison with the results of the averaged Lagrangian approach given in the next sections, may be found at the end of the chapter.

1.6 The Averaged Lagrangian method

In the following sections we will investigate the stability of capillary-gravity waves to slow modulations using the averaged Lagrangian technique. This technique was initiated by Whitham (1967) and has been later justified rigorously using a perturbation scheme by Luke (1967).

We first briefly recall the averaged lagrangian method (Whitham, 1974). Suppose there exists a variational principle

$$\delta J = \delta \iint_R L(\phi_t, \phi_x, \phi) dt\, d\underline{x} = 0 \tag{1.79}$$

for a function $\phi(\underline{x},t)$. The principle implies that the integral $J[\phi]$ over any finite region R should be stationary to small changes of ϕ as follows: Let $\phi(\underline{x},t)$ and $\phi(\underline{x},t)+h(\underline{x},t)$ be two "neighbouring" functions, where h is "small". The norm of h in which the "smallness" is measured is

$$\|h\| = \max|h| + \max|h_t| + \max|h_{x_i}| \;. \tag{1.80}$$

Both ϕ and h are taken to be continuously differentiable. Supposing that L has bounded continuous second derivatives, we obtain, by a Taylor expansion

$$J[\phi+h]-J[\phi] = \iint_R \left[L_{\phi_t} h_t + L_{\phi,_j} h_{xj} + L_\phi h \right] d\underline{x}dt + o(\|h^2\|) \tag{1.81}$$

where $\phi_{,j}$ denotes $\frac{\partial \phi}{\partial x_j}$. The expression linear in h is $\delta J[\phi,h]$, the first variation. The variational principle (1.79) requires that $\delta J[\phi,h] = 0$ for all admissible h. Choosing h to be a function that vanishes on the boundary of R, we get, on integration by parts, from (1.81)

$$\delta J[\phi,h] = \iint_R \left[-\frac{\partial}{\partial t} L_{\phi_t} - \frac{\partial}{\partial x_j} L_{\phi,_j} + L_\phi \right] h \, d\underline{x}dt \,. \qquad (1.82)$$

This implies that

$$\frac{\partial L_{\phi_t}}{\partial t} + \frac{\partial}{\partial x_j} L_{\phi,_j} - L_\phi = 0 \qquad (1.83)$$

by the continuity argument. (If (1.83) were not zero, say positive at some point, then it would be positive in a small neighbourhood about that point, and choosing h to be positive in this neighbourhood and zero elsewhere, one could violate the requirement that (1.82) should vanish).

The argument extends naturally if L involves second or higher order derivatives in ϕ. The general variational equation is

$$L_\phi - \frac{\partial}{\partial x_j} L_{\phi,_j} - \frac{\partial}{\partial t} L_{\phi_t} + \frac{\partial^2}{\partial t \partial x_j} L_{\phi_{t,j}}$$
$$+ \frac{\partial^2}{\partial t^2} L_{\phi_{tt}} + \frac{\partial^2}{\partial x_i \partial x_j} L_{\phi,ij} - \cdots = 0 \,. \qquad (1.84)$$

Our interest is in slowly varying wavetrains in which

$$\phi \sim a \cos(\theta + \eta) \tag{1.85}$$

where a, θ, η are slowly varying functions. The statement slowly varying implies that the amplitude a , wave number $k = \theta_x$, and frequency $\omega = -\theta_t$ are functions of x and t, but varies over distance and time scales large enough compared to the wavelength and period so that their derivatives can be neglected to a first approximation. This idea enables us to define an averaged Lagrangian. We substitute (1.85) into the expression for the Lagrangian L, neglect the derivatives of a, η, ω, k as being small and average over a period. This results in a function, the averaged Lagrangian, $\mathcal{L}(\omega,k,a)$. As an illustration consider the Lagrangian

$$L = \frac{1}{2}\phi_t^2 - \frac{1}{2}\alpha^2\phi_{x_i}^2 - \frac{1}{2}\beta^2\phi^2 \tag{1.86}$$

corresponding to the Klein-Gordon equation

$$\phi_{tt} - \alpha^2\nabla^2\phi + \beta^2\phi = 0 \; . \tag{1.87}$$

Substituting (1.85) into (1.87) and averaging, one gets

$$\mathcal{L} = \frac{1}{4}(\omega^2 - \alpha^2k^2 - \beta^2)a^2 \; . \tag{1.88}$$

The "averaged variational principle" postulates that

$$\delta \iint \mathscr{L}(-\theta_t, \theta_x, a)\,dt\,d\underline{x} = 0 \tag{1.89}$$

for the functions $\theta(\underline{x},t)$, $a(\underline{x},t)$.

The variational equation for a gives

$$\delta_a : \quad \mathscr{L}_a = 0 \tag{1.90}$$

and for θ,

$$\delta\theta : \quad \frac{\partial\mathscr{L}}{\partial t}\,\theta_t + \frac{\partial}{\partial x_j}\,(\mathscr{L}_{\theta,j}) = 0 \;. \tag{1.91}$$

These equations, along with the consistency conditions for the existence of θ, may be rewritten in terms of ω, $\underline{k}$, a as follows:

$$\mathscr{L}_a = 0 \tag{1.92}$$

$$\frac{\partial}{\partial t}\,\mathscr{L}_\omega - \frac{\partial}{\partial x_j}\,\mathscr{L}_{k_j} = 0 \tag{1.93}$$

$$\frac{\partial k_i}{\partial t} + \frac{\partial \omega}{\partial x_i} = 0, \quad \frac{\partial k_i}{\partial x_j} - \frac{\partial k_j}{\partial x_i} = 0 \;. \tag{1.94}$$

Equations (1.92)-(1.94) govern slow modulations. The first of these, (1.92), is the dispersion relation. The second, (1.93), can be shown, after some manipulation, to be equivalent to the amplitude modulation equation

$$\frac{\partial a^2}{\partial t} + \frac{\partial}{\partial x_j}(c_j a^2) = 0 \qquad (1.95)$$

where c_j is the group velocity,

$$c_j = \frac{\partial \omega(k)}{\partial x}. \qquad (1.96)$$

The averaged Lagrangian approach is a powerful tool for the investigation of slowly varying wavetrains since it can be adopted, with very little modification, to treat varied situations like non-uniform media and non-linear wavetrains (Whitham (1974)). The main drawback, however, is the difficulty in finding the appropriate Lagrangian for the system under consideration. In the following we will give a Lagrangian for capillary-gravity waves and use it to investigate the slow modulation of the waves.

1.7 Lagrangian for capillary-gravity waves

Luke (1967) has shown that the waterwave equations without surface tension effects follows from the variational principle

$$\delta \iint_R L \, d\underline{x}dt = 0 \qquad (1.97)$$

where

$$L = -\int_{-h}^{\eta} \left\{\phi_t + \frac{1}{2}(\nabla\phi)^2 + gy\right\} dy . \qquad (1.98)$$

Here R is an arbitrary region in $(\underline{x},t)$ space.

We now show that the equations for capillary-gravity waves (1.6)-(1.9) follows from the variational principle (1.97) if L is given by

$$L = -\int_{-h}^{\eta}\left[\Phi_t + \frac{1}{2}(\Phi_x^2+\Phi_y^2) + gy\right]dy - \tau\left[(1+\eta_x^2)^{1/2} - 1\right] . \quad (1.99)$$

For a small change $\delta\Phi$ in Φ, we have,

$$\begin{aligned}
-\delta\iint L\, d\underline{x}dt &= \iint_R \left\{\int_{-h}^{\eta}[\delta\Phi_t + (\underline{\nabla}\Phi\cdot\underline{\nabla}\delta\Phi)]dt\right\}d\underline{x}dt \\
&= \iint_R \left\{\frac{\partial}{\partial t}\int_{-h}^{\eta}\delta\Phi dy + \frac{\partial}{\partial x_i}\int_{-h}^{\eta}\Phi_{x_i}\delta\Phi dy\right\}d\underline{x}dt \\
&- \iint_R \left\{\int_{-h}^{\eta}(\Phi_{x_i x_i}+\Phi_{yy})\delta\Phi dy\right\}d\underline{x}dt \\
&- \iint_R \left[(\eta_t+\Phi_x\eta_x-\Phi_y)\delta\Phi\right]_{y=\eta} d\underline{x}dt \\
&- \iint_R \left[(\Phi_x h_x+\Phi_y)\delta\Phi\right]_{y=-h} d\underline{x}dt . \qquad (1.100)
\end{aligned}$$

The first term in (1.100) integrates out to the boundaries of R and vanishes if $\delta\Phi$ is chosen to vanish on the boundaries of R. If (1.100) were to vanish for all such $\delta\Phi$, it follows that

$$\Phi_{xx} + \Phi_{yy} = 0, \quad -h_0 < y < \eta \qquad (1.101)$$

$$\eta_t + \Phi_x\eta_x - \Phi_y = 0, \quad y = \eta \qquad (1.102)$$

$$\Phi_x h_x + \Phi_y = 0, \quad y = -h \qquad (1.103)$$

(1.101) is obtained by choosing $\delta\Phi = 0$ on $y = \eta$, $y = -h$ and using the usual variational argument. Then a choice of $\delta\Phi > 0$ on $y = \eta$, $\delta\Phi = 0$ on $y = -h$ gives (1.102) and a choice of $\delta\Phi > 0$ on $y = -h$, $\delta\Phi = 0$ on $y = \eta$ gives (1.103).

For a variation $\delta\eta$ in η,

$$\delta \iint\limits_R L \, dxdt = - \iint\limits_R \left[\Phi_t + \frac{1}{2}(\bar{\nabla}\Phi)^2 + gy\right]_{y=\eta} \delta\eta \, dxdt$$

$$- \tau \iint\limits_R \left[(1+\eta_x^2)^{-1/2} \eta_x \delta\eta_x\right] dxdt = 0 \; . \qquad (1.104)$$

Again a choice of $\delta\eta$ that vanishes on the boundary of R gives, by the divergence theorem and the continuity argument,

$$\Phi_t + \frac{1}{2}(\Phi_x^2+\Phi_y^2) + g\eta - \tau\eta_{xx}(1+\eta_x^2)^{-3/2} = 0 \text{ on } y = \eta \; . \qquad (1.105)$$

Thus we have shown that the complete set of water wave equations including surface tension (eqns. 1.6-1.9) can be derived from the Lagrangian (1.99) using the variational principle (1.97).

1.8 The Averaged Lagrangian

We now apply the variational principle to study periodic dispersive wavetrains. We take the following general form for a uniform periodic wavetrain:

$$\Phi = \beta x - \gamma t + \phi(\theta,y), \quad \theta = kx-\omega t \tag{1.106}$$

$$\eta = N(\theta) \ . \tag{1.107}$$

The phase function θ may be normalised to have a period 2π. The linear term $\beta x-\gamma t$ must be allowed in Φ because it is only the derivatives of Φ that represent periodic physical quantities. β is the mean horizontal velocity. The meaning of γ is less clear - it corresponds to absorbing the Bernoulli constant into the potential.

In the lowest order modulation approximation the Lagrangian is found by substituting the periodic wavetrain (1.106) and (1.107) into (1.99). Thus we get

$$\begin{aligned} L = &\int_0^{N(\theta)} \left\{\gamma+\omega\phi_\theta(\theta,y) - \frac{1}{2}(\beta^2+k^2\phi_\theta^2+2\beta k\phi_\theta) - \frac{1}{2}\phi_y^2-gy\right\}dy \\ &- \tau\left[(1+k^2N_\theta^2)^{1/2}-1\right] \\ \\ = &(\gamma - \frac{1}{2}\beta^2)N+(\omega-\beta k)\int_0^N \phi_\theta dy - \frac{1}{2}\int_0^N (k^2\phi_\theta^2+\phi_y^2)dy \\ &- \frac{1}{2}gN^2-\tau\left[(1+k^2N_\theta^2)^{1/2}-1\right]. \end{aligned} \tag{1.108}$$

Since the exact forms of $N(\theta)$ and $\phi(\theta,y)$ are not known one could either use the long wave approximation of the Boussinesq or Korteweg-de Vries type, or alternatively use the near-linear Stokes wave expansion. We use the latter approach to complement the Fourier mode analysis in the previous sections. Thus we let

$$N(\theta) = h + a\cos\theta + \sum_{n=2}^{\infty} a_n \cos n\theta \tag{1.109}$$

$$\phi(\theta,y) = \sum_{n=1}^{\infty} \frac{A_n}{n}\cosh(nky)\sin n\theta . \tag{1.110}$$

The form of ϕ ensures that it satisfies the Laplace equation. The Fourier coefficients A_n, a_n are assumed in advance to be $O(a^n)$ for small amplitude a; this is justified from our experience of Stokes expansion.

The procedure now is to substitute (1.109) and (1.110) into (1.100) to calculate L, keeping terms up to a^4 to include the first nonlinear effects. Then the averaged Lagrangian is calculated:

$$\mathcal{L} = \frac{1}{2\pi}\int_0^{2\pi} L d\theta . \tag{1.111}$$

In order to proceed up to terms of order a^4 the Fourier coefficients A_1, A_2 and a_2 will be required. These are obtained from the variational equations

$$\mathcal{L}_{A_1} = \mathcal{L}_{A_2} = \mathcal{L}_{a_2} = 0 . \tag{1.112}$$

These calculations are necessarily tedious, comparable to the labour involved in deriving the Stokes wave solutions in the Fourier mode analysis.

Let h_0 denote the undisturbed water height and choose the origin $y = 0$ at the bottom so that $h_0 = 0$. Then the averaged Lagrangian, after considerable labour, is obtained as

$$\begin{aligned} \mathcal{L} = {} & (\gamma - \tfrac{1}{2}\beta^2)h - \tfrac{1}{2} gh^2 - \tfrac{1}{4}(g+\rho k^2)a^2 \\ & + \frac{1}{4}\frac{(\omega-\beta k)^2}{kT} a^2 + \left[\frac{3}{64} - \frac{3}{4}U^2\right]\tau k^4 a^4 \\ & + \frac{1}{4}\left[g + \tau k^2\right]\left[T^2U^2 - \frac{3-T^2}{2T}U - \frac{2T^2-1}{4T^2}\right] \\ & \times k^2 a^4, \end{aligned} \tag{1.113}$$

where

$$T = \tanh(kh),$$

$$U = \frac{(g+\tau k^2)(3-T^2)}{4T[T^2(g+\tau k^2)-3\tau k^2]} .$$

This expression is not uniformly valid. It becomes singular when the relation

$$\tanh^2(kh)\cdot(g+\tau k^2) = 3\tau k^2 \tag{1.114}$$

is satisfied. This singularity is identical to that noted in (1.21) in the Stokes wave expansion. In the following we confine ourselves to nonsingular wave-numbers.

It is convenient to express the averaged Lagrangian (1.113) in terms of the energy density of capillary-gravity waves. The energy density E is given by (see Lighthill(1978) for derivation of this)

$$E = 1/2(g+\tau k^2)a^2, \tag{1.115}$$

the two terms making up the gravitational and capillary contribution respectively.

Consistent with the slow modulation theory we assume that <u>changes</u> in γ, β, h are $O(a^2)$. Thus in the coefficients of a^4 in (1.113), h may be replaced by the undisturbed water depth h_0. The resulting averaged Lagrangian, in terms of the energy density E is

$$\begin{aligned}\mathcal{L} = (\gamma - \tfrac{1}{2}\beta^2)h - \tfrac{1}{2}gh^2 + \tfrac{1}{2}E\left[\frac{(\omega-\beta k)^2}{(g+\tau k^2)kT} - 1\right] \\ + \frac{k^2E^2}{g+\tau k^2}\left[\frac{\frac{3}{16} - 3U^2}{g+\tau k^2}\tau k^2 + T_0^2U^2 - U\frac{3-T_0^2}{2T_0}\right. \\ \left. - \frac{2T_0^2-1}{4T_0^2}\right] \ldots\end{aligned} \tag{1.116}$$

where $T_0 = \tanh(kh_0)$.

The variational equation $\mathcal{L}_E = 0$ gives the dispersion relation

$$\frac{(\omega-\beta k)^2}{(g+\tau k^2)kT} = 1 - 4D_2\frac{k^2E}{g+\tau k^2}, \tag{1.117}$$

where

$$D_2 = \frac{\frac{3}{16} - 3U^2}{g+\tau k^2}\tau k^2 + T_0^2U^2 - U\frac{3-T_0^2}{2T_0} - \frac{2T_0^2-1}{4T_0^2}.$$

1.9 Dispersion relation and Stability analysis

Using the averaged Lagrangian (1.116) one can derive various expressions for mass, momentum and energy conservation and induced mean flow (see Whitham (1974), sec. 16.7-16.10). In most of these expressions the changes effected by including surface tension can be easily found by noting that the linear disperson relation $\omega^2 = gk \tanh(kh)$ has been replaced by $\omega^2 = (gk+\tau k^3)\tanh(kh)$. However, the most important consequences of the second order terms in (1.117) is in the modulational stability of the Stokes wavetrain. We now investigate this aspect and compare the results with the Fourier mode approach.

After some manipulation the dispersion relation (1.117) can be written as

$$\omega = \omega_0 + \frac{k^2 E}{c_0}\left[-\frac{2c_0\omega_0 D_2}{g+\tau k^2} - \frac{[2C_0 - \frac{1}{2}c_0(g+3\tau k^2)/(g+\tau k^2)]^2}{kh_0(gh_0 - C_0^2)} - \frac{1}{kh_0}\right] + O(E^2), \tag{1.118}$$

where $\omega_0 = [(g+\tau k^2)kT_0]^{1/2}$, and c_0, C_0 are the phase and group velocities respectively:

$$c_0 = [(g+\tau k^2)k^{-1}T_0]^{1/2}$$
$$C_0 = \frac{1}{2}c_0\left[\frac{g+3\tau k^2}{g+\tau k^2} + \frac{2kh_0}{\sinh(2kh_0)}\right].$$

Let us now look at the characteristic velocity of the modulation equations, (1.92)-(1.94). First assume that the dispersion relation is, up to second order,

$$\omega = \omega_0(k) + \omega_2(k)\, a^2 \tag{1.119}$$

Then the conservation law of the wave crests

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0$$

takes the form

$$\frac{\partial k}{\partial t} + \frac{\partial}{\partial k}(\omega_0 + \omega_2\, a^2)\, \frac{\partial k}{\partial x} + \omega_2\, \frac{\partial a^2}{\partial x} = 0 \tag{1.120}$$

The amplitude modulation equation (1.95) becomes

$$\frac{\partial a^2}{\partial t} + \frac{\partial}{\partial x}(\, \omega_0'\, a^2\,) = 0 \tag{1.121}$$

where a prime denotes $\frac{d}{dk}$.

The coupled set of equations (1.120) and (1.121) may be expressed in the form

$$A\, Y_x + I\, Y_t = 0 \tag{1.122}$$

where

$$Y = \begin{bmatrix} a^2 \\ k \end{bmatrix}, \tag{1.123}$$

$$A = \begin{bmatrix} \omega_0' & a^2\, \omega_0'' \\ \omega_2 & \omega_0' + a^2\, \omega_2' \end{bmatrix} \tag{1.124}$$

and

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The characteristic speeds V of (1.122) are found from the determinant

$$| A - V I | = 0 \qquad (1.125).$$

which expands to give

$$V = (\omega_0' + \omega_0 a^2/2)^2 \pm \left[(\omega_0' + \omega_2' a^2/2)^2 - \omega_0' (\omega_0' + a^2 \omega_2') + a^2 \omega_2 \omega_0'' \right]^{1/2} \qquad (1.126)$$

To the leading order in amplitude a we have

$$V \simeq \omega_0' \pm a (\omega_2 \omega_0'') \qquad (1.127)$$

Thus in our case the characteristic velocities are

$$V = C_0 \pm (\omega_0'' \Omega_2 k^2 E/c_0)^{1/2}.$$

where Ω_2 is the coefficient of k^2E/c_0 in (1.118).

It is remarkable that one can find all the stability characteristics just from the dispersion relation. In case $\omega_0''(k)\Omega_2 > 0$ the characteristics are real, the system is hyperbolic and stable to modulating perturbations. If $\omega_0''(k)\Omega_2 < 0$, on the other hand, the characteristics are imaginary and the system is elliptic. Thus small perturbations will grow in time and the wavetrain is unstable.

1.10. Discussion of results

The Fourier mode analysis gives a description of instabilities due to infinitesimally small perturbations, while the Lagrangian approach describes very gradual but not necessarily small modulations. Thus in some sense the two approaches complement each other in describing surface wave instabilities. It should be noted that the Fourier mode analysis is a linear perturbation analysis on the nonlinear water wave equations.

In figure 1.2 we have plotted the dimensionless groups $\tau k^2/g$ versus kh with the shaded regions indicating regions of instability given by the Fourier mode analysis. The result can be considered valid if ka is small, up to the first order in small quantities. The significant effect of surface tension is quite evident from the diagram. One observes that the neutral stability curve starting at kh = 1.363... for $\tau k^2/g = 0$ tends to smaller values of kh as $\tau k^2/g$ increases. The branch (a) corresponds to $Y = 0$ and the branch (b) corresponds to (1.21). In both situations the perturbation scheme breaks down and modified scales need to be introduced.

Figure 1.3 is a stability diagram based on the averaged Lagrangian approach. The branch (a) corresponds to $\omega_0'' = 0$ which is identical with $Y = 0$ giving rise to branch (a) in figure 1.2. The branch (b) in both figures result from the singularity in the Stokes wave expansion given by (1.21). There appear two additional branches (c) and (d) in figure 1.3. (c) corresponds to $gh_0 = C_0^2$, indicating the coincidence of group velocity C_0 with the phase velocity of long waves, $\sqrt{gh_0}$. In this situation Ω_2 becomes singular and once again the perturbation scheme has to be modified. The branch (d) corresponds to $\Omega_2 = 0$. The branches (c) and (d) are absent in figure 1.2 because as shown by (1.60) and (1.58) the dispersive effects balancing nonlinearity is taken only up to the first order in the Fourier mode analysis. As seen from (1.118), the branches (c) and (d) in figure 1.3 arise from the second order dispersive effects. The stability diagram in figure 1.3 is similar to that of Kawahara (1975) who uses a multiple scale formalism to derive a nonlinear Schrodinger equation governing the self modulation of capillary-gravity waves. It is interesting to note that if surface tension is not zero, there is always instability at some value of kh.

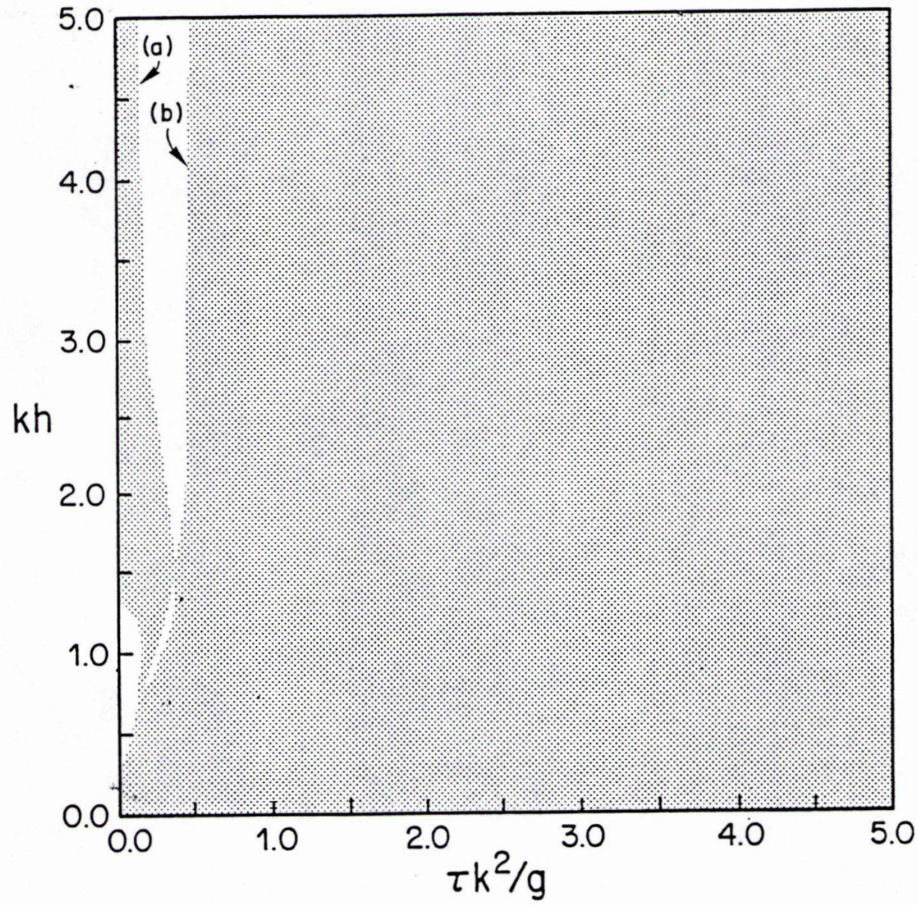

Figure 1.2. Stability diagram based on Fourier Mode Analysis. Shaded areas indicate instability.

