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A Classification of Kantowski–Sachs and Robertson–Walker Space–Times and Its Application to Space–times Transparent to Scalar Radiation

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled

A Classification of Kantowski–Sachs and
Robertson–Walker Space–times and
Its Application to Space–times
Transparent to Scalar Radiation

submitted by Shengli Wang in partial fulfillment of the requirements for the degree of Master of Science.

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ABSTRACT

A complete classification of Kantowski–Sachs space–times with the standard metric $ds^2 = dt^2 - A^2(t)dr^2 - B^2(t)d\Omega^2$ and of Robertson–Walker space–times with the standard metric $ds^2 = H(t)[\, dt^2 - dr^2 - g^{-1}(r)d\Omega^2 \,]$ is given. The classifications are applied to the transparent Kantowski–Sachs space–times and to a subclass of transparet Robertson–Walker space–times to give a complete classification of them. Also closed forms are given for all of transparent space–times on which the Einstein tensor has two double eigenvalues.

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TABLE OF CONTENTS

CHAPTER ONE
INTRODUCTION

§ 1.1 The Equivalence Problem in General Relativity

The entire content of general relativity may be summarized as follows. Space–time is a four dimensional manifold on which there is defined a Lorentz metric g_{ab}. This metric tensor describes the gravitational field in general relativity and it is found by solving the Einstein equations

$$G_{ab} := R_{ab} - \frac{1}{2} R g_{ab} = 8\pi \, T_{ab} \tag{1.1}$$

which relate the curvature of g_{ab} to the matter distribution in space–time. In (1.1) R_{ab} is the Ricci tensor, R is the Ricci scalar, and T_{ab} is the stress–energy tensor. Since both sides of (1.1) are symmetric these form a set of ten coupled nonlinear partial differential equations in the metric and its first and second derivatives. The explicit form of g_{ab} does of course depend on the choice of coordinate system. So two metrics are equivalent, i.e., they describe the same gravitational field if and only if there is coordinate transformation

$$x^a = x^a(\tilde{x}^{a'}) \tag{1.2}$$

such that the equation

$$\tilde{g}_{a'b'} = g_{ab} \frac{\partial x^a}{\partial \tilde{x}^{a'}} \frac{\partial x^b}{\partial \tilde{x}^{b'}} \tag{1.3}$$

holds [1]. We are using the summation convention on repeated indices. We write the metrics as

$$ds^2 = g_{ab} dx^a dx^b$$

and

$$d\tilde{s}^2 = \tilde{g}_{a'b'} d\tilde{x}^{a'} d\tilde{x}^{b'}$$

and as an abbreviation for (1.3) we write

$$ds^2 = d\tilde{s}^2 \tag{1.4}$$

under the coordinate transformation (1.2)

The so–called equivalence problem [1] consists of deciding whether or not two given metrics are equivalent. It is very hard to deal with this problem by solving the set (1.3) of nonlinear first order partial differential equations even if the metrics have very simple forms. This has motivated many mathematicians and relativists to try to find some practical procedures to deal with the equivalence problem. One of these is Karlhede's procedure [2] which claims that the equivalence problem can be reduced to finding out whether the finite set of equations

$$\begin{aligned}
&R_{\alpha\beta\rho\sigma} = \tilde{R}_{\alpha\beta\rho\sigma}, \\
&R_{\alpha\beta\rho\sigma;\gamma_1} = \tilde{R}_{\alpha\beta\rho\sigma;\gamma_1}, \\
&\cdots\cdots, \\
&R_{\alpha\beta\rho\sigma;\gamma_1\ldots\gamma_m} = \tilde{R}_{\alpha\beta\rho\sigma;\gamma_1\ldots\gamma_m},
\end{aligned} \tag{1.5}$$

is consistent or not, with $m \leq \frac{1}{2} n (n+1)$, and n is the dimension of the manifold. For the best estimate of m see [12]. The scalar quantities in (1.5) are the curvature tensor and its covariant derivatives projected onto a canonically chosen frame. In this work we will not use Karlhede's procedure, but instead our method will based on an analysis of a judiciously chosen set of scalar invariants. This method turns out to successfully classify the large sets of space–times considered here, but our initial attempts to use Karlhede's method indicate that its application to *sets* of metrics would be very difficult.

The motivation for the work presented in this thesis is the desire to have as detailed a classification as is possible of the set of so–called nonscattering or transparent space–times. The term transparent space–time means that it is possible for radiation to propagate on the space–time without scattering off the curvature. It is a quite special property, but it is not as restrictive as Huyguen's principle. Ideally the classification would be so detailed that given any two nonscattering metrics their equivalence or inequivalence would be easily decided, but this degree of precision may be obtainable only within certain subclasses. The set of nonscattering metrics has a large intersection with the set of Kantowski–Sachs (K–S) space–times and also with the set of Robertson–Walker (R–W) space–times, hence a start on the classification of nonscattering space–times would be the classification of these two special subclasses. We have accomplished this by finding a classification of the whole set of K–S space–times and also the whole set of R–W space–times which does provide a solution to the equivalence problem and is easily applicable to K–S and R–W nonscattering space–times. The application to nonscattering space–times is given in Chapter four.

We have established some results on the equivalence problem for nonscattering space–times which are not K–S or R–W. These are presented in Chapter four.

§ 1.2 The Scalar Invariants

The scalar invariants which we are going to use will be denoted by R, S, T, D, and w^2 where the first four are the coefficients of the characteristic polynomial of the Ricci tensor $R^a{}_b$, i.e.,

$$p(\lambda) := \det[\lambda I - (R^b{}_a)] = \lambda^4 - R\lambda^3 + S\lambda^2 - T\lambda + D \tag{1.6}$$

and w^2 is defined by

$$w^2 = w_{abcd}w^{abcd} \tag{1.7}$$

where w_{abcd} is the Weyl tensor. It is easy to see that R is the scalar curvature.

Later one can see that there is a class of K–S space–times on which there are no nonconstant scalar invariants depending on time alone just as is true for Minkowski and de Sitter space–times.

CHAPTER TWO
KANTOWSKI–SACHS SPACE–TIMES

§ 2.1 General Information on K–S Space–Times

It has been shown [3,4,5] that the standard K–S space–times are precisely those whose metrics can be expressed in the form

$$ds^2 = dt^2 - A^2(t)dr^2 - B^2(t)d\Omega^2 \tag{2.1}$$

for some functions A and B and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$.

For this class of space–times with the standard form (2.1) the Ricci tensor is:

$$R^1{}_1 = -\frac{BA_{tt} + 2B_{tt}A}{AB} \tag{2.2}$$

$$R^2{}_2 = -\frac{BA_{tt} + 2B_tA_t}{AB} \tag{2.3}$$

$$R^3{}_3 = -\frac{BB_tA_{tt} + A(BB_{tt} + B_t^2 + 1)}{AB^2} \tag{2.4}$$

$$R^3{}_3 = R^4{}_4 \tag{2.5}$$

$$R^b{}_a = 0 \text{ if } a \neq b \tag{2.6}$$

where t subscripts denote differentiation. Since $(R^b{}_a)$ is diagonal $R^a{}_a$ are necessarily the eigenvalues of $(R^b{}_a)$ and

$$R = a + b + 2c \tag{2.7}$$

$$S = ab + 2(a + b)c + c^2 \tag{2.8}$$

$$T = 2abc + (a + b)c^2 \tag{2.9}$$

$$D = abc^2 \tag{2.10}$$

where

$$a = R^1{}_1, \quad b = R^2{}_2, \quad c = R^3{}_3 \tag{2.11}$$

Now R, S, T, and D, are all constants if and only if a, b, and c are constants since a, b, and c are the roots of (1.6) and R, S, T, and D are expressed in terms of a, b, and c.

For the sake of conveniece of discussion we set

$$d\tilde{s}^2 = d\tilde{t}^2 - \tilde{A}^2(\tilde{t})d\tilde{r}^2 - \tilde{B}^2(\tilde{t})d\tilde{\Omega}^2 \tag{2.12}$$

where $d\tilde{\Omega}^2 = d\tilde{\theta}^2 + \sin^2\tilde{\theta}d\tilde{\phi}^2$.

It is easy to see that the two metrics (2.1) and (2.12) are equivalent if

$A^2(t) = p^2\tilde{A}^2(\tilde{t}),\ \ B^2(t) = \tilde{B}^2(\tilde{t})$ with the coordinate transformation

$$\begin{aligned} \tilde{t} &= t + c_1 \\ \tilde{r} &= pr + c_2 \\ \tilde{\theta} &= \theta + n\pi \\ \tilde{\phi} &= \phi + c_3 \end{aligned} \tag{2.13}$$

where c_1 , c_2 , c_3 , p are constants, $p \neq 0$, n is an integer.

§ 2.2 K–S Space–Times with a Nonconstant Scalar Invariant Depending on Time Alone

In this section we are going to investigate the K–S space–times on which at least one of the five scalar invariants that we have defined in section 2.1 is not a constant. For such K–S space–times we have the following

Theorem 2.1. Assuming that some $M \in \{ R, S, T, D, w^2 \}$ is not constant on an interval of t and the two metrics (2.1), (2.12) are equivalent to each other, then

$$A^2(t) = p^2\tilde{A}^2(\tilde{t}) \tag{2.14}$$

$$B^2(t) = \tilde{B}^2(\tilde{t}) \tag{2.15}$$

and

$$\begin{aligned} t &= \delta\tilde{t} + c_1 \\ r &= r(\tilde{r},\tilde{\theta},\tilde{\phi}) \\ \theta &= \theta(\tilde{r},\tilde{\theta},\tilde{\phi}) \\ \phi &= \phi(\tilde{r},\tilde{\theta},\tilde{\phi}) \end{aligned} \tag{2.16}$$

where p, c_1 are constants, $p \neq 0$, $\delta = \pm 1$.

NOTE. If (2.14) and (2.15) hold for $t = \delta\tilde{t} + c_1$ then it is easy to see that the two metrics (2.1) and (2.12) are equivalent.

