THE UNIVERSITY OF CALGARY

CLASSICAL CHARGED PARTICLE RADIATION

IN

ORTHOGONAL MAGNETIC AND ELECTRIC FIELDS

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

DEPARTMENT OF PHYSICS

CALGARY, ALBERTA

JUNE 1974

THE UNIVERSITY OF CALGARY

FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Classical Charged Particle Radiation in Orthogonal Magnetic and Electric Fields" submitted by David W. Hobill in partial fulfillment of the requirements for the degree of Master of Science.

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ABSTRACT

A review of the problems and solutions that arise from the Lorentz-Dirac equation is presented. The trajectories for both the non-relativistic and relativistic case are found for a particle influenced by external, orthogonal electric and magnetic fields constant in space and time. Energy losses for these particles are also calculated. Finally the limits to the applicability of the classical theory presented are discussed.

ACKNOWLEDGEMENT

The author would like to express his gratitude to:

Dr. C.J. Bland for his encouragement and assistance given during the work on this thesis.

Dr. S.R. Sreenivasan for his helpful discussions that proved useful for this work.

Mr. D. Kendall for his kind assistance with the PDP-8 computer.

Carol and Val von der Lin for the generous use of the typewriter upon which this thesis was typed.

This work was supported by the University of Calgary through graduate teaching and research assistantships during the fall and winter sessions and through intersession bursaries during the summer months. I would once again like to thank Dr. Bland for the final support needed to finish this thesis.

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CHAPTER I

INTRODUCTION

A) Toward an equation of motion for a radiating particle.

A fundamental property of any charged particle undergoing acceleration in an applied external field is that it will radiate electromagnetic energy. This radiation of energy will, in turn, affect the motion of the particle. The interaction between the radiation reaction of the particle motion with the electrodynamics of the particle may be represented by the following diagram.

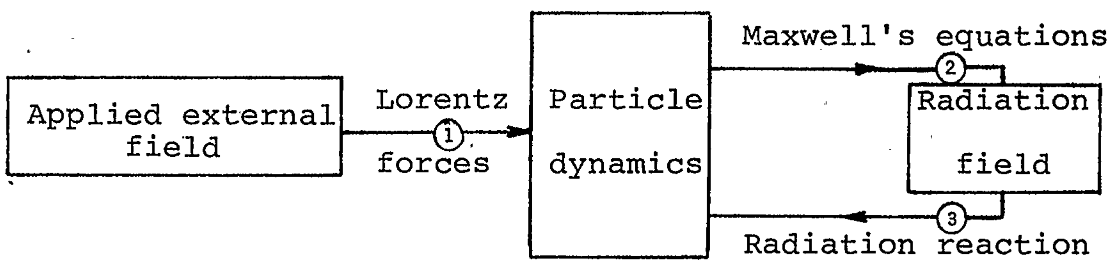

Figure 1-1. Schematic diagram representing the dynamics of a radiating charged particle.

In Fig. 1-1, process 1 represents the effect of the external field on the trajectory of the charged particle; process 2 specifies the field produced by the accelerated charge and process 3 indicates the resultant effect of radiation damping by the radiation field. The trajectory of a classical charged particle (effects due to quantum mechanics or possible finite size of the particle are not considered) must be described by Maxwell's equations (which describe the field resulting from accelerated motion) and by the equations of motion (which include the effects of both the external and self forces). In the case where the radiation reaction vanishes, the equations of motion should reduce to the Lorentz force equation. Dirac (1938) derived a relativistically covariant differential equation describing the trajectory of a relativistic classical charged particle emitting radiation while undergoing an acceleration:

$$\frac{du_\mu}{d\tau} = \frac{e}{mc} F_{\mu\nu} u^\nu + \frac{2e^2}{3mc^3}\left(\frac{d^2u_\mu}{d\tau^2} - \frac{1}{c^2} u_\mu \frac{du_\nu}{d\tau}\frac{du^\nu}{d\tau} \right) \qquad (1\text{-}1)$$

where e and m respectively represent the charge and mass of the particle, and $F_{\mu\nu}$ is the electromagnetic field tensor described by;

$$F_{\mu\nu} = \begin{bmatrix} 0 & H_z & -H_y & E_x \\ -H_z & 0 & H_x & E_y \\ H_y & -H_x & 0 & E_z \\ -E_x & -E_y & -E_z & 0 \end{bmatrix} \qquad (1\text{-}2)$$

and c is the speed of light.

The following notation is used in this work: the four velocity, u_μ is the derivative of the position with respect to proper time, τ ,

$$u_\mu = \frac{dx_\mu}{d\tau} \qquad (1\text{-}3)$$

as the velocity, v_i, is the derivative of the position with respect to ordinary time, t,

$$v_i = \frac{dx_i}{dt} \qquad (1\text{-}4)$$

Proper time and ordinary time are connected by the following relation:

$$\frac{d\tau}{dt} = \left(1- \frac{v^2}{c^2}\right)^{\frac{1}{2}} = \left(1+ \frac{u^2}{c^2}\right)^{\frac{1}{2}} \tag{1-5}$$

where u^2 and v^2 represent the sums of the squares of the spatial components of the corresponding velocities. The coordinate system is chosen such that the interval, ds, is defined by:

$$ds^2 = c^2dt^2-dx^2-dy^2-dz^2 \tag{1-6}$$

(i.e. $x^1=-x_1=x$, $x^2=-x_2=y$, $x^3=-x_3=z$, and $x^4=x_4=ct$). Derived from (1-6) the following relations (to be used later) are obtained:

$$\begin{aligned} u^\nu u_\nu &= c^2 \\ u^\nu \dot{u}_\nu &= 0 \\ \dot{u}^\nu \dot{u}_\nu &= -u^\nu \ddot{u}_\nu \end{aligned} \tag{1-7}$$

In the relativistic (four vector) equations, the dots over the variables indicate derivatives with respect to proper time; in non-relativistic (simple vector) equations they represent derivatives with respect to ordinary time. The Einstein summation convention is used for any repeated index. Quantities with Greek indicies are four vectors (include time components) where quantities with Latin indicies represent only the spatial components of the four vectors. All relativistic quantities have the same dimensions as their non-relativistic counterparts, i.e. u_i and τ have the dimensions of velocity and time respectively.

In general, equation (1-1) is usually written in component form. The fourth (time) component equation, though, is not an independent equation since it may be derived from the spatial component equations and relationships (1-7).

When $v/c \ll 1$, the non-relativistic equations of motion obtained from (1-1) and (1-2) are found to be:

$$\frac{dv_i}{dt} = \frac{e}{m}E_i + \frac{e}{mc}(v_jH_k - v_kH_j) + \frac{2e^2}{3mc^3}\frac{d^2v_i}{dt^2} \qquad (1\text{-}8)$$

This is the familiar Abraham-Lorentz equation which may also be derived from a plausibility argument based upon the principle of energy conservation for a non-relativistic charged particle. (See Appendix A).

Equations (1-1) and (1-8) are the exact equations of motion for an accelerated, radiating point charge within the framework of classical physics. Equation (1-1) known as the Dirac-Lorentz equation, is one of the most controversial equations in the history of physics. A number of different approaches [e.g. Dirac (1938), Wheeler and Feynman (1945) and Rohrlich (1965)] to the problem of radiating charges have resulted in the same equation, yet, the terms that represent the radiative reaction effects continue to present many physical difficulties.

B) Runaway solutions and pre-acceleration.

The obvious appearance of the third time derivative of the position in the Dirac-Lorentz equation sets the equation apart from all other classical dynamical equations which determine completely the trajectory of a particle (given the initial position and velocity).* In order to obtain the exact trajectory of a radiating charged particle a third known condition must be introduced.

Specification of say an initial acceleration is not the only problem resulting from the third time derivative term. So called "runaway" or "self-accelerated" solutions result from (1-1). Writing the non-relativistic equation in the familiar Abraham-Lorentz form (we use this equation since we are concerned only with the term $\ddot{u}_\nu$),

$$\vec{F}_{ext} = m\dot{\vec{v}} - m\tau_0\ddot{\vec{v}} \tag{1-9}$$

(where $\tau_0 = \frac{2}{3}\frac{e^2}{mc^3}$ and F_{ext} represents the total force on the particle resulting from the external electromagnetic fields) and set the external force equal to zero, it becomes obvious that the two possible solutions are:

$$\dot{\vec{v}}(t) = \begin{cases} 0 \\ \vec{a}_0 e^{t/\tau_0} \end{cases} \tag{1-10}$$

where $\vec{a}_0$ is the acceleration at time t=0.

Only the trivial solution is reasonable physically,

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* A new equation of motion in which the radiative reaction terms are dependent upon the applied external field and the particle acceleration has been suggested by Mo and Pappas (1970). While the equation is of second order and the solutions for certain cases are indistinguishable experimentally from the solutions to the Dirac-Lorentz equation, it has yet to be shown that the radiative reaction term is expressed by $[(2e^3)/(3m)]F_{\mu\nu}\dot{u}^\nu$ for all electromagnetic forces.

since it allows the velocity to be constant (this solution being a special case of the exponential solution when $\vec{a}_o=0$). The second solution, in physical terms, is clearly absurd since both the velocity and acceleration grow exponentially without limit as t increases. This non physical solution states that even without external forces present, the particle must gain energy only from itself.

Runaway solutions may be overcome by introducing asymptotic conditions. One such condition is that as the charge of the particle tends toward zero, the radiative effects become negligible, since the self fields tend toward zero. The second condition and the onemost often used is that in the limit of increasing time the acceleration must tend toward zero, or

$$\lim_{\tau\to\infty} \dot{u}_\mu(\tau) = 0 \tag{1-11}$$

It must be remembered that these conditions are not arbitrarily imposed. They are statements of physical reality and therefore are an essential part of the description of charged particle motion.

Defining the total force, $K_\mu(\tau)$ as

$$K_\mu(\tau) = \frac{e}{c}F_{\mu\nu}u^\nu - \frac{2}{3}\frac{e^2}{c^5}\dot{u}^\nu\dot{u}_\nu u_\mu \tag{1-12}$$

where the second term on the right hand side represents the rate at which the electromagnetic four momentum is emitted, equation (1-1) may be written as:

$$m(\dot{u}_\mu - \tau_o\ddot{u}_\mu) = K_\mu(\tau) \tag{1-13}$$

Multiplying (1-13) by the integrating factor $e^{-\tau/\tau_o}$ the equation of motion may be rewritten as:

$$\frac{d}{d\tau}\left(e^{-\tau/\tau_0}\dot{u}_\mu(\tau)\right) = \frac{1}{m\tau_0}\, e^{-\tau/\tau_0}\, K_\mu(\tau) \qquad (1\text{-}14)$$

Integrating (1-14) between the limits τ and ∞, the general solution for the force on the particle becomes

$$m\dot{u}_\mu(\tau) = \frac{e^{\tau/\tau_0}}{\tau_0}\int_\tau^\infty e^{-\tau'/\tau_0}\, K_\mu(\tau')\, d\tau' \qquad (1\text{-}15)$$

The asymptotic condition (1-11) has been used to imply the weaker condition

$$\lim_{\tau\to\infty} e^{-\tau/\tau_0}\, \dot{u}_\mu(\tau) = 0$$

The integro-differential equation of motion (1-15) differs from other equations of motion of classical mechanics in that the acceleration at time, τ, depends upon the weighted average of the force over all future time, rather than on the instantaneous value of the acting force. The presence of the factor, $\exp -(\tau'-\tau)/\tau_0$ indicates that time intervals of the order of τ_0 are involved. In order to more clearly view the behavior of the forces involved in the acceleration process, a new variable of integration will be introduced:

$$\sigma = \frac{1}{\tau_0}\,(\tau'-\tau).$$

Equation (1-15) may then be written as:

$$m\dot{u}_\mu(\tau) = \int_0^\infty K_\mu(\tau+\sigma\tau_0)\, e^{-\sigma} d\sigma \qquad (1\text{-}16)$$

Equation (1-16) can be regarded as a physically reasonable equivalent to the Dirac-Lorentz equation, (1-1). All solutions of (1-16) satisfy (1-1), but, runaway solutions do not occur. A new difficulty, however, is introduced by (1-16) and this is the violation of the traditional concept of causality. It is evident that the acceleration at time τ, depends upon the force acting at all times rather than at τ only. Also if the force K_μ is zero at some time τ_1, the particle still experiences an acceleration at times less than τ_1. Therefore the equation of motion (1-16) predicts a "pre-acceleration" of the particle before the time of the application of the force.

The time interval over which this pre-acceleration occurs for an electron is of the order of $\tau_0=6.27\times10^{-24}$ sec., which is the time that it takes light to travel two-thirds of the classical "radius" of the electron. (For other particles τ_0 would be smaller since the mass appears in the denominator). Such a short time interval is definitely beyond the limits of measurement, and, while microscopic causality is violated by the solutions to the integrodifferential equation, macroscopic causality is still satisfied since it is impossible to apply an external force within a time interval as short as τ_0.

