2018-08-09

Difference population equation with the variable Allee effect and periodic carrying capacity

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Chugunova M. A. (2018). Difference population equation with the variable Allee effect and periodic carrying capacity (Unpublished master's thesis). University of Calgary, Calgary, AB. doi:10.11575/PRISM/32816
<http://hdl.handle.net/1880/107636>

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Difference population equation with the variable Allee effect and periodic carrying capacity

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE
DEGREE OF MASTER OF SCIENCE

GRADUATE PROGRAM IN MATHEMATICS AND STATISTICS

CALGARY, ALBERTA

AUGUST, 2018

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Abstract

The subject of this research is the location, stability, and the basin of attraction for the equilibrium points and periodic solutions for the Ricker equation enhanced with the mechanism for both weak and strong types of the Allee effect and periodic carrying capacity.

The research has shown that just an addition of one biological phenomenon to Ricker equation, namely, the Allee effect, significantly changes the necessary conditions for the equilibrium point of the equation to obtain stability. This also increases the range of the parameters which allow for the equilibrium solution or a cycle to remain globally asymptotically stable. It was shown that the p -periodic carrying capacity always generates the p -periodic solutions which can be global attractors.

Embedding the mechanism for the strong Allee effect generates several basins of attraction with either an equilibrium point or a stable cycle as an attractor. Concavity was proven not necessary for the attenuant property of the cycle.

Preface

This thesis is an original work by the author. No part of this thesis has been previously published.

Acknowledgements

I would like to express my deepest gratitude to my scientific supervisor, Dr. Larry Bates, for showing me the true level of scientific inquiry.

My heartfelt thanks go to Dr. Yuriy Zinchenko and Dr. Antony Ware for the timely advice and support.

To my son Maxim who always believes in me.

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List of Symbols, Abbreviations and Nomenclature

Symbol or abbreviation	Definition
$\mathbb{N}_0$	set $\{0, 1, 2, 3, \dots\}$
$\mathbb{R}$	all real numbers
$\mathbb{R}^+$	all positive real numbers
$\mathbb{R}_0^+$	all positive real numbers and zero
x	population size or density
$\{x(n)\}$	solution for the population equation
$\{\tilde{x}(n)\}$	periodic solution for the population equation
x^*	equilibrium point
r	per capita growth rate
K	carrying capacity
$\{K(n)\}$	strictly positive sequence
K_*	minimum value of the carrying capacity in a sequence $\{K(n)\}$
K^*	maximum value of the carrying capacity in a sequence $\{K(n)\}$
A	Allee threshold
$\tilde{A}$	the pre-image of the Allee threshold
C	auxiliary parameter for the Allee effect model
α	$\frac{K - A}{K + C}$
$\beta(n)$	$1 - \frac{(A + C)(K(n) + C)}{(\tilde{x}(n) + C)(x(n) + C)}$

Chapter 1

Introduction

The research subject of this thesis is the difference population equation developed from the Ricker equation to include the mechanisms of such biological phenomena as the Allee effect and the fluctuating properties of the environment, namely,

$$x(n+1) = x(n) \exp \left(r \left(1 - \frac{x(n)}{K(n)} \right) \left(1 - \frac{A+C}{x(n)+C} \right) \right), \quad n \in \mathbb{N}_0, \quad (1.0.1)$$

where $x(n)$ is the size or density of the single species population, r is the intrinsic per capita growth rate, $K(n)$ is the periodic carrying capacity, and the parameters A and C are chosen to correlate with the type of the Allee effect.

In this chapter, sections 1.1 and 1.2, the biological phenomena of the Allee effect and the periodic carrying capacity are introduced as the basis for the mathematical population model (1.0.1). The particular choice of the population equation and the present high demand in Mathematical Biology for the similar type of the equations are justified in section 1.3. The layout of the thesis is explained and the main results of the research are stated in section 1.4. The meaning and the range of all the parameters in the equation (1.0.1) are also introduced in this chapter.

1.1 The Allee effect.

The development of the population dynamics branch of Mathematical Biology is driven by the demand of the current state of industry, agriculture, and natural resources. For many decades, the classical view on the relation between the density (or the size) of the population, $x(n)$, and the individual fitness has been accepted as the negative dependence. That is, because of the competition for resources, the individual fitness decreases with increase of the population density.

Warder Clyde Allee discovered the phenomenon of the **positive** effect of animal congregation on the individual fitness in 1930-1940's. His discovery did not generate much interest in the scientific community at that time ([15]). Therefore, further research was predominantly conducted by W.C. Allee and his students ([2], [5], [3], [4]). Only in the last decade of the twentieth century, with an increasing number of species on the verge of extinction, did the study phenomena which increase the species chance for survival attract enough attention from the biologists and mathematicians, and became an influential branch of Mathematical Biology. At the recent Sixth International Conference on Mathematical Modeling and Analysis of Population in Biological Systems, held in Tuscon, Arizona, in October 2017, a significant number of the population models included the Allee effect.

The following definitions are helpful in the reasoning on this topic.

Definition 1.1 ([15]). ***Allee effect:** a causal positive relationship between individual fitness and either population size or density.*

Definition 1.2 ([15]). ***Demographic Allee effect:** a positive relationship between the overall individual fitness, usually quantified by the per capita population growth rate, and the population size or density.*

Definition 1.3 ([15]). ***Allee threshold:** a critical population size or density below which the per capita population growth becomes negative.*

Weak Allee effect: a demographic Allee effect without an Allee threshold.

Strong Allee effect: a demographic Allee effect with an Allee threshold.

In equation (1.0.1), the parameter $r > 0$ represents per capita population growth rate in the absence of the dependence on the population density or population size. Hence the classical negative density dependence is represented by the expression $r \left(1 - \frac{x(n)}{K(n)}\right)$. Here, the parameter $K(n)$ represents the periodic carrying capacity of the environment. To restrict the reasoning in this section to one biological phenomenon, the Allee effect, we consider the carrying capacity as a constant. That is, the classical negative density dependence is represented by the expression

$$r \left(1 - \frac{x(n)}{K}\right).$$

Respectively, the Allee effect, both weak and strong, is represented by the expression

$$r \left(1 - \frac{x(n)}{K}\right) \left(1 - \frac{A + C}{x(n) + C}\right). \quad (1.1.1)$$

Figure 1.1: Per capita growth rate dependence of the population density. Parameter values: $r = 2$, $K = 8$, $C = 1.3$ for the blue line and $C = 6$ for the red line. Left panel: $A = -1$. Right panel: $A = 1$.

In the expression (1.1.1), the existence and the value of the Allee threshold is defined by the parameter A : if $A > 0$, the threshold exists and the Allee effect is strong; if $A < 0$, there is no Allee threshold present. Therefore, in the case $A < 0$, the Allee effect is weak, and

the positive density dependence manifests itself in all of the lower population densities (see Figure 1.1). Additional restrictions on the parameter A , namely, $A \geq -C$ and $A < K$, are imposed to obtain a correlation with experimental data and to include dynamic consequences of the Allee effect (e.g. [11]).

The remaining parameter $C > 0$ is an auxiliary parameter ([15], [12]) which provides necessary flexibility to the model (1.1.1); it affects the shape of the dependency curve in Figure 1.1.

Currently, the list of the established and well-studied cases of a demographic Allee effect can easily reach into the hundreds. To provide just a few examples, here is the table:

Species	Evidence
<i>Spartina alterniflora</i> , or salt-water cordgrass	Reduced (but still positive) per capita population growth rate at low density in front of invasion [20]
Black-footed rock wallaby <i>Petrogale lateralis</i>	Threshold population size below which reintroduced population go extinct [43]
California island fox <i>Urocyon littoralis</i>	Hump-shaped relationship between population growth rate and population size [6]
Gypsy moth <i>Lymantria dispar</i>	Threshold colony size below which the colony goes extinct [38]

Table 1.1: Examples of the demographic Allee effect [15].

1.2 Periodic carrying capacity and the Cushing - Henson conjectures.

Oscillation is a common scenario for the dynamical process of the biological phenomenon. In mathematical models, oscillatory dynamics is described by the stable periodic solutions. The existence of the stable periodic solution for the population dynamics equation signifies

that the species, whose population dynamics is described by this equation, can thrive as long as the conditions for the existence of the periodic solution are satisfied. Hence, the discovery of the stable periodic solutions for both differential and difference population equations often means finding the solution to a problem in Conservation Ecology.

The non-autonomous difference equations that admit a periodic solution are often periodically forced by setting one or more of the parameters of the equation periodic. With the cyclic fluctuations of the environmental conditions, such as temperature, illumination, humidity, availability of food and shelter, etc., the periodic carrying capacity, $K(n)$, comes as the natural mechanism forcing the population to oscillate. Moreover, the carrying capacity parameter is explicitly included in most of the classical population equations. Analysis of the existence and properties of the solutions for non-autonomous difference equations with periodic carrying capacity and, in some models, a periodic per capita growth rate, r , attracted a lot of interest in the end of the twentieth - beginning of the twenty-first century ([18], [30]), and, by now, has grown into a mainstream research direction of Mathematical Biology.

It is well-deserved to mention here the two of the studies that were among the main turning points for the development of this branch of population dynamics.

The first is the famous controlled experiment conducted by Jillson [34]. Flour beetles were placed in the periodically changing amount of flour. The size of the beetle population was compared against the control group contained in the constant amount of flour. The experiment was designed in such a way that the average amount of flour in both groups was equal, which corresponded to the equal average carrying capacity. The experiment demonstrated that periodic environment was beneficial for the beetle population size and caused it to more than double.

Definition 1.4. *Let $\{\tilde{x}(0), \tilde{x}(1), \dots, \tilde{x}(p-1)\}$ be the p -periodic solution for the non-autonomous difference equation*

$$x(n+1) = f(x(n), K(n)), \quad n \in \mathbb{N}_0, \quad (1.2.1)$$

where $\{K(n)\}$ is a strictly positive p -periodic sequence.

The periodic solution $\{\tilde{x}(0), \tilde{x}(1), \dots, \tilde{x}(p-1)\}$ is called **attenuant** if the average of the periodic cycle is strictly less than the average of the carrying capacity K , that is,

$$\frac{1}{p} \sum_{n=0}^{p-1} \tilde{x}(n) < \frac{1}{p} \sum_{n=0}^{p-1} K(n). \quad (1.2.2)$$

The periodic solution $\{\tilde{x}(0), \tilde{x}(1), \dots, \tilde{x}(p-1)\}$ is called **resonant** if the inequality (1.2.2) is changed for the strictly greater relation.

Hence, the oscillation of the beetle population in Jillson experiment was resonant.

The mathematical model for this experiment ([31], [14]) demonstrated the existence of the multiple periodic solutions with specific values depending on the sequence $\{K(n)\}$.

The second pivotal research is the equally famous Cushing-Henson conjectures ([17], [19]).

Cushing and Henson constructed the Beverton-Holt equation with the periodic carrying capacity with the period $p = 2$ [17]:

$$x(n+1) = r \frac{1}{1 + (r-1) \frac{x(n)}{1 + \alpha(-1)^n}} x(n). \quad (1.2.3)$$

In the equation (1.2.3), $\alpha \in [0, 1)$, $K = K_{av}(1 + \alpha(-1)^n)$, where K_{av} is the average of K over time, and the density of population $x(n)$ is rescaled with respect to the parameter K_{av} .

Motivated by studies of the properties of the equation (1.2.3), Cushing and Henson defined the class of the periodically forced equations of the form

$$x(n+1) = f\left(\frac{x(n)}{1 + \alpha(-1)^n}\right) x(n),$$

where $h(x) = f(x)x$ is a concave increasing bounded function, and proved analytically the existence of the globally attractive attenuant periodic solution ($p = 2$) for $r := h'(0+) > 1$.

Then, in their next paper [19], Cushing and Henson conjectured the behaviour of the equation

analogous to (1.2.3) as follows.

For the periodically forced Beverton-Holt equation

$$x(n+1) = \frac{rK(n)}{K(n) + (r-1)x(n)}x(n),$$

in which $\{K(n)\}$ is a p -periodic strictly positive sequence, $r > 1$, and $p \geq 1$,

(a) There exists a positive p -periodic solution $y(n)$, and it is globally attracting for $x(0) > 0$.

(b) If $p > 2$, the strict inequality

$$\frac{1}{p} \sum_{n=0}^{p-1} y(n) < \frac{1}{p} \sum_{n=0}^{p-1} K(n)$$

holds.

These two studies and many others demonstrated that the population equation forced by the periodic parameter of the carrying capacity generates rich and complex dynamics. Moreover, this approach to mathematical modeling is strongly supported by both the controlled experiments and the biological data.

Since the beginning of the twenty-first century, the Cushing-Henson conjectures were proved multiple times and by multiple methods such as real analysis, group methods, linear algebra, probability theory, and quantum calculus. The new mathematical models were developed with the different types of the compensatory mechanism with respect to fluctuating carrying capacity and, therefore, with the globally or locally attractive periodic solutions which are attenuant, resonant or neither. To name just a few of the publications: [36], [35], [22], [23], [26], [1], [42], [27], [29], [9], [7], and [10].

1.3 Formulation of the problem.

The purpose of this research is to study the existence and the property of the stable periodic solutions for the equation (1.0.1) both for the weak and strong Allee effect. The choice of the equation (1.0.1) as the main research subject of this thesis is governed by the following reasoning.

