#### THE UNIVERSITY OF CALGARY

Bifurcations and Chaos in the Froude Pendulum

by

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#### A THESIS

# SUBMITTED TO THE FACULTY OF GRADUATE STUDIES IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE IN ENGINEERING

DEPARTMENT OF MECHANICAL AND MANUFACTURING

### ENGINEERING

### CALGARY, ALBERTA

### DECEMBER, 1999

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0-612-49682-1



#### ABSTRACT

The Froude pendulum is a classical nonlinear mechanical system exhibiting friction induced, self excited oscillations. This system has not been studied completely from the analytical viewpoint. The nonlinearity arises from a cubic damping term and the sine function of the displacement in the equations. In this work, the averaging technique used by Sanders and Cushman is applied to the Froude pendulum and the planar bifurcations of the parameters are studied. This is achieved by averaging over orbits in the phase space of the unperturbed hamiltonian, deriving the Picard-Fuchs and Riccatti equations and numerically solving the latter. The bifurcation diagram enables the identification of limit cycles and various phase portraits. In the non-autonomous case, a Melnikov analysis yields a criterion for the onset of chaos. Thus this work provides interesting insights into the analytical aspects of the motion of the Froude pendulum.

#### ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my supervisor, Dr. M. C. Singh for his unfailing encouragement and support. I would also like to thank Prof. R. H. Cushman, without whose support and guidance, this work would not have been possible.

Thanks are due to Dr. M. Epstein and Dr. K. L. Chowdhury of the Department of Mechanical and Manufacturing Engineering for stimulating discussions and constant encouragement.

I would like to thank Mr. Khee Teek Wong and Mr. Marc Wrubelski, system administrators in the departments of Mechanical Engineering and Mathematics and Statistics respectively for their assistance. I also thank Ms. Nareeza Khan and Ms. Lynda Banach for all the support provided by them.

I have an immense debt of gratitude to my friends Srinivas Chari, Amitava Nandi and Pankaj Dhar who have stood firmly behind me at all times.

I am grateful beyond the power of mere words of expression to my parents and to my wife, for being there for me in every way possible.

Last but certainly not the least, I am thankful to the Almighty, whose blessings are behind the success of every endeavour.

iv

# Contents

1	Intr	oduction	1
2	Nor	nlinear Systems	11
	2.1	Introduction	11
	2.2	Linear Systems	16
	2.3	Critical Points And Their Classification	19
		2.3.1 Linearization	20
	2.4	Stability Of Solutions About Critical Points	26
	2.5	Nonlinear Systems	26
	2.6	Conclusion	29
3	Bifi	rcations and Chaos in Nonlinear Systems	30
	3.1	Introduction	30
	3.2	Cyclic Attractors (Limit cycles)	31

	3.3	Chaotic Attractors	34
	3.4	Bifurcations	38
		3.4.1 Local Bifurcations	39
		3.4.2 Global Bifurcations	41
	3.5	Averaging	42
4	The	Froude Pendulum	45
	4.1	Introduction	45
	4.2	The Froude Pendulum	47
	4.3	Analysis of The Froude Pendulum	49
	4.4	Conclusion	52
5	Bift	rcations and Limit Cycles in The Froude Pendulum	53
	5.1	Introduction	53
	5.2	The Froude Pendulum - A perturbation of the mathematical	
		pendulum	54
	5.3	The Averaged Equation	56
	5.4	The Picard-Fuchs and Riccatti Equations	60
	5.5	Plot of $\xi^0(h)$	66
	5.6	The Bifurcation Diagram	67

÷

vi

	5.7	Stability Of The Equilibrium Points	69
	5.8	The Phase Portrait and Limit Cycles from the Bifurcation	
		Diagram	70
6	Me	lnikov Analysis	92
	6.1	Introduction	92
	6.2	Perturbed Hamiltonian Systems	94
	6.3	The Melnikov Integral	97
	6.4	Melnikov Analysis of the Froude Pendulum	102
	6.5	Illustration of the Melnikov Analysis	107
7	Cor	nclusions and Further Work 1	17
	7.1	Conclusions	117
	7.2	Further Work	120

vii

# List of Figures

2.1	The Simple Pendulum	14
2.2	The Node	22
2.3	The Saddle Point	23
2.4	The Focus	24
2.5	The Centre	25
3.1	A Limit Cycle	32
3.2	Divergence from adjacent initial conditions - Chaos	36
4.1	The Froude pendulum - the rotating shaft is connected to an	
	engine	48
5.1	The Phase Space of the Mathematical Pendulum	58
5.2	The Plot of $\xi(h)$ vs $h$	67
5.3	The Bifurcation Diagram	68

5.4	Region I: Upper Saddle connection. $\epsilon = 0.0001$ . $a = 1$ . $c = 5$ .	74
5.5	Region I': Lower Saddle Connection, $\epsilon = 0.0001$ . $a = 1, c = -5$	75
5.6	Region II: Double Saddle Connection, $\epsilon = 0.0001, a = 1, c = 1$	76
5.7	Region II': Lower saddle connection, $\epsilon = 0.0001, a = 1, c = -1$	77
5.8	Region III: Upper Saddle Connection. $\epsilon = 0.0001$ . $a = -0.5$ , $c =$	
	4	78
5.9	Region III': Lower Saddle connection, $\epsilon = 0.0001, a = -0.5, c =$	
	-4	79
5.10	Limit Cycle for $a = -0.81$ , $c = 0$ , $\epsilon = 1$	80
5.11	Region IV: Limit Cycle. $\epsilon = 1$ . $a = -0.85$ . $c = 0$	81
5.12	Region IV: Double saddle connection. $\epsilon = 0.0001, a = -0.5, c =$	
	1	82
5.13	Region IX: Phase portrait for $\epsilon = 0.0001, a = -3, c = 0$	83
5.14	Region VIII: Phase portrait for $\epsilon = 0.0001, a = -3, c = 4$	84
5.15	Region VIII': Phase portrait for $\epsilon = 0.0001, a = -3, c = -4$ .	85
5.16	Phase portrait for $\epsilon = 1$ , $a = 1$ , $c = 0$ . A Node	86
5.17	The Node at $a = 1$ , $c = 0$ , for $\epsilon = 1$	87
5.18	Region V: Limit Cycle for $a = -1$ , $c = 0$ , $\epsilon = 1$	88
5.19	Region IV: Limit cycle for $a = -0.5$ , $c = 0, \epsilon = 1$	89

5.20	Limit Cycle for $a = 0.8$ . $c = 0.\epsilon = 1$
5.21	Candidate for Chaotic Attractor $a = c = \epsilon = F =$
	$\omega = \ldots \ldots \ldots 91$
6.1	Phase Space of Unperturbed Hamiltonian System 95
6.2	Phase Space of the perturbation of a Hamiltonian System 95
6.3	Homoclinic Tangle and the Melnikov Distance
6.4	The Melnikov Distance - The dashed curve represents the un-
	perturbed separatrix
6.5	Displacement-time diagram. No Chaos for $a = 1, b = 0.075$ . $Q =$
	0.0001. $h = 1, \ \omega = \frac{2}{\pi}$ . $F = 1.5$ . Initial conditions for two so-
	lutions $\phi(0) = 3$ and $\phi(0) = 3.2$
6.6	Phase Space. No Chaos for $a = 1$ , $b = 0.075$ , $Q = 0.0001$ , $h =$
	1. $\omega = \frac{2}{\pi}$ . $F = 1.5$ . Initial conditions for two solutions $\phi(0) =$
	3 and $\phi(0) = 3.2$
6.7	Displacement-time diagram. Indications of Chaos for $a =$
	1. $b = 0.075$ . $Q = 0.0001$ . $h = 1$ . $\omega = \frac{2}{\pi}$ . $F = 2.5$ Initial
	conditions for two solutions $\phi(0) = 3$ and $\phi(0) = 3.2$

x

6.10 Phase Space. No Chaos for F = 0.5, b = 0.075, Q = 0.0001, h =

1. 
$$\omega = \frac{2}{\pi}$$
.  $a = 0.5$  Initial conditions for two solutions  $o(0) = 3$ 

- 6.11 Displacement-time diagram. Indications of Chaos for F = 0.5. b = 0.075. Q = 0.0001. h = 1,  $\omega = \frac{2}{\pi}$ , a = 0.1 Initial conditions for two solutions  $\phi(0) = 3$  and  $\phi(0) = 3.2$ .... 115
- 6.12 Phase Space. Indications of Chaos for F = 0.5, b = 0.075, Q =

0.0001. h = 1.  $\omega = \frac{2}{\pi}$ , a = 0.1 Initial conditions for two solu-

xi

# Chapter 1

# Introduction

Nonlinear dynamical systems have attracted a great deal of attention since the early years of this century. Beginning with the pioneering work of Poincaré [33, 34] and followed by the seminal work of Andronov [1], Lyapunov [24, 25], Birkhoff [3, 4] and others, the theoretical developments gathered momentum with the fundamental contributions of Smale [37], Arnold [2] and others.

Even though the importance of applying the new results to problems in physics and engineering was recognised, the action was mostly confined to the theatre of mathematics. This situation changed drastically with the discovery of what has come to be known as *deterministic chaos*. Starting with the now famous discovery of Lorenz [23] in 1963 in his studies involving a simplified model of fluid convection related to the atmosphere, this phenomenon made its presence felt. almost in an ubiquitous fashion. in a broad class of non linear systems which model the real world.

Scientists were faced with the stark reality of unpredictability and extreme sensitivity to initial conditions even in the sacred territory of Newtonian mechanics [20]. This, inevitably, has led to a paradigm shift in the approach to nonlinear systems from the engineering point of view into an era when terms like experimental error are viewed through the prism of caution. The crucial insight that has been gained out of these intense efforts has been that nonlinear systems demonstrate fundamentally different behaviour in comparison with linear systems and demand treatment on a different footing. Hence, every linearization or neglect of nonlinear terms of any order in a problem needs rigorous justification and a cavalier approach in this respect can lead to highly erroneous conclusions.

Having observed that linear and nonlinear systems are fundamentally different, simultaneously it must be noted that there exists a remarkable correspondence between the two. There exists a battery of powerful theorems in the arsenal of the mathematician, such as the Hartman-Grobman theorem [15] and the Stable Manifold theorem [6], which affirm the correspondence between a linear system and its nonlinear cousin. In fact, it is this relationship which allows one to draw meaningful conclusions about a nonlinear system under a linearization. It is again, this feature that vitalizes the attack on nonlinear systems and makes the whole exercise worthwhile.

Thus, the series of developments in both the theoretical and observational aspects of nonlinear systems has ushered in a revolution in our understanding of physical phenomena, which ranks on par with the other two towering achievements of the human intellect in the present century, viz. the relativistic and quantum revolutions. But, perhaps, the omnipresent nature of nonlinear phenomena in the world around us, ranging from biological and social systems to quantum field theories and cosmological models make this area outstandingly unique.

It is quite pertinent now to examine the implications of these developments to engineering. Engineers have, for long, encountered apparently random effects in a wide variety of systems. The classical examples are mechanical, electrical, fluid, optical and control systems. In mechanical systems, nonlinear elastic or spring elements, friction and damping effects and the like contribute to the nonlinearities. In electrical circuits, nonlinear resistive, inductive or capacitative circuit elements and electromagnetic fields are a prime source. Turbulence, a purely nonlinear phenomenon is well known in fluid mechanics and its applications. The importance of nonlinear effects in servomechanisms and feedback control cannot be overemphasized.

Thus. the presence of nonlinearities in engineering applications has been long recognized. The interesting question is the implication of the theoretical and computational developments for these applications. It remains a fact that due to limitations in our understanding of nonlinear systems and also due to the inadequate percolation of ideas from the pure sciences to engineering, often these nonlinear effects were either ignored or swept under the rug during design or analysis. That this led sometimes to disastrous consequences is an unfortunate but valuable lesson of engineering history. The collapse of the Tacoma suspension bridge in the United States under self-excited oscillations is a case in point.

A more accurate understanding of nonlinear phenomena and application of the new developments to engineering problems is highly desirable. Even though the elimination of nonlinear effects in engineering problems borders on the impossible, a better insight undeniably leads to better analysis, design and control. c

Guided by this philosophy. this thesis examines the nonlinear effects in a classical system that exhibits friction induced, self-excited oscillations, viz. the Froude pendulum. We shall study this system in detail in the subsequent chapters but it is perhaps appropriate here to highlight the features of this system. The oscillations in the Froude pendulum are caused by friction. Thus this pendulum serves as an effective model in the analysis of friction induced motion. That, friction effects, albeit being of crucial import, have not been exhaustively explored in any approach to mechanics, adds to the mystery. The two features, which contribute to the nonlinear effects in this system are a cubic damping term (arising out of friction effects) and the sine function in the equations. In our analysis, we confront these terms as such, making no attempt to linearize them. That an effective analysis can be carried out and meaningful conclusions drawn with this approach is a highlight of the present work.

