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PREDICTION OF TRANSIENT OSCILLATIONS IN NONLINEAR CONTROL SYSTEMS

by

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ABSTRACT

The thesis presents a definition of a complex function which is exploited to introduce a unified theory for predicting the dynamic behavior of nonlinear control systems.

For symmetrical oscillations, a transient form of the Krylov-Bogoliubov asymptotic method is given. In addition, a new oscillatory transient describing function is derived using a finite Laplace transform. The oscillatory transient describing function is proven to be optimal in the sense that it minimizes the mean square error. A sampled data version of this function is also provided. A computational procedure is presented for predicting the oscillatory transient behavior in nonlinear systems and determining their dynamic stability.

A novel asymmetrical transient describing function is derived and its associated properties investigated. A sampled data version of this function is given. An auxiliary equation is introduced to describe the bias signal during the asymmetrical behavior. The new equation is utilized together with the quasi-linearized equation of the system to determine all possible asymmetrical oscillatory modes and their corresponding dynamic stability.

A comparison between the proposed theory and the other methods available in the literature is provided. The application of the theory to a practical system is demonstrated.

TO OUR PARENTS

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NOTATION

σ	Damping factor
ω_n	Real constant
μ	Small real constant
t	Time variable
$x(t)$	Input to nonlinear element
$f(x,\dot{x})$	Nonlinear continuous function
A_0	Instantaneous amplitude
ω_0	Natural frequency
θ	Phase shift
ψ	Time parameter
ω	Frequency of oscillations
A	Amplitude of oscillatory behavior
α	Relative damping
$N_p(A_0,\alpha,\sigma,\omega_0)$	Direct gain of nonlinearity
$N_q(A_0,\alpha,\sigma,\omega_0)$	Quadratic gain of nonlinearity
$y(t)$	Output of nonlinear element
$x(t,A_0)$	Oscillatory transient input to nonlinear element
$y(t,A_0)$	Oscillatory transient output of nonlinear element
s	Laplace transform complex variable
$N_h(A_0,s)$	Finite period complex function
$u_{-1}(t)$	Unit step function
$u_h(t)$	Finite period step function
h	Time interval
N	Nonlinear part of system
$x_h(t,A_0)$	Finite period oscillatory transient input to N

$y_h(t,A_0)$	Finite period oscillatory transient output of N
$X_h(A_0,s)$	Laplace transform of $x_h(t,A_0)$
$Y_h(A_0,s)$	Laplace transform of $y_h(t,A_0)$
$X(A_0,s)$	Laplace transform of $x(t,A_0)$
$Y(A_0,s)$	Laplace transform of $y(t,A_0)$
c_1, c_2	Real constants
$y(x,\dot{x};t)$	Nonlinear continuous function
OTDF	Oscillatory transient describing function
T	Complete period interval
$N_T(A_0,\alpha,\omega)$	Complete period OTDF
$N_{Tp}(A_0,\alpha,\omega)$	Direct component of $N_T(A_0,\alpha,\omega)$
$N_{Tq}(A_0,\alpha,\omega)$	Quadratic component of $N_T(A_0,\alpha,\omega)$
$y(\psi,A_0)$	Output of N
$y_a(\psi,A_0)$	Quasi-linearized output of N
$a(\psi)$	Amplitude of inphase component of $y_a(\psi,A_0)$
$b(\psi)$	Amplitude of quadratic component of $y_a(\psi,A_0)$
$e(\psi,A_0)$	Approximation error
$G(s)$, $G(\alpha\omega,j\omega)$	Transfer function of linear portion of system
$L_1(\alpha,\omega)$	Real component of $G(\alpha\omega,j\omega)$
$L_2(\alpha,\omega)$	Imaginary component of $G(\alpha\omega,j\omega)$
A_0'	Instantaneous amplitude value
$r(t)$	Reference input to system
$c(t)$	Output of system
$N_h(A_0,\alpha,\omega)$	Finite period function
$N_{hp}(A_0,\alpha,\omega)$	Direct component of $N_h(A_0,\alpha,\omega)$
$N_{hq}(A_0,\alpha,\omega)$	Quadratic component of $N_h(A_0,\alpha,\omega)$

$S_0(A_0', \alpha_0, \omega_0)$	Instantaneous oscillatory transient solution at A_0'
$S(A_0, \alpha, \omega)$	Instantaneous oscillatory transient solution at A_0
$U(S)$, $V(S)$	Real continuous functions of $S(A_0, \alpha, \omega)$
E	System operator
K_α	Relative damping gain
K_ω	Frequency gain
η	Stability factor
R_1, R_2	Small regions
δ_1, δ_2, ε_1, ε_2	Small real constants
ε	Belongs to
$\forall$	For every
A_{0q}	Limit cycle amplitude
S_q	Equilibrium state of system
$\mathcal{Z}$	Z-transform operator
p, q	Real constants
T_s	Sampling period
$(\ )^*$	Sampling function
τ	Time constant
n	Sampling instant
N_s	Sampling points in half period
$\phi(x_n, x_{n+1})$	Nonlinear sampled function
$g(x_n, x_{n+1})$	Nonlinear sampled function
z	Z - transform complex variable
$N_h(A_0, z)$	Finite period complex function
$N_T^*(A_0, \alpha, \omega)$	Sampled OTDF
$N_{Tp}^*(A_0, \alpha, \omega)$	Direct component of $N_T^*(A_0, \alpha, \omega)$
$N_{Tq}^*(A_0, \alpha, \omega)$	Quadratic component of $N_T^*(A_0, \alpha, \omega)$

$S^*(A_0,\alpha,\omega)$	Sampled oscillatory transient solution
η^*	Sampled stability factor
$N_{\frac{T}{2}}(A_0,\alpha,\omega)$	Half period OTDF
$N_{\frac{T}{2}p}(A_0,\alpha,\omega)$	Direct component of $N_{\frac{T}{2}}(A_0,\alpha,\omega)$
$N_{\frac{T}{2}q}(A_0,\alpha,\omega)$	Quadratic component of $N_{\frac{T}{2}}(A_0,\alpha,\omega)$
$\theta_T(A_0,\alpha,\omega)$	Phase of $N_T(A_0,\alpha,\omega)$
$Q_1(\alpha,\omega)$	Real component of $-1/G(\alpha\omega,j\omega)$
$Q_2(\alpha,\omega)$	Imaginary component of $-1/G(\alpha\omega,j\omega)$
$\theta_g(\alpha,\omega)$	Phase angle of $-1/G(\alpha\omega,j\omega)$
D	Real constant
θ_{m_1}	First maximum peak of $x(t)$
θ_{m_2}	First minimum peak of $x(t)$
$\psi_1,\ \psi_2,\ \psi_3$	Phase angles
K	Gain of system
$f(x,\dot{x};t)$	Nonlinear function
B_0	Instantaneous bias component
B	Bias amplitude of asymmetrical behavior
$N_0(A_0,B_0,\alpha,\sigma,\omega_0)$	Auxiliary direct gain of nonlinearity
$x(t,A_0,B_0)$	Asymmetrical oscillatory transient input to N
$y(t,A_0,B_0)$	Asymmetrical oscillatory transient output of N
$N_h(A_0,B_0,s)$	Finite period complex function
$N_T(A_0,B_0,\alpha,\omega)$	Asymmetrical OTDF
$N_{Tp}(A_0,B_0,\alpha,\omega)$	Direct component of $N_T(A_0,B_0,\alpha,\omega)$
$N_{Tq}(A_0,B_0,\alpha,\omega)$	Quadratic component of $N_T(A_0,B_0,\alpha,\omega)$
$N_0(A_0,B_0,\alpha,\omega)$	Auxiliary direct gain of N

$y(\psi,A_0,B_0)$	Asymmetrical output of N
$c(\psi)$	Output bias component of N
$y_a(\psi,A_0,B_0)$	Quasi-linearized output of N
$e(\psi,A_0,B_0)$	Approximation error
$W(A_0,B_0,\alpha,\omega)$	Auxiliary function of system
$S_0(A_0',B_0',\alpha_0,\omega_0)$	Instantaneous asymmetrical oscillatory transient solution at A'
$S(A_0,B_0,\alpha,\omega)$	Instantaneous asymmetrical oscillatory transient solution at A
E_a	Asymmetrical system operator
K_b	Bias gain
η_a	Asymmetrical stability factor
$N_h(A_0,B_0,z)$	Finite period complex function
$N_T^*(A_0,B_0,\alpha,\omega)$	Sampled asymmetrical OTDF
$N_0^*(A_0,B_0,\alpha,\omega)$	Sampled auxiliary direct gain of N
$W^*(A_0,B_0,\alpha,\omega)$	Sampled auxiliary function of system
$y'(\psi,A_0)$	Weighted output of N
$y_a'(\psi,A_0)$	Weighted quasi-linearized output of N
$\overline{e^2}$	Mean square error
a_T	Complete period in phase component
b_T	Complete period quadratic component
c_T	Complete period bias component
Iff	If and only if
$\blacksquare$	Q.E.D.

1. INTRODUCTION

1.1 INTRODUCTORY REMARKS

Systems that occur in real life are always nonlinear. Linear systems are approximate models of nonlinear systems and are considered usually for convenience. Nonlinearities are sometimes introduced to improve system performance. The behavior of nonlinear systems is complex since their characteristics are signal dependent. The main approaches for the study of nonlinear systems are the phase plane techniques, stability analysis and approximate solutions, [1] - [8].

The phase plane method has been utilized in the literature of nonlinear systems to determine their local and global behavior. It provides an exact topological account of all possible system trajectories with all possible initial conditions. However, this method is only convenient for low order systems as its application to higher order systems introduces formidable complications to the analysis.

Under stability analysis, Liapunov's theorem is usually used. This method is a powerful tool for obtaining a qualitative view of the system global behavior. It is mainly based upon investigating the given form of the differential equations without solving them. This comes by reformulating such equations. During reformulation certain information about the system characteristic is always lost and cannot be estimated. Consequently, most of the stability conditions are sufficient and not necessary. In addition, the stability analysis method cannot predict easily margins of stability nor the extent of any instability associated in the system.

Approximate methods of solution permit a direct and efficient way for the investigation of a wide class of nonlinear problems. They represent the first step in the design and synthesis of nonlinear systems. They give a simple estimation of how the structure and parameters of the system influence the system dynamic characteristics. System simulation can then provide the actual solution of the design problem. An important question can arise about the determination of the accuracy of solutions obtained using the approximation methods. Unfortunately, this problem is generally tedious to study and the designer is usually forced to use the approximate techniques despite incomplete knowledge of their accuracy.

Finally, it has been said very often that for a control system to be superior, its performance should be predicted precisely and should have a unique stable equilibrium. In the literature of nonlinear control systems, the problem of checking the uniqueness and stability of equilibrium has been always an extremely hazardous enterprise. In the following chapters an attempt is made to solve this problem by describing the oscillatory transient behaviors associated frequently with nonlinear control systems and determining their dynamic stability.

1.2 OBJECTIVES

This thesis is devoted to the analysis of transient oscillations in nonlinear control systems. The oscillations are classified as symmetrical or asymmetrical. The object is to describe such transient oscillations and to determine their dynamic stability. An approximate solution technique is used.

In Chapter 2, a unified theory for the investigation of symmetrical transient oscillations in nonlinear control systems is presented.

The analysis includes continuous and sampled data systems. A transient form of the Krylov-Bogoliubov asymptotic method is given. Also, a computational procedure for the prediction of symmetrical transient oscillations and determining their dynamic stability is described in detail.

In Chapter 3, the proposed theory given in Chapter 2 is applied to nonlinear control systems with symmetrical oscillatory behavior. A comparison between the presented theory and the other approximate methods available in the literature is also discussed.

In Chapter 4, the theory suggested has been generalized for nonlinear control systems with asymmetrical oscillatory transient behavior. The chapter follows similar steps as Chapter 2. The description of the dynamic stability of the behavior is also provided.

In Chapter 5, the theory presented is applied to a positional control system with symmetrical and nonsymmetrical nonlinearities.

2. SYMMETRICAL TRANSIENT OSCILLATIONS - THEORY

2.1 INTRODUCTION

Approximate techniques have been commonly used to analyze nonlinear control systems [1]-[8]. These techniques can mainly be applied due to the availability of digital computers. For the study of sustained oscillations in nonlinear systems, the describing function has proven to be particularly suitable. For the investigation of the transient or dynamic behaviour of nonlinear systems, various approaches have been reported [9]-[14]. The analytic description of transient oscillations in such systems is a matter of practical importance [7]. Such oscillations can be expressed in terms of exponentially damped (or divergent) sine waves with time varying amplitudes and frequencies. For lightly damped transients, the describing function can be applied with success [1], [9]-[14]. Grensted [15] proposed a technique for the analysis of transient oscillations restricted to second order differential equations. Popov [1] used some of the results by Grensted to extend the Krylov-Bogoliubov asymptotic method [2] to the transient case where the rate of change of the frequency and damping is small. Recently, Freeman and Cox [16], [17] introduced the concept of half period transient gain to describe the nonlinearities during the transient period. The approximation used minimizes the total square error. Freeman [18] showed that like the describing function, the quadratic component vanishes for single-valued nonlinearities. This property seems to yield inaccuracies in predicting oscillatory transient processes, since unlike sustained oscillations, transient oscillations undergo a phase shift upon passing through nonlinear elements. Furthermore, a half wave transient gain does not seem to be suitable for non-symmetrical nonlinearities. The use of two half period gains to study the latter nonlinearities can add formidable complications to the analysis.

In this work a unified theory is presented for studying transient oscillations in nonlinear systems. Continuous, also discrete, systems are considered.

2.2 THE TRANSIENT FORM OF THE KRYLOV-BOGOLIUBOV ASYMPTOTIC METHOD

In the study of transient oscillations in nonlinear systems, the transient oscillations are usually considered to be the solution of a second-order nonlinear differential equation of the form, [1],

$$\ddot{x} - 2\sigma\dot{x} + \omega_n^2 x + \mu\, f(x,\dot{x}) = 0 \qquad (2.2\text{-}1)$$

where σ, ω_n and μ are real constant coefficients. The value of μ is chosen to be small and $f(x,\dot{x})$ to be a nonlinear function of its arguments. Assume the general form of the solution of (2.2-1) to be

$$x(t) = A_0(t)\, e^{\sigma t} \sin[\omega_0 t + \theta(t)] \qquad (2.2\text{-}2)$$

where $A_0(t)$ and $\theta(t)$ are time dependent functions. For convenience we shall assume this time dependency without indicating it by appended arguments. We assume

$$\omega_0^2 = \omega_n^2 - \sigma^2 > 0. \qquad (2.2\text{-}3)$$

Let

$$\psi(t) = \omega_0 t + \theta \qquad (2.2\text{-}4)$$

so that

$$\dot{\psi}(t) = \omega$$
$$= \omega_0 + \dot{\theta}. \tag{2.2-5}$$

Furthermore, let

$$A = A_0 e^{\sigma t} . \tag{2.2-6}$$

We seek a solution of the form (2.2-2) and (2.2-7)

$$\dot{x}(t) = A_0 e^{\sigma t}(\sigma \sin\psi + \omega_0 \cos\psi). \tag{2.2-7}$$

Such a solution does not yield second order derivatives of A_0 and θ when $\ddot{x}(t)$ is determined. This requires that

$$\frac{\dot{A}_0}{A_0} \sin\psi + \dot{\theta} \cos\psi = 0. \tag{2.2-8}$$

Differentiating (2.2-7) with respect to time and using (2.2-2), (2.2-7) and (2.2-8), we obtain

$$\ddot{x}(t) = A_0 e^{\sigma t} \omega_0 \left(\frac{\dot{A}_0}{A_0} \cos\psi - \dot{\theta} \sin\psi \right) + 2\sigma\dot{x} - \omega_n^2 x. \tag{2.2-9}$$

Substituting (2.2-9) into (2.2-1), we get

$$\frac{\dot{A}_0}{A_0} \cos\psi - \dot{\theta} \sin\psi = - \frac{\mu}{A_0 \omega_0} e^{-\sigma t} f[A \sin\psi, A(\sigma \sin\psi + \omega_0 \cos\psi)]. \qquad \ldots(2.2\text{-}10)$$

From (2.2-8) and (2.2-10) and after some manipulations, we obtain

$$\dot{A}_0 = - \frac{\mu}{\omega_0} e^{-\sigma t} f[A \sin\psi, A(\sigma \sin\psi + \omega_0 \cos\psi)]\cos\psi \tag{2.2-11}$$

and

$$\dot{\theta} = \frac{\mu}{A_0 \omega_0} e^{-\sigma t} f[A \sin\psi, A(\sigma \sin\psi + \omega_0 \cos\psi)]\sin\psi . \tag{2.2-12}$$

Introduce the parameter α, to be denoted as the relative damping, such that

$$\alpha\psi = \sigma t. \tag{2.2-13}$$

Differentiating (2.2-13) yields

$$\alpha = \frac{\sigma}{\omega} . \tag{2.2-14}$$

Since μ is chosen to be small, then $\dot{A}_0$ and $\dot{\theta}$ are small, so that $A_0(t)$ and $\theta(t)$ are slowly varying functions of time. Hence we may consider $\dot{A}_0$ and $\dot{\theta}$ to be approximated by stepwise functions with the duration of each step equal to half a period [16] or to a complete period [7]. If the duration of each step is a complete period, then integrating (2.2-11) and (2.2-12), we get

$$\dot{A}_0 = -\frac{\mu}{2\pi\omega_0}\int_0^{2\pi} e^{-\alpha\psi}\, f[A\sin\psi,\, A(\sigma\sin\psi + \omega_0\cos\psi)]\cos\psi\, d\psi \qquad \ldots(2.2\text{-}15)$$

and

$$\dot{\theta} = \frac{\mu}{2\pi A_0\omega_0}\int_0^{2\pi} e^{-\alpha\psi}\, f[A\sin\psi,\, A(\sigma\sin\psi + \omega_0\cos\psi)]\sin\psi\, d\psi . \qquad \ldots(2.2\text{-}16)$$

Equations (2.2-15) and (2.2-16) will be denoted as the transient form of the Krylov-Bogoliubov asymptotic method. Upon comparing these formulae with those of Popov [1] two distinct differences appear. First, the introduction of the weighting function $e^{-\alpha\psi}$ under the integration and second upon evaluating the integrals, the amplitude A is expressed in its explicit form $A_0e^{\sigma t}$.

Equations (2.2-15) and (2.2-16) can be expressed as

$$\dot{A}_0 = -\frac{\mu}{2\omega_0} A_0 N_q \tag{2.2-17}$$

and

$$\dot{\theta} = \frac{\mu}{2\omega_0} N_p \tag{2.2-18}$$

where

$$\begin{aligned} N_p &= N_p(A_0, \alpha, \sigma, \omega_0) \\ &= \frac{1}{\pi A_0} \int_0^{2\pi} e^{-\alpha\psi} f[A \sin\psi, A(\sigma \sin\psi \\ &\qquad + \omega_0 \cos\psi)] \sin\psi \, d\psi \end{aligned} \tag{2.2-19}$$

and

$$\begin{aligned} N_q &= N_q(A_0, \alpha, \sigma, \omega_0) \\ &= \frac{1}{\pi A_0} \int_0^{2\pi} e^{-\alpha\psi} f[A \sin\psi, A(\sigma \sin\psi \\ &\qquad + \omega_0 \cos\psi)] \cos\psi \, d\psi. \end{aligned} \tag{2.2-20}$$

2.3 SYMMETRICAL OSCILLATORY TRANSIENT DESCRIBING FUNCTION

Let $x(t)$, (2.2-2), and $y(t)$ be the input and output to a nonlinear element. For $A_0(t)$, (2.2-2), a constant, let $x(t) = x(t,A_0)$ and $y(t) = y(t,A_0)$. Furthermore, let

$$x_h(t,A_0) = x(t,A_0)u_h(t) \tag{2.3-1}$$

and

$$y_h(t,A_0) = y(t,A_0)u_h(t) \tag{2.3-2}$$

where

$$u_h(t) = u_{-1}(t) - u_{-1}(t-h) \tag{2.3-3}$$

h is a constant.

Define the finite period complex function $N_h(A_0,s)$ as

$$N_h(A_0,s) = \frac{Y_h(A_0,s)}{X_h(A_0,s)} \ . \tag{2.3-4}$$

Hence,

$$N_h(A_0,s) = \frac{\int_0^\infty y_h(t,A_0)e^{-st}\,dt}{\int_0^\infty x_h(t,A_0)e^{-st}\,dt}$$

$$= \frac{\int_0^h y(t,A_0)e^{-st}\,dt}{\int_0^h x(t,A_0)e^{-st}\,dt} \ . \tag{2.3-5}$$

Consider a nonlinear element consisting of a nonlinearity $y(x,\dot{x};t)$. Let the input be

$$x(t,A_0) = A_0 e^{\sigma t}\sin\psi(t) \tag{2.3-6}$$

where $\psi(t) = \omega t$. Then the output can be expressed as

$$y(t,A_0) = y[A_0 e^{\sigma t} \sin\omega t,\ A_0 e^{\sigma t}(\sigma \sin\omega t + \omega \cos\omega t);\ t] \qquad \ldots(2.3\text{-}7)$$

where ω is assumed to be constant.

For $h=T=\frac{2\pi}{\omega}$ and expressing s as $s=\sigma+j\omega$, we obtain from (2.3-5), (2.3-6) and (2.3-7)

$$N_T(A_0,\sigma,\omega) = \frac{\int_0^{\frac{2\pi}{\omega}} y[A_0 e^{\sigma t} \sin\omega t,\ A_0 e^{\sigma t}(\sigma \sin\omega t + \omega \cos\omega t);\ t] e^{-(\sigma+j\omega)t}\, dt}{\int_0^{\frac{2\pi}{\omega}} A_0 e^{\sigma t}(\sin\omega t) e^{-(\sigma+j\omega)t}\, dt} . \qquad \ldots(2.3\text{-}8)$$

Changing the variable of integration in (2.3-8), introducing α, (2.2-14), and simplifying yields

$$N_T(A_0,\alpha,\omega) = \frac{j}{\pi A_0} \int_0^{2\pi} e^{-\alpha\psi}\, y[A_0 e^{\alpha\psi} \sin\psi,\ A_0 e^{\alpha\psi} \omega(\alpha \sin\psi + \cos\psi);\ \tfrac{\psi}{\omega}]\, e^{-j\psi}\, d\psi . \qquad (2.3\text{-}9)$$

Let $N_T(A_0,\alpha,\omega)$, (2.3-9), be the complete period oscillatory transient describing function. Express $N_T(A_0,\alpha,\omega)$ as

$$N_T(A_0,\alpha,\omega) = N_{Tp}(A_0,\alpha,\omega) + jN_{Tq}(A_0,\alpha,\omega) \qquad (2.3\text{-}10)$$

We have

$$N_{Tp}(A_0,\alpha,\omega) = \frac{1}{\pi A_0} \int_0^{2\pi} e^{-\alpha\psi}\, y[A_0 e^{\alpha\psi} \sin\psi,\ A_0 e^{\alpha\psi} \omega(\alpha \sin\psi + \cos\psi);\ \tfrac{\psi}{\omega}] \sin\psi\, d\psi \qquad (2.3\text{-}11)$$

and

$$N_{Tq}(A_0,\alpha,\omega) = \frac{1}{\pi A_0} \int_0^{2\pi} e^{-\alpha\psi}\, y[A_0 e^{\alpha\psi}\sin\psi,\ A_0 e^{\alpha\psi}$$

$$\omega(\alpha \sin\psi + \cos\psi);\ \tfrac{\psi}{\omega}] \cos\psi\, d\psi. \qquad (2.3\text{-}12)$$

If =0, (2.3-9) reduces to the describing function [7].

2.4 PROPERTIES OF THE OSCILLATORY TRANSIENT DESCRIBING FUNCTION

Let a nonlinear element $y(x,\dot{x};t)$ be injected with the transient component

$$x(\psi,A_0) = A_0 e^{\alpha\psi}\sin\psi \qquad (2.4\text{-}1)$$

where $\psi = \omega t$ and $\alpha = \frac{\sigma}{\omega}$. The output from the nonlinearity can be expressed as

$$y(\psi,A_0) = y[A_0 e^{\alpha\psi}\sin\psi,\ A_0 e^{\alpha\psi}\omega(\alpha\sin\psi + \cos\psi);\ \tfrac{\psi}{\omega}]\ . \qquad \ldots(2.4\text{-}2)$$

Assuming the nonlinear element can be approximated by a gain and a phase shift, the corresponding output can be expressed in the general form

$$y_a(\psi,A_0) = e^{\alpha\psi}(a\sin\psi + b\cos\psi) \qquad (2.4\text{-}3)$$

where a and b are functions of A_0, α and ψ. This dependency will not be indicated by appended arguments.

Let

$$y'(\psi,A_0) = e^{-\alpha\psi}y(\psi,A_0) \qquad (2.4\text{-}4)$$

and

$$y_a'(\psi,A_0) = e^{-\alpha\psi}y_a(\psi,A_0)\ . \qquad (2.4\text{-}5)$$

We seek to minimize the following error measure

$$\overline{e^2} = \frac{1}{2\pi} \int_0^{2\pi} e^2(\psi,A_0)\ d\psi \tag{2.4-6}$$

where

$$e(\psi,A_0) = y'(\psi,A_0) - y_a'(\psi,A_0)\ . \tag{2.4-7}$$

The weighting used to obtain y' and y_a', results in transforming the latter functions to sinusoids and hence permits the analysis to proceed similar to that of the describing function [7]. At the minimum of (2.4-6), we have

$$\frac{\partial\overline{e^2}}{\partial a} = \frac{\partial\overline{e^2}}{\partial b} = 0\ . \tag{2.4-8}$$

From (2.4-2) to (2.4-8), we obtain

$$a_T = \frac{1}{\pi} \int_0^{2\pi} e^{-\alpha\psi} y[A_0 e^{\alpha\psi}\sin\psi,\ A_0 e^{\alpha\psi}\omega(\alpha\ \sin\psi + \cos\psi);\ \frac{\psi}{\omega}] \sin\psi\ d\psi \qquad \ldots(2.4\text{-}9)$$

and

$$b_T = \frac{1}{\pi} \int_0^{2\pi} e^{-\alpha\psi} y[A_0 e^{\alpha\psi}\sin\psi,\ A_0 e^{\alpha\psi}\omega(\alpha\ \sin\psi + \cos\psi);\ \frac{\psi}{\omega}] \cos\psi\ d\psi. \qquad \ldots(2.4\text{-}10)$$

From (2.3-5), (2.4-1) and (2.4-3), it can be shown that

$$N_T(A_0,\alpha,\omega) = \frac{a_T}{A_0} + j\,\frac{b_T}{A_0} . \qquad (2.4\text{-}11)$$

Substituting (2.4-9) and (2.4-10) into (2.4-11), we obtain $N_T(A_0,\alpha,\omega)$ as given in (2.3-9). Hence $N_T(A_0,\alpha,\omega)$ is optimal in the sense that it minimizes the mean square error (2.4-6).

2.5 STABILITY STUDIES

Consider the nonlinear system in Fig. 2.1. $N_h(A_0,\alpha,\omega)$ describes the effect of a nonlinearity in the closed loop control system during a period h where A_0 is the instantaneous amplitude of the input. G(s) is the transfer function of a linear plant and is considered to be of the form

$$G(s) = \frac{a_m s^m + a_{m-1}s^{m-1} +\dots+ a_0}{b_n s^n + b_{n-1}s^{n-1} +\dots+ b_0} , \quad n-m \geq 2 \qquad (2.5\text{-}1)$$

where $s = \sigma + j\omega = \alpha\omega + j\omega$, so that

$$G(\alpha\omega,j\omega) \doteq L_1(\alpha,\omega) + jL_2(\alpha,\omega). \qquad (2.5\text{-}2)$$

$L_1(\alpha,\omega)$ and $L_2(\alpha,\omega)$ are real continuous functions of their arguments. For any transient oscillations to take place, it is necessary and sufficient that the quasi-linearized system satisfies

$$1 + N_h(A_0,\alpha,\omega)\, G(\alpha\omega,j\omega) = 0. \qquad (2.5\text{-}3)$$

Equation (2.5-3) implies

$$1 + N_{hp}(A_0,\alpha,\omega)L_1(\alpha,\omega) - N_{hq}(A_0,\alpha,\omega)L_2(\alpha,\omega) = 0 \qquad (2.5\text{-}4)$$

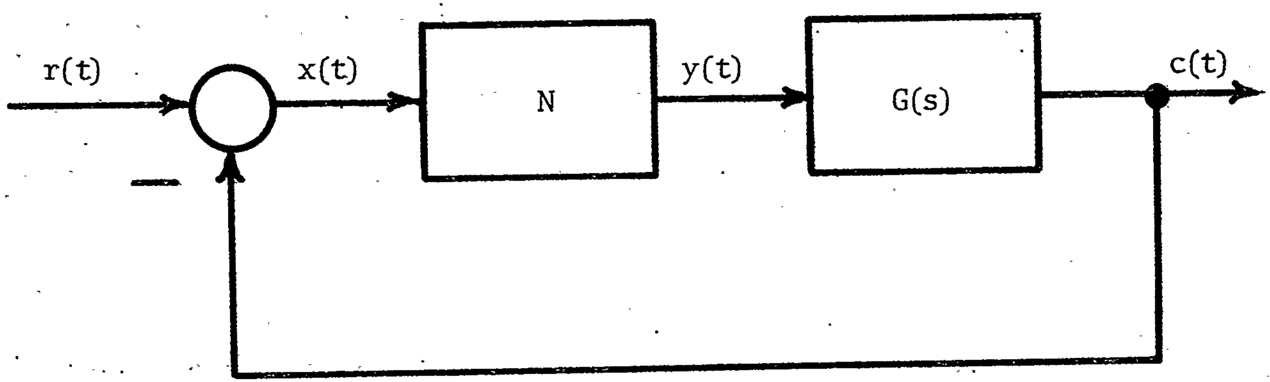

Fig. 2.1 A nonlinear control system.

and

$$N_{hp}(A_0,\alpha,\omega)L_2(\alpha,\omega) + N_{hq}(A_0,\alpha,\omega)L_1(\alpha,\omega) = 0. \qquad (2.5\text{-}5)$$

Equations (2.5-4) and (2.5-5) have an infinite number of solutions. For a given value of $A_0 = A_0'$, the solution will be denoted by $S_0 = S_0(A_0',\alpha_0,\omega_0)$ and represents the instantaneous oscillatory transient solution. Such a solution will be considered to be stable if any perturbations to A_0' tend to vanish as $t \to \infty$.