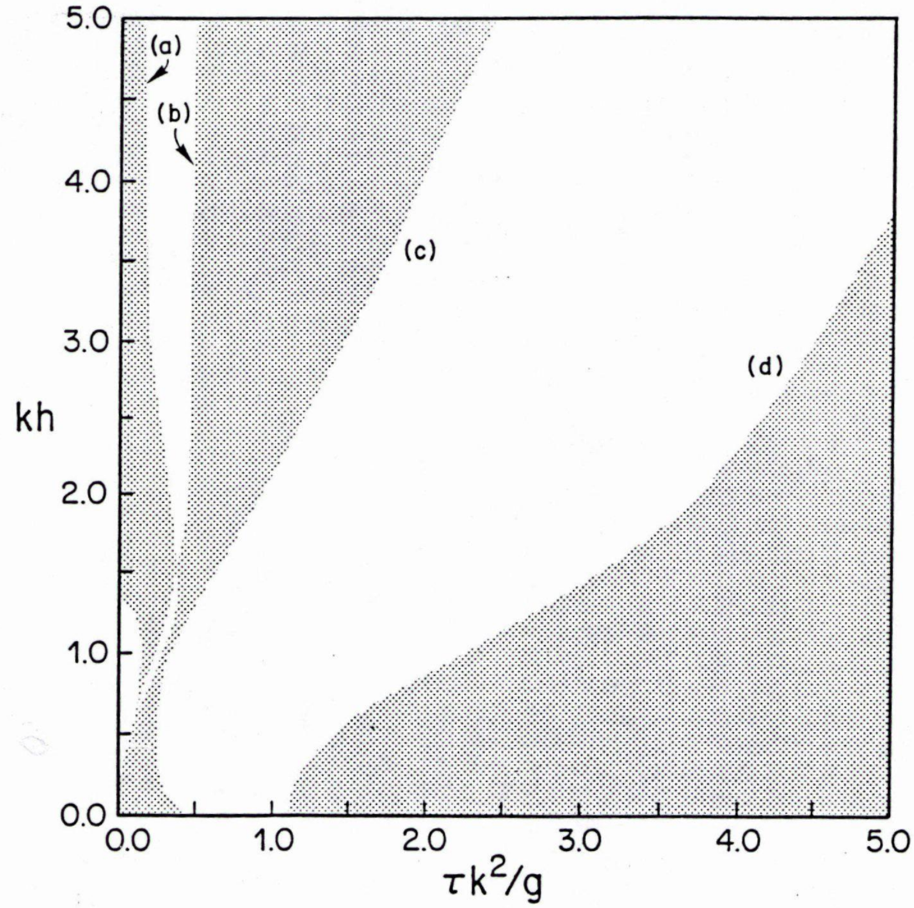

Figure 1.3. Stability diagram based on the averaged Lagrangian. Shaded areas indicate instability.

CHAPTER II

EVOLUTION OF PERTURBATIONS OF THE RENORMALISED LONG WAVE EQUATION *

2.1 Introduction

In recent years there has been considerable interest in the study of perturbed evolution equations. These equations govern physical phenomena such as wave propagation in a slowly varying medium, waves in a channel of varying cross section and solitary waves moving along a sloping beach. Karpman (1979) studied the perturbed KdV equation by using perturbations on the inverse scattering method, while Grimshaw (1979, 1981) and Johnson (1971) used series expansion to determine the evolution of the perturbed KdV equation. In the most recent work Smyth (1984) used two timing and matched asymptotic expansions to determine the evolution of the perturbed KdV equation

$$u_t + 6uu_x + u_{xxx} = \epsilon u \tag{2.1}$$

where ϵ is a small parameter. He finds two distinct regions behind the slowly varying solitary wave: (i) a near tail which eventually breaks up

---

* Contents of this chapter have been accepted for publication in the *Journal of Mathematical Physics*. Also presented at the Seventh Canadian Symposium on Fluid Dynamics held in Sackville, New Brunswick, June 1986.

into new solitary waves and which together with the soliton conserves the KdV mass, (ii) a far tail which makes no contribution, to $O(\epsilon)$, to the mass conservation.

Our aim in this chapter is to examine the evolution of solitary waves of the perturbed Renormalised Long Wave (RLW) equation (or the BBM equation after Benjamin et al. (1972))

$$u_t + 6uu_x - u_{xxt} = \epsilon u \tag{2.2}$$

using the asymptotic expansion techniques introduced by Smyth (1984). The RLW equation, with $\epsilon = 0$ in (2.2), is an alternative model equation for long waves and derives from the Boussinesq equation following the same assumptions used to derive the KdV equation (in fact one can use $u_t \simeq -cu_x$ in the dispersive correction term, remaining within the approximation for long waves, to obtain the RLW equation; here c is the wave velocity). Recent interest in the RLW equation comes from the numerical experiments showing the inelastic scattering properties of its solitary waves. In fact the RLW equation has only three non-trivial conservation laws depending smoothly on u and its derivatives whereas the KdV equation has an infinite number of conservation laws. However, in certain theoretical investigations, the RLW equation is superior as a model for long waves (see Benjamin et al. (1972) for a discussion of regularity properties of the RLW equation compared to the KdV equation for the same initial data). Since the perturbed evolution equations are crucial in many physical phenomena, it is worth investigating how a different model equation (in this case the RLW instead of KdV) affect the system properties. In this note we have chosen the form ϵu on the right of (2.2) simply for the sake of brevity. One could, for example, consider instead $-\epsilon u$ if a small damping is present or $-\epsilon u_{xx}$ in the presence of heat conduction. For all these latter cases, however, the analysis is essentially the same.

As in the case of the perturbed KdV equation it will be found that the slowly varying solitary wave solution of (2.2) does not conserve mass. It is then assumed that there is a 'near tail' region just behind the solitary wave caused by a mass flux from it. Behind the 'near tail' there will be another 'far tail' region governed by the linearised form of (2.2). The essential difference between the present analysis of the perturbed RLW equation and the perturbed KdV equation will be in the far tail region. The far tail will be found to be exponentially decaying as $x \to -\infty$ while for the perturbed KdV it is oscillatory, given by an Airy function.

2.2 The solitary wave

We will assume that the solution of (2.2) consists of a main solitary wave with slowly varying parameters given by the expansion

$$u = u^0(\theta,T) + \epsilon u^1(\theta,T) + \cdots \tag{2.3}$$

where

$$T = \epsilon t$$

$$\theta = x - \frac{C(T)}{\epsilon}$$

$$C_T = \omega_0(T) + \epsilon^2\omega_2(T) + \cdots \ . \tag{2.4}$$

Substituting (2.3) in (2.2) the zeroth order equation is

$$-\omega_0 u_\theta^0 + 6u^0 u_\theta^0 - \omega_0 u_{\theta\theta\theta}^0 = 0. \tag{2.5}$$

This has a solitary wave solution

$$u^0 = \eta(T)\mathrm{sech}^2\left[\frac{1}{2}\,\theta\right] \tag{2.6}$$

where

$$\eta(T) = \frac{1}{2}\,\omega_0(T). \tag{2.7}$$

It is worth observing that the solitary wave speed depends linearly on the amplitude unlike the KdV soliton for which $\omega_0(T) = 4(\eta_T^2)$.

One can show by simple conservation arguments that the solitary wave (2.6) does not conserve mass. For, the perturbed RLW equation (2.2) has energy conservation law

$$\frac{1}{2}\frac{d}{dt}\left[\int_{-\infty}^{\infty}(u^2+u_x^2)dx\right] = \epsilon \int_{-\infty}^{\infty} u^2 dx, \tag{2.8}$$

if $u \to 0$ as $x \to \pm\infty$.

Using (2.6) we have

$$\eta_T = \frac{5}{6}\,\eta$$

or

$$\eta = \eta_0 \, e^{\frac{5}{6}T}, \quad \text{where } \eta_0 \text{ is a constant.} \tag{2.9}$$

The equation (2.2) also has a mass conservation equation

$$\frac{d}{dt}\int_{-\infty}^{\infty} u \, dx = \epsilon \int_{-\infty}^{\infty} u \, dx. \tag{2.10}$$

But using (2.9) we have

$$\frac{d}{dt}\int_{-\infty}^{\infty} u \, dx = \frac{10}{3} \epsilon\eta \tag{2.11}$$

$$\epsilon \int_{-\infty}^{\infty} u \, dx = 4\epsilon\eta. \tag{2.12}$$

Thus we see that the slowly varying solitary wave (2.6) conserves energy, but not mass. To make up for this it will be assumed that there is a tail region behind the solitary wave.

We now take up a detailed formal asymptotic analysis of (2.2). First it will be shown that the expansion (2.3) is not uniformly valid as $x \to -\infty$. The $O(\epsilon)$ equation resulting from substituting (2.3) in (2.2) is

$$\omega_0 u^1_{\theta\theta\theta} + 6u^0 u^1_\theta - \omega_0 u^1_\theta + 6u^0_\theta u^1 - u^0_{T\theta\theta} + u^0_T = u^0. \tag{2.13}$$

This has adjoint

$$\omega_0 v_{\theta\theta\theta} + 6u^0 v_\theta - \omega_0 v_\theta = 0. \tag{2.14}$$

Multiplying (2.13) by v, (2.14) by u^0 and adding, we get

$$\omega_0\left[(v_{\theta\theta}u^1)_\theta + (u^1_{\theta\theta}v)_\theta - (v_\theta u^1_\theta)_\theta\right] + 6(u^0u^1v)_\theta - \omega_0(u^1v)_\theta$$
$$= (u^0 + u^0_T + u^0_{T\theta\theta})v \tag{2.15}$$

and integrating from $-\infty$ to ∞ w.r.t. θ,

$$-\omega_0\left[u^1v - u^1v_{\theta\theta} - u^1_{\theta\theta}v - u^1_\theta v_\theta\right]_{-\infty}^{\infty} = \int_{-\infty}^{\infty}(u^0 - u^0_T + u^0_{T\theta\theta})v\, d\theta. \tag{2.16}$$

We require that $u^1 \to 0$ as $\theta \to \infty$ and that u^1 is bounded as $\theta \to -\infty$. The bounded solutions of (2.14) are $v = u^0$ amd $v = 1$. When $v = u^0$, from (2.16) we have

$$0 = \int_{-\infty}^{\infty}(u^0 - u^0_T + u^0_{T\theta\theta})u^0\, d\theta . \tag{2.17}$$

Using (2.6) this gives

$$\eta_T = \frac{5}{6}\eta \tag{2.18}$$

as previously found.

Now consider the solution v = 1. If we assume that $u^1 \to 0$ as $\theta \to -\infty$ we would have, from (2.16),

$$0 = \int_{-\infty}^{\infty} (u^0 - u_T^0 + u_{T\theta\theta}^0)d\theta, \tag{2.19}$$

which would give an expression for η different from (2.18). We thus see that as $\theta \to -\infty$, $u^{(1)}$ does not tend to zero but tends to a constant value given by

$$\begin{aligned} 2\eta\, u^1 &= \int_{-\infty}^{\infty} (u^0 - u_T^0 + u_{T\theta\theta}^0)d\theta \\ &= \frac{2}{3}\eta. \end{aligned} \tag{2.20}$$

Thus as $\theta \to -\infty$, $u^{(1)}$ tends to a constant $\frac{1}{3}$, although $u^0 \to 0$, and the expansion (2.3) is not uniformly valid. This will be rectified by matching the expansion (2.3) with an outer expansion.

2.3 <u>The near tail</u>

The outer expansion for the near tail region just behind the soliton is assumed to depend on the slow scales $X = \epsilon x$, $T = \epsilon t$. Thus an expansion of the form

$$u = \epsilon V_1(X,T) + \epsilon^2 V_2(X,T) + \cdots \tag{2.21}$$

is considered.

From the perturbed RLW equation (2.2) we have

$$(V_1)_T = V_1 \ , \tag{2.22}$$

hence

$$V_1(X,T) = A(X)e^T \tag{2.23}$$

where A(X) is to be determined by matching with the inner solution. The matching has to be done with a moving solitary wave which has a speed

$$\omega_0' = 2\eta(T). \tag{2.24}$$

Hence we have, using (2.10),

$$\frac{dX}{dT} = 2\eta(T) = 2\eta_0 \ e^{\frac{5}{6}T} \ . \tag{2.25}$$

Integrating, we get

$$\epsilon x = \frac{12}{5}\eta_0 \ (e^{\frac{5}{6}T} - 1) \ . \tag{2.26}$$

Thus the solitary wave is at position X when

$$T = \frac{6}{5}\ln\left[1 + \frac{5}{12}\frac{X}{\eta_0}\right] \tag{2.27}$$

since $u_1 \to \frac{1}{3}$ as $\theta \to -\infty$, the matching requires that

$$\frac{1}{3} = A(X)e^T \tag{2.28}$$

when T is given by (2.27).

Thus we see that

$$A(X) = \frac{1}{3\left[1 + \frac{5}{12}\frac{X}{\eta_0}\right]^{\frac{6}{5}}} . \tag{2.29}$$

We notice that the near tail expansion cannot be valid for all times because of the exponential growth in time in (2.23). In the case of perturbed KdV equation the near tail breaks up eventually into new solitons and we expect the same behaviour to take place here too. We do not go into the details of the breakup since it has to be determined numerically. It can be checked at this stage that the near tail together with the solitary wave conserves the RLW mass to $O(\epsilon)$.

2.4 The far tail

Since the solitary wave started at $x = 0$ we expect that the near tail will extend from $x = 0$ till $x = x_s$, the position of the solitary wave. The region $x < 0$ will be called a far tail and it will be assumed to have an expansion of the form

$$u = \epsilon U_1(x,t,T) + \epsilon^2 U_2(x,t,T) + \cdots . \tag{2.30}$$

Using this in (2.2), we get the first order equation

$$(U_1)_t - (U_1)_{xxt} = 0 . \tag{2.31}$$

This has a similarity solution of the form

$$U_1 = B(t,T)e^x. \tag{2.32}$$

The function B(t,T) may be determined by matching with the near tail solution at x = 0. Using (2.23) and (2.29) we find

$$\frac{1}{3} e^T = B(t,T),$$

hence

$$U_1 = \frac{1}{3} e^{(T+x)} . \tag{2.33}$$

This far tail expression differs significantly from the KdV far tail found in Smyth (1984). There it is seen that

$$U_1 = \frac{e^T}{3\eta_0} \int_{-\infty}^{\frac{x}{(3t)^{1/2}}} Ai(s)ds \tag{2.34}$$

where Ai is the Airy function.

Figure 2.1 is a schematic diagram of the perturbed RLW solitary wave. Figure 2.2 shows the perturbed KdV solitary wave . We see that the significant difference occurs in the far tail - whereas the KdV far tail is oscillatory, the RLW far tail decays exponentially as $x \to -\infty$.

So far we have only examined $O(\epsilon)$ equations for the far tail. The long time evolution of the far tail is also undetermined at this stage. Nevertheless one can make certain observations without going into greater detail. Using the expansion (2.30) at the second order we get, for the perturbed KdV equation

$$(U_2)_{xx} + (U_2)_t + 6U_1(U_1)_x + (U_1)_T = U_1 \tag{2.35}$$

and for the RLW,

$$-(U_2)_{txx} + (U_2)_t \quad 6U_1(U_1)_x - (U_1)_{Txx} + (U_1)_T = U_1. \tag{2.36}$$

Thus we see that the derivatives of U_1 are involved at the second order. Because of the nature of the Airy function the expansion (2.30) leads to secularities at this order for the KdV equation. However, no such secularities arise for the RLW equation because of the exponential decay of the derivatives of (2.34) as $x \to -\infty$.

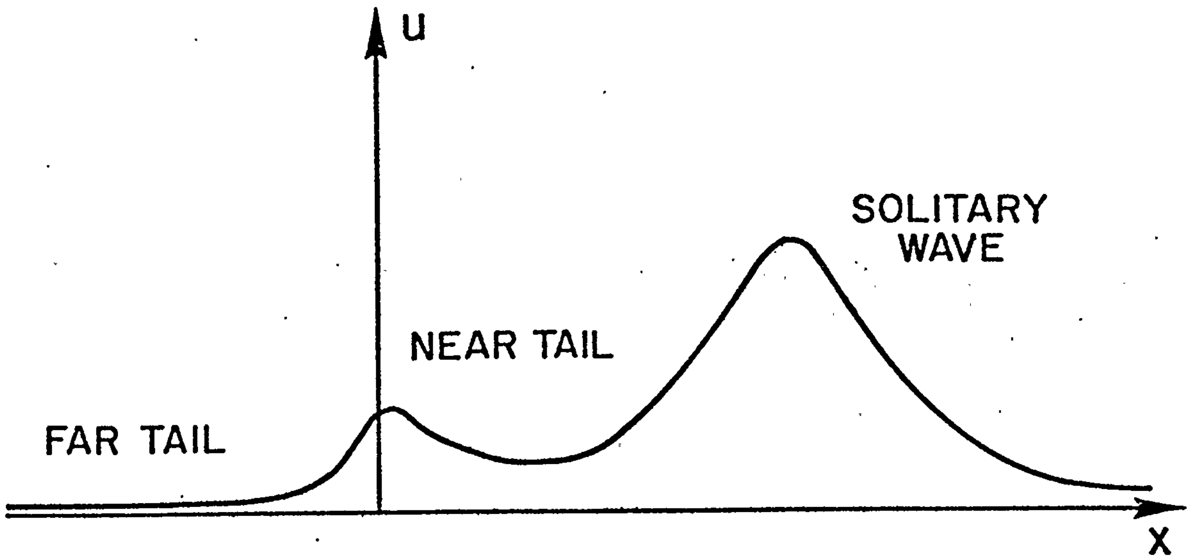

Figure 2.1. Perturbed RLW solitary wave

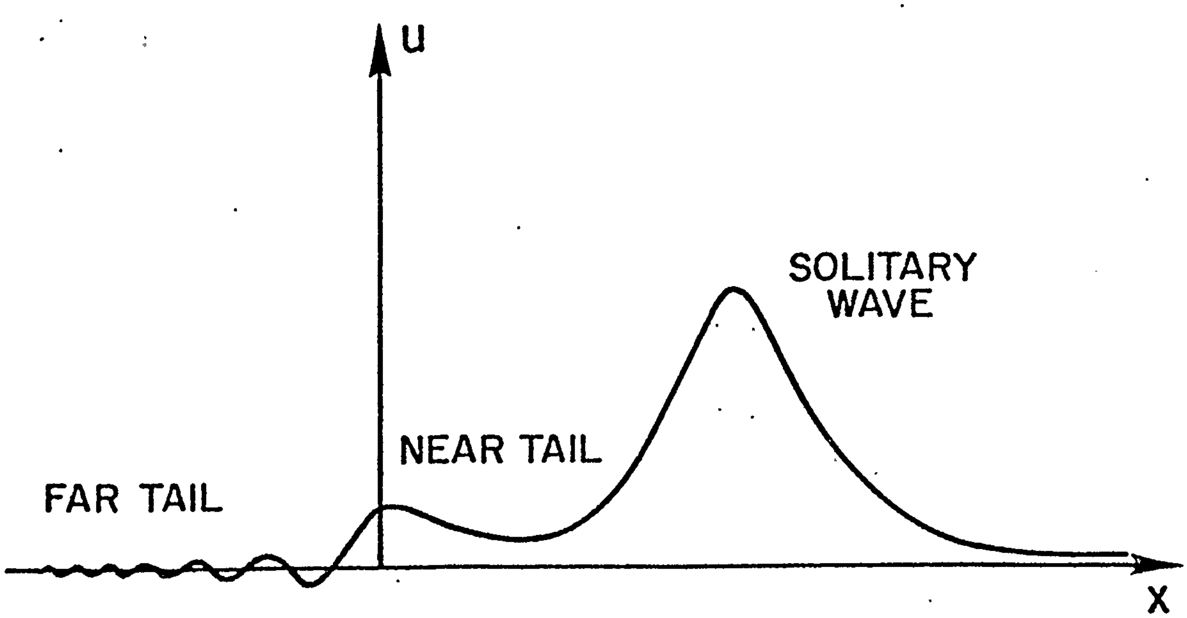

Figure 2.2. Perturbed KdV solitary wave

References for Part I

1. Barakat,R., 1984. Instability of periodic capillary-gravity waves in deep water. *Wave Motion*, 6, 155-165.

2. Barakat,R. and A.Houston, 1968. Nonlinear periodic capillary-gravity waves on a fluid of finite depth. *J. Geophy. Res.*,73, 6546-6554.

3. Benjamin,T.B. and J.E.Feir, 1967. The disintegration of wavetrains in deep water, I. Theory. *J. Fluid Mech.* 27, 417-430.

4. Benjamin,T.B., 1967. Instabiliy of periodic wavetrains in nonlinear dispersive systems. *Proc. Roy. Soc.* A299, 59-75.

5. Benjamin,T.B., J.L.Bona and J.T.Mahony, 1972. Model Equation for long waves in nonlinear dispersive systems. *Philo. Trans. Roy. Soc. London.*, *A* 272, 47-78.

6. Grimshaw,R., 1979. Slowly varying solitary waves, I - Korteweg-deVries Equation. *Proc.Roy.Soc.* A368, 359-375.

7. Johnson,R., 1971. On the asymptotic solution of the KdV equation with slowly varying coefficients. *J.Fluid Mech.*, 60, 813-824.

8. Karpman,V.I., 1979. Soliton Evolution in the presence of perturbations. *Physica Scripta.*, 20, 462-478.

9. Kawahara,T., 1975. Nonlinear self modulation of capillary-gravity waves on a liquid layer. *J. Phys. Soc. Japan.*, 38, 265-270.

10. Krasovskii,Y.P., 1960. On the theory of small waves of not small amplitude (in Russian). *Dokl. Akad. Nauk SSSR.* 130, 1237-1245.

11. Krasovskii,Y.P., 1961. On the theory of permanent waves of finite

amplitude (in Russian). *Zh. vychisl. Mat. mat. Fiz.*, 1,836-845.

12. Levi-Civita,T., 1925. Determination rigoureuse des ondes permanentes d'ampleur fine. *Math. Ann.*, 93, 264-
13. Lighthill,J., 1978. *Waves in Fluids*. Cambridge Univ. Press, New York.
14. Luke,J.C., 1966. A perturbation method for nonlinear dispersive wave problems. *Proc. Roy. Soc.*, A292, 403-412
15. Luke,J.C., 1967. A variational principle for a fluid with a free surface. *J. Fluid Mech.*, 27, 395-397.
16. MACSYMA 1983. MACSYMA Reference Manual, SYMBOLICS Inc., Cambridge Massachusetts.
17. Smyth, N.F., 1984. I: Soliton on a beach and related problems, II: Modulated Capillary Waves. Ph.D. Thesis, Caltech.
18. Stokes,G.B., 1847. On the theory of oscillatory waves. *Trans. Camb. Phil. Soc.*, 8, 441-472.
19. Struik,D.J., 1926. Determination rigoureuse des ondes irrotationnelles perodiues dans un canal à profoundeur finie. *Math. Ann.* 95, 595-610.
20. Whitham,G.B., 1965. A general approach to linear and nonlinear dispersive waves using a lagrangian. *J. Fluid Mech.* 22, 273-283.
21. Whitham,G.B., 1967. Nonlinear dispersion of water waves. *J. Fluid. Mech.*, 27, 399-412.
22. Whitham,G.B., 1974. *Linear and Nonlinear Waves*. John Wiley, New York.

RESULTS IN MICROPOLAR FLUID THEORY

CHAPTER III

INTRODUCTION TO MICROPOLAR FLUIDS

3.1 Introduction

The fundamental concepts of classical continuum mechanics have been known for over two hundred years, since the time of Euler. The classical picture of a continuous medium consists of a dense collection of point masses devoid of internal structure. The laws of motion and constitutive axioms are assumed to be valid for every part of the continuum.