The Froude pendulum has a considerably long history. It has been long recognized as an interesting mechanical system and has found mention in some of the classical treatises in nonlinear oscillations [28, 5]. It has also been treated as a paradigm for nonlinear friction in oscillatory systems [26]. But the treatment of this system in these works is far from being complete. With the possible exception of the work of the Soviet school, it does not seem to have enjoyed the extent of attention it deserves, from the analytical point of view. But recent work [9, 8] has drawn attention to the Froude pendulum and this served as the main motivation for the present work.

On the other hand, interesting strides have been taken in the development of averaging techniques applied to nonlinear differential equations. The exposition of Sanders and Verhulst [36] is a good survey of this area. Of particular interest in this context is the work of Sanders and Cushman [35] which develops and applies a unique averaging technique to the Josephson equation. In that work, the Josephson equation is treated as a perturbation of the mathematical pendulum. Averaging is then carried out on the system. Two appropriate functions are defined and the averaged equation is then studied using the properties of these functions. This leads to an interesting bifurcation picture and as a consequence, the limit cycles are classified and the entire phase portrait is generated for the Josephson equation.

The crucial element in the present work is the observation that the Froude pendulum too, can be treated as a perturbation of the mathematical pendulum, a well-understood hamiltonian system. It then becomes feasible to study the pendulum using the geometrical methods developed by Sanders and Cushman [35]. This leads to a bifurcation analysis and consequently to a classification of limit cycles and a clear view of the phase portrait.

The Froude pendulum is also found to be interesting from the point of view of chaos. Despite the amazing amount of work in the area of deterministic chaos. to this day, very few systems exist as paradigms for chaotic behaviour. The standard list of these systems, which includes the Lorenz system [23], the Van Der Pol oscillator [40] and the Duffing equation [11], almost exhausts the number of systems that have been extensively investigated from the chaos viewpoint. It is noteworthy that none of the above systems demonstrates friction induced, self-excited, oscillations. Also, none of the equations, which model these systems has a cubic damping term. In the work of Dai and Singh [9], it was shown that the Froude pendulum can behave chaotically.

Very few analytic criteria exist for predicting the onset of chaos in a nonlinear dynamical system. But among the techniques available, the Melnikov analysis [27] is a powerful method. This method shall be dealt with in detail in a later chapter. Suffice to say here that this yields an analytic criterion for the transversal intersection of the unstable and stable manifolds about a hyperbolic critical point, which leads to the creation of a homoclinic tangle and chaos.

In the present work, the Melnikov method is applied to the Froude pendulum in the non-autonomous case and a condition for the onset of homoclinic chaos is obtained. This condition is an inequality involving the parameters in the system viz. the damping coefficient, the stiffness coefficients and the amplitude and the frequency of the forcing function. Thus, respecting the inequality during design shall eliminate the possibility of chaos in the system.

The thesis is set as follows:

Chapter 2 introduces nonlinear systems in general. We survey the general properties using some examples. We also examine the important properties of linear systems, the process of linearization, and the Hartman-Grobman and Stable Manifold theorems [14] that establish the correspondence between linear and nonlinear systems. The concepts related to critical points and the stability types of critical points are introduced.

Chapter 3 is a deeper examination of the characteristics of nonlinear systems and their behaviour. Properties such as existence of limit cycles are studied. We also look into the interesting phenomena of bifurcations and chaos. The averaging technique, which is extremely useful in the analysis of nonlinear systems, is surveyed. This chapter focuses on the phenomena that we study in the context of the Froude pendulum.

Chapter 4 introduces the Froude pendulum. The physical system and the context in which it arises are described. The equations governing this system are established and the stage is set for the analysis that follows.

Chapter 5 forms the core of the thesis. Here, we apply the averaging technique to the autonomous Froude pendulum. The chapter begins by establishing that the Froude pendulum can be viewed as a perturbation of the mathematical pendulum. After appropriate scaling of the equation of the Froude pendulum. we identify two system parameters. We study the bifurcation phenomenon with respect to these parameters. The next step is averaging. This is carried out by defining two functions and then setting up Riccatti equations for these functions. These functions are then integrated numerically and the results are used in plotting the bifurcation diagram for the system. Further, numerical plots based on this diagram show the interesting phase space behaviour.

Chapter 6 deals with the Melnikov analysis. The method is introduced first in the framework of a general dynamical system and then applied to the non-autonomous Froude pendulum. The criterion for onset of chaos is then obtained. Further, we plot the phase portrait using numerical values for the system parameters obtained from the analysis and show that chaos does exist in a region where it is expected. This confirms the validity of the analysis.

Chapter  $\overline{i}$  highlights the conclusions and explores the possible directions of further research in this area.

The numerical work associated with this thesis and the graphs have been done using the software packages Mathematica © and MAPLE ©.

# Chapter 2

# Nonlinear Systems

### 2.1 Introduction

As we have mentioned in Chapter 1, the objective of this thesis is the analysis of a nonlinear, oscillatory, engineering system, viz. the Froude pendulum. This pendulum being a classical nonlinear dynamical system, techniques from the theory of nonlinear differential equations have to be applied to this system in order to achieve our objective. To this end, we survey some of the useful concepts and results from the mathematical theory in this chapter. The material presented here is available in all the standard treatments of nonlinear differential equations and nonlinear mechanics such as [14, 21, 31, 19]. Linear systems are well known in engineering. Due to the intimate relationship and the unique correspondence between a nonlinear system and its linear counterpart, any discussion of the former has to include a treatment of the latter as well. Hence, we also discuss linear systems in this chapter. In many practical considerations, the nonlinear system under question is transformed to a linear system via an approximation. This procedure, termed linearization deserves attention and hence also enters our discussion.

The theory of dynamical systems, which has found extensive applications in engineering in recent times. is the study of mathematically and physically interesting systems with respect to a parameter termed 'time'. Since differential equations offer the most convenient framework for such a study, the analysis of a dynamical system reduces to the study of the corresponding system of differential equations. In a discrete case, this goes over to the study of the associated map. Indeed, in most modern treatments, a dynamical system is identified with the differential equation or the map. Thus, the analysis of the physical system reduces to the study of the differential equation governing its evolution.

It follows from Newtonian mechanics that oscillatory mechanical systems are modeled using second order differential equations. The classical model in this context is that of the simple harmonic oscillator represented by a simple pendulum. Hence we shall introduce the essential features of the theory using this example.

Consider the simple pendulum (Fig 2.1).

The system comprises an inextensible string of length L. pivoted at point O. and carrying a bob of mass m. which is free to swing in the plane of the paper. In order to formulate the exact equation of motion, we consider the pendulum in a displaced position. as shown, where the angle x designates the deviation from the vertical equilibrium position. As shown, the forces on the mass are the vertical gravitational force mg and the tension T in the string.

Given this setting, neglecting frictional and other dissipative forces, the equation of motion of the pendulum can be written. applying Newton's second law as

$$mL\ddot{x} + mg\sin x = 0 \tag{2.1}$$

which can be rearranged as

$$\ddot{x} + k^2 \sin x = 0 \tag{2.2}$$

where  $k = \sqrt{\frac{q}{L}}$ 



The Simple Pendulum

Figure 2.1: The Simple Pendulum

Expanding  $\sin x$  as a power series, one obtains

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \dots \tag{2.3}$$

Substituting (2.3) into (2.2) we obtain

$$\ddot{x} + k^2 \left[ x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right] = 0$$
(2.4)

It is obvious that (2.4) is a nonlinear differential equation due to the presence of  $x^3$  and the higher order terms. It is also well known that, considering all the terms. (2.4) has closed form solutions only in terms of the jacobian elliptic functions. But, for the moment, we shall assume that the angle of oscillation is small. This allows us the approximation  $\sin x \approx x$  for small x and this leads us to the linear differential equation

$$\ddot{x} + k^2 x = 0 \tag{2.5}$$

It is a crucial step that we have taken here, with this assumption. We have used an approximation to linearize the system. Further, if one includes linear damping effects in the system and considers a pendulum driven by a harmonic forcing function, the equation of motion can be written as

$$\ddot{x} + k^2 x + c \dot{x} = F \cos \omega t \tag{2.6}$$

where c is the damping coefficient. F and  $\omega$  the amplitude and frequency of the forcing function respectively. The equation (2.6) is the basic equation for

periodic motion. We shall return to this equation as well as modifications of it. in detail at a later stage, but for the present, we consider (2.5) in order to study a linear system and to develop the theoretical framework.

### 2.2 Linear Systems

The equation (2.5) represents a simple linear oscillatory mechanical system. the undamped. unforced. harmonic oscillator. This system is readily integrable in terms of the standard functions and is conservative (hamiltonian).

The general solution to (2.5) can be written down as

$$x(t) = A\sin kt + B\cos kt \tag{2.7}$$

where A, B are arbitrary constants of integration. Now let us consider a more general linear system

$$\dot{x} = A \ x, \ x \in \mathbf{R}^{\mathbf{n}} \tag{2.8}$$

We note that x is a vector valued function with n components and. A is an  $n \times n$  matrix with constant coefficients. A solution of (2.8) is a vector valued

function  $x(x_0, t)$  depending on time t and the initial condition  $x(0) = x_0$ . Fundamental theorems of differential equations theory [13, 2] guarantee the existence and uniqueness of solutions to (2.8), for all  $t \in \mathbf{R}$  and  $x_0 \in \mathbf{R}^n$ . It is appropriate to mention here that such global existence of solutions in time does not, in general, hold for nonlinear systems.

However, for the linear system (2.8), the solution can be written down as

$$x(x_0, t) = e^{tA} x_0 (2.9)$$

where  $e^{tA}$  is the matrix obtained by exponentiating A. defined by the convergent series  $e^{tA} = [I + tA + \frac{t^2A^2}{2!} + \dots]$ . A general solution to the system (2.8) may be obtained by a linear superposition of 'n' linearly independent solutions  $x^1(t), x^2(t), \dots, x^n(t)$  as

$$x(t) = \sum_{j=1}^{n} c_j x^j(t)$$
 (2.10)

We note here that the superposition principle that led to the general solution is unique to linear equations. If A has 'n' linearly independent eigenvectors  $v^{j}. j = 1, 2, ..., n$ , then  $x^{j}(t) = e^{\lambda_{j}t}v^{j}$  where  $\lambda_{j}$  is the eigenvalue associated with  $v^{j}$ . This provides a basis set in the space of solutions.

Referring back to (2.9), we can consider  $e^{tA}$  as a mapping of  $\mathbb{R}^n$  onto itself. In other words,  $e^{tA}$  maps point  $x_0$  to  $x(x_0, t)$  after time 't'. Thus, the operator  $e^{tA}$  defines a flow on  $\mathbb{R}^n$  and this flow is generated by the vector field Ax.

If we look at the flow as the set of all solutions to the system (2.8), those solutions that lie in the linear subspaces spanned by the eigenvectors are invariant under the flow. For example, if  $v^j$  is a real eigenvector of A, then a solution based at a point  $c_j v^j \in \mathbb{R}^n$  remains on  $span\{v_j\}$  for all time, where  $span\{v_j\}$  is the vector space for which  $v_j$  is the set of basis vectors and so on. If this property is satisfied (that is, points in a certain subspace remain in the same subspace for all time under the flow), then the subspace is called an invariant subspace of the flow. Thus, the eigen spaces of A are invariant subspaces of the flow.

This leads to a classification of the subspaces spanned by the eigenvectors as follows:

- 1. Stable subspace,  $E^s = span\{v^1, ..., v^{n_s}\}$
- 2. Unstable subspace  $E^u = span\{u^1, ..., u^{n_u}\}$
- 3. Centre subspace,  $E^c = span\{w^1, ..., w^{n_c}\}$

where  $v^i$ ,  $u^j$ ,  $w^k$  are eigenvectors with negative, positive and zero real parts for eigenvalues. respectively. Also,  $n = n_s + n_u + n_c$ , where n is the dimensionality of the eigenspace.

We also note here that solutions in  $E^s$  exhibit exponential decay (monotonic or oscillatory), those in  $E^u$  grow exponentially and  $E^c$  is characterized by solutions which do neither.

The above classification provides the framework for the two important theorems for nonlinear systems that we take up in the following section.

### 2.3 Critical Points And Their Classification

In the analysis of dynamical systems, the idea of critical (equilibrium) points plays a key role. As we shall see in the sequel, these are points in phase space which represent solutions to the given equation for all times and the constant nature of these equilibria suggests these as good starting points for the study of the (often) complex behaviour in their vicinity. Also, the phenomenon of asymptotic convergence of nearby solutions to these points leads us to name them as attractors. The stability of solutions in the neighborhood of the critical points is another issue which adds to the importance of the analysis of equilibrium points. Consider a dynamical system

$$\dot{x} = f(x) \tag{2.11}$$

A point x = a such that f(a) = 0 is called a critical point (equilibrium point) of the system. We note that a critical point corresponds to an equilibrium solution. since x(t) = a satisfies the equation for all time. It is also useful to recall the fundamental uniqueness theorem [13, 2], which implies that there exists a unique solution curve corresponding to any given point in phase space. As a consequence, an equilibrium solution can never be reached by other solutions in finite time.