Theorem 2.1

Let U and V be the real and imaginary parts of (2.5-3). Consequently,

$$U(S) + jV(S) = 0 \qquad (2.5\text{-}6)$$

where $S = S(A_0,\alpha,\omega)$.

Assume that U and V and their first derivatives are continuous functions of S in a small domain around S_0. Define the operator E, such that

$$E = \frac{\partial}{\partial A_0} + K_\alpha(A_0)\frac{\partial}{\partial \alpha} + K_\omega(A_0)\frac{\partial}{\partial \omega} \qquad (2.5\text{-}7)$$

where K_α and K_ω are real functions of A_0.

Then a necessary and sufficient condition for S_0 to be stable is for a real positive number η to exist such that in the neighbourhood of S_0 the condition

$$\frac{\partial U}{\partial \alpha}E(U) + \frac{\partial V}{\partial \alpha}E(V) + \eta\left(\frac{\partial V}{\partial \alpha}\frac{\partial U}{\partial \omega} - \frac{\partial U}{\partial \alpha}\frac{\partial V}{\partial \omega}\right) = 0 \qquad (2.5\text{-}8)$$

is satisfied.

Proof

Equation (2.5-3) can be expressed as

$$U(A_0,\alpha,\omega) + jV(A_0,\alpha,\omega) = 0 \tag{2.5-9}$$

where U and V are the real and imaginary parts in (2.5-3). For $A_0 = A_0'$, we have $S_0 = S_0(A_0',\alpha_0,\omega_0)$ and it satisfies (2.5-9). Consider small perturbations around S_0 and such that $\dot{A}_0$ and $\dot{\theta}$ are small. We denote these perturbations as follows

$$\left.\begin{aligned} A_0' &\rightarrow A_0' + \Delta A_0 \\ \alpha_0 &\rightarrow \alpha_0 + \Delta\alpha_1 + j\Delta\alpha_2 \\ \omega_0 &\rightarrow \omega_0 + \Delta\omega_1 + j\Delta\omega_2 \end{aligned}\right\} \tag{2.5-10}$$

ΔA_0, $\Delta\alpha_1$ and $\Delta\omega_1$ are small real numbers. Furthermore,

$$\Delta\alpha_2 = \frac{\dot{\theta}}{\omega_0} \tag{2.5-11}$$

and

$$\Delta\omega_2 = -\frac{\dot{A}_0}{A_0} \quad . \tag{2.5-12}$$

Substituting the perturbed states into (2.5-9), expanding into a Taylor series and equating the real and imaginary parts after neglecting second order terms, we obtain in the neighbourhood of S_0

$$\frac{\partial U}{\partial A_0}\Delta A_0 + \frac{\partial U}{\partial \alpha}\Delta\alpha_1 + \frac{\partial U}{\partial \omega}\Delta\omega_1 = \frac{\partial V}{\partial \omega}\Delta\omega_2 + \frac{\partial V}{\partial \alpha}\Delta\alpha_2 \tag{2.5-13}$$

and

$$\frac{\partial V}{\partial A_0}\Delta A_0 + \frac{\partial V}{\partial \alpha}\Delta\alpha_1 + \frac{\partial V}{\partial \omega}\Delta\omega_1 = -\left(\frac{\partial U}{\partial \omega}\Delta\omega_2 + \frac{\partial U}{\partial \alpha}\Delta\alpha_2\right) . \qquad (2.5\text{-}14)$$

Dividing by ΔA_0 and taking the limit $\Delta A_0 \to 0$, we get

$$\frac{\partial U}{\partial A_0} + \frac{\partial U}{\partial \alpha}\frac{d\alpha_1}{dA_0} + \frac{\partial U}{\partial \omega}\frac{d\omega_1}{dA_0} = \frac{\partial V}{\partial \omega}\frac{d\omega_2}{dA_0} + \frac{\partial V}{\partial \alpha}\frac{d\alpha_2}{dA_0} \qquad (2.5\text{-}15)$$

and

$$\frac{\partial V}{\partial A_0} + \frac{\partial V}{\partial \alpha}\frac{d\alpha_1}{dA_0} + \frac{\partial V}{\partial \omega}\frac{d\omega_1}{dA_0} = -\left(\frac{\partial U}{\partial \omega}\frac{d\omega_2}{dA_0} + \frac{\partial U}{\partial \alpha}\frac{d\alpha_2}{dA_0}\right) . \qquad (2.5\text{-}16)$$

In the neighbourhood of S_0, let $K_\alpha(A_0') = \frac{d\alpha_1}{dA_0}$ and $K_\omega(A_0') = \frac{d\omega_1}{dA_0}$.

In terms of the operator E, (2.5-7), (2.5-15) and (2.5-16), can be expressed as

$$E(U) = \frac{\partial V}{\partial \omega}\frac{d\omega_2}{dA_0} + \frac{\partial V}{\partial \alpha}\frac{d\alpha_2}{dA_0} \qquad (2.5\text{-}17)$$

and

$$E(V) = -\left(\frac{\partial U}{\partial \omega}\frac{d\omega_2}{dA_0} + \frac{\partial U}{\partial \alpha}\frac{d\alpha_2}{dA_0}\right) . \qquad (2.5\text{-}18)$$

Eliminating $\frac{d\alpha_2}{dA_0}$, we get

$$\frac{\partial U}{\partial \alpha}E(U) + \frac{\partial V}{\partial \alpha}E(V) = \left(\frac{\partial U}{\partial \alpha}\frac{\partial V}{\partial \omega} - \frac{\partial U}{\partial \omega}\frac{\partial V}{\partial \alpha}\right)\frac{d\omega_2}{dA_0} . \qquad (2.5\text{-}19)$$

From (2.5-12) the instantaneous solution S_0 is stable iff

$$\eta = \frac{d\omega_2}{dA_0} > 0 . \qquad (2.5\text{-}20)$$

Hence, S_0 is stable iff in its neighbourhood there exists a real positive number η such that condition (2.5-8) is satisfied. ∎

Some corollaries will now be given. The proofs will be omitted since they are straightforward.

Let δ_1 and δ_2 be small nonnegative quantities and define the region R_1 around the instantaneous amplitude A_{0q} such that $R_1 = (A_{0q} - \delta_1, A_{0q} + \delta_2)$. Then $S_q = (A_{0q}, \alpha, \omega)$ is said to be an <u>equilibrium state of the nonlinear system</u>, (2.5-3), if $\eta > 0$ for $\forall A_0 \varepsilon R_1$. Let ε_1 and ε_2 be sufficiently small nonnegative quantities. Define the region $R_2 = (-\varepsilon_1, \varepsilon_2)$. A necessary and sufficient condition for the nonlinear system, (2.5-3), to have a <u>stable oscillatory mode (limit cycle)</u> is that $\eta > 0$ for $\forall \alpha \varepsilon R_2$. If α is a monotonically decreasing function of A_0, then a necessary and sufficient condition for the nonlinear system to possess a <u>unique stable limit cycle</u> is that $\eta > 0$ for $\forall \alpha \varepsilon R_2$, and $\eta \leq 0$ otherwise. Let $\alpha < 0$ for $\forall A_0 \geq 0$ and let ε_0 be a small positive quantity. If $\eta > 0$ for $\forall A_0 < \varepsilon_0$ and $\eta \leq 0$ for $\forall A_0 \geq \varepsilon_0$, then the nonlinear system, (2.5-3), is <u>exponentially asymptotically stable in the large</u>.

2.6 <u>NONLINEAR SAMPLED DATA SYSTEMS</u>

Transient form of the Krylov-Bogoliubov method.

We will now derive a sampled version of the nonlinear differential equation

$$\ddot{x} - 2\sigma\dot{x} + \omega_n^2 x + \varepsilon g(x,\dot{x}) = 0 \qquad (2.6\text{-}1)$$

where σ, ω_n and ε are constants and ε is required to be small. We consider a certain class of sampled data systems possessing the property

$$\omega T_s << 1 \tag{2.6-2}$$

where ω is the frequency of the oscillations and T_s is the sampling period. We assume that at time $t = nT_s$, the input to the nonlinear element can be expressed as

$$x(nT_s) = A_0(nT_s)e^{\sigma nT_s}\sin[\omega_0 nT_s + \theta(nT_s)] \ . \tag{2.6-3}$$

Since ε is selected to be small, then the variations in A_0 and θ with time will be small. ω_0 is chosen to satisfy

$$\omega_0^2 = \omega_n^2 - \sigma^2 > 0 \ . \tag{2.6-4}$$

Let

$$\begin{aligned}\psi^* &= \omega nT_s \\ &= \omega_0 nT_s + \theta(nT_s) \ .\end{aligned} \tag{2.6-5}$$

Let $\theta = \theta(nT_s)$, $\dot{\theta} = \dot{\theta}(nT_s)$, $A_0 = A_0(nT_s)$ and $\dot{A}_0 = \dot{A}_0(nT_s)$. Using finite difference approximations $\dot{x}(nT_s)$ and $\ddot{x}(nT_s)$ can be expressed as

$$\dot{x}(nT_s) = A_0 e^{\sigma nT_s}(\sigma \sin\psi^* + \omega_0 \cos\psi^*) \tag{2.6-6}$$

$$\ddot{x}(nT_s) = -\omega_n^2 x + 2\sigma\dot{x} + \omega_0\dot{A}_0 e^{\sigma nT_s}\cos\psi^* - \omega_0 A_0\dot{\theta}e^{\sigma nT_s}\sin\psi^* \tag{2.6-7}$$

where ωT_s satisfies (2.6-2) and, (2.2-7) and (2.2-8),

$$\frac{\dot{A}_0}{A_0}\sin\psi^* + \dot{\theta}\cos\psi^* = 0 \ . \tag{2.6-8}$$

Substituting (2.6-7) into (2.6-1) yields

$$\frac{\dot{A}_0}{A_0}\cos\psi^* - \dot{\theta}\sin\psi^* = -\frac{\varepsilon}{A_0\omega_0}e^{-\sigma nT_s}g[A\sin\psi^*,$$

$$A(\sigma\sin\psi^* + \omega_0\cos\psi^*)] \tag{2.6-9}$$

where

$$A = A_0e^{\sigma nT_s}. \tag{2.6-10}$$

$\dot{A}_0$ and $\dot{\theta}$ will be assumed to be constant in a complete period. Also let

$$N_s = \left[\frac{\pi}{\omega T_s}\right]. \tag{2.6-11}$$

From (2.6-8) and (2.6-9), the estimated rate of change in the instantaneous amplitude and phase can be shown to be

$$\dot{A}_0 = \frac{-\varepsilon}{2\omega_0N_s}\sum_{n=0}^{2N_s-1} e^{-\alpha\psi^*}g[A\sin\psi^*, A(\sigma\sin\psi^*$$

$$+ \omega_0\cos\psi^*)]\cos\psi^* \tag{2.6-12}$$

and

$$\dot{\theta} = \frac{\varepsilon}{2A_0\omega_0N_s}\sum_{n=0}^{2N_s-1} e^{-\alpha\psi^*}g[A\sin\psi^*, A(\sigma\sin\psi^*$$

$$+ \omega_0\cos\psi^*)]\sin\psi^*. \tag{2.6-13}$$

Equations (2.6-12) and (2.6-13) will be referred to as the sampled version of the Krylov-Bogoliubov asymptotic method in the transient case. These equations can also be applied to the nonlinear difference equation

$$x_{n+2} - 2px_{n+1} + q^2x_n + \delta\phi(x_n, x_{n+1}) = 0 \qquad (2.6\text{-}14)$$

where p, q and δ are constants and δ is small. The relationship between (2.6-1) and (2.6-14) is

$$\sigma = \frac{p-1}{T_s} \qquad (2.6\text{-}15)$$

$$\omega_n^2 = \frac{q^2+1-2p}{T_s^2} \qquad (2.6\text{-}16)$$

and

$$\varepsilon = \frac{\delta}{T_s^2} \; . \qquad (2.6\text{-}17)$$

Sampled oscillatory transient describing function.

Define the complex function $N_h(A_0, z)$ as

$$N_h(A_0, z) = \frac{Y_h(A_0, z)}{X_h(A_0, z)}$$

$$= \frac{\sum_{n=0}^{[h/T_s]} y(nT_s, A_0)\, z^{-n}}{\sum_{n=0}^{[h/T_s]} x(nT_s, A_0)\, z^{-n}} \qquad (2.6\text{-}18)$$

where h is a time interval, z is a complex variable and A_0 is assumed to be a constant. If $h \to \infty$, $N_h(A_0, z)$ reduces to the z-transform describing function [19].

For ω equal to a constant and $\omega T_s \ll 1$, substituting (2.3-6) and (2.3-7) into (2.6-18) and letting $z = e^{sT_s}$, $s = \sigma + j\omega$ and h=T, we obtain

$$N_T^*(A_0,\alpha,\omega) = \frac{j}{A_0 N_s} \sum_{n=0}^{2N_s-1} e^{-\alpha\psi^*} y[A \sin \psi^*, A\omega(\alpha \sin \psi^* + \cos \psi^*); \frac{\psi^*}{\omega}] e^{-j\psi^*} . \tag{2.6-19}$$

$N_T^*(A_0,\alpha,\omega)$ will denote the sampled oscillatory transient describing function, SOTDF with period T. The direct and quadratic components of $N_T^*(A_0,\alpha,\omega)$ are

$$N_{Tp}^*(A_0,\alpha,\omega) = \frac{1}{A_0 N_s} \sum_{n=0}^{2N_s-1} e^{-\alpha\psi^*} y[A \sin \psi^*, A\omega(\alpha \sin \psi^* + \cos \psi^*); \frac{\psi^*}{\omega}] \sin \psi^* \tag{2.6-20}$$

and

$$N_{Tq}^*(A_0,\alpha,\omega) = \frac{1}{A_0 N_s} \sum_{n=0}^{2N_s-1} e^{-\alpha\psi^*} y[A \sin \psi^*, A\omega(\alpha \sin \psi^* + \cos \psi^*); \frac{\psi^*}{\omega}] \cos \psi^* . \tag{2.6-21}$$

The sampled oscillatory transient describing function minimizes the mean square error. The proof is similar to that given in Section 2.4.

Stability analysis.

Consider the system in Fig. 2.2. The system contains a linear plant, a zero-order hold and nonlinear element. Let

$$G(z) = \mathcal{Z}\{ \frac{1-e^{-sT_s}}{s} L(s)\} \tag{2.6-22}$$

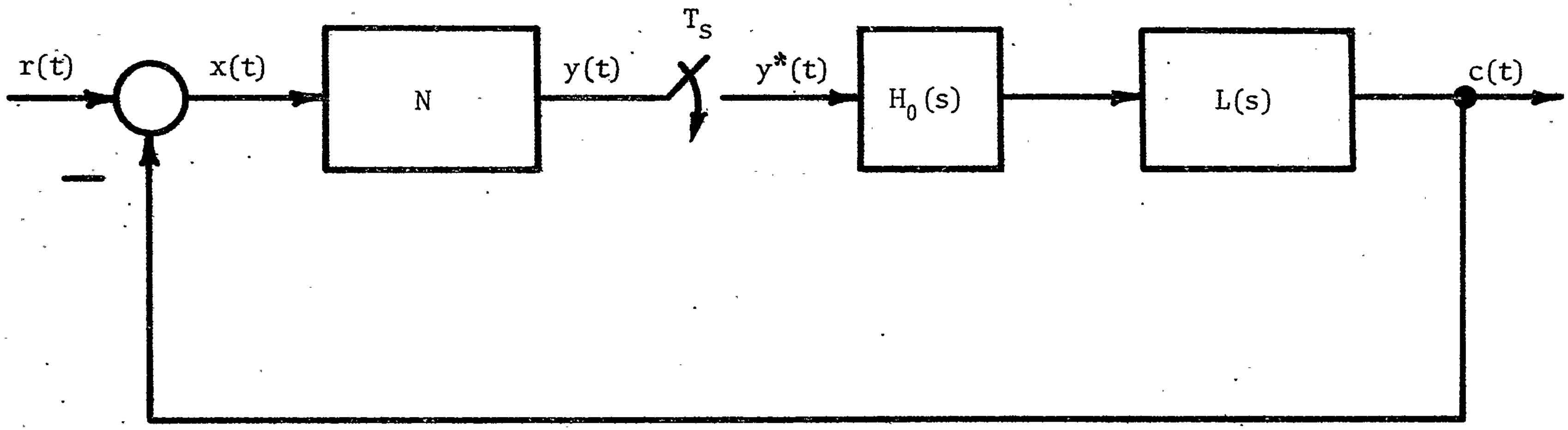

Fig. 2.2 A nonlinear sampled data control system.

$H_0(s) = \frac{1-e^{-sT_s}}{s}$, transfer function of zero-order hold.

L(s) : transfer function of linear plant.

N : nonlinear element.

The corresponding function $G^*(\alpha\omega, j\omega)$ can be expressed as

$$G^*(\alpha\omega, j\omega) = L_1^*(\alpha,\omega) + j\, L_2^*(\alpha,\omega) \tag{2.6-23}$$

where $L_1^*(\cdot)$ and $L_2^*(\cdot)$ are the real and imaginary components of $G^*(\alpha\omega, j\omega)$. For any transient oscillations to take place, it is necessary and sufficient that the quasi-transient system satisfies

$$1 + N_h^*(A_0,\alpha,\omega)\, G^*(\alpha\omega, j\omega) = 0 \tag{2.6-24}$$

where $N_h^*(A_0,\alpha,\omega)$ is the SOTDF with period h.
Equation (2.6-24) implies

$$1 + N_{hp}^*(A_0,\alpha,\omega)\, L_1^*(\alpha,\omega) - N_{hq}^*(A_0,\alpha,\omega)\, L_2^*(\alpha,\omega) = 0 \qquad \ldots(2.6\text{-}25)$$

and

$$N_{hp}^*(A_0,\alpha,\omega)\, L_2^*(\alpha,\omega) + N_{hq}^*(A_0,\alpha,\omega)\, L_1^*(\alpha,\omega) = 0. \tag{2.6-26}$$

For a given instantaneous amplitude $A_0 = A_0'$, the predicted instantaneous oscillatory transient state solution $S_0^* = S_0^*(A_0', \alpha_0, \omega_0)$ can be obtained by solving (2.6-25) and (2.6-26) simultaneously. A necessary and sufficient condition for S_0^* to be stable can be obtained as in Section 2.5, provided (2.6-2) is satisfied.

2.7 Extensions

In the previous sections we considered h=T. In Table 2-1, a summary of the results obtained is given for h=T and also for $h = \frac{T}{2}$.

Similar to the describing function, the analysis presented can be extended to nonlinear systems with several degrees of freedom [6] or with multiple nonlinearities [20]. For some classes of nonlinear elements possessing the characteristic

Table (2-1) Half and complete period formulae for oscillatory transient analysis

Method	Type		
Transient form of the Krylov-Bogoliubov asymptotic method	Half	$\dot{A}_0 = \frac{-\mu}{\pi\omega_0}\int_0^{\pi} e^{-\alpha\psi} f \cos\psi d\psi$	$\dot{\theta} = \frac{\mu}{\pi A_0 \omega_0}\int_0^{\pi} e^{-\alpha\psi} f \sin\psi d\psi$
	Complete	$\dot{A}_0 = \frac{-\mu}{2\pi\omega_0}\int_0^{2\pi} e^{-\alpha\psi} f \cos\psi d\psi$	$\dot{\theta} = \frac{\mu}{2\pi A_0 \omega_0}\int_0^{2\pi} e^{-\alpha\psi} f \sin\psi d\psi$
Oscillatory transient describing function	Half	$N_{\frac{T}{2}p}(A_0,\alpha,\omega) = \frac{2}{\pi A_0}\int_0^{\pi} e^{-\alpha\psi} y \sin\psi d\psi$	$N_{\frac{T}{2}q}(A_0,\alpha,\omega) = \frac{2}{\pi A_0}\int_0^{\pi} e^{-\alpha\psi} y \cos\psi d\psi$
	Complete	$N_{Tp}(A_0,\alpha,\omega) = \frac{1}{\pi A_0}\int_0^{2\pi} e^{-\alpha\psi} y \sin\psi d\psi$	$N_{Tq}(A_0,\alpha,\omega) = \frac{1}{\pi A_0}\int_0^{2\pi} e^{-\alpha\psi} y \cos\psi d\psi$
Sampled version of the transient form of the Krylov-Bogoliubov asymptotic method	Half	$\dot{A}_0 = \frac{-\varepsilon}{\omega_0 N_S}\sum_{n=0}^{N_S-1} e^{-\alpha\psi^*} g^* \cos\psi^*$	$\dot{\theta} = \frac{\varepsilon}{A_0 \omega_0 N_S}\sum_{n=0}^{N_S-1} e^{-\alpha\psi^*} g^* \sin\psi^*$
	Complete	$\dot{A}_0 = \frac{-\varepsilon}{2\omega_0 N_S}\sum_{n=0}^{2N_S-1} e^{-\alpha\psi^*} g^* \cos\psi^*$	$\dot{\theta} = \frac{\varepsilon}{2A_0 \omega_0 N_S}\sum_{n=0}^{2N_S-1} e^{-\alpha\psi^*} g^* \sin\psi^*$

Sampled Oscillatory transient describing function	Half	$N^*_{\frac{T}{2}p}(A_0,\alpha,\omega) = \frac{2}{A_0 N_s} \sum_{n=0}^{N_s-1} e^{-\alpha\psi^*} y^* \sin\psi^*$	$N^*_{\frac{T}{2}q}(A_0,\alpha,\omega) = \frac{2}{A_0 N_s} \sum_{n=0}^{N_s-1} e^{-\alpha\psi^*} y^* \cos\psi^*$
	Complete	$N^*_{Tp}(A_0,\alpha,\omega) = \frac{1}{A_0 N_s} \sum_{n=0}^{2N_s-1} e^{-\alpha\psi^*} y^* \sin\psi^*$	$N^*_{Tq}(A_0,\alpha,\omega) = \frac{1}{A_0 N_s} \sum_{n=0}^{2N_s-1} e^{-\alpha\psi^*} y^* \cos\psi^*$

$$y = y(A_0 e^{\alpha\psi} \sin \psi)$$
$$= e^{\alpha\psi} y(A_0 \sin \psi) \tag{2.7-1}$$

the describing function and the oscillatory transient describing function, OTDF, will yield the same results. Examples of such nonlinearities are given in Fig. 2.3.

2.8 Recommended Computational Procedure

In this section, we suggest a computational procedure for predicting the instantaneous state solutions and for determining the dynamic stability of the continuous nonlinear control system in Fig. 2.1. The procedure to be presented can be applied after some slight modifications to the nonlinear sampled data control systems in Fig. 2.2.

1. Determine the components $N_{hp}(A_0,\alpha,\omega)$ and $N_{hq}(A_0,\alpha,\omega)$ of the OTDF of the nonlinear element in the system.

2. Express the OTDF as an amplitude and a phase

$$N_h(A_0,\alpha,\omega) = \sqrt{N_{hp}^2(A_0,\alpha,\omega) + N_{hq}^2(A_0,\alpha,\omega)} \tag{2.8-1}$$

and

$$\theta_h(A_0,\alpha,\omega) = \tan^{-1}\left[\frac{N_{hq}(A_0,\alpha,\omega)}{N_{hp}(A_0,\alpha,\omega)}\right] . \tag{2.8-2}$$

3. For an arbitrary value of $A_0=A_0'$, plot a family of curves in the N_h-θ_h plane, to be denoted as the N-curves, by fixing α to be equal to arbitrary constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ and varying the parameter ω.

4. The aim is now to solve (2.5-3). From (2.5-3)

$$N_h(A_0,\alpha,\omega) = \frac{-1}{G(\alpha\omega,j\omega)} . \tag{2.8-3}$$

We equate the amplitude and phase of both sides of (2.8-3). From (2.8-1) and (2.8-2), the amplitude and phase of the left hand side of

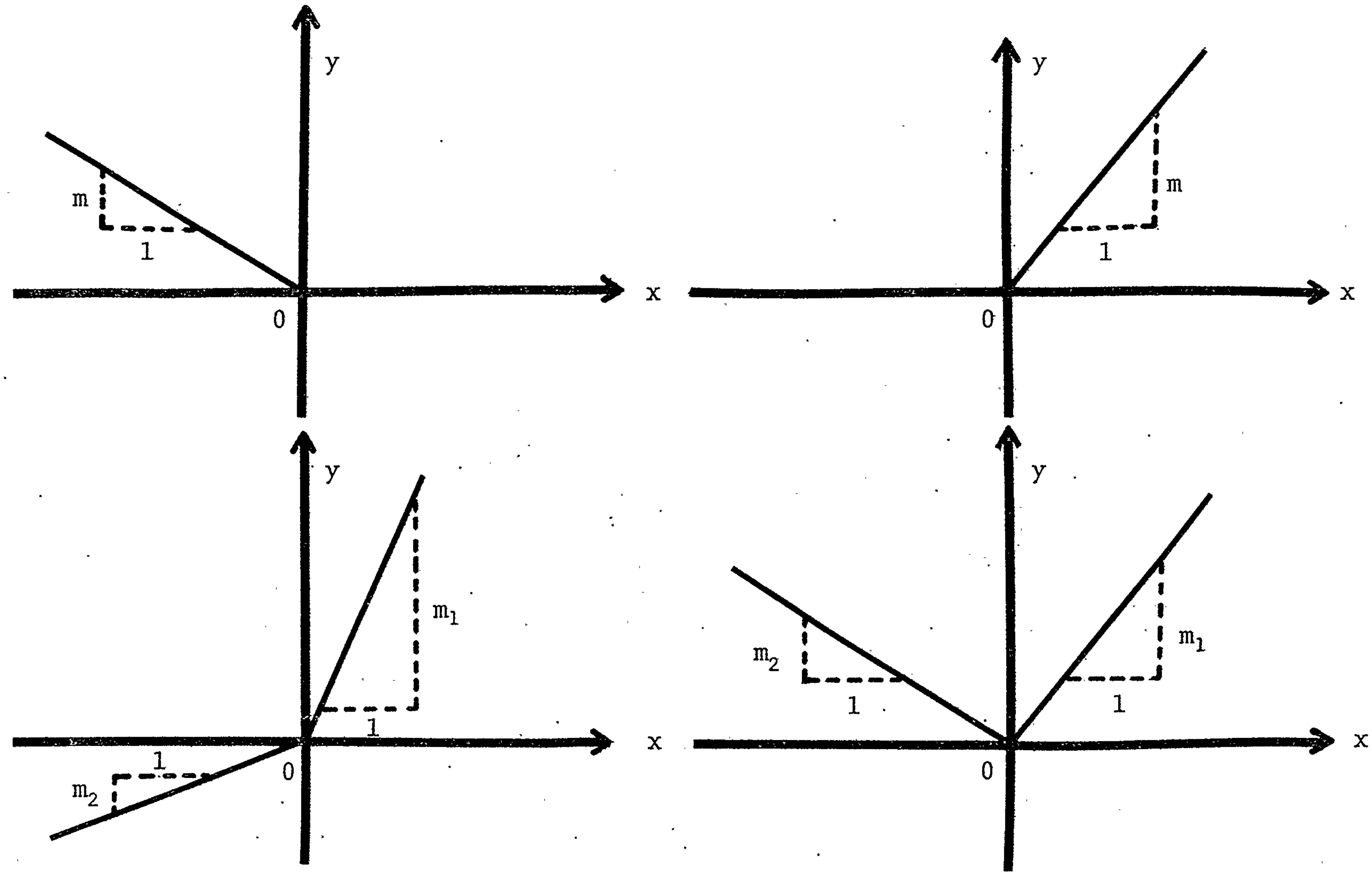

Fig. 2.3 Special class of nonlinear elements possessing the characteristic (2.7-1).