Proof of theorem 2.1. First we define another scalar invariant by

$$N = g^{ab}M_{,a}M_{,b} \,. \tag{2.17}$$

Since $M = M(t)$ and $g^{11} = 1$ (2.17) becomes

$$N = (M_t)^2 \,. \tag{2.18}$$

So if the two metrics are equivalent to each other one has

$$M(t) = \tilde{M}(\tilde{t}) \tag{2.19}$$

$$M_t = \tilde{M}_t \neq 0; \;\; M_{\tilde{t}} = \tilde{M}_{\tilde{t}} \neq 0 \tag{2.20}$$

$$N(t) = \tilde{N}(\tilde{t}) \,. \tag{2.21}$$

Equations (2.19) and (2.20) imply

$$t = f(\tilde{t}) \,. \tag{2.22}$$

By (2.18), (2.21), (2.22) one has

$$(M_t)^2 = (\tilde{M}_{\tilde{t}})^2 = (\tilde{M}_t)^2\left[\frac{dt}{d\tilde{t}}\right]^2 \tag{2.23}$$

and this implies

$$\left[\frac{dt}{d\tilde{t}}\right]^2 = 1 \,. \tag{2.24}$$

Hence we conclude

$$t = \pm\, \tilde{t} + c_1 \tag{2.25}$$

and

$$dt^2 = d\tilde{t}^2 \tag{2.26}$$

where c_1 is a constant. If we put $(t, r, \theta, \phi) = (x^1, x^2, x^3, x^4)$ by the equivalence condition (1.3) and (2.24) one finds

$$0 = A^2\left[\frac{\partial r}{\partial \tilde{t}}\right]^2 + B^2\left[\frac{\partial \theta}{\partial \tilde{t}}\right]^2 + B^2\sin^2\theta\left[\frac{\partial \phi}{\partial \tilde{t}}\right]^2 . \tag{2.27}$$

Combined with (2.25), (2.27) means that the possible coordinate transformation is of the form

$$
\begin{aligned}
t &= \delta\tilde{t} + c_1 \\
r &= r(\tilde{r},\tilde{\theta},\tilde{\phi}) \\
\theta &= \theta(\tilde{r},\tilde{\theta},\tilde{\phi}) \\
\phi &= \phi(\tilde{r},\tilde{\theta},\tilde{\phi})
\end{aligned}
\tag{2.28}
$$

where $\delta = \pm 1$. That the two metrics are equivalent means

$$ds^2 = d\tilde{s}^2 \tag{2.29}$$

under the coordinate transformation (2.28). With (2.26), (2.28), (2.29) we get

$$ds_1^2(t) = A^2(t)dr^2 + B^2(t)d\Omega^2 \tag{2.30}$$

is equal to

$$d\tilde{s}_1^2(\tilde{t}) = \tilde{A}^2(\tilde{t})d\tilde{r}^2 + \tilde{B}^2(\tilde{t})d\tilde{\Omega}^2 \tag{2.31}$$

under (2.28), which means that the metrics (2.30) and (2.31) are equivalent on the three space under (2.28). Since the scalar curvatures for $ds_1^2(t)$ and $d\tilde{s}_1^2(\tilde{t})$ are $\frac{2}{B^2(t)}$ and $\frac{2}{\tilde{B}^2(\tilde{t})}$ respctively we have

$$B^2(t) = \tilde{B}^2(\tilde{t}) \tag{2.32}$$

under (2.25). This implies

$$BB_t = \tilde{B}\tilde{B}_t \tag{2.33}$$

i.e.,

$$BB_t = \tilde{B}\tilde{B}_{\tilde{t}}\frac{d\tilde{t}}{dt} = \delta\tilde{B}\tilde{B}_{\tilde{t}} \tag{2.34}$$

hence

$$\frac{B_t}{B} = \delta\frac{\tilde{B}_{\tilde{t}}}{\tilde{B}} \tag{2.35}$$

and

$$\left(\frac{B_t}{B}\right)^2 = \left(\frac{\tilde{B}_{\tilde{t}}}{\tilde{B}}\right)^2 . \tag{2.36}$$

Also $(B^2)_{tt} = (\tilde{B}^2)_{tt}$ implies

$$B_t^2 + BB_{tt} = \tilde{B}_{\tilde{t}}^2 + \tilde{B}\tilde{B}_{\tilde{t}\tilde{t}} . \tag{2.37}$$

Using (2.32) and (2.36) one has

$$\frac{B_{tt}}{B} = \frac{\tilde{B}_{\tilde{t}\tilde{t}}}{\tilde{B}} . \tag{2.38}$$

Since

$$-\frac{1}{2} R = \frac{A_{tt}}{A} + 2\frac{A_t B_t}{AB} + \frac{2BB_{tt} + B_t^2 + 1}{B^2} \tag{2.39}$$

with (2.32), (2.36), (2.38) and (2.39) we have

$$\frac{A_{tt}}{A} + 2\frac{A_t B_t}{AB} = \frac{\tilde{A}_{\tilde{t}\tilde{t}}}{\tilde{A}} + 2\frac{\tilde{A}_{\tilde{t}}\tilde{B}_{\tilde{t}}}{\tilde{A}\tilde{B}} . \tag{2.40}$$

On the other hand since $R^1{}_1$ is an eigenvalue of $(R^a{}_b)$ we must have $R^1{}_1 = \tilde{R}^1{}_1$ under the special transformation (2.28), so

$$\frac{A_{tt}}{A} + 2\frac{B_{tt}}{B} = \frac{\tilde{A}_{\tilde{t}\tilde{t}}}{\tilde{A}} + 2\frac{\tilde{B}_{\tilde{t}\tilde{t}}}{\tilde{B}} . \tag{2.41}$$

Combining (2.38),(2.40) and (2.41) one has

$$\frac{A_t B_t}{AB} = \frac{\tilde{A}_{\tilde{t}}\tilde{B}_{\tilde{t}}}{\tilde{A}\tilde{B}} . \tag{2.42}$$

So here are two cases: $B \neq$ constant and $B =$ constant.

CASE 1. $B \neq$ constant.

In this case (2.42) implies

$$\frac{A_t}{A} = \delta\,\frac{\tilde{A}_{\tilde{t}}}{\tilde{A}} = \frac{\tilde{A}_t}{\tilde{A}} \tag{2.43}$$

because of (2.35). Eq. (2.43) implies

$$A^2(t) = p^2\tilde{A}^2(\tilde{t}) \tag{2.44}$$

where $p \neq 0$ is a constant. The theorem is thus established for this case.

CASE 2. $B =$ constant.

In this case we are going to use the relation

$$w_{a'b'c'd'} = w_{abcd}\frac{\partial x^a}{\partial \tilde{x}^{a'}}\frac{\partial x^b}{\partial \tilde{x}^{b'}}\frac{\partial x^c}{\partial \tilde{x}^{c'}}\frac{\partial x^d}{\partial \tilde{x}^{d'}}\,. \tag{2.45}$$

For K–S space–time with $B =$ constant in (2.1) we have

$$\begin{aligned} w_{1212} &= \frac{A^2}{3}\,\sigma \\ w_{1313} &= -\frac{B^2}{6}\,\sigma \\ w_{1414} &= -\frac{B^2}{6}\sin^2\theta\,\sigma \\ w_{1a1b} &= 0 \quad \text{if } a \neq b \end{aligned} \tag{2.46}$$

where $\sigma = \dfrac{A_{tt}}{A} + \dfrac{1}{B^2} \neq 0$ otherwise the five scalars must be constants by (2.2) – (2.6). So for this case one has

$$\tilde{w}_{1212} = w_{1a1a}\left[\frac{\partial x^a}{\partial \tilde{x}^2}\right]^2 \delta^2 \tag{2.47}$$

i.e.,

$$\tilde{A}^2\tilde{\sigma} = A^2\left[\frac{\partial r}{\partial \tilde{r}}\right]^2\sigma - \frac{B^2}{2}\omega\sigma$$

where $\omega = \left[\frac{\partial\theta}{\partial\tilde{r}}\right]^2 + \sin^2\theta\left[\frac{\partial\phi}{\partial\tilde{r}}\right]^2$. By (2.32) and (2.40) we have $\tilde{\sigma} = \sigma \neq 0$, hence

$$\tilde{A}^2 = A^2\left[\frac{\partial r}{\partial \tilde{r}}\right]^2 - \frac{B^2}{2}\omega\,. \tag{2.48}$$

On the other hand by the equivalence condition (1.3) we have

$$\tilde{A}^2 = A^2\left[\frac{\partial r}{\partial \tilde{r}}\right]^2 + B^2\omega \tag{2.49}$$

and comparing this with (2.48) one can see that $\omega = 0$ and that (2.49) is reduced to

$$\tilde{A}^2 = A^2\left[\frac{\partial r}{\partial \tilde{r}}\right]^2. \tag{2.50}$$

This and (2.28) imply that $\frac{\partial r}{\partial \tilde{r}}$ is a constant, so we have $A^2(t) = p^2\,\tilde{A}^2(\tilde{t})$ under (2.25). We have completed the proof.

It will be very easy to use this theorem to test whether two metrics (2.1) and (2.12) are equivalent or not when at least one of the five scalar invariants is not constant. The analysis will be complete if we can find all functions of A(t) and B(t) such that the five scalar invariants are all constants. We are going to do this in the next section.

§ 2.3 K–S Space–Times with the Five Scalar Invariants Being Constants

In this section we assume that the five scalar invariants are all constants. As we mentioned before in section 2.1 in this case we have that a, b, and c are constants. For the metric (2.1) the conformal invariant is

$$w^2 = \frac{1}{3}\frac{\sigma^2}{A^2B^4} \tag{2.51}$$

where

$$\sigma = B^2A_{tt} - BB_tA_t + A(-BB_{tt} + B_t^2 + 1)\ . \tag{2.52}$$

The metric (2.1) is conformally flat if and only if $\sigma = 0$, i.e., $w^2 = 0$, because the Weyl tensor vanishes if and only if w^2 vanishes. Without losing generality we suppose that A, $\sigma \geq 0$ so that $\rho = \sqrt{3w^2} = \dfrac{\sigma}{AB^2}$ is also a scalar invariant. By (2.2), (2.3), (2.4), and (2.11) we have

$$\frac{1}{B^2}(\,2BB_{tt} + B_t^2 + 1) = k \tag{2.53}$$

where $k = \frac{1}{2}(\,b - a - 2c\,)$ is a constant, hence $2BB_{tt} + B_t^2 + 1 - kB^2 = 0$. Multiplying on both sides by B_t we have $2BB_tB_{tt} + B_t^3 + B_t - kB^2B_t = 0$ and $(B_t^2B)_t + B_t - \frac{k}{3}(B^3)_t = 0$ it follows that $B_t^2B + B - \frac{k}{3}B^3 = d =$ constant or

$$\frac{B_t^2\ + 1}{B^2} = \frac{k}{3} + \frac{d}{B^3}\,. \tag{2.54}$$

Combining this with (2.53) we have

$$\frac{B_{tt}}{B} = \frac{k}{3} - \frac{d}{2B^3}\,. \tag{2.55}$$

Also since $\rho - k = \dfrac{1}{AB^2}(B^2A_{tt} - BB_tA_t - 3ABB_{tt}) =$ constant by (2.3) we have

$$-b - 3\left(\frac{A_tB_t}{AB} + \frac{B_{tt}}{B}\right) = \rho - k$$

hence

$$\frac{A_tB_t}{AB} + \frac{B_{tt}}{B} = \text{constant}$$

and

$$\frac{A_tB_t}{AB} - \frac{B_{tt}}{B} = \frac{a-b}{2} = \text{constant}\ .$$

Therefore

$$\frac{B_{tt}}{B} = \text{constant}\ .$$

By (2.55) we have

$$\left[\frac{B_{tt}}{B}\right]_t = \frac{3dB_t}{2B^4} = 0\ . \tag{2.56}$$

So there are two cases: $d = 0$, and $B =$ constant.