The subject of pre-acceleration has been discussed in detail by Wheeler and Feynman (1945) who find that over time intervals of the order of 10^{-24} sec. it is not possible to distinguish between the advanced and retarded interactions between particles in the universe. However, over longer time intervals the usual relations of physics (which contain only retarded reactions) are valid.

All of the shortcomings of the equation of motion for radiating classical charged particles have been studied extensively. Excellent reviews of the problems occuring in the radiative reaction equations may be found in Rohrlich (1965), Erber (1971) and Hughes (1971). It is fair to say that in the realm of classical electrodynamics the Dirac-Lorentz equation is "probably" the exact equation of motion for a point charge. Usage of the term "probably" is applied because the microscopic results of this equation have yet to be tested experimentally.

In the remainder of this thesis, further theoretical arguments in favor or against (1-1) will not be presented, but rather it will be treated (perhaps naively) as the basic equation describing the motion of a charged particle undergoing radiation reaction. From this assumption we shall proceed to derive observable results that may possibly be found experimentally. An analysis of both the non-relativistic and relativistic particle motions in orthogonal, uniform, static magnetic and electric fields is made. The motivation behind such a choice is two-fold. Firstly, from the experimental point of view, recent developments in particle accelerator technology make possible the generation of magnetic fields of the order of

10^7 gauss in the laboratory allowing electron beams of energies reaching to a few hundred GeV to be soon available [Herlach (1968), Herlach, et. al. (1971)]. Secondly, synchrotron radiation plays an important role in astrophysical applications, and the Dirac-Lorentz equation is the basis for describing this cosmic phenomenon [Sokolov and Ternov (1968)]. There are however limits to the applicability of classical theory and these will be discussed later.

CHAPTER II

SOME SOLUTIONS TO THE DIRAC-LORENTZ EQUATION

A) Existence and uniqueness of solutions

Having seen in Chapter I that the equations of motion for a charged particle give a number of solutions that are meaningless in physical terms, the major problem with the Dirac-Lorentz equation becomes one of isolating the physically valid solutions from the infinite number of non-physical solutions. It is therefore necessary to enquire into the existence and uniqueness of the solutions to the equations before attempting to solve them. More precisely, we are interested in the conditions that allow us to solve the Dirac-Lorentz equation meaningfully and whether our solutions are unique for the physically reasonable initial conditions that are specified.

The proof for the existence of the solutions (for certain weak conditions) was presented by Hale and Stokes (1962) who used extremely complicated mathematical techniques well beyond the exposition of this thesis. Only the essential features of their results will be presented. The starting point of the proof is with the third order differential equation of motion (1-1) together with the asymptotic condition (1-11). Here in lies the difficulty. If the asymptotic condition did not have to be satisfied the existence and uniqueness of the solutions to equation (1-1) would simply follow the standard theorems of ordinary differential equations. Being a third order differential equation, the Dirac-Lorentz equation will have a unique solution over any finite time interval when the <u>three</u> initial values, $x_\mu(0)$, $u_\mu(0)$, and $a_\mu(0)$ are specified and certain analyticity conditions are satisfied. Since there is a need to specify the initial acceleration, there is an indication that equation (1-1) cannot be an equation of motion. In Newtonian mechanics such equations provide knowledge of the acceleration at <u>all</u> times. The asymptotic condition is exactly what is needed to eliminate this difficulty. It has already been shown that the third order differential equation, together with the asymptotic condition is equivalent to the second order equation (1-16). Therefore only those initial accelerations that give solutions that satisfy (1-11) are the only admissable ones among all the possible initial accelerations. This is the much more difficult problem of the existence of solutions to the equations of motion with specific asympotic conditions.

The most important results obtained by Hale and Stokes concerning the existence of solutions to equation (1-1) with the asymptotic condition (1-11) can be stated in the following two theorems. Here the position, x, is defined by the relation; $\|x\| = (x_\mu x^\mu)^{\frac{1}{2}}$.

<u>Theorem 1.</u> There exists a solution $x(\tau)$ $(0 \le \tau < \infty)$ of the equation of motion (1-16) which satisfies the asymptotic condition (1-11) for any initial set $x(0)$ and $u(0)$

provided:

(a) $\|x(0)\| < \infty$ (b) $\|F(x,u,\tau)\| < \Phi(\tau)$ is continuous

(c) $\Phi(\tau) \to 0$ for $\tau \to \infty$ (d) $\int_0^\infty \Phi(\tau)\,d\tau < \infty$

Theorem 2. There exists a solution $x(\tau)$ $(\tau_i \le \tau < \infty)$ of the equation of motion (1-16) which satisfies the asymptotic condition (1-11) for any initial set $x(\tau_i)$ and $u(\tau_i)$ provided:

(a) $\|F(x,u,\tau)\| \le \dfrac{p+q\|u\|}{\|x\|^r}$, where $p \ge 0$, $q \ge 0$, $r > 1$

here $\|u\| = (\gamma^2 + \gamma^2 u_i u^i)^{\frac{1}{2}} = (2\gamma^2 - 1) > 1$

(b) τ_i is such that

$$\inf_{\tau > \tau_i} \left\| \frac{x(\tau_i)}{\tau} + \frac{u(\tau_i)(\tau - \tau_i)}{\tau} \right\| \ge \xi(p + q\|u(\tau_i)\|) + \left[\frac{(1+p\xi)}{1-\tau/\tau_0'}\right]^{1/\eta}$$

where $\xi = 1/(\eta - 1)\ \tau_i^{\eta-1}$

$$\text{and } \tau_0' = \frac{\tau_i^\eta}{(p+q)\|u(\tau_i)\|\,[\|u(\tau_i)\| + \xi(p + q\|u(\tau_i)\|)]} > \tau_0$$

Theorem 1 admits all forces that are integrable and bounded. Theorem 2 admits a larger class of forces but puts restrictions on the initial time, the initial velocity, and the initial position. Solutions satisfying more general conditions presumably exist, but these theorems nevertheless seem general enough to include all cases of physical interest.

The next general question concerns the uniqueness of of solutions. To a physicist it may seem intuitively obvious that given certain initial conditions $x(\tau_i)$ and $u(\tau_i)$

the solutions of any equation of motion are unique due to the principle of causality (causality here implying not only prediction but retrodiction as well). Yet, to remain mathematically consistant it is essential to have a proof of a solution's uniqueness. To date no such proof exists [Grandy (1970)]. The uniqueness problem must be considered one of the most important unsolved problems concerning the Dirac-Lorentz equation.

B) Non-relativistic solutions.

In most problems where a known external force is applied, an exact solution to the Dirac-Lorentz equation is impossible to obtain. For this reason approximation methods for obtaining the solutions must be used. The simplest and easiest approximation that can be made is to deal with a non-relativistic particle. In the lower velocity limit ($v/c \ll 1$) equation (1-1) becomes a simple vector equation (1-8). Since (1-8) does not contain the non-linear terms but only the second time derivative of the velocity, the non-relativistic equation may be solved using well-known techniques for solving ordinary, linear differential equations.

Equation (1-8) is applicable only to the extent that the damping force is small compared with the force exerted on the charge by the external field. The physical meaning of this condition (within the framework of classical electrodynamics) may be clarified as follows: The second time derivative of the velocity in the system of reference in which the charge is at rest at any given moment, (and neglecting the damping force) may be set to

$$\ddot{\vec{v}} = \frac{e}{m}\dot{\vec{E}} + \frac{e}{mc}\dot{\vec{v}}\times\vec{H} \tag{2-1}$$

In the second term substituting $\dot{\vec{v}}=\frac{e}{m}\vec{E}$ (to the same order of accuracy) one obtains:

$$\ddot{\vec{v}} = \frac{e}{m}\dot{\vec{E}} + \frac{e}{m^2c}\vec{E}\times\vec{H} \tag{2-2}$$

For the non-relativistic equation of motion the damping force is simply described by;

$$\vec{f} = \frac{2}{3}\frac{e^2}{mc^3}\frac{d^2\vec{v}}{dt^2} \tag{2-3}$$

Therefore the damping force (2-3), using equation (2-2) may be written (to the first order) as:

$$\vec{f} = \frac{2}{3}\frac{e^3}{mc^3}\dot{\vec{E}} + \frac{2}{3}\frac{e^4}{m^2c^4}\vec{E}\times\vec{H}$$

If one defines a frequency of motion, Ω, then $\dot{\vec{E}}$ is proportional to $\Omega\vec{E}$, and, consequently, the first term becomes of the order $\frac{e^3\Omega E}{mc^3}$ while the second term is of the order $\frac{e^4EH}{m^2c^4}$. Therefore, if the damping force is to be small compared to the external force exerted on the charge (of the order eE) the following condition must hold:

$$\frac{e^2}{mc^3}\Omega \ll 1 \text{ or introducing a wavelength}, \Lambda \sim c/\Omega$$

$$\Lambda \gg \frac{e^2}{mc^2} \tag{2-4}$$

Thus relation (2-4) states that the radiation damping in the non-relativistic case is applicable only when the wavelength of radiation incident on the charge is large compared to the classical radius of the particle. Here the classical limit e^2/mc^2 appears as the value where classical electrodynamics leads to internal contradictions.

Secondly, comparing the external field with the second term in the radiative force, sets the condition for the size of the magnetic field:

$$H \ll \frac{m^2 c^4}{e^3} \tag{2-5}$$

Having set limits on the physical quantities for which the Abraham-Lorentz is valid, the solutions to equation (1-8) may now be found. Plass (1961) has shown that as long as the applied external forces are both finite and continuous, analytic solutions to the non-relativistic equation of motion exist. Using a number of special cases Plass was able to obtain the trajectories of particles subjected to different applied forces. One such case was that for a charged particle moving in a constant magnetic field.

Assuming the magnetic field $\vec{H}$ to be directed along the z-axis in the normal Cartesian coordinate system, the equations of motion (1-8) may be written in component form as:

$$\frac{dv_x}{dt} - \tau_o \frac{d^2 v_x}{dt^2} = \omega v_y$$

$$\frac{dv_y}{dt} - \tau_o \frac{d^2 v_y}{dt^2} = -\omega v_x$$

$$\frac{dv_z}{dt} - \tau_o \frac{d^2 v_z}{dt^2} = 0 \qquad (2\text{-}6)$$

where $\omega = eH/(mc)$ is the cyclotron frequency of the particle. The exact non-divergent solution to these equations may be written in the form:

$$v_x(t) = v_x(0)\ e^{-\alpha_1 t} \cos \alpha_2 t$$

$$v_y(t) = v_x(0)\ e^{-\alpha_1 t} \sin \alpha_2 t$$

$$v_z(t) = v_z(0) \qquad (2\text{-}7)$$

where the phase factor is chosen such that $v_x(t=0) = v_x(0)$ and $v_y(t=0) = 0$. The values of the constants α_1 and α_2 were determined by substituting the solutions back into the original equations:

$$\alpha_1 = \frac{1}{2}\tau_o^{-1}\left\{\left[\tfrac{1}{2}+\tfrac{1}{2}(1+16\tau_o^2\omega^2)^{\frac{1}{2}}\right]^{\frac{1}{2}} - 1\right\}$$

$$\alpha_2 = \frac{1}{2}\tau_o^{-1}\left[-\tfrac{1}{2}+\tfrac{1}{2}(1+16\tau_o^2\omega^2)^{\frac{1}{2}}\right]^{\frac{1}{2}} \qquad (2\text{-}8)$$

The solutions above describe a motion that is an exponentially damped circular motion about the magnetic field.

C) Relativistic solutions.

It is not possible to solve the relativistic equations (other than in one dimension) exactly since they involve the cross products between the different velocity components. For the one-dimensional case Plass (1961) and Rohrlich (1965), using different methods have shown that as long as the external force is known explicitly as a bounded function function of proper time, exact solutions to the Dirac-Lorentz equation may be obtained.