(2.8-3) can be determined. Let the real and imaginary components of $\frac{-1}{G(\alpha\omega, j\omega)}$ be denoted by $Q_1(\alpha,\omega)$ and $Q_2(\alpha,\omega)$ respectively. The amplitude and phase of $\frac{-1}{G(\alpha\omega, j\omega)}$ can be expressed as

$$K_g(\alpha,\omega) = \sqrt{Q_1^2(\alpha,\omega) + Q_2^2(\alpha,\omega)} \qquad (2.8\text{-}4)$$

and

$$\theta_g(\alpha,\omega) = \tan^{-1}\left[\frac{Q_2(\alpha,\omega)}{Q_1(\alpha,\omega)}\right] \qquad (2.8\text{-}5)$$

respectively.

Let α assume arbitrary values $\alpha_1, \alpha_2, \ldots, \alpha_n$. Obtain a family of curves in the K_g-θ_g plane, to be denoted as the G-curves, for each α by varying ω. Since (2.5-3) must be satisfied, it follows that $N_h \equiv K_g$ and $\theta_h \equiv \theta_g$.

5. Superimpose the N and the G curves. Search for a point of intersection of these curves which has the same α and ω. Use interpolation if necessary. This point corresponds to $S_0 = S_0(A_0', \alpha_0, \omega_0)$.

6. Repeat the above for various values of A_0. This yields a curve $S = S(A_0, \alpha, \omega)$. The projection of S on the A_0-α plane gives a relative damping characteristic and on the A_0-ω plane a frequency characteristic.

7. From (2.5-3) determine $U(A_0,\alpha,\omega)$ and $V(A_0,\alpha,\omega)$, the real and imaginary parts of the characteristic equation. Determine $\frac{\partial U}{\partial A_0}$, $\frac{\partial V}{\partial A_0}$, $\frac{\partial U}{\partial \alpha}$, $\frac{\partial V}{\partial \alpha}$, $\frac{\partial U}{\partial \omega}$ and $\frac{\partial V}{\partial \omega}$. For different values of A_0, evaluate $K_\alpha(A_0)$ and $K_\omega(A_0)$ graphically from the projections of S on the A_0-α and A_0-ω planes, by determining the slopes of the curves at the selected values of A_0.

8. For the values of A_0 selected in 7, calculate $\eta(A_0)$ using (2.5-7), (2.5-8) and the information obtained in Step 7. The values of α and ω corresponding to the selected values of A_0 can be obtained

from the curve of S in (A_0,α,ω) space.

9. Plot the curve of η versus A_0 and determine the regions of stability using Theorem 2.1.

3. SYMMETRICAL TRANSIENT OSCILLATIONS - A STUDY

3.1 INTRODUCTION

In this chapter we consider two examples. The first is a continuous feedback control system with a rectangular hysteresis element and the second is a sampled data feedback control system with a nonlinear amplifier. We analyse these systems and determine the predicted values of the transient oscillations using different approximate methods including the one introduced in Chapter 2. The stability of these systems is also investigated.

3.2 CONTINUOUS FEEDBACK CONTROL SYSTEM WITH A RECTANGULAR HYSTERESIS ELEMENT

Consider the system shown in Fig. 3.1. Let the input to the nonlinear element be of the form

$$x(\psi) = A_0 e^{\alpha\psi} \sin\psi \tag{3.2-1}$$

where $\psi = \omega t$. Define ψ_1 and ψ_2 such that

$$e^{\alpha\psi_1} \sin\psi_1 = \frac{\delta}{A_0}, \qquad 0 < \psi_1 \le \theta_{m_1} \tag{3.2-2}$$

and

$$e^{\alpha\psi_2} \sin\psi_2 = -\frac{\delta}{A_0}, \qquad \pi < \psi_2 \le \theta_{m_2} \tag{3.2-3}$$

where θ_{m_1} and θ_{m_2} are defined as in Fig. 3.2. The complete period oscillatory transient describing function can be expressed as

$$N_T(A_0,\alpha,\omega) = N_{Tp}(A_0,\alpha,\omega) + jN_{Tq}(A_0,\alpha,\omega) \tag{3.2-4}$$

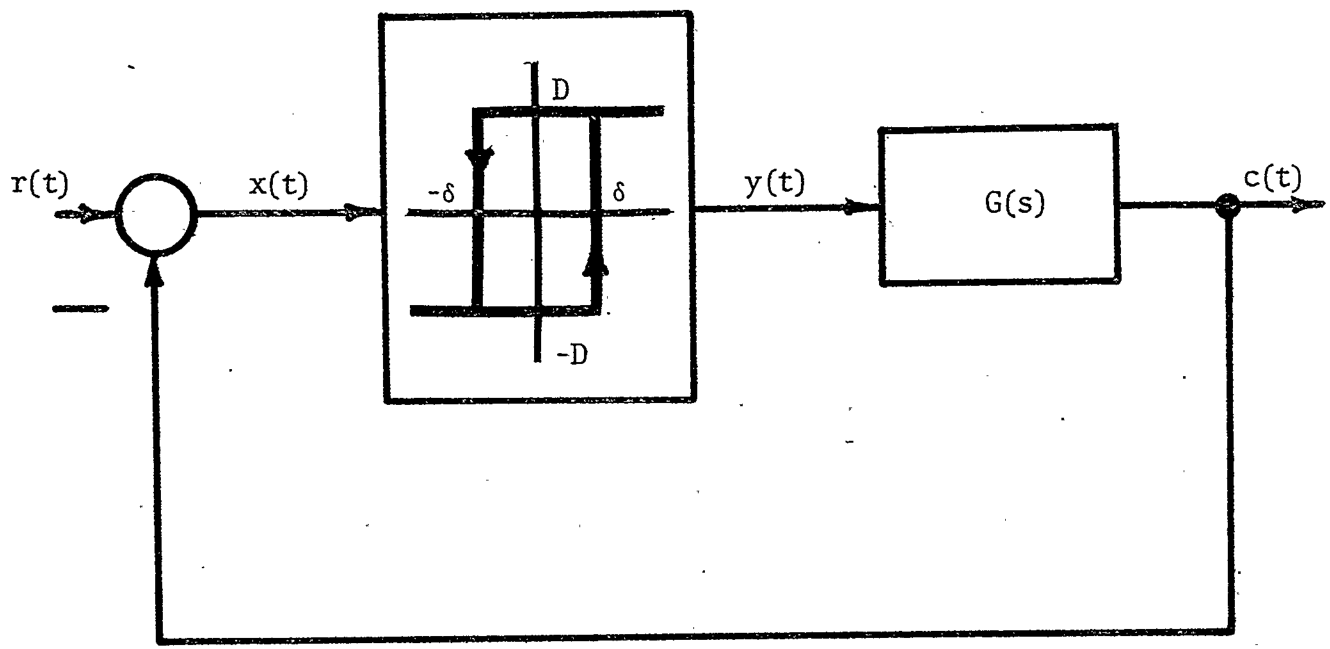

Fig. 3.1 A nonlinear control system with hysteresis.

$$G(s) = \frac{K}{s(\tau_1 s+1)(\tau_2 s+1)}$$

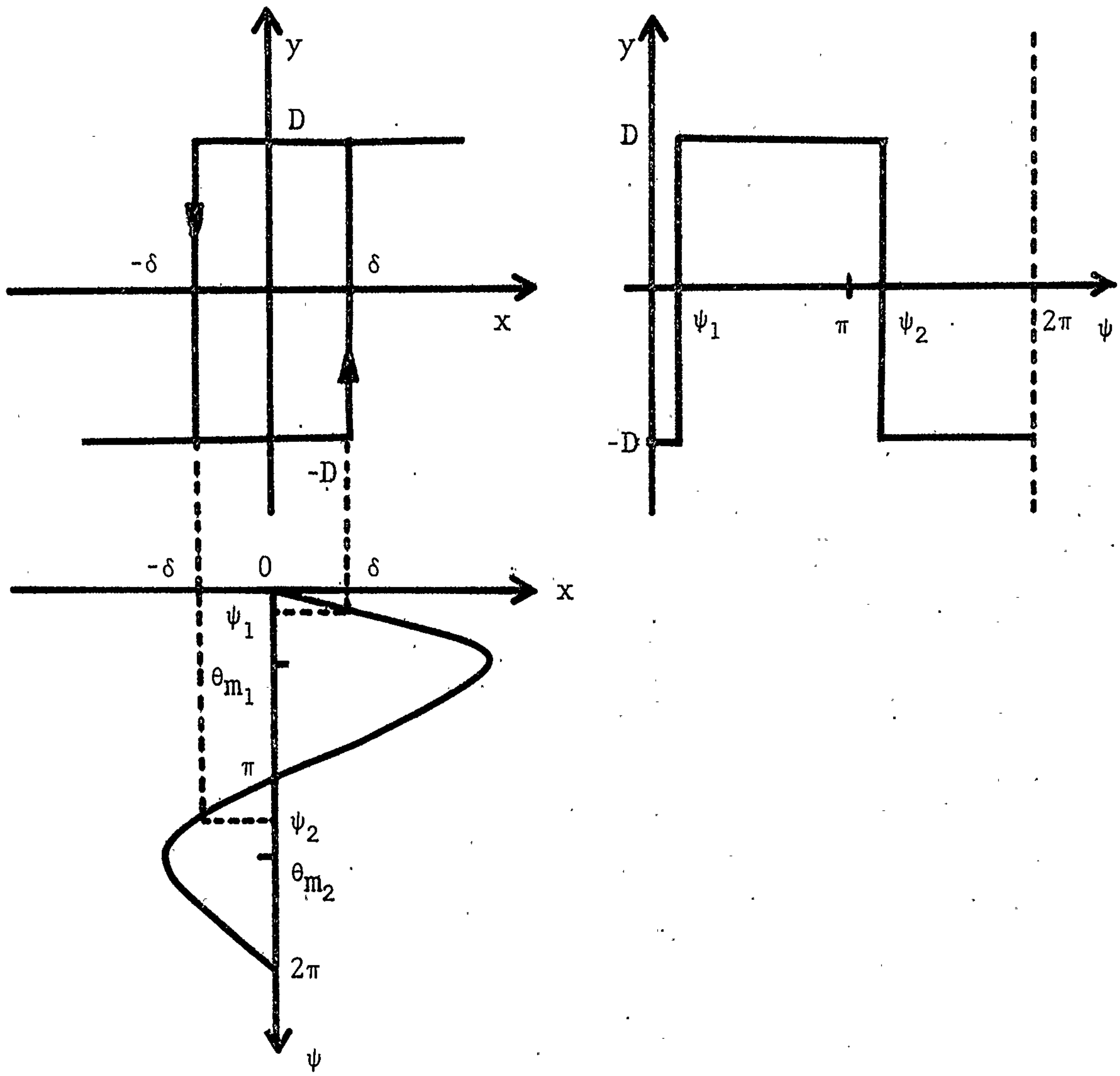

Fig. 3.2 Typical oscillatory transient performance of a rectangular hysteresis element.

where, (2.3-11),

$$N_{Tp}(A_0,\alpha,\omega) = \frac{1}{\pi A_0}\left[\int_0^{\psi_1} e^{-\alpha\psi}(-D)\ \sin\psi\ d\psi\right.$$

$$\left.+\int_{\psi_1}^{\psi_2} e^{-\alpha\psi}(D)\sin\psi\ d\psi + \int_{\psi_2}^{2\pi} e^{-\alpha\psi}(-D)\ \sin\psi\ d\psi\right] . \qquad \ldots(3.2\text{-}5)$$

This reduces to

$$N_{Tp}(A_0,\alpha) = \frac{D}{\pi\delta(1+\alpha^2)}\left[\sin 2\psi_1 + \sin 2\psi_2 + \frac{\delta}{A_0}(e^{-2\pi\alpha} - 1)\right.$$

$$\left.+ 2\alpha(\sin^2\psi_1 + \sin^2\psi_2)\right] . \qquad (3.2\text{-}6)$$

Also, (2.3-12),

$$N_{Tq}(A_0,\alpha,\omega) = \frac{1}{\pi A_0}\left[\int_0^{\psi_1} e^{-\alpha\psi}(-D)\ \cos\psi\ d\psi\right.$$

$$\left.+\int_{\psi_1}^{\psi_2} e^{-\alpha\psi}(D)\ \cos\psi\ d\psi + \int_{\psi_2}^{2\pi} e^{-\alpha\psi}(-D)\ \cos\psi\ d\psi\right] \qquad \ldots(3.2\text{-}7)$$

which reduces to

$$N_{Tq}(A_0,\alpha) = \frac{D\alpha}{\pi\delta(1+\alpha^2)}\left[\sin 2\psi_1 + \sin 2\psi_2 + \frac{\delta}{A_0}(e^{-2\pi\alpha} - 1)\right.$$

$$\left.- \frac{2}{\alpha}(\sin^2\psi_1 + \sin^2\psi_2)\right] . \qquad (3.2\text{-}8)$$

Using similar steps, the half period transient oscillatory describing function can also be derived. The results with other approximate gains reported in the literature are given in Table 3-1.

Let K=0.5, τ_1=1.0, τ_2=0.5, δ=0.5 and D=2.0. Then the transfer function of the plant in Fig. 3.1 becomes

$$G(s) = \frac{1}{s(s+1)(s+2)} \quad . \tag{3.2-9}$$

We will follow the procedure outlined in Section 2.8. For h=T we obtain $N_{Tp}(A_0,\alpha)$ and $N_{Tq}(A_0,\alpha)$ given in (3.2-6) and (3.2-8) respectively. We select $\frac{A_0}{\delta}$ = 16, also α =-0.25, -0.3, -0.35, ..., -0.7. Since N_{Tp} and N_{Tq} are independent of ω, then for each α selected we obtain a single point in the amplitude-phase plane, Fig. 3.3.

We determine $Q_1(\alpha,\omega)$ and $Q_2(\alpha,\omega)$. This yields

$$Q_1(\alpha,\omega) = -\alpha^3\omega^3 + 3\alpha\omega^3 - 3\alpha^2\omega^2 + 3\omega^2 - 2\alpha\omega \tag{3.2-10}$$

and

$$Q_2(\alpha,\omega) = -3\alpha^2\omega^3 + \omega^3 - 6\alpha\omega^2 - 2\omega \ . \tag{3.2-11}$$

Using (2.8-4), (2.8-5) and for the various values of α selected we determine the G-curves, Fig. 3.3. Searching for a point of intersection of the N and G curves which has the same α and ω, we obtain point a, Fig. 3.3. In Fig. 3.3 the N curve for $\frac{A_0}{\delta}$ = 16 and h = $\frac{T}{2}$ is also shown. The common point in this case is b. The above procedure is repeated for various values of A_0 and the results are plotted in the A_0-α and A_0-ω planes, Figs.3.4 and 3.5 respectively. In these figures the relationships between A_0-α and A_0-ω are shown for h = $\frac{T}{2}$, also using the describing function, the method of Freeman and Cox [16] and the estimated half period behavior obtained using simulation [31]. It can be seen from Figs. 3.4 and 3.5 that the best accuracy can be achieved using the half

Table (3-1) Equivalent complex gains of the rectangular hysteresis element during oscillatory transient behavior.

Method	Direct Component	Quadratic Component
Complete Period Oscillatory transient describing function	$\frac{D}{\pi\delta(1+\alpha^2)}[\sin 2\psi_1 + \sin 2\psi_2 + \frac{\delta}{A_0}(e^{-2\pi\alpha}-1) + 2\alpha(\sin^2\psi_1 + \sin^2\psi_2)]$	$\frac{D\alpha}{\pi\delta(1+\alpha^2)}[\sin 2\psi_1 + \sin 2\psi_2 + \frac{\delta}{A_0}(e^{-2\pi\alpha}-1) - \frac{2}{\alpha}(\sin^2\psi_1 + \sin^2\psi_2)]$
Half Period Oscillatory transient describing function	$\frac{2D}{\pi\delta(1+\alpha^2)}[\sin 2\psi_1 + \frac{\delta}{A_0}(e^{-\pi\alpha}-1) + 2\alpha\sin^2\psi_1]$	$\frac{2D\alpha}{\pi\delta(1+\alpha^2)}[\sin 2\psi_1 + \frac{\delta}{A_0}(e^{-\pi\alpha}-1) - \frac{2}{\alpha}\sin^2\psi_1]$
Describing function [7]	$\frac{4D}{\pi A_0}\sqrt{1-(\frac{\delta}{A_0})^2}$	$\frac{-4D\delta}{\pi A_0{}^2}$
Freeman and Cox half Period Complex gain [16]	$\frac{4D\alpha}{A_0(e^{2\pi\alpha}-1)}[2e^{\alpha\psi_1}\cos\psi_1 + e^{\pi\alpha} - 1 - 4\alpha\frac{\delta}{A_0}]$	$\frac{-8D\alpha\delta}{A_0{}^2(e^{2\pi\alpha}-1)}$

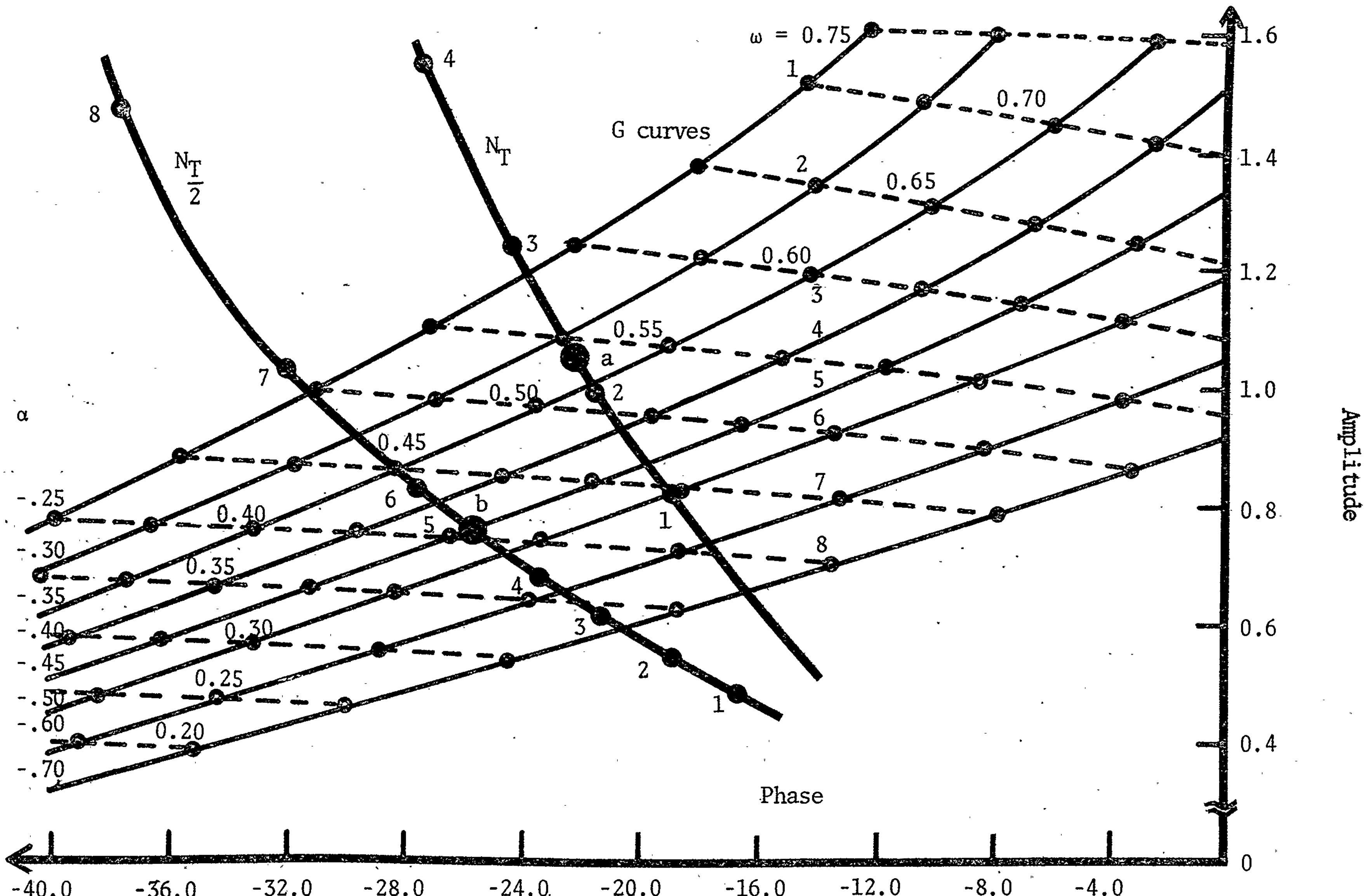

Fig. 3.3 Determination of the instantaneous transient solution at $\frac{A_0}{\delta} = 16$ for the system of Fig. 3.1.

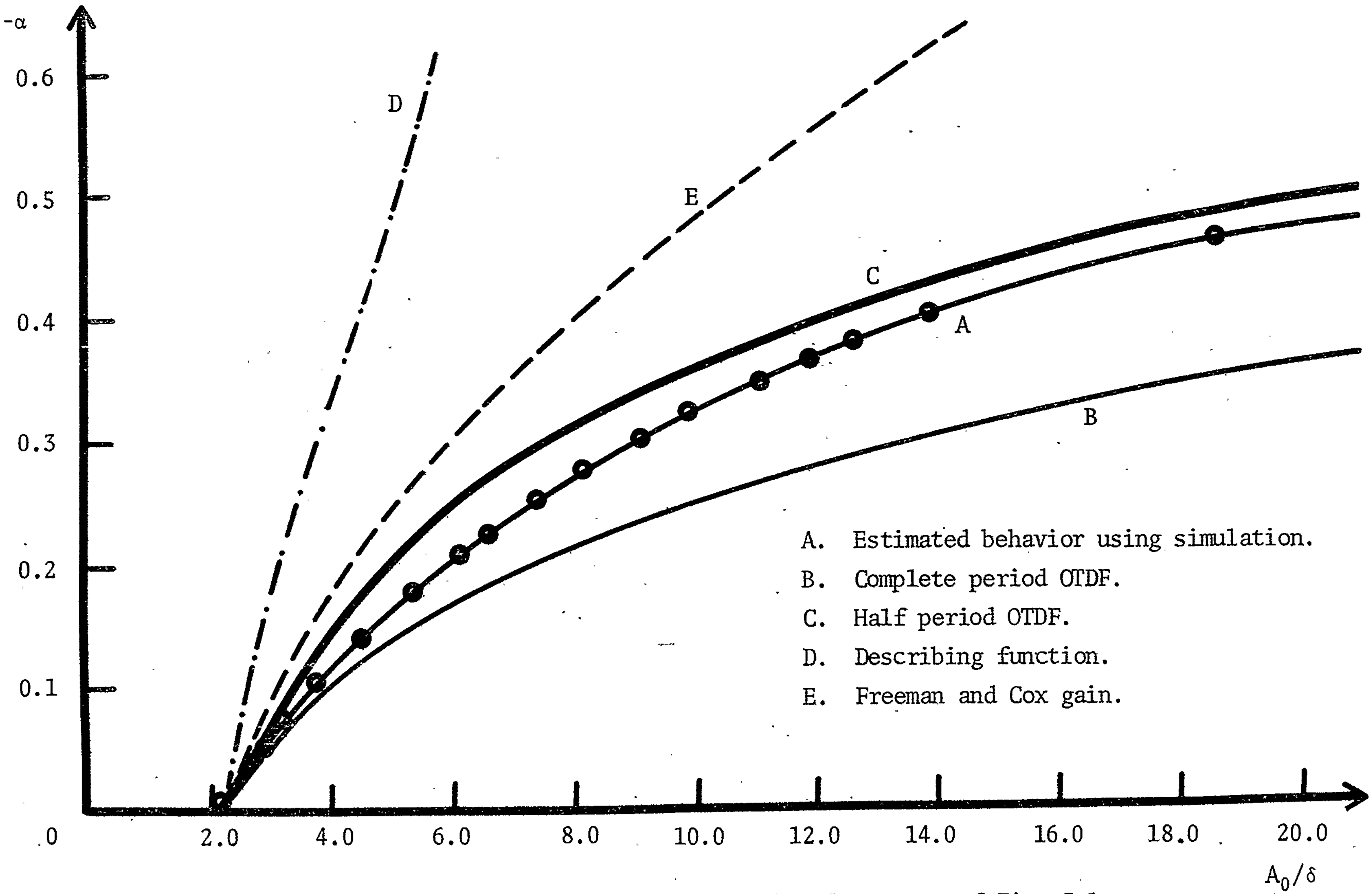

Fig. 3.4 Relative damping characteristics for the system of Fig. 3.1.