The limitations of such a view of the continuum are obvious. It does not, for example, do justice to the rheology of blood simply because it does not possess the mechanisms to characterize the kinematics and dynamics of cellular micromotions. Hence there is a need to extend the range of applicability of continuum mechanics to treat rheologically complex fluids such as liquid crystals, polymeric fluids, blood and fluids containing certain additives.

In the last few decades there has been a reformulation of the classical continuum concepts to account for local structural aspects and micromotions. In this context a continuous medium is regarded as sets of structured particles possessing not only mass and velocity but also a substructure with which is associated a moment of inertia density and a microdeformation tensor. The presence of the microscopic elements in a fluid gives rise to couple stresses arising from microelement interactions and to additional balance laws and constitutive relations.

Investigations during the last two decades have produced several approaches to the microcontinuum theories of fluids which are called by various names as simple micro-fluids, simple deformable directed fluids, micropolar fluids, polar fluids, dipolar fluids, etc. Some of these theories are general in nature while others are specialised to certain types of material structure and deformations.

Eringen (1964) initiated the study of general fluid microcontinua by considering the mechanics of fluids with deformable microelements, which he called simple microfluids. His model assigns each continuum particle a substructure - that is, each material volume called a macrovolume contains microvolume elements which can translate, rotate and deform independently of the motion of the macrovolume. However, each deformation of the macrovolume element can be expected to produce a subsequent deformation of the microvolume elements. The theory was based solely on the principles of continuum mechanics and not on molecular or statistical mechanics. Additional equations arise in the theory to account for the conservation of microinertia moments and balance of first stress moments.

The linear constitutive theory of these simple microfluids leads to a system of nineteen partial differential equations in nineteen unknowns and involves twenty-two viscosity coefficients. Admittedly this theory is too complicated and the underlying mathematical problem is not readily amenable to the solution of non-trivial problems. Subsequently Eringen

(1966) considered a subclass of these fluids called micropolar fluids which simplifies but restricts the general theory. In this theory, characterised by seven equations in seven unknowns, rigid particles contained in a small volume element can rotate about the centroid of the volume element in an average sense described by the microrotation vector, in addition to the rigid body motion of the entire volume element. Physically it represents fluids consisting of rigid, randomly oriented microelements suspended in a viscous medium.

The relative mathematical simplicity of the microcontinuum theory of micropolar fluids enabled it to be successfully applied in a variety of fluid mechanical problems. A review of various applications can be found in Ariman et al. (1973) and in the recent book by Stokes (1984). It is worth pointing out that Kolpashchikov et al. (1983) have devised a way of experimentally measuring the micropolar material coefficients. Turk et al. (1973) have applied the theory to model blood flow in arteries, and have obtained excellent agreement with the experimental blood flow data of Bugliarello and Sevilla (1970).

In this chapter we will briefly recall Eringen's theory of simple microfluids and the specialised micropolar fluids. In Chapter 4 we will prove two uniqueness theorems for the complete set of coupled non-linear partial differential equations governing the micropolar flow - one for flows inside a bounded region and the other for flows past a finite solid body. For this purpose we use some estimation techniques applied by Serrin (1959) and Dyer and Edmunds (1961) for classical viscous and Magnetohydrodynamic flows. In Chapter 5 we will explicitly construct the time dependent fundamental solutions for the Stokes-linearised equations of micropolar fluid theory. In Chapter 6 we will give an application of the theory to the problem of flow in a meandering channel.

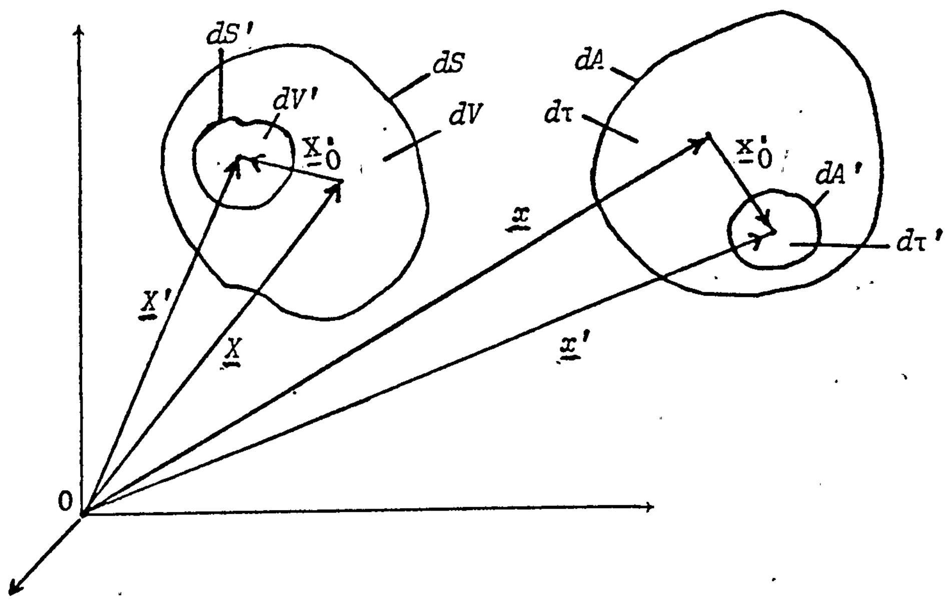

Figure 3.1. Deformed and undeformed volume elements

3.2 Deformation and Microdeformation

Let $\underline{X}'$ be a material point having rectangular coordinates X'_k ($k = 1,2,3$) of a body having volume V and surface S in the undisturbed state (refer to figure 3.1). After a deformation, this point will, at time t, occupy a position, say $\underline{x}'$, in the deformed body now occupying a region having volume τ and surface A. The mappings

$$\left.\begin{aligned} \underline{x}' &= \underline{x}(\underline{X}',t) \\ \underline{X}' &= \underline{X}(\underline{x}',t) \end{aligned}\right\} \qquad (3.1)$$

are called the motions and inverse motions respectively. We assume that the mappings (3.1) possess continuous partial derivatives with respect to their arguments to any required order and that the Jacobian

$$J = \left|\frac{\partial x'_\ell}{\partial X'_L}\right| = \begin{vmatrix} \dfrac{\partial x'_1}{\partial X'_1} & \dfrac{\partial x'_1}{\partial X'_2} & \dfrac{\partial x'_1}{\partial X'_3} \\ \dfrac{\partial x'_2}{\partial X'_1} & \dfrac{\partial x'_2}{\partial X'_2} & \dfrac{\partial x'_2}{\partial X'_3} \\ \dfrac{\partial x'_3}{\partial X'_1} & \dfrac{\partial x'_3}{\partial X'_2} & \dfrac{\partial x'_3}{\partial X'_3} \end{vmatrix} \qquad (3.2)$$

does not vanish for any X'_L and t. This ensures the existence of a unique inverse. In the following a repeated index will indicate summation over (1,2,3) and a comma followed by a subscript ℓ will indicate differentiation with respect to x_ℓ, and D/Dt will indicate material derivative.

Let $\underline{X}$ be the centre of mass of an arbitrary macroelement dV which is made up of microelements, a typical one of which is dV' with its centre of mass at $\underline{X}'$. Let us agree to denote properties associated with the material point $\underline{X}'$ by primed capital letters and those associated with spatial point $\underline{x}'$ by primed small letters. On deforming the macroelement, dV'+dS' centred at $\underline{X}'$ goes to $d\tau'$+dA'. Macroelements are composed of microelements and a simple averaging process is used to link up the properties of macroelements with those of microelements. For example if P, ρ are the average mass densities in dV and $d\tau$ respectively and P', ρ' those in $\underline{X}'$, $\underline{x}'$, then

$$\int_{dV} P'dV' = PdV, \quad \int_{d\tau} \rho' d\tau' = \rho d\tau \tag{3.3}$$

and

$$\int_{dV} P'\underline{X}'dV' = P\underline{X}dV. \tag{3.4}$$

We shall make the assumption of "microconservation" of mass,

$$P'dV' = \rho' d\tau' \tag{3.5}$$

which in turn implies "macroconservation" of mass

$$PdV = \rho d\tau. \tag{3.6}$$

Taking material derivative of (3.6), it follows that

$$\frac{D}{Dt}(\rho d\tau) = 0, \tag{3.7}$$

which is the spatial form of mass conservation.

From Figure 3.1,

$$\underline{X}' = \underline{X} + \underline{X}_0' \tag{3.8}$$

$$\underline{x}' = \underline{x} + \underline{x}_0' \ . \tag{3.9}$$

Also,

$$\underline{x} = \underline{x}(\underline{X},t) \tag{3.10}$$

$$\underline{X} = \underline{X}(\underline{x},t) \ . \tag{3.11}$$

Multiplying (3.8) by P'dV' and integrating over dV we get

$$\int_{dV} P'\underline{X}_0' dV' = 0 \ . \tag{3.12}$$

Recall that

$$\underline{x}' = \underline{x}(\underline{X}',t) = \underline{x}(\underline{X}+\underline{X}_0',t) = \underline{x}(\underline{X},t) + \underline{x}_0' \ . \tag{3.13}$$

Hence

$$\underline{x}_0' = \underline{x}_0(\underline{X},\underline{X}_0',t) \ . \tag{3.14}$$

If $\underline{x}_0'$ is an analytic function of $\underline{X}_0'$ we can write

$$\underline{x}'_0 = \underline{x}_0(\underline{X},0,t) + \left.\frac{\partial \underline{x}_0}{\partial X'_{0L}}\right|_{X'_{0L}=0} X'_{0L} + \cdots \quad . \tag{3.15}$$

But $\underline{x}_0(X,0,t) = 0$, hence if $|\underline{X}'_0|$ is assumed to be sufficiently small, then

$$x'_{0\ell} = x_{L\ell}(\underline{X},t)X'_{0L} \quad . \tag{3.16}$$

where

$$x_{L\ell}(\underline{X},t) = \left.\frac{\partial x_{0\ell}}{\partial X'_{0L}}\right|_{x'_{0L}=0} \quad . \tag{3.17}$$

In a similar manner the inverse micromotions are defined as

$$h_{\ell L} = \left.\frac{\partial X_{0L}}{\partial x'_{0\ell}}\right|_{x'_{0\ell}=0} \quad . \tag{3.18}$$

Thus it follows that

$$h_{kL}x_{L\ell} = \delta_{k\ell}, \quad h_{\ell M}x_{L\ell} = \delta_{ML} \quad . \tag{3.19}$$

Under the assumption (3.16), the motion carries the centre of mass of dV into the centre of mass of $d\tau$. For,

$$\begin{aligned}\int_{d\tau} \rho'\underline{x}'d\tau' &= \int_{d\tau} \rho'\cdot(\underline{x}+\underline{x}_L X'_{0L})d\tau' \\ &= \underline{x}\int_{d\tau} \rho' d\tau' + \underline{x}_L \int_{d\tau} X'_{0L}\rho' d\tau'\end{aligned}$$

$$= \underline{x}\ \rho d\tau .$$

The motion and inverse motion of a material point in a microlement is thus described as

$$x'_\ell = x_\ell(\underline{X},t) + x_{L\ell}(\underline{X},t)X'_{0L} \tag{3.20}$$

$$X'_L = X_L(\underline{x},t) + h_{\ell L}(\underline{x},t)x'_{0\ell} \ . \tag{3.21}$$

3.3 <u>Velocity, Acceleration, Microrotation and Deformation Rate Tensors</u>

The velocity of a material point is

$$v'_\ell = \dot{x}'_\ell = \left.\frac{dx'_\ell}{dt}\right|_{\underline{X}'} = \left.\frac{\partial x_\ell}{\partial t}\right|_{\underline{X}} + \left.\frac{\partial x_{L\ell}}{\partial t}\right|_{\underline{X}} X'_{0L} \tag{3.22}$$

with $\underline{X}$, X'_{0L} held fixed. This may be rewritten as

$$v'_\ell = v_\ell + \nu_{m\ell}x'_{0m} \tag{3.23}$$

where

$$\nu_{m\ell}(\underline{x},t) = h_{mL}\dot{x}_{L\ell} \tag{3.24}$$

and

$$\dot{x}_{L\ell} = \frac{D}{Dt}(x_{L\ell}) \ . \tag{3.25}$$

We have used the fact that

$$X'_{0L} = x'_{0\ell} h_{\ell L} \; . \tag{3.26}$$

In (3.23), v_ℓ is the mean velocity at $\underline{x}$ while $\nu_{m\ell} x'_{0m}$ is the "peculiar velocity" of microelements with respect to $\underline{x}$.

The acceleration of a material point is

$$a'_\ell = \frac{dv'_\ell}{dt} = \ddot{x}_\ell + \ddot{x}_{L\ell} X'_{0L} \; . \tag{3.27}$$

From (3.24),

$$\dot{x}_{L\ell} = \nu_{m\ell} + x_{Lm} , \tag{3.28}$$

thus

$$\begin{aligned} \ddot{x}_{L\ell} &= \dot{\nu}_{m\ell} x_{Lm} + \nu_{m\ell} \dot{x}_{Lm} \\ &= \dot{\nu}_{m\ell} x_{Lm} + \nu_{m\ell} \nu_{km} x_{Lk} \; . \end{aligned} \tag{3.29}$$

Using this in (3.27),

$$a'_\ell = a_\ell + \dot{\nu}_{m\ell} x_{Lm} X'_{0L} + \nu_{m\ell} \nu_{km} x_{Lk} X'_{0L} \; . \tag{3.30}$$

Using (3.26) in (3.30)

$$a'_\ell = a_\ell + (\dot{\nu}_{n\ell} + \nu_{m\ell} \nu_{nm}) x'_{0n} \; . \tag{3.31}$$

The tensor $\nu_{k\ell}$ plays a very important role in this theory and is called the gyration or microrotation tensor.

3.4 Balance Laws

The mechanical balance laws are obtained by applying an averaging process to a macroelement consisting of microelements. Let $t'_{k\ell}$, f'_ℓ be the stress tensor and body force respectively at a point $\underset{\sim}{x}'$ of the deformed microelement $d\tau'+dA'$. The momentum and moment of momentum balances of this microelement at $\underset{\sim}{x}'$ take the form

$$t'_{k\ell,k} + \rho'(f'_\ell - a'_\ell) = 0 \tag{3.32}$$

$$t'_{k\ell} = t'_{\ell k} \ . \tag{3.33}$$

To obtain the balance of moments for the arbitrary macroelement $d\tau+dA$ with mass centre at $\underset{\sim}{x}$, we multiply (3.32) by an arbitrary function $\phi'(\underset{\sim}{x}')$ and integrate over the material volume τ. Thus

$$\int_\tau \left[\int_{d\tau} [\phi' t'_{k\ell,k} + \phi'\rho'(f'_\ell - a'_\ell)]d\tau'\right] = 0 \ . \tag{3.34}$$

Using the divergence theorem, this becomes

$$\int_A \int_{dA} \phi' t_{k\ell} n'_k dA' + \int_\tau \int_{d\tau} [\phi'\rho'(f'_\ell - a'_\ell) - t'_{k\ell}\phi'_{,k}]d\tau' = 0. \tag{3.35}$$

When $\phi' \equiv 1$, we get

$$\int_A t_{k\ell} n_k dA + \int_\tau \rho(f_\ell - a_\ell) d\tau = 0 \tag{3.36}$$

where

$$\int_{dA} t'_{k\ell} n'_k dA' = t_{k\ell} n_k dA, \quad \int_{d\tau} \rho' f'_\ell d\tau' = \rho f_\ell d\tau, \quad \int_{d\tau} \rho' a'_\ell d\tau' = \rho a_\ell d\tau \qquad (3.37)$$

and $\underline{n}$, $\underline{n}'$ are unit outward normal to dA and dA' respectively. $t_{k\ell}$, f_ℓ, ρa_ℓ are to be interpreted as the stress tensor, body force and inertia associated with the macroelement $d\tau$. Using the divergence theorem in (3.36), we get

$$\int_\tau \left[t_{k\ell,k} + \rho(f_\ell - a_\ell) \right] d\tau = 0 \ .$$

It follows that whenever the integrand is continuous,

$$t_{k\ell,k} + \rho(f_\ell - a_\ell) = 0, \qquad (3.38)$$

which is the expression for momentum balance.

By taking $\phi'(\underline{x}') \equiv x'_m$, one can show, using the same procedure as above, that the equations for the balance of the "first stress moment" are

$$t_{m\ell} - \delta_{m\ell} + \lambda_{k\ell m,k} + \rho\left[\ell_{\ell m} - \dot{\sigma}_{\ell m} \right] = 0 \qquad (3.39)$$

where

$$\left.\begin{aligned}&\int_{dA} t'_{k\ell}x'_{0m}n'_k dA' = \lambda_{k\ell m}n_k dA, \quad \int_{d\tau}\rho' f'_\ell x'_{0m} d\tau' = \rho\ell_{\ell m} d\tau \\ &\int_{d\tau}\rho' a'_\ell x'_{0m} d\tau' = \rho\dot{\sigma}_{\ell m} d\tau, \quad \int_{d\tau} t'_{\ell m} d\tau' = s_{\ell m} d\tau\end{aligned}\right\} . \tag{3.40}$$

The first term in (3.39) is the moment of surface tractions on the microsurfaces about its centroid. Thus $\lambda_{k\ell m}n_k dA$ are called the "first stress moments". Similarly the second and third terms in (3.40) motivate calling $\ell_{\ell m}$ the "first body moment" and $\dot{\sigma}_{\ell m}$ the "inertial spin". The last term, $s_{m\ell}$, which is symmetric, is the "microstress average".

Using (3.31), we see that

$$\begin{aligned}\rho\dot{\sigma}_{\ell m} d\tau &= \int_{d\tau}\rho' x'_{0m}\left[a_\ell + (\dot{\nu}_{n\ell} + \nu_{k\ell}\nu_{nk})x'_{0n}\right]d\tau' \\ &= \rho i_{km}(\dot{\nu}_{k\ell} + \nu_{n\ell}\nu_{kn})d\tau\end{aligned} \tag{3.41}$$

where

$$\rho i_{km} d\tau = \int \rho' x'_{0k}x'_{0m} d\tau' . \tag{3.42}$$

Hence

$$\dot{\sigma}_{\ell m} = i_{km}(\dot{\nu}_{k\ell} + \nu_{n\ell}\nu_{kn}) . \tag{3.43}$$

Alternately, if we let

$$i_{km} = I_{LM}\chi_{Lk}\chi_{Mm}, \tag{3.44}$$

it can be checked that

$$\dot{\sigma}_{\ell m} = I_{LM}\ddot{x}_{L\ell}x_{Mm} \ . \tag{3.45}$$

Physically, I_{LM} resembles the "moment of inertia" tensor of the macroelement with respect to its centre of mass $\underline{X}$. i_{km} will be called the "microinertia moments" and by taking the material derivative of (3.44) it can be shown that the microinertia moments satisfy

$$\frac{\partial}{\partial t}(i_{km}) + i_{km,\ell}\nu_\ell - i_{\ell m}\nu_{\ell k} - i_{k\ell}\nu_{\ell m} = 0. \tag{3.46}$$

To complete the mechanical description of the microstructured fluids we need the energy balance equation. Without going into the details (see Ramkissoon (1975) for details) we summarize the energy balance equation:

$$\rho\dot{\epsilon} = t_{k\ell}\nu_{\ell,k} + \nu_{m\ell}(s_{m\ell}-t_{m\ell}) + \lambda_{k\ell m}\nu_{m\ell,k} + q_{k,k} + \rho h. \tag{3.47}$$

Here ϵ is the internal energy and h the heat source per unit mass of the macroelement and $\underline{q}$ is the heat flux vector.

Equations (3.38), (3.39) and (3.47) along with the mass conservation equation

$$\frac{\partial\rho}{\partial t} + (\rho v_k)_{,k} = 0 \tag{3.48}$$

provide the mechanical laws of motion for the fluid. Equations (3.48) and (3.38) are known from classical theory, (3.47) is an extension and

(3.39) is new, as it has no counterpart in classical continuum theory.

3.5 Micropolar Fluids

Till now we have considered the description of the rate of deformation and stress and the general laws of motion. Eringen (1966) has given the constitutive equations for a class of microstructured fluids called micropolar fluids. Our subject of study is this class of micropolar fluids. We now briefly reproduce the results of the linear constitutive theory of non-heat conducting micropolar fluids with isotropic structure (i.e., $i_{km} = i\delta_{km}$).

The micropolar fluids are assumed to have constitutive equations of the form

$$\begin{aligned}\underline{t} &= [-\pi + \lambda\ \mathrm{tr}(\underline{d}) + \lambda_0\ \mathrm{tr}(\underline{b}-\underline{d})]I \\ &\quad + 2\mu_0\ \underline{d} + 2\mu_1\ (\underline{b}-\underline{d}) + 2\mu_2\ (\underline{b}^t-\underline{d}),\end{aligned} \tag{3.49}$$

$$\begin{aligned}\underline{s} &= [-\pi + \lambda\ \mathrm{tr}(\underline{d}) + \eta_0\ \mathrm{tr}(\underline{b}-\underline{d})]I \\ &\quad + 2\mu_0\ \underline{d} + \mu_3\ (\underline{b}-\underline{b}^t-2\underline{d}),\end{aligned} \tag{3.50}$$

$$\begin{aligned}\lambda_{k\ell m} &= (\gamma_1 a_{mnn} + \gamma_2 a_{nmn} + \gamma_3 a_{nnm})\delta_{k\ell} \\ &\quad + (\gamma_4 a_{\ell nn} + \gamma_5 a_{n\ell n} + \gamma_6 a_{nn\ell})\delta_{km} \\ &\quad + (\gamma_7 a_{knn} + \gamma_8 a_{nkn} + \gamma_9 a_{nnk})\delta_{\ell m} \\ &\quad + (\gamma_{10} a_{k\ell m} + \gamma_{11} a_{km\ell} + \gamma_{12} a_{\ell km} \\ &\qquad + \gamma_{13} a_{mk\ell} + \gamma_{14} a_{\ell mk} + \gamma_{15} a_{m\ell k})\end{aligned} \tag{3.51}$$

and

$$\lambda_{k\ell m} = -\lambda_{km\ell},\ \nu_{k\ell} = -\nu_{\ell k}. \tag{3.52}$$

Here I is the unit tensor and tr denotes trace, and

$$b_{k\ell} = \nu_{k\ell} + v_{k,\ell}, \; d_{k\ell} = \frac{1}{2}(v_{k,\ell}+v_{\ell,k}), \; a_{k\ell m} \equiv \nu_{k\ell,m}, \qquad (3.53)$$

$(\lambda,\lambda_0,\mu_0,\mu_1,\mu_2,\mu_3,\eta_0,\gamma_1,\cdots,\gamma_{15})$ are viscosity coefficients and π is the thermodynamic pressure

$$\pi = - \left.\frac{\partial\epsilon}{\partial\rho^{-1}}\right|_{H,i} , \qquad (3.54)$$

H being the entropy per unit mass.