Since the exact solutions to the three dimensional equations are, in most cases not known, the solutions must be obtained in an approximate form. The most obvious and certainly the most often used method is the perturbation series approximation. This method uses the integrodifferential equation (1-16) and begins with the fact that the characteristic time τ_0 is small compared to our proper time scale, τ, and thus a Taylor series expansion about τ is made. If the force is slowly varying in time, the series may be expected to converge rapidly.

$$K_\mu(\tau+\sigma\tau_0) = \sum_{n=0}^{\infty} \frac{(\sigma\ \tau_0)^n}{n!} \frac{d^n}{d\tau^n} K_\mu(\tau) \tag{2-9}$$

Assuming also that the integral in (1-16) is absolutely and uniformly convergent, then the summation and integration processes are interchangable. Substituting (2-9) into (1-16) we obtain:

$$m\dot{u}_\mu(\tau) = \sum_{n=0}^{\infty} \frac{\tau_0{}^n}{n!} \frac{d^n}{d\tau^n} K_\mu(\tau) \int_0^\infty \sigma^n e^{-\sigma}\, d\sigma = \sum_{n=0}^{\infty} \tau_0{}^n \frac{d^n}{d\tau^n} K_\mu(\tau) \tag{2-10}$$

Using the definition (1-12) and defining the total radiation rate, R, as

$$R = \frac{2}{3}\frac{e^2}{c^3}\, \dot{u}^\nu \dot{u}_\nu$$

equation (2-10) may be written (keeping only the first few terms) as:

$$\dot{u}_\mu(\tau) = \frac{e}{mc} F_{\mu\nu}u^\nu - \frac{R(\tau)}{c^2}u_\mu + \tau_0\left[\frac{d}{d\tau}\left(\frac{e}{mc}F_{\mu\nu}u^\nu\right) - \frac{d}{d\tau}\left(\frac{R(\tau)}{c^2}u_\mu\right)\right] +$$

$$+ O\{\tau_0{}^2\} \tag{2-11}$$

Therefore the non-local second order equation (1-16) is equivalent to the local equation (2-10) with an infinite number of derivatives. Equation (2-10) in the zeroth order (i.e. neglecting the radiation terms) reduces to the well-known Lorentz force equation,

$$\dot{u}_\mu = \frac{e}{mc} F_{\mu\nu}u^\nu \tag{2-12}$$

The first order approximation is:

$$\dot{u}_\mu = \frac{e}{mc}F_{\mu\nu}u^\nu - \frac{1}{c^2}Ru_\mu + \tau_0\frac{e}{mc}\left(u^\nu\dot{F}_{\mu\nu} + F_{\mu\nu}\dot{u}^\nu\right) \tag{2-13}$$

This equation includes all terms of the order τ_0 (since R is of the order of τ_0, it must not be neglected). Unless the force varies rapidly in time, this approximation is the most often used due to the small magnitude of τ_0. Succesive terms of higher orders of τ_0 may be added but these are extremely small and add only minute corrections to the effects of radiation damping. In classical physics the expansion of equation (1-16) beyond terms of the order of τ_0 is not very interesting since there seems to be no classical process by which terms beyond the first order may be observed. Furthermore, it is well known that equations (2-12) and (2-13) are excellent approximations for the motion of a charged particle in an electromagnetic field. (Terms of the order $\tau_0{}^2$ have been included in the solutions obtained by Chand (1971) to describe the effects of radiation reaction on an energy spectrum of oscillating electrons as a means of illustrating the perturbation technique in classical physics.)

Gernet (1966) making use of equation (2-13) solved for the motion of an electron in a constant, static magnetic field. Since the field was constant in space and time the term representing the change in the magnetic field was neglected and the solutions were found to be (to the first order of τ_o):

$$u_x(\tau) = u_\perp(0)e^{-\alpha\tau}\sin(\omega\tau+\Theta_o)$$

$$u_y(\tau) = u_\perp(0)e^{-\alpha\tau}\sin(\omega\tau+\Theta_o) \qquad (2\text{-}14)$$

where $\alpha = \frac{2}{3}\frac{e^4H^2}{m^3c^5} = \omega^2\tau_o$. Gernet also made the assumption that $\omega\tau_o \ll 1$ (i.e. the damping term was extremely small).

Similar results have been obtained by Herrara (1973) who obtained the solutions to the sixth order of τ_o using the same approximation of the equation of motion that Gernet used. From the results of these authors it is quite obvious that the motion of a relativistic point charge in a constant, static magnetic field will be represented by an exponentially damped spiral and the damping is directly dependent upon the magnetic field strength squared.

Shen (1972) used the fact that extremely relativistic velocities and with strong radiation damping ($\omega\tau_o \simeq 1$) the dominant radiative term is $\frac{1}{c^2}Ru_\mu$ [See Landau and Lifshitz (1965)]. One simplifying approximation that can be made is that for this case is that $v \simeq c$.

Writing the equations of motion in the rest frame of the particle, (which would then travel under the influence of an "electric field" of the order γH) Shen obtained the following solutions:

$$u_x = u_o e^{-\alpha_1(\tau)} \cos \alpha_2(\tau) \sin \phi_o$$
$$u_y = u_o e^{-\alpha_1(\tau)} \cos \alpha_2(\tau) \sin \phi_o$$
$$u_z = u_o \cos \phi_o \tag{2-15}$$

where ϕ_o is the angle between the initial velocity, u_o and the magnetic field H at $\tau=0$. Also,

$$\alpha_1(\tau) = \frac{\omega^2\tau_o}{\gamma_o^3}\, \tau\left[1 + \frac{\omega^2\tau_o}{2\gamma} + O\{\tau_o{}^2\}\right]$$

$$\alpha_2(\tau) = \frac{\omega\tau_o}{\gamma_o}\left\{1 + \frac{\omega^2\tau_o}{\gamma_o}\tau - \frac{\omega^2\tau_o{}^2}{\gamma_o{}^2}\left(\frac{\omega^2\tau_o}{\gamma_o}\tau - 4\gamma_o \ln \frac{1+\omega^2\tau_o\tau\gamma_o{}^{-1}}{\omega^2\tau_o(1+(\tau_o\omega)^{-1})}\right)\right\}$$

$$\gamma_o = (1 - u_o{}^2/c^2)^{-\frac{1}{2}}$$

In these solutions, Shen maintained terms of the order of $(\omega^2\tau_o^2)$ suggesting that those terms of the order of $\omega\tau_o$ are nearly of the same magnitude as the terms describing the external Lorentz forces. The magnetic fields used were constsant in both space and time.

In 1971 Mitchell, et. al. (1971) solved the first order approximation equation for a particle influenced by constant aligned magnetic and electric fields. The equations of motion were written in the following form:

$$\dot{u}_x = \omega(u_y - \varepsilon u_z u_x - \chi u_x)$$

$$\dot{u}_y = \omega(-u_x - \varepsilon u_z u_y - \chi u_y)$$

$$\dot{u}_z = \omega\chi(1-u_z^2) \tag{2-16}$$

where $\varepsilon = E/H$ is the field strength and the dimensionless quantity χ, is defined as $\chi = \omega(1+\varepsilon^2)\tau_o$.

Since u_z could be solved for independently, and then substituted into the equations for u_x and u_y, the solutions may be obtained as follows:

$$u_x = u_\perp(0)e^{-\omega\chi\tau}\sin(\omega\tau_o + \theta_o)\ \mathrm{sech}(\omega\chi\tau + \phi_o)$$

$$u_y = u_\perp(0)e^{-\omega\chi\tau}\cos(\omega\tau_o + \theta_o)\ \mathrm{sech}(\omega\chi\tau + \phi_o)$$

$$u_z = u_\parallel(0)\ \tanh(\omega\chi\tau + \phi_o) \tag{2-17}$$

where θ_o and ϕ_o are constants to be determined from the initial conditions. Equations (2-17) show that u_x and u_y decay exponentially with proper time while the electric field accelerates the particle along the field line. What one would observe would be a particle travelling in a helical motion with and exponentially damped radius and with a decreasing pitch.

The problem of radiating charges travelling in orthogonal magnetic and electric fields has not yet been studied, and yet there exist a number of physical situations where this field configuration exists. The remainder of this thesis will therefore deal with the solutions to the equation of motion (1-1) in which the magnetic and electric fields are constant in space and time and are perpendicular to each other.

CHAPTER III

THE ABRAHAM-LORENTZ EQUATION FOR A CHARGE IN ORTHOGONAL MAGNETIC AND ELECTRIC FIELDS

This chapter will deal with the solutions to the non-relativistic equations of motion for a charged particle moving in orthogonal magnetic and electric fields that remain constant in space and time. Since the velocities are assumed to be much less than the speed of light, equation (1-8) may be applied to this problem. Observing that (1-8) is linear, the solutions may be found analytically. The spatial component equations obtained from (1-8) will not be linearly independent since they involve the cross-product of the velocity with the magnetic field, nor will the equations be homogeneous since the non-homogeneity is introduced by the addition of an electric field. Therefore (1-8) may be solved as a system of non-homogeneous second-order differential equations for the parameter, t.

Using a Cartesian coordinate system, the direction of the magnetic field will be chosen to be in the positive z-direction with magnitude H, while the electric field will be directed along the x-axis with magnitude E.

The equation of motion (1-8) may then be written in vector form as;

$$\dot{\vec{v}} = \frac{e}{m}\vec{E} + \frac{e}{mc}\vec{v}\times\vec{H} + \frac{2}{3}\frac{e^2}{mc^3}\ddot{\vec{v}}$$

Written in component form (1-8) becomes:

$$\dot{v}_x = \frac{e}{mc} Hv_y + \frac{2}{3}\frac{e^2}{mc^3}\ddot{v}_x + \frac{e}{m} E$$

$$\dot{v}_y = \frac{-e}{mc} Hv_x + \frac{2}{3}\frac{e^2}{mc^3}\ddot{v}_y$$

$$\dot{v}_z = \frac{2}{3}\frac{e^2}{mc^3}\ddot{v}_z \qquad (3\text{-}1)$$

where the dots above the velocity components now represent derivatives taken with respect to ordinary time (in the non-relativistic case proper time and ordinary time are equivalent).

The third equation is seen to de-couple from the equations involving the x and y components of the velocity. Solutions to the z component equation have already been discussed in Chapter I. Only the physically meaningful solution i.e. $v_z(t)$=constant will be accepted. The constant will be chosen from the initial conditions for v_z. In the case at hand if v_o is the magnitude of $\vec{v}(t)$ at t=0 and θ_o is the angle between the velocity and the magnetic field vector at the initial time, then the constant equals $v_o \cos \theta_o$. The solution that results in the exponential runaway has been discarded. Therefore, the velocity component directed parallel to the magnetic will be constant and consequently not be influenced by any external forces.

The equations concerned with the x and y components of the velocity must be solved as a system of equations. Firstly, the equation must be solved exactly neglecting the inhomogeneous term and then secondly using a variation of parameter technique, a particular solution may be obtained for the inhomogeneous system. Combining the exact non-divergent general solution of the homogeneous equation with the particulr solution will allow the velocity and hence the trajectory of the particle to be calculated at all times, t.

The homogeneous system of equations may be written as:

$$\begin{aligned} \ddot{v}_x - \omega_o \dot{v}_x + \omega\omega_o v_y &= 0 \\ \ddot{v}_y - \omega_o \dot{v}_y - \omega\omega_o v_x &= 0 \end{aligned} \qquad (3\text{-}2)$$

where $\omega_o = \tau_o^{-1} = (3/2)\frac{mc^3}{e^2}$ and $\omega = \frac{eH}{mc}$ are the fundamental radiation frequency and the Larmor frequency respectively.

Introducing two new variables, a_x and a_y, (which may be thought of in physical term as the x and y components of the acceleration) defined by:

$$a_x = \dot{v}_x \qquad\qquad a_y = \dot{v}_y \qquad (3\text{-}3)$$

the system of two second-order differential equations (3-2) may be written as a system of four first-order differential equations. Since a good deal of linear algebra will be employed, another simplifying proceedure will be to rename the velocity and acceleration components as follows:

$$u_1 = a_x \qquad u_2 = v_x \qquad u_3 = a_y \qquad u_4 = v_y \qquad (3\text{-}4)$$

The system of equations (3-2) may now be written as:

$$\dot{u}_1 - \omega_o u_1 + \omega\omega_o u_4 = 0$$

$$\dot{u}_2 - u_1 = 0$$

$$\dot{u}_3 - \omega_o u_1 - \omega\omega_o u_2 = 0$$

$$\dot{u}_4 - u_3 = 0 \qquad (3\text{-}5)$$

Assuming that each u_n is a solution of the exponential form: $\sum_{m=1}^{4} \kappa_{nm} e^{\lambda_m t} = u_n$ and substituting these back into the system (3-5) each λ_m may be found. Writng the equations out fully, is tedious but the proceedure may be simplified by writing the determinant of the coefficients of the linear equations in (3-5) as

$$\begin{vmatrix} (\lambda_m - \omega_o) & 0 & 0 & \omega\omega_o \\ 1 & -\lambda_m & 0 & 0 \\ 0 & -\omega\omega_o & (\lambda_m - \omega_o) & 0 \\ 0 & 0 & 1 & -\lambda_m \end{vmatrix} = 0$$

or, $$\lambda_m{}^4 - 2\omega_o\lambda_m{}^2 + \omega_o{}^2\lambda_m{}^2 + \omega_o{}^2\omega^2 = 0 \qquad (3\text{-}6)$$

From (3-6) the values of λ_m may be obtained. This may be accomplished by the substitution $\lambda_m = w_m + \frac{\omega_o}{2}$ in which case (3-6) may be rewritten as:

$$w_m{}^4 - \frac{\omega_o}{2}^2 w_m{}^2 + \frac{\omega_o}{16}^4 + (\omega\omega_o)^2 = 0 \qquad (3\text{-}7)$$

Solutions to (3-7) are the complex numbers:

$$w_m = \frac{\omega_o}{4}^2 \pm (\omega\omega_o)\dot{i}$$

defining $\xi = (\frac{\omega_o}{4}^2 + \omega\omega_o\dot{i})^{\frac{1}{2}}$ and $\eta = (\frac{\omega_o}{4}^2 + \omega\omega_o\dot{i})^{\frac{1}{2}}$ one obtains

$$w_1 = \xi \qquad w_2 = -\xi \qquad w_3 = \eta \qquad w_4 = -\eta \qquad (3\text{-}8)$$

The constants ξ and η may be determined easily from simple trignometric applications (see Appendix B) as:

$$\xi = \left(\frac{\omega_o^4}{16} + (\omega\omega_o)^2 \right)^{\frac{1}{4}} \left[\left(\frac{1}{2} + \frac{\omega_o^2}{8\sqrt{\omega_o^4/16+(\omega\omega_o)^2}} \right)^{\frac{1}{2}} + \right.$$

$$\left. + \; i \left(\frac{1}{2} + \frac{\omega_o^2}{8\sqrt{\omega_o^2/16+(\omega\omega_o)^2}} \right)^{\frac{1}{2}} \right] \qquad (3\text{-}9)$$

and η is the complex conjugate of ξ. Since $\lambda_m = w_m + \frac{\omega_o}{2}$, the values for the decay constants have now been found. The determinant of the coefficients, κ_{nm} will be found by another substitution of the assumed solution into (3-5).