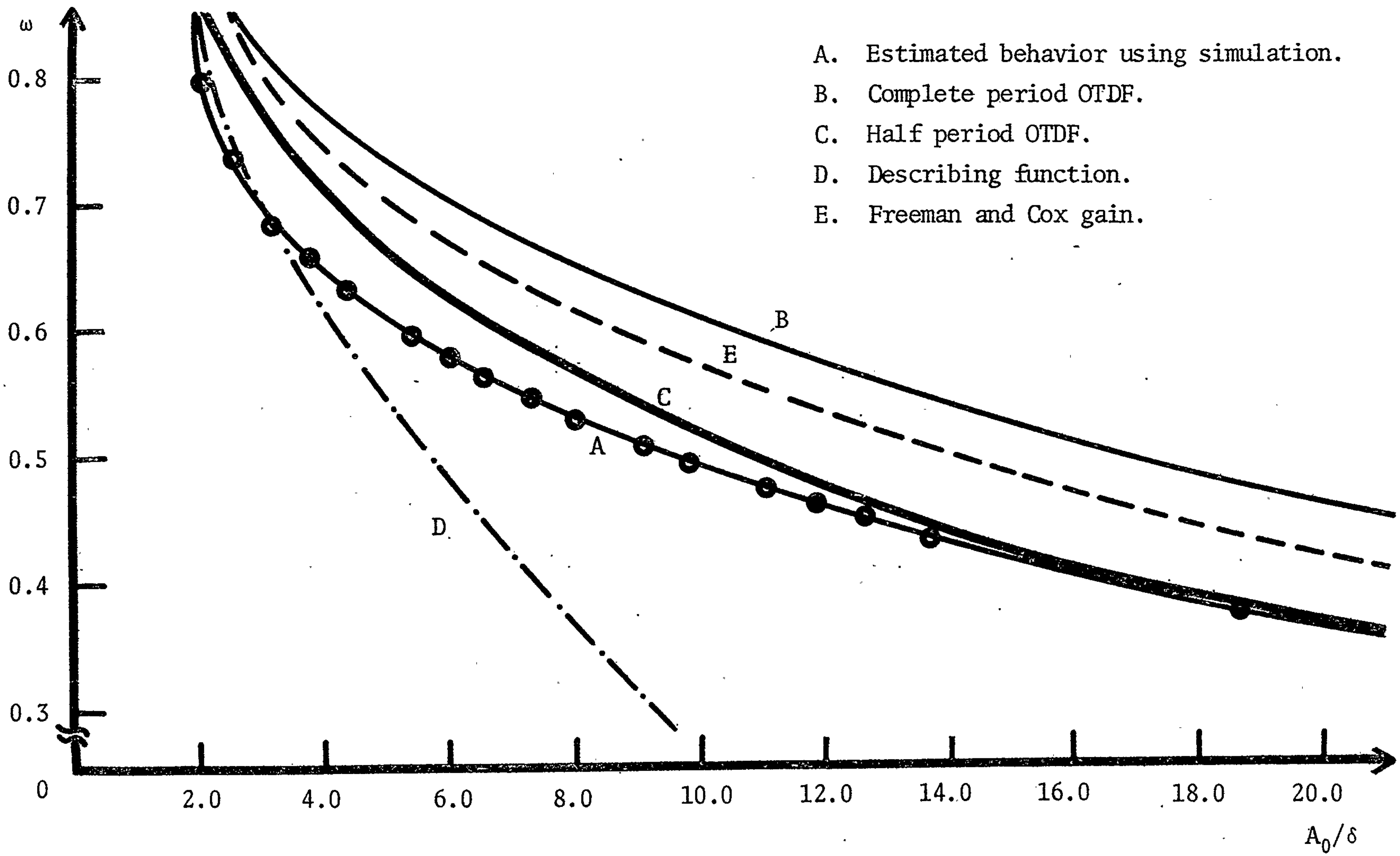

Fig. 3.5 Oscillatory transient frequency characteristics of the system of Fig. 3.1.

period OTDF. This can also be concluded from the responses in the time domain shown in Figs. 3.6 to 3.11. From Figs. 3.4 and 3.5 it can also appear that the describing function method fails completely when the oscillatory behavior starts to be heavily damped. To study the dynamic stability of the system, we shall use the half period OTDF to describe the nonlinearity during the oscillatory transient behavior. Following step 7 of Section 2.8 we find that $U(A_0,\alpha,\omega)$ and $V(A_0,\alpha,\omega)$ can be expressed as

$$U(A_0,\alpha,\omega) = -Q_1(\alpha,\omega) + N_{\frac{T}{2}p}(A_0,\alpha) \tag{3.2-12}$$

and

$$V(A_0,\alpha,\omega) = -Q_2(\alpha,\omega) + N_{\frac{T}{2}q}(A_0,\alpha) . \tag{3.2-13}$$

From (3.2-10) to (3.2-13), we obtain

$$\frac{\partial U}{\partial A_0} = \frac{\partial N_{\frac{T}{2}p}}{\partial A_0} \tag{3.2-14}$$

$$\frac{\partial V}{\partial A_0} = \frac{\partial N_{\frac{T}{2}q}}{\partial A_0} \tag{3.2-15}$$

$$\frac{\partial U}{\partial \alpha} = 3\alpha^2\omega^3 - 3\omega^3 + 6\alpha\omega^2 + 2\omega + \frac{\partial N_{\frac{T}{2}p}}{\partial \alpha} \tag{3.2-16}$$

$$\frac{\partial V}{\partial \alpha} = 6\alpha\omega^3 + 6\omega^2 + \frac{\partial N_{\frac{T}{2}q}}{\partial \alpha} \tag{3.2-17}$$

$$\frac{\partial U}{\partial \omega} = 3\alpha^3\omega^2 - 9\alpha\omega^2 + 6\alpha^2\omega - 6\omega + 2\alpha \tag{3.2-18}$$

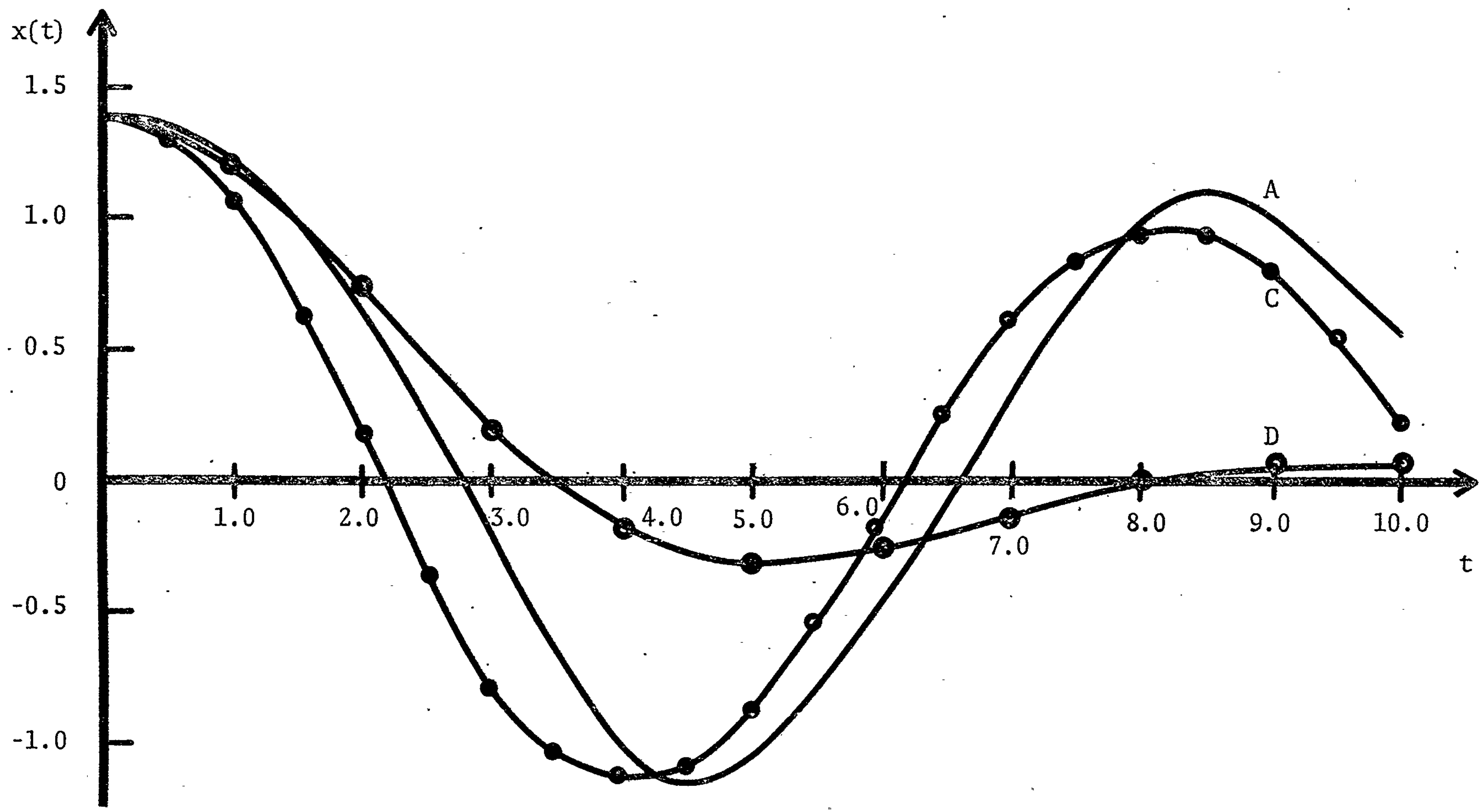

Fig. 3.6 Instantaneous oscillatory transient solution for the system of Fig. 3.1.
A. Using simulation. C. Half period OTDF. D. Describing function.

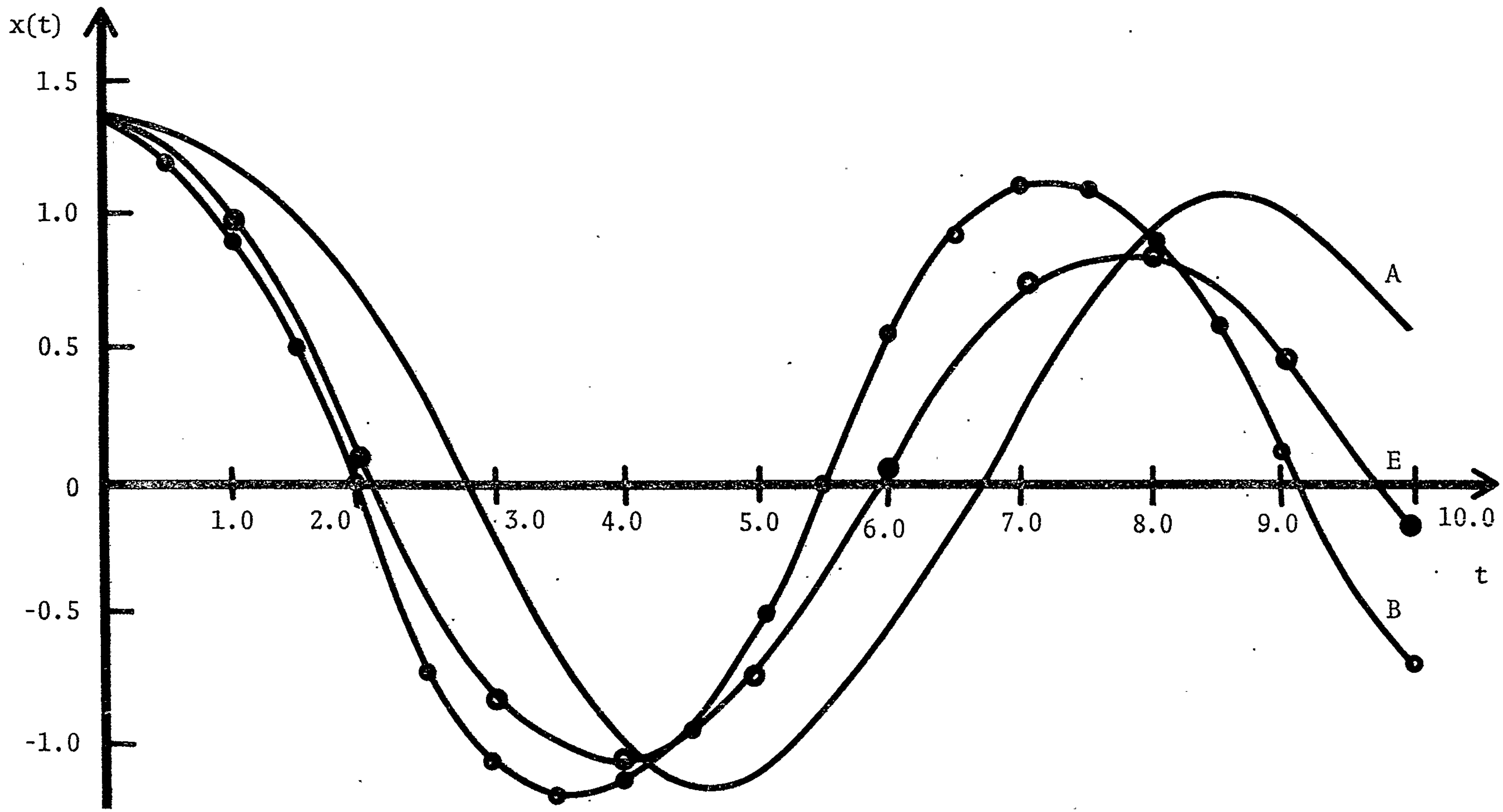

Fig. 3.7 Instantaneous oscillatory transient solution for the system of Fig. 3.1.
A. Using simulation. B. Complete period OTDF. E. Freeman and Cox gain.

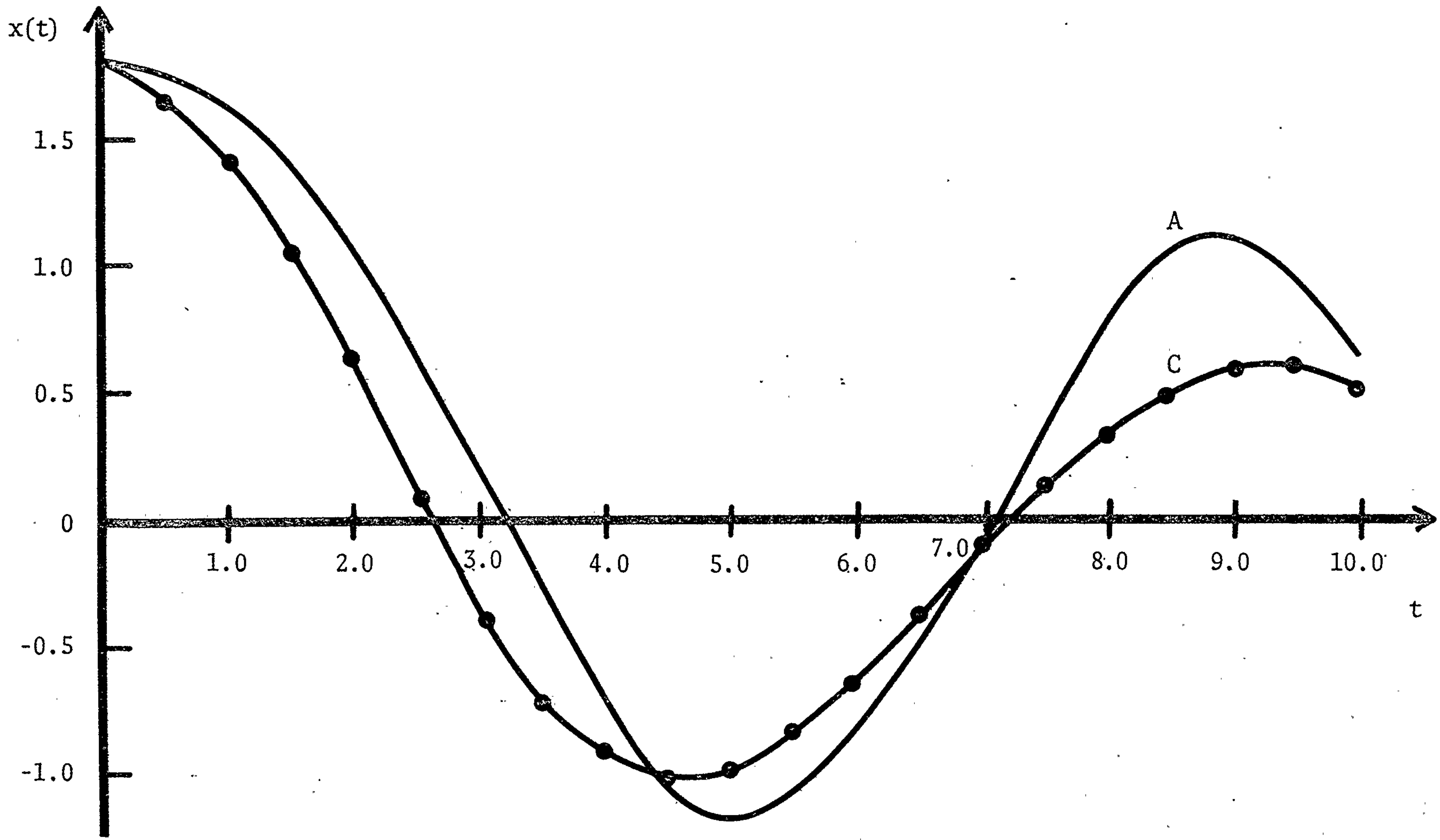

Fig. 3.8 Instantaneous oscillatory transient solution for the system of Fig. 3.1.
A. Using simulation. C. Half period OTDF.

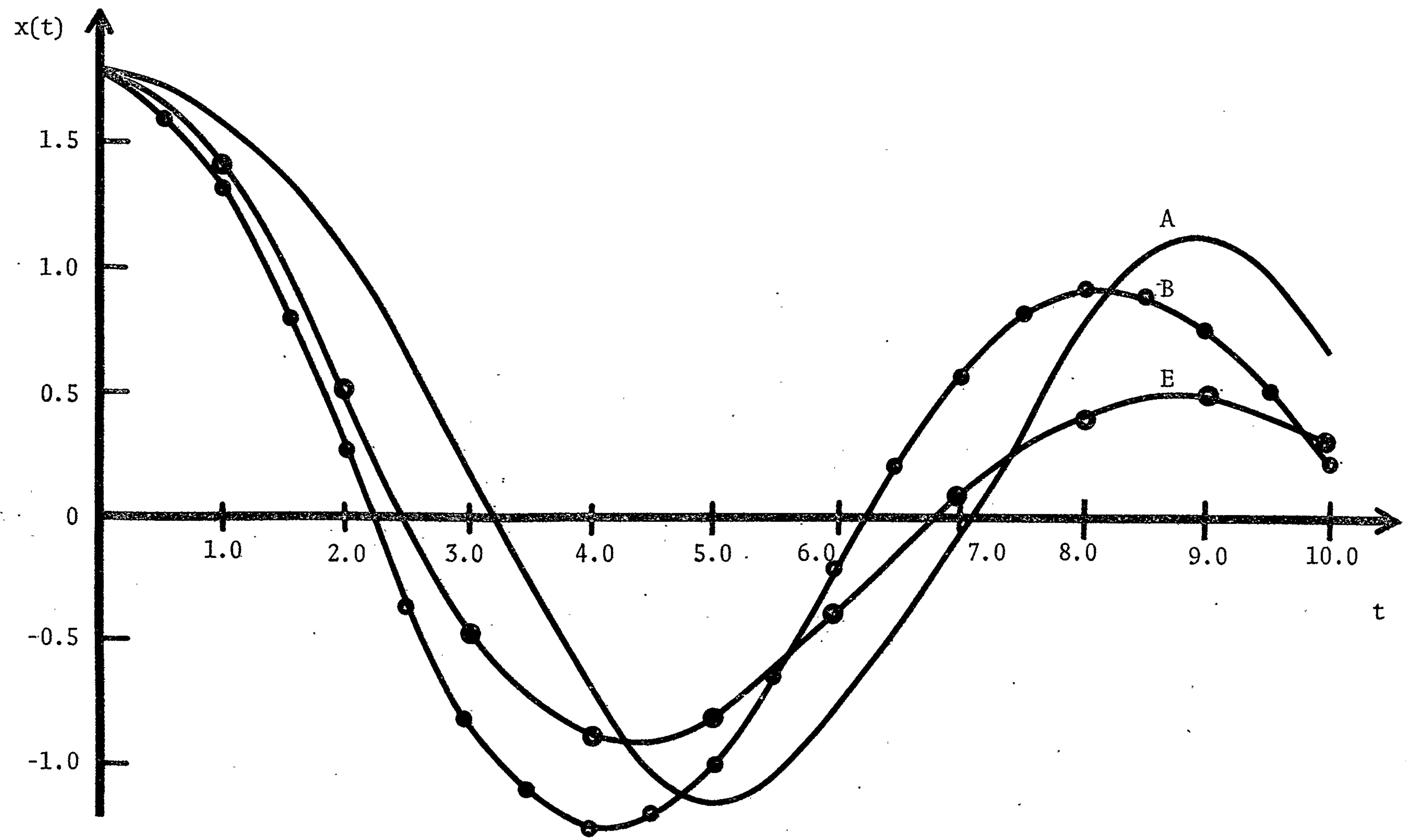

Fig. 3.9 Instantaneous oscillatory transient solution for the system of Fig. 3.1.
A. Using simulation. B. Complete period OTDF. E. Freeman and Cox gain.

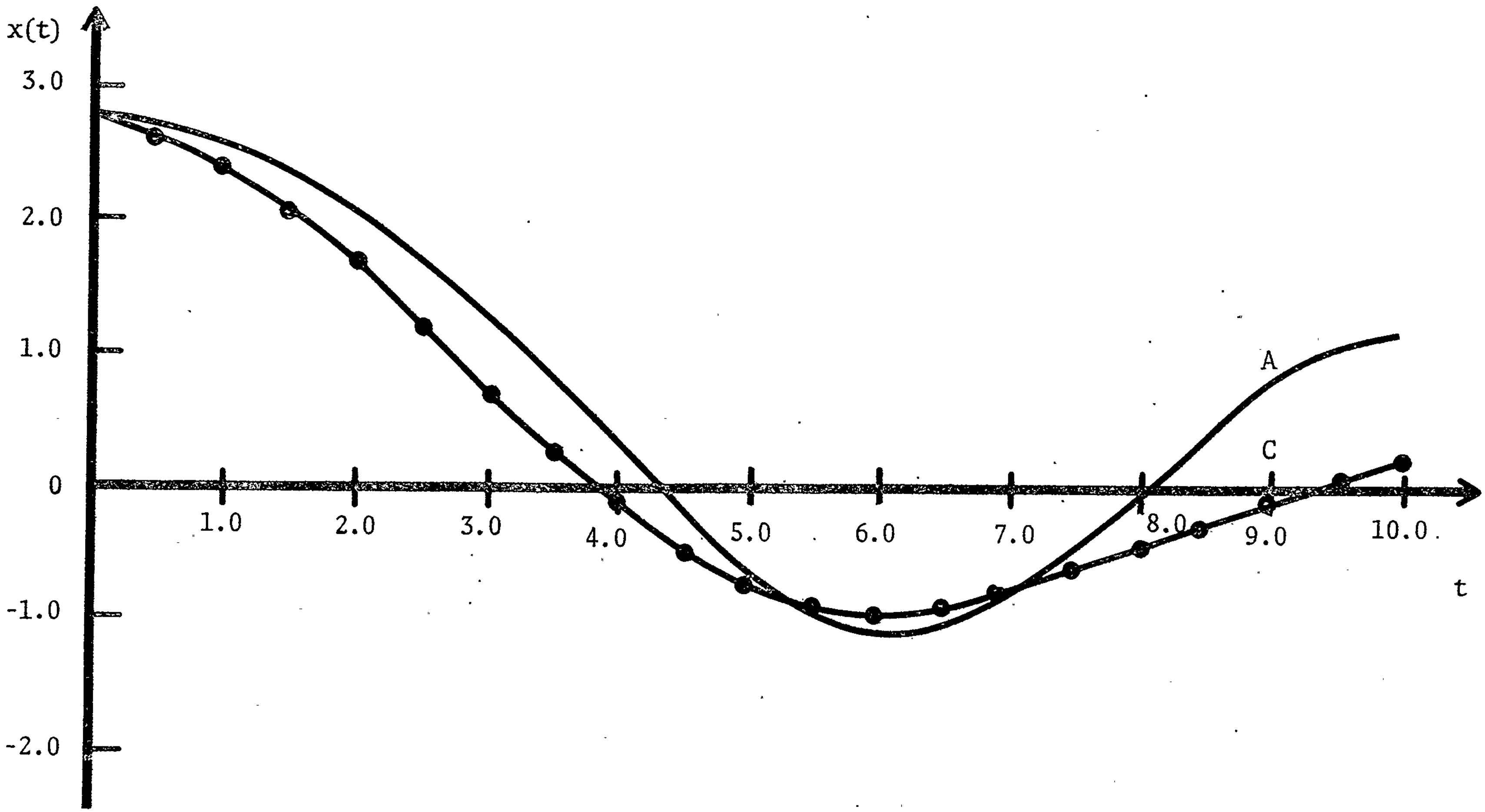

Fig. 3.10 Instantaneous oscillatory transient solution for the system of Fig. 3.1.
A. Using simulation. C. Half period OTDF.

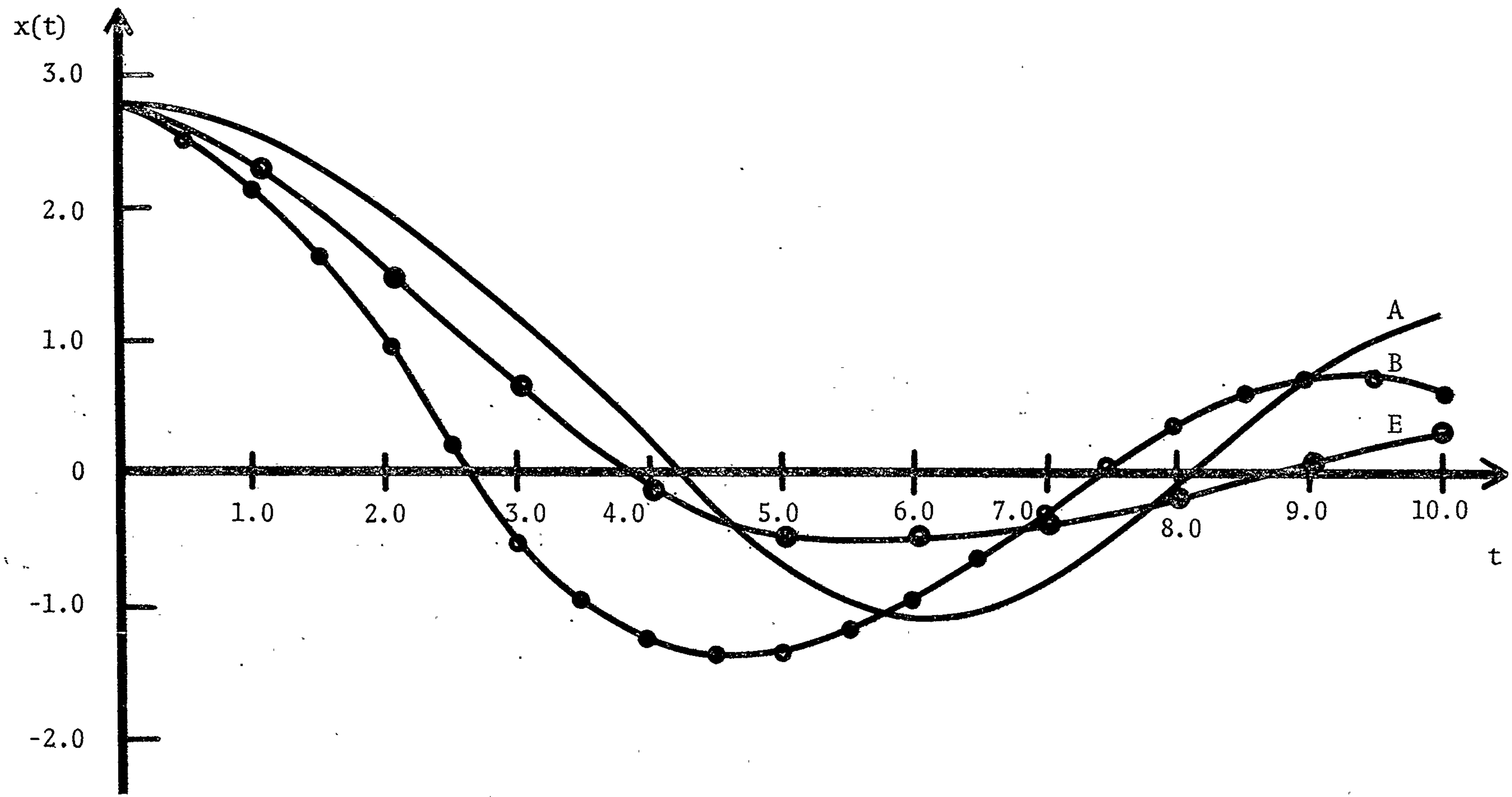

Fig. 3.11 Instantaneous oscillatory transient solution for the system of Fig. 3.1.
A. Using simulation. B. Complete period OTDF. E. Freeman and Cox gain.

and

$$\frac{\partial V}{\partial \omega} = 9\alpha^2\omega^2 - 3\omega^2 + 12\alpha\omega + 2 \ . \qquad (3.2\text{-}19)$$

$K_\alpha(A_0)$ and $K_\omega(A_0)$ are determined from Figs. 3.4 and 3.5 respectively. Following steps 8 and 9 of Section 2.8, we obtain the $\eta(A_0)$ curve shown in Fig. 3.12. The stability regions are indicated in the figure. The system has a unique limit cycle at $A_0 = 1.10$. System simulation indicated a unique limit cycle with amplitude $A_0 = 1.13$.

3.3 SAMPLED DATA CONTROL SYSTEM WITH A SQUARE ROOT ELEMENT

Consider the nonlinear sampled data system shown in Fig. 2.2. The nonlinear element will be assumed to have a square root characteristic, [7], that is

$$y = \sqrt{x} \ , \qquad x \geq 0 \qquad (3.3\text{-}1)$$

and

$$y = -\sqrt{-x} \ , \qquad x \leq 0 \qquad (3.3\text{-}2)$$

where x and y are the input and output of the nonlinear element respectively.

We select

$$L(s) = \frac{4}{s(s+1)(s+2)} \qquad (3.3\text{-}3)$$

and let $T_s = 0.1$ secs.

We have, [19],

$$G(z) = \mathcal{Z}[H_0(s)L(s)]$$

$$= \mathcal{Z}\left[\frac{4(1-e^{-0.1s})}{s^2(s+1)(s+2)}\right]$$

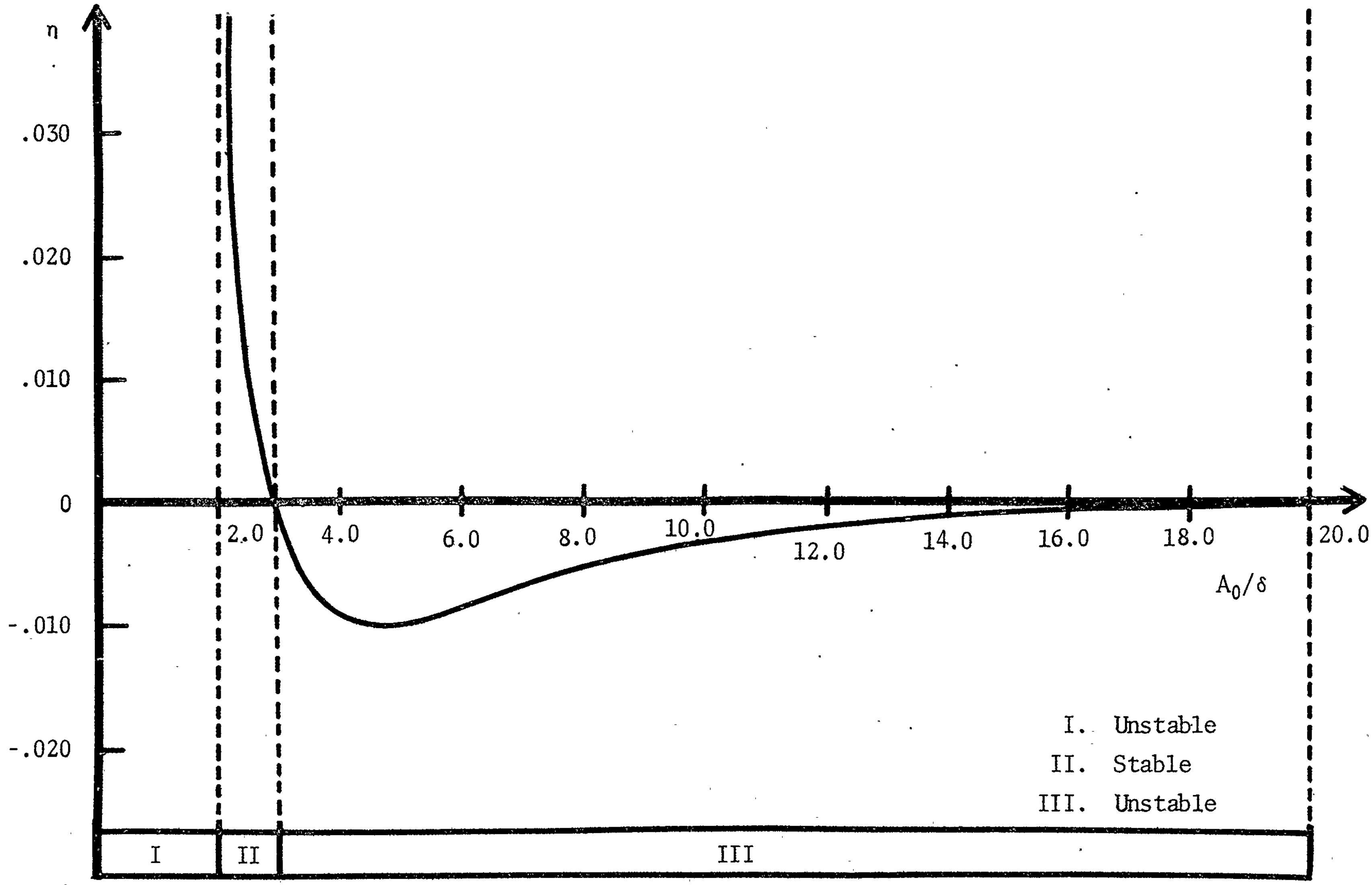

Fig. 3.12 Stability regions for the system of Fig. 3.1.

$$= \frac{0.2}{z-1} - 3.0 + \frac{4(z-1)}{(z-0.9048)} - \frac{z-1}{z-0.8187} \ . \qquad (3.3-4)$$

Substituting $z = e^{sT_s}$, with $s = \alpha\omega + j\omega$ and following the same steps as in Section 3.2, we obtain the instantaneous oscillatory transient solution in Figs. 3.13 and 3.14. From these figures it can also be seen that the best accuracy in the prediction is achieved using the half period sampled OTDF. The stability regions, determined as in Section 3.2, are shown in Fig. 3.15. It is seen that the system considered has a unique limit cycle at $A_0 = 0.55$. System simulation indicated a unique limit cycle with amplitude $A_0 = 0.5543$.