CASE 1. $d = 0$.

We will show that in this case the metric is conformally flat and is given by $A =$ constant or $A = \sinh(qt+c_2)$, $B = \frac{1}{q}\cosh(qt+c_1)$ where q, c_1 and c_2 are constants, $q \neq 0$.

From (2.54) and (2.55) we have

$$\frac{B_{tt}}{B} = \frac{B_t^2 + 1}{B^2} = \frac{k}{3} = q^2 > 0\ . \tag{2.57}$$

The unique solution of (2.57) is

$$B = \frac{1}{q}\cosh x \tag{2.58}$$

where $x = qt + c_1$, and q, and c are constants, $q \neq 0$. Since

$$\frac{1}{2}(a-b)+q^2=\frac{A_t B_t}{AB}$$

we have

$$\frac{A_t}{A}=\alpha\frac{B}{B_t}=\frac{\alpha}{q}\coth x \tag{2.59}$$

where

$$\alpha = q^2+\frac{1}{2}(a-b)\,. \tag{2.60}$$

From (2.2) and (2.57) we have $\dfrac{A_{tt}}{A}=-a-2q^2$, hence

$$\left[\frac{A_t}{A}\right]_t=-a-2q^2-\left[\frac{A_t}{A}\right]^2$$

which implies

$$\alpha\,\mathrm{sech}^2 x = a+2q^2+\frac{\alpha^2}{q^2}\coth^2 x$$

by (2.59). Therefore we have

$$\alpha\left(1-\frac{\alpha}{q^2}\right)\mathrm{sech}^2 x=\frac{\alpha^2}{q^2}+a+2q^2 \tag{2.61}$$

which implies either (i) $\alpha = 0$, or (ii) $\alpha = q^2$.

(i) $\alpha = 0$ implies $A =$ constant by (2.59) and $a + 2q^2 = 0$ by (2.61), this combined with (2.60) and $q^2 = \frac{k}{3} = \frac{1}{6}(b - a - 2c)$ yields

$$a = c = -2q^2, \ b = 0,$$

hence $R = -6q^2$, $D = 0$. Using (2.57) and A = constant in (2.52) gives $\sigma = 0$ which means that this class of space–times is conformally flat.

(ii) $\alpha = q^2$ implies $a = b$ by (2.60), hence $c = -k = -3q^2$. From (2.61) we have $a = -3q^2$ so

$$a = b = c = -3q^2, \quad R = -12q^2, \quad D = (3q^2)^4 \tag{2.62}$$

and from (2.59) we have $A = \sinh(qt + c_2)$, $B = \frac{1}{q}\cosh(qt + c_1)$. This implies $\sigma = 0$ and from (2.62) we have that $R_{ab} = \frac{1}{4} R\, g_{ab}$ therefore this class of space–times is de Sitter [6], while (i) is not since $a \neq b = 0$. We see that both can be labeled by q since the scalar curvatures are $-6q^2$ and $-12q^2$ respectively.

CASE 2. B = constant

In this case we have

$$a = b = -\frac{A_{tt}}{A} \text{ and } c = -\frac{1}{B^2} \tag{2.63}$$

by (2.2), (2.3) and (2.4). We distinguish several subcases.

(i) $a = 0$ implies that $A_{tt} = 0$ hence $A = c_1 + c_2 t$. Using the coordinate transformatiom

$$\tilde{t} = t \cosh r$$
$$\tilde{r} = t \sinh r$$
$$\tilde{\theta} = \theta$$
$$\tilde{\phi} = \phi$$

we have

$$d\tilde{t}^2 - d\tilde{r}^2 - B^2 d\tilde{\Omega}^2 = dt^2 - t\, dr^2 - B^2 d\Omega^2$$

and

$$R = -\frac{2}{B^2}\,, \quad T = D = 0\,, \quad \sigma = A = c_1 + c_2 t\,.$$

So this class of space–times can be labeled by the nonzero constant B , and is not conformally flat.

(ii) $a = -\dfrac{A_{tt}}{A} = -q^2 \neq 0$

Then $A_{tt} - q^2 A = 0$ hence $A = c_1 e^{qt} + c_2 e^{-qt}$

In this case we have the three following classes of space–times which are equivalent.

1) $c_1 = 0,\ c_2 \neq 0$ or $c_2 = 0,\ c_1 \neq 0$

It is clear by the discussion in section 2.1 and the coordinate transformation $t = -\tilde{t}$ that the K–S space–times with $A = c_1 e^{qt}$; $A = c_2 e^{-qt}$; $A = e^{qt}$ and $B =$ constant represent the same metric in different coordinates. We denote this metric by

$$ds_a^2 = dt^2 - e^{2qt} dr^2 - B^2 d\Omega^2\,. \tag{2.64}$$

2) $c_1 c_2 > 0$.

In this case we have

$$A = 2c_1 \cosh(\,qt + \alpha\,) \sim A = \cosh qt \tag{2.65}$$

where $e^{\alpha} = (\sqrt{c})^{-1}$, $c = \dfrac{c_2}{c_1}$ and "$\sim$" means that two A's with the same constant B represent the same metric. We denote this metric by

$$ds_b^2 = dt^2 - \cosh^2 qt\, dr^2 - B^2 d\Omega^2\,. \tag{2.66}$$

3) $c_1 c_2 < 0$

In this case we have

$$A = 2c_1 \sinh(qt + \alpha) \sim A = \sinh qt \tag{2.67}$$

where $e^{\alpha} = (\sqrt{c})^{-1}$,$c = -\dfrac{c_2}{c_1}$. We denote this metric by

$$ds_c^2 = dt^2 - \sinh^2 qt dr^2 - B^2 d\Omega^2 \,. \tag{2.68}$$

Next we will show that these three metrics in 1) 2) 3) are equivalent to each other. By putting $\tau = \dfrac{1}{q} e^{qt}$,the metric (2.64) becomes

$$ds_a^2 = \frac{1}{q^2\tau^2} (d\tau^2 - dr^2) - B^2 d\Omega^2 \tag{2.69}$$

and then by using null coordinates

$$\begin{aligned} \tau &= u + v \\ r &= u - v \end{aligned} \tag{2.70}$$

(2.70) becomes

$$ds_a^2 = \frac{4du\,dv}{q^2(u + v)^2} - B^2 d\Omega \,. \tag{2.71}$$

If we put $d\tau = \dfrac{dt}{\cosh qt}$ we have $e^{qt} = \tan\frac{q\tau}{2}$, then

$$\cosh qt = \frac{1}{2} (\tan\frac{q\tau}{2} + \mathrm{ctn}\frac{q\tau}{2}) = \frac{1}{\sin qt} \,.$$

Hence (2.66) becomes

$$ds_b^2 = \frac{1}{\sin^2 qt} (d\tau^2 - dr^2) - B^2 d\Omega^2 \,.$$

Using the coordinate transformation

$$\begin{aligned} \tau' &= qt \\ r' &= qr \end{aligned} \tag{2.72}$$

we have

$$ds_b^2 = \frac{1}{q^2 \sin^2\tau} (d\tau^2 - dr^2) - B^2 d\Omega^2$$

(after dropping primes on the coordinates). Hence we have

$$ds_b^2 = \frac{4dudv}{q^2 \sin^2(u+v)} - B^2 d\Omega^2 \tag{2.73}$$

by using null coordinates (2.70).

For the metric (2.68) if we use the transformation $d\tau = \frac{dt}{\sinh qt}$ we have that $\tau = \frac{1}{q} \ln (\tanh \frac{qt}{2})$, hence $e^{q\tau} = \tanh \frac{qt}{2}$ and then

$$\sinh q\tau = \frac{1}{2} (\tanh \frac{qt}{2} - \operatorname{ctnh} \frac{qt}{2}) = - \frac{1}{\sinh qt} .$$

Hence (2.68) becomes

$$ds_c^2 = \frac{1}{\sinh^2 qt} (d\tau^2 - dr^2) - B^2 d\Omega^2 .$$

Using (2.72) and then (2.70) we have

$$ds_c^2 = \frac{4dudv}{q^2 \sinh^2(u+v)} - B^2 d\Omega^2 . \tag{2.74}$$

It is easy to confirm that the transformations

$$u = \tan u' , \quad v = \tan v'$$

$$u = \tanh u' , \quad v = \tanh v'$$

carry the metric ds_a^2 with form (2.71) to the metrics ds_b^2 and ds_c^2 with the forms (2.73) and (2.74) respectively. Therefore we have that ds_a^2 , ds_b^2 , ds_c^2 , are the same metric in different coordinates. The five scalar invariants of this class are as follows:

$$
\begin{aligned}
R &= -2\left(q^2 + \frac{1}{B^2} \right) \\
S &= q^4 + \frac{4q^2}{B^2} + \frac{1}{B^4} \\
T &= -2\frac{q^2}{B^2}\left(q^2 + \frac{1}{B^2} \right) \\
D &= \frac{q^4}{B^4} \\
w^2 &= \frac{1}{3}\left(q^2 + \frac{1}{B^2} \right).
\end{aligned}
\tag{2.75}
$$

We see that all these five scalar invariants are symmetric with respect to q^2 and $\frac{1}{B^2}$ but we have the following

Theorem 2.2. The K–S space–times with $A = e^{qt} \sim A = e^{-qt} \sim A = \sinh qt \sim A = \cosh qt$, $B =$ constant can be labeled by the ordered pair (q^2, B^{-2}).