### 2.3.1 Linearization

Before advancing to the analysis of critical points and their classification, we show. in general. how nonlinear systems can be linearized. Consider (2.11). Assuming the existence of a Taylor's series expansion for f(x), in the neighbourhood of the critical point x = a, we write

$$\dot{x} = \left. \frac{\partial f(x)}{\partial x} \right|_{x=a} (x-a) + higher order terms.$$

The linearized equation is, then,

$$\dot{x} = \frac{\partial f(a)}{\partial x}(x-a)$$

We note here, once again, that the justification of this process of linearization yielding meaningful results in the case of nonlinear systems is a consequence of the Hartman-Grobman and Stable Manifold theorems [14] referred to in section 2.5 (pp. 27-28) of the thesis.

For simplicity, the point 'a' is shifted to the origin of phase space and putting  $\bar{x} = x - a$  yields

$$\dot{\bar{x}} = \frac{\partial f(a)}{\partial x} \bar{x}$$

If we abbreviate  $\frac{\partial f(a)}{\partial x} = A$ , an  $n \times n$  matrix with constant coefficients, and omit the bar over x. we get

$$\dot{x} = Ax$$

This is the system that we shall be dealing with. The characteristic equation for this system is

$$det(A - \lambda I) = 0 \tag{2.12}$$

Let the eigenvalues be denoted by  $\lambda_1, \lambda_2, ..., \lambda_n$ .

We now classify the critical points based on the nature of these eigenvalues at the points, assuming a two dimensional case, for simplicity. In this case, we will have only two eigenvalues,  $\lambda_1$  and  $\lambda_2$ . <u>.</u>-



Figure 2.2: The Node

#### Case 1 The Node

The eigenvalues are real and have the same sign. If  $\lambda_1 \neq \lambda_2$ , we have parabolic orbits in the phase space (Fig. 2.2).

This type of critical point is called a node. If  $\lambda_1$ ,  $\lambda_2 \leq 0$  we have an attractor while  $\lambda_1$ ,  $\lambda_2 \geq 0$  implies a repeller. If  $\lambda_1 = \lambda_2$ , the orbits are straight lines through the origin.



Figure 2.3: The Saddle Point

#### Case 2 Saddle Point(Hyperbolic Critical Point)

In this case, the eigenvalues are real and have different signs. The behaviour of the orbits is hyperbolic (Fig. 2.3 and there exist two solutions which converge to the point as  $t \to \infty$  and two solutions with the same property for  $t \to -\infty$ . The first two are called stable manifolds of the saddle point while the other two are called unstable manifolds.

#### Case (3): The Focus

The eigenvalues are complex conjugate, the orbits spiral in or out, de-



Figure 2.4: The Focus

pending on the sign of their real parts. with respect to the critical point (Fig. 2.4) and it is called a focus. In the case of an inward spiral, the point is an attractor and a repeller in the other case.

Case (4): The Centre

If the eigenvalues are pure imaginary, the point is called a centre. The orbits in the phase space are circles centred about the critical point (Fig. 2.5). The point is, obviously, not an attractor in this case.



Figure 2.5: The Centre
#### 2.4 Stability Of Solutions About Critical Points

The stability of the critical points was alluded to, in our discussion of attractors and repellers in the previous section. The general rule in this is the following. If all the eigenvalues of the coefficient matrix have negative real parts, the solutions are stable about the critical point. If, at least one eigenvalue has non-negative real part, there exists instability. This may also be intuitively understood as follows. In the case of a linear equation (or the linearized version of a nonlinear system), the solutions are of the form  $e^{\lambda_t t}$ where  $\lambda_t$  are the eigenvalues of the coefficient matrix. If  $\lambda_t$  has negative real part for all t, then we get solutions that die out in time. In other words, the solutions asymptotically approach an attractor. This implies stability. Alternatively, if at least one eigenvalue has positive real part, the solutions grow in time and one can expect instability.

### 2.5 Nonlinear Systems

Consider the nonlinear system

$$\dot{x} = f(x), x \in \mathbf{R}^{\mathbf{n}}, x(0) = x_0$$
 (2.13)

Invoking the basic existence and uniqueness theorems [[13, 2]] for differential equations, we can associate (at least locally) a flow  $\phi_t : \mathbf{R}^n \to \mathbf{R}^n$  defined by  $\phi_t(x_0) = x(t, x_0)$ , for the vector field (2.13).

Consider critical points of (2.13). Let  $\bar{x}$  be such a critical point. We linearize (2.13) about  $\bar{x}$  in the following way.

$$\dot{\xi} = Df(\bar{x})\xi, \xi \in \mathbf{R}^n \tag{2.14}$$

where  $Df = \left[\frac{\partial f_i}{\partial x_j}\right]$  is the Jacobian matrix of the first partial derivatives of the function f(x). Since (2.14) is a linear system, we can write the following equation for the flow

$$D\phi_t(\bar{x})\xi = e^{tDf(\bar{x})}\xi \tag{2.15}$$

Given this background. we state (without proof), the following two theorems that form the pillars of nonlinear analysis [14]

#### **Theorem 1** Hartman-Grobman:

If  $Df(\bar{x})$  has no zero or purely imaginary eigenvalues then there is a homeomorphism h defined on some neighbourhood U of  $\mathbb{R}^n$  locally taking the orbits of the nonlinear flow  $\phi_t$  of (2.13) to those of the linear flow  $e^{tDf(\bar{x})}$  of (2.15). The homeomorphism preserves the sense of orbits and can also be chosen to

#### preserve parameterization by time.

Now. we define the local stable and unstable manifolds of  $\bar{x}$ ,  $W^{s}_{loc}(\bar{x})$  and  $W^{u}_{loc}(\bar{x})$  as follows.

$$W^{s}_{loc}(\bar{x}) = x \in U | \phi_{t}(x) \to \bar{x}. as t \to \infty. and \phi_{t}(x) \in U \forall t \ge 0$$
$$W^{u}_{loc}(\bar{x}) = x \in U | \phi_{t}(x) \to \bar{x}. as t \to -\infty. and \phi_{t}(x) \in U \forall t \le 0 \quad (2.16)$$

where  $U \in \mathbb{R}^n$  is a neighbourhood of the fixed point  $\bar{x}$ . The invariant manifolds  $W^{s}_{loc}(\bar{x})$  and  $W^{u}_{loc}(\bar{x})$  provide nonlinear analogues of the flat stable and unstable eigenspaces  $E^{s}$  and  $E^{u}$  of the linear problem. The next theorem states that  $W^{s}_{loc}(\bar{x})$  and  $W^{u}_{loc}(\bar{x})$  are tangent to  $E^{s}$ .  $E^{u}$  at  $\bar{x}$ .

#### Theorem 2 Stable Manifold Theorem for a Fixed Point

Suppose that  $\dot{x} = f(x)$  has a hyperbolic fixed point  $\bar{x}$ . Then there exist local stable and unstable manifolds  $W^{s}_{loc}(\bar{x})$  and  $W^{u}_{loc}(\bar{x})$  of the same dimension  $n_{s}$  and  $n_{u}$  as those of the eigenspaces  $E^{s}$  and  $E^{u}$  of the linearized system, and tangent to  $E^{s}$ ,  $E^{u}$  at  $\bar{x}$ .  $W^{s}_{loc}(\bar{x})$ ,  $W^{u}_{loc}(\bar{x})$  are as smooth as the function f.

Concluding the section we note that no comments have been made in the case when the real parts of the eigenvalues vanish (i. e. the eigenvalues are

zero or pure imaginary). This involves the centre manifold theory which is not essential to our discussion and hence we leave it here with the note that the theory of centre manifolds via normal forms treats this important aspect of nonlinear systems.

#### 2.6 Conclusion

In this chapter, we took a closer look at nonlinear systems. Two important theorems that establish the connection between linear and nonlinear systems were stated. Further, the concept of linearization was stated. Also, ideas related to critical points of linear and nonlinear systems, their classification and stability issues were examined. On the basis of the framework so far, the next chapter explores the phenomena of bifurcations and chaos in nonlinear systems as well as the method of averaging.

## Chapter 3

# Bifurcations and Chaos in Nonlinear Systems

## 3.1 Introduction

In this chapter, we shall discuss some deeper aspects of nonlinear systems. We shall begin with cyclic attractors and go on to chaotic attractors in dynamical systems. This leads us to the interesting phenomenon of chaos. Bifurcation is another aspect that shall be dealt with. Finally, we briefly examine the method of averaging which is a powerful technique in the context of nonlinear systems. In the subsequent chapters the averaging method is applied to the Froude pendulum and this leads to a bifurcation analysis and a classification of the limit cycles thereby. The chaotic behaviour of the Froude pendulum is also explored. This explains our focus on these aspects in this chapter.

The general references pertinent to the material in this chapter are [39, 13, 41, 14, 36, 10, 22].

#### **3.2** Cyclic Attractors (Limit cycles)

In the previous chapter, the idea of point attractors of dynamical systems was discussed. These are fixed point (equilibrium) solutions that attract nearby solutions. It turns out that there exist other types of equilibrium solutions too viz. cyclic and chaotic attractors. We discuss the limit cycle (cyclic attractor) first.

A limit cycle is an isolated periodic solution of an autonomous system. represented in the phase plane by an isolated closed path, as shown in (Fig. (3.1)

The neighbouring paths are not closed but spiral into or away from the limit cycle C as shown. In the case illustrated here which is a stable limit cycle, the device represented by the system will spontaneously drift into the



Figure 3.1: A Limit Cycle

4

# CHAPTER 3. BIFURCATIONS AND CHAOS IN NONLINEAR SYSTEMS

corresponding periodic oscillation from a wide range of initial states. A stable limit cycle represents a stable stationary oscillation of a physical system, akin to the representation of a stable equilibrium by a stable critical point. The existence of limit cycles hence assumes great practical importance since they represent the stationary states of oscillations. The theory of limit cycles is also important in the study of self-sustained oscillations, the simplest example of which would be the motion of the pendulum of a clock [19]. The nearby trajectories approach the limit cycle. Thus, the initial conditions become immaterial as all motion approaches and settles on the cycle. In the case of the pendulum of a clock, the amplitude of oscillation at the start does not affect the final, stable, periodic motion. This is characteristic of all motion that approaches a stable limit cycle. This phenomenon is also found in the case of self-excited oscillations.

Linear systems with constant coefficients do not exhibit limit cycles. Since nonlinear equations cannot be solved in general, it is important to be able to establish the presence of limit cycles, if any, by other means.

#### 3.3 Chaotic Attractors

We now examine the phenomenon that has occupied the centre stage in nonlinear dynamics. in recent times.

To begin with a rather formal definition [39], a chaotic attractor may be geometrically identified as a stable structure of long term trajectories in a bounded region of phase space which folds the bundle of trajectories back onto itself, resulting in mixing and divergence of nearby states.

From a physical point of view. this means that. a system that exhibits chaotic behaviour can start off with two nearby initial states and end up in final states far away from each other after a certain period of time. In other words, the response of a chaotic system is highly sensitive to initial conditions.

In 1963, Lorenz [23] published an analysis of a simplified model of convection in the atmosphere of the earth which involved a set of nonlinear differential equations in three variables. A numerical approximation of any solution to this set of equations has the following interesting properties.

(a) The orbit is not closed.

(b) The orbit does not represent a transition stage to well known regular

behaviour, for some open regions of parameter space.

- (c) The orbit and the intricate geometrical structure it creates depend on the initial conditions in a very sensitive way. Thus, a slight perturbation of the initial conditions produces a very different picture.
- (d) The orbits with different initial conditions possess qualitative similarity in the sense that they are bounded within a certain region of phase space.
- (e) The system is very much deterministic. That is, if one were to start from identical initial conditions one would recover identical orbits.

A graphical representation of this phenomenon is given in Fig. (3.2].

Due to the bounded nature of the trajectories, the presence of an attracting region is quite evident in this case. But within the bounded region there exists an unpredictable, non-periodic pattern and this is termed chaotic behaviour. An attractor of this type is called a chaotic attractor.

The interesting discovery was that chaotic behaviour is generic to a class of nonlinear systems.

Here we consider the following nonlinear equation with a periodic forcing



Figure 3.2: Divergence from adjacent initial conditions - Chaos

function:

$$\ddot{x} + a \ \dot{x} + b \ x^3 = F \cos t \tag{3.1}$$

We focus on the phase space i. e. the  $(x, \dot{x})$  plane and observe two numerical solutions with arbitrarily small difference in the initial conditions. The solutions diverge exponentially with time. Continuing the solutions in time reveals the chaotic nature of the system.

The motion is non-periodic. The system is deterministic. Also, the exponential divergence makes it impossible to establish any long term correlation between the two solutions by reducing the difference in the initial conditions since each order of magnitude improvement in initial agreement vanishes in a fixed increment of time. In other words, solutions starting off with nearby initial conditions do not stay close to each other as they evolve in time. Yet, the set of trajectories exist in a bounded region of phase space and hence there does exist an attractor. Thus. (3.1) shows chaotic behaviour.