One then obtains from (3-5)

$$(\lambda_m - \omega_o)\kappa_{1m} + \omega\omega_o\kappa_{4m} = 0$$

$$\kappa_{1m} - \lambda_m\kappa_{2m} = 0$$

$$-\omega\omega_o\kappa_{2m} + (\lambda_m - \omega_o)\kappa_{3m} = 0$$

$$\kappa_{3m} - \lambda_m\kappa_{4m} = 0 \qquad (3\text{-}10)$$

Since the determinant of these equations is zero, the exists (according to Cramer's rule) non-unique solutions for the coefficients. Therefore an arbitrary choice for the value of one coefficient will determine the remaining coefficients. Choosing $\kappa_{4m}=1$, the remaining coefficients become;

$$\kappa_{3m} = \lambda_m$$

$$\kappa_{2m} = (\lambda_m - \omega_o)\lambda_m/\omega\omega_o$$

$$\kappa_{1m} = (\lambda_m - \omega_o)\lambda_m^2/\omega\omega_o \quad \text{which leads to} \qquad (3\text{-}11)$$

$\kappa_{4m} = -(\lambda_m - \omega_o)^2\lambda_m^2/(\omega\omega_o)^2 = 1$ is the same expression as eq. (3-6). Solving explicitly for each coefficient κ_{nm} (by substituting the appropriate values of λ_m into the expressions above) give the following result:

$$\kappa_{nm} = \begin{vmatrix} i(\frac{\omega_o}{2}+\xi) & i(\frac{\omega_o}{2}-\xi) & -i(\frac{\omega_o}{2}+\eta) & -i(\frac{\omega_o}{2}-\eta) \\ i & i & -i & -i \\ (\frac{\omega_o}{2}+\xi) & (\frac{\omega_o}{2}-\xi) & (\frac{\omega_o}{2}+\eta) & (\frac{\omega_o}{2}-\eta) \\ 1 & 1 & 1 & 1 \end{vmatrix} \tag{3-12}$$

The general solutions to the homogeneous equation have now been obtained. The constants of integration, k_{nm}, are to be determined from the initial conditions on the velocities (and accelerations if needed).

$$\begin{aligned} u_1 &= a_x = i\sum_m k_{1m}\,(-1)^p\,\lambda_m e^{\lambda_m t} \\ u_2 &= v_x = i\sum_m k_{2m}\,(-1)^p\,e^{\lambda_m t} \\ u_3 &= a_y = \sum_m k_{3m}\,\lambda_m\,e^{\lambda_m t} \\ u_4 &= v_y = \sum_m k_{4m}\,e^{\lambda_m t} \end{aligned} \tag{3-13}$$

where p satisfies the condition $p=\begin{cases} 0 \text{ when } m=1,2 \\ 1 \text{ when } m=3,4 \end{cases}$ (3-13a)

Since the solutions (3-13) are the parametric expressions for the trajectory of a particle travelling under the influence of an external constant magnetic field at non-relativistic velocities, the velocity components should be similar to those obtained by Plass(1961).

Using (3-13) and (3-8) the velocity components may simply be written:

$$v_x(t) = \frac{v_o}{4}\,e^{(\omega_o/2)t}\left[e^{\xi t}+e^{-\xi t}-e^{\eta t}-e^{-\eta t}\right] i\sin\theta_o$$

$$v_y(t) = \frac{v_0}{4}\, e^{(\omega_0/2)t} \left[e^{\xi t} + e^{-\xi t} + e^{\eta t} + e^{-\eta t}\right] \sin\theta_0 \tag{3-14}$$

where $v_0 = v(t=0)$ and θ_0 is the initial angle between the velocity and magnetic field vectors.

Upon defining $\nu = \frac{\omega_0^4}{16} + (\omega\omega_0)^2$ and then define the quantities, α_1 and α_2 as

$$\alpha_1 = \left(\frac{\nu^{\frac{1}{2}}}{2} + \frac{\omega_0^2}{8}\right)^{\frac{1}{2}} \quad \text{and} \quad \alpha_2 = \left(\frac{\nu^{\frac{1}{2}}}{2} - \frac{\omega_0^2}{8}\right)^{\frac{1}{2}} \tag{3-15}$$

equation (3-9) may be written as;

$$\xi = \alpha_1 + i\alpha_2 \qquad\qquad \eta = \alpha_1 - i\alpha_2$$

therefore (3-14) becomes

$$v_x(t) = v_0 e^{(\omega_0/2)t} \left(\frac{e^{\alpha_1 t} - e^{-\alpha_1 t}}{2}\right)\left(\frac{e^{i\alpha_2 t} - e^{-i\alpha_2 t}}{2}\right) i\sin\theta_0$$

$$= -v_0 e^{(\omega_0/2)t} \sinh(\alpha_1 t)\, \sin(\alpha_2 t)\, \sin\theta_0$$

$$v_y(t) = v_0 e^{(\omega_0/2)t} \left(\frac{e^{\alpha_1 t} + e^{-\alpha_1 t}}{2}\right)\left(\frac{e^{i\alpha_2 t} + e^{-i\alpha_2 t}}{2}\right) \sin\theta_0$$

$$= v_0 e^{(\omega_0/2)t} \cosh(\alpha_1 t)\, \cos(\alpha_2 t)\, \sin\theta_0 \tag{3-16}$$

In order to compare these results with those of Plass, α_1 and α_2 will be written in terms of ω and ω_0. Using (3-15) and remembering that $\tau_0\omega_0 = 1$:

$$\alpha_1 = \frac{\omega_0}{2}\left[\frac{1}{2} + \frac{1}{2}(1 + 16\tau_0^2\omega^2)^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

$$\alpha_2 = \frac{\omega_0}{2}\left[-\frac{1}{2} + \frac{1}{2}(1 + 16\tau_0^2\omega^2)^{\frac{1}{2}}\right]^{\frac{1}{2}} \tag{3-17}$$

Equation (3-16) is the most general solution to (3-2) and therefore will contain "runaway" solutions as well since the asymptotic condition (1-11) is not included in the equations of motion (3-1). Picking out the non-divergent terms gives as the physically valid solution;

$$v_x(t) = v_o e^{-\omega_1 t} \sin(\omega_2 t) \sin\theta_o$$

$$v_y(t) = v_o e^{-\omega_1 t} \cos(\omega_2 t) \sin\theta_o \qquad (3\text{-}18)$$

where $\omega_1 = \alpha_1 - \omega_o/2$ and $\omega_2 = \alpha_2$.

Equation (3-18) is found to agree with the results of Plass in that it also describes an exponentially damped circular motion about the magnetic field.

The effect of the orthogonal electric field on the particle trajectory must now be accounted for, and it was this field that introduced the inhomogeneity into the equations of motion. Therefore using a variation of parameter technique, a particular solution for the inhomogeneous system of equations may be found.

Substituting functions of the parameter, t, in place of the constants of integration in the general solution, will give one a particular solution to the inhomogeneous system of differential equations. These functions are chosen in such a way that they satisfy the following system of equations:

$$\sum_n \dot{k}_n(t)\ (-1)^p\ \lambda_n e^{\lambda_n t} = \frac{eE\omega_o}{m}$$

$$\sum_n \dot{k}_n(t)\ (-1)^p\ e^{\lambda_n t} = 0$$

$$\sum_n \dot{k}_n(t)\ \lambda_n e^{\lambda_n t} = 0$$

$$\sum_n \dot{k}_n(t)\ e^{\lambda_n t} = 0$$

where n runs from one to four and p satisfies the condition (3-13a) and $\dot{k}_n(t) = \frac{d}{dt} k_n(t)$. The determinant of the above system of equations is

$$\begin{vmatrix} i\lambda_1 e^{\lambda_1 t} & i\lambda_2 e^{\lambda_2 t} & i\lambda_3 e^{\lambda_3 t} & i\lambda_4 e^{\lambda_4 t} \\ ie^{\lambda_1 t} & ie^{\lambda_2 t} & ie^{\lambda_3 t} & ie^{\lambda_4 t} \\ \lambda_1 e^{\lambda_1 t} & \lambda_2 e^{\lambda_2 t} & \lambda_3 e^{\lambda_3 t} & \lambda_4 e^{\lambda_4 t} \\ e^{\lambda_1 t} & e^{\lambda_2 t} & e^{\lambda_3 t} & e^{\lambda_4 t} \end{vmatrix} = D$$

or

$$D = 4(\lambda_2-\lambda_1)(\lambda_3-\lambda_4)e^{2\omega_0 t}$$

Here the fact that $(\lambda_1+\lambda_2+\lambda_3+\lambda_4) = 2\omega_0$ has been used.

Therefore by Cramer's Rule;

$$\dot{k}_1 = D^{-1} \frac{\omega_0 eE}{m} \begin{vmatrix} 1 & -1 & -1 \\ \lambda_2 & \lambda_3 & \lambda_4 \\ 1 & 1 & 1 \end{vmatrix} ie^{(2\omega_0-\lambda_1)t} = \frac{ieE \exp(-\lambda_1 t)}{2m(\lambda_2-\lambda_1)}$$

$$\dot{k}_2 = D^{-1} \frac{\omega_0 eE}{m} \begin{vmatrix} 1 & -1 & -1 \\ \lambda_1 & \lambda_3 & \lambda_4 \\ 1 & 1 & 1 \end{vmatrix} ie^{(2\omega_0-\lambda_2)t} = \frac{-ieE \exp(-\lambda_2 t)}{2m(\lambda_2-\lambda_1)}$$

$$\dot{k}_3 = D^{-1} \frac{\omega_0 eE}{m} \begin{vmatrix} 1 & 1 & -1 \\ \lambda_1 & \lambda_2 & \lambda_4 \\ 1 & 1 & 1 \end{vmatrix} ie^{(2\omega_0-\lambda_3)t} = \frac{-ieE \exp(-\lambda_3 t)}{2m(\lambda_4-\lambda_3)}$$

$$\dot{k}_4 = D^{-1} \frac{\omega_0 eE}{m} \begin{vmatrix} 1 & 1 & -1 \\ \lambda_1 & \lambda_2 & \lambda_4 \\ 1 & 1 & 1 \end{vmatrix} ie^{(2\omega_0-\lambda_4)t} = \frac{ieE \exp(-\lambda_4 t)}{2m(\lambda_4-\lambda_3)}$$

Integrating the expressions above with respect to time gives the following functions:

$$k_1(t) = -\frac{\omega_0 eE}{m} \frac{\exp(-\lambda_1 t)}{2\lambda_1(\lambda_2-\lambda_1)} \qquad k_2(t) = \frac{\omega_0 eE}{m} \frac{\exp(-\lambda_2 t)}{2\lambda_2(\lambda_1-\lambda_2)}$$

$$k_3(t) = \frac{\omega_0 eE}{m} \frac{\exp(-\lambda_3 t)}{2\lambda_3(\lambda_4-\lambda_3)} \qquad k_4(t) = \frac{\omega_0 eE}{m} \frac{\exp(-\lambda_4 t)}{2\lambda_4(\lambda_4-\lambda_3)}$$

The particular solutions may then be obtained by substituting the expressions for $k_n(t)$ into the general solution expressions (3-13) as follows.

$$u_{1part} = \dot{\iota}\sum_n k_n(t)(-1)^p \lambda_n e^{\lambda_n t} = 0$$

$$u_{2part} = \dot{\iota}\sum_n k_n(t)(-1)^p e^{\lambda_n t} = 0$$

$$u_{3part} = \sum_n k_n(t) \lambda_n e^{\lambda_n t} = 0$$

$$u_{4part} = \sum_n k_n(t) e^{\lambda_n t} = \frac{E}{H}c \qquad (3\text{-}19)$$

The particular solution appears in the y component of the velocity only and is a constant proportional to the ratio of the electric field to the magnetic field. As expected the particular solutions for the acceleration components are zero since as $t \to \infty$ the acceleration components must approach zero. It will be noted that the constant velocity, $\frac{E}{H}c$, is simply the "drift" velocity of a particle travelling in orthogonal electric and magnetic fields and is perpendicular to the plane in which the fields lie.