We only need to prove the following

Lemma 2.1 The two metrics

$$ds^2 = dt^2 - \sinh^2 qt \; dr^2 - B^2 d\Omega^2 \tag{2.76}$$

$$d\tilde{s}^2 = d\tilde{t}^2 - \sinh^2 p\tilde{t} \; d\tilde{r}^2 - G^2 d\tilde{\Omega}^2 \tag{2.77}$$

are equivalent if and only if $q^2 = p^2$ and $B^2 = G^2$ where q, p, B, and G are constants, and B, and G are not zero.

Proof. By $R = \tilde{R}$ and $S = \tilde{S}$ in (2.75) we have

$$q^2 + \frac{1}{B^2} = p^2 + \frac{1}{G^2}$$

and

$$\frac{q^2}{B^2} = \frac{p^2}{G^2}.$$

Solving these equations we have either $q^2 = p^2$, hence $B^2 = G^2$, or $q^2 = \frac{1}{G^2}$, hence $p^2 = \frac{1}{B^2}$. We will prove that in the latter case we must have $q^2 = p^2 = \frac{1}{B^2} = \frac{1}{G^2}$

In this case we have

$$ds^2 = dt^2 - \sinh^2 qt\, dr^2 - B^2 d\Omega^2 \tag{2.78}$$

and

$$d\tilde{s}^2 = d\tilde{t}^2 - \sinh^2 \frac{\tilde{t}}{B}\, d\tilde{r}^2 - \frac{1}{q^2}\, d\tilde{\Omega}^2 \tag{2.79}$$

are equivalent under a transformation. Since the Ricci tensors of (2.78) and (2.79) are

$$\begin{aligned} (R^b{}_a) &= \mathrm{diag}\,(-q^2, -q^2, -\frac{1}{B^2}, -\frac{1}{B^2}) \\ (\tilde{R}^b{}_a) &= \mathrm{diag}\,(-\frac{1}{B^2}, -\frac{1}{B^2}, -q^2, -q^2) \end{aligned} \tag{2.80}$$

and by assumption we have

$$P\,(R^b{}_a)\,P^{-1} = (\tilde{R}^b{}_a) \tag{2.81}$$

where P is the Jacobi matrix of the transformation, if $q^2 \neq \frac{1}{B^2}$ then (2.81) would force the transformation to have the form

$$\begin{aligned} t &= t\,(\tilde{\theta},\tilde{\phi}) \\ r &= r\,(\tilde{\theta},\tilde{\phi}) \\ \theta &= \theta\,(\tilde{t},\tilde{r}) \\ \phi &= \phi\,(\tilde{t},\tilde{r})\,. \end{aligned}$$

This implies that the metric $dt^2 - \sinh^2 qt\, dr^2$ is equivalent to the metric $-\frac{1}{q^2} d\tilde{\Omega}^2$. But this is impossible because they have different signatures. Therefore we must have $q^2 = \frac{1}{B^2}$.

(iii) $-\frac{A_{tt}}{A} = q^2 \neq 0, \quad B = \text{constant}.$

In this case we have $A = c_1 \cos qt + c_2 \sin qt$.

1) $c_1 = 0,\ c_2 \neq 0$ or $c_2 = 0,\ c_1 \neq 0$.

By the transformation $qt = qt' + \frac{\pi}{2}$, $A = c_2 \sin qt \sim A = \sin qt \sim A = \cos qt$

2) $c_1 c_2 \neq 0$. We have $A \sim \sin (qt \pm \alpha) \sim \sin qt$ where $\tan\alpha = c$, $c = |\frac{c_1}{c_2}|$. The Ricci tensor is $(R^b{}_a) = \mathrm{diag} (q, q, -\frac{1}{B^2}, -\frac{1}{B^2})$. The five scalar invariants are:

$$R = 2 (q - \frac{1}{B^2})$$
$$S = (q^2 - \frac{1}{B^2})^2 - \frac{2q^2}{B^2}$$
$$T = \frac{2q^2}{B^2} (\frac{1}{B^2} - q^2)$$
$$D = \frac{q^4}{B^4}$$
$$w^2 = \frac{1}{3} (q^2 - \frac{1}{B^2})^2 .$$

So we can labeled this class of space–times by the ordered pair $(q^2, \frac{1}{B^2})$. The case (iii) cannot be equivalent to case (ii) since the Ricci tensors have different eigenvalues, unless $q = 0$ which results in case (i).

NOTE. For case (iii) if $q^2 - \frac{1}{B^2} = 0$ then $w^2 = 0$, $R = 0$, but the metrics are not de Sitter ones since $R_{ab} \neq \frac{1}{4} R g_{ab} = 0$ and $R_{3434} = -\frac{1}{B^2} \sin^2\theta \neq 0$.

§ 2.4 A Complete Classification of K–S Space–Times

Based on the discussion in section 2.2 and 2.3 for the K–S space–times (2.1) we have the following classification:

(1) The five scalar invariants are all constants:

I. $A^2 = p^2$, $B^2 = \frac{1}{a^2} \cosh^2 at$, where p can be any nonzero constant.

$$(R^a{}_b) = \mathrm{diag}\,(-2a^2, 0, -2a^2, -2a^2)$$

$$R = -6a^2$$

$$S = 12a^4$$

$$T = -8a^6$$

$$D = 0$$

$$w^2 = 0 .$$

The space–times are conformally flat and are labelled by the constant a.

II. $A^2 = p^2 \sinh^2 at$, $B^2 = \cosh^2 at$, where p can be any nonzero constant.

$$(R^b{}_a) = \mathrm{diag}\,(-3a^2, -3a^2, -3a^2, -3a^2)$$

$$R = -12a^2$$

$$S = 54a^4$$

$$T = -108a^6$$

$$D = 81a^8$$

$$w^2 = 0$$

$$R_{ab} = \frac{1}{4} R g_{ab} .$$

The space–times are conformally flat and are labelled by the constant a, they are de Sitter space–times.

III. $A^2 = p_1^2 e^{-2at} \sim A^2 = p_2^2 e^{2at} \sim A^2 = p_3^2 \cosh^2 at \sim A^2 = p_4^2 \sinh^2 at$, $B = \frac{1}{b}$ = constant , where p_1^2, p_2^2, p_3^2, and p_4^2 are any nonzero constants.

$$(R^b{}_a) = \mathrm{diag}\,(-a^2, -a^2, -b^2, -b^2)$$

$$R = -2\,(a^2 + b^2)$$

$$S = (a^2 + b^2)^2 + 2a^2b^2$$

$$T = -2a^2b^2(a^2 + b^2)$$

$$D = a^4b^4$$

$$w^2 = \frac{1}{3}(a^2 + b^2)^2 .$$

The space–times are labelled by the ordered pair (a^2, b^2) and are not conformally flat.

IV. $A^2 = p_1^2 \cos^2 at \sim A^2 = p_2^2 \sin^2 at$, $B = \frac{1}{b}$ = constant, where p_1 and p_2 are any nonzero constants.

$$(R^b{}_a) = \mathrm{diag}\,(a^2, a^2, -b^2, -b^2)$$

$$R = 2\,(a^2 - b^2)$$

$$S = (a^2 - b^2)^2 - 4a^2b^2$$

$T = -2a^2b^2(a^2 - b^2)$

$D = a^4b^4$

$w^2 = \frac{1}{3}(a^2 - b^2)^2$

$a^2 = b^2$ if and only if $w^2 = R = 0$.

The space–times are different from I and II because of different scalar cuvatures and different eigenvalues of the Ricci tensors. They are labelled by the ordered pair (a^2, b^2) .

V. $= \mathrm{III} \cap \mathrm{IV}$ $(a = 0)$ $A = p_1 t \sim A = p_2$, $B = \frac{1}{b} =$ constant , p_1 and p_2 are any nonzero constants.

$(R^b{}_a) = \mathrm{diag}(0, 0, -b^2, -b^2)$

$R = -2b^2$

$S = b^4$

$T = D = 0$

$w = \frac{b^4}{3}$.

The space–times are labelled by the constant b.

(2) At least one of the five scalar invariants is not a constant.

If $B^2(t) \neq \tilde{B}^2(\tilde{t})$ under $t = \delta\tilde{t} + c_1$, where $\delta = \pm 1$ and $c_1 =$ constant, or if there is no constant p such that $A^2(t) = p^2\tilde{A}^2(\tilde{t})$ then the two metrics (2.1) and (2.12) are inequivalent.

NOTE. From (2) we can see, as we mentioned in the section 1.2 , that there are no nonconstant scalar invariants depending on time t alone in class III since $\cosh^2 at \neq c_1^2 \sinh^2 a(\delta t + c)$ for any constants c and c_1.

CHAPTER THREE
ROBERTSON–WALKER SPACE–TIMES

§ 3.1 General Information on R–W Space–Times

It has been shown that the R–W space–times are precisely those whose metrics can be expressed in the form

$$ds^2 = H(t)\,[\,dt^2 - dr^2 - g^{-1}(r)\,d\Omega^2] \tag{3.1}$$

for the functions $H(t) > 0$ and

$$g^{-1}(r) = \begin{cases} r^2 & k = 0 \\ \sin^2 r & k = 1 \\ \sinh^2 r & k = -1 \end{cases} \tag{3.2}$$

$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$. Each g in (3.2) satisfies

$$g g_{rr} - g_r^2 - 2g^3 = 0 \tag{3.3}$$

and

$$\frac{4g^3 - g_r^2}{g^2} = 4k\,. \tag{3.4}$$

The scalar curvature and Ricci tensor are as follows:

$$R = -\frac{3}{2H^3}(2HH_{tt} - H_t^2 + 4kH^2) \tag{3.5}$$

$$R^1{}_1 = -\frac{3}{2H^3}(HH_{tt} - H_t^2)$$

$$R^2{}_2 = -\frac{g^2H_{tt} + H(2gg_{rr} - 3g_r^2)}{2g^2H^2}$$

$$R^3{}_3 = R^4{}_4 = -\frac{g^2H_{tt} + H(gg_{rr} - 2g_r^2 + 2g^3)}{2g^2H^2}$$

$$R^b{}_a = 0\,, \text{ if } a \neq b\,.$$

Using (3.3), (3.4), and (3.5) we have

$$R^1{}_1 = R + \frac{3}{2}H_k \tag{3.6}$$

$$R^2{}_2 = R^3{}_3 = R^4{}_4 = -\frac{1}{2}H_k \tag{3.7}$$

where

$$H_k = \frac{H_{tt} + 4kH}{H^2}\,. \tag{3.8}$$

We see that the four scalar invariants are all functions of t alone.