From an engineering point of view, unpredictability and chaos may be undesirable. There exists a correlation between this phenomenon and the system parameters. In other words, chaotic behaviour is seen only for certain ranges of values of the system parameters. Thus, for nonlinear systems amenable to analysis it is possible to identify chaotic regimes. The effort would then be to avoid these regimes. Hence, the identification of chaotic attractors in engineering systems is of prime importance.

A technique of identifying ranges of parameter values for which chaotic behaviour can be observed in a nonlinear system is the Melnikov analysis [27, 22, 14]. In chapter 6, we apply this method to the Froude pendulum and derive an analytic criterion for the onset of chaos.

#### 3.4 Bifurcations

The phenomenon of bifurcation refers to the significant qualitative changes that occur in the orbit structure of a dynamical system as the system parameters are varied [13, 17, 14, 7]. These changes have serious implications for the ultimate fate of the system and often this is a prelude to the onset of chaos.

In a broad class of systems it is observed that as the parameters go through a range of values, the qualitative nature of the phase space is drastically affected. These changes could range from anywhere between a change of stability type to fundamental variations in the topology of the phase space. Collectively, these phenomena are termed bifurcations. Bifurcations may be mainly classified as local and global and we briefly examine both, below.

#### **3.4.1** Local Bifurcations

The qualitative changes in the global structure of phase space, due to variations in the system parameters, that can be detected and studied locally (i. e. with respect to a critical point), are called local bifurcations.

In the previous chapter, it was observed that the nature of the eigenvalues under a linearization leads to a classification of the equilibrium points as saddles, nodes, foci and centres. Also, the sign of the eigenvalues dictates the stability type of the point. The principal idea in local bifurcations is the following. For parameter dependent systems, under a linearization, the eigenvalues would be functions of the parameters. It follows, then, that a change in the values of these parameters can affect the eigenvalue and hence the nature of the critical point may be altered. This is called a local bifurcation.

To make the idea more precise, let us consider a dynamical system  $\dot{x} = f(x, \mu)$  that depends on the parameter  $\mu$ . For the present discussion, we consider only a single parameter but it is evident that most practical problems involve more than one parameter. For example, the general oscillatory

mechanical system depends on the mass, stiffness. and damping coefficients as well as on the amplitude and frequency of the forcing function.

Let  $x_0(\mu)$  be a critical point of the above system. Hence,  $f(x_0(\mu): \mu) = 0$ For convenience, we shift  $x_0(\mu)$  to the origin by the following transformation.

$$z = x - x_0(\mu) \Rightarrow z' = A(\mu) z + O(z^2)$$
  
where  $A(\mu) = \frac{\partial f(x_0(\mu); \mu)}{\partial x}$ 

and  $O(z^2)$  represents the higher order terms in the expansion. The stability of  $x_0(\mu)$  is dependent on the eigenvalues of  $A(\mu)$ . Let the eigenvalues be  $\lambda_i(\mu)$ . Further,

1. if  $Re(\lambda_i(\mu)) < 0$ .  $\forall i, x_0(\mu)$  is uniformly and asymptotically stable.

2. if  $\exists$  at least one j such that  $Re(\lambda_j(\mu)) > 0$ ,  $x_0(\mu)$  is unstable.

Here, we note that A is a function of  $\mu$  and hence the eigenvalues are also functions of  $\mu$ . Thus, as  $\mu$  evolves, the nature of the eigen values may change affecting the stability of the critical point. A change in stability may be expected whenever, for some  $\mu = \mu_0$ ,  $Re\{\lambda_i(\mu_0)\} = 0$  where  $Re\{\lambda_i(\mu_0)\}$ represents the real part of the eigenvalue. The values of  $\mu_0$  for which the above condition is satisfied locate the bifurcation points of the system. Further, if we restrict ourselves to the condition that  $A(\mu)$  is a real valued matrix, that is, it can have either real values or complex conjugate pairs of eigenvalues only, bifurcation points can arise in the following ways:

- 1.  $\lambda_m(\mu_0) = 0$  and  $\lambda_n(\mu_0) < 0 \forall m \neq n$  and  $\lambda(\mu)$  is real.
- 2.  $\lambda_m(\mu) = \lambda_n(\mu) = \alpha(\mu_0) + i \beta(\mu_0)$  for some  $m, n; \alpha(\mu_0) = 0; \beta(\mu_0) \neq 0$ and  $Re\{\lambda_k(\mu_0)\} < 0 \forall k \neq m, n.$

Case 1 is called a one dimensional bifurcation and case 2 is called a Hopf bifurcation.

#### 3.4.2 Global Bifurcations

When a change in a parameter value alters the qualitative behaviour in phase space, a global bifurcation is said to have taken place. These are not local in the sense that their analysis cannot be restricted to the neighbourhood of a critical point. This behaviour is exceedingly complex and is yet to be understood exhaustively. Associated with global bifurcations are the appearance and disappearance of limit cycles, formation and destruction of homoclinic loops, saddle connections and so on.

Oscillatory engineering systems involve various parameters. Hence, the

analysis of bifurcations that result in abrupt changes in system behaviour are important in the analysis and design of these engineering systems.

Bifurcations may also be studied in parameter space. As we shall see in the sequel, the bifurcations of the Froude pendulum under our consideration occur in the space of parameters. Indeed, these effects in the parameter space are reflected in the phase space of the system, as well.

#### 3.5 Averaging

Averaging is an extremely powerful technique in asymptotic analysis [36, 30]. The starting point is a perturbed nonlinear system. If the system involves periodic functions, a corresponding averaged equation can be generated wherein the functions are integrated over the period.

The important result is that approximate solutions (often to any degree of accuracy) for the original equation may be written down by solving the simpler. averaged equation. Thus, averaging, in general, may be viewed as a technique of generating approximate solutions of perturbed nonlinear systems by solving the corresponding averaged equation, which, hopefully, admits simpler solutions. Once again, to make the ideas more precise, let us consider the following system:

$$\dot{x} = \epsilon f(t, x) + \epsilon^2 g(t, x, \epsilon); \ x(t_0) = x_0$$

where  $\epsilon$  is a perturbation parameter. If the function f is T-periodic in t. the averaged equation corresponding to the above equation can be written as

$$\dot{y} = \epsilon f^{0}(y); \ y(t_{0}) = x_{0}$$
where  $f^{0}(y) = \frac{1}{T} \int_{0}^{T} f(t, y) dt$ 

With respect to the above equations, the averaging theorem [36] states that, to any desired degree of approximation, the solutions of the original equation and the averaged version stay close enough to each other.

What makes this approach really powerful is the existence of theorems that guarantee a close correspondence between the original equation and the averaged version. Thus, properties like existence and stability of attractors, and bifurcations carry over to the averaged equation and vice versa. This makes it possible to deduce a great deal of information about the original system from the study of the averaged equation.

This technique is exceptionally useful in tackling nonlinear problems in engineering, since the averaged equation often admits simpler solutions. In chapter 5, we approach the Froude pendulum using this method.

## Chapter 4

## The Froude Pendulum

### 4.1 Introduction

The Froude pendulum is a classical mechanical system that exhibits frictioninduced. self-excited oscillations [28, 8, 29, 5, 26]. Apart from being unique in its own right in the class of interesting nonlinear systems. it has served as a paradigm for the treatment of friction-induced nonlinear motion. Despite the fact that this system has been known to be important for a considerably long time [28, 38], a survey of the literature confirms the existence of many open questions with respect to this system.

Self-excited oscillations have received considerable attention in the re-

search on nonlinear systems. the most important example being the Van der Pol oscillator. The dynamics of these systems being self generated, they are of great interest.

On the other hand, friction continues to be a grey area in the entire landscape of dynamics. The classical approach is to model friction by Coulomb's law,  $F_T = \mu F_N$ , where, the frictional force  $F_T$  equals the normal reaction  $F_N$ multiplied by the coefficient of friction  $\mu$ . The inadequacy of this ideal relationship as a satisfactory model both from the theoretical and applied points of view has been long recognized, for instance, in [32]. It is also observed that a broad class of engineering systems admit nonlinear frictional effects.

Nonlinear friction is the central theme in the discussion of the Froude pendulum and thus. along with the self-excited nature of the oscillations. the pendulum becomes an important object of study.

In the sequel, following standard analysis, the equations of the Froude pendulum are set up and then cast in a form that facilitates the bifurcation analysis that we intend to carry out via the averaging technique developed by Sanders and Cushman in the context of the Josephson equation [35].

#### 4.2 The Froude Pendulum

A schematic diagram of the Froude pendulum is shown in Fig (4.1). The shaft of the pendulum is connected to an engine which rotates freely in the bearing pivot at a constant angular velocity  $\Omega$ . The pendulum is fixed to the bearing pivot which swings on the rotating shaft. There arises friction in the contact surfaces between the shaft and the bearing pivot. It is standard procedure to treat frictional forces as functions of the slipping velocity [28, 26, 12, 18, 16]. If the angular displacement from the vertical by  $\phi$ , the frictional torque of the Froude pendulum is assumed to have a relation to the slipping angular velocity  $\phi$  and is expressed as a function  $M(\Omega - \phi)$  [28, 8, 26]. Thus the equation of motion can be written as

$$I\ddot{\phi} + c\dot{\phi} + mgl\sin\phi = M(\Omega - \dot{\phi}) \tag{4.1}$$

where *m* is the combined mass of the pendulum and the pivot, *I*, the total moment of inertia of all rotating components of the pendulum, *g*, the acceleration due to gravity, *c* the coefficient of damping, *l*, the distance from the axis of rotation to the centre of gravity of the pendulum. Expanding  $M(\Omega - \dot{\phi})$  as a power series about a given  $\Omega$  (chosen as a point of inflection of  $M(\Omega)$  implying  $M''(\Omega) = 0$ ) and considering only the first four terms of



Figure 4.1: The Froude pendulum - the rotating shaft is connected to an engine

the resulting series. we obtain the standard equation [28, 8]:

$$I\ddot{\phi} + c\dot{\phi} + mgl\sin\phi = M(\Omega) - M'(\Omega)\dot{\phi} - \frac{1}{6}M'''(\Omega)\dot{\phi}^3 \qquad (4.2)$$

Rearranging this equation, we get.

$$\ddot{l\phi} + (c + M'(\Omega))\dot{\phi} + \frac{1}{6}M'''(\Omega)\dot{\phi}^3 + mgl\sin\phi = M(\Omega)$$
(4.3)

Dividing both sides by I, and introducing constants, we obtain

$$\ddot{o} + a\dot{o} + b\dot{o}^3 + h\sin o = c \tag{4.4}$$

where 
$$a = \frac{c + M'(\Omega)}{I}$$
,  $b = \frac{M''(\Omega)}{6I}$ ,  $h = \frac{mgI}{I}$  and  $c = \frac{M(\Omega)}{I}$ .

Thus. (4.4) represents the unforced Froude pendulum. This is, obviously, a highly nonlinear equation with contributions from the cubic damping term and the sin  $\phi$  term.

#### 4.3 Analysis of The Froude Pendulum

As was observed before, the analysis of (4.4), in the literature is incomplete. One encounters treatments with the approximation  $\sin \phi \approx \phi$ , which reduces the system to a simple harmonic oscillator with a cubic damping term. Such a linearization (as has been emphasized before), is justified in certain cases, but, more often than not, obscures the essential features of the system. Even more disturbing is the fact that replacing  $\sin \phi$  with  $\phi$  leads to a completely different equation. a different system.

These considerations converge to the conclusion that, to the extent possible, (4.4) should be treated in its full generality, with  $\sin \phi$  and the cubic damping term receiving the attention they rightly deserve.

That such an approach. leading to meaningful results is possible, is the highlight of the present effort. The technique that we adapt here is the one adopted by Sanders and Cushman in the case of the Josephson equation. The Josephson junction is described by the equation [35]

$$\beta \ddot{\phi} + (1 + \gamma \cos \phi) \dot{\phi} + \sin \phi = \alpha \tag{4.5}$$

where  $\alpha$ .  $\beta$ .  $\gamma$  are constants.

The central idea is the following. The above equation can be treated as a perturbation of the following system known as the mathematical pendulum [41, 35] given by the equation,

$$\ddot{\phi} + \sin \phi = 0 \tag{4.6}$$

The mathematical pendulum is a well known Hamiltonian system and treating the Josephson equation as a perturbation of this system, averaging may be carried out over the level sets of the hamiltonian. Using this method, the bifurcations and the phase portrait of the Josephson equation can be studied. as shown in the work of Sanders and Cushman [35].

Consider the equation of the Froude pendulum (4.4). Rearranging terms. we get

$$\ddot{\phi} + h \sin \phi + a \dot{\phi} + b \dot{\phi}^3 = c$$

If we scale the above equation by setting h = 1, we get

$$\ddot{\phi} + \sin\phi + a\dot{\phi} + b\dot{\phi}^3 = c \tag{4.7}$$

this implies

$$\ddot{\phi} + \sin \phi = -a \dot{\phi} - b \dot{\phi}^3 + c \qquad (4.8)$$

Comparison with (4.6) shows that the difference between the two equations is in the extra damping terms and the constant term.