One can introduce a microrotation vector $\underline{\nu}$ and a couple stress tensor m whose components are defined as:

$$\nu_n = \frac{1}{2}\epsilon_{nk\ell}\nu_{k\ell}, \quad m_{kn} = -\epsilon_{n\ell m}\lambda_{k\ell m} \qquad (3.55)$$

where ϵ_{ijk} is the alternating tensor.
Similarly one can have

$$\dot{\sigma}_n = -\epsilon_{nk\ell}\dot{\sigma}_{k\ell}, \quad \dot{\sigma}_{k\ell} = -2\epsilon_{nk\ell}\dot{\sigma}_n \qquad (3.56)$$

$$\ell_n = -\epsilon_{nk\ell}\ell_{k\ell}, \quad \ell_{k\ell} = -2\epsilon_{nk\ell}\ell_n . \qquad (3.57)$$

Assuming that $i_{km} = i\delta_{km}$ (microisotropy) and using the fact that $\nu_{k\ell}$ is skew-symmetric, it follows from (3.46) that

$$\frac{Di}{Dt} = 0. \tag{3.58}$$

Thus i is constant along material lines, which we shall take as j/2. The kinematic relation (3.43) reduces to

$$\dot{\sigma}_k = j\dot{\nu}_k \; . \tag{3.59}$$

It follows (see Ramkissoon (1975) or Eringen (1966) for more details) that micropolar fluids are characterised by the constitutive equations:

$$t_{k\ell} = (-\pi + \lambda d_{mm})\delta_{k\ell} + (2\mu+\kappa)d_{k\ell} + \kappa\epsilon_{k\ell m}(\omega_m - \nu_m) \tag{3.60}$$

$$m_{k\ell} = \alpha_1 \nu_{m,m}\delta_{k\ell} + \beta\, \nu_{k,\ell} + \gamma\, \nu_{\ell,k} \; . \tag{3.61}$$

Here

$$\omega_m = \frac{1}{2}\,(\underline{\nabla} \times \underline{v})_m$$

is the vorticity vector and the coefficients, λ, μ, κ, α, β, γ are combinations of the viscosity coefficients introduced in (3.49) and (3.51). The equations of motion then become:

Mass Conservation

$$\frac{\partial\rho}{\partial t} + (\rho v_k)_{,k} = 0 \tag{3.62}$$

Momentum Balance

$$t_{\ell k,\ell} + \rho(f_k - \dot{v}_k) = 0 \qquad (3.63)$$

Balance of first stress moments

$$m_{nk,n} + \epsilon_{k\ell n} t_{\ell n} + \rho(\ell_k - \dot{\sigma}_k) = 0 \qquad (3.64)$$

Energy Equation

$$\rho\dot{\epsilon} = t_{k\ell}(v_{\ell,k} - \epsilon_{k\ell n}\nu_n) + m_{k\ell}\nu_{\ell,k} + q_{k,k} + \rho h. \qquad (3.65)$$

There are thermodynamic restrictions on the viscosity coefficients α, β, γ, λ, μ, κ. The second law of thermodynamics, which states that the rate of change of the total entropy is never less than the entropy influx through the surface of the body and the entropy production within the body, results in the Clausius-Duhem inequality

$$\rho\dot{H} - \left[\frac{q_k}{T}\right]_{,k} - \frac{\rho h}{T} \geq 0 \qquad (3.66)$$

where H is the entropy permit mass and T is the temperature. Application of this principle in micropolar fluid theory results in the following restrictions on the viscosity coefficients:

$$\left.\begin{aligned} &(3\lambda+2\mu+\kappa) \geq 0, \ (2\mu+\kappa) \geq 0, \ \kappa \geq 0 \\ &(3\alpha+\beta+\gamma) \leq 0, \ -\gamma \leq \beta \leq \gamma, \ \gamma \geq 0 \ . \end{aligned}\right\} \qquad (3.67)$$

3.6 <u>Field Equations</u>

The field equations for micropolar flow are obtained by inserting the constitutive laws (3.60) and (3.61) into the equations of motion. In vector notation these take the form:

Continuity Equation

$$\frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot\rho\underline{v} = 0 \tag{3.68}$$

Balance of Momentum

$$(\lambda+2\mu+\kappa)\underline{\nabla}\ \underline{\nabla}\cdot\underline{v}-(\mu+\kappa)\underline{\nabla}\times\underline{\nabla}\times\underline{v} + \kappa\ \underline{\nabla}\times\underline{\nu}-\underline{\nabla}\pi+\rho f = \rho\dot{\underline{v}} \tag{3.69}$$

Balance of First Stress Moments

$$(\alpha+\beta+\gamma)\underline{\nabla}\ \underline{\nabla}\cdot\underline{\nu}-\gamma\underline{\nabla}\times\underline{\nabla}\times\underline{\nu} + \kappa\underline{\nabla}\times\underline{v}-2\kappa\underline{\nu}+\rho\underline{\ell} = \rho j\dot{\underline{\nu}} \tag{3.70}$$

In the case of incompressible flow, (3.68) becomes

$$\underline{\nabla}\cdot\underline{v} = 0, \tag{3.71}$$

so that (3.69) takes the form

$$-(\mu+\kappa)\underline{\nabla}\times\underline{\nabla}\times\underline{v} + \kappa\underline{\nabla}\times\underline{\nu}-\underline{\nabla}p+\rho\underline{f} = \rho\dot{\underline{v}}\ . \tag{3.72}$$

If in addition the flow is steady and slow, then the equations (3.69) and (3.70) reduce to

$$-(\mu+\kappa)\underline{\nabla}\times\underline{\nabla}\times\underline{v} + \kappa\underline{\nabla}\times\underline{\nu}-\underline{\nabla}p+\rho\underline{f} = 0 \tag{3.73}$$

$$(\alpha+\beta+\gamma)\underline{\nabla}\ \underline{\nabla}\cdot\underline{\nu}-\gamma\underline{\nabla}\times\underline{\nabla}\times\underline{\nu} + \kappa\underline{\nabla}\times\underline{v}-2\kappa\underline{\nu}+\rho\underline{\ell} = 0. \tag{3.74}$$

It can be seen that the constant κ links the velocity and microrotation and is often termed the coupling constant, since its vanishing uncouples the differential equations. In this case the global motion is unaffected by microrotations.

3.7 Boundary Conditions

The general field equations given in the previous section represents seven scalar equations for seven unknown field parameters ρ, v_i and ν_i. Under appropriate boundary and initial conditions these differential equations should be capable of predicting the behaviour of micropolar fluids. As initial conditions we can prescribe the unknowns throughout the fluid at $t = 0$. As regards velocity, we could still insist on the no-slip condition on the boundary as in the classical situations.

There doesn't seem to be a universal agreement on the right boundary conditions for microrotation. Eringen (1966) suggested

$$\underline{\nu}(\underline{x}_B,t) = \underline{\nu}_B, \tag{3.75}$$

where $\underline{x}_B$ is a point on the boundary with prescribed microrotation $\underline{\nu}_B$. This condition, together with the no-slip condition is based on the assumption that the fluid adheres to the solid boundary, and has been used by most authors. The main criticism for this assumption is that microrotation and velocity are kinematically distinct, and thus the validity of the assumption is in doubt. Another boundary condition sometimes used is the Cauchy boundary condition where the microrotation has the same value as the fluid vorticity at the solid boundary. Turk et al. (1973), in their work on pulsatile blood flow, have applied yet another boundary condition, that the microrotation be constant on the boundary but requiring that microrotation gradients vanish there. But the problem of what is the best spin boundary condition for fluids with a microelement structure remains unsolved. In Chapter 6, in which we examine the flow of a micropolar fluid in a meandering channel, we will investigate the effect of two types of boundary conditions: (a) no-spin and (b) microrotation equals fluid vorticity.

CHAPTER IV

TWO UNIQUENESS THEOREMS FOR MICROPOLAR FLUIDS *

4.1 Introduction

In this chapter we establish two uniqueness theorems for the set of partial differential equations governing the motion of a micropolar fluid. First we prove that the motion of such a fluid within a bounded region of space, which could be time dependent, is unique. This is done under some general assumptions of initial and boundary conditions and the boundedness of the field variables and their derivatives. Next we prove a uniqueness theorem for the flow caused by a finite solid body situated within a fluid extending to infinity in space. Here again certain general boundary and initial conditions are assumed, together with certain behaviour of solutions far from the body.

The theorems we prove apply to the complete set of nonlinear partial differential equations given in the previous chapter. The only uniqueness theorems proved so far concerns steady,linearised flow as given by Ramkissoon (1984) and Cowin (1972).

---

* Contents of this chapter have been accepted for publication in *Acta Mechanica*. Also presented at the Tenth U.S. National Congress of Applied Mechanics held in Austin, Texas, June 1986.

It is natural to enquire about the existence of solutions of the equations of motion. Predictably, no existence results are available for the nonlinear equations of micropolar flow. This is not surprising, since even the classical Navier - Stokes equations lack existence theorems. For the linearised equations, however, existence results have been proved, under various boundary and initial conditions by Ramkissoon (1984).

4.2 Uniqueness of flow in bounded regions

Consider the motion of a compressible non-heat conducting micropolar fluid occupying a finite region $\mathcal{V} = \mathcal{V}(t)$ with sufficiently smooth boundary $\mathcal{B}$. We recall the constitutive equations for the fluid from Chapter 3:

$$t_{ij} = (-p+\lambda v_{k,k})\delta_{ij} + (2\mu+\kappa)d_{ij} + \epsilon_{ijk}\kappa(\omega_k-\nu_k) \tag{4.1}$$
$$m_{ij} = \alpha\nu_{k,k}\delta_{ij} + \beta\nu_{i,j} + \gamma\nu_{j,i} \,. \tag{4.2}$$

Here $\underline{v}$ is the velocity, $\underline{\nu}$ the microrotation, p the pressure and

$$d_{ij} = \frac{1}{2}(v_{i,j}+v_{j,i}) \tag{4.3}$$
$$\omega_i = \epsilon_{ijk}v_{j,k}. \tag{4.4}$$

Note that the viscosities satisfy $\mu \geq 0$, $\kappa \geq 0$, $3\lambda+2\mu+\kappa \geq 0$, $3\alpha+\beta+\gamma \geq 0$. The equations of motion are

$$\dot{\rho} + \rho v_{i,i} = 0 \tag{4.5}$$
$$t_{ji,j} + \rho f_i = \dot{\rho}_i \tag{4.6}$$
$$m_{ji,j} + \epsilon_{ijk}t_{jk} + \rho\ell_i = \rho j\dot{\nu}_i \,. \tag{4.7}$$

Here, as usual, $\underline{f}$ is the body force and $\underline{\ell}$ the body couple and a superposed dot indicates material derivatives. For a compressible fluid equations (4.5)-(4.7) have to be supplemented by the energy balance equation

$$\rho\dot{E} = t_{ij}d_{ij} + m_{ij}\nu_{j,i} + \epsilon_{ijk}(\omega_k - \nu_k)t_{ij} \tag{4.8}$$

where E is the specific internal energy.

It is convenient to recast equations (4.5)-(4.8) in an alternative form for the ensuing treatment. We thus define the second order tensors T, M, D, N, G and V as follows:

$$\begin{aligned} T_{ij} &= t_{ij}, & M_{ij} &= m_{ij}, \\ D_{ij} &= d_{ij}, & N_{ij} &= \epsilon_{ijk}(\omega_k - \nu_k), \\ G_{ij} &= \nu_{j,i}, & V_{ij} &= \lambda v_{k,k}\delta_{ij} + (2\mu+\kappa)d_{ij} + \kappa N_{ij}. \end{aligned}$$

Then the equations (4.5)-(4.8) may be rewritten in the form

$$\dot{\rho} + \rho \operatorname{div} \underline{v} = 0 \tag{4.9}$$

$$\rho\dot{\underline{v}} = \rho\underline{f} + \operatorname{div} T \tag{4.10}$$

$$\rho j\dot{\underline{\nu}} = \rho\underline{\ell} + \operatorname{div} M + 2\kappa(\underline{\omega} - \underline{\nu}) \tag{4.11}$$

$$\rho\dot{E} = T{:}D + M{:}G + T{:}N \tag{4.12}$$

where we have used the conventional notations

$$A{:}B = A_{ij}B_{ij}, \quad (\operatorname{div} A)_i = A_{ji,j}.$$

We assume the fluid is non-heat conducting and has equations of state

$$p = p(\rho,\Phi) \tag{4.13}$$

$$E = C_v \Phi \tag{4.14}$$

where Φ is the temperature and C_v is the specific heat at constant volume. C_v is taken to be a positive constant. The function p is assumed to be sufficiently differentiable.

Our aim is to determine if, given the initial velocity, microrotation, temperature and density distributions and prescribed boundary conditions at all times, they uniquely determine subsequent motion. The uniqueness of the corresponding steady, slow, incompressible flow under various boundary conditions has been established by Ramkissoon (1984). We show here the uniqueness of the flow governed by the complete set of equations of motion (4.9)-(4.12) and the boundary conditions prescribed below.

Let $\underline{n}$ denote the outward normal to $\mathscr{B}$ and G the outward normal velocity of $\mathscr{B}$. Then $U = \underline{v}\cdot\underline{n}-G$ is the relative normal velocity of particles on the boundary. Suppose that

a) At all points on the boundary $\mathscr{B}$ and at all times $\underline{v}$ is prescribed;

b) the initial values of the flow variables are prescribed at all points on the closure of $\mathscr{V}$;

c) at all points where $U < 0$ the temperature and density are prescribed.

Note that the condition (c) implies that at points where fluid is entering the region $\mathscr{V}$, the thermodynamic properties are prescribed.

By a solution of the initial value problem described by (4.9)-(4.12) and the boundary-initial conditions (a)-(c) we mean a set of continuously differentiable functions satisfying (4.9)-(4.12) and taking on the values prescribed in (a)-(c) in the closure of $\mathscr{V}$.

Our main conclusion is:

Theorem. Let the viscosities satisfy the inequalities $3\lambda+2\mu+\kappa > 0$, $2\mu+\kappa > 0$. Then there can be at most one solution of the above initial-boundary value problem.

Proof. Without loss of generality let us put $C_v = 1$. Let $(\rho,\underline{v},\underline{\nu},\phi)$ and $(\tilde{\rho},\tilde{\underline{v}},\tilde{\underline{\nu}},\tilde{\phi})$ be two solutions of the initial value problem. A tilde over a flow quantity shall be understood to mean a quantity evaluated for the second flow. Also let us denote $F' = \tilde{F}-F$ for any flow quantity F. For example $\underline{v}' = \tilde{\underline{v}}-\underline{v}$. The proof consists of showing that $v' = \nu' = \rho' = \phi' = 0$.

We first derive some identities which will be used frequently. If $\frac{d}{dt}$ denotes material derivative and if we define

$$\frac{dF'}{dt} = \frac{\partial F'}{\partial t} + (\underline{v} \cdot \mathrm{grad})F' \tag{4.15}$$

then it follows immediately that

$$\frac{d\tilde{F}}{dt} - \frac{dF}{dt} = \frac{dF'}{dt} + (\underline{v}' \cdot \mathrm{grad})\tilde{F} \tag{4.16}$$

for any flow quantity F. It is also easy to show by simple algebra that

$$\tilde{A}\tilde{B} - AB = AB' - A'\tilde{B} \ . \tag{4.17}$$

Next we derive the transport equation

$$\frac{d}{dt} \int_{\mathcal{V}} \rho F \, d\mathcal{V} = \int_{\mathcal{V}} \rho \frac{dF}{dt} \, d\mathcal{V} - \oint_{\mathcal{B}} \rho U F \, d\mathcal{B}. \tag{4.18}$$

To show this we start with the obvious identity

$$\frac{d}{dt} \int_{\mathcal{V}} \rho F \, d\mathcal{V} = \int_{\mathcal{V}} \frac{\partial}{\partial t} (\rho F) \, d\mathcal{V} + \oint_{\mathcal{B}} \rho F G \, d\mathcal{B}. \tag{4.19}$$

This says that the rate of change of $\int \rho F$ consists of two parts: the internal changes of ρF given by the first term on the right and changes due to motion of the boundary $\mathcal{B}$ which is moving with a normal velocity G.

Rewriting the right side of (4.19) as

$$\int_{\mathcal{V}} \frac{\partial}{\partial t} (\rho F) d\mathcal{V} + \oint_{\mathcal{B}} (\underline{v} \cdot \underline{n}) \rho F \, d\mathcal{B} + \oint_{\mathcal{B}} \rho F (G - \underline{v} \cdot \underline{n}) d\mathcal{B}$$

and using the divergence theorem,

$$\frac{d}{dt} \int_{\mathcal{V}} \rho F \, d\mathcal{V} = \int_{\mathcal{V}} \left[\frac{\partial}{\partial t} (\rho F) + \operatorname{div}(\rho F \, \underline{v})\right] d\mathcal{V} + \oint_{\mathcal{B}} \rho F (G - \underline{v} \cdot \underline{n}) d\mathcal{B}. \tag{4.20}$$

But

$$\frac{\partial}{\partial t}(\rho F) + \text{div}(\rho F\, \underline{v}) = \rho\left[\frac{\partial F}{\partial t} + (\underline{v}\cdot\text{grad})F\right] + F\left[\frac{\partial \rho}{\partial t} + \text{div}(\rho\, \underline{v})\right]$$

$$= \rho\,\frac{dF}{dt}\,, \tag{4.21}$$

using the equation of continuity and the definition of the material derivative. This proves (4.18), since $U = \underline{v}\cdot\underline{n} - G$ by definition.

In the following we shall agree to omit surface and volume infinitesimals in all expressions, since the context makes clear the nature of the integrals.

Since both the postulated flows satisfy (4.9), we get

$$\dot{\rho} + \rho\ \text{div}\ \underline{v} = 0 \tag{4.22}$$

$$\dot{\tilde{\rho}} + \tilde{\rho}\ \text{div}\ \underline{\tilde{v}} = 0\ . \tag{4.23}$$

Subtracting (4.23) from (4.22) and using the identities (4.16) and (4.17) we get

$$\frac{d\rho'}{dt} + \underline{v}'\cdot\text{grad}\ \tilde{\rho} + \rho\ \text{div}\ \underline{v}' + \rho'\ \text{div}\ \underline{\tilde{v}} = 0. \tag{4.24}$$

Multiplying this by ρ',

$$\frac{d}{dt}\left[\frac{1}{2}\rho'^2\right] = -\left\{\rho'\underline{v}'\cdot\text{grad }\tilde{\rho} + \rho\rho'\text{ div }\underline{v}' + \rho'^2\text{div }\tilde{\underline{v}}\right\}. \qquad (4.25)$$

Using the transport equation (4.18) in (4.25) we get

$$\frac{d}{dt}\int\frac{1}{2}\rho\rho'^2 = -\int\rho\left\{\rho'\underline{v}'\cdot\text{grad }\tilde{\rho} + \rho\rho'\text{ div }\underline{v}' + \rho'^2\text{div }\tilde{\underline{v}}\right\} - \oint\frac{1}{2}\rho U\rho'^2. \qquad (4.26)$$

Again since both flows satisfy (4.10), we have, by subtraction, scalar multiplication by $\underline{v}'$ and rearranging,

$$\rho\frac{d}{dt}\left[\frac{1}{2}v'^2\right] = -\left\{\rho'\underline{v}'\cdot(\tilde{\underline{a}}-\underline{f})+\rho\underline{v}'\cdot\tilde{D}\cdot\underline{v}'\right\} + \underline{v}'\cdot\text{div T} \qquad (4.27)$$

where $v'^2 = \underline{v}'\cdot\underline{v}'$ and $\tilde{\underline{a}} = \dfrac{d\tilde{\underline{v}}}{dt}$.

Let us now separate T into a symmetric and an antisymmetric part. Define

$$T = S+A \qquad (4.28)$$

where

$$S_{ij} = (-p+\lambda v_{k,k})\delta_{ij} + (2\mu+\kappa)d_{ij} \qquad (4.29)$$

and

$$A_{ij} = \epsilon_{ijk}\kappa(\omega_k-\nu_k). \qquad (4.30)$$

Then (4.27) can be rewritten as

$$\rho\frac{d}{dt}\left(\frac{1}{2}v'^2\right) = -\left\{\rho'\underline{v}'\cdot(\tilde{\underline{a}}-\underline{f})+\rho\underline{v}'\cdot\tilde{D}\cdot\underline{v}'+S':V'-\underline{v}'\cdot\text{div }A'\right\}$$

$$+\mathrm{div}(\underline{v}'\cdot S'). \tag{4.31}$$

Using the transport equation (4.18) and the fact that $\underline{v}' = 0$ on $\mathscr{B}$ we get

$$\frac{d}{dt}\int (\tfrac{1}{2}\,\rho v'^2) = -\int\left\{\rho'\underline{v}'\cdot(\underline{\tilde{a}}-\underline{f})+\rho\underline{v}'\cdot\tilde{D}\cdot\underline{v}'+S':V'-\underline{v}'\cdot\mathrm{div}\;A'\right\}. \tag{4.32}$$

The above procedure of subtraction, scalar multiplication and use of transport equation is now repeated in precisely the same manner with the remaining two equations of motion (4.11) and (4.12). The results are:

$$\frac{d}{dt}\int(\rho j\nu'^2) = -\int\left\{\rho'\underline{\nu}'\cdot(j\underline{\tilde{b}}-\underline{\ell})+\rho j\underline{\nu}'\cdot\tilde{G}\cdot\underline{v}'-2\kappa\underline{\omega}'\cdot\underline{\nu}'+2\kappa\nu'^2-\underline{\nu}'\cdot\mathrm{div}\;M'\right\} \tag{4.33}$$

where $\nu'^2 = \underline{\nu}'\cdot\underline{\nu}'$ and $\underline{\tilde{b}} = \dfrac{d\underline{\tilde{\nu}}}{dt}$, and

$$\frac{d}{dt}\int \frac{1}{2}\,\rho\phi'^2 = \int\phi'\Big\{T:D'+T':\tilde{N})+(-p+\lambda\theta')\tilde{\theta}+\kappa N':\tilde{D}$$

$$+\;((2\mu+\kappa)D':\tilde{D}+T:N')+M:G'+M':\tilde{G}\Big\} \tag{4.34}$$

$$+\int\left\{-\rho'\phi'\,\frac{d\tilde{\phi}}{dt}-\rho\phi'\underline{v}'\cdot\mathrm{grad}\;\tilde{\phi}\right\} - \oint\frac{1}{2}\,\rho U\phi'^2$$

where $\theta = \mathrm{div}\;\underline{v} = I:D$ (I being identity).

Our next task is to estimate the various quantities occurring on the right sides of equations (4.26), (4.32), (4.33) and (4.34). For this purpose we restrict our attention to a fixed time interval $0 \le t \le \tau$, where τ is arbitrary but fixed. If the flow is known to exist for time t_0, then $\tau \le t_0$. In what follows let P denote an upper bound; P will be different at each estimate but it will be possible to determine its size at each stage. We use this convention to avoid introducing too many symbols at the estimation stage. Let ϵ be an arbitrary positive number to be fixed later. Using Cauchy's inequality $2ab \le a^2+b^2$ we have

$$|\rho\rho'\underline{v}'\cdot\mathrm{grad}\tilde{\rho}+\rho^2\rho'\mathrm{div}\ \underline{v}'+\rho\rho'^2\mathrm{div}\ \tilde{\underline{v}}| \le P(\rho'^2+v'^2) + \epsilon\theta'^2. \quad (4.35)$$

Here P depends on ϵ and the bounds for the magnitudes of $(\rho,\underline{v},\underline{\nu},\phi)$ and $(\tilde{\rho},\tilde{\underline{v}},\tilde{\underline{\nu}},\tilde{\phi})$ and their derivatives during $0 \le t \le \tau$.

It is convenient to introduce the following notation:

$$I = \int \frac{1}{2}\rho\rho'^2$$
$$J = \int \frac{1}{2}\rho v'^2$$
$$K = \int \frac{1}{2}\rho j\nu'^2$$
$$L = \int \frac{1}{2}\rho\phi'^2 .$$

Using (4.35) and the fact that $\rho \cup \rho'^2 \ge 0$ because of the boundary conditions, we get from (4.26),

$$\frac{dI}{dt} \le \int P(\rho'^2+v'^2) + \int \epsilon\theta'^2, \quad 0 \le t \le \tau. \quad (4.36)$$

Turning to (4.32), we observe that

$$S':V' = -p'\theta'+V':D'. \tag{4.37}$$

Then it is easily seen that

$$\frac{dJ}{dt} \leq \int P(\rho'^2+v'^2+p'^2) + \int[\epsilon(\theta'^2+(\text{div } A')^2)-V':D'], \quad 0 \leq t \leq \tau$$
(4.38)

where $(\text{div}A)^2 = (A_{ij,i})(A_{kj,k})$.