R–W space–times are conformally flat, hence $w^2 = 0$, in addition $(R^b{}_a) = \mathrm{diag}(a, b, b, b,)$ where $a = R^1{}_1$, $b = R^2{}_2 = R^3{}_3 = R^4{}_4$, hence a and b must be the eigenvalues of the Ricci tensor. If R, S, T, and D are constants then a and b must be constants and hence H_k is also a constant.

§ 3.2 A Complete Classification of R–W Space–Times

For the convenience of discussion we set

$$d\tilde{s}^2 = \tilde{H}(\tilde{t})\,[\,d\tilde{t}^2 - d\tilde{r}^2 - \tilde{g}^{-1}(\tilde{r})\,d\tilde{\Omega}^2] \tag{3.9}$$

where $d\tilde{\Omega}^2 = d\tilde{\theta}^2 + \sin^2\tilde{\theta}^2 d\tilde{\varphi}^2$.

For the metric (3.1) one would like to ask whether we can specify the space–times by the ordered pair (H(t) , k) or not . The answer is no since the de Sitter and Minkowski space–times are the counterexamples [7]. The question then becomes whether we can specify the non–de Sitter R–W space–times with standard form (3.1) by the ordered pair (H(t) , k). The answer is almost yes based on the following

__Theorem 3.1__ Assume that two metrics (3.1) and (3.9) which are not de Sitter space–times are equivalent. Then $k = \tilde{k}$, $p^2H(t) = \tilde{H}(\tilde{t})$, and $t = p\tilde{t} + c$, where p and c are constants, and $p \neq 0$. Moreover if $k \neq 0$ then $p = \pm 1$.

The theorem is based on the following lemmas .

__Lemma 3.1.__ If two metrics (3.1) and (3.9) are equivalent and some $M \in \{ R, S, T, D \}$ is not a constant then

$$\left[\frac{d\tilde{t}}{dt}\right]^2 = \frac{H}{\tilde{H}} \tag{3.10}$$

$$\frac{k}{H} = \frac{\tilde{k}}{\tilde{H}} . \tag{3.11}$$

If in addition $k = 0$, then $\dfrac{\tilde{H}}{H} = p^2 =$ constant and $t = p\tilde{t} + c$.

Since k and $\tilde{k}$ can only be 0 or ± 1 and H and $\tilde{H}$ are positive, (3.11) implies $k = \tilde{k}$. Then if $k \neq 0$ we must have $H = \tilde{H}$ and $t = \delta\tilde{t} + c$, where c is a constant and $\delta = \pm 1$.

To make the classification complete one would like to find all R–W space–times for which all four scalar invariants are constants. To answer this question we have the following

Lemma 3.2. If the four scalar invariants of the metric are all constants then either H = constant or (3.1) are de Sitter space–times.

NOTE. In the case of H = constant we know that $R = -\frac{6k}{H}$ from (3.5) hence if $k \neq 0$ we have $H = \tilde{H}$ by $R = \tilde{R}$ and if $k = 0$ the metric is just Minkowski space–time for any constant H so theorem 3.1 is still valid for this case.

Proof of Lemma 3.1.

Eq.(3.10) is established by an argument similar to that which proved (2.24). By the equivalence condition (1.3) we conclude the coordinate transformation has the form

$$\begin{aligned} \tilde{t} &= \tilde{t}(t) \\ \tilde{r} &= \tilde{r}(r,\theta,\varphi) \\ \tilde{\theta} &= \tilde{\theta}(r,\theta,\varphi) \\ \tilde{\varphi} &= \tilde{\varphi}(r,\theta,\varphi) \,. \end{aligned} \tag{3.12}$$

Under this transformation we have

$$ds^2 = d\tilde{s}^2 \tag{3.13}$$

and combining this with (3.10) we have

$$ds_1^2 = d\tilde{s}_1^2 \tag{3.14}$$

under (3.12) , where

$$ds_1^2 = H(t)\,[\,dr^2 + g^{-1}(r)\,d\Omega^2] \tag{3.15}$$

$$d\tilde{s}_1^2 = \tilde{H}(\tilde{t})\,[\,d\tilde{r}^2 + \tilde{g}^{-1}(\tilde{r})\,d\tilde{\Omega}^2]\,. \tag{3.16}$$

Eq.(3.14) means that the two metrics (3.15) and (3.16) are equivalent on the three space. Since the scalar curvatures of (3.15) and (3.16) are

$$\frac{1}{2g^2H}(4gg_{rr} - 7g_r^2 + 4g^3) = \frac{6k}{H} \tag{3.17}$$

and

$$\frac{1}{2\tilde{g}^2\tilde{H}}(4\tilde{g}\tilde{g}_{\tilde{r}\tilde{r}} - 7\tilde{g}_{\tilde{r}}^2 + 4\tilde{g}^3) = \frac{6\tilde{k}}{\tilde{H}} \tag{3.18}$$

respectively and they must equal to each other at corresponding points we have obtained (3.11) . If $k = 0$, in cartesian coordinates(3.13) becomes

$$H(t)\,[\,dt^2 - dx^2 - dy^2 - dz^2] = \tilde{H}(\tilde{t})\,[\,d\tilde{t}^2 - d\tilde{x}^2 - d\tilde{y}^2 - d\tilde{z}^2]$$

under the coordinate transformation

$$\begin{aligned} t &= t(\tilde{t}) \\ x &= x(\tilde{x},\tilde{y},\tilde{z}) \\ y &= y(\tilde{x},\tilde{y},\tilde{z}) \\ z &= z(\tilde{x},\tilde{y},\tilde{z})\,. \end{aligned}$$

With this form and the equivalence condition (1.3) we have

$$H(t) = \tilde{H}(\tilde{t})\left(\left[\frac{\partial\tilde{x}}{\partial x}\right]^2 + \left[\frac{\partial\tilde{y}}{\partial y}\right]^2 + \left[\frac{\partial\tilde{z}}{\partial z}\right]^2\right)$$

which implies $\frac{H}{\tilde{H}} =$ constant. We have completed the proof of Lemma 3.1.

Proof of Lemma 3.2. Assuming that the four scalar invariants are all constants we know that H_k is also a constant as mentioned before in section 3.1. From (3.5) and (3.8) we have

$$6HH_{tt} - 3H_t^2 + 12kH^2 + 2RH^3 = 0 \tag{3.19}$$

and

$$H_{tt} + 4kH - H^2H_k = 0 \tag{3.20}$$

i.e.,

$$6HH_{tt} + 24kH^2 - 6H^3H_k = 0 \,. \tag{3.21}$$

Combining (3.19) and (3.21) one has

$$3H_t^2 + 12kH^2 - 2H^3 \,(\, R + 3H_k \,) = 0 \,. \tag{3.22}$$

On the other hand multiplying the both sides of (3.20) by H_t we have

$$H_{tt}H_t + 4kHH_t - H_kH^2H_t = 0$$

i.e.,

$$\frac{1}{2}\,(H_t^2)_t + 2k\,(H^2)_t - \frac{1}{3}\,H_k\,(H^3)_t = 0$$

hence we have

$$3H_t^2 + 12kH^2 - 2H_kH^3 = \text{constant} \,. \tag{3.23}$$

Combining (3.22) and (3.23) We have

$$H^3 (R + 2H_k) = \text{constant} \tag{3.24}$$

which means that either $H = \text{constant}$ or $R + 2H_k = 0$. In the case of $R + 2H_k = 0$ we have that $R^1{}_1 = R^2{}_2 = R^3{}_3 = R^4{}_4 = \frac{1}{4} R$ by (3.6) and (3.7) and hence we have

$$R_{ab} = \frac{1}{4} R\, g_{ab} \,. \tag{3.25}$$

Combined with $w_{abcd} = 0$ (3.25) means that the metric is de Sitter [6]. We have completed the proof of Lemma 3.2.

NOTE. From the proof we have seen that $R = -2H_k = \text{constant}$ is equivalent to the condition that all four invarinats are constants.

Next we will find all functions H(t) for each k in the metric (3.1) such that the four scalar invariants are all constants. So we have to solve

$$R = -2H_k = \text{constant} \,. \tag{3.26}$$

Once we have done this it will be very easy to tell whether two metrics (3.1) and (3.9) are equivalent or not just by checking whether we have $(p^2H(t) , k) = (\tilde{H}(\tilde{t}) , \tilde{k})$ under the coordinate transformation $t = pt + c$ except for functions which satisfy (3.26). It is also very easy to apply the results to the transparent space–times in the next section.

Using the expressions (3.5) and (3.8) for R and H_k respectively, $R + 2H_k = 0$ becomes $2HH_{tt} - 3H_t^2 - 4kH^2 = 0$ or equivalently

$$2\,\frac{HH_{tt} - H_t^2}{H^2} - \left[\frac{H_t}{H}\right]^2 - 4k = 0 \,.$$

We can rewrite this as

$$2\left[\frac{H_t}{H}\right]_t = \left[\frac{H_t}{H}\right]^2 + 4k\,. \tag{3.27}$$

Combining this with $R = -2H_k =$ constant we have the following results:

1)

(i) $k = 0$ $H(t) = \dfrac{c}{(t+c_1)^2} \sim H(t) = \dfrac{c}{t^2}$.

(ii) $k = 1$ $H(t) = \dfrac{c}{\sin^2(t+c_1)} \sim H(t) = \dfrac{c}{\sin^2 t}$.

(iii) $k = -1$ $H(t) = \dfrac{c}{\sinh^2(t+c_1)} \sim H(t) = \dfrac{c}{\sinh^2 t}$.

It has been shown that the metrics (3.1) with (i) , (ii) , (iii) , respectively represent the same space–times, namely the de Sitter ones [7]. For this kind of space–times the scalar curvature $R = -\dfrac{12}{c}$

2)

(i) $k = 0$ $H(t) =$ constant $\sim H(t) = 1$.

(ii) $k = 1$ $H(t) = \dfrac{c}{\cos^2(t+c_1)} \sim H(t) = \dfrac{c}{\sin^2 t}$.

(iii) $k = -1$ $H(t) = 4ce^{2t} \sim H(t) = 4ce^{-2t} \sim H(t) = 4e^{2t}$.