Thus, the critical observation here is that if we treat a and b to be small, the Froude pendulum can be treated as a perturbation of the mathematical pendulum. Taking this approach, we are able to study the bifurcations, limit cycles and the entire phase portrait of the Froude pendulum. The details are worked out in the following chapter.

#### 4.4 Conclusion

The Froude pendulum is an example of friction-induced, self-excited, nonlinear oscillations. This system has not been analyzed exhaustively. From the form of the equations, it is observed that this system can be treated as a perturbation of the mathematical pendulum. One such system that has been studied from this angle is the Josephson equation [35]. It turns out to be possible to apply the same techniques to the Froude pendulum. This leads to a bifurcation analysis, a classification of limit cycles and interesting phenomena in the phase space. This is carried out in the next chapter.

## Chapter 5

# Bifurcations and Limit Cycles in The Froude Pendulum

## 5.1 Introduction

This chapter forms the core of the thesis. Here, we apply the averaging technique mentioned in [35], the method of Sanders and Cushman to the Froude pendulum. In [35], this technique is applied to the Josephson equation and the crucial observation here is that both the Josephson equation and the Froude pendulum can be treated as different perturbations of the well known Hamiltonian system viz. the mathematical pendulum. The latter system is the starting point of the discussion in [35]. and we apply the same method to the Froude pendulum. This leads to a bifurcation analysis via the Ricatti equations and as a consequence, significant comments can be made on the limit cycles in the system.

As we have observed earlier, a knowledge of the limit cycles and their classification is extremely useful in the analysis of nonlinear systems and the case of the Froude pendulum is no exception. A highlight of this approach, as we shall see in the sequel is that no attempt is made to linearize the system. The nonlinearities presented by  $\sin \phi$  and the cubic damping term in the equations are genuine and we treat them as such.

# 5.2 The Froude Pendulum - A perturbation of the mathematical pendulum

Let us recall the Froude pendulum given by the equation [4.4]:

$$\ddot{\phi} + a\dot{\phi} + b\dot{\phi}^3 + h\sin\phi = c$$

where  $a = \frac{c+M'(\Omega)}{I}$ ,  $b = \frac{M'''(\Omega)}{6I}$ ,  $h = \frac{mgl}{I}$  and  $c = \frac{M(\Omega)}{I}$ .

There are four parameters a. b. h, c associated with the system. For the bifurcation analysis that follows, we consider the two parameters a and c. Thus, we need to scale the above equation such that we are left with these two parameters. Setting mgl = I and  $M'''(\Omega) = 6I$ , we get the following equation.

$$\ddot{\phi} + a\dot{\phi} + \dot{\phi}^3 + \sin\phi = c \tag{5.1}$$

It is noted here that this choice is motivated by the need to carry out bifurcation analysis with respect to the parameters a and c.

Before proceeding further, we remark that the analysis that follows is not restricted to the set of parameters that we have chosen. Our choice is guided by the fact that a is a coefficient of the leading damping term and c can be viewed as a constant value of the forcing function. It is possible to carry out the same bifurcation analysis for a different set of parameters.

The analysis carried out here closely follows [35].

Transforming to the first order system which we shall call  $X_{a,c}$ 

$$\mathbf{X}_{a,c}: \phi = y \qquad (a)$$
$$\dot{y} = -\sin\phi + \epsilon [c - (ay + y^3)] \qquad (b) \qquad (5.2)$$

where  $\epsilon$  is a perturbation parameter.

Holding  $\epsilon$  fixed.  $\mathbf{X}_{a,c}$  is a two parameter family of vector fields on the cylinder  $TS^1$ , which we shall study by the averaging method.

If we set  $\epsilon = 0$  in (5.2), we see that it gives us the following system

$$\dot{\phi} = y$$
 (a)  
 $\dot{y} = -\sin\phi$  (b) (5.3)

This system of equations represents the mathematical pendulum and hence the statement that the Froude pendulum can be treated as a perturbation of the mathematical pendulum is validated.

#### 5.3 The Averaged Equation

We now derive the averaged equation. As stated before, when  $\epsilon = 0$ , the unperturbed system is the Hamiltonian vector field  $\mathbf{X}_H$  describing the mathematical pendulum where the Hamiltonian function is

$$H(\phi, y) = \frac{y^2}{2} - \cos\phi \qquad (5.4)$$

Instead of variables  $(\phi, y)$  we use  $(\phi, h)$ , where h is defined as

$$h = \frac{y^2}{2} - \cos\phi \tag{5.5}$$

Averaging, as we have seen in an earlier chapter, can be broadly called as a method of constructing periodic solutions of perturbed equations from the known solutions of the unperturbed problem. In this case, as we shall see, this procedure reduces to the integration of the perturbed part of the equation over level sets of the unperturbed system. That this process reveals an enormous amount of information about the perturbed system is the central theme of the story.

Differentiating (5.5) with respect to  $\phi$  and using (5.2), we get

But

$$h = \frac{y^2}{2} - \cos \phi$$
  

$$\Rightarrow y^2 = 2(h + \cos \phi)$$
  

$$y = \pm \sqrt{2(h + \cos \phi)}$$
(5.7)

Before we take the next step of averaging, we need to comment on the

CHAPTER 5. BIFURCATIONS AND LIMIT CYCLES IN THE FROUDE PENDULUM



Figure 5.1: The Phase Space of the Mathematical Pendulum

phase space of the mathematical pendulum.

We note that in Fig. (5.1) the phase space is actually a cylinder with the points  $(\pm \pi, 0)$  identified. This diagram is a planar representation of the surface of this cylinder. There exist three distinct families of closed cycles  $\Gamma_h$ on the cylinder:

1.  $\Gamma_h^0$ , when -1 < h < 1. The level set (which is a periodic solution of  $X_h$  except when h = 1) is smooth, connected, compact and contractable

to a point.

- 2.  $\Gamma_h^+$ , when h > 1 and y > 0.  $\Gamma_h^+$  is the component of the level set given by the graph of the function  $\sqrt{2(h + \cos \phi)}$ .  $\Gamma_h^+$  is not contractable to a point and it winds around the cylinder.
- 3.  $\Gamma_h^-$ , when h > 1 and y < 0. Here the equation to the curve is  $y = -\sqrt{2(h + \cos \phi)}$ .

Averaging (5.6) over a compact, connected component  $\Gamma_h$  of the level set leads to the averaged equation

$$\frac{dh}{d\phi} = \epsilon \left[ c \int_{\Gamma_h} d\phi - a \int_{\Gamma_h} y d\phi + \int_{\Gamma_h} y^3 d\phi \right]$$
(5.8)

Non degenerate zeroes of the right hand side of (5.8) correspond to limit cycles of  $X_{a,c}$ . Define the path integrals involved in the problem as

$$A(h) = \int_{\Gamma_h} d\phi \tag{5.9}$$

$$B(h) = \int_{\Gamma_h} y d\phi \tag{5.10}$$

$$C(h) = \int_{\Gamma_h} y^3 d\phi \tag{5.11}$$

From here on, when required, we use superscripts 0 and  $\pm$  on A, B and C to denote the  $\Gamma_h$  family being used. For a fixed value of the parameter c,

those values of a which give rise to zeroes of the averaged equation in (5.8) are exactly the values of the function

$$\eta(h) = c \frac{A}{B} - \xi(h)$$

$$where$$

$$\xi(h) = \frac{C}{B}$$
(5.12)

Setting the right hand side of (5.8) equal to zero gives, using (5.11)

$$cA - aB - C = 0 \Rightarrow aB = cA - C \Rightarrow a = c\frac{A}{B} - \frac{C}{B}$$
 (5.14)

since A. B and C are functions of h, it follows that  $\eta(h)$  is as defined in (5.12).

#### 5.4 The Picard-Fuchs and Riccatti Equations

The next step in studying the averaged equation is to find the Picard-Fuchs equations satisfied by the functions A, B and C and then to analyze the solutions of the resulting 'Riccatti' equations. As we shall see in the sequel, the bifurcation picture emerges as a result of solving the 'Riccatti' equations.

From (5.11),  $C(h) = \int_{\Gamma_h} y^3 d\phi$ . Hence

$$\frac{dC}{dh} = \int_{\Gamma_h} 3y^2 \frac{dy}{dh} d\phi$$

$$= 3 \int_{\Gamma_h} y(y \frac{dy}{dh}) d\phi$$
$$= 3 \int_{\Gamma_h} y d\phi$$
$$= 3B(h) \qquad (5.15)$$

Here, we have used the result  $y \frac{dy}{dh} = 1$ , which is obtained by differentiating (5.5) with respect to h. Also we have used (5.11) in obtaining (5.15). From (5.10).  $B = \int_{\Gamma_h} y d\phi$ . Hence

$$\frac{dB}{dh} = \int_{\Gamma_h} \frac{dy}{dh} d\phi \tag{5.16}$$

Differentiating both sides of (5.5) with respect to h

$$y\frac{dy}{dh} = 1$$

$$\frac{dy}{dh} = \frac{1}{y}$$
(5.17)

Using (5.17) in (5.16), we get

$$\frac{dB}{dh} = \int_{\Gamma_h} \frac{1}{y} d\phi \tag{5.18}$$

Again. from (5.11).

$$C = \int_{\Gamma_{h}} (y^{3}) d\phi = \int_{\Gamma_{h}} (y^{2}) y d\phi$$
  
=  $\int_{\Gamma_{h}} 2(h + \cos \phi) y d\phi$  (from (5.5)) (a)  
=  $2h \int_{\Gamma_{h}} y d\phi + 2 \int_{\Gamma_{h}} y \cos \phi d\phi$  (b)

$$= 2hB + 2\int_{\Gamma_h} y\cos\phi d\phi \qquad (c) \qquad (5.19)$$
From [35. (3.9).p.502]. integrating by parts,

$$\int y\cos\phi \ d\phi = -\int \frac{dy}{d\phi}\sin\phi \ d\phi$$

Differentiating (5.5) with respect to  $\phi$  on both sides. (keeping in mind that

h is treated as a separate variable), we get

$$\frac{dy}{d\phi} = -\frac{\sin\phi}{y}$$

Using this result.

$$\int y \cos \phi \, d\phi = -\int \frac{dy}{d\phi} \sin \phi \, d\phi = \int \frac{\sin^2 \phi}{y} d\phi$$

Again. from (5.5).

$$\sin^2\phi = 1 - \left(\frac{y^2}{2} - h\right)^2$$

Thus.

$$\int y \cos \phi \, d\phi = \int \frac{1}{y} \left[ 1 - \left( \frac{y^2}{2} - h \right)^2 \right] \, d\phi$$

Hence. we get

$$\int_{\Gamma_h} y \cos \phi \, d\phi = \int_{\Gamma_h} \left[ \frac{1}{y} - \frac{1}{4} (y^3) + hy - \frac{h^2}{y} \right] \, d\phi \tag{5.20}$$

Substituting into (5.19), we get

$$C = 2hB + 2\int_{\Gamma_h} \left[\frac{1}{y} - \frac{1}{4}(y^3) + hy - \frac{h^2}{y}\right] d\phi$$
 (5.21)

Hence, using (5.11) and (5.18), we get

$$C = 2hB + 2\frac{dB}{dh} - \frac{C}{2} + 2hB - 2(h^2)\frac{dB}{dh}$$
(5.22)

Thus.

$$2(1-(h^2))\frac{dB}{dh} + 4hB - \frac{3}{2}C = 0$$
 (5.23)

Or.

$$(1 - (h^2))\frac{dB}{dh} + 2hB - \frac{3}{4}C = 0$$
(5.24)

From (5.15).  $\frac{dC}{dh} = 3B$ .