Since the particle undergoes damped motion as a result of the magnetic field only, the trajectory of the particle will be a damped trochoid. Integrating the velocity components from (3-18) and (3-19) over time gives the parametric expression for the position of the particle at time, t. The expressions therefore are:

$$x(t) = \frac{-v_0 \sin\theta_0 e^{-\omega_1 t}}{\omega_1{}^2 + \omega_2{}^2}\left(\omega_1 \sin(\omega_2 t) + \omega_2 \sin(\omega_2 t)\right)$$

$$y(t) = \frac{-v_0 \sin \theta_0 e^{-\omega_1 t}}{\omega_1{}^2 + \omega_2{}^2} \left(\omega_2 \sin(\omega_2 t) + \omega_1 \cos(\omega_2 t)\right) +$$

$$+ \frac{E}{H} ct \tag{3-20}$$

The damped trochoid motion may be described in three different forms; prolate (or looping trajectory), curtate (or smoothly rounded), or cycloidal depending upon the rate at which energy is lost by the particle. Since the rate of energy loss is dependent upon the field strengths, the type of particle trajectory will also be specified by the size of the electromagnetic field. A more detailed description of particle trajectories and energy losses under varying field configurations and strengths will be handled in Chapter V.

CHAPTER IV

THE DIRAC-LORENTZ EQUATION FOR A CHARGE IN ORTHOGONAL MAGNETIC AND ELECTRIC FIELDS

The relativistic equations of motion (1-1) cannot be solved exactly, since they involve the cross products between the different velocity components. Non-linear expressions are therefore introduced into the radiative reaction and approximation techniques must be relied upon in order to obtain a solution to the equations of motion. Thus far two different methods have been used. Plass (1961) using Picard's method assumed the non-linear terms (which are of the same order as the third time derivative of the velocity components) to be small compared with the velocities, and the first and second time derivatives of the velocities. This method of successively approximating the solution to a quasi-linear differential equation in powers of τ_0, allowed Plass to obtain approximate solutions to the equation (1-1) which were very similar to the solutions to the non-relativistic equations of motion. Once again, motion was restricted to being in a uniform, static magnetic field. The second method of solution, the perturbation series approximation of the equation of motion, has already been described in Chapter II.

Upon treating the motion of a relativistic charged particle in orthogonal electric and magnetic fields, the latter method shall be used to obtain approximate solutions to equation (1-1). Since terms of the order $\tau_o{}^2$ and cannot be observed classically, the first order approximation, (2-13), of equation (1-1) will be used as the equation that fully describes the trajectory of a radiating particle. It will be remembered also that equation (2-13) was derived using the asymptotic condition (1-11) and therefore solutions to (2-13) will not contain the divergent terms that appeared in the solutions found in the previous chapter.

In order to write equation (2-13) in a usable form, the assumption that $\omega\tau_o \ll 1$ must be employed. Since τ_o is such a small quantity the magnetic fields that require $\omega\tau_o \simeq 1$ are of the order 10^{15} Gauss and therefore the assumption that the Larmor frequency, ω, is much smaller than the fundamental radiation frequency, ω_o, is valid physically.

Writing equation (1-1) in the form

$$\dot{u}_\mu = \frac{e}{mc} F_{\mu\nu} u^\nu + \Gamma_\mu \tag{4-1}$$

where $\Gamma_\mu = \frac{2}{3} \frac{e^2}{mc^3} \ddot{u}_\mu - \frac{1}{c^2} u_\mu \dot{u}^\nu \dot{u}_\nu$ is known as the Abraham four vector which describes the radiative damping force. The Abraham four vector may be written, to the first order (making use of (2-12)) as

$$\Gamma_\mu = \tau_o \frac{e}{mc^2} \left(u^\nu \frac{d}{d\tau} F_{\mu\nu} + F_{\mu\nu} \dot{u}^\nu \right) - \frac{1}{c^2} R u_\mu \tag{4-2}$$

Applying the condition $\omega\tau_o \ll 1$ allows the acceleration to be of the same order as the Lorentz force divided by the particle's mass. Equation (4-2) then becomes:

$$\Gamma_\mu = \tau_o \frac{e}{mc^2} \left(-u^\nu u^\lambda \frac{d}{dx_\lambda} F_{\mu\nu} + \frac{e}{mc} F_{\mu\lambda} F^{\nu\lambda} u_\nu \right) - \frac{2}{3} \frac{e^2 \tau_o}{mc^5} \left(F_{\nu\lambda} u^\lambda \right) \left(F^{\nu\eta} u_\eta \right) u_\mu \tag{4-3}$$

where the definition of the four velocity $u_\lambda = \frac{dx_\lambda}{d\tau}$ and the chain rule have been employed. Equation (4-3) has also been obtained in a different manner by Landau and Lifshitz (1965). The first term vanishes only if the field is constant in both space and time due to the interdependency of space and time in relativistic theory.

Both the electric and magnetic fields considered in this problem will be constant in space and time, therefore the first order approximation of equation (1-1) may be written as,

$$\dot{u}_\mu = \frac{e}{mc}F_{\mu\nu}u^\nu - \tau_o\frac{e^2}{m^2c^5}F_{\mu\lambda}F^{\nu\lambda}u_\nu + \tau_o\frac{e^2}{mc^5}\left(F_{\nu\lambda}F^{\nu\eta}u^\lambda u_\eta\right)u_\mu \qquad (4\text{-}4)$$

Upon choosing the directions of the electric and magfields to be along the z and x axes respectively, the spatial components of equation (4-4) become (see Appendix C);

$$\dot{u}_x = \frac{e}{mc}Hu_y - \frac{2}{3}\frac{e^3}{m^2c^5}\left(E^2-H^2\right)u_x + \frac{2e^3\gamma^2}{3m^2c^3}\left(H^2\left(u_x{}^2+u_y{}^2\right)+e^2E^2\left(1+u_x\right)+2eEHu_y\right)u_x$$
$$+ (e/m)E$$

$$\dot{u}_y = \frac{-e}{mc}Hu_x + \frac{2e^3\gamma^2}{3m^2c^3}\left(H^2\left(u_x{}^2+u_y{}^2\right)+e^2E^2\left(1+u_x\right)+2eEHu_y\right)u_x +$$
$$-\left(\frac{2}{3}\frac{e^2}{mc^4}EH + \frac{2}{3}\frac{e^3}{m^2c^5}H^2\right)u_y \qquad (4\text{-}5)$$

While the first order equation does not involve the second time derivatives of the velocity components, it does remain complicated by the presence of the squares and cross products of the velocity components thenmselves. Equation (4-5) may be further simplified if a Lorentz transformation to a moving frame in which the electric field vanishes is made and equation (4-5) is written in that frame.

Since the quantities $\vec{E}\cdot\vec{H}$ and E^2-H^2 are invariant under all Lorentz transformations and in the particular problem at hand $\vec{E}\cdot\vec{H}=0$, there must exist a Lorentz frame in which either of the fields vanishes. The present problem also restricts the quantity E^2-H^2 to be less than zero, hence the only field that would be allowed to vanish would be the electric field. Using the well known Lorentz tranformation formulae for electromagnetic fields and setting the transformed electric field to zero gives as the velocity of the moving frame:

$$\vec{v} = \frac{\vec{E}\times\vec{H}}{H^2}\, c \tag{4-6}$$

which is the "drift" velocity of a particle in an orthogonal electric and magnetic field configuration and is directed perpendicular to the plane described by the electromagnetic fields. Applying the field transformation equations to the magnetic field, the transformed magnetic field becomes:

$$\vec{H}' = \left(\frac{H^2-E^2}{H^2}\right)^{\frac{1}{2}} \vec{H} = \frac{1}{\gamma_0}\vec{H} \tag{4-7}$$

(All quantities measured in the Lorentz frame travelling with the drift velocity are represented by the primed quantities). In the moving frame the only field acting on the particle is a static magnetic field which has the same direction as the magnetic field in the laboratory frame but is reduced by a factor of γ_0^{-1}. Thus the motion of the particle in the drifting frame will be thought of as occuring in a static magnetic field only. Setting $E'=0$

equation (4-5) may be written in the moving frame as;

$$\dot{u}_x' = \omega' u_y' + \tau_0 \omega'^2 (1-\beta'^2\gamma^2) u_x'$$

$$\dot{u}_y' = -\omega' u_x' + \tau_0 \omega'^2 (1-\beta'^2\gamma^2) u_y' \qquad (4-8)$$

where $\beta'^2 = u'^2/c^2$ and ω' is the Larmor frequency of the particle in the moving frame. The assumption that the energy does not change appreciably over one cycle (approximately $2\pi/\omega$) can be made easily as long $\omega\tau_0 << 1$. Energy losses in this case are extremely small (i.e. a variation in γ of 10^{-2} would occur for 100 cycles for a magnetic field strength of 1 Gauss) and the trajectory of the particle is not altered drastically. Calculation of the trajectory may be easily made by assuming that γ remains constant over one cycle. Subsequent cycles may be obtained by calculating the small energy change that occurred in the previous cycle and then using that value as the initial γ for the following cycle. This method of successive approximation of the trajectory works quite well when the magnetic field is as large as 10^{13}G (see Chapter V). All previous work dealing with the relativistic equations of motion with the exception of that by Shen (1972) have maintained that γ will remain constant indefinitely. For these cases the requirement that $\omega\tau_0 << 1$ must be adhered to rigidly and therefore the radiative reaction will have little effect upon the particle dynamics. Over the period of one revolution for which the initial energy is represented by γ the equations of motion in the drift frame are:

$$\dot{u}_x' = \omega' u_y' + \tau_0 \omega'^2 \gamma^2 u_x'$$
$$\dot{u}_y' = -\omega' u_x' + \tau_0 \omega'^2 \gamma^2 u_y' \qquad (4-9)$$

The solutions to equations (4-9) may be found very simply by using the same technique as employed in Chapter III for the non-relativistic equations. Equations (4-9) are two first order homogeneous equations. Assuming solutions of the form:

$$u_1 = u_x' = \kappa_{11} e^{\lambda_1 \tau} + \kappa_{12} e^{\lambda_2 \tau}$$
$$u_2 = u_y' = \kappa_{21} e^{\lambda_1 \tau} + \kappa_{22} e^{\lambda_2 \tau}$$

and substituting these back into (4-9) allows one to solve for the exponential arguements, which are complex.

$$\lambda_1 = -\gamma^2 \omega'^2 \tau_0 + i\omega' \qquad \lambda_2 = -\gamma^2 \omega'^2 \tau_0 + i\omega' \qquad (4-10)$$

The solutions for the coefficients are not unique, therefore assuming without loss of generality that $\kappa_{21}=1$ the remaining coefficients become:

$$\kappa_{11} = i \qquad \kappa_{12} = 1$$
$$\kappa_{21} = 1 \qquad \kappa_{22} = i \qquad (4-11)$$

Having solved for the coefficients above, the most general solutions for the equations of motion (4-9) are:

$$u_1 = a_1 i \exp -(\gamma^2 \omega'^2 \tau_0 + i\omega')\tau + a_2 \exp -(\gamma^2 \omega'^2 \tau_0 - i\omega')\tau$$
$$u_2 = a_1 \exp -(\gamma^2 \omega'^2 \tau_0 + i\omega')\tau + a_2 i \exp -(\gamma^2 \omega'^2 \tau_0 - i\omega')\tau \qquad (4-12)$$

where the integration factors, a_1 and a_2 are arbitrary constants to be determined from initial conditions.

Rewriting the exponential terms with the complex arguments in terms of trignometric functions and specifying the initial value of the velocity vector to be u_o and at an initial angle θ_o to the magnetic field vector, the equations of motion have the solution;

$$u_x' = u_o \exp -(\gamma^2 \omega'^2 \tau_o)\tau \ \sin \omega'\tau \ \ \sin \theta_o$$

$$u_y' = u_o \exp -(\gamma^2 \omega'^2 \tau_o)\tau \ \sin \omega'\tau \ \ \sin \theta_o \qquad (4\text{-}13)$$

In the moving reference frame the particle executes a spiral trajectory resulting from the exponential damping of the original circular motion that arises without radiative reaction present. This motion is exactly the same that occurs for a particle moving in a static constant magnetic field under the influence of radiation damping.

In the non-moving frame, the solutions are much more complicated since a Lorentz transformation must be made to express the velocity in terms of the quantities in that frame. The solutions to the equations of motion in the observer's frame then become;

$$u_x = \gamma u_o (H^2 - E^2)^{\frac{1}{2}} \sin(\omega\tau/\gamma_o) \ \sin \theta_o \times$$
$$\times \left(H \exp(\gamma^2 \omega^2 \tau_o \tau/\gamma_o^2) + E u_o \cos(\omega\tau/\gamma_o) \ \sin \theta_o\right)^{-1}$$

$$u_y = \gamma u_o H \cos(\omega\tau/\gamma_o) \ \sin \theta_o \times$$
$$\times \left(H \exp(\gamma^2 \omega^2 \tau_o \tau/\gamma_o{}^2) + E u_o \cos(\omega\tau/\gamma_o) \ \sin \theta_o\right)^{-1} +$$
$$+ \frac{E}{c}\left(H - \gamma u_o \exp -(\gamma^2 \omega^2 \tau_o \tau/\gamma_o{}^2) \ \cos(\omega\tau/\gamma_o) \ \sin \theta_o\right)^{-1}$$

(4-14)

These equations represent the exact solutions to the first order relativistic equations of motion for a charged particle undergoing radiative reaction in orthogonal magnetic and electric fields.