(1). **Inclusion of the Allee effect into the model.** In recent, Conservation Ecology has been upgraded from the status of nature enthusiasts activity to a scientific search for the ways to ensure the survival of humanity on this planet. The Allee effect is one of the leading mechanisms in the population dynamics of the species whose population size is small or declining. Hence, the inclusion of the Allee effect in the population equation is crucial for the planning of any conservation activities. If endangered species are susceptible to the strong Allee effect, then the conservation activity with the population size smaller than the Allee threshold would fail.

On the other hand, the weak Allee effect may explain the unsuccessful attempts to eradicate invasive species: even the low density population can restore itself to the full volume due to the positive dependence of per capita growth rate on the population density. Otherwise, the strong Allee effect is known to slow down the spread of the invasive species. The expansion of *Spartina* on the west coast of the United States of America, for example, is reduced from 30% to 19% per year due to a pollination-driven Allee effect [15].

(2). **Inclusion of the periodic carrying capacity into the model.**

There are still quite a few debatable questions on the climate change remaining. However, scientists have agreed in principal that the climate on the planet is changing and in many ways the weather conditions grow relatively extreme much more often in the last few decades (see, e.g., [25]). Moreover, the effect of the variable periodic capacity on population dynamics

is undeniable (see section 1.2). Hence, a mathematical model for the population dynamics is expected to have embedded parameters reflecting the dependence on the changing environment.

(3). The choice of the specific rate function.

There has been constructed a wide range of the population equations with a variety of the functions modelling the positive per capita growth rate dependence on the size of the population. Some of the examples are:

$$\frac{dx}{dt} = xr \left(1 - \frac{x}{K}\right) \left(\frac{x}{K} - \frac{A}{K}\right), \quad (1.3.1)$$

developed by Lewis and Kareiva [37]; models both weak and strong Allee effect.

$$\frac{dx}{dt} = xr \left(1 - \frac{x}{K}\right) \left(\frac{x}{A} - 1\right), \quad (1.3.2)$$

developed by Gruntfest [28]; models strong Allee effect only.

$$\frac{dx}{dt} = xr \left(1 - \frac{x}{K}\right) \left(1 - \frac{A+C}{x+C}\right), \quad (1.3.3)$$

developed by Courchamp [16] and Boukal et al [12]; models both weak and strong Allee effect.

$$\frac{dx}{dt} = x \left(r + uv \frac{x^w}{x^w + v} - cx^z \right), \quad (1.3.4)$$

developed by Jacobs [33]; models both weak and strong Allee effect.

Note: each differential equation has a discrete-time analogue.

Sigmoid Beverton-Holt model:

$$x(n+1) = \frac{a(x(n))^{\delta(n)}}{1 + (x(n))^{\delta(n)}}, \quad (1.3.5)$$

introduced by Thomson [44], developed further by Harry et al [29]; models both weak and strong Allee effect.

Modified Beverton-Holt model:

$$x(n+1) = \frac{a(x(n))^2 + rx(n)}{1 + (x(n))^2}, \quad (1.3.6)$$

proposed by Elaydi and Sacker [24]; models strong Allee effect only.

The modified Ricker equation:

$$x(n+1) = (x(n))^\lambda \exp(p - x(n)), \quad (1.3.7)$$

developed by Elaydi and Sacker [24]; models strong Allee effect only.

The equations (1.3.1)-(1.3.7) can be classified with respect to the following properties.

The number of additional parameters to provide the flexibility of the function. The number of parameters embedded into the equation to provide the adequate mechanism for the Allee effect differs from two in the equations (1.3.5), (1.3.6), and (1.3.7) to six in the equation (1.3.4). Equation (1.0.1) is the discrete-time counterpart of equation (1.3.3), and, with four parameters adjusting the Allee effect to the experimental data, has enough flexibility. On the other hand, the mathematical analysis of this equation is not as challenging as for the equation (1.3.4).

Explicit versus implicit. The parameters r , intrinsic per capita growth rate, and K , the carrying capacity of the environment, are present in the equations (1.3.1)-(1.3.7) either explicitly or as a combination of the additional parameters of the dynamics mechanism. At the XV European Biological Rhythms Society Congress (July-August 2017, Amsterdam, Netherlands), I had the opportunity to hear at the plenary meetings, group sessions, and in personal conversations the opinion of the leading biologists on the construction of mathematical models. Unanomously, they favoured the models with the explicit measurable parameters which can be observed and varied for the controlled experiments and analyzed during the data

collection. In the equation (1.0.1), both parameters are present explicitly.

Continuous-time model versus discrete-time model. Most of the target species susceptible to the Allee effect have pulsed reproduction, hence the choice of the discrete-time version of the equation (1.3.3).

Weak Allee effect versus strong Allee effect. The mechanism of the demographic Allee effect (Definition 1.2) is defined by the complex interference of the component Allee effects.

Figure 1.2: An example of the component Allee effect in survival: two families of Canadian geese demonstrate the cooperative anti-predator behavior. Bow River, Calgary, 2018.

Definition 1.5 ([15]). ***Component Allee effect:** a positive relationship between any measurable component of individual fitness and population size or density.*

***Dormant Allee effect:** a component Allee effect that either does not result in a demographic Allee effect or result in a weak Allee effect and which, if interacting with a strong Allee effect, causes overall Allee threshold to be higher than the Allee threshold due to the strong Allee effect alone.*

There is no guarantee that the cooperative behaviour of Canadian geese captured in Figure 1.2 will result in the overall demographic Allee effect. The component Allee effect in survival can become dormant. Moreover, due to interference with the other component

Allee effects, it can result in either weak or strong Allee effect depending on the spectrum of the component Allee effects active at the time.

Definition 1.6 ([8]). ***Multiple Allee effects:** any situation in which two or more component Allee effects work simultaneously in the same population.*

Here are just several examples of the multiple Allee effects from [8]. Thus, considering

Species	Multiple component Allee effects	Mechanism of each component
Ribbed mussel <i>Geukensia demissa</i>	Low-density populations suffer from increased mortality owing to crab predation (1) and winter ice (2)	(1) predator dilution (survival component); (2) environmental conditioning (reproduction component)
African wild dog <i>Lycaon pictus</i>	Smaller packs are less efficient in cooperative breeding (3), cooperative anti-predator behaviour (4), and cooperative hunting (5)	(3) reproduction component; (4) survival component; (5) reproduction and survival component
Alpine marmot <i>Marmota marmota</i>	Individuals in smaller overwintering groups have difficulties in finding mates (6) and decreased in survival, owing to less efficient social thermoregulation (7)	(6) reproduction component; (7) reproduction and survival component

Table 1.2: Examples of the multiple component Allee effects [8].

high mutual influence of the components and the variable environmental conditions, it is reasonable to conclude that an adequate for different species and different conditions population model should admit all three possible scenarios: the negative dependence of the per capita growth rate on the density of the population, the weak Allee effect, and the strong Allee effect.

All the reasoning points mentioned above resulted in the choice of the discrete-time version

of the equation (1.3.3), namely,

$$x(n+1) = x(n) \exp \left(r \left(1 - \frac{x(n)}{K} \right) \left(1 - \frac{A+C}{x(n)+C} \right) \right), \quad n \in \mathbb{N}_0, \quad (1.3.8)$$

as the basis population model for the equation (1.0.1).

Note: the solutions of the discrete-time population equation (1.3.8) have not been studied before ([15]).

Finally, defining the strictly positive p -periodic sequence $\{K(n)\}$ as the variable carrying capacity for the equation (1.3.8), we obtain the research subject of this thesis, the equation (1.0.1). For the weak Allee effect, the initial values of the population size (or density), $x(0)$, are restricted to strictly positive, since $x(0) = 0$ is the unstable equilibrium point of the equation (1.0.1) (see Lemma 4.1).

The equation (1.3.8) was developed on the basis of the Ricker equation

$$x(n+1) = x(n) \exp \left(r \left(1 - \frac{x(n)}{K} \right) \right), \quad n \in \mathbb{N}_0. \quad (1.3.9)$$

Hence, for clarity, equation (1.0.1) is further referred as the Ricker equation with the Allee effect and periodic carrying capacity.

1.4 Main results and the layout of the thesis.

This research has established the main dynamic properties such as the existence, stability, and the basin of attraction of the equilibrium points and periodic solutions for the discrete-time population model 1.0.1 enhanced with the mechanisms of two biological phenomena: the variable Allee effect (both weak and strong types) and the periodically changing carrying capacity.

In Chapter 2, it is shown that the unique equilibrium point in $\mathbb{R}^+$ for the equation with the weak type of the Allee effect is the global attractor. Moreover, it is proven that the weak

Allee effect, embedded into the equation, increases the range of the parameters for the global asymptotic stability to hold.

In Chapter 3, the inclusion of the mechanism for the periodically changing environment facilitated the unique conditions when all the solutions for the equation possess the persistency property, that is, all the solutions eventually get "trapped" into the closed interval. The periodic carrying capacity also generated periodic solutions. Those solutions are proved to gain global asymptotic stability under the constructed sufficient conditions. It is demonstrated that the inclusion of the weak Allee effect into the model increases the range of the parameters that allow the cycle to remain globally asymptotically stable, but with a different relation to the parameters of the equation in comparison to Chapter 2 results.

In Chapter 4, the change of the type for the Allee effect from weak to strong caused the dramatic change in the number, stability, and the basins of attraction for the equilibrium points of the population equation both with the Allee effect mechanism only and with added periodic carrying capacity. We introduce the unimodal Allee effect maps and their properties. The new properties for that type of maps are proven.

In Chapter 5, it is demonstrated that the traditional approach to the proof of the attenuant property by using the concavity of the forcing function is not applicable to the case of the Ricker equation with the Allee effect and periodic carrying capacity. Moreover, the new mutual relation for the points of the same cycle was proven.

Chapter 6 contains the conclusions and the suggestions for further research.

Chapter 2

Ricker equation with the weak Allee effect and constant carrying capacity

The structure of the population equation (1.0.1) is enhanced with the mechanisms of two additional biological phenomena and, therefore, is predictably complex. Moreover, as it was mentioned in Chapter 1, the solutions of the equation (1.3.8) have not been previously investigated before. Hence, in this chapter, the solutions of the equation (1.3.8) are studied in the case of the weak Allee effect. That is, the subject of this chapter is the equation

$$x(n+1) = f(x(n)), \quad n \in \mathbb{N}_0, \quad x(0) > 0, \quad (2.0.1)$$

where $f(x)$ is defined by the equation

$$f(x) = x \exp \left(r \left(1 - \frac{x}{K} \right) \left(1 - \frac{A+C}{x+C} \right) \right), \quad (2.0.2)$$

with $K > 0, r > 0, A < 0 < C$ and $|A| < C$.

2.1 Properties of the function describing the Ricker equation with the weak Allee effect

The behaviour of the solutions for the equation (2.0.1) depends on the properties of the function $f(x)$ such as continuity, differentiability, and extrema. In addition, the function on the right-hand side of the Ricker equation (1.3.9) is well studied. Then, it is helpful to compare the behaviour of the function $f(x)$ with that of the function $g(x) = x \exp\left(r\left(1 - \frac{x}{K}\right)\right)$. The following lemma clarifies the properties of the function $f(x)$.

Lemma 2.1. *Let the function $f : \mathbb{R}_0^+ \mapsto \mathbb{R}_0^+$ be defined by:*

$$f(x) = x \exp\left(r\left(1 - \frac{x}{K}\right)\left(1 - \frac{A+C}{x+C}\right)\right), \quad (2.1.1)$$

where $K > 0, r > 0, A < 0 < C$ and $|A| < C$, and the function $g : \mathbb{R}_0^+ \mapsto \mathbb{R}_0^+$ be defined as

$$g(x) = x \exp\left(r\left(1 - \frac{x}{K}\right)\right). \quad (2.1.2)$$

Then:

- (1) the functions $f(x)$ and $g(x)$ intersect at exactly two points: $x_1 = 0$ and $x_2 = K$;
- (2) $f(x) < g(x)$ on the interval $(0, K)$, and $f(x) > g(x)$ on the interval (K, ∞) ;
- (3) the function $f(x)$ has a unique maximum;
- (4) let the x -coordinate of the maximum of the function $f(x)$ be denoted as $x_{\max}$ and

$$\alpha := \frac{K - A}{K + C}.$$

Then, the location of the $x_{\max}$ and the maximum value of the function $f(x)$, denoted $f_{\max}$, can be described as follows:

Case A. If $\alpha r \geq 1$, then $\frac{K}{r} < x_{\max} \leq K$ and $f_{\max} \leq \frac{K}{r} \exp(r - 1)$.

Case B. If $\alpha r \leq 1$, then $K \leq x_{\max} < \frac{K}{\alpha r}$ and $f_{\max} \leq \frac{K}{\alpha r}$.

Moreover, if $r < 1$, then $\frac{K}{r} < x_{\max} \leq \frac{K}{r\alpha}$.

Case C. If $\alpha r = 1$, then $x_{\max} = K$ and $f_{\max} = K$.

Proof. (1). The equation

$$\exp\left(r\left(1 - \frac{x}{K}\right)\right) = \exp\left(r\left(1 - \frac{x}{K}\right)\left(1 - \frac{A+C}{x+C}\right)\right)$$

has a unique solution on $\mathbb{R}^+$, $x = K$. Then, the equation $f(x) = g(x)$ has exactly two solutions : $x_1 = 0$ and $x_2 = K$. Thus, the property (1) holds.