In a similar fashion we obtain from (4.33),

$$\frac{dK}{dt} \leq \int P(\rho'^2+v'^2+\nu'^2) + \int [\epsilon|\nu'\omega'|(\text{div } M')^2-2\kappa\nu'^2], \ 0 \leq t \leq \tau. \tag{4.39}$$

Again from (4.34), noting that $\rho U\rho' \geq 0$ on the boundary because of the condition (c), we get

$$\frac{dL}{dt} \leq \int P(\rho'^2 + v'^2 + p'^2 + \gamma'^2) + \int \epsilon(\theta'^2 + D':D' + G':G' + M':M') \tag{4.40}$$

Adding inequalities (4.36),(4.38),(4.39) and (4.40),we obtain

$$\frac{d}{dt}(I + J + K + L) \leq P(I + J + K + L) + \int [\ \epsilon\ (3\ \theta'^2 + D':D') - V':D'\ ]$$

$$\int [\ \epsilon\ (\ |\nu'\omega'|(\text{div } M')^2 + G':G' + M':M' + (\text{div } A)^2\ ) - 2\ \kappa\ \nu'^2\ ] \tag{4.41}$$

For a symmetric 3x3 matrix D one can show that D:D, which is the sum of squares of all the nine elements in the matrix, satisfies the relation

$$3\text{D: } D = \theta^2 + \Delta^2, \tag{4.42}$$

where $$\Delta^2 = (d_1-d_2)^2+(d_2-d_3)^2+(d_3-d_1)^2, \tag{4.43}$$

the d_i's being the eigenvalues of D. This is most easily proved by observing that D:D is invariant under a similarity transform of D. For, let $(P)_{ij} = p_{ij}$ and $(P^{-1})_{ij} = q_{ij}$; then

$$\begin{aligned}
(P^{-1}DP):(P^{-1}DP) &= (P^{-1}DP)_{ij}(P^{-1}DP)_{ji} \\
&= (q_{ik}d_{k\ell}p_{\ell j})(q_{jk'}d_{k'\ell'}p_{\ell' i}) \\
&= q_{ik}d_{k\ell}\delta_{\ell k'}d_{k'\ell'}p_{\ell' i} \\
&= q_{ik}d_{k\ell}d_{k\ell'}p_{\ell' i} \\
&= \delta_{k\ell'}d_{k\ell}d_{k\ell'} = d_{k\ell}d_{k\ell} \\
&= d_{k\ell}d_{\ell k} = D:D \ .
\end{aligned}$$

The assertion (4.42) now easily follows by a simple computation on a diagonalised matrix.

Using (4.42), we have that

$$\begin{aligned}
V':D' &= \lambda\theta'^2+(2\mu+\kappa)D':D' \\
&= \frac{1}{3}\left\{(3\lambda+2\mu+\kappa)\theta'^2+(2\mu+\kappa)\Delta'^2\right\} \ .
\end{aligned} \tag{4.44}$$

In addition, we also have

$$\epsilon(3\theta'^2+D':D') = \frac{\epsilon}{3}(10\theta'^2+\Delta'^2). \tag{4.45}$$

Thus

$$\begin{aligned}\epsilon(3\theta'^2+D':D')-V':D' &= \frac{\epsilon}{3}(10\theta'^2+\Delta'^2) - \frac{1}{3}((3\lambda+2\mu+\kappa)\theta'^2+(2\mu+\kappa)\Delta'^2)\\ &\leq 0,\end{aligned} \tag{4.46}$$

if ϵ is chosen sufficiently small. Note that the conditions on viscosities stated in the theorem were used to make this conclusion.

Once again by choosing ϵ sufficiently small we have,

$$\epsilon\left[|\nu'\omega'|(\text{div } M')^2+(\text{div } A')^2+G':G'+M'M'\right] - 2\kappa\nu'^2 \leq 0. \tag{4.47}$$

Combining the observations (4.46) and (4.47) we finally obtain from (4.41),

$$\frac{d}{dt}(I+J+K+L) \leq P(I+J+K+L), \quad 0 \leq t \leq \tau \tag{4.48}$$

On integration of this inequality we get

$$I+J+K+L \leq (I+J+K+L)\Big|_{t=0} e^{Pt}, \quad 0 \leq t \leq \tau. \tag{4.49}$$

Since I, J, K, L are zero initially, they remain zero throughout $0 \leq t \leq \tau$. But then $v' = \nu' = \rho' = \phi' = 0$ and the two postulated flows are identical till $t = \tau$. It follows that the two flows are identical as long as they exist and the theorem is proved.

As a simple corollary one can state that under the conditions of the theorem the flow of an incompressible micropolar fluid is unique. For, in this case $\tilde{\rho} = \rho$ and hence $\rho' \equiv 0$ and the result follows immediately from the theorem.

One can consider other combinations of restrictions on the viscosities and the proof carries through without difficulty. An important exception occurs when $3\lambda+2\mu+\kappa = 0$ and $2\mu+\kappa = 0$ (which makes $\lambda = 0$). When this happens, the crucial step (4.46) no longer holds, and the theorem cannot be proved.

4.3 Uniqueness of flow past a solid body

In this section we prove a uniqueness theorem for an incompressible micropolar fluid in the presence of a solid body. Specifically, given a finite solid body within a mass of incompressible micropolar fluid that extends to infinity, we seek to determine if the motion of the fluid is uniquely determined by the motion of the solid body. An affirmative result is proved subject to certain smoothness and boundedness conditions on the velocity, microrotation and their derivatives together with a certain convergence condition on pressure at large distances from the solid body.

We recall the equations of motion of an incompressible micropolar fluid in component form

$$v_{i,i} = 0 \tag{4.50}$$

$$(\mu+\kappa)v_{i,jj}+\kappa\epsilon_{ijk}\nu_{j,k}-p_{,i}+\rho f_i = \rho\left[\frac{\partial v_i}{\partial t} + v_k v_{i,k}\right] \tag{4.51}$$

$$\gamma\,\nu_{i,jj}+(\alpha+\beta)\nu_{j,ij}-2\kappa\nu_i+\kappa\epsilon_{ijk}v_{j,k}+\rho\ell_i = \rho_j\left[\frac{\partial \nu_i}{\partial t} + v_k\nu_{i,k}\right]. \tag{4.52}$$

The symbols are explained in Chapter 3. A suffix following a comma indicates partial derivative.

Consider a solid body S with sufficiently smooth boundary ∂S immersed in a micropolar fluid of infinite extent. Let E denote the set of points of space exterior to S and T be a time interval $(0,t_0)$ where t_0 is arbitrary but fixed. To completely specify the problem we assume that

i) The flow variables $\underline{v}$, $\underline{\nu}$ are prescribed throughout $E \cup S$ at $t = 0$.

ii) The v_i and ν_i are prescribed on ∂S at all times $t \geq 0$.

iii) The f_i and ℓ_i are prescribed at all times and all points of space.

In addition, we require the following boundedness and continuity conditions:

iv) The velocity components v_i and their first partial derivatives with respect to space and time are continuous bounded functions of these variables in $E \times T$ and the second order spatial derivatives are continous in $E \times T$.

v) The microrotation components ν_i and their first partial derivatives with respect to space and time are bounded continuous functions and the second order space derivatives of ν_i are continuous in $(E \cup \partial S) \times T$.

vi) The pressure p is continuous and has continuous first order spatial derivatives in $E \times T$.

We further assume the following convergence condition at infinity:

Let $r^2 = x_i x_i$ (using summation convention). At infinity p converges to a constant p_0 such that for all t in T

$$p = p_0 + O(r^{-1/2-\epsilon}) \text{ as } r \to \infty, \ \epsilon \text{ being an arbitrary small positive constant.}$$

Our main result is

Theorem. There can be at most one solution of the equations (4.50)-(4.51) satisfying the conditions (i)-(vi) and the convergence conditions on pressure.

Proof: Assume that the conditions (i)-(vi) as well as the convergence condition on pressure holds. Let $\{v_i, \nu_i, p\}$ and $\{v_i+v_i', \nu_i+\nu_i', p+p'\}$ be two possible solutions of the problem. The proof will consist of showing that v_i', ν_i' and p' are identically zero.

Since both solutions satisfy the basic equations (4.50)-(4.52) we have, by subtraction,

$$v'_{i,i} = 0 \tag{4.53}$$

$$(\mu+\kappa)v'_{i,jj}+\kappa\epsilon_{ijk}\nu'_{j,k}-p'_{,i} = \rho v'_i\left[\frac{\partial v'_i}{\partial t} + v'_k(v'_i+v_i)_{,k} + v_k v'_{i,k}\right] \tag{4.54}$$

$$\gamma\nu'_{i,jj}+(\alpha+\beta)\nu'_{j,ij}-2\kappa\nu'_i+\kappa\epsilon_{ijk}v'_{j,k} = \rho j\left[\frac{\partial \nu'_i}{\partial t} + v'_k(\nu'_i+\nu_i)_{,k} + v_k\nu'_{i,k}\right]. \tag{4.55}$$

These are the equations governing the perturbation quantities v_i', ν_i' and p'.

Let B_r be a closed ball centred at the origin, of radius r and having surface C_r. Let us choose r large enough so that the solid body S remains within B_r for $t \in T$. Let $B_r' = B_r \cap E$.

On multiplying (4.54) by v_i' and integrating over B_r', using Green's theorem and the boundary conditions $v_i' = \nu_i' = 0$ on ∂S, we get

$$\int_{B_r'} -(\mu+\kappa)v_{i,j}'v_{i,j}'dB_r+(\mu+\kappa)\int_{C_r} v_i'v_{i,j}'n_j dC_r+\int_{B_r'} \kappa\epsilon_{ijk}v_i'\nu_{j,k}'dB_R$$

$$-\int_{C_r} v_i'p'n_i dC_R = \int_{B_r'} \frac{\rho}{2}\frac{\partial}{\partial t}(v_i'v_i')dB_R+\int_{B_r'} \rho v_i'v_k'(v_i+v_i')_{,k}dB_R+\int_{C_r} \frac{\rho}{2} v_i'v_i'v_k n_k dC_r. \tag{4.56}$$

In this expression n_i denotes an outward drawn unit normal on C_r. Similarly, multiplying (4.55) by ν_i' and integrating over B_r', one gets, using Green's theorem and the boundary conditions on S,

$$-\gamma\int_{B_r'} \nu_{i,j}'\nu_{i,j}'dB_r+\gamma\int_{C_r} \nu_{i,j}'\nu_i'n_j dC_r-(\alpha+\beta)\int_{B_r} \nu_{i,j}'\nu_{j,i}'dB_r$$

$$+(\alpha+\beta)\int_{C_r} \nu_i'\nu_{j,i}'n_i dC_r-2\kappa\int_{B_r'} \nu_i'\nu_i'dB_r-\int_{B_r'} \kappa\epsilon_{ijk}\nu_{i,k}'v_j'dB_r+\int_{C_r} \kappa\epsilon_{ijk}\nu_i'v_j'n_k dC_r$$

$$+ \gamma\int_{\partial S} \nu_{i,j}'\nu_i'n_j dS + (\alpha+\beta)\int_{\partial S} \nu_i'\nu_{j,i}'n_i dS$$

$$= \int_{B_r'} \frac{\rho j}{2}\frac{\partial}{\partial t}(\nu_i'\nu_i')dB_r+\int_{B_r'} \rho j\nu_i'v_k'(v_i+v_i')_{,k}dB_r+\int_{C_r} \frac{\rho j}{2} \nu_i'\nu_i'v_k n_k dC_r$$

$$+ \frac{\rho j}{2}\int_{\partial S} \nu_i'\nu_i'v_k n_k dS, \tag{4.57}$$

where dS denotes a surface element on ∂S.

Adding (4.56) and (4.57) and rearranging, we get

$$\int_{B_r'}\left[\frac{1}{2}\rho\frac{\partial}{\partial t}(v_i'v_i') + \frac{1}{2}\rho j\frac{\partial}{\partial t}(\nu_i'\nu_i')+(\mu+\kappa)v_{i,j}'v_{i,j}'+\gamma\nu_{i,j}'\nu_{i,j}'\right]dB_r =$$

$$-\int_{B_r'}\left[\rho v_i'v_k'(v_i+v_i')_{,k}+\rho j\nu_i'v_k'(\nu_i'+\nu_i)_{,k}+2\kappa\nu_i'\nu_i'+(\alpha+\beta)\nu_{i,j}'\nu_{j,i}'+2\kappa\epsilon_{ijk}\nu_{i,k}'v_j'\right]dB_r$$

$$+\int_{C_r}\left[(\mu+\kappa)v_i'v_{i,j}'n_j-\kappa\epsilon_{ijk}v_i'\nu_j'n_k-\frac{\rho}{2}v_i'v_i'v_kn_k-v_i'p'n_i+(\alpha+\beta)\nu_i'\nu_{j,i}'n_j+\right.$$

$$\left.+\gamma\,\nu_i'\nu_{i,j}'n_j-\frac{\rho j}{2}\nu_i'\nu_i'v_kn_k\right]dC_r$$

$$+\int_{\partial S}[\gamma\nu_{i,j}'\nu_i'+(\alpha+\beta)\nu_i'\nu_{j,i}'-\frac{\rho j}{2}\nu_i'\nu_i'v_j]n_j dS. \qquad (4.58)$$

Integrating (4.58) with respect to t from 0 to t_1 and then again with respect to t_1 from 0 to a, where a ($0<a\le t_0$) is to be prescribed later, it follows that

$$\int_0^a dt_1\left\{\int_{B_r'}\left[\frac{1}{2}\rho v_i'v_i'+\frac{1}{2}\rho j\nu_i'\nu_i'\right]dB_r\right\}+\int_0^a dt_1\left\{\int_0^{t_1}dt\int_{B_r'}\left[(\mu+\kappa)v_{i,j}'v_{i,j}'+\gamma\nu_{i,j}'\nu_{i,j}'\right]dB_r\right\}$$

$$=-\int_0^a dt_1\left\{\int_0^{t_1}dt\int_{B_r'}\left[\rho v_i'v_k'(v_i+v_i')_{,k}+\rho j\nu_i'v_k'(\nu_i+\nu_i')_{,k}+2\kappa\nu_i'\nu_i'+(\alpha+\beta)\nu_{i,j}'\nu_{j,i}'+\right.\right.$$

$$+ 2\kappa\epsilon_{ijk}\nu'_{i,k}v'_j\Big]dB_r\Bigg\}$$

$$+ \int_0^a dt_1\Bigg\{\int_0^{t_1} dt\int_{C_r}\Big[(\mu+\kappa)v'_iv'_{i,j}n_j - \kappa\epsilon_{ijk}v'_i\nu'_jn_k$$

$$- \frac{\rho}{2}v'_iv'_iv_kn_k - v'_ip'n_i + (\alpha+\beta)\nu'_i\nu'_{j,i}n_j +$$

$$+ \gamma\,\nu'_i\nu'_{i,j}n_j - \frac{\rho j}{2}\nu'_i\nu'_iv_kn_k\Big]dC_r\Bigg\}$$

$$+ \int_0^a dt_1\Bigg\{\int_0^{t_1}\int_{\partial S}[\gamma\nu'_{i,j}\nu'_j + (\alpha+\beta)\nu'_i\nu'_{j,i} - \frac{\rho j}{2}\nu'_i\nu'_iv_j]n_j dS\Bigg\}\ . \qquad (4.59)$$

Let us call the left side of (4.59), $P(r)$. Then because μ, κ, γ, ρ, j are all non negative it is easy to see that $P(r) \geq 0$.

Now we need estimates for the various quantities on the right of (4.59). Using Cauchy's inequality $2ab \leq a^2+b^2$ and conditions (iii) and (iv), we find that

$$|v'_iv'_k(v_i+v'_i)_{,k}| \leq N_1v'_iv'_i$$

$$|\nu'_iv'_k(\nu'_i+\nu_i)_{,k}| \leq N_2(v'_iv'_i+\nu'_i\nu'_i)$$

$$|v'_iv'_{i,j}n_j| \leq \frac{3}{2}v'_iv'_i + \frac{1}{2}v'_{i,j}v'_{i,j}$$

$$|v'_iv'_iv_kn_k| \leq N_3v'_iv'_i$$

$$|\nu'_i\nu'_{j,i}n_j| \leq \frac{3}{2}\nu'_i\nu'_i + \frac{1}{2}\nu'_{i,j}\nu'_{i,j}$$

$$|\nu'_i\nu'_{i,j}n_j| \leq \frac{3}{2}\nu'_i\nu'_i + \frac{1}{2}\nu'_{i,j}\nu'_{i,j}$$

$$|\nu'_i\nu'_iv_kn_k| \leq N_4\nu'_i\nu'_i$$

$$|\nu'_{j,i}\nu'_{i,j}| \le N_5\nu'_{i,j}\nu'_{i,j}$$

$$|\epsilon_{ijk}v'_j\nu'_{i,k}| \le N_6(v'_iv'_i+\nu'_{i,j}\nu'_{i,j})$$

where $N_1,\ldots,N_6$ are positive constants. For example $N_3 = \frac{1}{3}\sup|v_k|$, the supremum taken in $E \times T$, over all indices k.

Using the convergence condition on p, that $p \sim O(r^{-1/2-\epsilon})$ as $r \to \infty$, we get

$$\left|\iint_{C_r} p'v'_in_i dC_r\right| \le \left[\left[\iint_{C_r} p'^2 dC_r\right]\left[\iint_{C_r} v'_iv'_i dC_r\right]\right]^{1/2}$$

$$\le N_7 r^{1/2-\epsilon} \left|\iint_{C_r} v'_iv'_i dC_r\right|^{1/2} .$$

Using these inequalities in (4.59), we have

$$0 \le P(r) \le (N_1+\rho N_2+2\kappa N_6)a \int_0^a dt_1 \int_{B'_r} v'_iv'_i dB_r + (2\kappa+\rho N_2)a \int_0^a dt_1 \int_{B'_r} \nu'_i\nu'_i dB_r +$$
$$\left[(\alpha+\beta)N_5+2\kappa N_6\right]\int_0^a dt\left\{\int_0^{t_1} dt_1 \int_{B'_r} \nu'_{i,j}\nu'_{i,j} dB_r\right]\right\}$$
$$+ \int_0^a dt_1\left\{\iint_{C_r} \left[\frac{3}{2}(\mu+\kappa)+\frac{\rho}{2}N_3+\kappa\right]v'_iv'_i dC_r\right\} + \int_0^a dt_1\left\{\iint_{C_r} \left[\frac{3}{2}(|\alpha+\beta|+\gamma)+\frac{\rho j}{2}N_4+\kappa\right]\nu'_i\nu'_i dC_r\right\}$$

$$+\int_0^a dt_1\left\{\int_0^{t_1} dt\int_{C_r} \frac{\mu+\kappa}{2}\, v'_{i,j}v'_{i,j}dC_r\right\}+\int_0^a dt_1\left\{\int_0^{t_1} dt\int_{C_r} \frac{1}{2}(|\alpha+\beta|+\gamma)\nu'_{i,j}\nu'_{i,j}dC_r\right\}$$

$$+ N_7 r^{1/2-\epsilon}\int_0^a dt_1\left\{\int_0^{t_1} dt\left[\int_{C_r} v'_i v'_i dC_r\right]^{1/2}\right\}$$

$$+ \int_0^a dt_1\left\{\int_0^{t_1} dt\int_{\partial S} [\gamma\nu'_{i,j}\nu'_i+(\alpha+\beta)\nu'_i\nu'_{j,i} - \frac{\rho j}{2}\, \nu'_i\nu'_i v_j]n_j dS\right\} . \qquad (4.60)$$

Let us now put

$$m = \max\left\{ \frac{2(N_1+\rho N_2+2\kappa N_6)}{\rho} , \frac{2(2\kappa+\rho N_2)}{\rho j} \right\} \qquad (4.61)$$

and choose

$$a = \max\left\{ \frac{t_0}{n} : \frac{t_0}{n} \le \frac{1}{2m} , \text{ n integer} \right\} . \qquad (4.62)$$

Then (4.60) becomes

$$0 \le P(r) \le \frac{P(r)}{2} +$$

$$+ \int_0^a dt_1\int_{C_R} \left[\frac{3}{2}(\mu+\kappa)+N_3+\kappa\right]v'_i v'_i dC_r + \int_0^a dt_1\int_{C_r} \left[\frac{3}{2}(|\alpha+\beta|+\gamma)+\frac{\rho j}{2}N_4+\kappa\right]\nu'_i\nu'_i dC_r$$

$$+ \int_0^a dt_1\left\{\int_0^{t_1} dt\int_{C_r} \frac{\mu+\kappa}{2}\, v'_{i,j}v'_{i,j}dC_r\right\}+ \int_0^a dt_1\left\{\int_0^{t_1} dt\int_{C_r} \frac{1}{2}(|\alpha+\beta|+\gamma)\nu'_{i,j}\nu'_{i,j}dC_r\right\}$$

$$+ a^{3/2}N_7 r^{1/2-\epsilon}\left[\int_0^a dt\int_{C_r} v_i'v_i'dC_r\right]^{1/2}$$

$$+\int_0^a dt_1\left\{\int_0^{t_1} dt\int_{\partial S}[\gamma\nu_{i,j}'\nu_i'+(\alpha+\beta)\nu_i'\nu_{j,i}' - \frac{\rho j}{2}\nu_i'\nu_i'v_j]n_j dS\right\}. \quad (4.63)$$

Because of the boundedness conditions on ν_i and its derivatives on the surface of the body ∂S, it follows that the last term on the right of the above equation is bounded. Thus it is clear that we can find a positive number ℓ, by taking the appropriate suprema, such that

$$0 \leq P(r) \leq \ell\left[P'(r) + r^{1/2-\epsilon}\sqrt{P'(r)} + 1\right] \quad (4.64)$$

where $P'(r) \equiv \frac{dP(r)}{dr}$.

We now proceed to show that $P(r) \equiv 0$. First we observe that because of the boundedness condition (iii) and (iv), $P(r) \sim O(r^3)$ as $r \to \infty$. Suppose $P(r_0) \neq 0$ for some $r_0 > 0$. Then $P(r_0) > 0$. Also $P'(r) > 0$ for $r > r_0$ (because $P(r)$ is the volume integral of non-negative quantities), hence $P(r)$ is monotonic increasing and

$$0 < P(r_0) \leq P(r) \leq \ell\left[P'(r) + r^{1/2-\epsilon}\sqrt{P'(r)} + 1\right],\ r > r_0 . \quad (4.65)$$

Thus

$$P'(r) + \sqrt{P'(r)}\, r^{1/2-\epsilon} - \left[\frac{P(r_0)}{\ell} - 1\right] \geq 0 \quad (4.66)$$

and hence

$$\sqrt{P'(r)} = \frac{1}{2}\, r^{1/2-\epsilon}\left[-1 + \sqrt{1+4r^{2\epsilon-1}\left[\frac{P(r_0)}{\ell} - 1\right]}\right] > 0, \quad r > r_0 \;. \tag{4.67}$$

Thus it follows that

$$\sqrt{P'(r)} \geq \Delta\, r^{1/2-\epsilon}\, r^{2\epsilon-1}, \tag{4.68}$$

where $\Delta = \frac{P(r_0)}{\ell} - 1 > 0$.

Thus we have, from (4.65), using (4.68),

$$P(r) \leq \ell P'(r)\left[1 + \frac{r^{1-2\epsilon}}{\Delta}\right] + \ell \;. \tag{4.69}$$

Assuming that $1-2\epsilon > 0$, we have, for sufficiently large r,

$$P(r) \leq \frac{2\ell}{\Delta}\, P'(r)\, r^{1-2\epsilon} + \ell \tag{4.70}$$

or

$$\frac{P'(r)}{P(r)} \geq \frac{\Delta}{2\ell}\, r^{2\epsilon-1} + \frac{\Delta}{2}\,\frac{r^{2\epsilon-1}}{P(r)} \;. \tag{4.71}$$

This gives, upon integration,

$$P(r) \geq P(r_0)\, \exp\left\{\frac{\Delta}{2\ell}\,(r^{2\epsilon} - r_0^{2\epsilon}) + f(r)\right\} \tag{4.72}$$

where $f(r) \sim O(r^{2\epsilon-3})$ as $r \to \infty$.

But this contradicts the assertion that $P(r) = O(r^3)$ as $r \to \infty$. Hence $P(r) \equiv 0$. It follows that $v_i' = 0$, $\nu_i' = 0$ throughout $E \times (0,a)$. Integrating (4.59) with respect to t from a to $a+t_1$ and with respect to t_1 from 0 to a, and repeating the above arguments it follows that $v_i' = \nu_i' = 0$ throughout $E \times (a,2a)$. In this manner we can cover the whole interval T in steps of length a and it follows that $v_i' = \nu_i' = 0$ in $E \times T$. Finally from (4.54) and the conditions on p it follows that $p' = 0$ in $E \times T$. This proves the theorem, since t_0 was arbitrary.

CHAPTER V

CAUSAL FUNDAMENTAL SOLUTIONS FOR THE SLOW FLOW OF A MICROPOLAR FLUID *

5.1 Introduction

One fruitful method of dealing with the linearised equations of fluid dynamics is the construction of fundamental singular solutions. These are solutions of the flow equations corresponding to a delta function external force. Apart from giving insight into how the various parameters affect the nature of the flow, these fundamental solutions also serve as the kernels in an integral representation of the flow variables. Panico (1979) has constructed such fundamental solutions for classical Navier-Stokes fluids and applied them to a variety of flow problems. In the case of micropolar fluids Ramkissoon (1975) has constructed fundamental solutions for the steady Stokes-linearised equations in two and three dimensions, while Olmstead and Majumdar (1983) have constructed solutions to the steady two dimensional Oseen-linearised equations.