Writing the velocity components in a Taylor series form in powers of the denominator, $(1+vu_y/c^2)$ where $|\frac{vu_y}{c^2}|<1$ equations (4-14) may be approximated to:

$$u_x = u_o\gamma/\gamma_o \exp(-\gamma^2\omega^2\tau_o\tau/\gamma_o{}^2)\cos(\omega\tau/\gamma_o)\sin\theta_o \times$$
$$\times\left(1-\frac{u_oE}{H}\gamma \exp(-\gamma^2\omega^2\tau_o\tau/\gamma_o{}^2)\sin(\omega\tau/\gamma_o)\sin\theta_o+\ldots\right)$$

$$u_y = \left(u_o\gamma/\gamma_o \exp(-\gamma^2\omega^2\tau_o\tau/\gamma_o{}^2)\sin(\omega\tau/\gamma_o)\sin\theta_o+\frac{E}{H}c\right)\times$$
$$\times\left(1-\frac{u_oE}{H}\gamma \exp(-\gamma^2\omega^2\tau_o\tau/\gamma_o{}^2)\sin(\omega\tau/\gamma_o)\sin\theta_o+\ldots\right)$$

(4-15)

The non-relativistic solutions may be represented by the first term in the expansion and by setting γ and γ_o to unity. These equations then reduce to equations (3-18). By setting E=0, equation (4-15) becomes equivalent to the first order expression for the relativistic solutions to the Lorentz-Dirac equation obtained by Shen (1972).

An extremely important result of the solutions to the relativistic equations of motion deals with the velocity component parallel to the magnetic field vector. In Chapter III it was shown that the longitudinal velocity component remains constant. This would be even so in the reference frame moving with the drift velocity, but, as a result of the Lorentz transformation (i.e. in effect, a result of of the Doppler shift) the z component will not remain constant. Applying the Lorentz transformation of velocities to the z component of the velocity in the drift frame the velocity component in the observer's frame becomes:

$$u_z = \frac{\gamma u_0 (H^2 - E^2)^{\frac{1}{2}} \cos\theta_0}{H + u_0 \gamma E \exp(-\gamma^2 \omega^2 \tau_0 \tau/\gamma_0{}^2) \cos(\omega\tau/\gamma_0) \sin\theta_0}$$

(4-16)

Therefore unlike the non-relativistic particle for which the radiation gathers its energy from the transverse components of the velocity only, the relativistic particle will lose energy from both the transverse and longitudinal velocity components.

This result is a consequence of the drift motion of the particle since the electric field plays an important part in the longitudinal energy loss. Setting the electric field to zero in equation (4-16) results in $u_z = u_0 \cos\theta_0$ which is a constant. This was the same result obtained from the non-relativistic problem. The implications of (4-16) in terms of the radiative energy loss and the trajectory of the particle will be discussed more fully in Chapter V.

Computation of the actual trajectories of the relativistic particle is more complex than that for the non-relativistic particle. The velocities above are written in terms of proper time, τ, and in order to obtain the trajectory of the particle for ordinary time, t, one must integrate equation (1-5) over all past time in order to determine the passage of ordinary time. This is a much more involved process than simply varying the parameter in the velocity expressions. But, in the problem at hand if γ remains constant for at least one revolution the relationship between the intervals of ordinary and proper time is linear and the transformation may be made more easily.

Qualitatively though, the trajectories will be similar to the non-relativistic trajectories. Departures resulting from the Lorents transformation from the drifting frame to the observer's frame will occur, and these will be noticed especially in the longitudinal velocity component already discussed. Further discussion of the relativistic effects will be reserved for the next chapter.

CHAPTER V

PHYSICAL IMPLICATIONS OF THE THEORY

A) Trajectories and energy losses

In Chapters II, III, and IV detailed calculations on the solutions of the equations of motion for a radiating charged particle have been presented. The details have been mainly of a mathematical nature and physical interpretations have been made in passing. This chapter will be devoted to the applicabilty of the equations of motion of certain physical situations to the solutions obtained in previous chapters. Trajectories of individual particles and energy losses will also be discussed in this chapter. Two parameters which are crucial in a study of physical applicability of the equations of motion are the initial energy and the magnetic field strength. As will be shown later, the strength of the electric field becomes an important factor in the relativistic case.

The non-relativistic equations will be studied first since these solutions give the gross behavior of the particle. For most practical problems the magnetic field strengths are sufficiently small enough that the Larmor frequency will be smaller than the fundamental frequency of radiation, ω_0, or in other words, $\omega\tau_0 << 1$. The arguments α_1 and α_2 (3-17) of the exponential and trignometric functions in Chapter III may be written as a Taylor series in powers of $\omega\tau_0$. These become:

$$\alpha_1 = \omega_0(\tfrac{1}{2} + \tau_0{}^2\omega^2 - 5\tau_0{}^4\omega^4 + 42\tau_0{}^6\omega^6 + \ldots)$$

$$\alpha_2 = \omega(1 - 2\tau_0{}^2\omega^2 + 14\tau_0{}^4\omega^4 + \ldots) \tag{5-1}$$

Since the argument of the exponential decay is $\omega_1 = \alpha_1 - \omega_0/2$ this constant becomes:

$$\omega_1 = \tau_0\omega^2(1 - 5\tau_0{}^2\omega^2 + 42\tau_0{}^4\omega^4 + \ldots) \tag{5-2}$$

Therefore $-\omega_1$ which determines the exponential decay of the motion increases in the first approximation as the square of the Larmor frequency. Beyond this first order approximation, the actual frequency of circular motion, described by equation (3-17) diminshes by powers of $\tau_0{}^2$ and and higher orders. Thus to the first order, the parametric expressions for the velocities become;

$$v_x(t) = v_0 e^{-\tau_0\omega^2 t}\sin\omega t \sin\theta_0$$

$$v_y(t) = v_0 e^{-\tau_0\omega^2 t}\sin\omega t \sin\theta_0 + \frac{E}{H}c \tag{5-3}$$

These equations may also be obtained from the first terms of (4-15) by setting γ and γ_0 to one and equating proper and ordinary time.

The only requirement placed upon the electric field is that $|E|<|H|$ in order to ensure that the drift velocity does not exceed the velocity of light. For the drift to be non-relativistic the electric field strength must be atleast one order of magnitude smaller than the magnetic field strength.

For a complete understanding of the dependence of the solutions upon the magnetic field, it is also instructive to consider the limiting value when $\omega\tau_0 >> 1$. In this case;

$$\omega_1 = \omega_2 = \sqrt{\tfrac{1}{2}\omega_0\omega} \tag{5-4}$$

the decay constant and the frequency of motion are equal. The particle will then radiate most of its energy in a single revolution, and will then travel with a constant velocity depending upon the drift rate and the magnitude of the z velocity component.

Before describing the energy losses, it will be helpful to view the trajectories in the limiting cases discussed above. The concept of the center of gyration will be used to simplify matters. This point is the instantaneous center of circular motion see Alfven and Falthammar (1963) and moves with a velocity $v_g = v_\perp + v_\parallel$ where $v_\perp$ represents the drift velocity of the particle and $v_\parallel$ represents the velocity due to the radiative reaction. In the case where only the Lorentz force acts (i.e. radiation reaction is negligible) the velocity of the center of gyration is:

$$v_g = v_d = Ec/H$$

The radius of curvature about this point at any time, t, will be given by;

$$\rho_g(t) = \frac{v_0 \sin\theta_0}{(\omega_1{}^2 + \omega_2{}^2)} \exp(-\omega_1 t) \tag{5-5}$$

Since the radius of cuvature is decreasing in time, the motion of the particle will tend to follow a curve that becomes increasingly curtate as time increases (i.e.

the oscillations about the line describing the motion of the center of gyration will decrease). The form of cycloidal motion will depend upon the radius of curvature per revolution to the drift velocity. A critical time defined as the time at which the motion is simply cycloidal (i.e. the time when this ratio is unity) can be found from (5-5) to be;

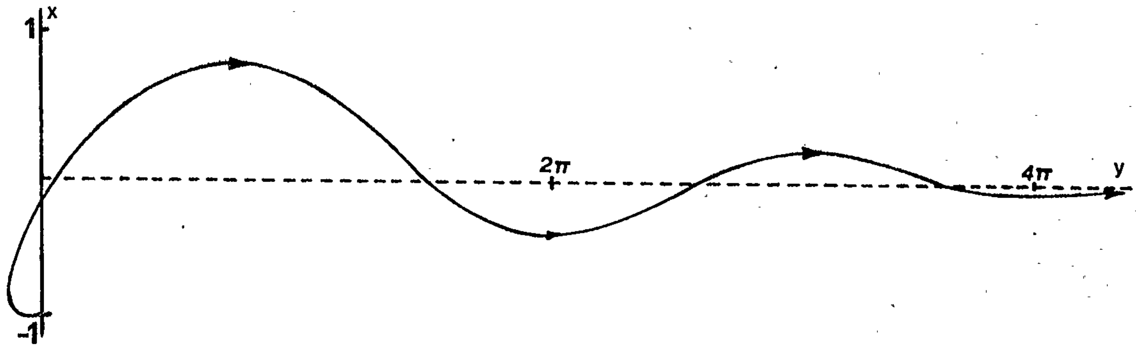
$$t_c = \omega_2^{-1} \ln \frac{Ec}{v_0 \sin \theta_0} \left(\frac{\omega_1^2 + \omega_2^2}{\omega_2^2} \right)^{\frac{1}{2}} \qquad (5\text{-}6)$$

When $t<t_c$ the motion will be prolate, and curtate when $t>t_c$. Should $t_c<0$, the prolate motion will be unable to occur. In the case where $\omega\tau_0 \gg 1$, t_c will be of the order τ_0 and therefore the damping (proportional to the inverse of t_c) will be very large.

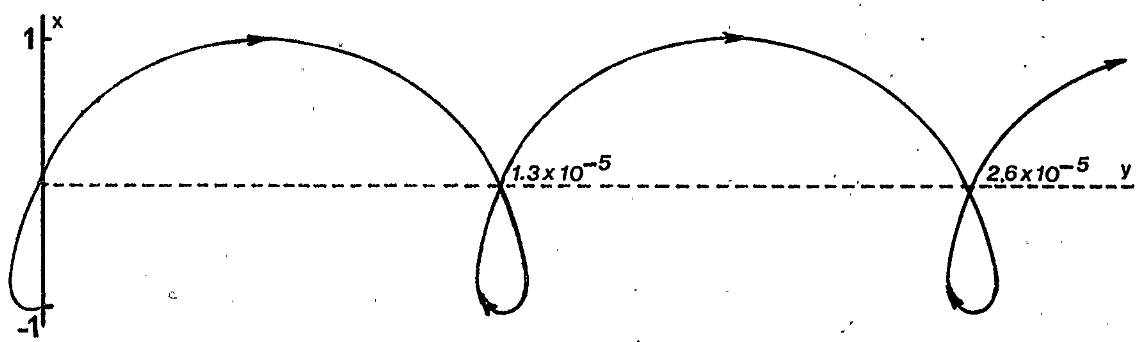
In Figure (5-1) the trajectory of a single electron in the x-y plane is represented for the cases where the parameter $\omega\tau_0$ is greater than, less than, and approximately equal to unity. It can be seen quite readily that even in the intermediate case of $\omega\tau_0=0.937$ the particle loses an appreciable amount of its initial energy in one period, $2\pi/\omega_2$.

For the case $\omega\tau_0 \ll 1$, the motion of the particle deviates only slightly from the motion of a particle influenced by the Lorentz-forces only.