The metrics (i) or (iii) are equivalent to Minkowski space–time [7] while (ii) are de Sitter ones again.

NOTE. $H(t) = \dfrac{c}{\cosh^2(t+c_1)}$ is another solution of (3.27) with $k = -1$ but it is not a solution of H_k = constant.

Based on the above discussion we have the following complete classification of R–W space–times:

(1) R , H_k are constants.

I. $H(t) = p^2 =$ constant , $k = \pm 1$.

$$R = -\frac{6k}{p^2}$$

$$(R^b{}_a) = \text{diag}\left(0 , -\frac{2k}{p^2} , -\frac{2k}{p^2} , -\frac{2k}{p^2} \right) .$$

The space–times are labelled by the pair (p^2 , k).

II. Minkowski space–time

($H(t)$, k) can be any one of (p^2, 0), ($4ce^{2t}$, -1), ($4ce^{-2t}$, -1)

where $p \neq 0$, c are constants.

III. de Sitter (and anti–de Sitter) space–times.

($H(t)$, k) can be any one of $\left(\frac{c}{t^2} , 0 \right)$, $\left(\frac{c}{\sin^2 t} , 1 \right)$, $\left(\frac{c}{\sinh^2 t} , -1 \right)$

$$R = -\frac{12}{c},$$

$(R^b{}_a) = \text{diag}\left(-\frac{3}{c} , -\frac{3}{c} , -\frac{3}{c} , -\frac{3}{c} \right)$, where $c \neq 0$ is a constant ,

The space–times are labelled by the constant c .

(2) At least one of R and H_k is not a constant.

$$(R^b{}_a) = \mathrm{diag}\,(R + \tfrac{3}{2} H_k \,, -\tfrac{1}{2} H_k \,, -\tfrac{1}{2} H_k \,, -\tfrac{1}{2} H_k) \,.$$

The space–times are labelled by the ordered pair ($p^2H(t)$, k) under the coordinate transformation $t = p\tilde{t} + c$, where c is a constant and if $k \neq 0$ $p^2 = 1$, and if $k = 0$ p can be any nonzero constant.

NOTE. From (2) we can see, as we mentioned in section 1.2 that there are no nonconstant scalar invariants depending on time t alone in the de Sitter space–times (Minkowski space–time is the special case $R = 0$) because these space–times have two or three expressions, i.e., different pairs (H(t) , k) which do not satisfy the conditions in (2).

CHAPTER FOUR

APPLICATION TO TRANSPARENT SPACE–TIMES

§ 4.1 Discussion of Classification of the Transparent Space–Times

In 1986 and 1988 Dr. W.E. Couch and Dr. R.J. Torrence [8,9,10] presented a large class of spherically symmetric space–times which are transparent to scalar multipole waves in the same sense that flat space–time is. In this section we prove some results on the classification of transparent space–times. These results will show that the set of inequivalent transparent space–times is very large. Similarly the complete classifications of transparent K–S and R–W metrics given in sections 4.2 and 4.3 show that those classes of inequivalent transparent space–times are also large.

In general, transparent space–times have metrics with the form

$$ds^2 = H(t,r)\,[\,dt^2 - dr^2 - g^{-1}(t,r)\,d\Omega^2] \tag{4.1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$. A scalar field Ψ obeys the covariant wave equation $g^{ab}\Psi_{;ab} = 0$ on the background (4.1). Upon separation of variables by $\Psi = \chi^{-1}S(r,t)\,Y_{\tilde{\ell}m}(\theta,\varphi)$ the equation governing S is

$$S_{rr} - S_{tt} - V\,S - \tilde{\ell}(\tilde{\ell} + 1)\,g\,S = 0 \tag{4.2}$$

where

$$\chi_{rr} - \chi_{tt} - V\,\chi = 0 \tag{4.3}$$

and

$$\chi^2 = H/g. \tag{4.4}$$

The space–times are transparent for special choices of the functions H(t,r) and g(t,r) which make (4.2) solvable in the sense defined in [8] and [9]. The class is large, being labelled by several functions of one variable and several real valued parameters.

The metric (4.1) is necessarily spherically symmetric and is conformally flat if and only if

$$C = gg_{rr} - g_r^2 - 2g^3 - gg_{tt} + g_t^2 = 0 \tag{4.5}$$

where t and r subscripts denote partial differentiation. The scalar curvature of (4.1) is given by

$$R = \frac{6}{g\chi^3}\left(\chi_{rr} - \chi_{tt} + \frac{C\chi}{6\,g^2} \right) = \frac{6\,V}{H} + \frac{C}{g^3\chi^2} \tag{4.6}$$

If we use null coordinates $u = \frac{1}{2}\,(t+r)$ and $v = \frac{1}{2}\,(t-r)$ then (4.3) becomes

$$\chi_{uv} + V\chi = 0\,. \tag{4.3$'$}$$

If V and g are given such that (4.2) is a nonscattering equation and χ is determined by the requirement that it be a solution of (4.3) then from (4.4) we obtain an H, therefore we have a transparent space–time with the form (4.1). For the choices of $g(t,r)$ and $V(t,r)$ we refer reader to [8], [9], and [10].

To start the discussion we set

$$I = \{ N, M, D, \ldots \} \tag{4.7}$$

where the elements in the set I are scalar invariants which are functions of t and r. In this case all of the five scalar invariants defined in section 1.2 are functions of t and r. Let the matrix A be defined by

$$A = \begin{pmatrix} N_t & M_t & D_t \ldots \\ N_r & M_r & D_r \ldots \end{pmatrix} \tag{4.8}$$

and let

$$d\tilde{s}^2 = \tilde{H}(\tilde{t},\tilde{r}) \, [\, d\tilde{t}^2 - d\tilde{r}^2 \, \tilde{g}^{-1}(\tilde{t},\tilde{r}) \, d\tilde{\Omega}^2] \, . \tag{4.9}$$

Then we have the following

<u>Theorem 4.1</u> If the two metrics (4.1) and (4.9) are equivalent and the rank $(A) = 2$ then the coordinate transformation has the form

$$\begin{aligned} t &= t(\tilde{t},\tilde{r}) \\ r &= r(\tilde{t},\tilde{r}) \\ \theta &= \theta(\tilde{\theta},\tilde{\varphi}) \\ \varphi &= \varphi(\tilde{\theta},\tilde{\varphi}) \, . \end{aligned} \tag{4.10}$$

Consequently we have

$$\chi^2 = \tilde{\chi}^2 \tag{4.11}$$

under the coordinate transformation (4.10), and if we use null coordinates we have

$$\begin{aligned} u &= u(\tilde{u}) \\ v &= v(\tilde{v}) \end{aligned} \tag{4.12}$$

and

$$(\, H, V, g\,) = \frac{d\tilde{u}}{du}\frac{d\tilde{v}}{dv}(\, \tilde{H}, \tilde{V}, \tilde{g}\,) \tag{4.13}$$

under (4.12).

<u>Proof.</u> Assuming that rank (A) = 2 then there are at least two scalar invariants, say N and M, in I such that

$$\det \begin{pmatrix} N_t & M_t \\ N_r & M_r \end{pmatrix} \neq 0\, . \tag{4.14}$$

Differentiate

$$\begin{aligned} N(t,r) &= \tilde{N}(\tilde{t},\tilde{r}) \\ M(t,r) &= \tilde{M}(\tilde{t},\tilde{r}) \end{aligned} \tag{4.15}$$

on both sides with respect to $\tilde{\theta}$, we have

$$\begin{aligned} N_t \frac{\partial t}{\partial \tilde{\theta}} + N_r \frac{\partial t}{\partial \tilde{\theta}} &= 0 \\ M_t \frac{\partial t}{\partial \tilde{\theta}} + M_r \frac{\partial r}{\partial \tilde{\theta}} &= 0\, . \end{aligned} \tag{4.16}$$

By (4.14), Eq.(4.16) imply

$$\frac{\partial t}{\partial \tilde{\theta}} = \frac{\partial r}{\partial \tilde{\theta}} = 0 \,. \tag{4.17}$$

Similarly we can show

$$\frac{\partial t}{\partial \tilde{\varphi}} = \frac{\partial r}{\partial \tilde{\varphi}} = 0 \,. \tag{4.18}$$

By the equivalence condition (1.3) we obtain

$$\frac{\partial \theta}{\partial \tilde{t}} = \frac{\partial \theta}{\partial \tilde{r}} = \frac{\partial \varphi}{\partial \tilde{t}} = \frac{\partial \varphi}{\partial \tilde{r}} = 0 \,. \tag{4.19}$$

Eqs.(4.17), (4.18) and (4.19) imply (4.10). By (4.10) we have

$$H\,(\,dt^2 - dr^2) = \tilde{H}\,(\,d\tilde{t}^{\,2} - d\tilde{r}^{\,2}) \tag{4.20}$$

and

$$\chi^2 d\Omega^2 = \tilde{\chi}^2 d\tilde{\Omega}^2 \tag{4.21}$$

under (4.10). The scalar curvatures for the metrics $\chi^2 d\Omega^2$ and $\tilde{\chi}^2 d\tilde{\Omega}^2$ are $\dfrac{2}{\chi^2}$ and $\dfrac{2}{\tilde{\chi}^2}$ respectively, hence we have obtained (4.11). If we use null coordinates (4.20) becomes

$$H du dv = \tilde{H} d\tilde{u} d\tilde{v} \,. \tag{4.22}$$

Using the equivalence condition (1.3) on (4.22) we have either

$$\begin{aligned} u &= u(\tilde{u}) \\ v &= v(\tilde{v}) \end{aligned} \tag{4.23}$$

or

$$u = u(\tilde{v}) \tag{4.24}$$

$$v = v(\tilde{u}) .$$

Without loosing generality we suppose that (4.23) holds. Then from (4.22) we have

$$H = \tilde{H} \frac{d\tilde{u}}{du} \frac{d\tilde{v}}{dv} . \tag{4.25}$$

By (4.11) we have

$$g = \tilde{g} \frac{d\tilde{u}}{du} \frac{d\tilde{v}}{dv} . \tag{4.26}$$

Using (4.11) and (4.23) $\tilde{\chi}_{\tilde{u}\tilde{v}} + \tilde{V}\tilde{\chi} = 0$ is equivalent to

$$\chi_{uv} + \tilde{V} \frac{d\tilde{u}}{du} \frac{d\tilde{v}}{dv} \chi = 0 . \tag{4.27}$$

Comparing this with (4.3′) we have

$$V = \tilde{V} \frac{d\tilde{u}}{du} \frac{d\tilde{v}}{dv} . \tag{4.28}$$

We have now proved (4.13) and theorem 4.1.