3.4 FILTER HYPOTHESIS

If transient oscillations are present in the nonlinear system, the frequency spectrum of the output of the linear plant will be continuous, for example as in Fig. 3.16. Those results were obtained using the Hewlett-Packard Fourier Analyzer 5450A, [32]. The amplitude spectrum of this output predicted using the describing function consists of an impulse at the fundamental frequency, Fig. 3.16. If the OTDF is used, the spectrum of the output is predicted to be as shown in the same figure. The amplitude spectrum of the prediction error using the OTDF improves as the order of the plant increases. This is demonstrated in Fig. 3.16 using Butterworth filters. It is also obvious from Fig. 3.16 that the predicted output using the OTDF has a better amplitude spectrum than that obtained using the describing function. It should be noted that the results reported here are qualitative. An exact error analysis is formidable.

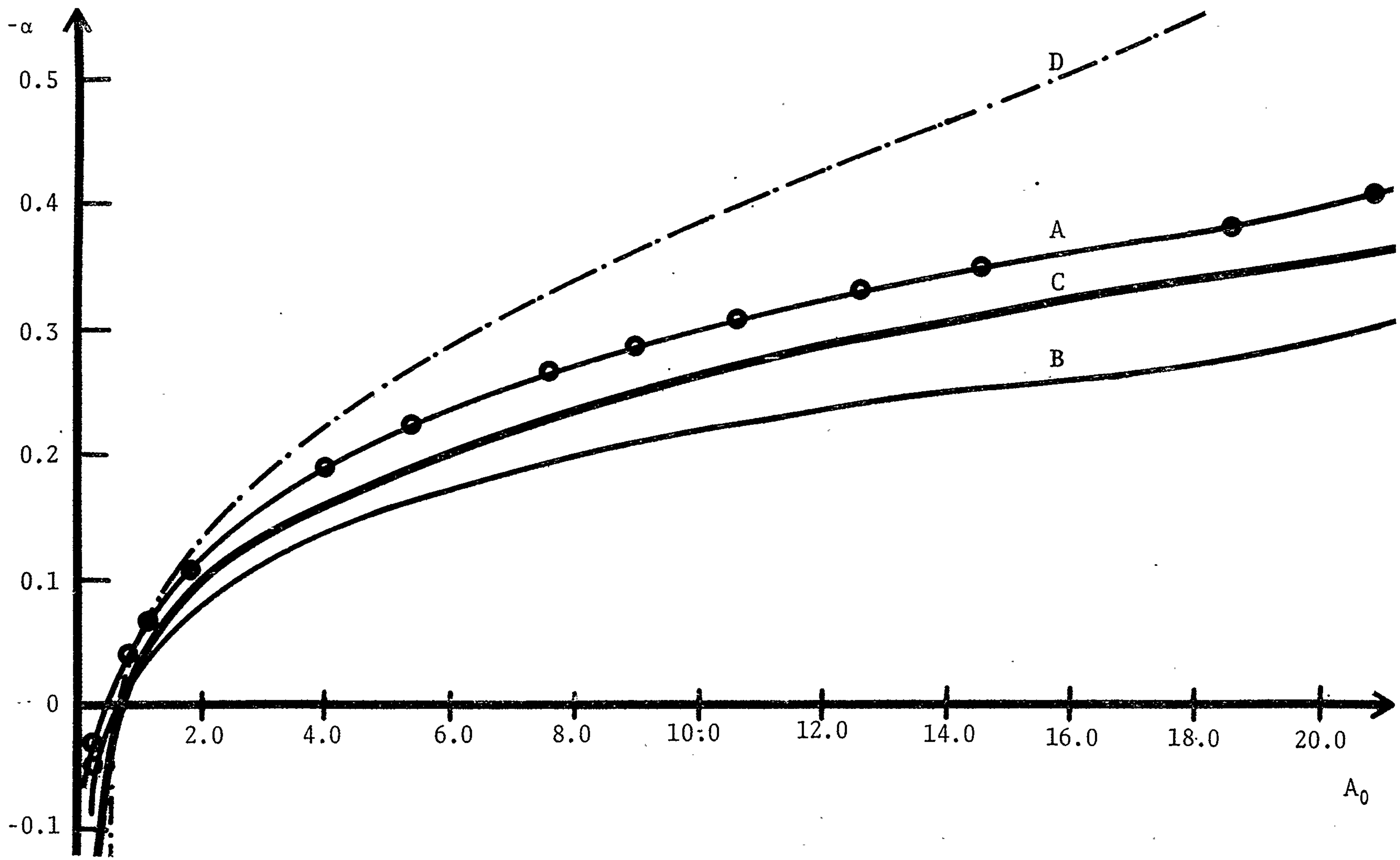

Fig. 3.13 Relative damping characteristics of the system given in Section 3.3.
A. Estimated behavior using simulation. B. Complete period sampled OTDF.
C. Half period sampled OTDF. D. Sampled describing function.

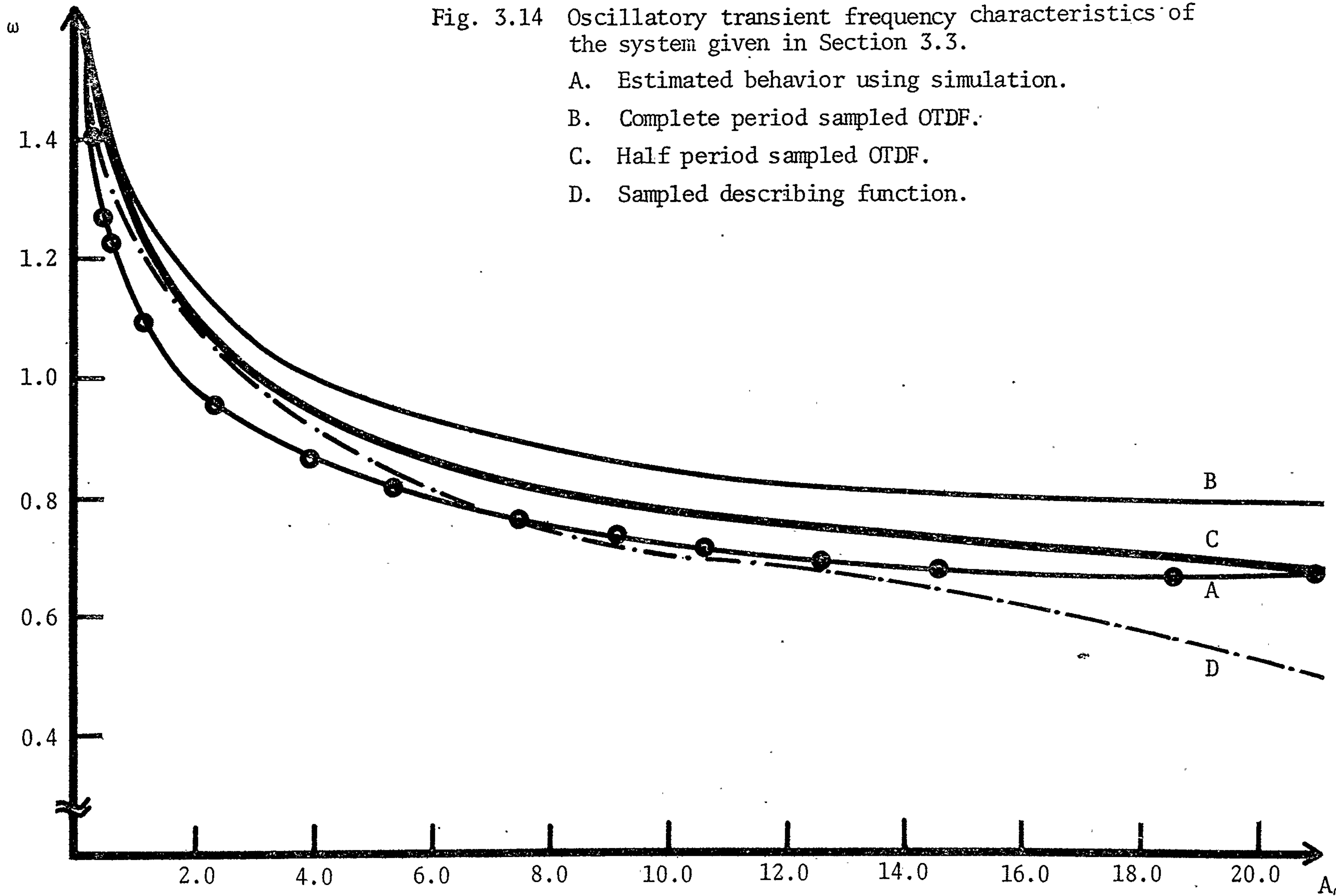

Fig. 3.14 Oscillatory transient frequency characteristics of the system given in Section 3.3.

A. Estimated behavior using simulation.

B. Complete period sampled OTDF.

C. Half period sampled OTDF.

D. Sampled describing function.

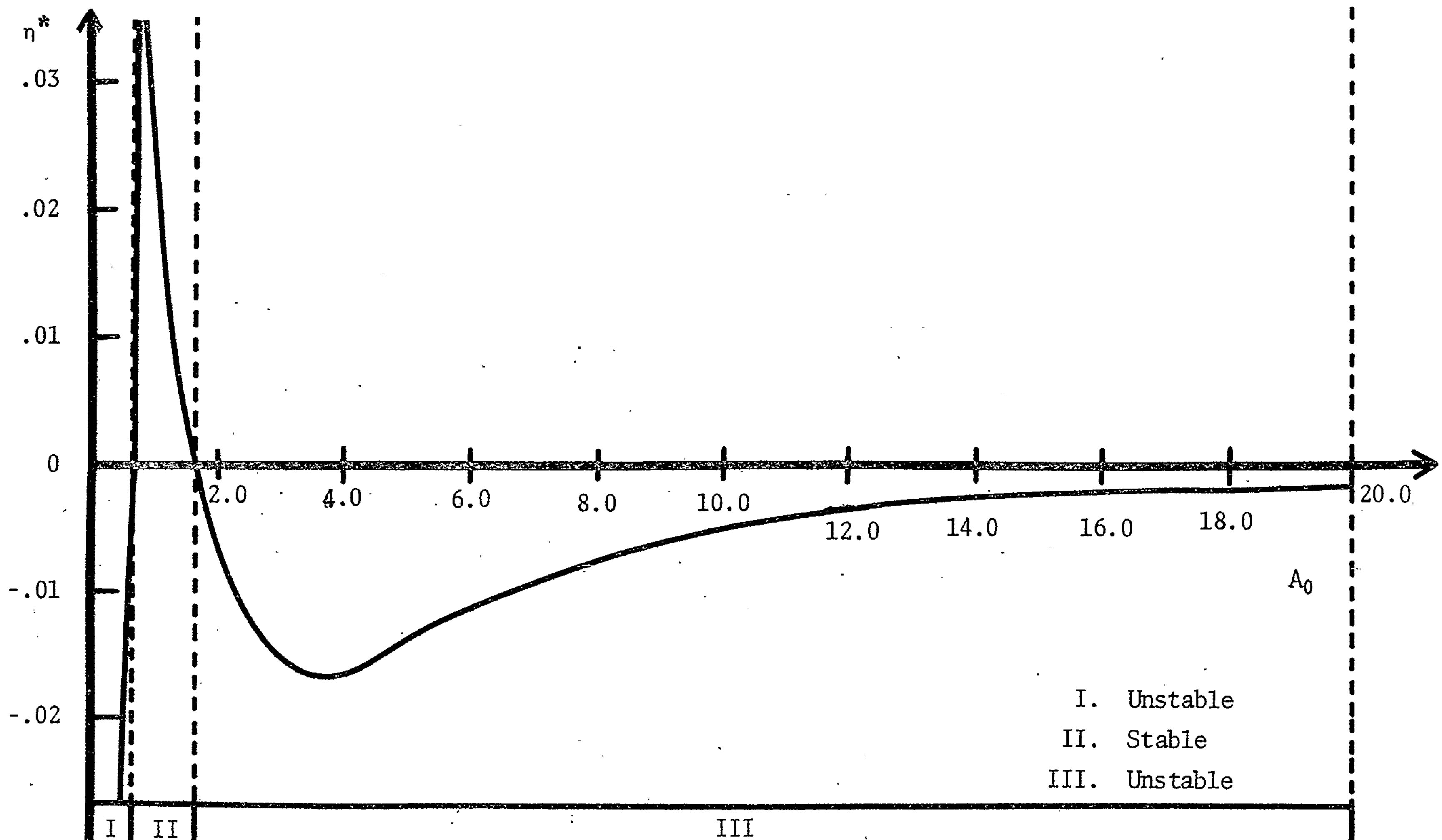

Fig. 3.15 Stability regions for the system given in Section 3.3.

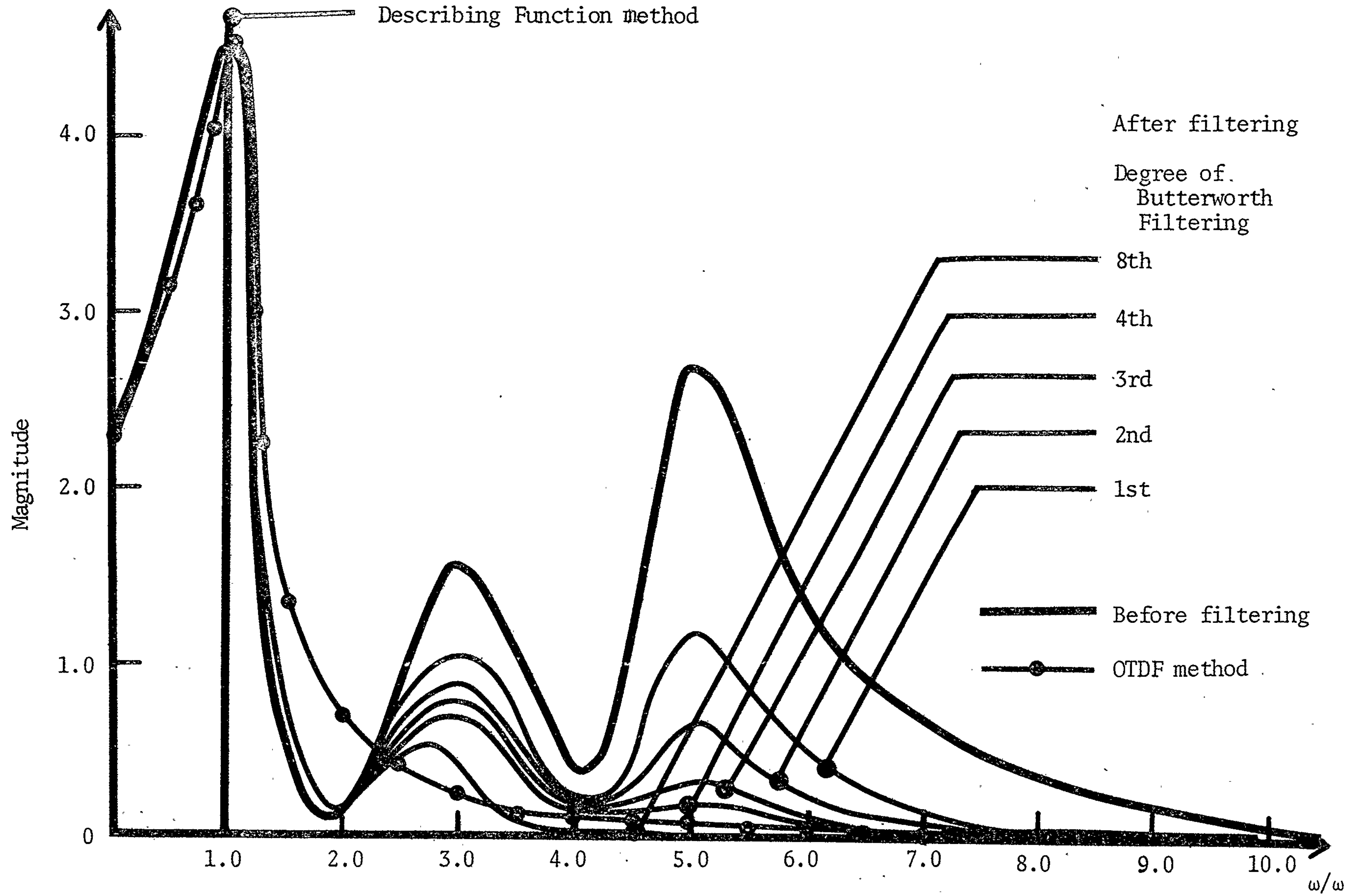

Fig. 3.16 Filtering effect on the frequency spectrum of a transient signal.

4. ASYMMETRICAL TRANSIENT OSCILLATIONS

4.1 INTRODUCTION

Asymmetrical oscillatory transient processes are common in some classes of nonlinear control systems where the oscillatory behavior is superimposed on aperiodic slowly varying signals. This type of oscillation is encountered when the input-output characteristics of the nonlinearity is nonsymmetrical, or when the input to the nonlinear element consists of an oscillatory signal and an aperiodic or bias signal. In most cases, it is required that the frequency of the bias signal be sufficiently lower than the oscillatory one [8]. For lightly damped transients, many writers have used the concept of dual describing function to represent the oscillatory behavior [1], [12], [21] - [23]. They considered the input to the nonlinearity to be a sinusoidal component superimposed upon a constant bias signal. The sinusoidal signal effectively linearized the nonlinearity to the bias signal. As the basic assumption in that approach is only true for lightly damped transients, desirable responses have not been predicted with sufficient accuracy when the oscillations are damped. The following sections present a unified theory for the investigation of asymmetrical transient oscillations in nonlinear systems. Continuous, also discrete, systems are considered.

4.2 THE ASYMMETRICAL TRANSIENT FORM OF THE KRYLOV-BOGOLIUBOV ASYMPTOTIC METHOD

In the study of asymmetrical transient oscillations in nonlinear systems, the asymmetrical behavior is usually considered to be the solution of the second-order nonautonomous nonlinear differential equation of the form

$$\ddot{x} - 2\sigma\dot{x} + \omega_n^2 x + \mu f(x,\dot{x};t) = 0 \qquad (4.2\text{-}1)$$

where σ, ω_n and μ are real constant coefficients. The value of μ is restricted to be small and $f(x,\dot{x};t)$ is a nonlinear function.

We seek a solution for equation (4.2-1) of the form

$$x(t) = e^{\sigma t}\{B_0(t) + A_0(t).\sin[\omega_0 t + \theta(t)]\} \tag{4.2-2}$$

where $A_0(t)$, $B_0(t)$ and $\theta(t)$ are time dependent functions. For convenience we shall assume this time dependency without indicating it by appended arguments. ω_0 is chosen such that

$$\omega_0{}^2 = \omega_n{}^2 - \sigma^2 > 0. \tag{4.2-3}$$

Let

$$\psi(t) = \omega_0 t + \theta \tag{4.2-4}$$

so that

$$\begin{aligned}\dot{\psi}(t) &= \omega \\ &= \omega_0 + \dot{\theta}.\end{aligned} \tag{4.2-5}$$

Also assume $A = A_0\, e^{\sigma t}$, $B = B_0\, e^{\sigma t}$ and $\alpha = \frac{\sigma}{\omega}$. Consequently, $\dot{x}(t)$ and $\ddot{x}(t)$ can be expressed using an approach similar to that followed in Section 2.2 as

$$\dot{x}(t) = \sigma(B + A \sin\psi) + \omega_0 A \cos\psi \tag{4.2-6}$$

and

$$\begin{aligned}\ddot{x}(t) = \omega_0\, e^{\sigma t}(\dot{A}_0 \cos\psi - \dot{\theta}A_0 \sin\psi) + \omega_0{}^2\, e^{\sigma t}\, B_0 \\ + 2\sigma\dot{x} - \omega_n^2 x.\end{aligned} \tag{4.2-7}$$

This requires, (2.2-8),

$$\dot{B}_0 + \dot{A}_0 \sin\psi + \dot{\theta}A_0 \cos\psi = 0. \tag{4.2-8}$$

Substituting (4.2-8) into (4.2-1), we get

$$\begin{aligned}B_0 + \frac{\dot{A}_0}{\omega_0}\cos\psi - \frac{\dot{\theta}A_0}{\omega_0}\sin\psi = -\frac{\mu e^{-\alpha\psi}}{\omega_0{}^2} f[B + A \sin\psi, \\ \sigma(B + A \sin\psi) + \omega_0 A \cos\psi; \frac{\psi}{\omega}] .\end{aligned} \tag{4.2-9}$$

Also, from (4.2-8) and (4.2-9) and after some manipulations, we obtain

$$\dot{B}_0 \sin\psi + \omega_0 B_0 \cos\psi + \dot{A}_0 = -\frac{\mu}{\omega_0} e^{-\alpha\psi} f[B + A\sin\psi,$$

$$\sigma(B + A\sin\psi) + \omega_0 A\cos\psi; \frac{\psi}{\omega}] \cos\psi \qquad (4.2\text{-}10)$$

and

$$\frac{\dot{B}_0}{A_0}\cos\psi - \frac{\omega_0 B_0}{A_0}\sin\psi + \dot{\theta} = \frac{\mu e^{-\alpha\psi}}{A_0\omega_0} f[B + A\sin\psi,$$

$$\sigma(B + A\sin\psi) + \omega_0 A\cos\psi; \frac{\psi}{\omega}] \sin\psi. \qquad (4.2\text{-}11)$$

Since μ is restricted to be small, then $\dot{A}_0$, $\dot{B}_0$ and $\dot{\theta}$ are small, so that $A_0(t)$, $B_0(t)$ and $\theta(t)$ are slowly varying functions of time. Consequently, by averaging equations (4.2-9) to (4.2-11) over a complete period, the estimated values of $\dot{A}_0$, $\dot{\theta}$ and B_0 turn to be

$$\dot{A}_0 = -\frac{\mu}{2\pi\omega_0}\int_0^{2\pi} e^{-\alpha\psi} f[B + A\sin\psi, \sigma(B + A\sin\psi) +$$

$$\omega_0 A\cos\psi; \frac{\psi}{\omega}] \cos\psi \, d\psi \qquad (4.2\text{-}12)$$

$$\dot{\theta} = \frac{\mu}{2\pi A_0\omega_0}\int_0^{2\pi} e^{-\alpha\psi} f[B + A\sin\psi, \sigma(B + A\sin\psi) +$$

$$\omega_0 A\cos\psi; \frac{\psi}{\omega}] \sin\psi \, d\psi \qquad (4.2\text{-}13)$$

and

$$B_0 = -\frac{\mu}{2\pi\omega_0^2}\int_0^{2\pi} e^{-\alpha\psi} f[B + A\sin\psi, \sigma(B + A\sin\psi) +$$

$$\omega_0 A\cos\psi; \frac{\psi}{\omega}] \, d\psi . \qquad (4.2\text{-}14)$$

Equations (4.2-12), (4.2-13) and (4.2-14) will be referred to as the asymmetrical transient form of the Krylov-Bogoliubov asymptotic method. When evaluating the integrals in those equations, A and B should be expressed in their explicit form as $A_0e^{\sigma t}$ and $B_0e^{\sigma t}$ respectively. Define the function $N_0(A,B_0,\alpha,\sigma,\omega_0)$ as

$$N_0(A_0, B_0,\alpha,\sigma,\omega_0) = \frac{1}{2\pi B_0}\int_0^{2\pi} e^{-\alpha\psi} f[B + A \sin\psi, \sigma(B + A \sin\psi) + \omega_0 A \cos\psi; \frac{\psi}{\omega}] \, d\psi \quad . \tag{4.2-15}$$

Then, equation (4.2-14) can be written as

$$\omega_0^2 + \mu N_0(A_0,B_0,\alpha,\sigma,\omega_0) = 0 \quad . \tag{4.2-16}$$

Equation (4.2-16) will be denoted as the auxiliary equation of the system described by (4.2-1).

4.3 ASYMMETRICAL OSCILLATORY TRANSIENT DESCRIBING FUNCTION

Let x(t), (4.2-2), and y(t) be the input and the output to a nonlinear element. For $A_0(t)$ and $B_0(t)$, (4.2-2), equal to constants, let $x(t) = x(t,A_0,B_0)$ and $y(t) = y(t,A_0,B_0)$. Furthermore, let

$$x_h(t,A_0,B_0) = x(t,A_0,B_0)\, u_h(t) \tag{4.3-1}$$

and

$$y_h(t,A_0,B_0) = y(t,A_0,B_0)\, u_h(t) \tag{4.3-2}$$

where $u_h(t)$ is given in (2.3-3).

Define the finite period complex function $N_h(A_0,B_0,s)$ as

$$N_h(A_0,B_0,s) = \frac{Y_h(A_0,B_0,s)}{X_h(A_0,B_0,s)}$$

$$= \frac{\int_0^\infty y_h(t,A_0,B_0)\ e^{-st}\ dt}{\int_0^\infty x_h(t,A_0,B_0)\ e^{-st}\ dt}$$

$$= \frac{\int_0^h y(t,A_0,B_0)\ e^{-st}\ dt}{\int_0^h x(t,A_0,B_0)\ e^{-st}\ dt} . \tag{4.3-3}$$

Consider a nonlinearity $y(x,\dot{x};t)$. Let the input to this nonlinearity be

$$x(t,A_0,B_0) = e^{\sigma t}\ [B_0 + A_0\ \sin\psi(t)] \tag{4.3-4}$$

where $\psi(t) = \omega t$ and ω is assumed to be constant. Then, the output $y(t,A_0,B_0)$ can be written as

$$\begin{aligned} y(t,A_0,B_0) = y\{&e^{\sigma t}\ (B_0 + A_0\ \sin\omega t),\ e^{\sigma t}\ [\sigma(B_0 + A_0\ \sin\omega t) \\ &+ \omega A_0\ \cos\omega t];\ t\} \ . \end{aligned} \tag{4.3-5}$$

In dealing with asymmetrical types of oscillations, we select $h = T = \frac{2\pi}{\omega}$.