Very few time dependent flow problems have been solved in micropolar fluids. This is not surprising, considering the complexity of the

---

* Contents of this chapter were presented at the Twenty Third Annual Meeting of the Society for Engineering Science held in Buffalo, New York, August 1986.

governing equations. In this chapter we will explicitly construct fundamental singular solutions for the non-steady (causal) Stokes-linearised two dimensional micropolar equations. The fundamental solutions could be used to obtain integral representations of the flow field. Because of the complex nature of the fundamental solutions it is not expected that they will lead to the exact solution of time dependent problems. However one could analyse the resulting integral equations asymptotically, for small physical parameters, for example. This could lead to an understanding of how the physical parameters affect the characteristics of the flow.

5.2 Formulation

The equations of motion of an incompressible micropolar fluid are, in vector notation,

$$\underline{\nabla}\cdot\underline{u}' = 0 \tag{5.1}$$

$$(\mu+\kappa)\underline{\nabla}^2\underline{u}' - \rho(\underline{u}'\cdot\underline{\nabla})\underline{u}' - \rho\frac{\partial u'}{\partial t'} + \kappa\,\underline{\nabla}\times\underline{\nu}' - \underline{\nabla}\,p' = \underline{F}' \tag{5.2}$$

$$\gamma\,\underline{\nabla}^2\underline{\nu}' + (\alpha+\beta)\underline{\nabla}(\underline{\nabla}\cdot\underline{\nu}') - \rho j(\underline{u}'\cdot\underline{\nabla})\underline{\nu}' - \rho j\frac{\partial\nu'}{\partial t'} - 2\kappa\underline{\nu}'+\kappa\underline{\nabla}\times\underline{u}' = \underline{L}'. \tag{5.3}$$

Here all the variables are in dimensional form, with $\underline{u}'$ being the velocity, $\underline{\nu}'$ the microrotation and p' the pressure. The constants α, β, γ, μ, κ, ρ and j are characteristic of the fluid. The body force $\underline{F}'$ and the body couple $\underline{L}'$ are assumed to be known.

The equations (5.1)-(5.3) are nondimensionalised through the scalings

$$\underline{u}' = U\underline{u},\ \underline{\nu}' = V\underline{\nu},\ p' = \frac{U(\mu+\kappa)}{\ell}\,p,\ \underline{x}' = \ell\underline{x},\ t' = \frac{\ell}{U}\,t,$$

$$\underline{F}' = \frac{U(\mu+\kappa)}{\ell^2}\,\underline{F},\ \underline{L}' = \frac{V\gamma}{\ell^2}\,\underline{L}\ ,$$

where U, V, ℓ are some reference velocity, microrotation and length, respectively.

Confining our interest to two dimensional flow in the x_1-x_2 plane we take

$$\underline{u}(\underline{x},t) = \hat{i}_1 u_1(\underline{x},t) + \hat{i}_2 u_2(\underline{x},t);\quad \underline{\nu}(\underline{x},t) = \hat{i}_3 \nu(\underline{x},t)$$

$$\underline{F}(\underline{x},t) = \hat{i}_1 F_1(\underline{x},t) + \hat{i}_2 F_2(\underline{x},t);\quad \underline{L}(\underline{x},t) = \hat{i}_3 L(\underline{x},t),$$

$\hat{i}_1$, $\hat{i}_2$, $\hat{i}_3$ being unit vectors in the three coordinate directions.

In conformity with the Stokes linearisation we now neglect the convective operator $\underline{u}\cdot\underline{\nabla}$ in (5.2) and (5.3) and obtain the following four scalar equations:

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0 \tag{5.4}$$

$$\left[\nabla^2 - m\,\frac{\partial}{\partial t}\right]u_1 + b\,\frac{\partial \nu}{\partial x_2} - \frac{\partial p}{\partial x_1} = F_1 \tag{5.5}$$

$$\left[\nabla^2 - m\frac{\partial}{\partial t}\right]u_2 - b\frac{\partial \nu}{\partial x_1} - \frac{\partial p}{\partial x_2} = F_2 \tag{5.6}$$

$$\left[\nabla^2 - n\frac{\partial}{\partial t} - c\right]\nu + a\frac{\partial u_2}{\partial x_1} - a\frac{\partial u_1}{\partial x_2} = L \tag{5.7}$$

where

$$a = \frac{\kappa \ell U}{\gamma V},\; b = \frac{\kappa \ell V}{(\mu+\kappa)U},\; c = \frac{2\kappa \ell^2}{\gamma},\; m = \frac{\rho \ell U}{\mu+\kappa},\; n = \frac{j\rho \ell U}{\gamma}\,.$$

In slow two-dimensional flows, one is required to solve the system of equations (5.4)-(5.7) under appropriate initial and boundary conditions. This is generally a formidable task. However, much insight into the nature of the flow can be obtained by considering the fundamental singular solutions, which are solutions of the governing equations obtained by replacing the body force $\underline{F}$ and the body couple $\underline{L}$ by appropriate delta functions. Moreover the fundamental singular solutions can be used to reduce the problem of solving (5.4)-(5.7) into solving a set of integral equations, which, in general, are easier to handle. We thus define the following fundamental solution problem:

$$\frac{\partial E_{i1}}{\partial x_1} + \frac{\partial E_{i2}}{\partial x_2} = 0 \tag{5.8}$$

$$\left[\nabla^2 - m\frac{\partial}{\partial t}\right]E_{i1} + b\frac{\partial Q_i}{\partial x_2} - \frac{\partial e_i}{\partial x_1} = -\delta_{i1}\delta(\underline{x})\delta(t-t_0) \tag{5.9}$$

$$\left[\nabla^2 - m\frac{\partial}{\partial t}\right]E_{i2} - b\frac{\partial Q_i}{\partial x_1} - \frac{\partial e_i}{\partial x_2} = -\delta_{i2}\delta(\underline{x})\delta(t-t_0) \tag{5.10}$$

$$\left[\nabla^2 - n\frac{\partial}{\partial t} - c\right]Q_i + a\frac{\partial E_{i2}}{\partial x_1} - a\frac{\partial E_{i1}}{\partial x_2} = -\delta_{i3}\delta(\underline{x})\delta(t-t_0) \tag{5.11}$$

$$i = 1,2,3$$

where δ_{ij} is the Kronecker delta and $\delta(\underline{x}) = \delta(x_1,x_2)$ is the two dimensional delta function. The singularities of the delta functions have been located at $x_1 = x_2 = 0$, $t = t_0$ for convenience.

Our aim is to determine the solutions of (5.8)-(5.11) which tends to zero at infinity since this is the one of usual interest. It should be noted that physically $(E_{i1}(\underline{x},t), E_{i2}(\underline{x},t), e_i(\underline{x},t), Q_i(\underline{x},t))$ is the response to a concentrated force at $\underline{x} = 0$, $t = t_0$ in the i^{th} direction for $i = 1,2$ and for $i = 3$ this is the response to a concentrated couple at the origin at $t = t_0$ acting in the x_1-x_2 plane.

5.3 Solution by Laplace transform

To solve the system of equations (5.8)-(5.11) we adopt a Laplace transform procedure. Taking the transform $\int_0^\infty e^{-st}(\ )dt$ with respect to time of the system (5.8)-(5.11) we obtain

$$\frac{\partial \bar{E}_{i1}}{\partial x_1} + \frac{\partial \bar{E}_{i2}}{\partial x_2} = 0 \tag{5.12}$$

$$\left[\nabla^2 - ms\right]\bar{E}_{i1} + b\,\frac{\partial \bar{Q}_i}{\partial x_2} - \frac{\partial \bar{e}_i}{\partial x_1} = -e^{-st_0}\delta_{i1}\delta(\underline{x}) \tag{5.13}$$

$$\left[\nabla^2 - ms\right]\bar{E}_{i2} - b\,\frac{\partial \bar{Q}_i}{\partial x_1} - \frac{\partial \bar{e}_i}{\partial x_2} = -e^{-st_0}\delta_{i2}\delta(\underline{x}) \tag{5.14}$$

$$\left[\nabla^2 - ns - c\right]\bar{Q}_i + a\,\frac{\partial \bar{E}_{i2}}{\partial x_1} - a\,\frac{\partial \bar{E}_{i1}}{\partial x_2} = -e^{-st_0}\delta_{i3}\delta(\underline{x}) \tag{5.15}$$

$$i = 1,2,3.$$

Here $\bar{E}_{ij}$, $\bar{Q}_i$, $\bar{e}_i$ denote the Laplace transform with respect to time of E_{ij}, Q_i, e_i respectively.

To solve (5.12)-(5.15) we have used a method based on the matrix representation of the system. The method is somewhat laborious and we briefly outline the procedure. Similar procedures for constructing solutions can be found in Panico (1978), Ramkissoon (1975).

The first step is to write the systems (5.12)-(5.15) in the following matrix form:

$$\begin{bmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & 0 & 0 \\ \Delta_1 & 0 & -\frac{\partial}{\partial x_1} & b\,\frac{\partial}{\partial x_2} \\ 0 & \Delta_1 & -\frac{\partial}{\partial x_2} & -b\,\frac{\partial}{\partial x_1} \\ -a\,\frac{\partial}{\partial x_2} & a\,\frac{\partial}{\partial x_1} & 0 & \Delta_2 \end{bmatrix} \begin{bmatrix} \bar{E}_{i1} \\ \bar{E}_{i2} \\ \bar{e}_i \\ \bar{Q}_i \end{bmatrix} = \begin{bmatrix} 0 \\ -\delta_{i1}\delta e^{-st_0} \\ -\delta_{i2}\delta e^{-st_0} \\ -\delta_{i3}\delta e^{-st_0} \end{bmatrix} \tag{5.16}$$

where

$$\Delta_1 = \nabla^2 - ms$$
$$\Delta_2 = \nabla^2 - ns - c$$

and δ denotes $\delta(x_1,x_2)$, the two dimensional delta function.

So long as we do not carry out any division, we can treat the above system as a linear algebraic system. This enables us to uncouple the equations. We find that

$$\Delta\bar{E}_{11} = -\frac{\partial^2}{\partial x_2^2}\Delta_2 \delta e^{-st_0} \tag{5.17}$$

$$\Delta\bar{E}_{21} = -\frac{\partial^2}{\partial x_1 \partial x_2}\Delta_2 \delta e^{-st_0} \tag{5.18}$$

$$\Delta\bar{E}_{31} = b\,\frac{\partial}{\partial x_2}\nabla^2 \delta e^{-st_0} \tag{5.19}$$

$$\Delta\bar{E}_{12} = \frac{\partial^2}{\partial x_1\ \partial x_2}\Delta_2 \delta e^{-st_0} \tag{5.20}$$

$$\Delta\bar{E}_{22} = -\frac{\partial^2}{\partial x_1^2}\Delta_2 \delta e^{-st_0} \tag{5.21}$$

$$\Delta\bar{E}_{32} = -b\,\frac{\partial}{\partial x_1}\nabla^2 \delta e^{-st_0} \tag{5.22}$$

$$\Delta\bar{e}_1 = \frac{\partial}{\partial x_1}\left[\Delta_1\Delta_2 + ab\nabla^2\right]\delta e^{-st_0} \tag{5.23}$$

$$\Delta\bar{e}_2 = \frac{\partial}{\partial x_2}\left[\Delta_1\Delta_2 + ab\nabla^2\right]\delta e^{-st_0} \tag{5.24}$$

$$\Delta\bar{e}_3 = 0 \tag{5.25}$$

$$\Delta\bar{Q}_1 = -a \frac{\partial}{\partial x_2} \nabla^2 \delta e^{-st_0} \tag{5.26}$$

$$\Delta\bar{Q}_2 = a \frac{\partial}{\partial x_1} \nabla^2 \delta e^{-st_0} \tag{5.27}$$

$$\Delta\bar{Q}_3 = - \Delta_1 \nabla^2 \delta e^{-st_0} \tag{5.28}$$

where $\Delta = \nabla^2\left[\Delta_1\Delta_2 + ab\ \nabla^2\right]$.

We have found that the solutions to (5.17)-(5.28) can be expressed, in Laplace space, in terms of two scalar functions $\bar{\Theta}(x_1,x_2,s)$, $\bar{\phi}(x_1,x_2,s)$ as follows:

$$\bar{E}_{11} = \frac{\partial^2}{\partial x_2^2}\left\{\bar{\Theta} - \frac{1}{ms}\nabla^2\bar{\Theta} + \frac{1}{ms}(\nabla^2 - ns - c)\bar{\phi} + \frac{ab}{ms}\,\bar{\phi}\right\} \tag{5.29}$$

$$\bar{E}_{12} = \bar{E}_{21} = -\frac{\partial^2}{\partial x_1 \partial x_2}\left\{\bar{\Theta} - \frac{1}{ms}\nabla^2\bar{\Theta} + \frac{1}{ms}(\nabla^2 - ns - c)\ \bar{\phi} + \frac{ab}{ms}\,\bar{\phi}\right\} \tag{5.30}$$

$$\bar{E}_{22} = \frac{\partial^2}{\partial x_1^2}\left\{\bar{\Theta} - \frac{1}{ms}\nabla^2\bar{\Theta} + \frac{1}{ms}(\nabla^2 - ns - c)\bar{\phi} + \frac{ab}{ms}\,\bar{\phi}\right\} \tag{5.31}$$

$$\bar{E}_{31} = -\ b\,\frac{\partial\bar{\phi}}{\partial x_2} \tag{5.32}$$

$$\bar{E}_{32} = b\,\frac{\partial\bar{\phi}}{\partial x_1} \tag{5.33}$$

$$\bar{e}_1 = -\frac{\partial}{\partial x_1}(\nabla^2 - ms)\bar{\Theta} \tag{5.34}$$

$$\bar{e}_2 = -\frac{\partial}{\partial x_2}(\nabla^2 - ms)\bar{\Theta} \tag{5.35}$$

$$\bar{e}_3 = 0 \tag{5.36}$$

$$\bar{Q}_1 = a\,\frac{\partial\bar{\phi}}{\partial x_2} \tag{5.37}$$

$$\bar{Q}_2 = -a \frac{\partial \bar{\Phi}}{\partial x_1} \tag{5.38}$$

$$\bar{Q}_3 = (\nabla^2 - ms)\bar{\Phi} . \tag{5.39}$$

The scalar functions $\bar{\Theta}$ and $\bar{\Phi}$ are such that they satisfy the following relations:

$$(\nabla^2 - ms)\nabla^2\bar{\Theta} = - e^{-st_0}\delta(\bar{x}) \tag{5.40}$$

$$\left[(\nabla^2 - ms)(\nabla^2 - ns - c) + ab\ \nabla^2\right]\bar{\Phi} = - e^{-st_0}\delta(\bar{x}). \tag{5.41}$$

The transforms of the fundamental solutions are determined once the solutions of the equations (5.40) and (5.41) are known. (5.40) has the solution

$$\bar{\Theta} = \frac{e^{-st_0}}{4\pi ms}\left\{\log(r^2) + K_0(\sqrt{ms}\ r)\right\}, \tag{5.42}$$

where $r = (x_1^2 + x_2^2)^{1/2}$ and K_0 denotes modified Bessel function of the second kind. This is the solution that tends to zero as $r \to \infty$.

The solution of (5.41) presents a few difficulties. In general the inversion of a fourth order partial differential operator is readily achieved only if it can be factored into a product of two quadratic operators. This requirement imposes restrictions on the constants a, b, c, m, n. We now derive those conditions.

Let

$$L = (\nabla^2 - ms)(\nabla^2 - ns - c) + ab\ \nabla^2. \tag{5.43}$$

We need L to be factorable such that

$$L = (\nabla^2 + A_1 s + B_1)(\nabla^2 + A_2 s + B_2), \tag{5.44}$$

where A_i, B_i $(i = 1,2)$ are constants independent of s.

We have chosen the above form because we would like to obtain conditions on the parameters of the problem, independent of the Laplace transform variables. Compatability between (5.43) and (5.44) requires that

$$\begin{aligned} A_1 + A_2 &= -(n+m) \\ A_1 A_2 &= mn \\ B_1 + B_2 &= ab-c \\ B_1 B_2 &= 0 \\ A_1 B_2 + A_2 B_1 &= mc. \end{aligned}$$

These requirements are satisfied if

$$ab = \frac{n-m}{n} c > 0 \tag{5.45}$$

in which case

$$A_1 = -n,\ A_2 = -m,\ B_1 = 0,\ B_2 = -\frac{mc}{n}.$$

If, on the other hand, we choose $A_1 = -m$, $A_2 = -n$, then we have $ab = 0$, which is not physically interesting. Similarly the choice $ab = c$ which makes $B_1 = B_2 = 0$ also leads to $mc = 0$, a physically uninteresting case. We will thus require that the condition (5.45) holds. In terms of the original micropolar parameters this condition implies that

$$j = \frac{2\gamma}{2\mu+\kappa} \,. \tag{5.46}$$

This is the condition obtained by Olmstead and Majumdar (1983) for the Oseen flow problem. Similar conditions were also obtained by Smith and Guram (1974), in considering Taylor flows.

Assuming that the condition (5.45) holds we proceed to determine $\bar{\Phi}$ which now satisfies

$$(\nabla^2 - ns)(\nabla^2 - ms - \frac{mc}{n})\bar{\Phi} = -e^{-st_0}\delta(\underline{x}). \tag{5.47}$$

To solve (5.47) we first observe that if f_1 and f_2 are two functions satisfying

$$(\nabla^2 - a_1^2)f_1 = -e^{-st_0}\delta(\underline{x}) \tag{5.48}$$

and

$$(\nabla^2 - a_2^2)f_2 = -e^{-st_0}\delta(\underline{x}) \tag{5.49}$$

then the function

$$g = \frac{1}{a_1^2 - a_2^2} f_1 - f_2 \tag{5.50}$$

satisfies

$$(\nabla^2 - a_1^2)(\nabla^2 - a_2^2) g = -e^{-st_0} \delta(\underline{x}). \tag{5.51}$$

Since

$$(\nabla^2 - a^2) f = -\delta(\underline{x}) \tag{5.52}$$

has the solution

$$f = \frac{1}{2\pi} K_0(ar), \tag{5.53}$$

(where K_0 denotes the modified Bessel function of the second kind) we immediately obtain

$$\bar{\phi}(x_1, x_2, s) = \frac{e^{-st_0}}{2\pi(n-m)(s-\lambda)} \left\{ K_0(\sqrt{ns}\, r) - K_0\left[\left(ms + \frac{mc}{n}\right)^{1/2} r \right] \right\} \tag{5.54}$$

Here $r = (x_1^2 + x_2^2)^{1/2}$.

This essentially completes the solution in Laplace space, only the appropriate differentiations and substitutions in the expressions for $(\bar{E}_{ij}, \bar{e}_i, \bar{Q}_i)$ remain to be done.

To determine the fundamental solution in real space we would require the inverse Laplace transforms of $\bar{\Theta}$, $\bar{\Theta}/s$, $\bar{\Phi}$ and $\bar{\Phi}/s$. Using the convolution theorem for Laplace transforms, these are found to be

$$\Theta = L^{-1}(\bar{\Theta}, s \to t) = \frac{H(t-t_0)}{4\pi m}\left\{\log(r^2) + E1\left[\frac{mr^2}{4(t-t_0)}\right]\right\} \tag{5.55}$$

$$L^{-1}\left[\frac{\bar{\Theta}}{s}\right] = \frac{H(t-t_0)}{4\pi m}\left\{(t-t_0)\log(r^2) + \int_0^{t-t_0} E1\left[\frac{mr^2}{4z}\right]dz\right\} \tag{5.56}$$

$$L^{-1}(\bar{\Phi}) = \frac{H(t-t_0)e^{\lambda(t-t_0)}}{4\pi(n-m)}\int_0^{t-t_0}\frac{e^{-\lambda v - \frac{nr^2}{4v}} - e^{-\left[\lambda + \frac{c}{n}\right]v - \frac{mr^2}{4v}}}{v}\,dv \tag{5.57}$$

$$L^{-1}\left(\frac{\bar{\Phi}}{s}\right) = \frac{H(t-t_0)}{4\pi\lambda(n-m)}\left\{e^{\lambda(t-t_0)}\int_0^{t-t_0}\frac{e^{-\lambda n - \frac{nr^2}{4v}} - e^{-\left[\lambda + \frac{c}{n}\right]v - \frac{mr^2}{4v}}}{v}\,dv\right.$$
$$\left. + \int_0^{t-t_0}\frac{e^{-\frac{c}{n}v - \frac{mr^2}{4v}} - e^{-\frac{nr^2}{4v}}}{v}\,dv\right\} \tag{5.58}$$

where $H(t)$ is the Heaviside unit function and

$$E1(x) = \int_x^{\infty}\frac{e^{-y}}{y}\,dy \tag{5.59}$$

is the exponential integral.

In order to facilitate writing explicit expressions for the solutions we introduce a set of notations.