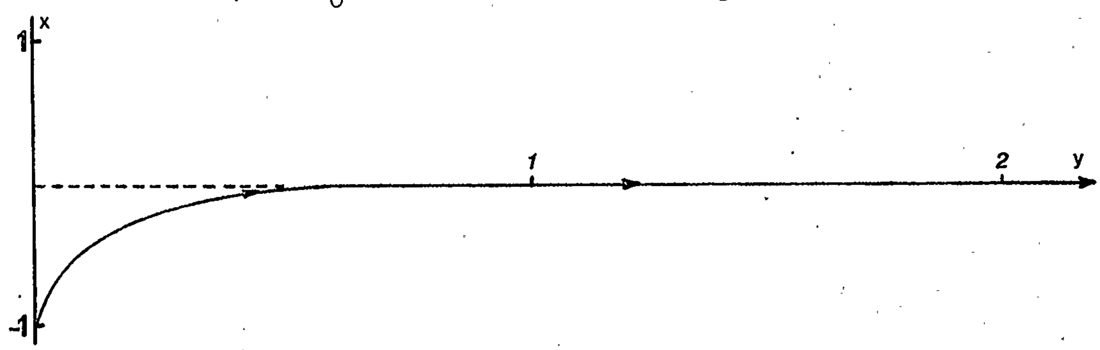
The energy loss per unit time may be calculated, readily from the expressions for the velocity:



a) $\omega\tau_0=0.937$ $t_c=0.498\ \omega_2^{-1}$



b) $\omega\tau_0=6\times10^{-5}$ $t_c=1.6\times10^{10}\ \omega_2^{-1}$



c) $\omega\tau_0=4\times10^{2}$ $t_c=-1.5\ \omega_2^{-1}$

Figure (5-1). Various trajectories of a radiating classical particle in orthogonal electric and magnetic fields for different values of the parameter, $\omega\tau_0$. The units of distance are chosen such that the initial 'undamped' position is unity. The path of the trajectory is represented by the solid line and the motion of the center of gyration is represented by the dashed line.

$$\frac{d\varepsilon}{dt} = 2\omega_1 v_o{}^2 e^{-2\omega_1 t} \sin^2\theta_o \qquad (5\text{-}7)$$

In the two extreme cases the energy loss per unit time becomes;

$$\frac{d\varepsilon}{dt} = -v_o{}^2\omega^2\tau_o \sin^2\theta_o \qquad : \omega\tau_o \ll 1$$

$$\frac{d\varepsilon}{dt} = -v_o{}^2(2\omega\omega_o)^{\frac{1}{2}} \sin^2\theta_o \exp\left(-(2\omega\omega_o)^{\frac{1}{2}}t\right) \qquad : \omega\tau_o \gg 1$$

(5-8)

For the case $\omega\tau_o \ll 1$ the energy loss may be written in the familiar form;

$$-\frac{d\varepsilon}{dt} = \frac{2}{3}\left(\frac{e^2}{mc^2}\right)^2 H^2 v_o{}^2 \sin^2\theta_o \qquad (5\text{-}9)$$

which is just the non-relativistic energy loss expression for a particle losing energy in a constant, static magnetic field. On the other hand when $\omega\tau_o \gg 1$ the energy of the particle is lost in a time of the order of τ_o and the particle will then move with a constant velocity neither gaining or losing energy.

Finally it will be observed that the pitch angle of the particle motion will decrease to a constant value:

$$\lim_{t\to\infty} \theta = \tan^{-1}\left(\frac{Ec}{Hv_o \cos\theta_o}\right) \qquad (5\text{-}10)$$

due to the decrease in the transverse component of the velocity. The decay of the pitch angle is dependent simply upon the exponential damping term and its rapidity of decay depends inversely upon the value of $\omega\tau_o$.

The relativistic problem is not as simple as that discussed above. The Lorentz transformation from the drifting frame to the laboratory frame introduces departures from the trajectories of the non-relativistic particle.

The instantaneous radiation rate for the non-relativistic particle may be obtained from the time-like component of equation (4-4), since $u_4=\gamma c$:

$$\frac{d\gamma}{d\tau} = \frac{-\tau_0 e^2}{mc^3}\left(\frac{\varepsilon(\tau)}{mc^2}\right)^2\left(H^2(u_x{}^2+u_y{}^2)-E^2\right) +2EHu_y + \frac{eEu_x}{H} \tag{5-11}$$

Substituting equations (4-15) into (5-11), the first order relativistic energy loss per unit rest mass per unit time for a single charged particle becomes;

$$\begin{aligned}\frac{d\gamma}{d\tau} &= -\frac{2}{3}\,\frac{e^2}{mc^2}\;\frac{\varepsilon(\tau)^2}{m^2c^4}H^2u_o{}^2\exp(-2\gamma^2\omega^2\tau_0\tau/\gamma_o{}^2)\sin^2\theta_o\, + \\ &+ \frac{u_0\gamma}{\gamma_0}\exp(-\gamma^2\omega^2\tau_0\tau/\gamma_0{}^2)\sin\,\theta_o\,\times \\ &\times\left(\frac{e}{m}E\,\cos(\omega\tau/\gamma_0)-2\tau_0\,\frac{e^2}{mc^3}\;EH\,\sin(\omega\tau/\gamma_0)\right)\end{aligned} \tag{5-12}$$

In the absense of the electric field and in the case where the magnetic field strength is extremely small, the energy loss of the particle is simply the ordinary synchrotron radiation loss expression. Successive terms for the instantaneous radiation rate are of the order $(\omega\tau_0/\gamma_0)^2$ and modify the energy loss only slightly if the Larmor frequency is not too large.

The non-constancy of the "drift" velocity complicates the trajectory of the particle since the velocity of the center of gyration is affected by the Lorentz transformations. With a center of gyration velocity component perpendicular to the magnetic field and increasing with time, the trajectory of the particle in the relativistic case will be flatter than those depicted in Figure (5-1). The damping of the circular motion therefore becomes slightly more pronounced for the relativistic particle.

Another difference between the relativistic and non-relativistic cases is that the pitch angle in the relativistic case decreases more rapidly than in the non-relativistic case. Firstly, the damping of the transverse component of velocity is more rapid. The decrease of the velocity components goes as

$$\left[\exp(\gamma^2\omega^2\tau_0\tau/\gamma_0{}^2) + \frac{E}{H}u_0\gamma\cos(\omega\tau/\gamma_0)\sin\theta_0\right]^{-1}$$

relativistically where as non-relativistically the decrease (for $\omega\tau_0 << 1$) is $\exp(-\omega^2\tau_0\tau)$. In most cases this effect is small if $E << H$. Secondly the overall increase in the longitudinal velocity component to the value Ec/H also causes a more rapid decrease in the pitch angle. However the limiting value of the pitch angle (5-10) still remains the same. It must be remembered that these relativistic effects are generally quite small and add only minor corrections to the motion described non-relativistically as long as $E << H$ and $\omega\tau_0 << 1$. Only in this case can the perturbation method used give meaningful results.

B) The applicability of the classical theory

Thus far very loose limits have been placed upon the applicability of the classical theory of radiation from accelerated charges used in the previous chapters. One must enquire whether certain astrophysical or laboratory conditions are satisfied by this theory. First quantum mechanical effects will become important when the deBroglie wavelength of the electron is of the same order as the characteristic length described by the Larmor radius. Therefore in order to treat the electron as a point charge in a well defined orbit without wave interference;

$$\frac{mc^2\gamma\beta}{eH} >> \frac{h}{\gamma mc\beta} \qquad \text{or}$$

$$\gamma^3\beta^2 >> \frac{2}{3}\,\gamma H\frac{eh}{m^2c^3} \tag{5-13}$$

where $\beta=\frac{v}{c}$, $\gamma=(1-\beta^2)^{-\frac{1}{2}}$ and h is Planck's constant. Here the factor 2/3 has been introduced without disturbing the order of magnitude estimation. In a quantum mechanical treatment of the radiative reaction using a modified Dirac equation, Sokolov, Klepikov, and Ternov (1952) have shown that the parameter, $\frac{2}{3}\,\frac{\gamma H}{H_q}$ (where $H_q= \frac{m^2c^3}{eh} =4.4\times10^{13}$ G) must satisfy even more demanding requirements than those of (5-13). Their results show that quantum mechanical effects occur at lower energies and thus the limiting requirement on clasical theory is:

$$\frac{2}{3}\,\frac{\gamma H}{H_q} << 1 \tag{5-14}$$

rather than criterion (5-13). Experimental evidence obtained from particle accelerator work [see Sokolov and Ternov (1968)] has shown that (5-14) is a much better estimate to the limits of classical theory.

Having placed a maximum value upon the magnetic field (i.e. 6.6×10^{13} G in the non-relativistic case) one must determine the limits to the energies that may be described by classical theory. Returning to equation (1-1), it can be seen that the order of magnitude of the three terms on the right hand side (in terms of the Lorentz force) is given by the ratio, $1:\omega\tau_0:\gamma^2\beta^2\omega\tau_0$. Since the radiation reaction is described by the sum of the last two terms, the overall magnitude of the radiation reaction force in terms of the Lorentz force is given by the parameter, $\gamma^2\omega\tau_0$. The theory presented in Chapters II, III, and IV holds on the condition that this parameter is less than unity (i.e. the radiation reaction forces are smaller in magnitude than the applied forces from the external field) or that;

$$\gamma^2 H << \frac{m^2c^4}{e^3} = H_c = 6\times10^{15}\ G \tag{5-15}$$

Figure (5-2) is a plot of the range of validity of different radiative reaction theories determined by criteria (5-14) and (5-15). Obviously astrophysical situations dealing with synchrotron radiation fall well within the applicability of the theory presented. Many laboratory situations including the recent megagauss experiments with the Stanford Linear Accelerator [Herlach, et. al. (1971)] also may be described by the first order relativistic theory. The intense magnetic fields and high particle energies associated with pulsars, however, must be describedby all levels of radiative reaction theories and therefore cannot be solely explained classically.

An application of the relativistic theory presented in Chapter IV has been carried out by Bland and Hobill (1974)

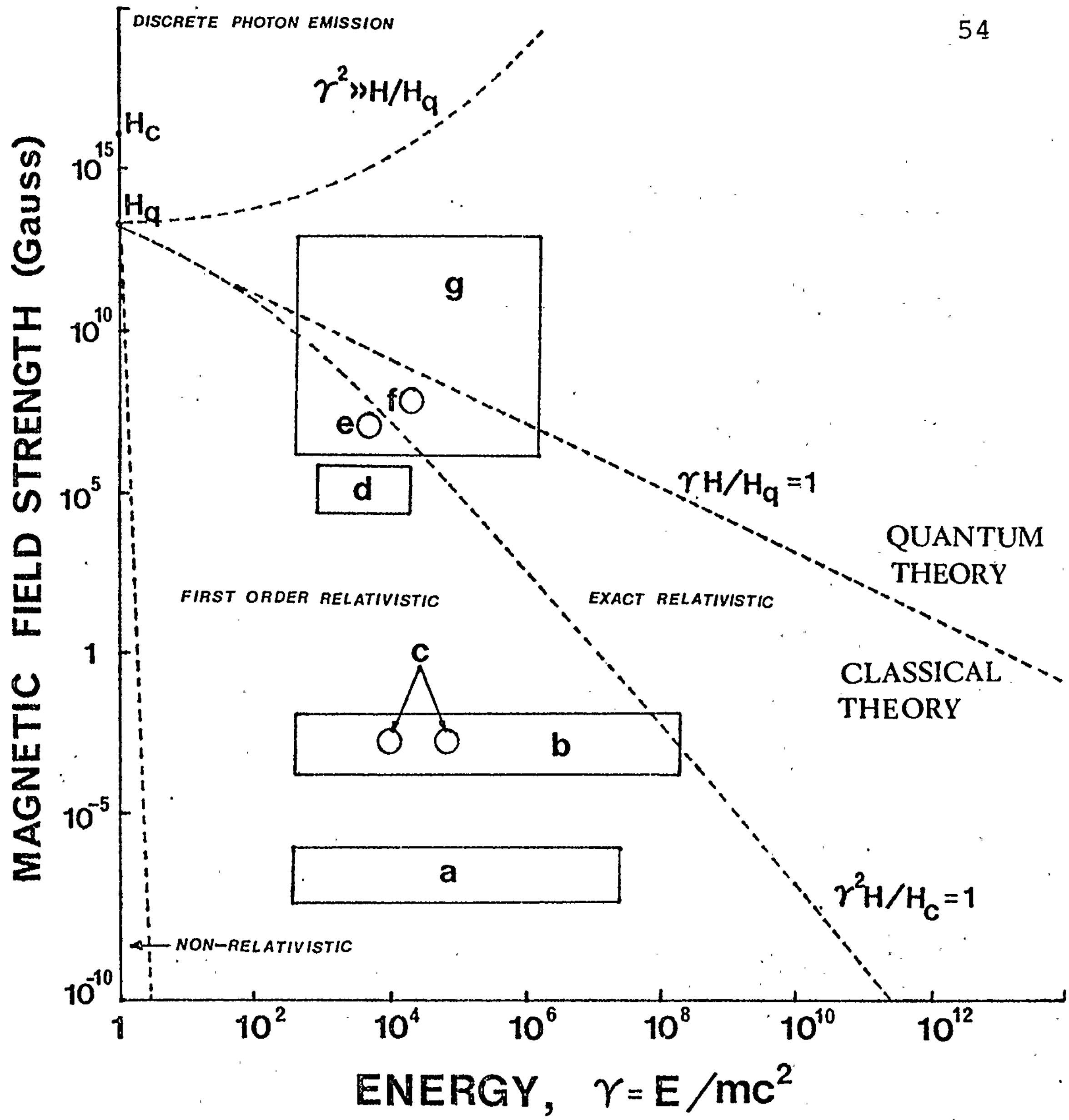


Figure (5-2). The validity of different levels of radiation reaction theory. One must be aware that higher level effects may be observed in a lower level theory. The lettered regions indicate several astrophysical and laboratory situations.

a) cosmic ray electrons in the galaxy
b) relativistic electrons in the Crab Nebula
c) neutral sheet electrons [Bland and Hobill (1974)]
d) synchrotron electron accelerators
e) SLAC electron beams [Herlach, et. al. (1971)]
f) NAL electron beams (proposed)
g) pulsar electrodynamics

for an idealized infinitely thin neutral sheet (thereby implying orthogonal electric and magnetic fields). Particles with energies ranging from $\gamma=10^2$ to 10^5 were introduced into neutral sheets with magnetic field strenghths ranging from 10^{-5} to 10^{-3} gauss. The instantaneous position and energy of the particle were calculated by updating the value of γ over each period and this was accomplished for times of the order of 10^7 sec. The relation between proper and ordinary time was assumed to be linear , therefore allowing a direct correlation between these two times. Since the order of magnitude change of γ over 50 revolutions was only 10^{-2}, γ was assumed to remain constant for fewer revolutions.