Using (5.16) and (5.24), the Picard-Fuchs equations for the system can be written as

$$(1-h^2)\frac{d}{dh}\begin{pmatrix}B\\C\end{pmatrix} = \begin{pmatrix}-2h & \frac{3}{4}\\3(1-h^2) & 0\end{pmatrix}\begin{pmatrix}B\\C\end{pmatrix}$$
(5.25)

Having obtained the Picard-Fuchs system of equations, the next step is to derive the Ricatti equations for  $\xi(h)$  and  $\eta(h)$ . We have defined  $\xi(h) = \frac{C}{B}$  and  $\eta(h) = c\frac{A}{B} - \xi(h)$ . Hence, using (5.13), (5.16) and (5.24),

$$\frac{d\xi(h)}{dh} = \frac{\frac{B\frac{dC}{dh} - C\frac{dB}{dh}}{B^2}}{= \frac{1}{B}3B - \frac{\xi(h)}{B}\frac{dB}{dh}} \qquad (using (5.13) and (5.16))$$
$$= 3 - \frac{1}{B}\xi(h) \left[\frac{1}{(1-h^2)}\frac{3}{4}C - 2hB\right] \qquad (from (5.24))$$
$$= 3 - \frac{1}{(1-h^2)}\left[\frac{3}{4}\xi^2(h) - 2h\xi(h)\right] \qquad (using (5.13))$$

63

Thus.

$$(1-h^2)\frac{d\xi}{dh} = 3(1-h^2) - \frac{3}{4}\xi^2(h) + 2h\xi(h)$$
(5.26)

This is the Ricatti equation satisfied by  $\xi(h)$ . Since  $\eta(h) = c\frac{A}{B} - \xi(h)$ ,

$$\frac{d\eta}{dh} = c\frac{d}{dh}\frac{A}{B} - \frac{d\xi}{dh}$$
(5.27)

Consider

$$\frac{d}{dh}\frac{A}{B} = \frac{B\frac{dA}{dh} - A\frac{dB}{dh}}{B^2}$$
(5.28)

From (5.9),  $A = \int_{\Gamma_h} d\phi$ , and hence

$$\frac{dA}{dh} = 0 \tag{5.29}$$

Using (5.29) in (5.28).

$$\frac{d}{dh}\frac{A}{B} = -\frac{A}{B^2}\frac{dB}{dh} \qquad (a)$$

$$= -\frac{A}{B}\frac{1}{B}\frac{dB}{dh} \qquad (b)$$

$$= -\frac{1}{c}(\xi + \eta)\frac{1}{B}\left[\frac{1}{1-h^2}\frac{3}{4}C - 2hB\right] \quad (c) \tag{5.30}$$

from (5.24) and (5.27).

Using (5.13) in the above expression, we get

$$\frac{d}{dh}\frac{A}{B} = -\frac{1}{c}(\xi + \eta)\left[\frac{1}{1 - h^2}\frac{3}{4}\xi(h) - 2h\right]$$
(5.31)

Using (5.31) in (5.27), we get

$$\frac{d}{dh}\eta(h) = -\frac{(\xi+\eta)}{1-h^2} \left[\frac{3}{4}\xi - 2h\right] - \frac{1}{1-h^2} \left[-\frac{3}{4}\xi^2 + 2h\xi + 3(1-h^2)\right](5.32)$$
$$\Rightarrow (1-h^2)\frac{d\eta}{dh} = -\frac{3}{4}\xi^2 - \frac{3}{4}\xi\eta + 2h\left(\eta + \xi\right) + \frac{3}{4}\xi^2 - 2h\xi - 3(1-h^2)\left(5.33\right)$$

This gives

$$(1-h^2)\frac{d\eta}{dh} = -\frac{3}{4}\xi\eta + 2h\eta - 3(1-h^2)$$
(5.34)

This is the Ricatti equation satisfied by  $\eta(h)$ .

We now comment on the relationship between  $\xi^0(h)$  and  $\eta^0(h)$ . Here the superscripts indicate that these functions are considered over  $\Gamma_h^0$ , as defined in section [5.3]. We recall,  $\eta(h) = c\frac{A}{B} - \xi(h)$ . For  $\Gamma^0(h)$ ,  $A = \int_{\Gamma_h} d\phi = 0$ , since the cycle is contractable to a point. Hence

$$\eta^{0}(h) = -\xi^{0}(h) \tag{5.35}$$

Before we proceed to integrate the Ricatti equations for  $\xi(h)$  and  $\eta(h)$  numerically, we need the initial conditions for both. Deriving these shall be our next task. We use a simple argument for the following derivation. From (5.26).

$$h = 1 \Rightarrow -\frac{3}{4}(\xi)^2 + 2\xi = 0 \quad (a)$$
  
$$\Rightarrow \xi^0(1) = \frac{8}{3}, \ \xi^0(-1) = 0 \quad (b) \quad (5.36)$$

Since  $\eta(h) = c\frac{A}{B} - \xi(h)$ , where  $A = \int_{\Gamma_h} d\phi$  and  $B = \int_{\Gamma_h} y \, d\phi$ , and since  $\phi$ varies between  $-\pi$  and  $\pi$ .

$$\eta(1) = \frac{c \int_{-\pi}^{\pi} d\phi}{\int_{-\pi}^{\pi} y d\phi} - \xi(1)$$
(5.37)

Here.

$$\int_{-\pi}^{\pi} y d\phi = \int_{-\pi}^{\pi} \sqrt{2(1 + \cos \phi)} \, d\phi, \text{ (setting } h = 1 \text{ in } (5.7)\text{)} \quad (a)$$

$$= \int_{-\pi}^{\pi} \sqrt{2(2\cos\frac{\phi}{2}^2)} d\phi \qquad (b)$$

$$= 2\int_{-\pi}^{\pi} \cos\frac{\phi}{2} d\phi = 8 \qquad (c) \quad (5.38)$$

$$2\int_{-\pi}^{\pi} \cos\frac{\phi}{2}d\phi = 8 \qquad (c) \ (5.38)$$

Using (5.38) and (5.36(b)) in (5.37) we get

$$\eta(1) = \frac{\pi c}{4} - \frac{8}{3} \tag{5.39}$$

#### Plot of $\xi^0(h)$ 5.5

Given the initial condition  $\xi^0(-1) = 0$ , we solve numerically, the differential equation for  $\xi(h)$  (5.26). The numerical integration was carried out using the software package Mathematica  $\bigcirc$ . As can be seen from the plot.  $\xi(1) = \frac{8}{3}$ . which is exactly the calculated value. We observe that in this range [-1, 1)for h,  $\xi(h)$  has a unique maximum given by  $\xi^0_{max} = 2.668$ .



Figure 5.2: The Plot of  $\xi(h)$  vs h

#### 5.6 The Bifurcation Diagram

The next step is the crucial one in this analysis, one that results in the important bifurcation diagram that we seek. In this, we plot the bifurcation curve (Fig. 5.3) between the two parameters a and c, of the system. The starting point is (5.39),  $\eta(1) = \frac{\pi c}{4} - \frac{8}{3}$ . This gives us an initial condition  $\eta(1)$ , for every value of c. Using this, the differential equation for  $\eta(h)$  (5.34) is solved numerically. The heavily mathematical arguments in [35] give that the bifurcations occur at  $\eta_{max}(h) = a$ . Using this result, we pick the maximum value of  $\eta(h)$  from the numerical solution and plot it against c. This gives the



Figure 5.3: The Bifurcation Diagram

## CHAPTER 5. BIFURCATIONS AND LIMIT CYCLES IN THE FROUDE PENDULUM

bifurcation diagram. As can be seen from the phase portraits, for different regions of parameter values in the bifurcation diagram, we observe different behaviour in the phase space. Using the results  $\xi^0(1) = 2.666$  and  $\xi^0_{max} = 2.668$  from (Fig. 5.2), and from (5.35), we expect significant changes at a = -2.666 and a = -2.668. Hence these points are also important in the bifurcation diagram. This is established by the rigorous arguments in [35]. The numerical work associated with this section was carried out using Mathematica  $\mathbb{C}$ .

### 5.7 Stability Of The Equilibrium Points

The critical points of the vector field (5.2) are given by  $\sin \phi = \epsilon c$  and y = 0. Linearization of (5.2) about the equilibrium points gives the Jacobian matrix

$$\begin{pmatrix} 0 & 1 \\ -\cos\phi & -\epsilon a \end{pmatrix}_{\sin\phi=\epsilon c} (a)$$

$$= \begin{pmatrix} 0 & 1 \\ \pm\sqrt{1-\epsilon^{2}c^{2}} & -\epsilon a \end{pmatrix} (b)$$

$$\approx \begin{pmatrix} 0 & 1 \\ \pm(1+\frac{1}{2}\epsilon^{2}c^{2}) & -\epsilon a \end{pmatrix} (c) (5.40)$$

Eigenvalues can be calculated from the characteristic equation of (5.40)

$$\lambda^{2} + \epsilon a \lambda - (1 + \frac{1}{2} \epsilon^{2} c^{2})(\pm 1) = 0 \quad (a)$$
  
$$\lambda_{\pm} = \frac{-\epsilon a \pm \sqrt{\epsilon^{2} a^{2} + 4\eta (1 + \frac{1}{2} \epsilon^{2} c^{2})}}{2}, \text{ where } \eta = \pm 1 \quad (b) \qquad (5.41)$$

For given values of a and c this gives the stability type of the critical point.

# 5.8 The Phase Portrait and Limit Cycles from the Bifurcation Diagram

Before we discuss the details of the phase portrait, we note that the computational work in this section was carried out using the *DE Tools* subpackage of the software Maple ©. This facilitates the plotting of the phase portrait for different parameter values.

We shall examine the phase portrait in some regions of the bifurcation diagram. There exist difficulties in this exercise for two reasons. In a numerical plot of the equations, we need an estimate of the perturbation parameter  $\epsilon$  which is unobtainable from averaging theory. The averaging arguments all hold good for 'sufficiently' small epsilon and it is not straightforward to obtain a range of numerical values for this  $\epsilon$  and this creates difficulties in

the computational part of the analysis. Nonetheless, we do plot the phase portrait for an arbitrarily chosen small value of the perturbation parameter and still are able to see the existence of limit cycles.

As we shall discuss in the next chapter, this knowledge of the existence of limit cycles is quite important from the point of view of further work. It is well known that a non autonomous nonlinear system is extremely hard to deal with and in this context, if one were to study the Froude pendulum driven by a forcing function, the existence of limit cycles and their behaviour with respect to the unforced problem becomes important. Here, due to significant computational difficulties, we treat this exercise just as an indicator of the different types of system behaviour. These difficulties are compounded by the complex behaviour of nonlinear systems. The initial conditions, step size in the numerical integration, time period for which the solutions are traced, and the inherent capability of the software package are the other major constraints in this context. However, the regions we examine are quite rich in structure and the numerical results that we get do possess a high degree of clarity.

The important features that we see from the phase portraits drawn on the basis of the bifurcation diagram are the following. First of all, we see

## CHAPTER 5. BIFURCATIONS AND LIMIT CYCLES IN THE FROUDE PENDULUM

the presence of limit cycles. Interestingly, we see the existence of limit cycles under large perturbations as well. Averaging is essentially an approximation method and hence, the fundamental averaging theorems guarantee results only under small perturbations. But in this case, we see that the limit cycles survive under large perturbations. At once, we should add that in nonlinear systems limit cycles can make sudden appearances and disappearances due to different reasons and hence, we need to be cautious in drawing conclusions. Yet, the presence of limit cycles is always valuable information.

A closed trajectory joining the saddle points in the phase portrait is called a double saddle connection. If the connection exists in the upper (lower) half of the phase plane alone, it is known as an upper (lower) saddle connection. We see the appearance of saddle connections of all three types.

The last figure in the series (Fig. 5.21), suggesting a chaotic attractor for the Froude pendulum under the action of a forcing function needs special mention. Here the pendulum is driven by a forcing amplitude F = 0.4 and a forcing frequency  $\omega = 2$ . Chaotic behaviour in the Froude pendulum has been observed in recent times [9] and the winding of the limit cycle around the annular region suggests chaotic behaviour. This can be expected since the time dependent forcing function adds another dimension to the phase space and the presence of invariant tori is possible.

Concluding the chapter, it is noted that the nonlinear averaging technique as used by Cushman and Sanders [35] was applied successfully to the Froude pendulum. The bifurcation diagram for 'a' vs 'c' and the phase portrait have been obtained. the phase portraits have been plotted using (5.2). Parameter values for a and c have been taken based on the regions in the bifurcation diagram (Fig. 5.3). The actual values of a and c, the regions to which they correspond and values of the perturbation parameter  $\epsilon$  are detailed in the phase portrait.



Figure 5.4: Region I: Upper Saddle connection,  $\epsilon = 0.0001$ , a = 1, c = 5

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Figure 5.5: Region I': Lower Saddle Connection,  $\epsilon = 0.0001, a = 1, c = -5$ 







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Figure 5.7: Region II': Lower saddle connection,  $\epsilon = 0.0001, a = 1, c = -1$ 









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CHAPTER 5. BIFURCATIONS AND LIMIT CYCLES IN THE FROUDE PENDULUM



Figure 5.11: Region IV: Limit Cycle,  $\epsilon = 1$ , a = -0.85, c = 0

CHAPTER 5. BIFURCATIONS AND LIMIT CYCLES IN THE FROUDE PENDULUM



Figure 5.12: Region IV: Double saddle connection,  $\epsilon = 0.0001, a = -0.5, c =$ 

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Figure 5.13: Region IX: Phase portrait for  $\epsilon = 0.0001, a = -3, c = 0$ 





BIFURCATIONS AND LIMIT CYCLES IN THE CHAPTER 5. BIFURC FROUDE PENDULUM



Figure 5.15: Region VIII': Phase portrait for  $\epsilon = 0.0001, a = -3, c = -4$ 

CHAPTER 5. BIFURCATIONS AND LIMIT CYCLES IN THE



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Figure 5.17: The Node at a = 1, c = 0, for  $\epsilon = 1$ 











CHAPTER 5. BIFURCATIONS AND LIMIT CYCLES IN THE FROUDE PENDULUM



Figure 5.20: Limit Cycle for a = 0.8,  $c = 0, \epsilon = 1$ 





Figure 5.21: Candidate for Chaotic Attractor a = 0.4 c = 0  $\epsilon = 1$  F = 0.4 $\omega = 1$ 

## Chapter 6

## Melnikov Analysis

### 6.1 Introduction

Despite the extensive investigations of the phenomenon of chaos in recent times. there still exist very few analytic criteria capable of predicting chaos in a nonlinear system. From the point of view of engineering, the power to predict is extremely important. Typically, engineering systems involve various parameters that naturally enter the governing differential equations. For example, as we have seen before, in the case of an oscillatory mechanical system, the parameters would be mass, coefficients of stiffness and damping, the amplitude and the frequency of the forcing function. In this case, if the system is chaotic in certain regimes, it would indeed be quite desirable to obtain a criterion which predicts the onset of chaotic behaviour for certain ranges of values of the system parameters.