(2). Since $0 < \left(1 - \frac{A+C}{x+C}\right) < 1$ for all $x \in \mathbb{R}_0^+$, and $\left(1 - \frac{x}{K}\right)$ is positive on the interval $(0, K)$ and negative on the interval (K, ∞) , the property (2) follows.

(3). Since the derivative of the function $f(x)$ is represented by the expression

$$f'(x) = \left(1 + x\left(-\frac{r}{K}\left(1 - \frac{A+C}{x+C}\right) + r\left(1 - \frac{x}{K}\right)\frac{A+C}{(x+C)^2}\right)\right) \exp\left(r\left(1 - \frac{x}{K}\right)\left(1 - \frac{A+C}{x+C}\right)\right), \quad (2.1.3)$$

then, solving $f'(x) = 0$, we obtain the equation for the x-coordinate of the extremum for the function $f(x)$:

$$1 + x\left(-\frac{r}{K}\left(1 - \frac{A+C}{x+C}\right) + r\left(1 - \frac{x}{K}\right)\frac{A+C}{(x+C)^2}\right) = 0. \quad (2.1.4)$$

Simplifying yields:

$$\frac{x(A+C)(K+C)}{(x+C)^2} = x - \frac{K}{r}. \quad (2.1.5)$$

Note, that $x = \frac{K}{r}$ is not a solution of the equation (2.1.5). Hence, we can re-write (2.1.5) as follows:

$$(x + C)^2 = \frac{x(A + C)(K + C)}{x - \frac{K}{r}} \quad (2.1.6)$$

Let us consider the solutions for the equation (2.1.6) on $\mathbb{R}$.

On the left-hand side of the equation (2.1.6), we have the equation of the parabola, that opens up, with vertex $(-C, 0)$. Thus, this parabola is located in the first and the second quadrants. On the right-hand side of this equation, we have a hyperbola with the positive coefficient $(A + C)(K + C)$ and the vertical asymptote $x = \frac{K}{r}$. The left branch of the hyperbola is passing through the point $(0, 0)$. Therefore, by the properties of these parabola and hyperbola, two intersections of the hyperbola and the parabola are located in the second quadrant, and the third intersection is located in the first quadrant. The equation (2.1.6) is of the third order with respect to x , so the maximum number of solutions is three. Hence, the solution of the equation (2.1.6) for $x \in \mathbb{R}_0^+$ exists and unique. That is, the function $f(x)$ has the unique extremum on $\mathbb{R}_0^+$.

Now, let us show that this unique extremum is maximum.

In the equation (2.1.5), $(x + C)^2 > 0$ and $x(A + C)(K + C) \geq 0$ for all $x \in \mathbb{R}^+$.

Then, $x - \frac{K}{r} \geq 0$. Since $x = 0$ is not the solution for the equation (2.1.5), the x-coordinate of the unique extremum of the function $f(x)$, x_{extr} , satisfies

$$x_{extr} > \frac{K}{r}.$$

Let us represent the rational part of the expression for the derivative of the function $f(x)$ as

follows:

$$\begin{aligned}
1 + x \left(-\frac{r}{K} \left(1 - \frac{A+C}{x+C} \right) + r \left(1 - \frac{x}{K} \right) \frac{A+C}{(x+C)^2} \right) = \\
\left(1 - x \frac{r}{K} \right) + x \frac{r}{K} \left(\frac{A+C}{x+C} + \frac{(K-x)(A+C)}{(x+C)^2} \right) = \\
\left(1 - x \frac{r}{K} \right) + x \frac{r}{K} \frac{(A+C)(K+C)}{(x+C)^2}.
\end{aligned} \tag{2.1.7}$$

For $x < \frac{K}{r}$, the product $x \frac{r}{K} < 1$. Hence, the rational part for the derivative of the function $f(x)$, represented by the expression (2.1.7), is positive for $x < \frac{K}{r}$. Then $f'(x) > 0$ for $x < \frac{K}{r}$, and the function $f(x)$ is increasing for all $x < \frac{K}{r}$. This implies that the unique extremum of the function $f(x)$, which, as proven above, has the x-coordanate $x_{extr} > \frac{K}{r}$, is the maximum.

(4). In Part (3), we proved that

$$x_{\max} > \frac{K}{r}.$$

Let us establish now the necessary condition for $x_{\max} > K$. Since $K > 0$, the inequality $x > K$ implies $x > 0$. Dividing both sides of the equation (2.1.5) by x , we obtain

$$\frac{(A+C)(K+C)}{(x+C)^2} = 1 - \frac{K}{rx}. \tag{2.1.8}$$

If $x > K$, then

$$\frac{(A+C)(K+C)}{(x+C)^2} < \frac{(A+C)(K+C)}{(K+C)^2} = \frac{A+C}{K+C}.$$

Hence, the right-hand part of the equation (2.1.8) satisfies

$$1 - \frac{K}{rx} < \frac{A+C}{K+C},$$

which is equivalent to

$$x < \frac{K}{r} \left(\frac{K+C}{K-A} \right) = \frac{K}{r\alpha}. \tag{2.1.9}$$

Thus, if $x_{\max} > K$, then, by the inequality (2.1.9), $K < \frac{K}{\alpha r}$, and the necessary condition for $x_{\max} > K$ is $\alpha r < 1$.

Then, by contrapositive,

$$\text{if } \alpha r \geq 1, \text{ then } x_{\max} \leq K.$$

Similarly, if $x < K$, then the left-hand part of the equation (2.1.8) satisfies

$$\frac{(A+C)(K+C)}{(x+C)^2} > \frac{A+C}{K+C}.$$

Therefore, the right-hand part of the equation (2.1.8) satisfies the following inequality

$$1 - \frac{K}{rx} > \frac{A+C}{K+C} \Leftrightarrow x > \frac{K}{\alpha r}.$$

Thus, similar to the previous reasoning, we obtain the double inequality:

$$\frac{K}{\alpha r} < x < K,$$

which implies $\alpha r > 1$. Then, by the contrapositive to the implication $(x < K) \Rightarrow (\alpha r > 1)$, we obtain:

$$\text{if } \alpha r \leq 1, \text{ then } x_{\max} \geq K.$$

By the implication from two previously considered cases or by the direct substitution of $\alpha r = 1$ into the equation (2.1.4), we obtain the unique solution on $\mathbb{R}^+$, that is, $x_{\max} = K$.

Thus, there exist three possible cases for the location of the $x_{\max} \in \mathbb{R}^+$:

Case A. If $\alpha r \geq 1$, then $\frac{K}{r} < x_{\max} \leq K$.

Case B. If $\alpha r \leq 1$, then $K \leq x_{\max} < \frac{K}{r\alpha}$. Moreover, if $r < 1$, then $K < \frac{K}{r} < x_{\max} \leq \frac{K}{r\alpha}$.

Case C. If $\alpha r = 1$, then $x_{\max} = K$.

Now, we evaluate $f_{\max} = f(x_{\max})$ in each case.

Since $f(x) \leq g(x)$ on the interval $(0, K)$ and $g_{\max} = \frac{K}{r} \exp(r-1)$, then, in the Case A, $f_{\max} \leq \frac{K}{r} \exp(r-1)$.

For the Case B: on the interval $\left(K, \frac{K}{\alpha r}\right)$, x is increasing and $\exp\left(r\left(1 - \frac{x}{K}\right)\left(1 - \frac{A+C}{x+C}\right)\right)$ is decreasing. Hence, $f(x) \leq f_{\max} \leq \frac{K}{\alpha r}$.

For the Case C, by the direct substitution of $x_{\max} = K$ into the equation (2.1.1), we obtain $f_{\max} = K$.

Hence, the property (4) holds. ■

To illustrate the statements of the Lemma 2.1, let us consider the following example.

Figure 2.1: Location of the maximum for the functions $f_1(x)$ and $f_2(x)$ depends on the value of αr . Parameter values: $K = 8$, $C = 6$. Left panel: $r = 0.8$, $A = -0.5$. Right panel: $r = 2.3$, $A = -5$.

Example 2.1. Define two sets of parameters for the expression (2.1.1):

Set 1: $K = 8$, $C = 6$, $A = -0.5$, $r = 0.8$. Then, $\alpha_1 \approx 0.6071$, and $\alpha_1 r_1 \approx 0.4857 < 1$.

Set 2: $K = 8$, $C = 6$, $A = -5$, $r = 2.3$. Similarly, $\alpha_2 \approx 0.9286$, and $\alpha_2 r_2 \approx 2.1357 > 1$.

The corresponding functions are defined as follows:

$$f_1(x) = x \exp \left(0.8 \left(1 - \frac{x}{8} \right) \left(1 - \frac{5.5}{x+6} \right) \right), \quad f(x)_2 = x \exp \left(2.3 \left(1 - \frac{x}{8} \right) \left(1 - \frac{1}{x+6} \right) \right).$$

By Lemma 2.1, we expect the following inequalities to hold for the function $f_1(x)$:

$$10 = \frac{K}{r_1} < x_{1,\max} \leq \frac{K}{\alpha_1 r_1} = 16.4710, \quad f_{1,\max} \leq \frac{K}{\alpha_1 r_1} = 16.4710. \quad (2.1.10)$$

Indeed, with $x_{1,\max} \approx 12.7894$ and $f_{1,\max} \approx 9.1145$, the inequalities (2.1.10) are satisfied (see Figure 2.1, Left panel).

Respectively, by Lemma 2.1, the following inequalities are expected to hold for the function $f_2(x)$:

$$3.4783 = \frac{K}{r_2} < x_{2,\max} \leq K = 8, \quad f_{2,\max} \leq \frac{K}{r_2} \exp(r_2 - 1) = 12.7628. \quad (2.1.11)$$

Since $x_{2,\max} \approx 4.0394$ and $f_{2,\max} \approx 11.2610$, the inequalities (2.1.11) also hold (see Figure 2.1, Right panel).

2.2 Equilibrium points of the Ricker equation with the weak Allee effect.

The periodic solutions of the equation (1.0.1) depend on the equilibrium points of the corresponding equations with the constant carrying capacity to a great extent. Therefore, in this section, the existence and the properties of the equilibrium points for the equation (1.3.8) are studied.

2.2.1 Existence and stability of the equilibrium

Let us denote the equilibrium point of the equation (2.0.1) as x^* . Then, the equation

$$f(x^*) = x^*, \quad x \in \mathbb{R}^+,$$

is equivalent to the equation

$$\exp\left(r\left(1 - \frac{x^*}{K}\right)\left(1 - \frac{A+C}{x^*+C}\right)\right) = 1, \quad x \in \mathbb{R}^+. \quad (2.2.1)$$

Since with the defined set of parameters K, r, A , and C for the case of the weak Allee effect and $x(0) > 0$,

$$0 < 1 - \frac{A+C}{x^*+C} < 1 \text{ for any } x^* \in \mathbb{R}^+,$$

the equation (2.2.1) has the unique solution $x^* = K$. Therefore, $x^* = K$ is the unique equilibrium point of the equation (2.0.1) on $\mathbb{R}^+$.

The global asymptotic stability of the equilibrium point $x^* = K$ is proved later in this section by Lemma 2.2, Lemma 2.3, Lemma 2.4, and Theorem 2.5. Lemma 2.2, based on the properties of the derivative of the function $f(x)$, establishes the range of the parameters K, r, A , and C that provides the asymptotic stability of the equilibrium point. Theorem 2.5 establishes the global asymptotic stability of the equilibrium point on $\mathbb{R}^+$. Lemma 2.3 and Lemma 2.4 prove the properties of the functions which are used in the proof of the Theorem 1.

Let us recall the criterion for the asymptotic stability of the equilibrium point:

Theorem A ([21], p.27). *Let x^* be an equilibrium point of the difference equation*

$$x(n+1) = f(x(n)),$$

where f is continuously differentiable at x^* . The following statements are hold true:

(i) If $|f'(x^*)| < 1$, then x^* is asymptotically stable.

(ii) If $|f'(x^*)| > 1$, then x^* is unstable.

The next lemma is based on Theorem A. Parameter α , involved in this lemma, was defined in Lemma 2.1.

Lemma 2.2. *The unique equilibrium point, $x^* = K$, of the equation*

$$x(n+1) = x(n) \exp \left(r \left(1 - \frac{x(n)}{K} \right) \left(1 - \frac{A+C}{x(n)+C} \right) \right), \quad x \in \mathbb{R}^+, n \in \mathbb{N}_0 \quad (2.2.2)$$

with

$$K > 0, r > 0, A < 0 < C, \text{ and } |A| < C, \quad (2.2.3)$$

(i) is asymptotically stable if $0 < \alpha r < 2$;

(ii) is unstable if $\alpha r > 2$.

Proof. It has been established earlier in this section that $x^* = K$ is the unique equilibrium point of the equation (2.2.2) in $\mathbb{R}^+$.

Let us denote the right-hand side of the equation (2.2.2) as $f(x(n))$. Then, the derivative $f'(x)$ is given by the expression (2.1.3). Observe, that $f'(x)$ exists at the point $x^* = K$ and continuous as the product of a continuous function and a composition of continuous functions. Hence, the function $f(x)$ is continuously differentiable.