Expressing s as $s = \sigma + j\omega$, we obtain from (4.3-3), (4.3-4) and (4.3-5)

$$N_T(A_0,B_0,\alpha,\omega) = \frac{\int_0^{\frac{2\pi}{\omega}} y\{e^{\sigma t}(B_0+A_0 \sin\omega t),\ e^{\sigma t}[\sigma(B_0+A_0 \sin\omega t)}{\int_0^{\frac{2\pi}{\omega}} e^{\sigma t}(B_0+A_0 \sin\omega t)\ e^{-(\sigma+j\omega)t}\ dt}$$

$$+ \omega A_0 \cos\omega t];\ t\}\ e^{-(\sigma+j\omega)t}\ dt. \qquad (4.3\text{-}6)$$

Changing the variable of integration in (4.3-6), introducing α, and simplifying yields

$$N_T(A_0,B_0,\alpha,\omega) = \frac{j}{\pi A_0}\int_0^{2\pi} e^{-\alpha\psi}\ y\{e^{\alpha\psi}\ (B_0 + A_0 \sin\psi),$$

$$e^{\alpha\psi}\omega[\alpha(B_0 + A_0 \sin\psi) + A_0 \cos\psi];\ \frac{\psi}{\omega}\ \}\ e^{-j\psi}\ d\psi\ . \qquad \ldots(4.3\text{-}7)$$

Let $N_T(A_0,B_0,\alpha,\omega)$, (4.3-7), be the asymmetrical oscillatory transient describing function. Express $N_T(A_0,B_0,\alpha,\omega)$ as

$$N_T(A_0,B_0,\alpha,\omega) = N_{Tp}(A_0,B_0,\alpha,\omega) + jN_{Tq}(A_0,B_0,\alpha,\omega) \qquad \ldots(4.3\text{-}8)$$

We have

$$N_{Tp}(A_0,B_0,\alpha,\omega) = \frac{1}{\pi A_0}\int_0^{2\pi} e^{-\alpha\psi}\ y\{e^{\alpha\psi}\ (B_0 + A_0 \sin\psi),$$

$$e^{\alpha\psi}\omega[\alpha(B_0 + A_0 \sin\psi) + A_0 \cos\psi];\ \frac{\psi}{\omega}\ \}\ \sin\psi\ d\psi \qquad \ldots(4.3\text{-}9)$$

and

$$N_{Tq}(A_0,B_0,\alpha,\omega) = \frac{1}{\pi A_0}\int_0^{2\pi} e^{-\alpha\psi}\, y\{e^{\alpha\psi}(B_0+A_0\ \sin\psi),$$

$$e^{\alpha\psi}\omega[\alpha(B_0+A_0\ \sin\psi) + A_0\ \cos\psi];\ \frac{\psi}{\omega}\}\ \cos\psi\ d\psi \qquad (4.3\text{-}10)$$

If we let $B_0 = 0$, (4.3-7) reduces to the complete period OTDF (2.3-9).

To complete the analysis, let us define $N_0(A_0,B_0,\alpha,\omega)$, to be the auxiliary direct gain, as

$$N_0(A_0,B_0,\alpha,\omega) = \lim_{s\to\alpha\omega} N_T(A_0,B_0,s)$$

$$= \lim_{s\to\alpha\omega} \frac{\int_0^T y(t,A_0,B_0)\ e^{-st}\ dt}{\int_0^T x(t,A_0,B_0)\ e^{-st}\ dt}\ . \qquad (4.3\text{-}11)$$

Substituting $x(t,A_0,B_0)$ and $y(t,A_0,B_0)$, (4.3-4) and (4.3-5), into (4.3-11), we obtain after some manipulations

$$N_0(A_0,B_0,\alpha,\omega) = \frac{1}{2\pi B_0}\int_0^{2\pi} e^{-\alpha\psi}\, y\{e^{\alpha\psi}(B_0+A_0\ \sin\psi),$$

$$e^{\alpha\psi}\omega[\alpha(B_0+A_0\ \sin\psi) + A_0\ \cos\psi];\ \frac{\psi}{\omega}\}\ d\psi. \qquad (4.3\text{-}12)$$

4.4 PROPERTIES OF THE ASYMMETRICAL OSCILLATORY TRANSIENT DESCRIBING FUNCTION

Let a nonlinear element $y(x,\dot{x};t)$ be injected with the transient component

$$x(\psi,A_0,B_0) = e^{\alpha\psi}(B_0+A_0\ \sin\psi). \qquad (4.4\text{-}1)$$

Then, the output can be expressed as

$$y(\psi,A_0,B_0) = y\{e^{\alpha\psi}(B_0+A_0 \sin\psi),\ e^{\alpha\psi}\omega[\alpha(B_0+A_0 \sin\psi) + A_0 \cos\psi];\ \frac{\psi}{\omega}\}\ . \tag{4.4-2}$$

Due to the presence of the bias signal, the output of the nonlinear element will be assumed to take the form, (2.4-3),

$$y_a(\psi,A_0,B_0) = e^{\alpha\psi}(a \sin\psi + b \cos\psi + c) \tag{4.4-3}$$

where a, b and c are functions of A_0, B_0, α and ω. This dependency will not be indicated by appended arguments.

As in Section 2.4, let

$$y'(\psi,A_0,B_0) = e^{-\alpha\psi}y(\psi,A_0,B_0) \tag{4.4-4}$$

and

$$y_a'(\psi,A_0,B_0) = e^{-\alpha\psi}y_a(\psi,A_0,B_0). \tag{4.4-5}$$

We minimize the mean error square over a complete period $\overline{e^2}$, where

$$e(\psi,A_0,B_0) = y'(\psi,A_0,B_0) - y_a'(\psi,A_0,B_0)\ . \tag{4.4-6}$$

At the minimum,

$$\frac{\partial \overline{e^2}}{\partial a} = \frac{\partial \overline{e^2}}{\partial b} = \frac{\partial \overline{e^2}}{\partial c} = 0\ . \tag{4.4-7}$$

From (4.4-2) to (4.4-6), we obtain

$$a_T = \frac{1}{\pi} \int_0^{2\pi} e^{-\alpha\psi}\, y\{e^{\alpha\psi}(B_0+A_0 \sin\psi),\ e^{\alpha\psi}\omega[\alpha(B_0+A_0 \sin\psi)$$

$$+ A_0 \cos\psi];\ \frac{\psi}{\omega}\} \sin\psi \, d\psi \tag{4.4-8}$$

$$b_T = \frac{1}{\pi} \int_0^{2\pi} e^{-\alpha\psi}\, y\{e^{\alpha\psi}(B_0+A_0 \sin\psi),\ e^{\alpha\psi}\omega[\alpha(B_0+A_0 \sin\psi)$$

$$+ A_0 \cos\psi];\ \frac{\psi}{\omega}\} \cos\psi \, d\psi \tag{4.4-9}$$

and

$$c_T = \frac{1}{2\pi} \int_0^{2\pi} e^{-\alpha\psi}\, y\{e^{\alpha\psi}(B_0+A_0 \sin\psi),\ e^{\alpha\psi}\omega[\alpha(B_0+A_0 \sin\psi)$$

$$+ A_0 \cos\psi];\ \frac{\psi}{\omega}\} \, d\psi . \tag{4.4-10}$$

From (4.3-3), (4.4-1) and (4.4-3), it can be shown that

$$N_T(A_0,B_0,\alpha,\omega) = \frac{a_T}{A_0} + j\,\frac{b_T}{A_0} . \tag{4.4-11}$$

Moreover, it appears from (4.3-11), (4.4-1) and (4.4-3) that

$$N_0(A_0,B_0,\alpha,\omega) = \frac{c_T}{B_0} . \tag{4.4-12}$$

Substituting (4.4-8), (4.4-9) and (4.4-10) into (4.4-11) and (4.4-12), we obtain $N_T(A_0,B_0,\alpha,\omega)$ and $N_0(A_0,B_0,\alpha,\omega)$ as given in (4.3-7) and (4.3-12). Hence, they are optimal in the sense that they minimize the mean square error.

4.5 STABILITY STUDIES

Consider the nonlinear system in Fig. 2.1. Let $N_T(A_0,B_0,\alpha,\omega)$ describe the effect of the nonlinearity in the closed loop system during a period T, where A_0 and B_0 are the instantaneous amplitude and biased component of the input respectively. For any asymmetrical transient oscillations to take place, it is necessary and sufficient that the quasi-linearized system satisfies

$$1 + N_T(A_0,B_0,\alpha,\omega) \cdot G(\alpha\omega,j\omega) = 0. \qquad (4.5\text{-}1)$$

Also, to describe the biased component let, (4.3-3), (4.3-11) and (4.3-12),

$$\begin{aligned} W(A_0,B_0,\alpha,\omega) &= \lim_{s\to\alpha\omega} [1 + N_T(A_0,B_0,s)\ G(s)] \\ &= 1 + N_0(A_0,B_0,\alpha,\omega)\ G(\alpha\omega) \\ &= 0. \end{aligned} \qquad (4.5\text{-}2)$$

Equation (4.5-2) will be referred to as the auxiliary equation of the system in Fig. 2.1.

The solution to equations (4.5-1) and (4.5-2) for a given value of $A_0 = A_0'$ will be denoted by $S_0 = S_0(A_0',B_0',\alpha_0,\omega_0)$ and represents the instantaneous asymmetrical oscillatory transient solution. Such a solution will be considered to be stable if any perturbations to A_0' tend to vanish as $t\to\infty$.

<u>Theorem 4.1</u>

Let U and V be the real and imaginary parts of (4.5-1). Consequently,

$$U(S) + j\ V(S) = 0. \qquad (4.5\text{-}3)$$

Also let, (4.5-2),

$$W(S) = 0 \qquad (4.5\text{-}4)$$

where $S = S(A_0,B_0,\alpha,\omega)$.

Assume that U and V and their first derivatives are continuous functions of S in a small domain around S_0. Define the operator E_a, such that

$$E_a = \frac{\partial}{\partial A_0} + K_b(A_0)\frac{\partial}{\partial B_0} + K_\alpha(A_0)\frac{\partial}{\partial \alpha} + K_\omega(A_0)\frac{\partial}{\partial \omega} . \quad (4.5\text{-}5)$$

Then a necessary and sufficient condition for S_0 to be stable is for a real positive number η_a to exist such that in the neighborhood of S_0 the condition

$$\frac{\partial U}{\partial \alpha} E_a(U) + \frac{\partial V}{\partial \alpha} E_a(V) + \eta_a\left(\frac{\partial V}{\partial \alpha}\frac{\partial U}{\partial \omega} - \frac{\partial U}{\partial \alpha}\frac{\partial V}{\partial \omega}\right) = 0 \quad (4.5\text{-}6)$$

is satisfied.

Proof

Equation (4.5-1) can be expressed as

$$U(A_0,B_0,\alpha,\omega) + j\, V(A_0,B_0,\alpha,\omega) = 0 \qquad (4.5\text{-}7)$$

where U and V are the real and imaginary parts in (4.5-1). For $A_0 = A_0'$, we have $S_0 = S_0(A_0',B_0',\alpha_0,\omega_0)$ and it satisfies (4.5-2) and (4.5-7).

Consider small perturbation around S_0 such that $\dot{A}_0$, $\dot{B}_0$ and $\dot{\theta}$ are small. We denote these perturbations as follows

$$\left.\begin{aligned} A_0' &\to A_0' + \Delta A_0 \\ B_0' &\to B_0' + \Delta B_0 \\ \alpha_0 &\to \alpha_0 + \Delta\alpha_1 + j\,\Delta\alpha_2 \\ \omega_0 &\to \omega_0 + \Delta\omega_1 + j\,\Delta\omega_2 \end{aligned}\right\} \qquad (4.5\text{-}8)$$

ΔA_0, ΔB_0, $\Delta\alpha_1$ and $\Delta\omega_1$ are real numbers. $\Delta\alpha_2$ and $\Delta\omega_2$ are given by equations (2.5-11) and (2.5-12) respectively.

Following similar steps as in Theorem 2.1, we get in the neighborhood of S_0 the relationships

$$\frac{\partial U}{\partial A_0} + \frac{\partial U}{\partial B_0}\frac{dB_0}{dA_0} + \frac{\partial U}{\partial \alpha}\frac{d\alpha_1}{dA_0} + \frac{\partial U}{\partial \omega}\frac{d\omega_1}{dA_0} = \frac{\partial V}{\partial \omega}\frac{d\omega_2}{dA_0} + \frac{\partial V}{\partial \alpha}\frac{d\alpha_2}{dA_0} \tag{4.5-9}$$

and

$$\frac{\partial V}{\partial A_0} + \frac{\partial V}{\partial B_0}\frac{dB_0}{dA_0} + \frac{\partial V}{\partial \alpha}\frac{d\alpha_1}{dA_0} + \frac{\partial V}{\partial \omega}\frac{d\omega_1}{dA_0} = -\left(\frac{\partial U}{\partial \omega}\frac{d\omega_2}{dA_0} + \frac{\partial U}{\partial \alpha}\frac{d\alpha_2}{dA_0}\right). \tag{4.5-10}$$

In the neighborhood of S_0, let $K_b(A_0') = \frac{dB_0}{dA_0}$, $K_\alpha(A_0') = \frac{d\alpha_1}{dA_0}$ and $K_\omega(A_0') = \frac{d\omega_1}{dA_0}$.

In terms of the operator E_a, (4.5-5), equations (4.5-9) and (4.5-10) can be expressed as

$$E_a(U) = \frac{\partial V}{\partial \omega}\frac{d\omega_2}{dA_0} + \frac{\partial V}{\partial \alpha}\frac{d\alpha_2}{dA_0} \tag{4.5-11}$$

and

$$E_a(V) = -\left(\frac{\partial U}{\partial \omega}\frac{d\omega_2}{dA_0} + \frac{\partial U}{\partial \alpha}\frac{d\alpha_2}{dA_0}\right). \tag{4.5-12}$$

Eliminating $\frac{d\alpha_2}{dA_0}$, we get

$$\frac{\partial U}{\partial \alpha}E_a(U) + \frac{\partial V}{\partial \alpha}E_a(V) = \left(\frac{\partial U}{\partial \alpha}\frac{\partial V}{\partial \omega} - \frac{\partial U}{\partial \omega}\frac{\partial V}{\partial \alpha}\right)\frac{d\omega_2}{dA_0}. \tag{4.5-13}$$

From (2.5-12), the instantaneous solution S_0 is stable iff

$$\eta_a = \frac{d\omega_2}{dA_0} > 0. \tag{4.5-14}$$

Hence, S_0 is stable iff in its neighborhood there exists a real positive number η_a such that the condition (4.5-6) is satisfied. ∎ Similar corollaries can be obtained as those given in Section 2.5.

4.6 NONLINEAR SAMPLED DATA SYSTEMS

Asymmetrical transient form of the Krylov-Bogoliubov method

We will now present a sampled version of the nonlinear differential equation

$$\ddot{x} - 2\sigma\dot{x} + \omega_n^2 x + \varepsilon g(x,\dot{x};t) = 0 \qquad (4.6\text{-}1)$$

where σ, ω_n and ε are constants and ε is required to be small. We consider only the class of systems possessing the property $\omega T_s << 1$, (2.6-2).

The general solution of (4.6-1) can be expressed as

$$x(nT_s) = e^{\sigma nT_s}\{B_0(nT_s) + A_0(nT_s) \cdot \sin[\omega_0 nT_s + \theta(nT_s)]\} \qquad \ldots(4.6\text{-}2)$$

where ω_0 satisfies (2.6-4).

Since ε is selected to be small, then the variation in A_0, B_0 and θ will be small.

We seek a solution for $\dot{x}(nT_s)$ in the form

$$\dot{x}(nT_s) = \sigma(B + A \sin\psi^*) + \omega_0 A \cos\psi^*. \qquad (4.6\text{-}3)$$

This requires, (2.2-8),

$$\dot{B}_0 + \dot{A}_0 \sin\psi^* + \dot{\theta}A_0 \cos\psi^* = 0. \qquad (4.6\text{-}4)$$

Following the same steps as in Sections 2.6 and 4.2 we can write the sampled version of the asymmetrical transient form of the Krylov-Bogoliubov method as, (2.6-9) to (2.6-11),

$$\dot{A}_0 = -\frac{\varepsilon}{2\omega_0 N_s} \sum_{n=0}^{2N_s-1} e^{-\alpha\psi^*} g[B + A \sin\psi^*, \sigma(B + A \sin\psi^*) +$$

$$\omega_0 A \cos\psi^*; \frac{\psi^*}{\omega}] \cos\psi^* \qquad (4.6\text{-}5)$$

$$\dot{\theta} = \frac{\varepsilon}{2A_0\omega_0 N_S} \sum_{n=0}^{2N_S-1} e^{-\alpha\psi^*} g[B + A\sin\psi^*, \sigma(B + A\sin\psi^*) +$$

$$\omega_0 A\cos\psi^*; \frac{\psi^*}{\omega}] \sin\psi^* \tag{4.6-6}$$

and

$$B_0 = -\frac{\varepsilon}{2N_S\omega_0^2} \sum_{n=0}^{2N_S-1} e^{-\alpha\psi^*} g[B + A\sin\psi^*, \sigma(B + A\sin\psi^*)$$

$$+ \omega_0 A\cos\psi^*; \frac{\psi^*}{\omega}] . \tag{4.6-7}$$

Sampled asymmetrical oscillatory transient describing function

Define the finite period complex function $N_h(A_0,B_0,z)$ as

$$N_h(A_0,B_0,z) = \frac{Y_h(A_0,B_0,z)}{X_h(A_0,B_0,z)}$$

$$= \frac{\sum_{n=0}^{[h/T_S]} y(nT_S,A_0,B_0)z^{-n}}{\sum_{n=0}^{[h/T_S]} x(nT_S,A_0,B_0)z^{-n}} . \tag{4.6-8}$$

From (4.3-4), (4.3-5) and (4.6-8), and letting $t = nT_S$, $z = e^{sT_S}$, $s = \sigma + j\omega$ and $h = T$, we can write the sampled asymmetrical transient describing function as

$$N_T^*(A_0,B_0,\alpha,\omega) = \frac{j}{A_0N_S} \sum_{n=0}^{2N_S-1} e^{-\alpha\psi^*} y\{B + A\sin\psi^*,$$

$$\omega[\alpha(B + A\sin\psi^*) + A\cos\psi^*]; \frac{\psi^*}{\omega}\} e^{-j\psi^*} . \tag{4.6-9}$$

Let us also define $N_0^*(A_0,B_0,\alpha,\omega)$, to be denoted as the sampled auxiliary direct gain, as follows

$$N_0^*(A_0,B_0,\alpha,\omega) = \lim_{z\to e^{\alpha\omega T_S}} N_T(A_0,B_0,z)$$

$$= \lim_{z\to e^{\alpha\omega T_S}} \frac{\sum_{n=0}^{[T/T_S]} y(nT_S,A_0,B_0)z^{-n}}{\sum_{n=0}^{[T/T_S]} x(nT_S,A_0,B_0)z^{-n}}$$

$$= \frac{1}{2N_SB_0} \sum_{n=0}^{2N_S-1} e^{-\alpha\psi^*} y\{B + A \sin\psi^*,$$

$$\omega[\alpha(B + A \sin\psi^*) + A \cos\psi^*]; \frac{\psi^*}{\omega}\}.$$

...(4.6-10)

Similar to Section 4.4, the complex function $N_T^*(A_0,B_0,\alpha,\omega)$ and the gain $N_0^*(A_0,B_0,\alpha,\omega)$ are optimal in the sense that they minimize the mean square error.

Stability analysis

The stability of the sampled data system can be investigated in a similar way as in Section 4.5. The instantaneous asymmetrical solution must also satisfy the auxiliary equation (4.6-11)

$$W^*(A_0,B_0,\alpha,\omega) = \lim_{z\to e^{\alpha\omega T_S}} [1 + N_T(A_0,B_0,z)\ G(z)]$$

$$= 1 + N_0^*(A_0,B_0,\alpha,\omega)\ G^*(\alpha\omega)$$

$$= 0. \tag{4.6-11}$$

4.7 CONTINUOUS FEEDBACK CONTROL SYSTEM WITH NONSYMMETRICAL PRELOAD ELEMENT

Consider the system shown in Fig. 4.1. The nonlinear element will be assumed to have a nonsymmetrical preload characteristic, [9], that is,

$$y = D_1 + m_1 x , \quad x \geq 0 \tag{4.7-1}$$

and

$$y = -D_2 + m_2 x, \quad x < 0. \tag{4.7-2}$$

Where x and y are the input and the output of the nonlinear element respectively. Let x be of the form

$$x(\psi) = e^{\alpha\psi}(B_0 + A_0 \sin\psi) \tag{4.7-3}$$

where $\psi = \omega t$. Define ψ_1 and ψ_2 such that

$$A_0 \sin\psi_1 + B_0 = 0, \quad \theta_{m_1} \leq \psi_1 \leq \theta_{m_2} \tag{4.7-4}$$

and

$$A_0 \sin\psi_2 + B_0 = 0, \quad \theta_{m_2} \leq \psi_2 \leq 2\pi \tag{4.7-5}$$

where θ_{m_1} and θ_{m_2} are defined as in Fig. 4.2.

Let $\delta_0 = \dfrac{B_0}{A_0}$, $\delta_1 = \dfrac{D_1}{A_0}$ and $\delta_2 = \dfrac{D_2}{A_0}$.

The asymmetrical oscillatory transient describing function can be expressed as

$$N_T(A_0,B_0,\alpha,\omega) = N_{Tp}(A_0,B_0,\alpha,\omega) + j\, N_{Tq}(A_0,B_0,\alpha,\omega) \qquad \ldots(4.7\text{-}6)$$

where, (4.3-9),

$$N_{Tp}(A_0,B_0,\alpha,\omega) = \frac{1}{\pi A_0} \{\int_0^{\psi_1} e^{-\alpha\psi}[D_1 + m_1 e^{\alpha\psi}(B_0 + A_0 \sin\psi)]$$

$$\sin\psi \, d\psi + \int_{\psi_1}^{\psi_2} e^{-\alpha\psi} [-D_2 + m_2 e^{\alpha\psi}(B_0 + A_0 \sin\psi)]\sin\psi \, d\psi +$$

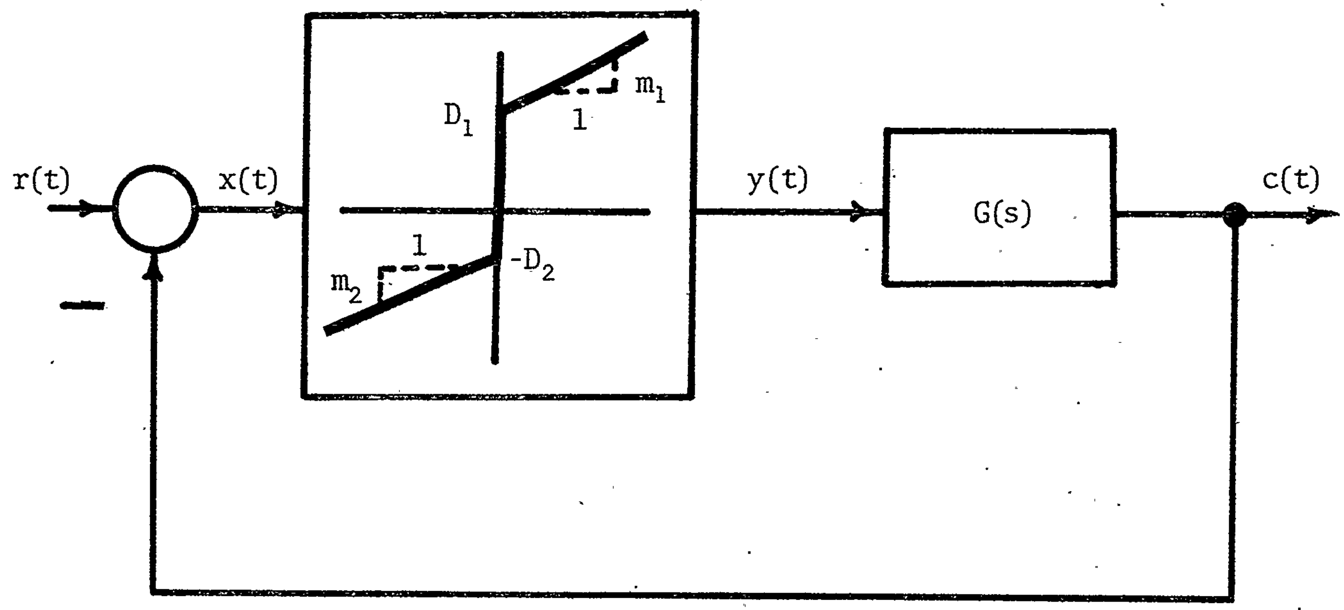

Fig. 4.1 A nonlinear control system with a preload element.

$$G(s) = \frac{K}{(\tau_1 s+1)(\tau_2 s+1)(\tau_3 s+1)}$$

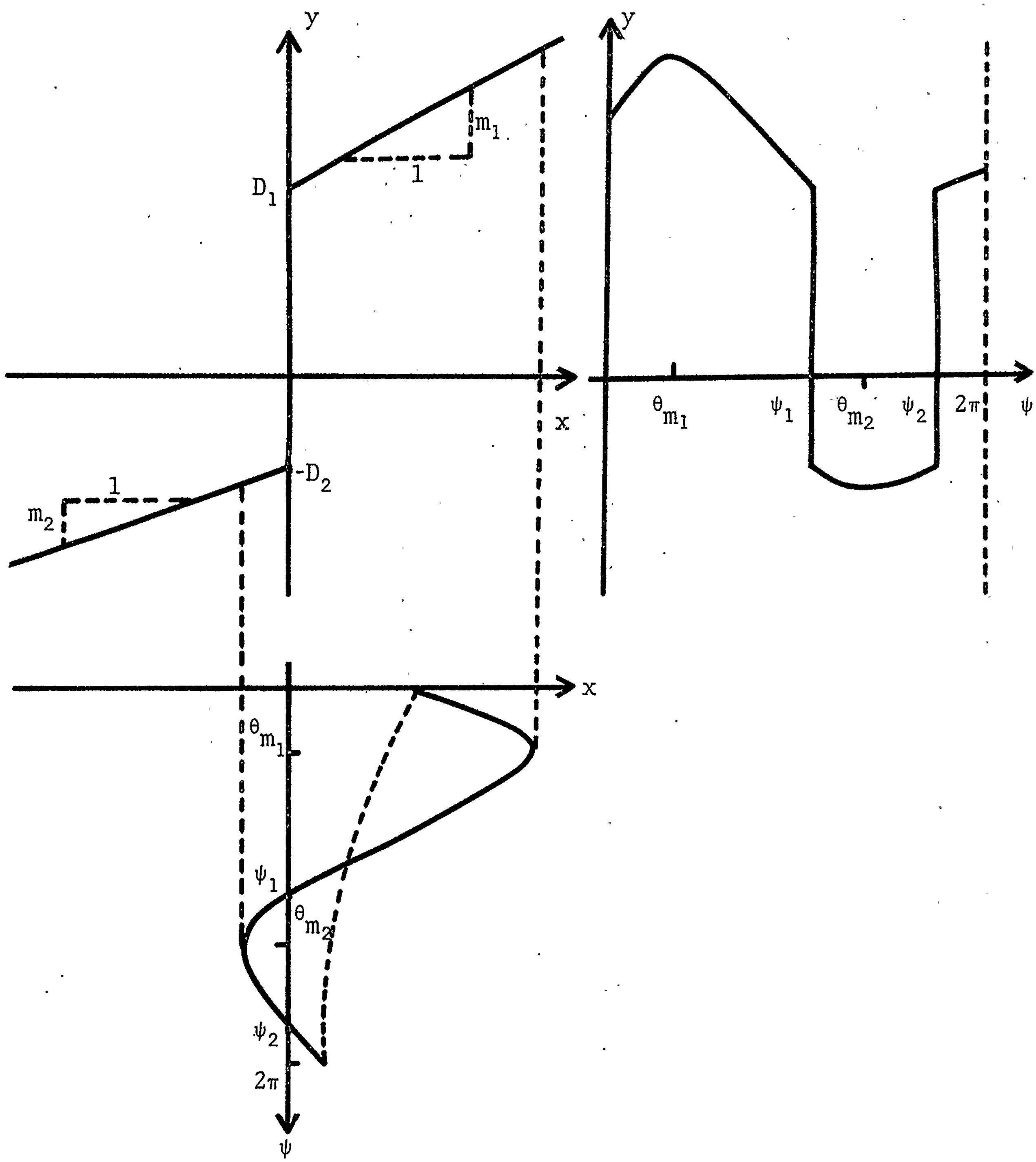

Fig. 4.2 Typical asymmetrical transient performance of a preload element.

$$\int_{\psi_2}^{2\pi} e^{-\alpha\psi} \; [D_1 + m_1 e^{\alpha\psi}(B_0 + A_0 \sin\psi)] \sin\psi \; d\psi\}. \quad (4.7\text{-}7)$$

This reduces to

$$N_{Tp}(A_0,B_0,\alpha) = \frac{1}{\pi} \; \{\frac{m_1}{2} (2\pi + \psi_1 - \psi_2) - \frac{m_2}{2} (\psi_1 - \psi_2) +$$

$$\frac{1}{4} (m_2 - m_1) (\sin 2\psi_1 - \sin 2\psi_2) - \frac{1}{1+\alpha^2} \; [e^{-\alpha\psi_1}(\delta_1 + \delta_2)$$

$$(\alpha\sin\psi_1 + \cos\psi_1) - e^{-\alpha\psi_2}(\delta_1 + \delta_2)(\alpha\sin\psi_2 + \cos\psi_2) -$$

$$\delta_1(1 - e^{-2\pi\alpha})]\}. \quad (4.7\text{-}8)$$

Also, (4.3-10),

$$N_{Tq}(A_0,B_0,\alpha,\omega) = \frac{1}{\pi A_0} \; \{\int_0^{\psi_1} e^{-\alpha\psi}[D_1 + m_1 e^{\alpha\psi}(B_0 + A_0 \sin\psi)]$$

$$\cos\psi \; d\psi + \int_{\psi_1}^{\psi_2} e^{-\alpha\psi}[-D_2 + m_2 e^{\alpha\psi}(B_0 + A_0 \sin\psi)] \cos\psi \; d\psi$$

$$+ \int_{\psi_2}^{2\pi} e^{-\alpha\psi}[D_1 + m_1 e^{\alpha\psi}(B_0 + A_0 \sin\psi)] \cos\psi \; d\psi\} \quad (4.7\text{-}9)$$

which reduces to

$$N_{Tq}(A_0,B_0,\alpha) = \frac{1}{\pi} \{ \frac{1}{4} (m_2 - m_1)(\cos 2\psi_1 - \cos 2\psi_2) -$$

$$\frac{1}{1+\alpha^2} [e^{-\alpha\psi_1}(\delta_1+\delta_2)(\alpha\cos\psi_1 - \sin\psi_1) - e^{-\alpha\psi_2}(\delta_1+\delta_2)$$

$$(\alpha\cos\psi_2 - \sin\psi_2) - \alpha\delta_1(1-e^{-2\pi\alpha})]\}. \quad (4.7\text{-}10)$$

Furthermore, for the auxiliary direct gain we have, (4.3-12),

$$N_0(A_0,B_0,\alpha,\omega) = \frac{1}{2\pi B_0}\{\int_0^{\psi_1} e^{-\alpha\psi}[D_1 + m_1 e^{\alpha\psi}(B_0 + A_0 \sin\psi)]d\psi$$

$$+ \int_{\psi_1}^{\psi_2} e^{-\alpha\psi}[-D_2 + m_2 e^{\alpha\psi}(B_0 + A_0 \sin\psi)]\, d\psi$$

$$+ \int_{\psi_2}^{2\pi} e^{-\alpha\psi}[D_1 + m_1 e^{\alpha\psi}(B_0 + A_0 \sin\psi)]\, d\psi\} \qquad (4.7\text{-}11)$$

which reduces to

$$N_0(A_0,B_0,\alpha) = \frac{1}{2\pi}\{m_1(2\pi + \psi_1 - \psi_2) - m_2(\psi_1-\psi_2) +$$

$$\frac{(m_2-m_1)}{\delta_0}(\cos\psi_1 - \cos\psi_2) - \frac{\delta_2}{\alpha\delta_0}(e^{-\alpha\psi_1} - e^{-\alpha\psi_2}) +$$

$$\frac{\delta_1}{\alpha\delta_0}(1 - e^{-\alpha\psi_1} + e^{-\alpha\psi_2} - e^{-2\pi\alpha})\}. \qquad (4.7\text{-}12)$$

Let $D_1 = 2.0$, $D_2 = 1.0$, $m_1 = 0.2$, $m_2 = 0.5$, $\tau_1 = 0.2$, $\tau_2 = 0.5$, $\tau_3 = 1.0$ and $K = 5.0$. Then the transfer function of the plant in Fig. 4.1 becomes

$$G(s) = \frac{5.0}{(0.2s+1)(0.5s+1)(s+1)}\,. \qquad (4.7\text{-}13)$$

Following similar steps as those outlined in Section 2.8, we obtain the asymmetrical instantaneous oscillatory transient solution shown in Figs. 4.3, 4.4 and 4.5. From these figures, it can be seen that the accuracy of prediction is less than that of the symmetrical case. Stability regions are also given in Fig. 4.6 using the analysis given in Section 4.5.