We now apply this result to transparent space–times. By (4.13) we have

$$\frac{V}{H} = \frac{\tilde{V}}{\tilde{H}} . \tag{4.29}$$

Combining this with $R = \dfrac{6V}{H} + \dfrac{C}{g^3 \chi^2}$ we have

$$\frac{C}{g^3} = \frac{\tilde{C}}{\tilde{g}^3} . \tag{4.30}$$

Multiplying (4.29) on both sides of (4.20) we have

$$V\ (\ dt^2 - dr^2) = \tilde{V}\ (\ d\tilde{t}^2 - d\tilde{r}^2)\ . \tag{4.31}$$

The scalar curvature of (4.31) is very important, e.g., if we choose one of the functions in (4.45) as h in V then the scalar curvature of (4.31) is $\frac{2}{\ell(\ell+1)}$. This means that different ℓ with the same g and h will generate different space–times. For any choice of $V \neq 0$ in [8] and [9] the scalar curvature of (4.31) is a constant in terms of ℓ, ℓ' etc., so this gives a restriction to the equivalent transparent space–times.

We have seen that the general situation is complicated. So far we have not classified the class of space–times of (4.1) completely. We have not found the closed form of space–times with the condition of rank (A) < 2, and secondly we have not proved that (4.12) and (4.13) are linear. But we still can use this method to test whether the two given concrete transparent space–times are equivalent or not by checking the consistency of (4.11), (4.13) and (4.30) for all solutions of (4.12) satisfying (4.26) or (4.28).

§ 4.2 Application to Transparent K–S Space–times

In this section we apply the results from section 2.4 to obtain a complete classification of transparent K–S space–times.

Using the coordinate transformation

$$dt = \frac{d\tau}{A(\tau)} \tag{4.32}$$

the K–S metric

$$ds^2 = d\tau^2 - A^2(\tau)\, dr^2 - B^2(\tau)\, d\Omega^2 \tag{4.33}$$

is carried to

$$ds^2 = H(t)\,[\, dt^2 - dr^2 - g^{-1}(t)\, d\Omega^2] \tag{4.34}$$

where

$$H(t) = A^2(\tau) \tag{4.35}$$

$$g(t) = \frac{A^2(\tau)}{B^2(\tau)}. \tag{4.36}$$

Hence

$$\chi(t) = B(\tau) \tag{4.37}$$

under (4.32).

If the metric (4.34) is transparent then [10]

$$g(t) = \frac{1}{\cosh^2 t} \tag{4.38}$$

$$V(t) = -\frac{\ell(\ell+1)}{\sinh^2 t}. \tag{4.39}$$

Since g(t) in (4.38) satisfies

$$C = g_t^2 - g g_{tt} - 2g^3 = 0\,. \tag{4.40}$$

the transparent K–S psace–times are conformally flat. By (4.37), (4.5) becomes

$$B_{tt} + V\,B = 0\,. \tag{4.41}$$

In this case the scalar curvature of (4.34) is

$$R = \frac{6\,V}{H} = \frac{6\,V}{A^2} \tag{4.42}$$

(1) The five scalar invariants are constants.

From section 2.4 for the conformally flat space–times we have

(i) $A^2 = p^2 =$ constant, $B^2 = \frac{1}{a^2}\cosh^2 a\tau$.

By (4.32) we have $\tau = pt + c$. But B does not satisfy (4.41), so this is not a transparent space–time.

(ii) $A^2 = p^2\sinh^2 a\tau$, $B^2 = \cosh^2 a\tau$.

By (4.38), (4.39) and (4.41) we have $a = 1$ and $\ell = 1$ and $B(\tau) = \frac{\cosh t}{\sinh t}$. This is a transparent space–time.

(iii) $A^2 = p^2\cos^2 a\tau$, $B = \frac{1}{a} =$ constant. These are transparent space–times.

By (4.38), (4.39) and (4.41) we have $ap = 1$, $\ell = 0$.

(2) At least one of the five scalar invariants is not a constant.

By (4.42) and Theorem 2.1 we have

$$V(t) = p^2\,\tilde{V}(\tilde{t}) \tag{4.43}$$

under $\tilde{t} = \delta t + c$ which comes from (4.32). From (4.43) we have $p^2 = 1$ and $c = 0$, hence we have $\ell = \tilde{\ell} \geq 2$.

Let $B_1(t)$ and $B_2(t)$ denote two linearly independent solutions of (4.41) for an integer $\ell \geq 2$ then $B^2(t) = [\,c_1B_1(t) + c_2B_2(t)\,]^2$. By the theorem 2.1 it is easy to see that we can specify this subclass of space–times by $\ell \times (c_1, c_2)$ with the redundancies $\ell \times (c_1, c_2) = \ell \times (-c_1, -c_2)$ for $\ell \geq 2$.

The results presented above provide the following scheme which decides the equivalence or otherwise of any two nonflat transparent K–S metrics defined by (4.38) , (4.39) and (4.41). The space–times are inequivalent if and only if one of the following holds:

(a) $\ell \neq \tilde{\ell}$

(b) $\ell = \tilde{\ell}$, $(c_1, c_2) \neq (\tilde{c}_1, \tilde{c}_2)$ except for the redundancies specified in (2).

§ 4.3 Application to Transparent R–W Space–times

A subclass of the transparent space–times which is within the class of R–W space–times is defined by the condition that H(t) in the metric (3.1) satisfies

$$(\sqrt{H})_{tt} + [\, k + V(a,t)\,]\sqrt{H} = 0 \tag{4.44}$$

where $V(a,t) = \ell(\ell+1)h(a,t)$, ℓ is a nonegative integer and h(a,t) is one of the four functions

$$\left\{ -\frac{1}{t^2}, -\frac{1}{a^2\sin^2(t/a)}, -\frac{1}{a^2\sinh^2(t/a)}, \frac{1}{a^2\cosh^2(t/a)} \right\} \tag{4.45}$$

where a is a nonzero constant. For the k = 0 case a may be set equal to one by a scale transformation on t in (4.44) hence for that case (4.44) becomes

$$(\sqrt{H})_{tt} + V(1,t)\sqrt{H} = 0$$

so we can rewrite (4.44) as

$$(\sqrt{H})_{tt} + [\, k + V(1+ak,t)]\sqrt{H} = 0\,. \tag{4.46}$$

The scalar curvature of this subclass is

$$R = \frac{6V}{H}. \tag{4.47}$$

We now give the solution to equivalence problem for these transparent R–W space–times in terms of the quantities k, ℓ, h(a,t), and H(t) which generate them. All these space–times are contained in the following three cases.

(1) $\ell = 0$ ($\Longrightarrow V = R = 0$).

$$(R^b{}_a) = \mathrm{diag}\left(\tfrac{3}{2} H_k , -\tfrac{1}{2} H_k , -\tfrac{1}{2} H_k , -\tfrac{1}{2} H_k \right) \tag{4.48}$$

(i) $k = 0$.

From (4.46) we have

$H(t) = (c_1 t + c_2)^2 \sim H(t) = ct^2$ if $c_1 \neq 0$

$H_k = \dfrac{2c_1}{(c_1 t + c_2)^2} \neq 0 = -\frac{1}{2} R$, if $c_1 \neq 0$.

If $c_1 = 0$ the metric (3.1) becomes Minkowski.

(ii) $k = 1$.

Eq.(4.46) becomes $(\sqrt{H}\,)_{tt} + \sqrt{H} = 0$, hence

$H(t) = (c_1 \cos t + c_2 \sin t)^2 \sim H(t) \doteq c \sin^2(t+\alpha) \sim H(t) = c \sin^2 t$

$H_k = \dfrac{2}{c \sin^4 t} \neq$ constant .

(iii) $k = -1$.

Eq.(4.46) becomes $(\sqrt{H}\,)_{tt} - \sqrt{H} = 0$ and we have three cases:

a) $H(t) = ce^{2t} \sim H(t) = ce^{-2t}$.

$H_k = 0$, this is Minkowski space–time .

b) $H(t) = c\sinh^2(t+\alpha) \sim H(t) = c\sinh^2 t$.

$$H_k = \frac{2}{c\sinh^4 t} \neq \text{constant} .$$

c) $H(t) = c\cosh^2(t+\alpha) \sim H(t) = c\cosh^2 t$.

$$H_k = \frac{2}{c\cosh^4 t} \neq \text{constant} .$$

Apart from the Minkowski metric here we have just shown that all the space–times above have a nonconstant H_k and the transformation $t = p\tilde{t} + c$ cannot carry one to another. So they are all different space–times.

(2) $R = -2\,H_k = \text{constant}$, and $\ell \neq 0$.

Since for the Minkowski metric we must have $\ell = 0$ by the discussion in section 3.2 we know that in this case we only have de Sitter space–times.

(i) $(\,H(t)\,,\,k\,) = (\,\frac{c}{t^2}\,,\,0\,)$.

Substituting this in (4.46) we have $\ell = 1$ and $h = -\frac{1}{t^2}$

(ii) $(\,H(t)\,,\,k\,) = (\,\frac{c}{\sin^2 t}\,,\,1\,)$.

Substituting this in (4.46) we have $\ell = 1$ and $h = -\frac{1}{\sin^2 t}$

(iii) $(\,H(t)\,,\,k\,) = (\,\frac{c}{\sinh^2 t}\,,\,-1\,)$.

Substituting this in (4.46) we have

$\ell = 1,\ h = -\frac{c}{\sinh^2 t}$.

In this case ℓ can only be 1, and besides de Sitter ones we also have other transparent space–times, e.g., $\sqrt{H(t)} = t^2$ for $V = -\frac{2}{t^2}$ and $k = 0$ [8] .

(3) $\ell \neq 0$ and at least one of R and H_k is not a constant.