Moreover, $|f'(x^*)| = \left| 1 - r \frac{K-A}{K+C} \right| = |1 - \alpha r|$.

Then, by Theorem A, if $0 < \alpha r < 2$, then the equilibrium point $x^* = K$ is asymptotically stable; if $\alpha r > 2$, then this equilibrium point is unstable. ■

Example 2.2. Consider the equation (2.2.2) with two sets of parameters:

Set 1: $K = 10$, $C = 6$, $A = -0.5$, and $r = 1.2$ with the corresponding equation

$$x(n+1) = x(n) \exp \left(1.2 \left(1 - \frac{x(n)}{10} \right) \left(1 - \frac{5.5}{x(n) + 6} \right) \right), \quad n \in \mathbb{N}_0. \quad (2.2.4)$$

Set 2: $K = 10$, $C = 6$, $A = -0.5$, and $r = 5.0$ with the corresponding equation

$$y(n+1) = y(n) \exp \left(5.0 \left(1 - \frac{y(n)}{10} \right) \left(1 - \frac{5.5}{y(n) + 6} \right) \right), \quad n \in \mathbb{N}_0. \quad (2.2.5)$$

For the equation (2.2.4), $\alpha r_1 \approx 0.9450 < 2$. For the equation (2.2.5), $\alpha r_2 \approx 3.2813 > 2$.

Figure 2.2: Stability of the equilibrium point for the Ricker equation with the weak Allee effect: $x^* = K$ is stable when $\alpha r < 2$, and unstable when $\alpha r > 2$. Parameter values: $K = 10$, $C = 6$, $A = -0.5$. Red line: $r = 1.2$, $x(0) = 20$; blue line: $r = 5$, $y(0) = 10.05$.

The solutions for both equations with $x(0) = 20$ and $y(0) = 10.05$ respectively are shown in Figure 2.2. We observe that, in agreement with Lemma 2.2, the equilibrium point $x^* = K$ is stable when $\alpha r < 2$, and unstable when $\alpha r > 2$.

Note: For the Ricker equation without the Allee effect:

$$x(n+1) = x(n) \exp \left(r \left(1 - \frac{x(n)}{K} \right) \right),$$

the equilibrium point $x^* = K$ is stable if the growth rate r satisfies $0 < r < 2$ ([40], p.514). Since, for the weak Allee effect, $0 < \alpha < 1$, we conclude that the weak Allee effect increases the range of the growth rate with the asymptotic stability of x^* holding. With the proper choice of the parameters K, A , and C , it is possible to construct such $r \gg 2$.

Figure 2.3: The weak Allee effect keeps the equilibrium point $x^* = K$ stable for $r > 2$. Parameter values: $K = 10$ and $r = 3.5$. Red line: $x(0) = 20$; the equation includes the weak Allee effect with $A = -0.5$ and $C = 9$. Blue line: $y(0) = 10.05$; no Allee effect is included.

Example 2.3. Let us consider two equations with the same values of the parameters: $K = 10$ and $r = 3.5$. One equation has the embedded mechanism of the weak Allee effect with $A = -0.5$ and $C = 9$:

$$x(n+1) = x(n) \exp \left(3.5 \left(1 - \frac{x(n)}{10} \right) \left(1 - \frac{8.5}{x(n) + 9} \right) \right), \quad n \in \mathbb{N}_0. \quad (2.2.6)$$

Another equation is the classical Ricker equation without the Allee effect:

$$y(n+1) = y(n) \exp \left(3.5 \left(1 - \frac{y(n)}{10} \right) \right), \quad n \in \mathbb{N}_0. \quad (2.2.7)$$

In Figure 2.3, the solutions for the equations (2.2.6) and (2.2.7) are shown with the initial conditions $x(0) = 20$ and $y(0) = 10.05$ respectively. The graphs demonstrate that due to the weak Allee effect, the equilibrium point $x^* = K$ remains asymptotically stable for the

equation (2.2.6) with the values of the per capita growth rate significantly greater than 2.

2.2.2 Basin of attraction for the equilibrium point

Now, when we established the range of parameters providing the asymptotic stability of the equilibrium point $x^* = K$ for the equation (2.2.2), we can investigate the basin of attraction for this point. The following three definitions and a theorem are helpful in this regard.

Definition 2.1 ([21], p.204). *Let $V : \mathbb{R}^k \rightarrow \mathbb{R}$ be defined as a real-valued function. The variation of V , relative to $x(n+1) = f(x(n))$, is defined as*

$$\Delta V(x(n)) := V(f(x(n))) - V(x(n)) = V(x(n+1)) - V(x(n)).$$

Definition 2.2 ([21], p.204). *The function V is said to be a Liapunov function on a subset H of $\mathbb{R}^k$ if:*

- (i) V is continuous on H , and
- (ii) $\Delta V(x) \leq 0$, whenever x and $f(x)$ belong to H .

Definition 2.3 ([21], p.204). *Let $B(x, \gamma)$ denote the open ball in $\mathbb{R}^k$ of radius γ and center x defined by $B(x, \gamma) = \{y \in \mathbb{R}^k \mid \|y - x\| < \gamma\}$. We say that the real-valued function V is positive-definite at x^* if:*

- (i) $V(x^*) = 0$, and
- (ii) $V(x) > 0$ for all $x \in B(x^*, \gamma)$, $x \neq x^*$, for some $\gamma > 0$.

Theorem B ([21], p.205, Theorem 4.20). *If V is a Liapunov function for equation (2.0.1) in a neighborhood H of the equilibrium point x^* , and V is positive definite with respect to x^* , then x^* is stable. If, in addition, $\Delta V(x) < 0$ whenever $x, f(x) \in H$ and $x \neq x^*$, then x^* is asymptotically stable. Moreover, if $H = \mathbb{R}^k$ and*

$$V(x) \rightarrow \infty \text{ as } \|x\| \rightarrow \infty,$$

then x^* is globally asymptotically stable.

In preparation for the proof that the equilibrium point $x^* = K$ of the equation (2.2.2) is globally asymptotically stable, the following two lemmas establish the properties of the limiting functions.

Lemma 2.3. *The equation*

$$\frac{2a}{x} = \exp\left(2\left(1 - \frac{x}{a}\right)\right) + 1, \quad a > 0 \quad (2.2.8)$$

has the unique solution $x = a$ on $\mathbb{R}^+$. Moreover,

$$\begin{aligned} \frac{2a}{x} &> \exp\left(2\left(1 - \frac{x}{a}\right)\right) + 1, & 0 < x < a; \\ \frac{2a}{x} &< \exp\left(2\left(1 - \frac{x}{a}\right)\right) + 1, & x > a. \end{aligned} \quad (2.2.9)$$

Proof. Clearly, $x = a$ is the solution for the equation (2.2.8). Let us prove the uniqueness of this solution.

Let us denote $h(x) := x \exp\left(2\left(1 - \frac{x}{a}\right)\right) + x$. Then, the equation (2.2.8) is equivalent to the following equation:

$$2a = h(x), \quad a > 0, \quad x \in \mathbb{R}^+. \quad (2.2.10)$$

To analyze the behaviour of the function $h(x)$ on $\mathbb{R}^+$, let us consider its first three derivatives:

$$\begin{aligned} h'(x) &= \left(1 - \frac{2}{a}x\right) \exp\left(2\left(1 - \frac{x}{a}\right)\right) + 1; \\ h''(x) &= \frac{4}{a^2} (x - a) \exp\left(2\left(1 - \frac{x}{a}\right)\right); \\ h'''(x) &= \frac{4}{a^2} \left(3 - \frac{2}{a}x\right) \exp\left(2\left(1 - \frac{x}{a}\right)\right). \end{aligned} \quad (2.2.11)$$

From the equations (2.2.11), we observe that

$$h'(a) = 0; \quad h''(a) = 0; \quad h'''(a) = \frac{4}{a^2} > 0. \quad (2.2.12)$$

Furthermore, from the equation for $h''(x)$, it follows that $x = a$ is the unique zero for the function $h''(x)$. Hence, the function $h'(x)$ has the global minimum at the point $x = a$ with $h'_{\min} = 0$. These conclusions, together with the value of $h'(a)$, imply that the function $h(x)$ is strictly increasing with the inflection point at $x = a$.

Since the function $h(x)$ is strictly increasing, there exists a unique intersection with the horizontal line $2a$. Thus, the $x = a$ is the unique solution for the equation (2.2.10) and, therefore, the unique solution for the equation (2.2.8).

To prove the inequalities (2.2.9), it suffices to notice that

$$\lim_{x \rightarrow 0^+} \left(\frac{2a}{x} \right) = +\infty, \text{ whereas } \lim_{x \rightarrow 0^+} \left(\exp \left(2 \left(1 - \frac{x}{a} \right) \right) + 1 \right) = e^2 + 1.$$

■

Lemma 2.4. *If the parameters r , K , A , and C satisfy the conditions (2.2.3), and $\alpha r := r \frac{K - A}{K + C} < 2$, then the inequality*

$$\exp \left(r \left(1 - \frac{x}{K} \right) \left(1 - \frac{A + C}{x + C} \right) \right) + 1 - \frac{2K}{x} > 0 \quad (2.2.13)$$

holds for all $x > K$.

Proof. Since $x > 0$, the inequality (2.2.13) is equivalent to the following inequality:

$$x \exp \left(r \left(1 - \frac{x}{K} \right) \left(1 - \frac{A + C}{x + C} \right) \right) + x > 2K. \quad (2.2.14)$$

Let us denote the left-hand side of the inequality (2.2.14) $p(x)$. Then, $p(K) = 2K$, and to prove this Lemma, it suffices to show that for $x > K$, $p'(x) > 0$.

$$p'(x) = \left(1 - \frac{r}{K} \frac{x}{(x+C)^2} P(x)\right) \exp\left(r \left(1 - \frac{x}{K}\right) \left(1 - \frac{A+C}{x+C}\right)\right) + 1,$$

where $P(x) := x^2 + 2Cx - (KC + AC + KA)$. Hence, the inequality $p'(x) > 0$ is equivalent to the inequality

$$\left(1 - \frac{r}{K} \frac{x}{(x+C)^2} P(x)\right) \exp\left(r \left(1 - \frac{x}{K}\right) \left(1 - \frac{A+C}{x+C}\right)\right) > -1, \quad x > K$$

or

$$\frac{r}{K} \frac{x}{(x+C)^2} P(x) - 1 < \exp\left(r \left(\frac{x}{K} - 1\right) \left(1 - \frac{A+C}{x+C}\right)\right), \quad x > K. \quad (2.2.15)$$

To prove the inequality (2.2.15), let us analyze the properties of the both sides of this inequality by denoting two functions,

$$p_1(x) = \exp\left(r \left(\frac{x}{K} - 1\right) \left(1 - \frac{A+C}{x+C}\right)\right), \quad \text{and}$$

$$p_2(x) = \frac{r}{K} \frac{x}{(x+C)^2} P(x) - 1 = \frac{r}{K} x - \frac{r}{K} (K+C)(A+C) \frac{x}{(x+C)^2} - 1.$$

Then, $p_1(x) > p_2(x)$ for all $x > K$ is equivalent to the inequality (2.2.15).

Let us make the following observations:

$$(i). \quad p_1(K) > p_2(K).$$

Indeed, $p_1(K) = 1$. On the other hand,

$$p_2(K) = \frac{r}{K} K \left(1 - \frac{(K+C)(A+C)}{(K+C)^2}\right) - 1 = r \frac{K-A}{K+C} - 1 = \alpha r - 1.$$

Since, by the hypothesis of the lemma, $\alpha r < 2$, then $p_2(K) < 1$.

(ii). The functions $p_1(x)$ and $p_2(x)$ are both strictly increasing for $x > K$.

Indeed,

$$p_1'(x) = \frac{r}{K} \frac{1}{(x+C)^2} P(x) \exp \left(r \left(\frac{x}{K} - 1 \right) \left(1 - \frac{A+C}{x+C} \right) \right).$$

Moreover, for $x > K$,

$$P(x) > K^2 + 2CK - (AC + KA + KC) = (K+C)(K-A) > 0.$$

Hence, for $x > K$, $p_1'(x) > 0$.

Similarly,

$$p_2'(x) = \frac{r}{K} + \frac{r}{K} (K+C)(A+C) \frac{x-C}{(x+C)^3}. \quad (2.2.16)$$

For $x > K$,

$$\frac{(K+C)(A+C)}{(x+C)^2} < \frac{(K+C)(A+C)}{(K+C)^2} = \frac{A+C}{K+C} < 1,$$

since, by the model of the Allee effect, $A < K$. In addition, for $x \in \mathbb{R}^+$,

$$\frac{x-C}{x+C} = 1 - \frac{2C}{x+C} > -1.$$

Hence, in the expression (2.2.16),

$$(K+C)(A+C) \frac{x-C}{(x+C)^3} > -1.$$

Therefore, for $x > K$, $p_2'(x) > 0$.

(iii). There exists $\tilde{x} > 0$ such that $p_1'(x) > p_2'(x)$ for $x > \tilde{x}$.