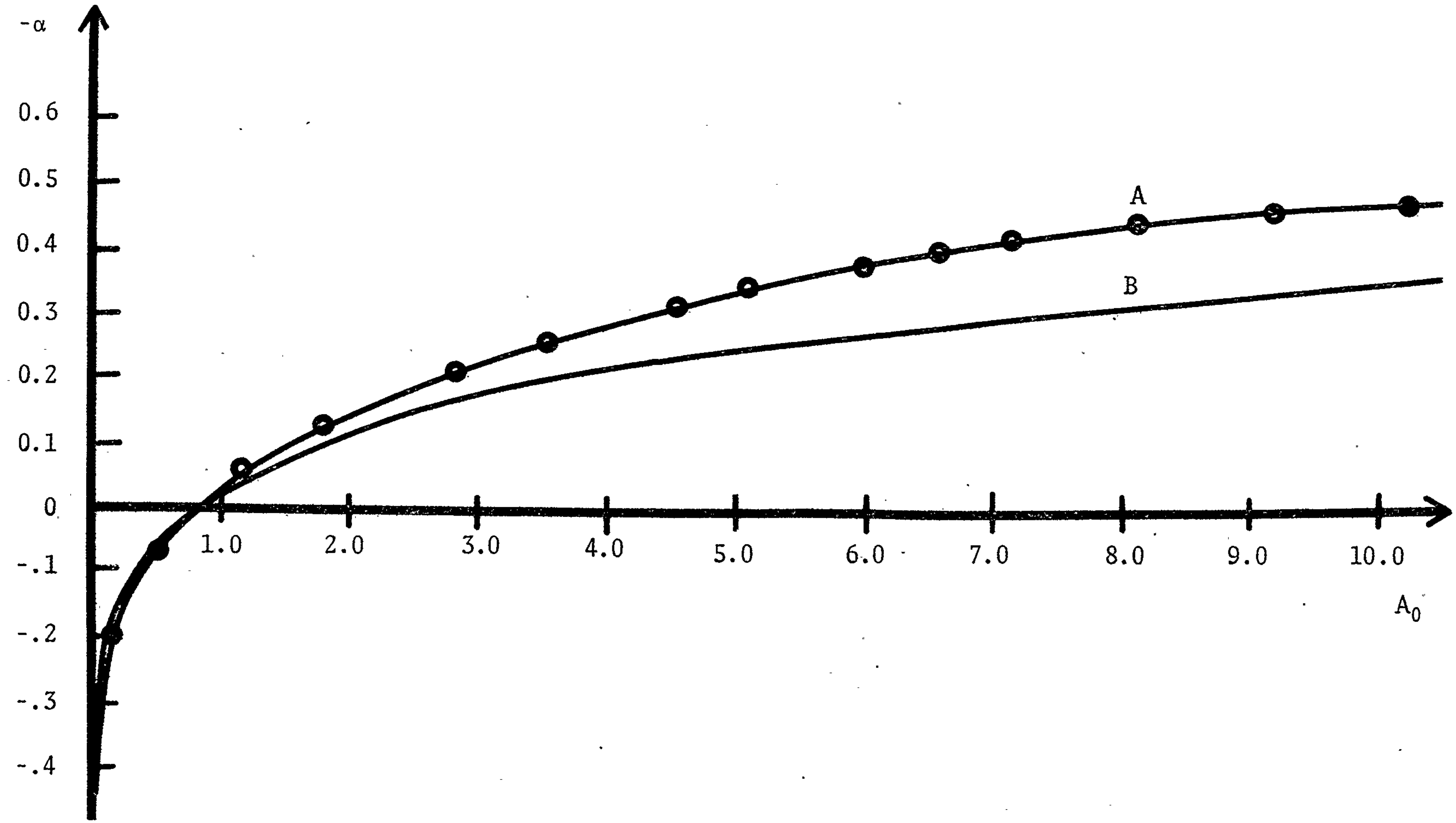

Fig. 4.3 Asymmetrical relative damping characteristics of the system given in Section 4.7.

A. Estimated behavior using simulation.

B. Asymmetrical OTDF.

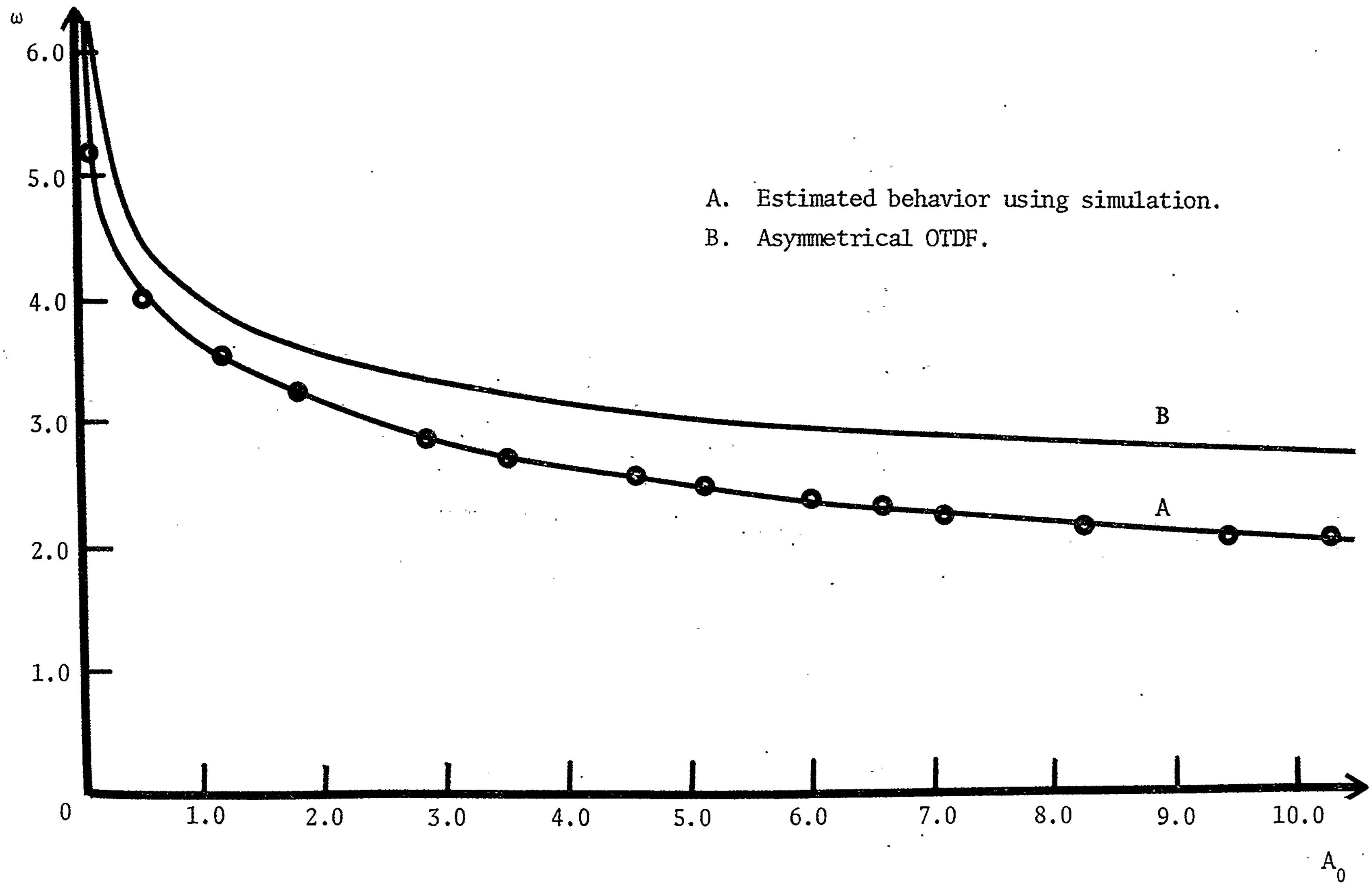

Fig. 4.4 Asymmetrical oscillatory transient frequency characteristics of the system given in Section 4.7.

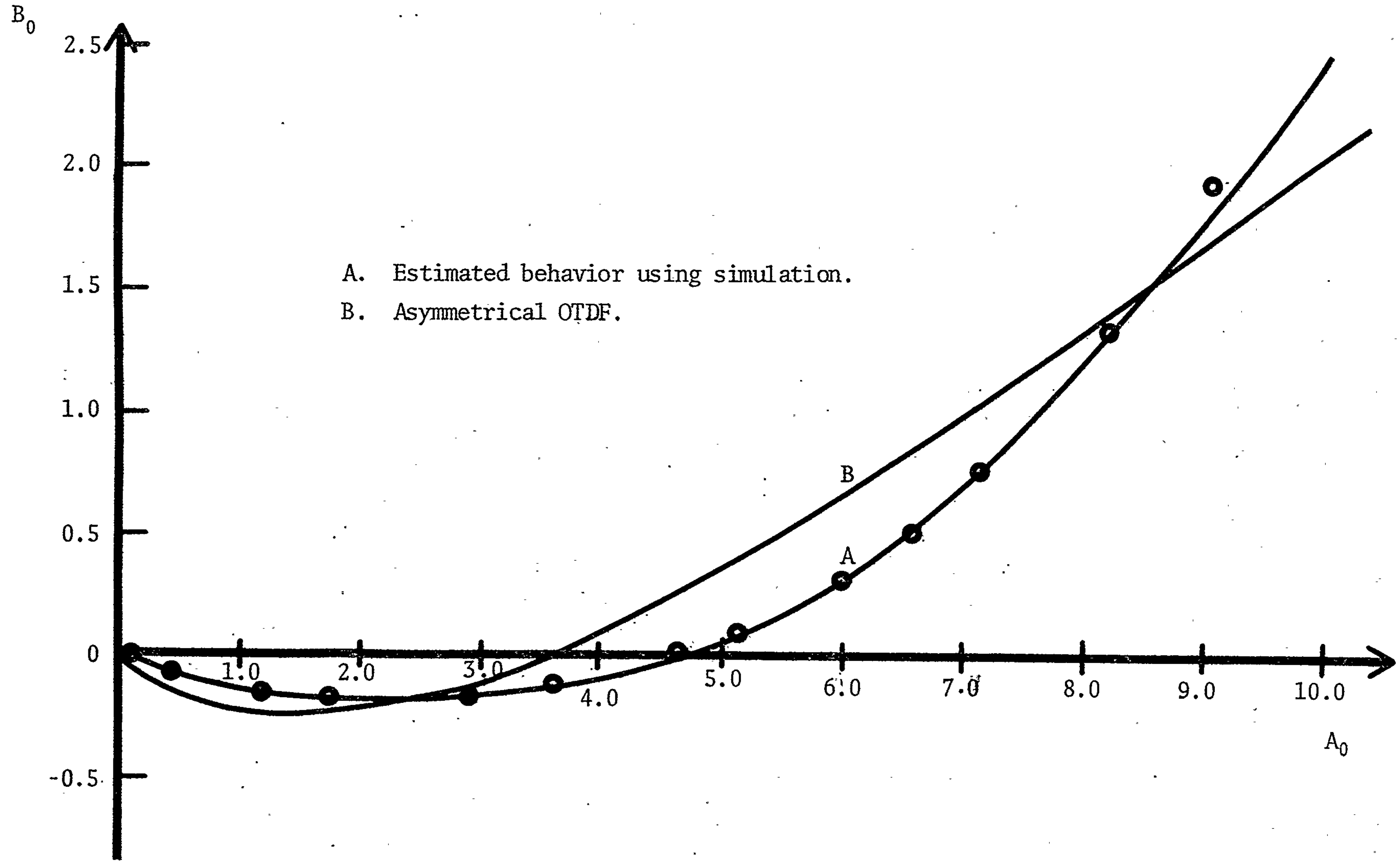

Fig. 4.5 Asymmetrical oscillatory transient bias characteristics of the system given in Section 4.7.

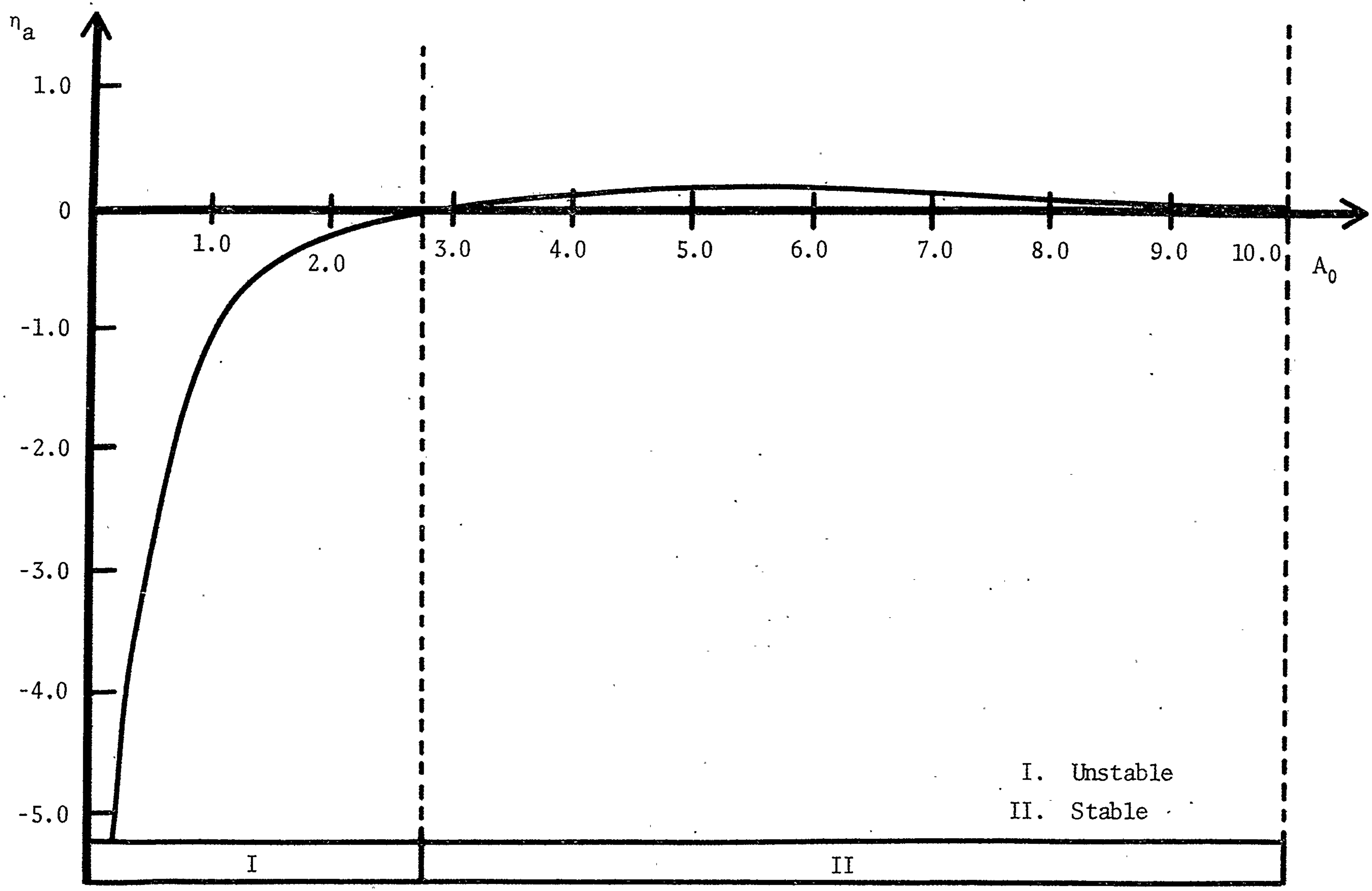

Fig. 4.6 Stability regions for the system given in Section 4.7.

5. APPLICATIONS

5.1 INTRODUCTION

Interest in the study of nonlinear oscillations arose due to their occurrence in practical systems, such as in automatic regulatory systems and in follow-up systems, [1, 7, 8, 19, 24-30]. Oscillatory transient behavior appears in many practical systems for example in the stabilization of powered gyroscopes with several degrees of freedom [1]. This problem can be studied using the transient form of the Krylov-Bogoliubov asymptotic method as given in Section 2.2. The same analysis can also be applied to many types of electronic oscillators [6].

In this chapter we give two examples for the application of the OTDF and the asymmetrical OTDF.

5.2 POSITIONAL CONTROL SYSTEM

Consider the positional control system shown in Fig. 5.1, [33]. The nonlinear friction in the rotating parts of the servomotor is neglected. The motor is represented by a first-order linear system and the tachometer characteristics are assumed to be linear. The backlash element and the preamplifier reduce to a rectangular hysteresis element, Fig. 3.1, with D=1.0 and δ=0.05. The transfer function of the linear system can be written as

$$G(s) = \frac{4.91}{s(0.688s+1)\left(\frac{s^2}{100} + \sqrt{2}\ \frac{s}{10} + 1\right)} . \qquad (5.2\text{-}1)$$

Using the half period OTDF given in Table 3-1 to represent the nonlinear element, the instantaneous oscillatory transient solutions can be

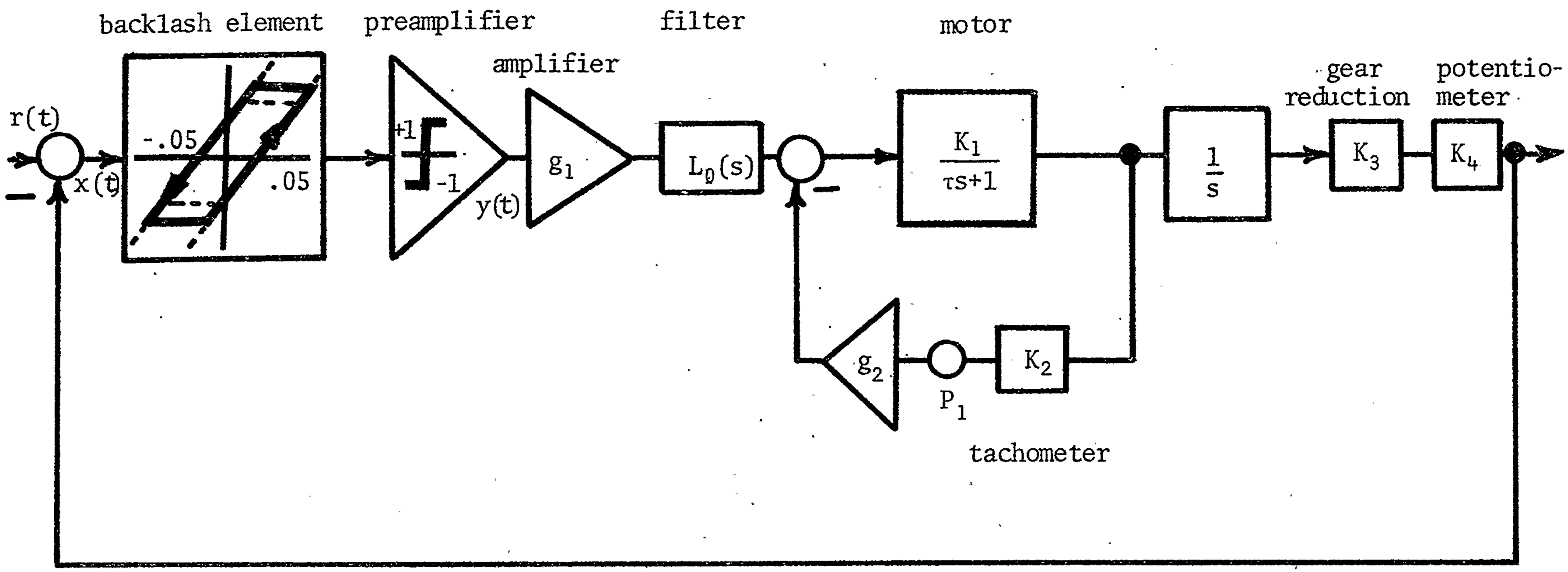

Fig. 5.1 Schematic diagram of the positional control system given in Section 5.2.

K_1	= 12.66 rad./sec. voltage		τ	= 0.85 sec.	
K_2	= 0.027	voltage sec./rad.	P_1	= 0.03	
K_3	= 1:30		g_1	= 2.88	voltage/voltage
K_4	= 5.03	voltage/rad.	g_2	= 23.2	voltage/voltage

$$L_0(s) = \frac{1}{\left(\frac{s}{10}\right)^2 + \sqrt{2}\,\frac{s}{10} + 1}$$

obtained by following the procedure outlined in Section 2.8. The results together with those obtained using measurements on the physical system are shown in Figs. 5.2 and 5.3. The latter results were obtained by introducing step inputs, measuring the corresponding $x(t)$ and from it computing A_0, α and ω. The stability regions, computed using the procedure described in Section 2.8, are shown in Fig. 5.4. The system is seen to have a unique limit cycle at $A_0 = 1.0$. This was also confirmed by the practical system. From Figs. 5.2 and 5.3 it is seen that the accuracy of the prediction method is very satisfactory due to surplus filtering in the system.

5.3 SYSTEM WITH NONSYMMETRICAL NONLINEARITY

Consider the positional control system shown in Section 5.2. Let us investigate the case when the nonlinearity is a biased rectangular relay. Let us denote the input and the output of the nonlinearity by x and y respectively. We get

$$y = mD \quad , \qquad x \geq b \tag{5.3-1}$$

and

$$y = -D \quad , \qquad x < b \tag{5.3-2}$$

where m, D and b are constants.

Let the input to the nonlinearity be an asymmetrical oscillatory transient signal, (4.4-1). Define ψ_1, ψ_2 and ψ_3 such that

$$e^{\alpha\psi_1}(A_0 \sin\psi_1 + B_0) = b \; , \quad 0 < \psi_1 < \theta_{m_1} \tag{5.3-3}$$

$$e^{\alpha\psi_2}(A_0 \sin\psi_2 + B_0) = b \; , \quad \theta_{m_1} \leq \psi_2 \leq \theta_{m_2} \tag{5.3-4}$$

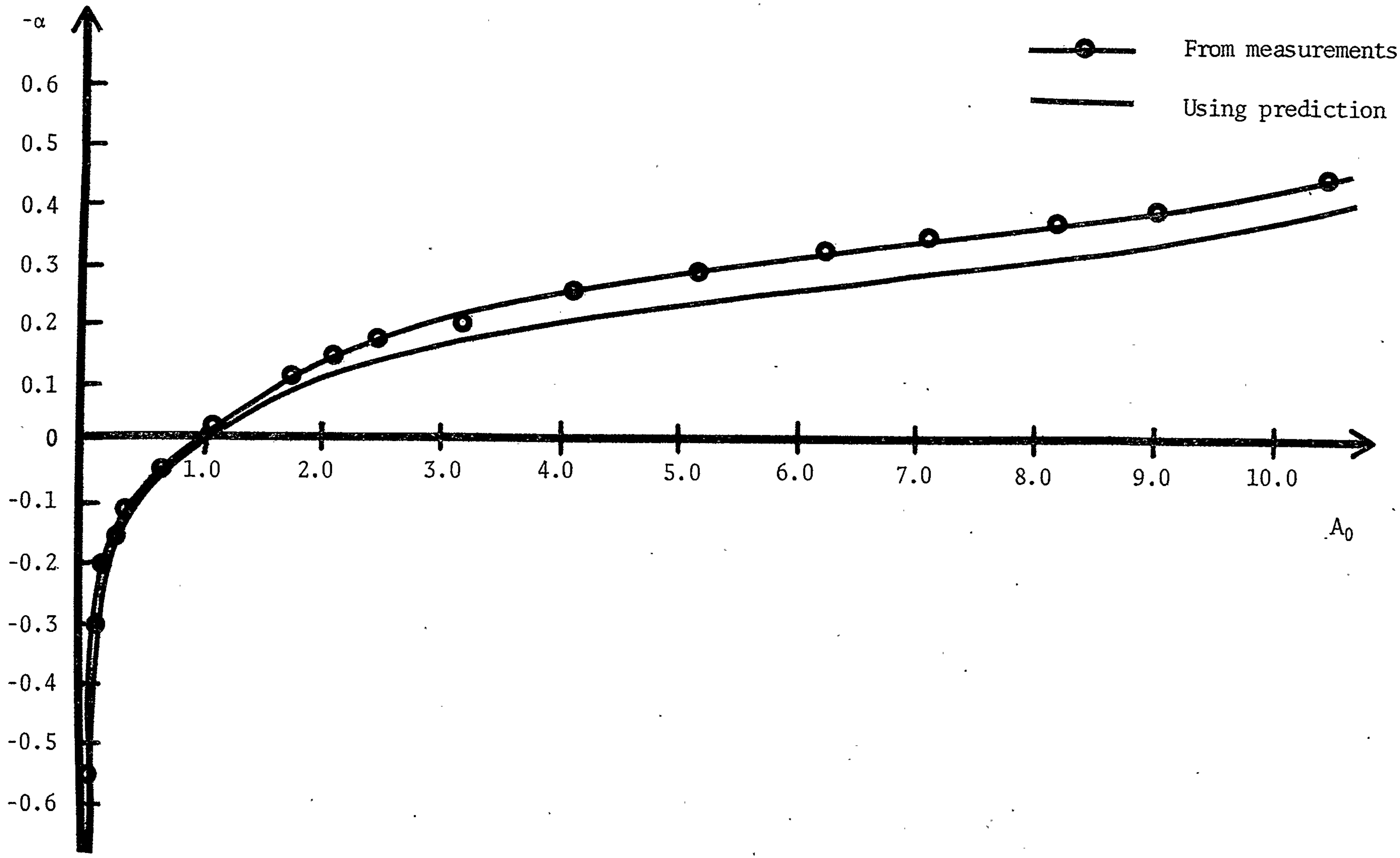

Fig. 5.2 Predicted and measured relative damping characteristics for the system given in Section 5.2.