In this case by application of theorem 3.1 we have that if the two metrics (3.1) and (3.9) are equivalent then

$$p^2H(t) = \tilde{H}(\tilde{t}), \;\; k = \tilde{k} \tag{4.49}$$

and

$$t = p\tilde{t} + c \, . \tag{4.50}$$

By $R = \tilde{R}$ and (3.31) we have

$$p^2V(1+ak,t) = \tilde{V}(1+\tilde{a}k,\tilde{t}) \tag{4.51}$$

under (4.50). Since (4.50) cannot carry one function to another in (4.45), (4.51) implies that V and $\tilde{V}$ must be the same function in (4.45) and $\ell = \tilde{\ell}$; furthermore $p^2 = 1$ if $h \neq -\frac{1}{t^2}$, and $c = n\pi(1+ak)$ for $h = \frac{-1}{(1+ak)^2\sin^2[t/(1+ak)]}$ while $c = 0$ for the others, where n is an integer. Hence (4.50) becomes $t = \pm\, t + c$. Let $H_1(t)$ and $H_2(t)$ denote two linearly independent solutions of (4.46) for a fixed triple (ℓ,h,k), then $H(t) = (\, c_1H_1 + c_2H_2)^2$. It is easy to see that we can identify this subclass of space–times by $(\,\ell\,,h\,,k\,)\times(\,c_1\,,c_2\,)$ with the redundancies $(\,\ell\,,h\,,k\,)\times(\,c_1\,,c_2\,) = (\,\ell\,,h\,,k\,)\times(\,-c_1\,,-c_2\,)$ for $h \neq -\frac{1}{t^2}$. In the case of $h = -\frac{1}{t^2}$, (4.44) gives $H_1(t) = t^n$ and $H_2(t) = t^m$ where $n = \frac{1}{2}(\,1 + \sqrt{1+4\ell(\ell+1)}\,)$; $m = \frac{1}{2}(\,1 - \sqrt{1+4\ell(\ell+1)}\,)$, hence in this case the redundancies are $(\,\ell\,,-\frac{1}{t^2}\,,0\,)\times(\,c_1\,,c_2\,) = (\,\ell\,,-\frac{1}{t^2}\,,0\,)\times(\,p^{n+1}c_1\,,p^{m+1}c_2\,)$.

The results presented above provide the following scheme which decides the equivalence or otherwise of any two nonflat transparent R–W metrics defined by (4.44) and (4.45). The space–times are inequivalent if and only if one of the following holds:

(a) $\ell \neq \tilde{\ell}$.

(b) $\ell = \tilde{\ell} = 0$ or 1, the distinctions listed in (1) or (2) are satisfied.

(c) $\ell = \tilde{\ell} \geq 2$, $k \neq \tilde{k}$.

(d) $\ell = \tilde{\ell} \geq 2$, $k = \tilde{k}$, h and $\tilde{h}$ are different ones of the four functions in (4.45).

(e) $\ell = \tilde{\ell} \geq 2$, $k = \tilde{k}$, h and $\tilde{h}$ are the same, $(c_1, c_2) \neq (\tilde{c}_1, \tilde{c}_2)$ except for the redundancies specified in (3).

A similar analysis will apply to a more general class of nonscattering V's based on the results in [10].

§ 4.4 The Conformally Flat Transparent Space–Times with the Einstein Tensor Having Two Double Eigenvalues

In this section we give closed forms for all of the conformally flat transparent space–times with two double eigenvalues and h being any one of the functions in (4.45).

The Einstein tensor of the metric (4.1) has the form

$$(G^b{}_a) = \begin{pmatrix} a & -k & 0 & 0 \\ k & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{pmatrix} . \tag{4.52}$$

This shows that the Einstein tensor of the metric always has one double eigenvalue which we denote by c in (4.52). In fact the spherically symmetric Einstein tensor always has one double eigenvalue [11] .

If the metric (4.1) is conformally flat we can always write it as

$$H(t,r)\,[\,dt^2 - dr^2 - r^2 d\Omega^2]$$

or

$$H(u,v)\,[\,4\,du\,dv - (u-v)^2 d\Omega^2]. \tag{4.53}$$

In this case we have

$$a = \frac{1}{4H}(-4\,\alpha_{rr} - \alpha_r^2 + 3\,\alpha_t^2\,) - \frac{4\alpha_r}{rH} \tag{4.54}$$

$$b = \frac{1}{4H}(\,4\,\alpha_{tt} + \alpha_t^2 - 3\,\alpha_r^2) - \frac{4\alpha_r}{rH} \tag{4.55}$$

$$k = \frac{1}{2H}(\,2\,\alpha_{tr} - \alpha_t\alpha_r) \tag{4.56}$$

where

$$H = e^{\alpha}. \tag{4.57}$$

The condition for the metric (4.1) having two double eigenvalues is $(\,a - b\,)^2 = 4\,k^2$ which is equivalent to

$$(\,a - b - 2\,k\,)\,H = 0 \tag{4.58}$$

or

$$(\,a - b + 2\,k\,)\,H = 0\,. \tag{4.59}$$

Substituting (4.54), (4.55) and (4.56) into (4.58) and (4.59) respectively we have

$$(\,\alpha_t + \alpha_r\,)_t + (\,\alpha_t + \alpha_r\,)_r = \frac{1}{2}(\,\alpha_t + \alpha_r\,)^2 \tag{4.60}$$

$$(\,\alpha_t - \alpha_r\,)_t - (\,\alpha_t - \alpha_r\,)_r = \frac{1}{2}(\,\alpha_t - \alpha_r\,)^2\,. \tag{4.61}$$

Setting

$$\alpha_t + \alpha_r = \delta\,e^{\sigma} \tag{4.62}$$

where $\delta = \pm 1$. Then (4.60) becomes

$$\delta(\sigma_t + \sigma_r) = -e^{\sigma}. \tag{4.63}$$

Using null coordinates (4.63) becomes

$$\delta\,\sigma_u = \frac{1}{2}e^{\sigma}. \tag{4.64}$$

Solving (4.64) we have

$$\delta e^{\sigma} = \alpha_t + \alpha_r = \alpha_u = -\frac{2}{u + \phi(v)}. \tag{4.65}$$

Solving this we have $\alpha = \ln \dfrac{\Psi^2(v)}{(u + \phi(v))^2}$, hence

$$H = e^{\alpha} = \frac{\Psi^2(v)}{(u + \phi(v))^2} \tag{4.66}$$

where $\Psi(v)$ and $\phi(v)$ are functions of v. Similarly from (4.61) we have

$$H = e^{\alpha} = \frac{\Psi^2(u)}{(v + \phi(u))^2}. \tag{4.67}$$

In fact (4.66) and (4.67) will generate the same metric (4.53) by renaming u and v. Note that if $\alpha_t + \alpha_r = 0$ in (4.60) or $\alpha_t - \alpha_r = 0$ in (4.61) we have

$$H = \Psi^2(v) \tag{4.68}$$

and

$$H = \Psi^2(u). \tag{4.69}$$

For H in (4.67) we have

$$\chi = \frac{\Psi(u)\,(u-v)}{v + \phi(u)}. \tag{4.70}$$

Next we will find that for which functions $\Psi(u)$ and $\phi(u)$ (4.53) are transparent space–times, that is, χ satisfies (4.6). It is easy to calculate

$$\frac{\chi_{uv}}{\chi} = -\frac{k(u)\ (\phi + v) + f(u)}{(u-v)\ (\phi + v)^2} \tag{4.71}$$

where

$$k(u) = \frac{\Psi'(u)}{\Psi(u)}(\phi(u) + u) + (\phi'(u) + 1) \tag{4.72}$$

$$f(u) = -2\,\phi'(u)\,(\phi(u) + u)\,. \tag{4.73}$$

By (4.6) we have

$$\frac{k(u)\,(\phi(u)\ + v) + f(u)}{V} = (u - v)\,(\phi(u) + v)^2\,. \tag{4.74}$$

Now we look at some examples of $V = \ell(\ell+1)h$ where h is one of the functions in (4.45). Inserting $h = -\frac{1}{t^2}$ into (4.74) reorganizing the both sides of (4.74) to be polynomials in v with the coefficients being functions of u, then comparing the the coefficients on the both sides of (4.74) we obtain that $\phi(u) = u$ and $\Psi(u) = c\,u^{\alpha}$ where c and α are constants and $\alpha = \frac{\ell(\ell+1) - 2}{2}$, hence we have

$$H = \frac{c^2 u^{2\alpha}}{(u + v)^2} \sim H = \frac{u^{2\alpha}}{(u + v)} \tag{4.75}$$

under the transformation $c\,u^{\alpha} = \tilde{u}^{\alpha}$, $c\,v^{\alpha} = \tilde{v}^{\alpha}$. Note that $\ell = 1$ if and only if $\alpha = 0$ and $H = \frac{c^2}{(u + v)^2}$, these are just de Sitter space–times and in this case the above transformation is not valid.

Inserting the $h \neq -\frac{1}{t^2}$ will give us $\ell = 0$, hence $f(u) = k(u) = 0$ by $f(u) = 0$ we have two cases: $\phi(u) = c =$ constant and $\phi(u) = -u$

(i) $\phi(u) = c$.

By $k(u) = 0$ we have $\Psi(u) = \frac{a}{u + c}$ where a is a constant, hence

$$H = \frac{a^2}{(u + c)^2(v + c)^2} \sim H = \frac{1}{u^2v^2} \tag{4.76}$$

by the transformation $u = a\,\tilde{u}$, $v = a\,\tilde{v}$. It is easy to see that this is Minkowski space–time since using the transformation $\tilde{u} = \frac{1}{u}$, $\tilde{v} = \frac{1}{v}$ carries $H = 1$ in (4.53) with H in (4.76).

(ii) $\phi(u) = -u$.

By $k(u) = 0$ we have that $\Psi(u)$ can be any function of u, hence

$$H = \frac{\Psi^2(u)}{(u - v)^2}\ . \tag{5.77}$$

The Einstein tensor of this space–time has the form

$G^1{}_1 = G^2{}_2 = -G^3{}_3 = -G^4{}_4 = \Psi^{-2}(u)$,
$G^1{}_2 = -\frac{1}{\Psi^4}(v - u)\,(v\,\Psi\,\Psi_{uu} - 2\,\Psi_u^2\,v - \Psi\,\Psi_{uu}\,u + 2\,\Psi_u^2\,u - 2\,\Psi\,\Psi_u)$, $G^b{}_a = 0$, if $a \neq b$.

For $H = \Psi^2(u)$, $\chi = \Psi(u)\,(u - v)$. Then

$$-\frac{\chi_{uv}}{\chi} = \frac{\Psi'(u)}{\Psi(u)\ (u - v)} = V\ . \tag{5.78}$$

This forces $\ell = 0$ and $\Psi(u) =$ constant, this is Minkowski space–time again.

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