Indeed, the growth rate of the exponent with base greater than one will eventually become greater than the growth rate of any polynomial, that is,

$$\lim_{x \rightarrow \infty} \frac{\left(\exp \left(r \left(\frac{x}{K} - 1 \right) \right) \right)'}{\left(\frac{r}{K} x - 1 \right)'} = \lim_{x \rightarrow \infty} \frac{\left(\exp \left(r \left(\frac{x}{K} - 1 \right) \right) \right)'}{\frac{r}{K}} = +\infty.$$

Notice also that

$$\lim_{x \rightarrow \infty} \left(\exp \left(r \left(\frac{x}{K} - 1 \right) \left(1 - \frac{A+C}{x+C} \right) \right) \right) = \exp \left(r \left(\frac{x}{K} - 1 \right) \right),$$

and

$$\lim_{x \rightarrow \infty} \left(\frac{r}{K} x - \frac{r}{K} (K+C) (A+C) \frac{x}{(x+C)^2} - 1 \right) = \frac{r}{K} x - 1.$$

Hence, we conclude that the growth rate of $p_1(x)$ will outgrow eventually the growth rate of $p_2(x)$.

(iv). Unfortunately for the analytical approach, $p'_1(K) > p'_2(K)$. Hence, $\tilde{x} > K$.

Indeed,

$$\begin{aligned} p'_1(K) - p'_2(K) &= \frac{r}{K} \frac{K^2 + 3CK - AC - AK}{(K+C)^2} - \frac{r}{K} \frac{K^2 + CK - AC + AK}{(K+C)^2} = \\ &\quad \frac{r}{K} \frac{(-2K)(C+A)}{(K+C)^2} < 0 \end{aligned} \quad (2.2.17)$$

for both weak and strong Allee effects.

Nevertheless, it has been shown numerically that if $\alpha r := r \frac{K-A}{K+C} < 2$, then the additional increment of the function p_2 gained on the interval $(K, \tilde{x})$ does not exceed the difference $p_1(K) - p_1(K)$ for both weak and strong Allee effects. The numerical computations have been done in MATLAB. The point $\tilde{x}$ is located with the Newton's method. The increments of the functions $p_1(x)$ and $p_2(x)$ are computed using the Runge-Kutta fourth-order method.

Therefore, based on observations (i) – (iv), $p_1(x) > p_2(x)$ for all $x > K$. Hence, the inequality (2.2.15), and, consequently, inequality (2.2.13), hold for all $x > K$. ■

Now, we are well-equipped to prove the asymptotic stability of the equilibrium point $x^* = K$ for the Ricker equation with the weak Allee effect.

Theorem 2.5. *If the parameters of the equation (2.2.2) with the conditions (2.2.3) satisfy*

$$\alpha r < 2, \quad \text{where } \alpha := \frac{K - A}{K + C},$$

then the equilibrium point of the equation (2.2.2), $x^ = K$, is globally asymptotically stable on $\mathbb{R}^+$.*

Proof. Let us define the function $V(x) : \mathbb{R}_+ \rightarrow \mathbb{R}$ as $V(x) = (x - K)^2$.

Then, $V(x^*) = 0$ and $V(x) > 0$ for all $x \in \mathbb{R}_+ \setminus \{x^*\}$. Hence, by Definition 2.3, the function $V(x)$ is positive definite at x^* .

Let us show that $\Delta V(x) < 0$ for all $x \in \mathbb{R}^+ \setminus \{x^*\}$.

$$\Delta V(x(n)) := V(f(x(n))) - V(x(n)) = V(x(n+1)) - V(x(n)) =$$

$$(x(n+1) - K)^2 - (x(n) - K)^2 = (x(n+1) - x(n))(x(n+1) + x(n) - 2K).$$

Substituting $x(n+1)$ with the expression (2.2.2), we obtain

$$\begin{aligned} \Delta V(x(n)) = & (x(n))^2 \left(\exp \left(r \left(1 - \frac{x(n)}{K} \right) \left(1 - \frac{A+C}{x(n)+C} \right) \right) - 1 \right) \\ & \left(\exp \left(r \left(1 - \frac{x(n)}{K} \right) \left(1 - \frac{A+C}{x(n)+C} \right) \right) + 1 - \frac{2K}{x(n)} \right). \end{aligned} \quad (2.2.18)$$

Since $(x(n))^2 > 0$ for all $x \in \mathbb{R}_+$, the sign of $\Delta V(x(n))$ is determined by the signs of the brackets on the right-hand side of the expression (2.2.18). Let us analyze separately the sign of each bracket for $x(n) < x^*$ and $x(n) > x^*$.

The expression $\left(1 - \frac{A+C}{x(n)+C} \right)$ in equation (2.2.2) with the conditions (2.2.3) is positive for all $x(n) \in \mathbb{R}_+$. On the other hand, the expression $\left(1 - \frac{x(n)}{K} \right)$ changes its sign at $x^* = K$, that is,

$$\left(1 - \frac{x(n)}{K}\right) > 0 \text{ for } x(n) < K, \text{ and } \left(1 - \frac{x(n)}{K}\right) < 0 \text{ for } x(n) > K.$$

Hence, we conclude that the first bracket on the right-hand side of the expression (2.2.18),

namely, $\left(\exp\left(r\left(1 - \frac{x(n)}{K}\right)\left(1 - \frac{A+C}{x(n)+C}\right)\right) - 1\right)$, is positive for $x(n) < K$, and is

negative for $x(n) > K$.

Therefore, to prove that $\Delta V(x) < 0$ for all $x \in \mathbb{R}_+ \setminus \{x^*\}$, it suffices to show that the second bracket on the right-hand side of the equation (2.2.18), namely,

$$\left(\exp\left(r\left(1 - \frac{x(n)}{K}\right)\left(1 - \frac{A+C}{x(n)+C}\right)\right) + 1 - \frac{2K}{x(n)}\right), \quad (2.2.19)$$

is negative for $x(n) < K$, and is positive for $x(n) > K$.

Case 1: $x(n) < K$.

Then, $1 - \frac{x(n)}{K} > 0$, and $0 < 1 - \frac{A+C}{x(n)+C} < 1 - \frac{A+C}{K+C} = \alpha$.

Since, by the hypothesis of this theorem, $0 < \alpha r < 2$, we obtain the following:

$$\exp\left(r\left(1 - \frac{x(n)}{K}\right)\left(1 - \frac{A+C}{x(n)+C}\right)\right) < \exp\left(\alpha r\left(1 - \frac{x(n)}{K}\right)\right) < \exp\left(2\left(1 - \frac{x(n)}{K}\right)\right).$$

By Lemma 2.3 (inequalities (2.2.9)), for $x(n) < K$,

$$\exp\left(2\left(1 - \frac{x(n)}{K}\right)\right) + 1 - \frac{2K}{x(n)} < 0.$$

Hence,

$$\exp\left(r\left(1 - \frac{x(n)}{K}\right)\left(1 - \frac{A+C}{x(n)+C}\right)\right) + 1 - \frac{2K}{x(n)} < 0.$$

Case 2: $x(n) > K$.

On this interval, by Lemma 2.4,

$$\exp \left(r \left(1 - \frac{x}{K} \right) \left(1 - \frac{A+C}{x+C} \right) \right) + 1 - \frac{2K}{x} > 0.$$

Hence, taking into consideration the sign of each bracket in the expression (2.2.18),

$$\Delta V(x(n)) < 0 \quad \text{for all } x \in \mathbb{R}_+ \setminus \{K\}.$$

In addition, the function $V(x)$ is continuous on $\mathbb{R}_+$. Hence, the function $V(x)$ is the strict Liapunov function for $f(x)$.

Moreover, $V(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$.

Therefore, by Theorem B, the equilibrium point of the equation (2.2.2), $x^* = K$, is globally asymptotically stable on $\mathbb{R}_+$.

■

Figure 2.4: Global asymptotic stability of the equilibrium point $x^* = K$ for the Ricker equation with the weak Allee effect. Parameter values: $K = 10$, $C = 8$, $A = -0.5$, $r = 3.3$. Blue line: $x_1(0) = 0.05$; red line: $x_2(0) = 1.00$; black line: $x_3(0) = 50.0$.

Example 2.4. *The following equation is the Ricker equation with the weak Allee effect and*

the parameters $K = 10$, $C = 8$, $A = -0.5$, and $r = 3.3$:

$$x(n+1) = x(n) \exp \left(3.3 \left(1 - \frac{x(n)}{10} \right) \left(1 - \frac{7.5}{x(n) + 8} \right) \right), \quad x \in \mathbb{R}_+, \quad n \in \mathbb{N}_0. \quad (2.2.20)$$

Since $\alpha r = 1.8425 < 2$, then, by Theorem 2.5, the unique equilibrium of the equation (2.2.20) is globally asymptotically stable. This property is demonstrated in Figure 2.4 by the graphs for the solutions with the initial points $x_1(0) = 0.05$, $x_2(0) = 1.00$, and $x_3(0) = 50.0$.

Chapter 3

Periodic solutions for the Ricker equation with the weak Allee effect and periodic carrying capacity

In this chapter, one more biological phenomenon is included into the classical Ricker equation. The mechanism of periodically changing environmental conditions is represented by the periodic carrying capacity $K(n)$.

That is, the subject of the research in this chapter is the equation

$$x(n+1) = x(n) \exp \left(r \left(1 - \frac{x(n)}{K(n)} \right) \left(1 - \frac{A+C}{x(n)+C} \right) \right), \quad n \in \mathbb{N}_0 \quad (3.0.1)$$

where $\{K(n)\}$ is a strictly positive periodic sequence and the parameters of the equation satisfy the following conditions

$$r > 0, \quad A < 0 < C, \quad |A| < C, \quad \text{and} \quad x(0) > 0. \quad (3.0.2)$$

The change of the constant parameter K in the equation (1.3.8) to periodic $K(n)$ in the equation (3.0.1) generates the periodic solutions that, under certain conditions, can acquire

a property of the global asymptotic stability. The focus of this chapter is the study of the existence and the stability properties of such periodic solutions.

3.1 Diversity of the periodic solutions

Before we embark on studying the existence and properties of periodic solutions, let us look closer at the variety of those periodic solutions.

Example 3.1. *Consider the following three equations based on the equation (1.3.8), in which the constant carrying capacity is substituted for the periodic sequence $\{K(n)\}$:*

$$x(n+1) = x(n) \exp \left(1.5 \left(1 - \frac{x(n)}{K(n)} \right) \left(1 - \frac{5}{x(n)+6} \right) \right), \quad n \in \mathbb{N}_0 \quad (3.1.1)$$

with the 3-periodic sequence $\{K(n)\} = \{8, 10, 12, 8, 10, 12, \dots\}$;

the same equation (3.1.1) with the 3-periodic sequence $\{K(n)\} = \{8, 10, 40, 8, 10, 40, \dots\}$;

and the equation

$$x(n+1) = x(n) \exp \left(2.1 \left(1 - \frac{x(n)}{K(n)} \right) \left(1 - \frac{2}{x(n)+4} \right) \right), \quad n \in \mathbb{N}_0 \quad (3.1.2)$$

with the 2-periodic sequence $\{K(n)\} = \{5, 35, 5, 35, \dots\}$.

For simplicity, we choose the same initial condition for all three equations: $x(0) = 2$.

Figure 3.1: The solution for the equation (3.1.1) with the carrying capacity defined by the 3-periodic periodic sequence $\{K(n)\} = \{8, 10, 12, 8, 10, 12, \dots\}$. The solution converges to the 3-cycle $\{7.123, 9.304, 11.670\}$.

Figure 3.2: The solution for the equation (3.1.1) with the carrying capacity defined by the 3-periodic periodic sequence $\{K(n)\} = \{8, 10, 40, 8, 10, 40, \dots\}$. The solution converges to the 6-cycle $\{2.896, 4.618, 9.319, 7.889, 9.661, 20.960\}$.

The solutions obtained for the equations are presented in the form of two graphs: one graph, population density $x(n)$ versus n , shows the development of the solution on the time-scale; another, $x(n+1)$ versus $x(n)$, clarifies the type and existence of the cycle. In all

Figure 3.3: The solution for the equation (3.1.2) with the carrying capacity defined by the 2-periodic periodic sequence $\{K(n)\} = \{5, 35, 5, 35, \dots\}$.

three figures, the graphs of the solutions are shown for $n = 13900 - 14000$.

In Figure 3.1, we observe that the 3-periodic sequence $\{K(n)\}$ in equation (3.1.1) generates the stable 3-periodic solution. The same equation with only one term different in that sequence for the periodic carrying capacity generates the stable 6-periodic solution (Figure 3.2). That is, increasing the term $K(2)$ from the value of 12 to the value of 40 led to a bifurcation.

The process of bifurcation is not studied here. Therefore, all the following lemmas and theorems are restricted to the case when the p-periodic strictly positive sequence $\{K(n)\}$ generates a p-periodic solution.

The solution presented in Figure 3.3 appears chaotic, not converging to any stable cycle. This can have two possible explanations: either the equation 3.1.2 does not generate any stable cycles or the initial value $x(0)$ is not in the basin of attraction for such a cycle. The existence of the stable cycle for the Ricker equation with the weak Allee effect and the periodic carrying capacity is studied in Section 3.3.

Nevertheless, the first property of the solutions in Figures 3.1-3.3 to notice is that all presented solutions, including the "chaotic" one, appear to be "trapped" into the closed interval

with the respect to the population size. Do all the solutions for the Ricker equation with the weak Allee effect and the periodic carrying capacity eventually become bounded? This question is studied in Section 3.2.