One technique that turns out to be extremely useful in this context is the Melnikov method [27]. It is applicable to behaviour in the neighbourhood of hyperbolic critical points alone and the reason for this becomes apparent once we are acquainted with the theory behind this technique. That this analytical method is applicable to a broad class of systems is indicative of its utility.

We shall begin by examining the behaviour of separatrices about hyperbolic critical points under a perturbation and proceed to describe the theory behind the Melnikov approach. Here we closely follow the treatment in [22] and refer to the same for more details. After deriving the criterion for the onset of chaos, we shall apply the analysis to the system of our prime concern. the Froude pendulum.

### 6.2 Perturbed Hamiltonian Systems

Consider a Hamiltonian system given by

$$H = H_0 + \epsilon H_1, \tag{6.1}$$

perturbed under an area preserving mapping.

It is interesting to examine the consequences of the perturbation given by (6.1). Let the unperturbed system corresponding to (6.1) have the phase space structure of an elliptical critical point flanked by two hyperbolic critical points (Fig. 6.1). The standard example of such a system is the harmonic oscillator.

A comparison of (Fig. 6.1) with (Fig. 6.2) leads to the conclusion that under the perturbation, the stable and unstable manifolds  $(H^+ \text{ and } H^-)$  of the two critical points are not likely to join together smoothly. This is the central point of the discussion.

At each of the hyperbolic critical points, four curves join, corresponding to the two incoming trajectories of the stable manifold  $H^+$  and the two outgoing trajectories of the unstable manifold  $H^-$ . A point x is said to lie on  $H^+$  if the repeated transformation  $T^n x$  brings x to the critical point as n tends to infinity. Similarly, the point lies on  $H^-$  if the inverse transformation



Figure 6.1: Phase Space of Unperturbed Hamiltonian System



Figure 6.2: Phase Space of the perturbation of a Hamiltonian System

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brings x to the singular point as n tends to infinity. The period being infinite on the separatrix, the movement of x towards the critical point becomes increasingly slow as the saddle point is approached.

We again observe that under the perturbation. the  $H^-$  curve leaving one critical point generically intersects the  $H^+$  curve arriving at the neighbouring critical point. This intersection is called a homoclinic point, as it connects outgoing and incoming trajectories of the topologically same hyperbolic point. The presence of a single intersection implies the presence of infinitely many more all of which are homoclinic points. The existence of these homoclinic points leads to what is known as a homoclinic tangle.

Let us examine this phenomenon a bit more closely with reference to (Fig. 6.2). The first intersection occurs at X, the second at X', the third at X'' and so on. The distance between successive points decreases as one moves closer to the critical point. Successive points are a result of an area preserving mapping  $T^n$ . As a consequence, the fluctuation of the trajectory gets increasingly wild as it gets closer to the critical point. This creates the homoclinic tangle, leading to chaotic behaviour.

### 6.3 The Melnikov Integral

In this section, we derive the Melnikov integral which provides the criterion for the transversal intersection of the stable and unstable manifolds which leads to the formation of the homoclinic tangle and signals the onset of chaos. The Melnikov integral measures the distance between the stable and unstable manifolds under the perturbation (Fig. 6.4). The idea is the following: if this distance 'd' changes from positive to negative. or vice versa, then in between, at some point, 'd' is 0. That is, the stable and unstable manifolds intersect creating the homoclinic tangle.

The discussion in this section closely follows the treatment in [22].

For a simple illustration of the theory we consider a two dimensional autonomous system that has a single hyperbolic critical point and is perturbed by a periodic function of time. Thus this is a time dependent perturbation of a Hamiltonian system. However it is to be noted that the argument carries over to higher dimensions. Let the two dimensional system be denoted by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \epsilon \mathbf{f}_1(\mathbf{x}, t) \tag{6.2}$$

where  $\mathbf{x} = (x_1, x_2)$  and  $\mathbf{f_1}$  is periodic in t with period T. The unperturbed


Figure 6.3: Homoclinic Tangle and the Melnikov Distance

system is taken to be integrable and is assumed to possess a hyperbolic fixed point  $X_0$  and an integrable separatrix orbit  $x_0(t)$  such that  $\lim_{t\to\infty} x_0 = X_0$ .

The system is illustrated in Fig. (6.3(a)). The stable and unstable orbits  $x^{s}(t)$  and  $x^{u}(t)$  are labelled and smoothly joined to each other. There is, in general, an elliptic fixed point within the separatrix orbit.

To find the condition for intersection, we calculate, using perturbation theory, the distance D from the unstable to the stable orbit at time  $t_0$ . For D < 0 for all  $t_0$ , we have Fig. (6.3(b)). For D > 0 we have Fig. (6.3(c)) and if D changes sign for some  $t_0$ , we have the chaotic motion of Fig. (6.3(d)).



Figure 6.4: The Melnikov Distance - The dashed curve represents the unperturbed separatrix

To calculate D, we need the stable and unstable orbits  $x^s$  and  $x^u$  to first order in  $\epsilon$ . Writing

$$\mathbf{x}^{s,u}(t,t_0) = \mathbf{x}_0(t-t_0) + \epsilon \, \mathbf{x}_1^{s,u}(t,t_0) \tag{6.3}$$

where  $t_0$  is an arbitrary initial time and inserting (6.3) into (6.2), we obtain to first order

$$\frac{d}{dt}\mathbf{x}_1^{s,u} = \mathbf{M}(x_0) \mathbf{x}_1^{s,u} + \epsilon \mathbf{f}_1 (\mathbf{x}_0 (t - t_0), t)$$
(6.4)

where

$$\mathbf{M}(\mathbf{x_0}) = \begin{pmatrix} f_{01}; x_{01} & f_{01}; x_{02} \\ f_{02}; x_{01} & f_{02}; x_{02} \end{pmatrix}$$

is the Jacobian matrix of  $f_0$  evaluated at  $\mathbf{x}_0(t - t_0)$  and where the second subscripts denote the components of  $f_0$  and  $\mathbf{x}_0$ . Also,  $f_{01}$ ;  $x_{01} = \frac{\partial f_{01}}{\partial x_0^{-1}}$  and so on. We must solve (6.4) for  $\mathbf{x}^s$  for  $t > t_0$ , and for  $\mathbf{x}^u$  for  $t < t_0$ , with the condition that

 $\mathbf{x}^{\mathbf{s}}(t \to \infty) = \mathbf{x}^{\mathbf{u}}(t \to -\infty) = \mathbf{X}_{\mathbf{p}}$ , where  $\mathbf{X}_{\mathbf{p}}$  is the perturbed position of the hyperbolic fixed point. The two solutions differ by

$$\mathbf{d}(t, t_0) = \mathbf{x}^{\mathbf{s}}(t, t_0) - \mathbf{x}^{\mathbf{u}}(t, t_0) = \mathbf{x}_1^{\mathbf{s}}(t, t_0) - \mathbf{x}_1^{\mathbf{u}}(t, t_0)$$
(6.5)

The Melnikov distance  $D(t, t_0)$  is defined as

$$D(t,t_0) = \mathbf{N}.\mathbf{d} \tag{6.6}$$

which is the projection of d along a normal N to the unperturbed orbit  $\mathbf{x}_0$ at t (Fig. 6.4). From (6.2) (with  $\epsilon = 0$ ), a normal to  $\mathbf{x}_0(t - t_0)$  is

$$\mathbf{N}(t,t_0) = \begin{pmatrix} -f_{02}(x_0) \\ f_{01}(x_0) \end{pmatrix}$$
(6.7)

Introducing the wedge operator

 $\mathbf{x} \wedge \mathbf{y} = x_1 y_2 - x_2 y_1$  and substituting (6.7) into (6.6), we can write

$$D(t, t_0) = \mathbf{f_0} \wedge \mathbf{d} \tag{6.8}$$

To find an expression for D, we use (6.5) to write

$$D = D^s - D^u \tag{6.9}$$

with

$$D^{(s.u)}(t,t_0) = \mathbf{f_0} \wedge \mathbf{x_1}^{(s.u)}$$

$$(6.10)$$

Taking the time derivative of (6.10)

$$\dot{D}^{s} = \dot{f}_{0} \wedge \mathbf{x}_{1}^{s} + \mathbf{f}_{0} \wedge \dot{\mathbf{x}}_{1}^{s} = \mathbf{M}(\mathbf{x}_{0}).\dot{\mathbf{x}}_{0} \wedge \mathbf{x}_{1}^{s} + \mathbf{f}_{0} \wedge \dot{x}_{1}^{s}$$
(6.11)

Using  $\mathbf{x}_0 = \mathbf{f}_0$  and also (6.4) in (6.11)

$$\dot{D}^{s} = \mathbf{M}(\mathbf{x}_{0}) \cdot f_{0} \wedge \mathbf{x}_{1}^{s} + \mathbf{f}_{0} \wedge \mathbf{M}(\mathbf{x}_{0}) \cdot \mathbf{x}_{1}^{s} + \mathbf{f}_{0} \wedge \mathbf{f}_{1}$$
(6.12)

The first two terms in (6.12) combine to give

$$\dot{D}^{s} = T \tau \mathbf{M}(\mathbf{x}_{0}) \mathbf{f}_{0} \wedge \mathbf{x}_{1}^{s} + \mathbf{f}_{0} \wedge \mathbf{f}_{1}$$
(6.13)

where  $Tr\mathbf{M}$  is the trace of the Jacobian matrix of  $\mathbf{f_0}$ . since  $D^s$  follows the stable orbit, we must integrate (6.13) from t to  $\infty$ . Rather than treat this general case, we focus on an unperturbed Hamiltonian system, for which  $Tr\mathbf{M} = 0$  on the separatrix. Integrating (6.13) then yields

$$D^{s}(\infty, t_0) - D^{s}(t_0, t_0) = \int_{t_0}^{\infty} \mathbf{f}_0 \wedge \mathbf{f}_1 dt$$

But

$$D^{s}(\infty, t_{0}') = \mathbf{f}_{0}(\mathbf{x}_{0}(\infty - t_{0})) \wedge \mathbf{x}_{1}^{s} = 0$$

because  $f_0(X_0) = 0$ . Thus

$$D^{s}(t_{0}, t_{0}) = -\int_{t_{0}}^{\infty} \mathbf{f}_{0} \wedge \mathbf{f}_{1} dt \qquad (6.14)$$

Proceeding similarly to calculate  $D^u$ , we obtain  $\dot{D}^u = Tr\mathbf{M}(x_0)D^u + \mathbf{f_0} \wedge \mathbf{f_1}$ since  $D^u$  follows the unstable orbit, we integrate from  $-\infty$  to  $t_0$  to obtain. for an unperturbed hamiltonian system,

$$D^{u}(t_0, t_0) = \int_{-\infty}^{t_0} \mathbf{f}_0 \wedge \mathbf{f}_1 dt$$
(6.15)

Using (6.14) and (6.15) in (6.9), we obtain finally

$$D(t_0, t_0) = -\int_{-\infty}^{\infty} \mathbf{f_0} \wedge \mathbf{f_1} dt$$
(6.16)

If D changes sign at some  $t_0$ , the case in Fig. (6.3(d)) occurs and chaotic motion is present near the separatrix.