It was found that not only is there a noticable decrease in the pitch angle, but that there is also along with this, after an initial energy loss, an appreciable acceleration resulting from a stronger coupling of the electric field. The reversal of the magnetic field at the neutral sheet is just what is needed to overcome the energy losses that would occur in a simple orthogonal field situation. This reversal of fields also produces a gradient of the magnetic field at the boundary which along with the electric field serves to accelerate the particle. Assuming that the magnetic fields described above exist at neutral sheets associated with the Crab Nebula, this process of accelerating electrons may describe the infra-red spectrum of the Crab.

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APPENDIX A

RADIATION REACTION FORCE FROM CONSERVATION OF ENERGY

The following plausibility argument is not a rigorous proof but is only used as an example of the origin of a radiative reaction force obtained from the principle of energy conservation. In concidering only the non-relativistic limit, it is known that if an external force is exerted upon a particle, it accelerates according to Newton's equation of motion.

$$m\dot{\vec{v}} = \vec{F}_{ext} \qquad \text{(A-1)}$$

Since the particle is accelerated, the Larmor power formula will give the total instantaneous power radiated by the charge.

$$P(t) = \frac{2}{3}\frac{e}{c}(\dot{\vec{v}})^2 \qquad \text{(A-2)}$$

To account for this energy loss, the existence of a radiative force, $\vec{F}_{rad}$, is assumed and Newton's second law becomes;

$$m\dot{\vec{v}} = \vec{F}_{ext} + \vec{F}_{rad} \qquad \text{(A-3)}$$

Now equating the radiatve energy loss to the action of the radiative force over a time interval $t_1 < t < t_2$ one obtains:

$$\int_{t_1}^{t_2} \frac{2}{3}\frac{e^2}{c^3}\dot{\vec{v}}^2\, dt = -\int_{t_1}^{t_2} \vec{F}_{rad}\cdot\vec{v}\, dt \qquad \text{(A-4)}$$

Where the work done by F_{rad} is the negative of the energy radiated in the same time interval. The first integral may be integrated by parts to yield:

$$\int_{t_1}^{t_2} \frac{2}{3}\frac{e^2}{c^3}\dot{\vec{v}}^2 dt = \frac{2}{3}\frac{e^2}{c^3}\left[\vec{v}\cdot\dot{\vec{v}}\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \vec{v}\cdot\ddot{\vec{v}}\, dt \right]$$

$$= \int_{t_1}^{t_2} \vec{F}_{rad}\cdot\vec{v}\, dt \qquad \text{(A-5)}$$

If the motion is periodic and $t_2 - t_1$ is some multiple of the period, or if $\vec{v}\cdot\dot{\vec{v}}(t_1) = \vec{v}\cdot\dot{\vec{v}}(t_2)$ for reasons other than mentioned above, or if the average value of $\vec{v}\cdot\dot{\vec{v}}$ over some appropriate time scale is zero then

$$\frac{2}{3}\frac{e^2}{c^3}\vec{v}\cdot\dot{\vec{v}}\Big|_{t_1}^{t_2} = 0$$

and equation (A-5) becomes

$$\int_{t_1}^{t_2} \left[\vec{F}_{rad} - \frac{2}{3}\frac{e^3}{c^2}\ddot{\vec{v}}\right]\cdot\vec{v}\, dt = 0 \qquad \text{(A-6)}$$

Equation (A-6) is satisfied if the integrand is zero;

$$\vec{F}_{rad} = \frac{2}{3}\frac{e^2}{c^3}\ddot{\vec{v}} \qquad \text{(A-7)}$$

which leads to equation (1-9).

Obviously two flaws exist in the above reasoning; 1) sufficient justificaion for eliminating $\vec{v}\cdot\dot{\vec{v}}$ has not been provided, and 2) even if $\vec{v}\cdot\dot{\vec{v}}$ vanishes the radiation reaction component perpendicular to $\vec{v}$ has not been found since only the parallel component arises from the dot product in equation (A-6).

APPENDIX B

DETERMINATION OF THE PARAMETERS, ξ AND η

Writing the complex numbers ξ^2 and η^2 in exponential form, one obtains from De Moivre's theorem;

$$\xi^2 = \frac{\omega_o^2}{4} + \omega\omega_o i = \rho e^{i\theta} = \rho(\cos\theta + i\sin\theta) \qquad \text{(B-1)}$$

$$\eta^2 = \frac{\omega_o^2}{4} - \omega\omega_o i = \rho e^{i\theta} = \rho(\cos\theta + i\sin\theta)$$

Taking the square root of each of these gives:

$$\pm\xi = \pm\rho^{\frac{1}{2}}(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}) \; ; \; \pm\eta = \pm\rho^{\frac{1}{2}}(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}) \qquad \text{(B-2)}$$

To determine the trignometric functions of $\frac{\theta}{2}$, $\cos\theta$ must be found from Fig. B-1.

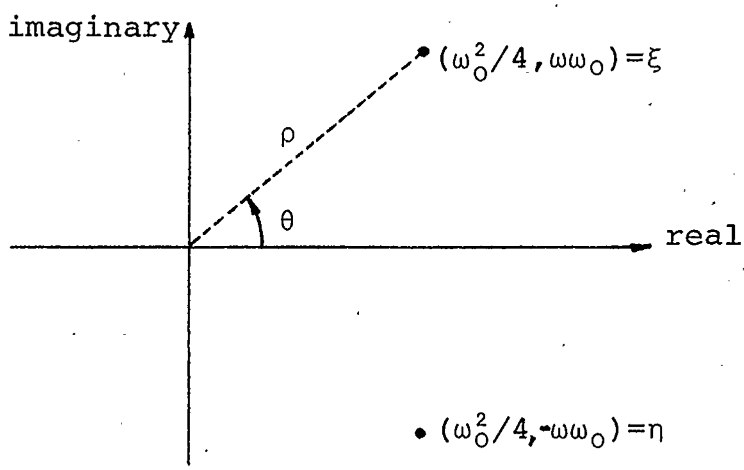

Fig. B-1. Representation of the imaginary numbers, ξ and η on a Cartesian co-ordinate system.

Therefore if $\rho = \left(\frac{\omega_o^4}{16} + (\omega\omega_o)^2\right)^{\frac{1}{2}}$

$$\text{then } \cos\theta = \frac{\omega_0^2}{4}\left(\frac{\omega_0^2}{16} + (\omega\omega_0)^2\right)^{-\frac{1}{2}}.$$

Using the identities:

$$\cos\frac{\theta}{2} = \sqrt{\tfrac{1}{2}(1+\cos\theta)} \quad \text{and} \quad \sin\frac{\theta}{2} = \sqrt{\tfrac{1}{2}(1-\cos\theta)}$$

and substituting these into (B-2) one obtains:

$$+\xi = \left(\frac{\omega_0^4}{16} + (\omega\omega_0)^2\right)^{\frac{1}{4}} \left[\left(\frac{1}{2} + \frac{\omega_0^2}{8\sqrt{\omega_0^4/16 + (\omega\omega_0)^2}}\right)^{\frac{1}{2}} + \right.$$
$$\left. + i\left(\frac{1}{2} - \frac{\omega_0^2}{8\sqrt{\omega_0^4/16 + (\omega\omega_0)^2}}\right)^{\frac{1}{2}}\right] \qquad \text{(B-3)}$$

and η is simply the complex conjugate of equation (B-3).

APPENDIX C

DETERMINATION OF THE FIRST APPROXIMATION OF THE LORENTZ-DIRAC EQUATION IN COMPONENT FORM

The first approximation of the Lorentz-Dirac equation may be written in tensor form as (see Chapter IV);

$$\dot{u}_\mu = \frac{e}{mc} F_{\mu\nu} u^\nu - \tau_o \frac{e^2}{m^2 c^3} F_{\mu\lambda} F^{\nu\lambda} u_\nu + \tau_o \frac{e^2}{m c^5} \left(F_{\nu\lambda} F^{\nu\eta} u^\lambda u_\eta \right) u_\mu \qquad \text{(C-1)}$$

The anti-symmetric electromagnetic field tensor $F_{\mu\nu}$ may be written as follows (assuming $\vec{H}$ to be directed along the z-axis and $\vec{E}$ along the x-axis):

$$F_{\mu\nu} = \begin{bmatrix} 0 & H & 0 & E \\ -H & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -E & 0 & 0 & 0 \end{bmatrix} \qquad \text{(C-2)}$$

Therefore the contravariant tensor, $F^{\mu\nu}$, becomes:

$$F^{\mu\nu} = \begin{bmatrix} 0 & H & 0 & -E \\ -H & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E & 0 & 0 & 0 \end{bmatrix} \qquad \text{(C-3)}$$

To determine the contribution from the Lorentz force we find

$$F_{\mu\nu} u^\nu = \begin{bmatrix} 0 & H & 0 & E \\ -H & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -E & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} Hu_2 + Eu_1 \\ -Hu_1 \\ 0 \\ -Eu_1 \end{bmatrix} = -F^{\mu\nu} u_\nu \qquad \text{(C-4)}$$

In other words the distinction between the covariant and contravariant tensor products for the Lorentz force vanishes except for the sign.

Now the second term on the right hand side of (C-1) must be evaluated. First the mixed tensor $F_{\mu\lambda}F^{\nu\lambda}$ is found to be

$$F_{\mu\lambda}F^{\nu\lambda} = \begin{bmatrix} E^2-H^2 & 0 & 0 & 0 \\ 0 & -H^2 & 0 & EH \\ 0 & 0 & 0 & 0 \\ 0 & -EH & 0 & E^2 \end{bmatrix} \qquad \text{(C-5)}$$

Multiplying (C-5) by the vector u_ν one obtains:

$$F_{\mu\lambda}F^{\nu\lambda}u_\nu = \begin{bmatrix} (E^2-H^2)u_1 \\ -H^2u_2+EHu_4 \\ 0 \\ -EHu_2+E^2u_4 \end{bmatrix} \qquad \text{(C-6)}$$

Finally the product $F_{\nu\lambda}F^{\nu\eta}u^\lambda u_\eta$ may be written as $\left(F_{\nu\lambda}u^\lambda\right)^2$ since $F_{\nu\lambda}u^\lambda = -F^{\nu\lambda}u_\lambda$ so that $\left(F_{\nu\lambda}F^{\nu\eta}u^\lambda u_\eta\right)\cdot u_\mu$ becomes after using (C-4):

$$\left(F_{\nu\lambda}u^\lambda\right)^2 u_\mu = \left[\left(Hu_2+Eu_4\right)^2+H^2u_1^2+E^2u_1^2\right] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \qquad \text{(C-7)}$$

Now all that is needed is to choose the appropriate components from (C-4), (C-6), and (C-7) and substitute them into (C-1). The first order Lorentz-Dirac equation may be written in component form as:

$$\dot{u}_1 = \frac{e}{mc}\left(Hu_2+Eu_1\right) - \tau_o\frac{e^2}{m^2c^3}\left(E^2-H^2\right)u_1 - \tau_o\frac{e^2\gamma^2}{mc^5}\left[(Hu_2+Eu_4)^2 + H^2u_1^2+E^2u_1^2\right]u_1$$

$$\dot{u}_2 = \frac{-e}{mc}Hu_1 - \tau_o\frac{e^2}{m^2c^3}\left[EHu_2 - H^2u_2\right] - \tau_o\frac{e^2\gamma^2}{mc^5}\left[(Hu_2+Eu_4)^2 + H^2u_1^2+E^2u_1^2\right]u_2$$

$$\dot{u}_3 = -\tau_o\frac{e^2\gamma^2}{mc^5}\left[(Hu_2+Eu_4)^2+H^2u_1^2+E^2u_1^2\right]u_3$$

$$\dot{u}_4 = -\frac{e}{m}Eu_1 + \tau_o\frac{e^2}{m^2c^3}(E^2u_4-EHu_2) - \tau_o\frac{e^2\gamma^2}{mc^5}\left[(Hu_2+Eu_4)^2 + H^2u_1^2+E^2u_1^2\right] u_4.$$