Let us define

$$I_j(a,b) = \int_0^{t-t_0} \frac{e^{-av-b/v}}{v^j}\,dv$$

and introduce the following symbols:

$$r_{1,2} = \exp\left[\frac{-mr^2}{4(t-t_0)}\right]\left\{\frac{mx_i^2}{(t-t_o)r^2} - \frac{2}{r^2} + \frac{4x_i^2}{r^4}\right\} + \frac{2}{r^2} - \frac{4x_i^2}{r^4},\ i = 1,2$$

$$r_3 = \exp\left[\frac{-mr^2}{4(t-t_0)}\right]\left\{\frac{mx_1x_2}{(t-t_0)r^2} + \frac{4x_1x_2}{r^4}\right\} - \frac{4x_1x_2}{r^4}$$

$$r_{4,5} = \frac{mx_i^2}{r^2} I_1\left[0,\frac{mr^2}{4}\right] + I_0\left[0,\frac{mr^2}{4}\right]\left\{\frac{4x_i^2}{r^4} - \frac{2}{r^2}\right\} + 2(t-t_0)\left\{\frac{1}{r^2} - \frac{2x_i^2}{r^4}\right\}$$

$$(i = 1,2)$$

$$r_6 = \frac{mx_1x_2}{r^2} I_1\left[0,\frac{mr^2}{4}\right] + \frac{4x_1x_2}{r^4} I_0\left[0,\frac{mr^2}{4}\right] - \frac{4x_1x_2(t-t_0)}{r^4}$$

$$r_{7,8} = -\frac{m^2}{2} I_2\left[0,\frac{mr^2}{4}\right] + \frac{m^3}{4} x_i^2\, I_3(0,ar^2),\ i = 1,2$$

$$r_9 = \frac{m^3}{4} x_1x_2\, I_3\left[0,\frac{mr^2}{4}\right]$$

$$r_{10,11} = -\frac{mx_i}{2(t-t_0)^2} \exp\left[\frac{mr^2}{4(t-t_0)}\right],\ i = 1,2$$

$$r_{12,13} = -\frac{1}{2}\frac{m^2x_i}{(t-t_0)^2}\exp\left[\frac{mr^2}{4(t-t_0)}\right],\ i = 1,2$$

$$s_{1,2} = \exp\{\lambda(t-t_0)\}\left\{-\frac{1}{4}m^2x_i^2\ I_3\left[\frac{c}{n}+\lambda,\frac{mr^2}{4}\right] + \frac{1}{4}n^2x_i^2\ I_3\left[\lambda,\frac{nr^2}{4}\right]\right.$$

$$\left.+\frac{1}{2}m\ I_2\left[\frac{c}{n}+\lambda,\frac{mr^2}{4}\right] - \frac{1}{2}n\ I_2\left[\lambda,\frac{nr^2}{4}\right]\right\},\ i = 1,2$$

$$s_3 = \exp\{\lambda(t-t_0)\}\left\{-\frac{1}{4}m^2x_1x_2\ I_3\left[\frac{c}{n}+\lambda,\frac{mr^2}{4}\right] + \frac{1}{4}n^2x_1x_2\ I_3\left[\lambda,\frac{nr^2}{4}\right]\right\}$$

$$s_{4,5} = \frac{\exp\{\lambda(t-t_0)\}}{\lambda}\left\{-\frac{1}{4}m^2x_i^2\ I_3\left[\frac{c}{n}+\lambda,\frac{mr^2}{4}\right] + \frac{1}{4}\frac{n^2x_i^2}{}\ I_3\left[\lambda,\frac{nr^2}{4}\right]\right.$$

$$\left.+\frac{m}{2}\ I_2\left[\frac{c}{n}+\lambda,\frac{mr^2}{4}\right] - \frac{n}{2}\ I_2\left[\lambda,\frac{nr^2}{4}\right]\right\}$$

$$+\frac{1}{\lambda}\left\{\frac{1}{4}m^2x_i^2\ I_3\left[\frac{c}{n},\frac{mr^2}{4}\right] - \frac{1}{4}\frac{n^2x_i^2}{}\ I_3\left[0,\frac{nr^2}{4}\right]\right.$$

$$\left.-\frac{m}{2}\ I_2\left[\frac{c}{n},\frac{mr^2}{4}\right] + \frac{n}{2}\ I_2\left[0,\frac{nr^2}{4}\right]\right\},\ i = 1,2$$

$$s_6 = \frac{\exp\{\lambda(t-t_0)\}}{\lambda}\left\{-\frac{1}{4}m^2x_1x_2\ I_3\left[\frac{c}{n}+\lambda,\frac{mr^2}{4}\right] + \frac{1}{4}n^2x_1x_2\ I_3\left[\lambda,\frac{nr^2}{4}\right]\right\}$$

$$+\frac{1}{\lambda}\left\{+\frac{1}{4}m^2x_1x_2\ I_3\left[\frac{c}{n},\frac{mr^2}{4}\right] - \frac{1}{4}n^2x_1x_2\ I_3\left[0,\frac{nr^2}{4}\right]\right\}$$

$$s_{7,8} = \frac{\exp\{\lambda(t-t_0)\}}{16x}\left\{-m^4x_i^2r^2\ I_5\left[\frac{c}{n}+\lambda,\frac{mr^2}{4}\right] + n^4r^2\ I_5\left[\lambda,\frac{nr^2}{4}\right]\right.$$

$$+ 2m^3(r^2 + 6x_i^2)I_4\left[\frac{c}{n} + \lambda, \frac{mr^2}{4}\right] - 2n^3(r^2 + 6x_i^2)I_4\left[\lambda, \frac{nr^2}{4}\right]$$

$$- 16m^2\ I_3\left[\frac{c}{n} + \lambda, \frac{mr^2}{4}\right] + 16n^2\ I_3\left[\lambda, \frac{nr^2}{4}\right] \Bigg\}$$

$$+ \frac{1}{16\lambda} \Bigg\{ m^4x_i^2r^2\ I_5\left[\frac{c}{n}, \frac{mr^2}{4}\right] - n^4x_i^2r^2\ I_5\left[0, \frac{nr^2}{4}\right]$$

$$- 2m^3(r^2+6x_i^2)I_4\left[\frac{c}{n}, \frac{mr^2}{4}\right] + 2n^3(r^2 + 6x_i^2)I_4\left[0, \frac{nr^2}{4}\right]$$

$$+ 16m^2\ I_3\left[\frac{c}{n}, \frac{mr^2}{4}\right] - 16n^2\ I_3\left[0, \frac{nr^2}{4}\right] \Bigg\}, \quad i = 1,2$$

$$s_9 = \frac{x_1x_2\exp\{\lambda(t-t_0)\}}{16\lambda} \Bigg\{- m^4r^2\ I_5\left[\frac{c}{n} + \lambda, \frac{mr^2}{4}\right] + n^4r^2\ I_5\left[\lambda, \frac{nr^2}{4}\right]$$

$$+ 12m^3\ I_4\left[\frac{c}{n} + \lambda, \frac{mr^2}{4}\right] - 12n^3\ I_4\left[\lambda, \frac{nr^2}{4}\right] \Bigg\}$$

$$+ \frac{x_1x_2}{16\lambda} \Bigg\{ m^4r^2\ I_5\left[\frac{c}{n}, \frac{mr^2}{4}\right] - n^4r^2\ I_5\left[0, \frac{nr^2}{4}\right]$$

$$- 12m^3\ I_4\left[\frac{c}{n}, \frac{mr^2}{4}\right] + 12n^3\ I_4\left[0, \frac{nr^2}{4}\right] \Bigg\}$$

$$s_{10,11} = \frac{x_i}{2} \Bigg\{ mI_2\left[\frac{c}{n} + \lambda, \frac{mr^2}{4}\right] - nI_2\left[\lambda, \frac{nr^2}{4}\right] \Bigg\}, \quad i = 1,2$$

$$s_{12} = \lambda\ \exp(\lambda(t-t_0)) \Bigg\{ I_1\left[\lambda, \frac{nr^2}{4}\right] - I_1\left[\frac{c}{n} + \lambda, \frac{mr^2}{4}\right] \Bigg\}$$

$$+ \frac{1}{t-t_0} \left\{ \exp\left[- \frac{nr^2}{4(t-t_0)}\right] - \exp\left[- \frac{c}{n} (t-t_0) - \frac{mr^2}{4(t-t_0)}\right] \right\} .$$

In terms of these symbols the fundamental solution is given below:

$$E_{11} = \frac{H(t-t_0)}{4\pi} \left\{ \frac{1}{m} (r_2 - r_8) + \frac{1}{n-m} \left[s_8 - \frac{mc}{n} s_5 - ns_2\right] \right\} \tag{5.60}$$

$$E_{12} = E_{21} = \frac{H(t-t_0)}{4\pi} \left\{ \frac{1}{m} (-r_3 + r_9) + \frac{1}{n-m} \left[- s_9 + \frac{mc}{n} s_6 - ns_3\right] \right\} \tag{5.61}$$

$$E_{22} = \frac{H(t-t_0)}{4\pi} \left[\frac{1}{m} (r_1 - r_7) + \frac{1}{n+m} \left[s_7 - \frac{mc}{n} s_4 - ns_1\right] \right] \tag{5.62}$$

$$E_{31} = - b \frac{H(t-t_0)}{4\pi(n-m)} s_{11}, \tag{5.63}$$

$$E_{32} = - b \frac{H(t-t_0)}{4\pi(n-m)} s_{10} \tag{5.64}$$

$$e_1 = - \frac{H(t-t_0)}{4\pi m} (- r_{12} + mr_{10}) \tag{5.65}$$

$$e_2 = \frac{H(t-t_0)}{4\pi m} (- r_{13} + mr_{11}) \tag{5.66}$$

$$e_3 = 0 \tag{5.67}$$

$$Q_1 = a \frac{H(t-t_0)}{4\pi(n-m)} s_{11} \tag{5.68}$$

$$Q_2 = -a \frac{H(t-t_0)}{4\pi m} s_{10} \tag{5.69}$$

$$Q_3 = \frac{H(t-t_0)}{4\pi(n-m)} (s_1+s_2-ms_{12}) \ . \tag{5.70}$$

This completes the determination of the fundamental solution.

It is worth comparing these solutions with the corresponding fundamental solutions for the classical Navier-Stokes equations (see Panico(1978)). In that case

$$E_{ij}(\underline{x} \ , \ t) = (\delta_{ij} \nabla^2 - \frac{\partial^2}{\partial x_i \partial x_j}) S(\underline{x} \ , \ t) \tag{5.71}$$

$$e_i \ (\underline{x} \ , \ t) = - \frac{\partial}{\partial x_i} (\nabla^2 - R \frac{\partial}{\partial t}) S(\underline{x} \ , \ t) \tag{5.72}$$

Here R is the Reynolds number and S should satisfy

$$\nabla^2 (\nabla^2 - R \frac{\partial}{\partial t}) S(\underline{x} \ ,t) = - \delta(\underline{x}) \ \delta(t). \tag{5.73}$$

The solution for S is

$$S = \frac{H(t-t_0)}{4 \pi R} \left[\log(r^2) + E1\left\{\frac{Rr^2}{4(t-t_0)}\right\} \right] \tag{5.74}$$

which compares with (5.55). There is no classical analogue for the function ϕ occuring in the micropolar theory.

5.4 Integral Representations

One of the principal uses of the fundamental singular solutions is to obtain integral representations of flow variables. Consider two dimensional micropolar flow past a finite object occupying a region A with smooth contour Γ in the presence of body force $\underline{F} = (F_1, F_2)$ and body couple $F_3 \hat{i}_3$. Then the flow field has the following integral representation:

$$u_i(\underline{x},t) = \int_0^\infty \iint_A E_{ji}\,(\underline{x} - \underline{\xi}\,, t - \tau)\; F_j\,(\underline{x} - \underline{\xi}\,, t - \tau)\; dA(\underline{\xi})\; d\tau$$
$$+ \int_0^\infty \int_\Gamma E_{ji}\,(\underline{x} - \underline{\xi}\,, t - \tau)\; \sigma_j\,(\underline{x} - \underline{\xi}\,, t - \tau)\; d\Gamma(\underline{\xi})\; d\tau \qquad (5.75)$$

$$\nu\,(\underline{x},t) = \int_0^\infty \iint_A Q_j\,(\underline{x} - \underline{\xi}\,, t - \tau)\; F_j\,(\underline{x} - \underline{\xi}\,, t - \tau)\; dA(\underline{\xi})\; d\tau$$
$$+ \int_0^\infty \int_\Gamma Q_j\,(\underline{x} - \underline{\xi}\,, t - \tau)\; \sigma_j\,(\underline{x} - \underline{\xi}\,, t - \tau)\; d\Gamma(\underline{\xi})\; d\tau \qquad (5.76)$$

$$p\,(\underline{x},t) = \int_0^\infty \iint_A e_j\,(\underline{x} - \underline{\xi}\,, t - \tau)\; F_j\,(\underline{x} - \underline{\xi}\,, t - \tau)\; dA(\underline{\xi})\; d\tau$$
$$+ \int_0^\infty \int_\Gamma e_j\,(\underline{x} - \underline{\xi}\,, t - \tau)\; \sigma_j\,(\underline{x} - \underline{\xi}\,, t - \tau)\; dA(\underline{\xi})\; d\tau \qquad (5.77)$$

with the repeated index j implying sum over j = 1,2,3. The unknown functions σ_j are related to the stress function on the body. Once the values of $\underline{u}, \nu$ and p are prescribed on the body, together with the body force and couple, (5.75) - (5.77) lead to a set of integral equations for the unknown functions σ_j. Knowing σ_j, the flow variables could be determined from (5.75) - (5.77). The complicated nature of E_{ij} , Q_i , and e_i virtually rules out exact solutions in most cases. However an asymptotic analysis based on small parameter approximation could be carried out on the integral equations. An extensive review of such applications in classical viscous fluids can be found in Olmstead and Gautesen (1976).

CHAPTER VI

MICROPOLAR FLOW IN A MEANDERING CHANNEL *

6.1 Introduction

Interest in viscous flow in channels and conduits with irregular surfaces and curving centrelines has been widespread in recent years. A prime motivation for such studies comes from a need to understand flow characteristics in blood vessels that lead to various pathological conditions. Lee and Fung (1970) have numerically studied flow in tubes with a bell shaped constriction and determined wall stresses for low Reynolds numbers. Chow and Soda (1972) used a perturbation technique to obtain solutions of flow in tubes with a continuous constriction. However, all these studies consider blood as a homogeneous Newtonian fluid. In recent years it has come to be known that the micropolar fluid theory, which allows for micromotions within the continuum, serves as a better model for such rheologically complex fluids as blood. Ariman et al. (1974) have used this microcontinuum approach to study steady and pulsatile flow of blood in circular conduits. They have obtained results for velocity profiles and cell rotational velocities which are in good

---

* Contents of this chapter have been accepted for publication in the *Canadian Journal of Chemical Engineering*.

agreement with the experimental results of Bugliarello and Sevilla (1970). Radhakrishnama Charya (1977) has extended the studies of Chow and Soda (1972) to the case of micropolar fluids.

In this chapter we study the flow of a micropolar fluid in a meandering channel of constant width. The corresponding problem for viscous fluids in three dimensions has been first investigated by Wang (1980). He considers small centreline curvatures $K(s)$, setting $K(s) = \epsilon k(s)$ where s is length measured along the centreline and ϵ is a small perturbation parameter and k is of order 1. In order to effectively use a perturbation scheme he also assumes small Reynolds numbers, $Re = O(\epsilon)$. In a recent paper Van Dyke (1983) has significantly improved upon Wang's scheme by introducing the further assumption that the centreline curvature is not only small but also slowly varying in the sense that the variation of the channel takes place over distances large compared with the channel width. With this assumption Van Dyke could remove Wang's restriction that $Re = O(\epsilon)$ and proceed as far as the fourth approximation in a systematic perturbation scheme. We consider the corresponding problem for a micropolar fluid. Assuming that the channel meanders slowly and slightly we use a perturbation series for the stream-function and microrotation to study the effect of curvature and micropolarity on the shear stresses on the wall and the downstream pressure gradient. The nature of the exact form of boundary condition for microrotation is not yet settled; therefore we investigate the effect of two types of boundary conditions: i) that the microrotation vanishes on the boundary (the no-spin boundary condition), ii) that the microrotation on the boundary is equal to the local fluid angular velocity. In finding exact solutions for arbitrary curvatures and micropolar parameters we could not go beyond the second approximation because the expressions become too big to write down. As a result we could not determine how the micropolar Reynolds numbers B and D (defined later on) affect the flow, since their effect comes into play only in the third approximation. However we have found strong dependence of important flow properties on the parameter $A = \kappa/\mu{+}\kappa$ which effectively measures the micropolarity of the fluid. Although the general way in which curvature and micropolarity affects the flow characteristics has been found to be independent of the type of boundary condition on microrotation, the magnitude of these effects has been found to be enhanced by the no-spin boundary condition.

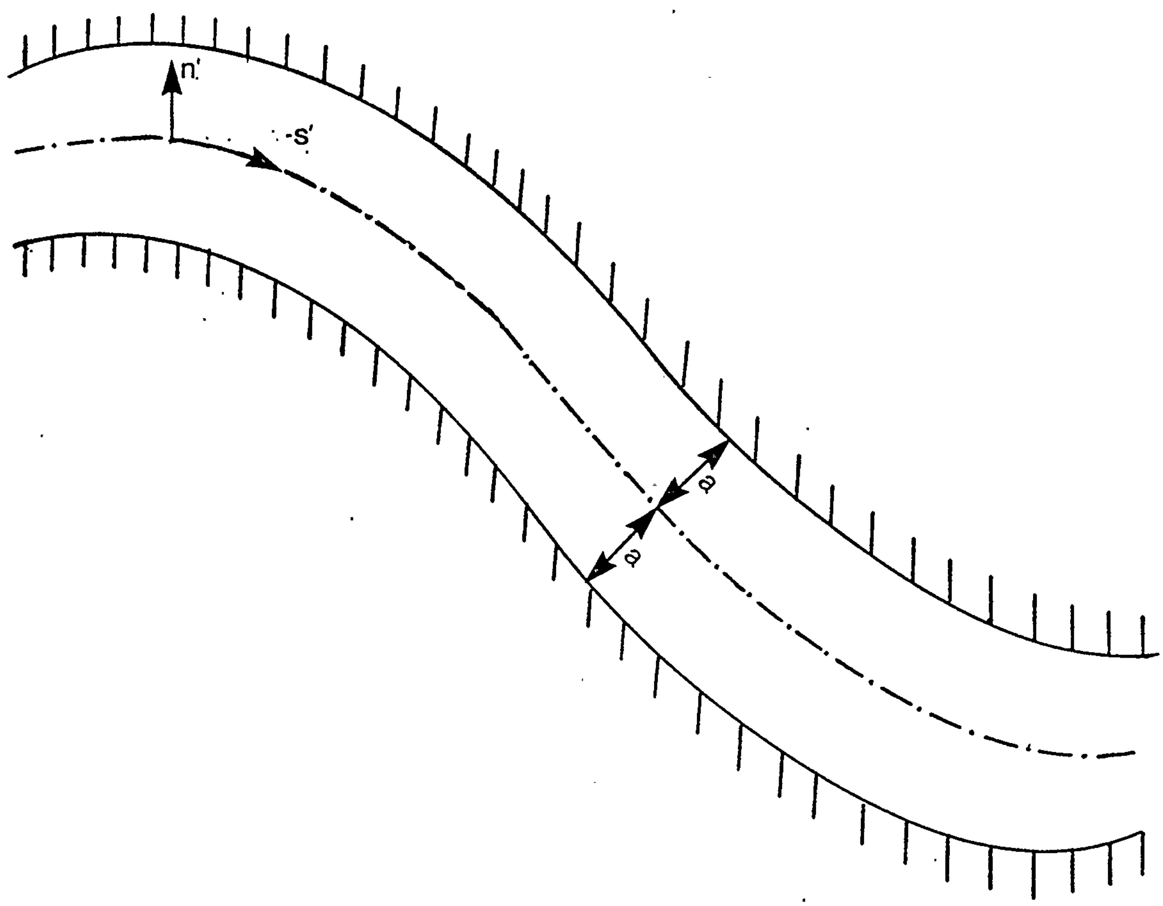

Figure 6.1. Meandering channel and coordinate system

6.2 Statement of the Problem

Consider the steady two-dimensional flow of an incompressible micropolar fluid through a smooth curvilinear channel of constant width 2a (Figure 6.1). Following Wang (1980) we introduce a coordinate system (s',n',z') consisting of a distance s' measured along the centreline of the channel and a distance n' normal to it and z' taken parallel to the generators of the channel. The curvature of the centreline is assumed to be given by a smooth function k'(s) of the length s' along the centreline. The curvature is reckoned positive if the channel is turning to the left.

From chapter III the equations governing the steady motion of an incompressible micropolar fluid in the absence of body couples are

$$\underline{\nabla} \cdot \underline{u}' = 0 \tag{6.1}$$

$$-(\mu+\kappa)\underline{\nabla}\times(\underline{\nabla}\times\underline{u}') + \kappa\underline{\nabla}\times\underline{\nu}' - \rho(\underline{u}'\cdot\underline{\nabla})\underline{u}' - \underline{\nabla}p' - \rho\underline{f}' = 0 \tag{6.2}$$

$$\gamma[\underline{\nabla}(\underline{\nabla}\cdot\underline{\nu}') - \underline{\nabla}\times(\underline{\nabla}\times\underline{\nu}')] + (\alpha+\beta)\underline{\nabla}(\underline{\nabla}\cdot\underline{\nu}') - \rho j(\underline{u}'\cdot\underline{\nabla})\underline{\nu}' - 2\kappa\underline{\nu}' + \kappa\underline{\nabla}\times\underline{u}' = 0. \tag{6.3}$$

Here all variables are in dimensional form, with $\underline{u}'$ being the velocity, p' the pressure and $\underline{\nu}'$ the microrotation. The constants α, β, γ, μ, κ, j, ρ characterize the properties of the fluid, and f' is the body force. For the problem under discussion we assume that

$$\underline{u}'(s',n') = (u'(s',n'),v'(s',n'),0),$$

$$\underline{\nu}'(s',n') = (0,0,\nu'(s',n')).$$

Then in terms of the coordinate system introduced above the equations of motion (6.1)-(6.2) become

$$\frac{\partial u'}{\partial s'} + \frac{\partial}{\partial n'}[(1-k'n')v'] = 0 \qquad (6.4)$$

$$-(\mu+\kappa)\frac{\partial}{\partial n'}\left\{\frac{1}{1-k'n'}\left[\frac{\partial v'}{\partial s'} - \frac{\partial}{\partial n'}(1-k'n')u'\right]\right\} + \kappa\frac{\partial \nu'}{\partial n'}$$

$$-\rho\left[\frac{u'}{1-k'n'}\frac{\partial u'}{\partial s'} + v'\frac{\partial u'}{\partial n'}\right] - \frac{1}{1-k'n'}\frac{\partial p'}{\partial s'} + \frac{\rho k'}{1-k'n'}u'v' = 0 \qquad (6.5)$$

$$\frac{(\mu+\kappa)}{1-k'n'}\frac{\partial}{\partial s'}\left\{\frac{1}{1-k'n'}\left[\frac{\partial v'}{\partial s'} - \frac{\partial}{\partial n'}(1-k'n')u'\right]\right\} - \frac{\kappa}{1-k'n'}\frac{\partial \nu'}{\partial s'}$$

$$-\rho\left[\frac{u'}{1-k'n'}\frac{\partial v'}{\partial s'} + v'\frac{\partial v'}{\partial n'}\right] - \frac{\partial p'}{\partial n'} + \rho k'\frac{u'^2}{1-k'n'} = 0 \qquad (6.6)$$

$$\frac{-\gamma}{1-k'n'}\left\{\frac{\partial}{\partial s'}\left[\frac{-1}{1-k'n'}\frac{\partial \nu'}{\partial s'}\right] - \frac{\partial}{\partial n'}\left[(1-k'n')\frac{\partial \nu'}{\partial n'}\right]\right\} - 2\kappa\nu'$$

$$-\rho j\left[\frac{u'}{1-k'n'}\frac{\partial \nu'}{\partial s'} + v'\frac{\partial \nu'}{\partial n'}\right] + \frac{\kappa}{1-k'n'}\left[\frac{\partial v'}{\partial s'} - \frac{\partial}{\partial n'}(1-k'n')u'\right] = 0. \qquad (6.7)$$

The last term in (6.5) and (6.6) arises from the centrifugal effects due to curvature.

Next we introduce a stream function ψ' defined by $u' = \dfrac{\partial \psi'}{\partial n'}$,

$v' = \frac{1}{1-k'n'} \frac{\partial \psi'}{\partial s'}$, so that the equation of continuity (6.4) is identically satisfied. Let the mean volume flux rate per unit distance normal to the s-n plane be 2M. Then the variables can be nondimensionalised as follows:

$$\psi = \psi'/M, \ \nu = \nu'/\left[\frac{M}{a^2}\right], \ p = p'/\frac{(\mu+\kappa)M}{a^2}, \ s = s'/a, \ n = n'/a, \ k = k'/(1/a),$$
$$z = z'/a.$$

Eliminating pressure from (6.6) and (6.7) we get the following equations of motion in nondimensional form:

$$\nabla^4\psi + A\nabla^2\nu = \frac{B}{1-kn}\left[\frac{\partial\psi}{\partial n}\frac{\partial}{\partial s} - \frac{\partial\psi}{\partial s}\frac{\partial}{\partial n}\right]\nabla^2\psi \qquad (6.8)$$

$$C\nabla^2\nu = 2\nu + \nabla^2\psi + \frac{D}{1-kn}\left[\frac{\partial\psi}{\partial n}\frac{\partial}{\partial s} - \frac{\partial\psi}{\partial s}\frac{\partial}{\partial n}\right]\nu \qquad (6.9)$$

where

$$A = \frac{\kappa}{\mu+\kappa}, \qquad B = \frac{M\rho}{\mu+\kappa},$$

$$C = \frac{\gamma}{a^2\kappa}, \qquad D = \frac{M\rho j}{a^2\kappa}$$

are dimensionless parameters and

$$\nabla^2 \equiv \frac{1}{1-kn}\left[\frac{\partial}{\partial s}\left[\frac{1}{1-kn}\frac{\partial}{\partial s}\right] + \frac{\partial}{\partial n}\left[(1-kn)\frac{\partial}{\partial n}\right]\right] \qquad (6.10)$$

We restrict our attention to channels whose centrelines meander slowly and slightly. That is to say, locally the centrelines deviate only slightly from a straightline and the slow variation of the centreline takes place over distances large compared with the channel width. Following Van Dyke (1983) this is expressed by setting

$$k(s) = \epsilon f(\delta s)$$

where $\epsilon << 1$, $\delta << 1$ are two small positive perturbation parameters. For simplicity of analysis we set $\epsilon = \delta$ and introduce a new stretched coordinate $S = \epsilon s$, so that $k(s) = \epsilon f(S)$. With this assumption on the curvature the equations of motion (6.8) and (6.9) become

$$\nabla^4\Psi + A\nabla^2\nu = \frac{\epsilon B}{1-kn}\left[\frac{\partial\Psi}{\partial n}\frac{\partial}{\partial S} - \frac{\partial\Psi}{\partial S}\frac{\partial}{\partial n}\right]\nabla^2\Psi \tag{6.11}$$

$$C\nabla^2\nu = 2\nu + \nabla^2\Psi + \frac{\epsilon D}{1-kn}\left[\frac{\partial\Psi}{\partial n}\frac{\partial}{\partial S} - \frac{\partial\Psi}{\partial S}\frac{\partial}{\partial n}\right]\nu \tag{6.12}$$

where

$$\nabla^2 \equiv \frac{1}{1-kn}\left[\epsilon^2\frac{\partial}{\partial S}\left[\frac{1}{1-kn}\frac{\partial}{\partial S}\right] + \frac{\partial}{\partial n}\left[(1-kn)\frac{\partial}{\partial n}\right]\right] . \tag{6.13}$$

The complicated nature of equations (6.11) and (6.12) rules out any analytic solution. In order to obtain some information about the flow behaviour we resort to a regular perturbation method.