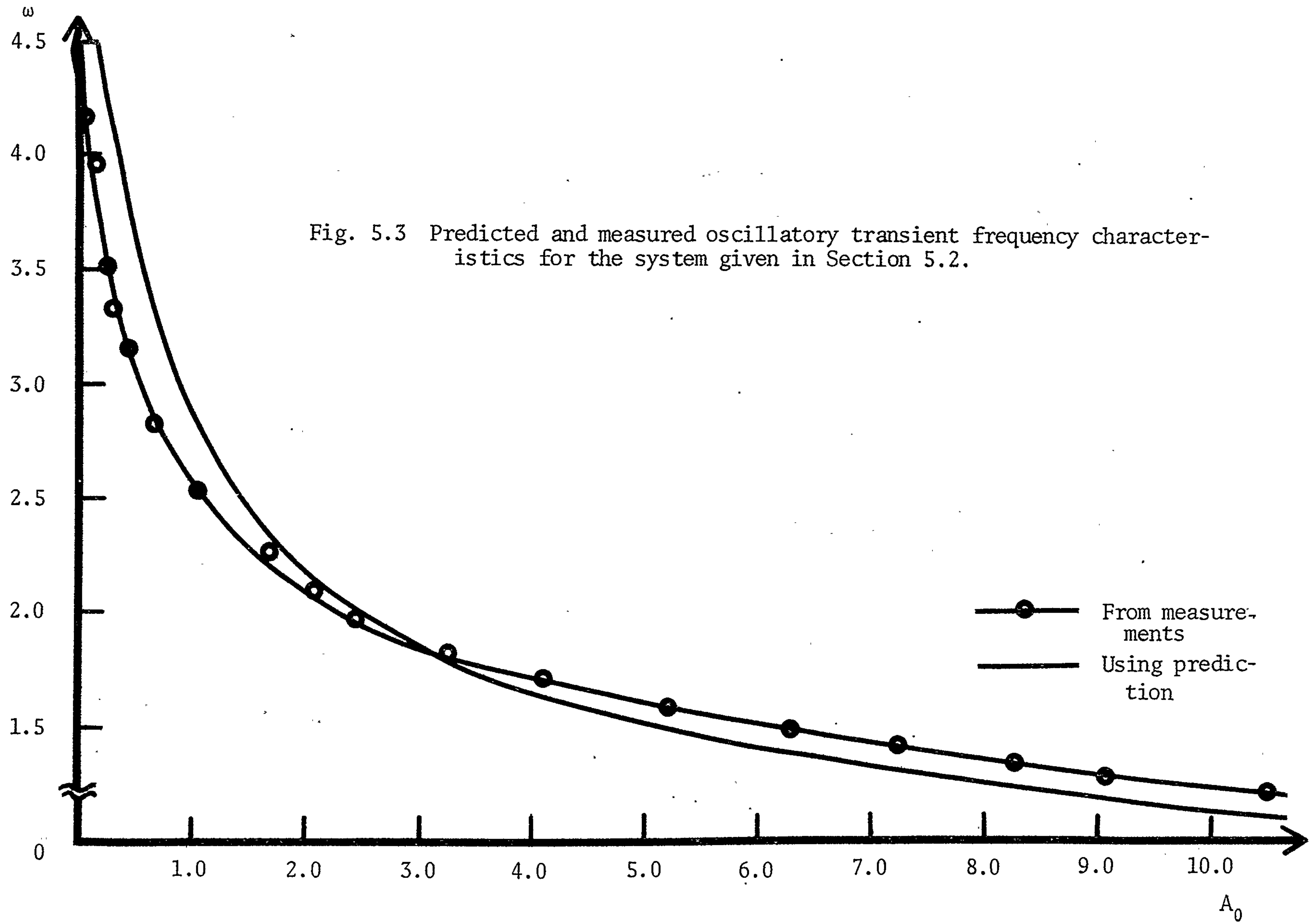

Fig. 5.3 Predicted and measured oscillatory transient frequency characteristics for the system given in Section 5.2.

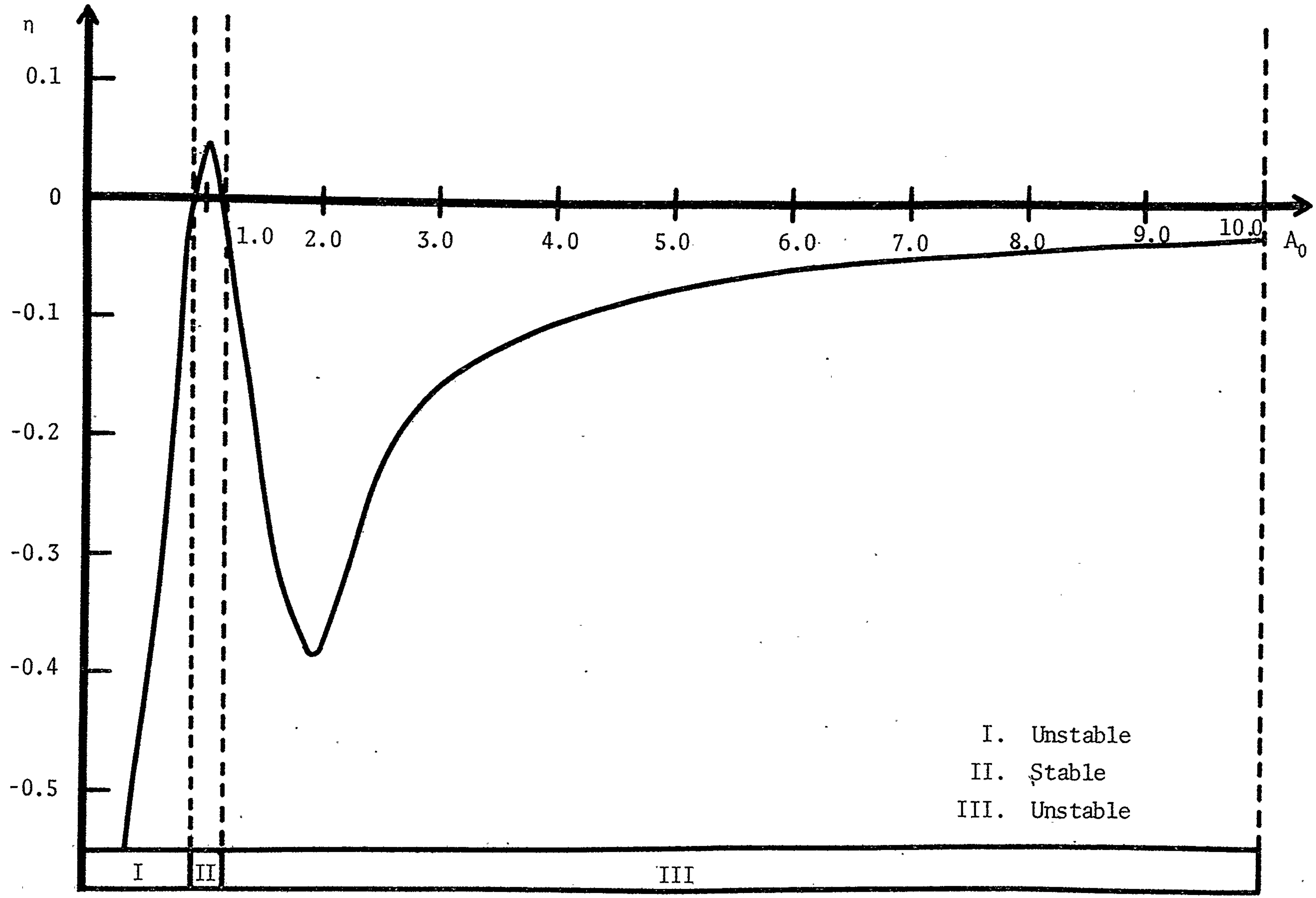

Fig. 5.4 Stability regions for the system given in Section 5.2.

and

$$e^{\alpha\psi_3}(A_0 \sin\psi_3 + B_0) = b \ , \quad \theta_{m_2} < \psi_3 < 2\pi \tag{5.3-5}$$

where θ_{m_1} and θ_{m_2} are given in Fig. 4.2.

The direct component of the asymmetrical oscillatory transient describing function can be expressed as, (4.3-9),

$$N_{Tp}(A_0, B_0, \alpha, \omega) = \frac{1}{\pi A_0}\left[\int_0^{\psi_1} e^{-\alpha\psi}(-D)\sin\psi d\psi + \int_{\psi_1}^{\psi_2} e^{-\alpha\psi} mD\sin\psi d\psi + \int_{\psi_2}^{\psi_3} e^{-\alpha\psi}(-D)\sin\psi d\psi + \int_{\psi_3}^{2\pi} e^{-\alpha\psi} mD\sin\psi d\psi\right] \ . \tag{5.3-6}$$

This reduces to

$$N_{Tp}(A_0, B_0, \alpha) = \frac{D}{\pi A_0(1+\alpha^2)}\{(m+1)\left[e^{-\alpha\psi_1}(\alpha\sin\psi_1 + \cos\psi_1) - e^{-\alpha\psi_2}(\alpha\sin\psi_2 + \cos\psi_2) + e^{-\alpha\psi_3}(\alpha\sin\psi_3 + \cos\psi_3)\right] - (me^{-2\pi\alpha} + 1)\} \ . \tag{5.3-7}$$

Also, for the quadratic component, we have, (4.3-10),

$$N_{Tq}(A_0, B_0, \alpha, \omega) = \frac{1}{\pi A_0}\left[\int_0^{\psi_1} e^{-\alpha\psi}(-D)\cos\psi d\psi + \right.$$

$$\int_{\psi_1}^{\psi_2} e^{-\alpha\psi}\, mD\, \cos\psi d\psi + \int_{\psi_2}^{\psi_3} e^{-\alpha\psi}(-D)\, \cos\psi d\psi \; +$$

$$\left.\int_{\psi_3}^{2\pi} e^{-\alpha\psi}\, mD\, \cos\psi d\psi \quad \right] \qquad (5.3\text{-}8)$$

which reduces to

$$N_{Tq}(A_0,\, B_0,\, \alpha) = \frac{D}{\pi A_0(1+\alpha^2)} \{(m+1) \; [\, e^{-\alpha\psi_1}(\alpha \cos\psi_1 -$$

$$\sin\psi_1) - e^{-\alpha\psi_2}(\alpha \cos\psi_2 - \sin\psi_2) + e^{-\alpha\psi_3}(\alpha \cos\psi_3 -$$

$$\sin\psi_3)] - \alpha(me^{-2\pi\alpha} + 1)\} \; . \qquad (5.3\text{-}9)$$

For the auxiliary direct gain of the nonlinearity, we have, (4.3-12),

$$N_0(A_0,\, B_0,\, \alpha,\, \omega) = \frac{1}{2\pi B_0}\left[\int_0^{\psi_1} e^{-\alpha\psi}(-D)\; d\psi + \int_{\psi_1}^{\psi_2} e^{-\alpha\psi} mD\; d\psi\right.$$

$$\left. + \int_{\psi_2}^{\psi_3} e^{-\alpha\psi}(-D)\; d\psi + \int_{\psi_3}^{2\pi} e^{-\alpha\psi}\, mD\; d\psi \quad \right] \qquad (5.3\text{-}10)$$

or after simplification, we get

$$N_0(A_0,\, B_0,\, \alpha) = \frac{D}{2\pi B_0 \alpha} \; [\; (m+1)\,(e^{-\alpha\psi_1} - e^{-\alpha\psi_2} + e^{-\alpha\psi_3})$$

$$- \; (me^{-2\pi\alpha} + 1)] \; . \qquad (5.3\text{-}11)$$

We consider the same linear plant described by (5.2-1), and select m=1.5, D=1 and b=0.5. By following the same steps as in Section 4.7, the asymmetrical instantaneous oscillatory transient solutions can be determined. The results together with the measured quantities from the practical system are shown in Figs. (5.5), (5.6), and (5.7). The stability regions can be computed and are plotted in Fig. (5.8). The system has a unique asymmetrical limit cycle at A_0 = 1.10 and B_0 = 0.173. The practical system indicated a unique asymmetrical limit cycle at A_0 = 1.163 and B_0 = 0.220.

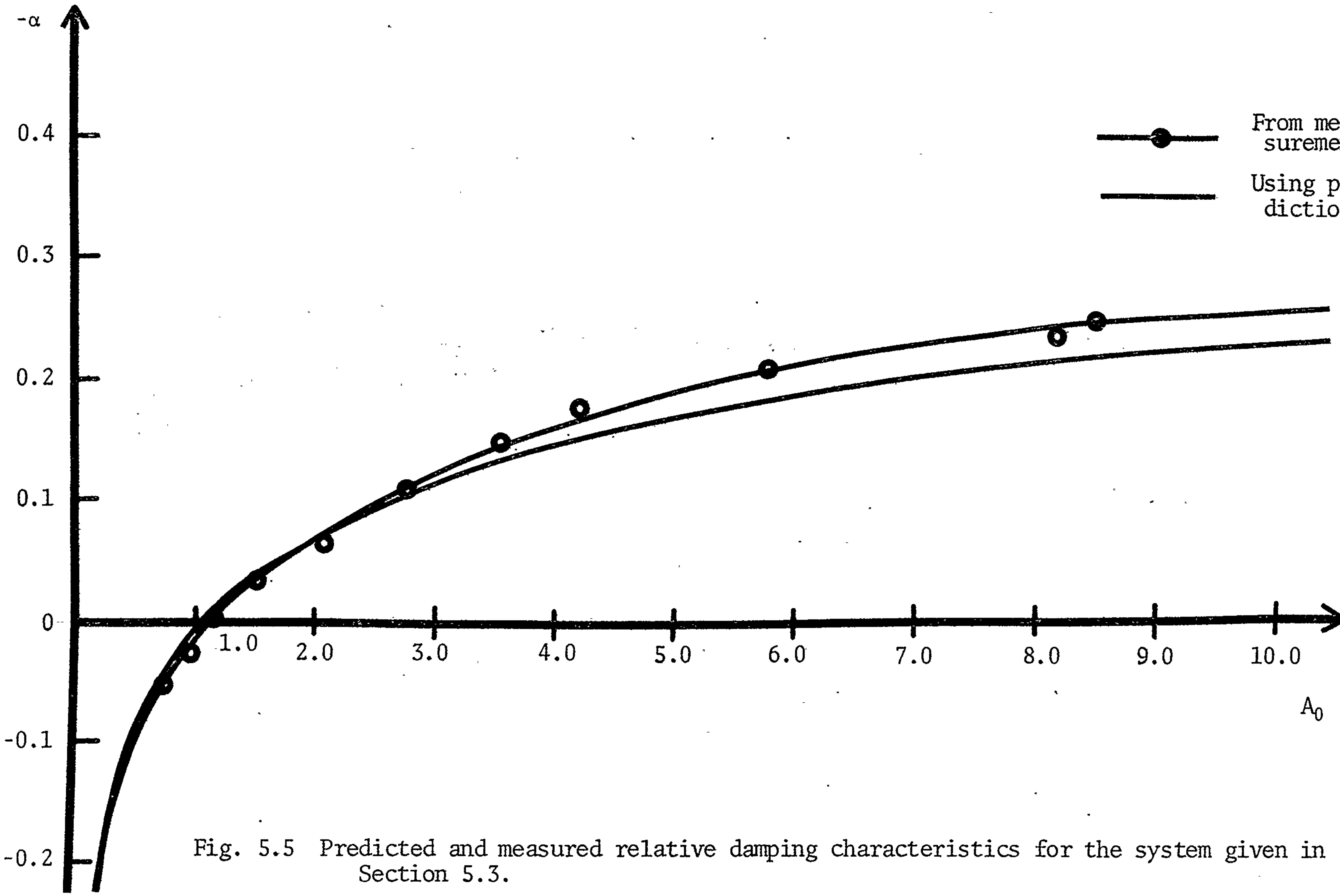

Fig. 5.5 Predicted and measured relative damping characteristics for the system given in Section 5.3.

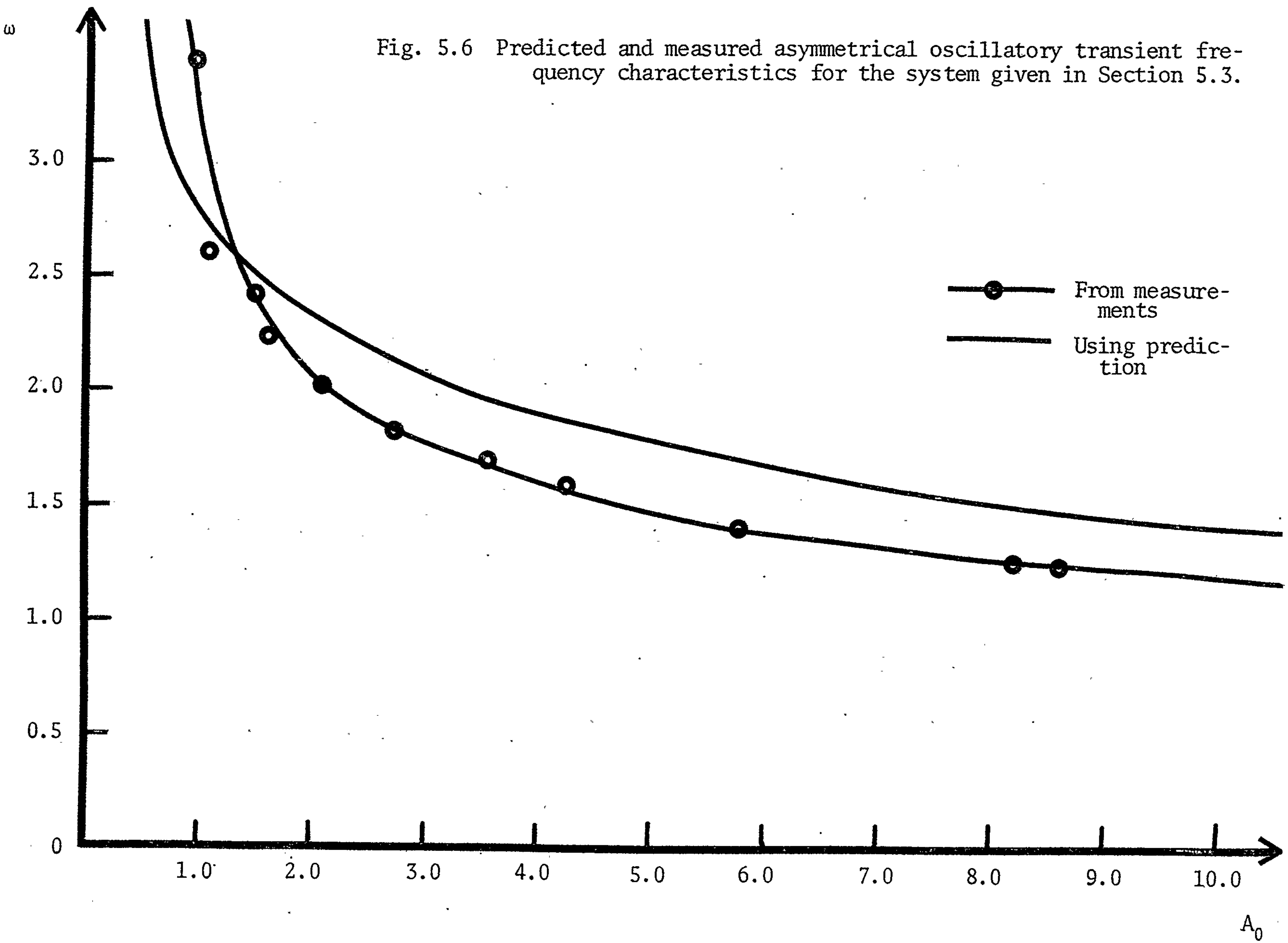

Fig. 5.6 Predicted and measured asymmetrical oscillatory transient frequency characteristics for the system given in Section 5.3.

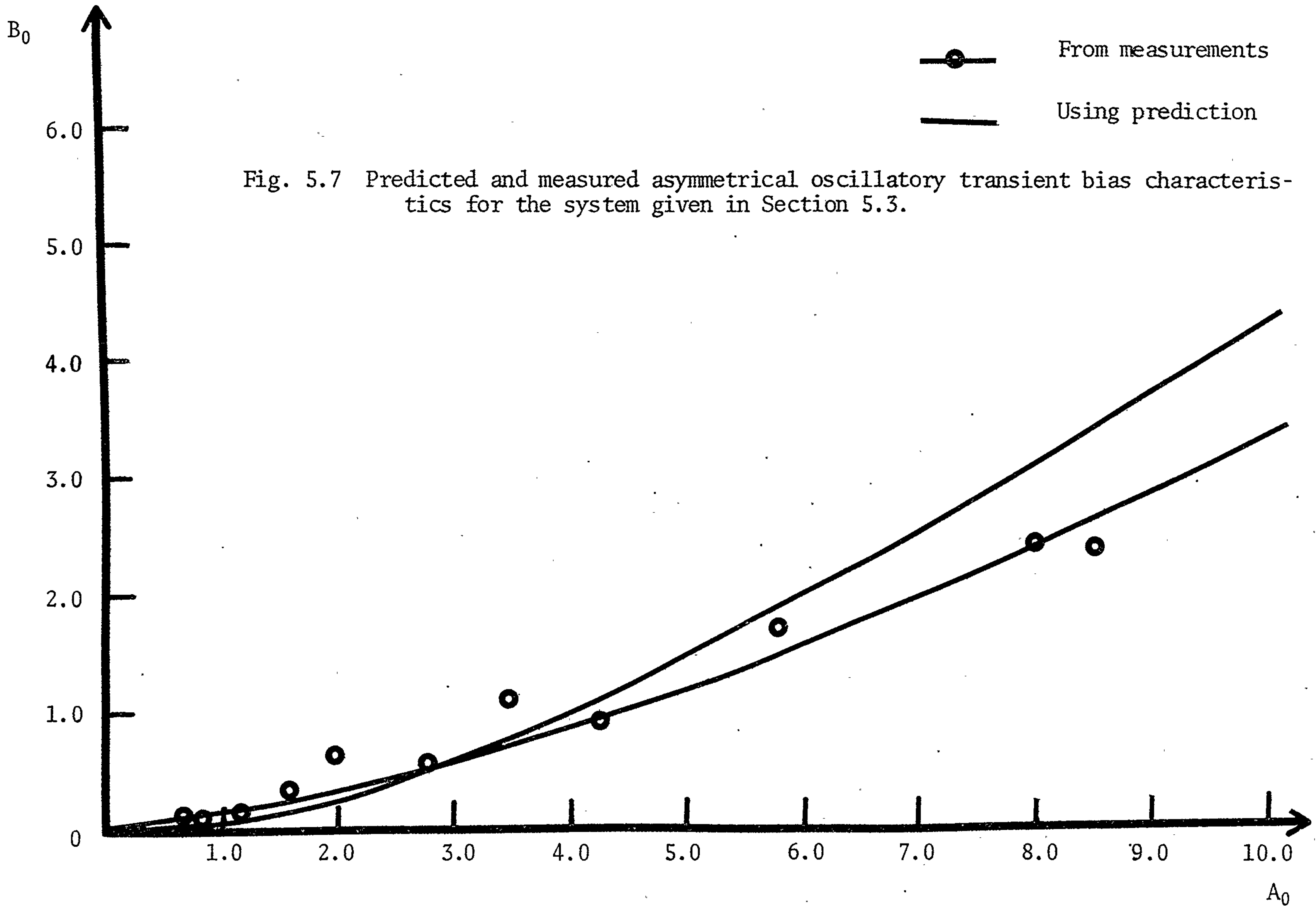

Fig. 5.7 Predicted and measured asymmetrical oscillatory transient bias characteristics for the system given in Section 5.3.

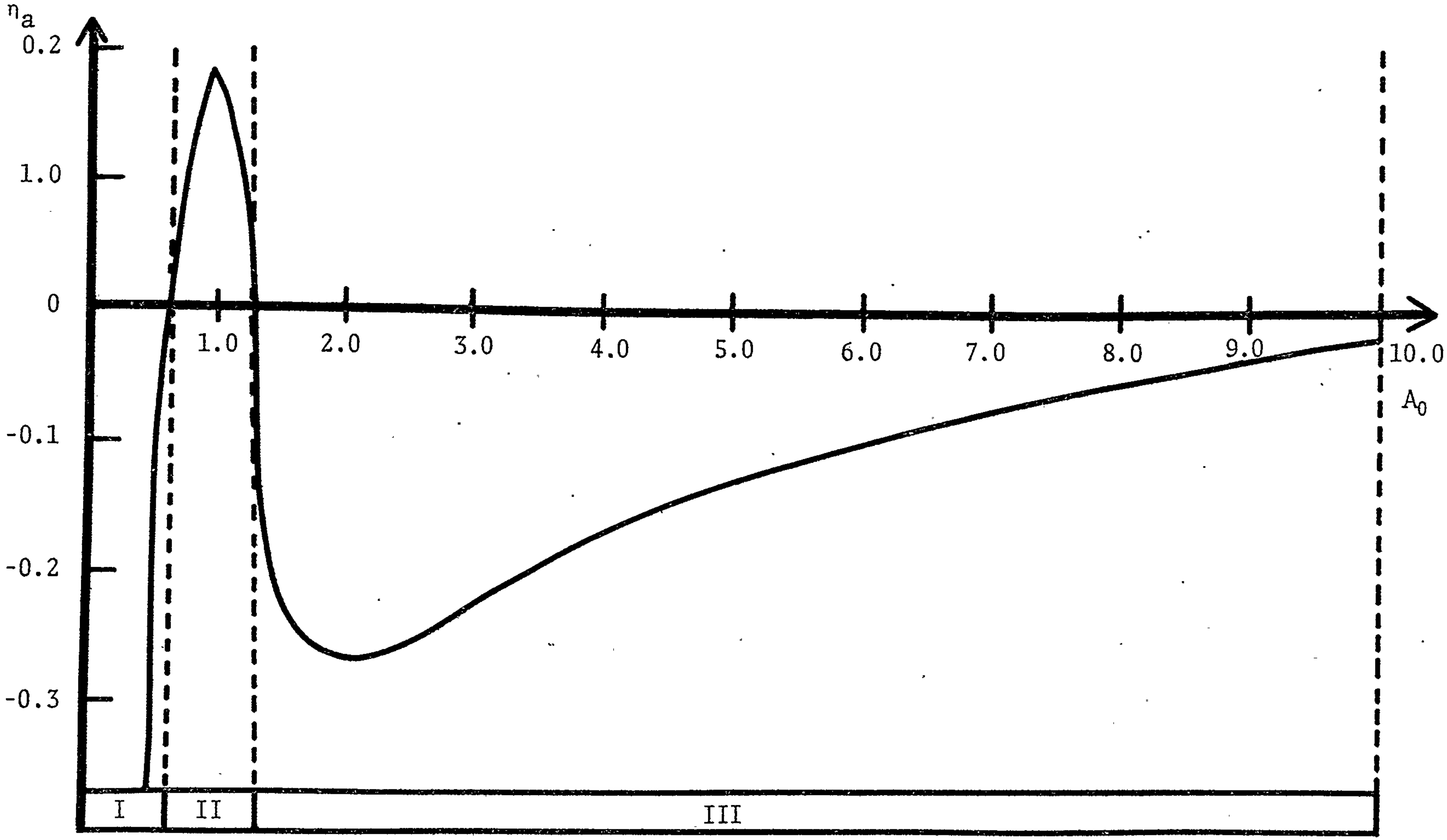

Fig. 5.8 Stability regions for the system given in Section 5.3.
I. Unstable II. Stable III. Unstable

6. CONCLUSION

A proposed theory for the investigation of the dynamic behavior of nonlinear control systems has been presented. It represents the generalization of the describing function method in the transient case.

For the investigation of symmetrical transient oscillations, a transient form of the Krylov-Bogoliubov asymptotic method has been introduced. In addition, a new complex function based on a finite period of time has been defined. The new definition has been utilized to derive a new oscillatory transient describing function. It has been proven that the OTDF is optimal in the sense that it minimizes the mean square error in the approximation. In estimating the OTDF, half and complete period intervals have been considered. It has been shown that the half period OTDF results in high accuracy with systems employing symmetrical nonlinearities. Moreover, by comparing the new OTDF with the other approximate gains reported in the literature it has been found that the OTDF yields the best results especially when the transient oscillations are heavily damped. Furthermore, using the approach suggested, which is not a step by step approach, it is possible to determine the dynamic stability of nonlinear systems during the oscillatory transient behavior. The proposed theory has been extended to a wide class of sampled data systems.

Asymmetrical transient oscillations in nonlinear systems have been investigated. A new asymmetrical transient form of the Krylov-Bogoliubov asymptotic method has been advanced. A finite period complex gain has been utilized to derive the asymmetrical OTDF. This function generalizes the dual describing function to the transient case and minimizes the mean square error. The thesis presented also a new auxiliary equation to be used with the quasi-linearized characteristic equation to describe the asymmetrical oscillatory transient behavior of the system. For both continuous and sampled data systems, it has been possible to predict the asymmetrical behavior and to determine the system dynamic stability.

It has been demonstrated that the accuracy of prediction is improved by using surplus filtering in the system. A position control system was used to show the applicability of the technique proposed. Finally, the thesis has succeeded in predicting accurately the oscillatory transient behaviors associated with nonlinear control systems and determining their dynamic stability without using any step by step approach. This it is felt represents the main contribution of the thesis.

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