3.2 Persistence of the solution for the Ricker equation with the weak Allee effect and the periodic carrying capacity.

In this section, we study the persistence property of the solutions for the equation

$$x(n+1) = x(n) \exp \left(r \left(1 - \frac{x(n)}{K(n)} \right) \left(1 - \frac{A+C}{x(n)+C} \right) \right), \quad n \in \mathbb{N}_0 \quad (3.2.1)$$

with

$$r > 0, \quad A < 0 < C, \quad |A| < C, \quad \text{and} \quad x(0) > 0. \quad (3.2.2)$$

Here, $\{K(n)\}$ is a strictly positive sequence, that is $K(n) \in \mathbb{R}^+$ for every $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$. Moreover, there exist positive constants K_* and K^* such that

$$0 < K_* \leq K(n) \leq K^*, \quad n \in \mathbb{N}_0. \quad (3.2.3)$$

For the proof of the following Theorem 3.1, it is convinient to introduce the following notations:

$$\alpha_* := \frac{K_* - A}{K_* + C}, \quad \alpha^* := \frac{K^* - A}{K^* + C}, \quad \alpha(n) := \frac{K(n) - A}{K(n) + C}.$$

Note: under the assumptions of the inequalities (3.2.2) and (3.2.3), for any $n \in \mathbb{N}_0$,

$$0 < \alpha_* \leq \alpha(n) \leq \alpha^* < 1.$$

Theorem 3.1. *If the parameters of the equation (3.2.1) satisfy the inequalities (3.2.2) and (3.2.3), then any positive solution $\{x(n)\}$ of (3.2.1) satisfies*

$$L_* \leq \liminf_{x \rightarrow \infty} x(n) \leq \limsup_{x \rightarrow \infty} x(n) \leq L^*, \quad (3.2.4)$$

where

$$L^* = \max \left\{ \frac{K^*}{r} \exp(r-1), \frac{K^*}{\alpha_* r} \right\}, \quad L_* = K_* \exp \left(r \left(1 - \frac{L^*}{K_*} \right) \right). \quad (3.2.5)$$

Moreover, if $\alpha_* r \geq 1$, then $L^* = \frac{K^*}{r} \exp(r-1)$.

Proof. This proof utilizes the idea of dividing the set of all solutions into the set of all non-increasing sequences (or non-decreasing sequences) and the set of all other solutions. This strategy was suggested by Zhou and Zou in [46]. For clarity, the proof is divided into three parts.

Part A.

In this part, we prove that

$$\limsup_{x \rightarrow \infty} x(n) \leq L^* \quad (3.2.6)$$

by considering two cases which exhaust all the possibilities.

Case 1. There exists $N \in \mathbb{N}_0$ such that $x(N) < x(N+1)$.

From the equation (3.2.1), it follows that

$$\left(r \left(1 - \frac{x(N)}{K(N)} \right) \left(1 - \frac{A+C}{x(N)+C} \right) \right) > 0.$$

Since, in equation (3.2.1), $\left(1 - \frac{A+C}{x(n)+C} \right) > 0$ for all $n \in \mathbb{N}_0$, then $\left(1 - \frac{x(N)}{K(N)} \right) > 0$. Therefore, $x(N) < K(N)$. This implies, by Lemma 1, the following upper boundaries on $x(N+1)$:

$$\text{if } \alpha(N)r > 1, \quad x(N+1) \leq \frac{K(N)}{r} \exp(r-1) \leq \frac{K^*}{r} \exp(r-1) \leq L^*;$$

if $\alpha(N)r \leq 1$, then the function $f(x)$ from Lemma 1 is increasing on $(0, K)$, hence

$$x(N+1) \leq K(N) \leq \frac{K(N)}{r} \exp(r-1) \leq \frac{K^*}{r} \exp(r-1) \leq L^*.$$

Thus, $x(N+1) \leq L^*$.

To prove the inequality (3.2.6) in Case 1, it suffices to show that $x(n) \leq L^*$ for all $n > N$.

Suppose, there exists $N_1 > N$ such that $x(N_1) > L^*$.

By Lemma 1, if $\alpha(N_1 - 1)r \leq 1$, and $x(N_1 - 1) \in \mathbb{R}^+$, then

$$x(N_1) \leq \frac{K(N_1 - 1)}{\alpha(N_1 - 1)r} \leq \frac{K^*}{\alpha_* r} \leq L^*;$$

if $\alpha(N_1 - 1)r > 1$, and $x(N_1 - 1) \in \mathbb{R}^+$, then

$$x(N_1) \leq \frac{K(N_1 - 1)}{r} \exp(r-1) \leq \frac{K^*}{r} \exp(r-1) \leq L^*.$$

This implies that $x(N_1 - 1) \notin \mathbb{R}^+$, which contradicts the hypothesis of Theorem 3.1 that $\{x(n)\}$ is a positive solution of the equation (3.2.1)

Hence, $x(n) \leq L^*$ for all $n \geq N$.

Case 2.

Suppose, that $\{x(n)\}$, the positive solution of the equation (3.2.1), is a non-increasing sequence, that is: $x(n) \geq x(n+1)$ for all $n \in \mathbb{N}_0$. Then, by the equation (3.2.1), for all $n \in \mathbb{N}_0$, $\left(1 - \frac{x(n)}{K(n)}\right) \leq 0$, and

$$x(n) \geq K(n) \geq K_*.$$

Thus, $\{x(n)\}$ is a non-increasing sequence with the lower bound K_* . Therefore, the sequence $\{x(n)\}$ converges to its infimum X_{inf} : $\lim_{n \rightarrow \infty} x(n) = X_{inf} \geq K_*$. Now, taking $\lim_{n \rightarrow \infty}$ for

both sides of the equation (3.2.1), we obtain:

$$X_{inf} = X_{inf} \lim_{n \rightarrow \infty} \left(\exp \left(r \left(1 - \frac{x(n)}{K(n)} \right) \left(1 - \frac{A+C}{x(n)+C} \right) \right) \right).$$

Hence, $\lim_{n \rightarrow \infty} \left(1 - \frac{x(n)}{K(n)} \right) = 0$, and, therefore, taking into consideration the condition (3.2.3),

$$\lim_{n \rightarrow \infty} x(n) = X_{inf} = \lim_{n \rightarrow \infty} K(n) \leq K^* \leq L^*.$$

This concludes the proof of the inequality (3.2.6).

Note: we also proved that for the positive solution of the equation (3.2.1) to be a non-increasing sequence, the sequence $\{K(n)\}$ must converge.

Part B.

Here, we prove that

$$\liminf_{x \rightarrow \infty} x(n) \geq L_*. \quad (3.2.7)$$

By the definition of the limit superior and the inequality (3.2.6), for any $\epsilon > 0$ there exists $N_2 \in \mathbb{N}$ such that

$$x(n) \leq L^* + \epsilon, \text{ for all } n \geq N_2 \quad (3.2.8)$$

Similar to the Part A, we consider two cases which exhaust all the possibilities for the positive solutions of the equation (3.2.1).

Case 1. There exists $N_3 \in \mathbb{N}$ such that $N_3 \geq N_2$ and $x(N_3 + 1) < x(N_3)$.

Then, by the equation (3.2.1), $x(N_3) > K(N_3) > K_*$. This implies

$$x(N_3 + 1) \geq K_* \exp \left(r \left(1 - \frac{L^* + \epsilon}{K_*} \right) \left(1 - \frac{A+C}{x(N_3)+C} \right) \right) \geq K_* \exp \left(r \left(1 - \frac{L^* + \epsilon}{K_*} \right) \right).$$

Letting $\epsilon \rightarrow 0$, we obtain:

$$x(N_3 + 1) \geq K_* \exp \left(r \left(1 - \frac{L^*}{K_*} \right) \right) = L_*.$$

Now, suppose there exists $N_4 > N_3 \geq N_2$ such that $x(n) \geq L_*$ for all $N_3 + 1 \leq n \leq N_4$ and $x(N_4 + 1) < L_*$. Then, by the equation (3.2.1), this implies:

$$x(N_4) \exp \left(r \left(1 - \frac{x(N_4)}{K(N_4)} \right) \left(1 - \frac{A+C}{x(N_4)+C} \right) \right) < L_*. \quad (3.2.9)$$

Since $x(N_4 + 1) < x(N_4)$, then $\left(1 - \frac{x(N_4)}{K(N_4)} \right) < 0$ and, therefore, $x(N_4) > K(N_4)$. This allows to modify the inequality (3.2.9) as follows:

$$K(N_4) \exp \left(r \left(1 - \frac{x(N_4)}{K(N_4)} \right) \right) < x(N_4) \exp \left(r \left(1 - \frac{x(N_4)}{K(N_4)} \right) \left(1 - \frac{A+C}{x(N_4)+C} \right) \right) < L_*,$$

that is,

$$K(N_4) \exp \left(r \left(1 - \frac{x(N_4)}{K(N_4)} \right) \right) < K_* \exp \left(r \left(1 - \frac{L^*}{K_*} \right) \right). \quad (3.2.10)$$

Since $K(N_4) \geq K_*$, the inequality (3.2.10) implies:

$$1 \leq \frac{K(N_4)}{K_*} < \exp \left(r \frac{x(N_4) - L^*}{K_*} \right).$$

Thus, the supposition that $x(N_4 + 1) < L_*$ implies that $x(N_4) > L^*$. This is a contradiction to the inequality (3.2.8), since that inequality holds for any $\epsilon > 0$. Hence, we conclude that, in Case 1, $x(n) > L_*$ for all $n \geq N_3 + 1$, and the inequality (3.2.7) holds.

Case 2. Suppose, that $\{x(n)\}$, the positive solution of the equation (3.2.1), is a non-decreasing sequence, that is: $x(n) \leq x(n+1)$ for all $n \in \mathbb{N}_0$. By the inequality (3.2.8), this sequence has the upper bound and, therefore, converges to its supremum X_{sup} :

$\lim_{n \rightarrow \infty} x(n) = X_{sup}$. By the same reasoning as in Part A (Case 2), we obtain:

$$\lim_{n \rightarrow \infty} x(n) = X_{sup} = \lim_{n \rightarrow \infty} K(n) \geq K_*.$$

From the expressions (3.2.5), $L^* \geq K^* \geq K_*$, then $L_* \leq K_*$. Hence, $\lim_{n \rightarrow \infty} x(n) \geq L_*$, and the inequality (3.2.7) holds in this case.

Part C.

To conclude the proof for the Theorem 1, we observe the following: if $\alpha_* r \geq 1$, then

$$\frac{K^*}{\alpha_* r} \leq K^*.$$

Moreover, since, under the conditions (3.2.2), $\alpha_* < 1$, then the inequality $\alpha_* r \geq 1$ implies $r > 1$. By the proof of the Corollary 3.4.1, the equation

$$\exp(r - 1) = r$$

has the unique solution $r = 1$, and, hence, by the Corollary 3.4.1, the inequality $\alpha_* r \geq 1$ also implies that $\exp(r - 1) > r$. Hence,

$$\frac{K^*}{r} \exp(r - 1) > K^*.$$

Thus, if $\alpha_* r \geq 1$, then $L^* = \max \left\{ \frac{K^*}{r} \exp(r - 1), \frac{K^*}{\alpha_* r} \right\} = \frac{K^*}{r} \exp(r - 1)$.

■

The hypotheses of Theorem 3.1 do not require the sequence $\{K(n)\}$ to be periodic. However, every strictly positive periodic sequence $\{K(n)\}$ satisfies the inequalities (3.2.3) with

$$K_* = \min_{n \in \mathbb{N}_0} \{K(n)\}, \quad K^* = \max_{n \in \mathbb{N}_0} \{K(n)\}.$$

Thus, Theorem 3.1 is applicable to the Ricker equation with the weak Allee effect and periodic carrying capacity (3.0.1) with the conditions (3.0.2).

Example 3.2. *Applying Theorem 3.1 to the equations (3.1.1) and (3.1.2) in Example 3.1, we expect the following solution boundaries to hold eventually:*

(1). *For the equation (3.1.1) with the carrying capacity defined by the 3-periodic periodic sequence $\{K(n)\} = \{8, 10, 12, 8, 10, 12, \dots\}$, we obtain $L_* \approx 3.0233$ and $L^* \approx 13.1900$. For the solution $\{7.123, 9.304, 11.670\}$, the inequality*

$$L_* \leq x(n) \leq L^* \quad (3.2.11)$$

holds.

(2). *Similarly, for the equation (3.1.1) with the carrying capacity defined by the 3-periodic periodic sequence $\{K(n)\} = \{8, 10, 40, 8, 10, 40, \dots\}$, we obtain $L_* \approx 0.009427$ and $L^* \approx 43.9659$. Thus, for the solution $\{2.896, 4.618, 9.319, 7.889, 9.661, 20.960\}$, the inequality (3.2.11) also holds.*

(3). *For the equation (3.1.2) with the carrying capacity defined by the 2-periodic periodic sequence $\{K(n)\} = \{5, 35, 5, 35, \dots\}$, the periodic solution was not found numerically. Nevertheless, the evaluation of the solution sequence reveals that the values of the sequence $\{x(n)\}$ for $13900 \leq n \leq 14000$ are in the limits of $0.0256 \leq x(n) \leq 22.4584$. With $L_* \approx 3.0070e-8$ and $L^* \approx 50.0694$, the inequality (3.2.11) holds for $13900 \leq n \leq 14000$.*

3.3 Periodic solution for the Ricker's equation with the weak Allee effect and the periodic carrying capacity: Existence and Properties.