# 6.4 Melnikov Analysis of the Froude Pendulum

Consider the forced Froude pendulum given by the following equation

$$\ddot{\phi} + a\dot{\phi} + b\dot{\phi}^3 + hsin\phi = Q(\Omega) + Fcos(\omega t)$$
(6.17)

Transforming to a first order system

$$\dot{\phi} = y$$
  
$$\dot{y} = -hsin\phi - ay - by^3 + Q(\Omega) + Fcos\omega t \qquad (6.18)$$

The unperturbed system is

$$\dot{\phi} = y$$
  
 $\dot{y} = -hsin\phi$ 

For  $\phi = 0$ .  $\pm \pi$  and y = 0, the right hand side of the unperturbed system vanishes and hence the critical points are given by  $(0,0), (\pm \pi, 0)$ . The hamiltonian for the unperturbed system is given by

$$H = \frac{y^2}{2} - h\cos\varphi + h \tag{6.19}$$

From (6.19),

$$H(\pi, 0) = 2h \tag{6.20}$$

Substituting (6.20) into (6.19), we get

$$\frac{y^2}{2} - h\cos\phi + h = 2h \qquad (a)$$

$$\Rightarrow \frac{y^2}{2} = h \cos\phi + h = h(1 + \cos\phi) \qquad (b)$$

$$= h(2\cos^2\frac{\phi}{2}) \qquad (c)$$

$$\Rightarrow y = \pm 2\sqrt{h}\cos\frac{\phi}{2} \qquad (d) \qquad (6.21)$$

Again (0,0) is a fixed point. Hence

$$H(0,0) = 0 \tag{6.22}$$

Substituting (6.22) into (6.19),

$$\frac{y^2}{2} - h\cos\phi + h = 0$$

$$y = \pm \sqrt{2h(\cos\phi - 1)}$$

$$\dot{\phi} = \pm \sqrt{2h(\cos\phi - 1)}d\phi = \pm \sqrt{2h(\cos\phi - 1)}dt$$

This can be integrated to give

$$\phi = 2 \arcsin(\tanh(\sqrt{h})t) \tag{6.23}$$

$$y = \dot{\phi} = 2\sqrt{hsech} \sqrt{h} t \tag{6.24}$$

Referring back to (6.2), the equations can be written in explicit form as

$$\begin{pmatrix} \dot{\phi} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f_{01} \\ f_{02} \end{pmatrix} + \epsilon \begin{pmatrix} f_{11} \\ f_{12} \end{pmatrix}$$
(6.25)

From (6.18), for the Froude pendulum, these equations can be written as

$$\begin{pmatrix} \dot{\phi} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -h\sin\phi \end{pmatrix} + \epsilon \begin{pmatrix} 0 \\ -ay - by^3 + Q(\Omega) + F(\cos\omega t) \end{pmatrix}$$
(6.26)

Comparison of (6.25) and (6.26) gives

$$f_{01} = y$$
 (a)  
 $f_{11} = 0$  (b)  
 $f_{02} = -h \sin \phi$  (c)

$$f_{12} = -ay - by^{3} + Q(\Omega) + F \cos \omega t \qquad (d)$$
(6.27)

From the definition of the wedge product.

$$f_0 \wedge f_1 = f_{01}f_{12} - f_{02}f_{11}$$
  
=  $y(ay - by^3 + Q(\Omega) + F\cos\omega t$  (6.28)

From (6.16) and (6.28), the Melnikov integral can then be written as

$$D = -\int_{-\infty}^{\infty} \left[-ay^2 - by^4 + yQ(\Omega) + yF\cos(\omega)t\right] dt$$
 (6.29)

Now. from (6.24)

 $y = 2 \sqrt{h} \operatorname{sech}(\sqrt{h}) t$ 

Substituting the above equation into (6.29) and denoting the four integrals on the right hand side of (6.29) by  $D_1$ ,  $D_2$ ,  $D_3$  and  $D_4$ , one obtains

$$D_1 = \int_{-\infty}^{\infty} 4ah \ sech^2 \sqrt{h} \ t \ dt \tag{6.30}$$

This can be evaluated as

$$D_1 = 8ah \tag{6.31}$$

$$D_2 = \int_{-\infty}^{\infty} 16 \ bh^2 \ sech^4 \sqrt{h} \ t \ dt$$
 (6.32)

$$D_3 = -Q(\Omega) \int_{-\infty}^{\infty} 2\sqrt{h} \, \operatorname{sech}\sqrt{h} \, t \, dt \tag{6.33}$$

$$D_4 = -\int_{-\infty}^{\infty} 2\sqrt{h} \, \operatorname{sech}\sqrt{h} \, t \, F \cos \omega t dt \tag{6.34}$$

where

$$D = \sum_{i=1}^{4} D_i \tag{6.35}$$

These can be evaluated as

$$D_{1} = 8ah$$

$$D_{2} = \frac{32}{3}bh^{2}$$

$$D_{3} = -Q(\Omega)4\sqrt{h}\frac{\pi}{2}$$

$$D_{4} = -2\pi F sech\frac{\pi\omega}{2\sqrt{h}}$$
(6.36)

Finally, we obtain

•

$$D = 8ah + \frac{32}{3}bh^2 - 2\pi[\sqrt{h}Q(\Omega) + Fsech\frac{\pi\omega}{2\sqrt{h}}]$$
(6.37)

We know that a, b and h do not change sign. Hence D changes sign depending on  $Q(\Omega)$ , F and  $\omega$ . Thus, from (6.37) a combination of the parameter values  $Q(\Omega)$ , F,  $\omega$  which changes the sign of D leads to homoclinic chaos in the Froude pendulum.

#### 6.5 Illustration of the Melnikov Analysis

Based on the analysis above, we examine the phase portrait for values of the parameters which indicate chaotic behaviour. In all the figures, we trace the evolution of two solutions with initial conditions  $\phi(0) = 3, y(0) = 4$  and  $\phi(0) = 3.2, y(0) = 4.2$ .

1. Variation of forcing amplitude, F. other parameters fixed.

Consider a = 1. b = 0.075.  $Q(\Omega) = 0.0001$ . h = 1.  $\omega = \frac{2}{\pi}$ . For these values, from (6.37), the value of F at which the Melnikov distance D changes sign is F = 2.16088. Below we plot the t,  $\phi(t)$  diagram and the o(t). y(t) diagram (the phase space) for F = 1.5 and F = 2.5. That is, for one value below and the other above the critical value. As we can clearly see, for F = 2.5, there are indications of chaotic behaviour.

- (a) Non Chaotic Case (Figures 6.5, 6.6)
- (b) Indication of Chaotic Behaviour (Figures 6.7, 6.8)

2. Variation of damping coefficient, a. other parameters fixed

Consider F = 0.5, b = 0.075,  $Q(\Omega) = 0.0001$ , h = 1,  $\omega = \frac{2}{\pi}$ . For these values, from (6.37), the value of a at which the Melnikov distance D changes sign is, a = 0.2545868. Below we plot the t. o(t) diagram and o(t). y(t) diagram (the phase space) for a = 0.1 and a = 0.5. That is, for one value below and the other above the critical value. As we can clearly see, for a = 0.1, there are indications of chaotic behaviour.

- (a) Non Chaotic Case (Figures 6.9, 6.10)
- (b) Indication of Chaotic Behaviour (Figures 6.11, 6.12)

In conclusion, we note that the following diagrams are only indicative of chaotic behaviour, based on the Melnikov analysis. Further numerical investigations have to be carried out to obtain a more detailed understanding of chaos in the Froude pendulum.



Figure 6.5: Displacement-time diagram. No Chaos for a = 1, b = 0.075, Q = 0.0001, h = 1,  $\omega = \frac{2}{\pi}$ , F = 1.5. Initial conditions for two solutions  $\phi(0) = 3$  and  $\phi(0) = 3.2$ 



Figure 6.6: Phase Space. No Chaos for a = 1, b = 0.075, Q = 0.0001, h = 1.  $\omega = \frac{2}{\pi}$ , F = 1.5. Initial conditions for two solutions  $\phi(0) = 3$  and  $\phi(0) = 3.2$ 



Figure 6.7: Displacement-time diagram. Indications of Chaos for a = 1, b = 0.075, Q = 0.0001, h = 1,  $\omega = \frac{2}{\pi}$ , F = 2.5 Initial conditions for two solutions  $\phi(0) = 3$  and  $\phi(0) = 3.2$ 

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Figure 6.8: Phase Space. Indications of Chaos for a = 1, b = 0.075, Q = 0.0001, h = 1,  $\omega = \frac{2}{\pi}$ , F = 2.5 Initial conditions for two solutions  $\phi(0) = 3and\phi(0) = 3.2$ 



Figure 6.9: Displacement-time diagram. No Chaos for F = 0.5, b = 0.075, Q = 0.0001, h = 1,  $\omega = \frac{2}{\pi}$ , a = 0.5 Initial conditions for two solutions  $\phi(0) = 3$  and  $\phi(0) = 3.2$ 



Figure 6.10: Phase Space. No Chaos for F = 0.5, b = 0.075, Q = 0.0001, h = 1,  $\omega = \frac{2}{\pi}$ , a = 0.5 Initial conditions for two solutions  $\phi(0) = 3$ and  $\phi(0) = 3.2$ 



Figure 6.11: Displacement-time diagram. Indications of Chaos for F = 0.5, b = 0.075, Q = 0.0001, h = 1,  $\omega = \frac{2}{\pi}$ , a = 0.1 Initial conditions for two solutions  $\phi(0) = 3$  and  $\phi(0) = 3.2$ 



Figure 6.12: Phase Space. Indications of Chaos for F = 0.5, b = 0.075, Q = 0.0001. h = 1,  $\omega = \frac{2}{\pi}$ , a = 0.1 Initial conditions for two solutions  $\phi(0) = 3$  and  $\phi(0) = 3.2$ 

### Chapter 7

### **Conclusions and Further Work**

The averaging method applied to the autonomous Froude pendulum and the Melnikov analysis applied to the non-autonomous case of the same system have yielded useful and interesting results. In this chapter, we draw conclusions from these results and suggest directions for further work.

#### 7.1 Conclusions

1. With respect to the analysis and design of nonlinear engineering systems, the work presented in this thesis confirms that the neglect of nonlinear terms, unless justified completely in the context, can lead to highly erroneous conclusions. For instance, had one treated the Froude pendulum as a perturbation of the simple harmonic oscillator, i.e.  $\ddot{x} + x = 0$ , the analysis would have been much simpler, but the essential features would have been completely missed.

- 2. The Froude pendulum. being a classical example of friction induced nonlinear oscillations. the present work highlights the richness of structure and phenomena inherent in this class of systems. Also, we have shown that a considerable amount of precise analysis is possible in engineering systems which exhibit nonlinear frictional effects. The focus on the two parameters 'a' and 'c' in the present work is by no means unique and one could carry our similar analyses in the case of different sets of parameters.
- 3. An important observation in this thesis is the presence of limit cycles in the Froude pendulum and the bifurcation phenomena associated with the two parameters. Limit cycles, as we have seen before, are of vital importance in nonlinear analysis.
- 4. In this thesis, we have analyzed a system with a nonlinear damping term. This facilitates the analysis of other engineering systems with nonlinear damping factors.

- 5. From the point of view of chaos, the present work confirms, by the technique of Melnikov analysis and computational verification, that the Froude pendulum exhibits chaotic behaviour. Thus, we confirm that yet another important engineering system demonstrates chaos.
- 6. The analysis of most oscillatory systems involves the small angle approximation, i.e., sin(φ) ≈ φ. Since we do not make any such approximation, the technique used in this thesis applies to large oscillations. The present work also shows that the averaging method can be applied to a wide class of nonlinear systems that can be treated as perturbations of the mathematical pendulum. These are highlights of the present work.
- 7. From the analytical point of view, the present work is an example of nonlinear averaging. The neglect of nonlinearities has also serious consequences from the mathematical point of view since it leads to a completely different set of mathematical results. Seen from the perspective of perturbation techniques, the work in this thesis reminds us that we are not constrained to treat all nonlinear oscillators as perturbations of the simple harmonic oscillator. In other words, effective

analysis is possible even as we retain the nonlinear terms and consider large oscillations and nonlinear damping.

8. Finally, we need to look at the conclusions that could be drawn from the numerical perspective. While the importance of numerical tools in nonlinear analysis is too well known to warrant further mention, the present work highlights the advantage of having analytical results before the system is approached from the computational perspective. For instance, it was the analysis that pointed to the different regions in phase space, where interesting phenomena were observed. In the case of the Melnikov analysis, the analytical results gave an indication of parameter values where chaotic behaviour could be found. But we must note that the present work would not have been complete without computational tools

#### 7.2 Further Work

The present work suggests various directions for further research.

1. The direct extension of the present work will be a similar analysis of the forced Froude pendulum. Non-autonomous nonlinear systems are notorious for serious difficulties and the Froude pendulum cannot be expected to be an exception. Yet, the bifurcation diagram, the limit cycles and the different phase portraits provide valuable information for further work in the non-autonomous case.

- 2. The bifurcations with respect to sets of parameters other than the one considered in this thesis can be carried out, within the framework that we have used. The behaviour of the system under a scaling different from the one used in this thesis should also be interesting.
- 3. The cubic nonlinearity in the damping term is a consequence of expanding the 'friction function',  $F(M - \omega)$  as a Taylor series. The present technique can be applied in the case of any general function representing friction.
- 4. Under a broader scope, the averaging technique used in this thesis can be applied to various nonlinear systems similar to the Froude pendulum, which so far have been treated under the small angle approximation.
- 5. The Melnikov analysis may be applied to other important engineering systems to obtain analytical criteria for the onset of chaos.

6. Experimental work needs to be carried out on the Froude pendulum to obtain a relation between the frictional torque acting on the system and relative angular velocity,  $M(\Omega - \dot{\theta})$  and to verify the results obtained in this thesis.

These observations conclude the chapter and the thesis.

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