Consistent with our assumptions that the channel deviates only slightly from straightline we expect that the Poiseuille flow in a straight channel should remain the first approximation to the solution of (6.11) and (6.12). Hence we expand the solutions ψ and ν in a perturbation series

$$\psi(S,n) = \psi_0(n) + \epsilon\psi_1(S,n) + \epsilon^2\psi_2(S,n)+\cdots \qquad (6.14)$$

$$\nu(S,n) = \nu_0(n) + \epsilon\nu_1(S,n) + \epsilon^3\nu_2(S,n)+\cdots \qquad (6.15)$$

where $\psi_0(n)$, $\nu_0(n)$ are the Poiseuille flow in a straight channel. Substituting (6.14) and (6.15) into (6.11) and (6.12) and collecting terms we obtain the following equations for successive approximations (these are generated using MACSYMA (1983)):

ϵ^0 order:

$$A\,\frac{d^2\nu_0}{dn^2} + \frac{d^4\psi_0}{dn^4} = 0 \qquad (6.16)$$

$$C\,\frac{d^2\nu_0}{dn^2} = 2\nu_0 + \frac{d^2\psi_0}{dn^2} \qquad (6.17)$$

ϵ^1 order:

$$A \frac{\partial^2 \nu_1}{\partial n^2} + \frac{\partial^4 \Psi_1}{\partial n^4} = - A \left[\frac{d\nu_0}{dn}\right] + 2 \left[\frac{d^3 \Psi_0}{dn^3}\right] f(S) \tag{6.18}$$

$$C \frac{\partial^2 \nu_1}{\partial n^2} - C \frac{d\nu_0}{dn} f(S) = \frac{\partial^2 \Psi_1}{\partial n^2} + 2\nu_1 - \frac{d\Psi_0}{dn} f(S) \tag{6.19}$$

<u>ϵ^2 order:</u>

$$A \frac{\partial^2 \nu_2}{\partial n^2} - A \frac{\partial \nu_1}{\partial n} f(S) + \frac{\partial^4 \Psi_2}{\partial n^4} - 2 \frac{\partial^3 \Psi_1}{\partial n^3} f(S) + \left[- An \frac{d\nu_0}{dn} - 2n \frac{d^3 \Psi_0}{dn^3} \right.$$

$$\left. - \frac{d^2 \Psi_0}{dn^2} f^2(S)\right] = - B \left\{\frac{d^3 \Psi_0}{dn^3} \frac{\partial \Psi_1}{\partial S} - \frac{d\Psi_0}{dn} \frac{\partial^3 \Psi_1}{\partial n^2 \partial S} + \left[\frac{d\Psi_0}{dn}\right]^2 f'(S)\right\} \tag{6.20}$$

$$C \frac{\partial^2 \nu_2}{\partial n^2} - C \frac{\partial \nu_1}{\partial n} f(S) - Cn \frac{d\nu_0}{dn} f^2(S) = D \left\{\frac{d\Psi_0}{dn} \frac{\partial \nu_1}{\partial S} - \frac{\partial \nu_0}{\partial n} \frac{\partial \Psi_1}{\partial S}\right\}$$

$$- \frac{\partial \Psi_1}{\partial n} f(S) + 2\nu_2 - n \frac{d\Psi_0}{dn} f^2(S) + \frac{\partial^2 \Psi_2}{\partial n^2} . \tag{6.21}$$

The boundary conditions on Ψ are $\Psi(n = \pm 1) = \pm 1$ and $\frac{\partial \Psi}{\partial n}$ $(n = \pm 1) = 0$. There is, however, no universal agreement on the boundary conditions for $\underline{\nu}$. Many authors use the condition of no spin on the boundary. However, as pointed out by Ariman, Turk and Sylvester (1974), there is some evidence suggesting that particles could actually be rotating on the boundary, tumbling along the wall. A physically reasonable assumption under such circumstances would be to take microrotation on the boundary as equal to the local fluid angular velocity.

In the following we will investigate the effect of both kinds of boundary conditions on the flow.

6.3 Solution with No-Spin Condition

Imposing the boundary conditions

$$\Psi_0(n = \pm 1) = \pm 1, \; \frac{d\Psi_0}{dn}(n = \pm 1) = 0, \; \nu_0(n = \pm 1) = 0, \tag{6.22}$$

the solution to (6.16) and (6.17) is readily found to be

$$\nu_0(n) = gn + h \sinh(mn) \tag{6.23}$$

$$\Psi_0(n) = in + j\, n^3 + \ell \sinh(mn) \tag{6.24}$$

where

$$g = \frac{3m^2 \sinh(m)}{d}, \; h = \frac{3m^2}{d}, \; i = \frac{-[3m^2 \sinh(m) + 3m(Cm^2-2)\cosh(m)]}{d},$$

$$j = \frac{m^2 \sinh(m)}{d}, \; \ell = \frac{3(Cm^2-2)}{d};$$

$$d = 3(Cm^2-2)(\sinh(m) - m\cosh(m)) - 2m^2 \sinh(m)$$

and

$$m^2 = \frac{2-A}{C} = \frac{2\mu+\kappa}{\mu+\kappa} \frac{a^2 \kappa}{\gamma} .$$

This solution for a straight channel may be found in a number of works (see Radhakrishnama Charya (1977)).

It is worth observing that when the fluid becomes Newtonian,

$$\left.\begin{array}{l} g \to -1/2,\ h \to -\dfrac{3}{2\sinh(m)} \\[2ex] i \to 3/2,\ j \to -1/2,\ \ell \to 0 \\[2ex] Cm^2 \to 2,\quad A \to 0,\quad \nu \to 0 \end{array}\right\} \tag{6.25}$$

and we recover the result $\psi_0(n) = 3/2\ n - 1/2\ n^3$ as given by Van Dyke (1983).

Using the known expressions for $\psi_0(n)$ and $\nu_0(n)$, the ϵ^1 order equations (6.18) and (6.19) become

$$A\,\frac{\partial^2\nu_1}{\partial n^2} + \frac{\partial^4\psi_1}{\partial n^4} = [(Ahm+2\ell m^3)\cosh(mn)+Ag+12j]f(S) \tag{6.26}$$

$$C\,\frac{\partial^2\nu_1}{\partial n^2} = 2\nu_1 + \frac{\partial^2\psi_1}{\partial n^2} = [(Chm-\ell m)\cosh(mn)-3jn^2-i+Cg]f(S). \tag{6.27}$$

Again, using the limits (6.25), these equations reduce, in the case of Newtonian fluids, to $\dfrac{\partial^4\psi_1}{\partial n^4} = -6f(S)$, which agrees with Van Dyke (1983).

The solutions to (6.26) and (6.27) subject to homogeneous boundary conditions are found to be

$$\begin{aligned} \psi_1(S,n) = {} & n^2\left[\frac{w}{2} - \frac{Cq}{2m^2}\right] + n^4\left[\frac{t}{12} - \frac{q}{12m^2}\right] + \left[\frac{r}{m^2} - \frac{5Cp}{4m^4} + \frac{9p}{2m^6}\right]\cosh(mn) \\ & + \left[\frac{Cp}{2m^3} - \frac{p}{m^5}\right]n\sinh(mn) + \frac{Cm^2-2}{m^4}\,(k_1e^{mn}+k_2e^{-mn}) - k_3n^2 + k_4 \end{aligned} \tag{6.28}$$

$$\nu_1(S,n) = -\frac{n^2q}{2m^2} + \frac{npsinh(mn)}{2m^3} - \frac{5pcosh(mn)}{4m^4} + \frac{k_1e^{mn}+k_2e^{-mn}}{m^2} + k_3 \qquad (6.29)$$

where

$$p = \frac{1}{C}\,[Chm^3+\ell m^3+Ahm]f(S)$$

$$q = \frac{1}{C}\,[6j+Ag]f(S)$$

$$r = [\ell m-Chm]f(S)$$

$$t = 3j\ f(S)$$

$$w = [-Cg+i]f(S)$$

and k_1, k_2, k_3, k_4 are constants depending on the various parameters. Evaluation of these constants using MACSYMA shows that k_1 and k_2 have 10 terms each, k_3 and k_4 have 30 and 88 terms respectively when written out in terms of the basic parameters. They are not given as they are too lengthy to be written down.

We have not attempted to go to the next approximation, equations (6.20) and (6.21), as the expressions involved are too big.

6.4 Solution with Boundary Condition $\underline{\nu} = 1/2\ \underline{\nabla} \times \underline{u}$

The condition that the microrotation is equal to the local fluid angular velocity on the wall is given by $\underline{\nu} = 1/2\ \underline{\nabla} \times \underline{u}$ on the wall. In terms of the streamfunction Ψ this condition becomes

$$\nu = - 1/2 \nabla^2 \Psi, \quad n = \pm 1. \tag{6.30}$$

From this we readily obtain the following boundary conditions on ν_0, ν_1, ν_2:

$$\nu_0 = - 1/2 \frac{d^2 \Psi_0}{dn^2}, \quad n = \pm 1 \tag{6.31}$$

$$\nu_1 = - 1/2 \left\{ \frac{\partial^2 \Psi_1}{\partial n^2} - \frac{d\Psi_0}{dn} f(S) \right\}, \quad n = \pm 1 \tag{6.32}$$

$$\nu_2 = - 1/2 \left\{ \frac{\partial^2 \Psi_2}{\partial n^2} - \frac{\partial \Psi_1}{\partial n} f(S) - n \frac{d\Psi_0}{dn} f^2(S) \right\}, \quad n = \pm 1. \tag{6.33}$$

Solving (6.16) and (6.17) with the boundary conditions (6.32) and $\Psi_0(n = \pm 1) = \pm 1$, $\frac{d\Psi_0}{dn}(n = \pm 1) = 0$, we obtain the Poiseuille flow

$$\nu_0(n) = 3/2\ n \tag{6.34}$$

$$\Psi_0(n) = 3/2\ n - 1/2\ n^3. \tag{6.35}$$

It is interesting to note that the solution for $\Psi_0(n)$ is the same as for Newtonian fluids.

Using (6.34) and (6.35), the equations for second approximations, (6.18) and (6.19) becomes

$$A \frac{\partial^2 \nu_1}{\partial n^2} + \frac{\partial^4 \Psi_1}{\partial n^4} = \left[\frac{3}{2} A - 6\right] f(S) \tag{6.36}$$

$$C \frac{\partial^2 \nu_1}{\partial n^2} = \frac{\partial^2 \Psi_1}{\partial n^2} + 2\nu_1 + \frac{3C}{2} f(S) - \frac{3}{2} (1-n^2) f(S). \tag{6.37}$$

The solution to (6.36) and (6.37) with the boundary conditions (6.32) together with $\Psi_1(n = \pm 1) = 0$, $\frac{\partial \Psi_1}{\partial n} (n = \pm 1) = 0$ is found to be

$$\nu_1(S,n) = - \frac{n^2 p^1}{2m^2} + a_1 \cosh(mn) + a_2 \tag{6.38}$$

$$\begin{aligned} \Psi_1(S,n) = &- \frac{n^4}{8} f(S) - \frac{n^2 q^1}{2} + \left[\frac{n^4}{12m^2} - \frac{Cn^2}{2m^2}\right] p^1 \\ &+ \frac{Cm^2-2}{m^2} (a_1 \cosh(mn)) - a_2 n^2 + a_3 \end{aligned} \tag{6.39}$$

where

$$p^1 = \frac{3}{2C} (A-1) f(S)$$

$$q^1 = \frac{3}{2} (C-1) f(S)$$

$$a_1 = \frac{3m^2 f(S) + 2m^2 q^1 + 2Cp^1}{2Cm^4 \cosh(m)}$$

$$a_2 = \left[\frac{3}{4} \frac{\sinh(m)}{m\cosh(m)} - \frac{3}{2} \frac{\sinh(m)}{2Cm^3 \cosh(m)} - \frac{1}{4}\right] f(S)$$

$$+ \left[\frac{\sinh(m)}{2m\cosh(m)} - \frac{\sinh(m)}{Cm^3\cosh(m)} - \frac{1}{2}\right] q^1$$

$$+ \left[\frac{C\sinh(m)}{2m^3\cosh(m)} - \frac{\sinh(m)}{m^5\cosh(m)} - \frac{C}{2m^2} + \frac{1}{6m^2}\right] p^1$$

$$a_3 = \left\{\frac{3}{4}\frac{\sinh(m)}{m\cosh(m)} - \frac{3}{2}\frac{\sinh(m)}{Cm^3\cosh(m)} - \frac{3}{2m^2} + \frac{3}{Cm^4} - \frac{1}{8}\right\} f(S)$$

$$+ \left\{\frac{\sinh(m)}{2m\cosh(m)} - \frac{\sinh(m)}{Cm^3\cosh(m)} - \frac{1}{m^2} + \frac{2}{Cm^4}\right\} q^1$$

$$+ \left\{\frac{\sinh(m)}{2m^3\cosh(m)} - \frac{\sinh(m)}{m^5\cosh(m)} + \frac{1}{12m^2} - \frac{C}{m^4} + \frac{2}{m^6}\right\} p^1 .$$

Two quantities that are of prime importance in the channel flow are the longitudinal pressure gradient and the shear stress on the walls. From the streamwise momentum equation (6.5) we find that the non-dimensional longitudinal pressure gradient has the expression

$$-\frac{\partial p}{\partial s} = \left[-\frac{d^3\Psi_0}{dn^3} - A\nu_0\right] + \epsilon\left[-\frac{\partial^3\Psi_1}{\partial n^3} - A\nu_1 + n\left[\frac{d^3\Psi_0}{dn^3} + \frac{d^2\Psi_0}{dn^2}\right] f(S)\right]$$

$$+ \epsilon^2\left[-\frac{\partial^3\Psi_2}{\partial n^3} - B\frac{d^2\Psi_0}{dn^2}\frac{\partial\Psi_1}{\partial S} + B\frac{d\Psi_0}{dn}\frac{\partial^2\Psi_1}{\partial n\partial S} - A\nu_2 + \frac{d\Psi_0}{dn} f^2(S)\right.$$

$$+ nf(S) \frac{\partial^3 \Psi_1}{\partial n^3} + \frac{\partial^2 \Psi_1}{\partial n^2} f(S) \Bigg] + O(\epsilon^3). \qquad (6.40)$$

The non-dimensional shear stress t_{Sn} on the wall is found to be given by

$$t_{Sn} = \left[(1-A) \frac{d^2\Psi_2}{dn^2} - 2A\nu_0\right] + \epsilon\left[(1-A) \frac{\partial^2\Psi_1}{\partial n^2} - 2A\nu_1 + (1+A) \frac{d\Psi_0}{dn} f(S)\right]$$

$$+ \epsilon^2\Bigg[(1-A) \frac{\partial^2\Psi_2}{\partial n^2} - 2A\nu_2 + (1+A) \frac{d\Psi_0}{dn} nf(S)$$

$$+ (1+A) \frac{\partial\Psi_1}{\partial n} f(S)\Bigg] + O(\epsilon^3). \qquad (6.41)$$

We use these expressions and the solutions for the first and second approximations derived above to calculate numerical values of pressure gradient and wall stress in the next section.

	$\underline{\nu} = 0$ Boundary condition		$\underline{\nu} = 1/2\ \underline{\nabla} \times \underline{u}$ Boundary condition	
	ε^0	ε^1	ε^0	ε^1
$A = 0.1$	3.0237	$-0.0136\ f(S)$	3	$-0.0171\ f(S)$
$A = 0.3$	3.0719	$-0.0412\ f(S)$	3	$-0.05005\ f(S)$
$A = 0.5$	3.1211	$-0.0698\ f(S)$	3	$-0.08143\ f(S)$
$A = 0.7$	3.1716	$-0.0980\ f(S)$	3	$-0.1109\ f(S)$
$A = 0.9$	3.2231	$-0.1272\ f(S)$	3	$-0.1382\ f(S)$

Table 6.1. Centreline pressure gradient for C = 1

	$\underline{\nu} = 0$ Boundary condition		$\underline{\nu} = 1/2\ \underline{\nabla} \times \underline{u}$ Boundary condition	
	ε^0	ε^1	ε^0	ε^1
$A = 0.1$	-2.685	$-1.784\ f(S)$	-3	$-1.489\ f(S)$
$A = 0.3$	-2.064	$-1.363\ f(S)$	-3	$-1.464\ f(S)$
$A = 0.5$	-1.458	$-0.939\ f(S)$	-3	$-1.438\ f(S)$
$A = 0.7$	-0.864	$-0.562\ f(S)$	-3	$-1.408\ f(S)$
$A = 0.9$	-0.285	$-0.184\ f(S)$	-3	$-1.375\ f(S)$

Table 6.2. Shear stress on the wall n = 1 for C = 1

6.5 Numerical Results and Discussion

Much of the simplicity gained by introducing the concept of slow and slight variations in the channel flow of viscous fluids is lost in the case of micropolar fluids. For viscous Newtonian fluids the successive terms of the perturbation series could be determined by simple quadrature (see Van Dyke (1983)). For micropolar fluids, however, one needs to solve two coupled differential equations at each stage of approximation. There are at least four basic parameters describing micropolar flow and a multiplicity of various combinations of these parameters at each step rapidly blows up the amount of labour required. In principle, one can go to higher approximations using a computer program that manipulates algebraic symbols. We have used the symbol manipulation system MACSYMA to generate the successive perturbation equations (6.16)-(6.21), the expressions for pressure gradient and stresses and to solve the differential equations.

Here we have carried out the solution to the second approximation. Although we have made no assumptions about the smallness of the two micropolar Reynolds' numbers B and D, the inertial terms did not affect either the first or the second order solutions.

Table 6.1 gives the first and second approximations to $-\frac{\partial p}{\partial s}$ along the centreline for various values of A and fixed C = 1, computed using the two kinds of boundary conditions.

The corresponding result for Newtonian fluids is a value 3 for ϵ^0 term and no ϵ^1 terms. We observe that if we use the no-spin boundary condition, the pressure gradient down the centreline is increased for micropolar fluids. The secondary effect of curvature is to reduce the pressure gradient on positively curving portions of the channel and increase it on negatively curving portions, the amount of increase or decrease being larger for fluids of higher micropolarity. In contrast, for the second type of boundary condition the ϵ^0 terms are unaffected by micropolarity while the secondary effects are exactly as above although slightly enhanced in magnitude. Table 6.2 gives the first and second approximations to the shear stress on the wall $n = 1$ for various values of A and fixed $C = 1$.

For the wall $n = -1$, the ϵ^0-order terms are opposite in sign to those given above while ϵ^1-order terms are the same. The corresponding results for Newtonian fluids are $-3-2\epsilon f(S)$ for the wall $n = 1$ and $3-2\epsilon f(S)$ for $n = -1$. We observe a substantial reduction in shear stress on the walls for micropolar fluids using the no-spin boundary condition. On positively curving portions of the upper wall the shear stresses are increased due to secondary curvature effects while it is decreased on negatively curving portions, the increase or decrease becoming smaller with increasing micropolarity. The opposite effect applies to the lower wall. We note that the secondary effects may be explained by what Wang (1980) calls "the streamlines taking a less tortuous path than the centreline." In sharp contrast, the second type of boundary condition leads to less pronounced secondary effects and no reduction of shear stress for the first approximation.

Finally we would like to make some concluding remarks regarding the relevance of this study to the field of blood rheology. The results in this chapter are of some interest to the investigators in blood rheology. It is known that the behaviour of blood as measured by viscometric techniques is markedly non-Newtonian; for example the presence of a yield stress and the dependence of apparent viscosity on shear rate despite the fact that blood plasma is a Newtonian fluid tends to indicate non-Newtonian character. It is believed that the suspended blood cells are responsible for the observed non-Newtonian nature of blood rheology through such mechanisms as erythrocyte deformation and erythrocyte aggregation. Another interesting anomalous viscous property exhibited by blood is the Fahraens-Lindqvist effect - the apparent viscosity decrease with decreasing tube diameter and/or cell volume fraction and the inverse variation with shear rate. It is known that artherosclerosis which causes heart attacks and strokes among other things is related to deposition of fat on artery walls. The haemodynamic forces causing these depositions are not fully understood (see Roach, 1980). In this context, a better understanding of stresses on the walls of curving conduits and channels would be vital. The present study has been motivated by the belief that despite its limitations, the micropolar fluid model would be better suited to describe blood flow in meandering channels because it allows for certain micromotions within the fluid. Such models would be of increasing importance for understanding the behaviour of blood and physiological fluids in artificial organs. More experimental and theoretical work is required in this area.

References for Part II

1. Ariman,T., M.A.Turk and N.D.Sylvester, 1973. Microcontinuum Fluid Mechanics - A review. *Int. J. Engng. Sci.* 11, 905-930.

2. Bugliarello,G. and D.Sevilla, 1970. Velocity Distribution and other characterestics of steady and pulsatile blood flow in fine glass tubes. *Biorheology*, 7, 85-95.

3. Chow,J.C.F. and K.Soda, 1972. Laminar flow in tubes with constriction. *Phys. Fluids*, 15, 1700-1706.

4. Cowin,S.C., 1974. Theory of polar fluids. *Adv. in Appl. Mech.* 14, 279-347.

5. Dyer,R.H. and D.E.Edmunds, 1961. A uniqueness theorem in Magnetohydrodynamics. *Arch. Rat. Mech. Anal.*, 8, 254-262.

6. Eringen,A.C. 1964. Simple microfluids. *Int. J. Engng. Sci.*, 2, 205-222

7. Eringen,A.C. 1966. Theory of micropolar fluids. *J. Math. Mech.*, 16, 1-18.

8. Kopashchikov,V., N.P.Migun and P.P.Prokhorenko, 1983. Experimental determination of material micropolar coeffficients. *Int. J. Engng. Sci.*, 21, 401-411.

9. Lee,J.S. and Y.C.Fung, 1970. Flow in locally constricted tubes at low Reynolds numbers, *J. Appl. Mech.*, 37, 9-16.

10. MACSYMA 1983. MACSYMA Reference Manual, SYMBOLICS, Inc., Cambridge, Massachusetts.

11. Olmstead,W.E. and S.R.Majumdar, 1983 Fundamental Oseen solution for the two dimensional flow of a micropolar fluid. *Int. J.Engng.*

Sci., 21, 423-430.

12. Olmstead,W.E. and A.K.Gautesen, 1976. Integral representation and the Oseen flow problem. *Mechanics Today* (Edited by S.Nemat-Nasser), Vol.3, Pergamon Press, New York.

13. Panico,V.D., 1978. Fundamental solution of the linearised Navier-Stokes equation and applications. Ph.D. thesis, Northwestern University, Evanston, Il.

14. Radhakrishnama Charya, 1977. Flow of a micropolar fluid through a constricted channel. *Int. J. Engng. Sci.* 15, 719-725.

15. Ramkissoon,H., 1984. Boundary Value probems in microcontinuum fluid mechanics. *Quart. Appl. Math.* 42, 1-13.

16. Ramkissoon,H., 1975. Fundamental solutions and some flow problems in micropolar fluids. Ph.D. thesis, University of Calgary.

17. Roach,M.R., 1980. Canadian Symposium on fluid dynamics, Proceedings. University of Calgary.

18. Smith,A.C. and G.S.Guram, 1974. Kampé de Fériet and Taylor motions of a micropolar fluid. *Lett. Appl. Engng. Sci.*, 2, 279-292.

19. Serrin,J. 1959. On the uniqueness of compressible fluid motions. *Arch. Rat. Mech. Anal.*, 3, 271-288.

20. Stokes,V.K., 1984. *Theories of Fluids with Microstructure.* Springer Verlag, New York.

21. Turk,M.A., N.D.Sylvester and T.Ariman, 1973. On the pulsatile flow of blood. *Trans. Soc. Rheol.*, 17, 1-24.

22. Van Dyke,M., 1983. Laminar flow in a meandering channel. *SIAM J. Appl. Math.*, 43, 696-702.

23. Wang,C.Y., 1980. Flow in narrow curved channels. *J. Appl. Mech.*, 47, 7-10