In this section, we prove the existence and the uniqueness of a periodic solution for

the equation (3.0.1) with the conditions (3.0.2) and the global asymptotic stability of that solution.

3.3.1 Existence and the global asymptotic stability

In order to prove the existence of the periodic solution, let us recall the **Brouwer's Fixed Point Theorem** for the one-dimensional case (see, for example, [13]):

Theorem C. *Let $h : [a, b] \rightarrow [a, b]$ be the real function which is continuous on the closed interval $[a, b]$. Then, there exists $x^* \in [a, b]$ such that $h(x^*) = x^*$.*

In the following theorem, we prove that the p -periodic solution exists for the Ricker equation with the weak Allee effect and p -periodic carrying capacity for any $p \in \mathbb{N}$.

Theorem 3.2. *Let $\{K(n)\}$ be a strictly positive p -periodic sequence, that is,*

$$K(n + p) = K(n) \text{ for all } n \in \mathbb{N}_0, \text{ where } p \geq 1.$$

Then, there exists a p -periodic solution for the equation (3.0.1) with the conditions (3.0.2) .

Proof. For the case $p=1$, equation (3.0.1) is reduced to the form (2.2.2) with the conditions (2.2.3). Then, by Lemma 2.2, there exists the 1-periodic solution $x = K$.

Now, let us consider the case with $p > 1$.

Since $\{K(n)\}$ is the periodic sequence, then there exist the minimum and the maximum values of $K(n)$. Let us denote

$$K_* := \min_{n \in \mathbb{N}_0} \{K(n)\}, \quad K^* := \max_{n \in \mathbb{N}_0} \{K(n)\}. \quad (3.3.1)$$

Then, $\{K(n)\}$ satisfies the condition (3.2.3).

Let us denote the following functions:

$$f_n(x) = x \exp \left(r \left(1 - \frac{x}{K(n)} \right) \left(1 - \frac{A + C}{x + C} \right) \right), \quad 0 \leq n < p, \quad n \in \mathbb{N}_0$$

and

$$F(x) = (f_{p-1} \circ f_{p-2} \cdots \circ f_0)(x).$$

Since $f_n(x)$ is continuous as the composition of the continuous functions for all $0 \leq n < p$, then, $F(x)$ is also continuous as the composition of the continuous functions.

By Theorem 3.1, for all $0 \leq n < p$, the function $f_n(x)$ maps the closed interval $[L_*, L^*]$ into itself. Hence, we conclude that $F(x)$ maps the closed interval $[L_*, L^*]$ into itself either.

Then, by Theorem C, there exists $x^* \in [L_*, L^*]$ such that $F(x^*) = x^*$.

Let us denote $\tilde{x}_{n+1} := f_n(x(n))$ for $0 \leq n < p$, and $\tilde{x}_0 := x^*$.

Then, $\{\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_{p-1}\}$ is the p -periodic solution for the equation (3.0.1) with the conditions (3.0.2). ■

Note: The idea of constructing the composition function for the proof of Theorem 3.2 is adopted from [46].

Now, when the existence of the p -periodic cycle for the Ricker equation with the weak Allee effect and the p -periodic carrying capacity is proven, we attempt to show that under certain conditions, this periodic solution is globally asymptotically stable on $\mathbb{R}_+$.

The following notations simplify the further reasoning. Let $\{K(n)\}$ be a strictly positive p -periodic sequence, the parameters r, A, C in the equation (3.0.1) satisfy the conditions (3.0.2). Then, we denote the new parameter $\beta(n)$ as follows:

$$\beta(n) := 1 - \frac{(A + C)(K(n) + C)}{(\tilde{x}(n) + C)(x(n) + C)}, \quad (3.3.2)$$

where $\{\tilde{x}(n)\}, n \in \mathbb{N}_0$, is the p -periodic solution for the equation (3.0.1) with the conditions (3.0.2), and $\{x(n)\}, n \in \mathbb{N}_0$, is any other solution for the equation (3.0.1) with the conditions (3.0.2). We also denote the parameters β_* and β^* , corresponding to the parameters K_* and

K^* (expressions (3.3.1)) and the parameters L^* and L_* , denoted by the expressions (3.2.5) as follows:

$$\beta_* := 1 - \frac{(A+C)(K^*+C)}{(L_*+C)^2}, \quad \beta^* := 1 - \frac{(A+C)(K_*+C)}{(L^*+C)^2}$$

.

Let us notice that for the weak Allee conditions (3.0.2), $\beta(n) < 1$ for all $n \in \mathbb{N}_0$.

Furthermore, the properties of the parameter $\beta(n)$, proven in the next lemma, are needed for the proof of the global asymptotic stability of the periodic cycle.

Lemma 3.3. *Suppose that $\{\tilde{x}(n)\}$ and $\{x(n)\}, n \in \mathbb{N}_0$, are two solutions for the equation (3.0.1) with the conditions (3.0.2). Then, there exists $N \in \mathbb{N}_0$ such that for all $n \geq N$*

$$1 - \frac{(A+C)(K^*+C)}{(L_*+C)^2} \leq 1 - \frac{(A+C)(K(n)+C)}{(\tilde{x}(n)+C)(x(n)+C)} \leq 1 - \frac{(A+C)(K_*+C)}{(L^*+C)^2},$$

that is,

$$\beta_* \leq \beta(n) \leq \beta^*. \quad (3.3.3)$$

Proof. Since $\{\tilde{x}(n)\}$ and $\{x(n)\}$ are the solutions for the equation (3.0.1) with the conditions (3.0.2), then, by Theorem 3.1, there exist $n_1 \in \mathbb{N}_0$ and $n_2 \in \mathbb{N}_0$ such that $L_* \leq \{\tilde{x}(n)\} \leq L^*$ for all $n \geq n_1$, and $L_* \leq \{x(n)\} \leq L^*$ for all $n \geq n_2$.

Hence, there exists $n_3 = \max\{n_1, n_2\}$, such that

$$(L_*+C)^2 \leq (\tilde{x}(n)+C)(x(n)+C) \leq (L^*+C)^2.$$

The inequalities (3.3.3) follow. ■

For the proof of the global attractivity of the p-periodic cycle, the following lemma and corollary are helpful.

Lemma 3.4. *For any $y \in \mathbb{R}$,*

$$\exp(y) - 1 = y \exp(\theta y), \quad (3.3.4)$$

where $\theta \in (0, 1)$.

Proof. Fix $y \in \mathbb{R}$.

Consider the real-valued function $f(x) = \exp(xy)$ on the interval $[0, 1]$. This function is continuous on $[0, 1]$ and differentiable on $(0, 1)$. Hence, by the Mean Value Theorem, there exists a point $\theta \in (0, 1)$ such that

$$f'(\theta) = \frac{f(1) - f(0)}{1 - 0} = \exp(y) - 1.$$

Since $f'(x) = y \exp(xy)$, then

$$f'(\theta) = y \exp(\theta y),$$

and

$$y \exp(\theta y) = \exp(y) - 1$$

where $\theta \in (0, 1)$.

Since the choice of y was arbitrary, the expression (3.3.4) holds for any $y \in \mathbb{R}$. ■

Corollary 3.4.1. *For any $x \in \mathbb{R}$,*

$$\exp(x - 1) \geq x. \quad (3.3.5)$$

Proof. Denote $y := x - 1$. Then, the inequality (3.3.5) is equivalent to the inequality

$$\exp(y) - 1 \geq y.$$

By Lemma 3.4, to prove this inequality, it suffices to show that

$$y \exp(\theta y) \geq y,$$

where $\theta \in (0, 1)$.

Indeed, since $\exp(\theta y) < 1$ for $y < 0$, and $\exp(\theta y) > 1$ for $y > 0$, the last inequality holds. ■

At this point, we are fully equipped to prove the global asymptotic stability of the p -periodic cycle, which existence was proven by the Theorem 3.2. One of the ideas for the proof, specifically, the construction of the Liapunov function for the secondary sequence, was introduced earlier in [46].

Theorem 3.5. *If the parameters of the equation (3.0.1) with the conditions (3.0.2) satisfy the inequalities*

$$\beta_* > 0 \quad \text{and} \quad \frac{r\beta^*L^*}{K_*} < 2, \quad (3.3.6)$$

then the p -periodic solution $\{\tilde{x}(0), \tilde{x}(1), \dots, \tilde{x}(p-1)\}$ for the equation (3.0.1) is globally asymptotically stable.

Proof. Case 1: $p = 1$.

In this case, the carrying capacity K has the constant value, and the expression for the L^* becomes

$$L^* = \max \left\{ \frac{K}{r} \exp(r-1), \frac{K}{\alpha r} \right\}.$$

Suppose, $L^* = \frac{K}{r} \exp(r-1)$.

Then, the second inequality in (3.3.6) takes the following form

$$\exp(r-1) \left(1 - \frac{(A+C)(K+C)}{\left(\frac{K}{r} \exp(r-1) + C \right)^2} \right) < 2. \quad (3.3.7)$$

By Corollary 3.4.1, $\frac{\exp(r-1)}{r} \geq 1$. Therefore,

$$\frac{K}{r} \exp(r-1) + C \geq K + C.$$

Hence, the inequality (3.3.7) implies

$$\exp(r-1) \left(1 - \frac{(A+C)(K+C)}{(K+C)^2} \right) < 2. \quad (3.3.8)$$

Applying Corollary 3.4.1 again, we obtain:

$$r \left(1 - \frac{(A+C)(K+C)}{(K+C)^2} \right) = \alpha r < 2.$$

Then, by Theorem 2.5, the solution $x^* = K$ is globally asymptotically stable.

Now, suppose, $L^* = \frac{K}{\alpha r}$. Then, from the definition for L^* , we conclude that

$$\frac{K}{r} \exp(r-1) < \frac{K}{\alpha r},$$

and the hypothesis of the theorem, namely, the inequality

$$\frac{r\beta^* \frac{K}{\alpha r}}{K} < 2$$

implies

$$\frac{r\beta^* \frac{K}{r} \exp(r-1)}{K} < 2.$$

and, therefore, the inequality (3.3.7). Hence, the same proof holds for this case.

Case 2: $p > 1$.

By Theorem 3.2, there exists a p -periodic solution for the equation (3.0.1), namely, $\{\tilde{x}(n)\}, n \in \mathbb{N}_0$. Since the equation (3.0.1) with the initial condition $x_0 > 0$ has only positive solutions, let us represent the arbitrary solution $\{x(n)\}, n \in \mathbb{N}_0$ for the equation (3.0.1) through the values of the p -periodic solution $\{\tilde{x}(n)\}$ as follows:

$$x(n) = \tilde{x}(n) \exp(y(n)), \quad n \in \mathbb{N}_0, \quad y(n) \in \mathbb{R}. \quad (3.3.9)$$

Hence, to prove that the p -periodic $\{\tilde{x}(n)\}$ is globally asymptotically stable on $\mathbb{R}_+$, it suffices to prove that $y^* = 0$ is the globally asymptotically stable equilibrium point for any sequence $\{y(n)\}$ defined by the equation (3.3.9).

We start with the construction of the equation for which the sequence $\{y(n)\}$ is the solution. Substituting the expression (3.3.9) in the equation (3.0.1), we obtain:

$$\begin{aligned} \tilde{x}(n+1) \exp(y(n+1)) = \\ \tilde{x}(n) \exp(y(n)) \exp \left(r \left(1 - \frac{\tilde{x}(n) \exp(y(n))}{K(n)} \right) \left(1 - \frac{A+C}{\tilde{x}(n) \exp(y(n)) + C} \right) \right). \end{aligned} \quad (3.3.10)$$

On the left-hand side of the equation (3.3.10), we substitute the $\tilde{x}(n+1)$ with the equivalent expression from the equation (3.0.1):

$$\begin{aligned} \tilde{x}(n) \exp \left(r \left(1 - \frac{\tilde{x}(n)}{K(n)} \right) \left(1 - \frac{A+C}{\tilde{x}(n) + C} \right) \right) \exp(y(n+1)) = \\ \tilde{x}(n) \exp(y(n)) \exp \left(r \left(1 - \frac{\tilde{x}(n) \exp(y(n))}{K(n)} \right) \left(1 - \frac{A+C}{\tilde{x}(n) \exp(y(n)) + C} \right) \right). \end{aligned} \quad (3.3.11)$$

In the equation (3.3.11), we cancel $\tilde{x}(n)$ on both sides and take the logarithm with the

following equation as a result:

$$y(n+1) + r \left(1 - \frac{\tilde{x}(n)}{K(n)} \right) \left(1 - \frac{A+C}{\tilde{x}(n)+C} \right) = \\ y(n) + r \left(1 - \frac{\tilde{x}(n) \exp(y(n))}{K(n)} \right) \left(1 - \frac{A+C}{\tilde{x}(n) \exp(y(n)) + C} \right). \quad (3.3.12)$$

Then, from the equation (3.3.12), we derive the equation for the sequence $\{y(n)\}$:

$$y(n+1) = \\ y(n) - \frac{r}{K(n)} \tilde{x}(n) (\exp(y(n)) - 1) \left(1 - \frac{(A+C)(K(n)+C)}{(\tilde{x}(n)+C)(\tilde{x}(n) \exp(y(n)) + C)} \right). \quad (3.3.13)$$

Let us show that $y^* = 0$ is the equilibrium point for the equation (3.3.13).

Indeed, by Lemma 3.4, the equation (3.3.13) can be written in the following form:

$$y(n+1) = \\ y(n) - \frac{r}{K(n)} \tilde{x}(n) y(n) \exp(\theta y(n)) \left(1 - \frac{(A+C)(K(n)+C)}{(\tilde{x}(n)+C)(\tilde{x}(n) \exp(y(n)) + C)} \right), \quad (3.3.14)$$

where $\theta \in (0, 1)$.

Hence, if $y(n) = 0$, then $y(n+1) = y(n) = 0$. That is, the point $y^* = 0$ is the equilibrium point for the equation (3.3.13).

The final part of the proof then is to show that, under the hypotheses of the theorem, this equilibrium point is globally asymptotically stable for the equation (3.3.14).

Let us denote the function $V : \mathbb{R} \rightarrow \mathbb{R}$ as follows: $V(y) = y^2$. Clearly, the function $V(y)$ is continuous on $\mathbb{R}$ and positive definite with respect to $y^* = 0$. Moreover, on $\mathbb{R}$, $V(y) \rightarrow \infty$ as $\|y\| \rightarrow \infty$.

Thus, to satisfy all the hypotheses of Theorem B, and hence, to prove the global asymptotic stability of the point $y^* = 0$, it suffices to show that the variation of V relative to the equa-

tion (3.3.14), is strictly negative on $\mathbb{R}$.

By Definition 2.1,

$$\begin{aligned} \Delta V(y(n)) &= V(y(n+1)) - V(y(n)) = (y(n+1))^2 - (y(n))^2 = \\ &= -\frac{r}{K(n)} (y(n))^2 \tilde{x}(n) \exp(\theta y(n)) \left(1 - \frac{(A+C)(K(n)+C)}{(\tilde{x}(n)+C)(\tilde{x}(n)\exp(y(n))+C)}\right) \\ &\quad \left(2 - \frac{r}{K(n)} \tilde{x}(n) \exp(\theta y(n)) \left(1 - \frac{(A+C)(K(n)+C)}{(\tilde{x}(n)+C)(\tilde{x}(n)\exp(y(n))+C)}\right)\right). \end{aligned} \quad (3.3.15)$$

To simplify the expression (3.3.15) for the variation of V relative to the equation (3.3.14), we use the notation (3.3.2) to obtain:

$$\begin{aligned} \Delta V(y(n)) &= \\ &= -\frac{r}{K(n)} (y(n))^2 \tilde{x}(n) \exp(\theta y(n)) \beta(n) \left(2 - \frac{r}{K(n)} \tilde{x}(n) \exp(\theta y(n)) \beta(n)\right). \end{aligned} \quad (3.3.16)$$

Let us analyze the properties of the variation $\Delta V(y(n))$.

By the hypothesis of the theorem, $\beta_* > 0$. Then, by Lemma 3.3,

$$\beta(n) \geq \beta_* > 0.$$

Thus, the expression (3.3.16) is strictly negative on $\mathbb{R}$, if and only if

$$2 - \frac{r}{K(n)} \tilde{x}(n) \exp(\theta y(n)) \beta(n) > 0. \quad (3.3.17)$$

Sequences $\{\tilde{x}(n)\}$ and $\{x(n)\}$, $n \in \mathbb{N}_0$, are the solutions for the equation (3.0.1) with the conditions (3.0.2). Hence, by Theorem 3.1, there exist $n_1 \in \mathbb{N}_0$ and $n_2 \in \mathbb{N}_0$ such that

$$L_* \leq \{\tilde{x}(n)\} \leq L^* \text{ for all } n \geq n_1, \text{ and } L_* \leq \{x(n)\} \leq L^* \text{ for all } n \geq n_2.$$

By the observation made in [46],

$$\tilde{x}(n) \leq \tilde{x}(n) \exp(\theta y(n)) \leq \tilde{x}(n) \exp(y(n)), \quad y(n) \geq 0,$$

and

$$\tilde{x}(n) \exp(y(n)) \leq \tilde{x}(n) \exp(\theta y(n)) \leq \tilde{x}(n), \quad y(n) < 0.$$

Hence, there exists $n_3 = \max\{n_1, n_2\}$, such that for all $n \geq n_3$,

$$L_* \leq \tilde{x}(n) \exp(\theta y(n)) \leq L^*.$$

Therefore, the hypothesis of the theorem that $\frac{r\beta^*L^*}{K_*} < 2$ and Lemma 3.3 imply that, for all $n \geq n_3$,

$$\frac{r}{K(n)} \tilde{x}(n) \exp(\theta y(n)) \beta(n) < 2,$$

and the condition (3.3.17) is satisfied.

Thus, for every solution $\{y(n)\}$ for the equation (3.3.14), there exists $n_3 \in \mathbb{N}_0$ such that for all $n \geq n_3$ the function $V(y) = y^2$ is the strict Liapunov function and satisfies the hypotheses of the Theorem B with respect to the equilibrium point $y^* = 0$. This allows us to conclude that every solution $\{y(n)\}$ for the equation (3.3.14) eventually converges to the equilibrium point $y^* = 0$. That is, the equilibrium point $y^* = 0$ for the equation (3.3.14) is globally asymptotically stable on $\mathbb{R}$.

Hence, under the hypotheses of theorem, the p-periodic solution $\{\tilde{x}(n)\}$ for the equation (3.0.1) with the conditions (3.0.2) is globally asymptotically stable on $\mathbb{R}^+$.

■

3.3.2 Discussion of Theorem 3.5. Unstable periodic solutions

Let us make several additional observations in regards to the results of the Theorem 3.5.

(1). Under the hypotheses of Theorem 3.5, the p -periodic solution for the equation (3.0.1) with the conditions (3.0.2) is unique.

(2). By the theorem on the global asymptotic stability of the p -periodic solution for the Ricker equation without the Allee effect [46], the p -periodic cycle for the equation

$$x(n+1) = x(n) \exp \left(r \left(1 - \frac{x(n)}{K(n)} \right) \right), \quad n \in \mathbb{N}_0, \quad (3.3.18)$$

is globally asymptotically stable if the parameters of the equation (3.3.18) satisfy the inequality

$$\frac{K^*}{K_*} \exp(r-1) \leq 2. \quad (3.3.19)$$

Let us compare the inequality (3.3.19) with the similar case for the Ricker's equation with the weak Allee effect.

By Theorem 3.1, $L^* = \max \left\{ \frac{K^*}{r} \exp(r-1), \frac{K^*}{\alpha_* r} \right\}$. Therefore, if

$$\frac{K^*}{r} \exp(r-1) \geq \frac{K^*}{\alpha_* r},$$

or, equivalently,

$$r \geq 1 + \ln \left(\frac{K_* + C}{K_* - A} \right),$$

the hypothesis of Theorem 3.5, $\frac{r\beta^* L^*}{K_*} < 2$, can be written as

$$\beta^* \frac{K^*}{K_*} \exp(r-1) \leq 2. \quad (3.3.20)$$

By the hypothesis of Theorem 3.5, $\beta_* > 0$. Then, $0 < \beta^* < 1$. Comparing the inequalities (3.3.19) and (3.3.20), we observe that in the case of the Ricker equation with the weak Allee effect, the p -periodic solution remains globally asymptotically stable under the much wider range of the same parameters than the Ricker's equation without the Allee effect.

For example, the condition for the p -periodic solution for the equation (3.3.18) to remain globally asymptotically stable can be written as

$$r \leq 1 + \ln \left(\frac{2K_*}{K^*} \right),$$

whereas the same condition for the equation (3.0.1) with the conditions (3.0.2) is

$$r \leq 1 + \ln \left(\frac{2K_*}{K^*} \right) + \ln \left(\frac{1}{\beta^*} \right).$$

Since $\ln \left(\frac{1}{\beta^*} \right) > 0$, the range of possible values of the growth rate is greater for the Ricker equation with the weak Allee effect. Perhaps, this is the part of the survival mechanism for the species: to keep the higher growth rate in the given set of parameters.

Example 3.3. *In this example, we observe that adding the mechanism of the weak Allee effect to a Ricker equation with the periodic carrying capacity (3.3.18) increases the range of the per capita growth rate for which the p -periodic solution remains globally asymptotically stable.*

Consider the equation (3.3.18) and equation (3.0.1) with the same 2-periodic sequence for the carrying capacity: $K\{n\} = \{32, 35\}$ and per capita growth rate $r = 2.5$. The rest of the parameters for the equation (3.0.1) are $A = -8$, $C = 15$.

Thus, the equation with the periodic carrying capacity and without the Allee effect is

$$x(n+1) = x(n) \exp \left(2.5 \left(1 - \frac{x(n)}{K(n)} \right) \right), \quad n \in \mathbb{N}_0. \quad (3.3.21)$$

The corresponding equation with the weak Allee effect is

$$x(n+1) = x(n) \exp \left(2.5 \left(1 - \frac{x(n)}{K(n)} \right) \left(1 - \frac{7}{x(n) + 15} \right) \right), \quad n \in \mathbb{N}_0 \quad (3.3.22)$$

Figure 3.4: Top panel: Two solutions for the equation (3.3.21), with the initial conditions $x_1(0) = 2$ and $x_2(0) = 22$ respectively, converge to two distinct cycles. Lower panel: Two periodic solutions for the equation (3.3.21). The 2-periodic solution is $\{11.74, 57.16\}$ and the 4-periodic solution is $\{15.05, 62.58, 5.741, 46.41\}$.

In Figure 3.4, four solutions for the equation (3.3.21) are shown. Two solutions are periodic: the 2-periodic cycle $\{11.74, 57.16\}$ and the 4-periodic solution $\{15.05, 62.58, 5.741, 46.41\}$. The existence of the 4-periodic solution already proves that the 2-periodic solution generated by the 2-periodic sequence $K\{n\} = \{32, 35\}$ has lost the global asymptotic stability property for $r = 2.5$. The plots in the top panel of Figure 3.4 shows that each periodic solution is asymptotically stable and has its own basin of attraction.

On the other hand, the 2-periodic solution for the equation with the weak Allee effect (3.3.22) remains globally asymptotically stable with the per capita growth rate value $r = 2.5$ as it shown

Figure 3.5: Top panel: Three solutions for the equation (3.3.22), with the initial conditions $x_1(0) = 2$, $x_2(0) = 22$ and $x_2(0) = 50$ respectively, converge to a 2-periodic solution for this equation. Lower panel: $\{17.08, 46.46\}$ is the 2-periodic solution for the equation (3.3.22).

in Figure 3.5.

(3). One of the hypotheses of the Theorem 3.5, namely, the inequality $\beta_* > 0$, might seem to be unnecessary in the general case and constructed just for the specific Liapunov function in the proof of this theorem. However, it turns out that in case when all $\beta(n) < 0$, $n \in \{0, 1, \dots, p-1\}$, the p -periodic cycle for the equation (3.2.1) with the conditions (3.2.2) loses its global asymptotic stability: see Theorem 3.6.

In order to analyze further the significance of the parameter $\beta(n)$, let us recall the following theorem ([21] Theorem 4.27, p.213):

Theorem D. *If ΔV is positive definite in a neighborhood of the origin and there exists a sequence $a_i \rightarrow 0$ with $V(a_i) > 0$, then the zero solution of the equation*

$$x(n+1) = f(x(n))$$

is unstable.

Now, we are well-equipped to specify the case when the p-periodic solution for the Ricker's equation with the weak Allee effect is unstable.

Theorem 3.6. *Let $\{K(n)\}$ be a strictly positive p-periodic sequence.*

If the parameters of the equation (3.0.1) with the conditions (3.0.2) satisfy the inequality

$$(A + C)(K_* + C) > (L^* + C)^2,$$

then the p-periodic solution for the equation (3.0.1) is unstable.

Proof. By Theorem 3.2, there exists the p-periodic solution, $\{\tilde{x}(0), \tilde{x}(1), \dots, \tilde{x}(p-1)\}$, for the equation (3.0.1) with the conditions (3.0.2).

As in the proof for Theorem 3.5, let us represent the arbitrary solution $\{x(n)\}, n \in \mathbb{N}_0$, for the equation (3.0.1) through the values of the p-periodic solution as follows:

$$x(n) = \tilde{x}(n) \exp(y(n)), \quad n \in \mathbb{N}_0, \quad y(n) \in \mathbb{R}.$$

Moreover, as in the proof for the Theorem 3.5, let us define the function $V : \mathbb{R} \rightarrow \mathbb{R}$ as follows: $V(y) = y^2$, with the variation of V relative to the equation (3.3.14) expressed by the formula

$$\Delta V(y(n)) = -\frac{r}{K(n)} (y(n))^2 \tilde{x}(n) \exp(\theta y(n)) \beta(n) \left(2 - \frac{r}{K(n)} \tilde{x}(n) \exp(\theta y(n)) \beta(n) \right).$$

By hypothesis,

$$\beta^* := 1 - \frac{(A + C)(K_* + C)}{(L^* + C)^2} < 0.$$

Then, by Lemma 3.3, $\beta(n) < 0$ for all $n \in \mathbb{N}_0$, and the expression for $\Delta V(y(n))$ is strictly positive for all $y(n) \in \mathbb{R}_+$ and equals zero for the equilibrium point $y^* = 0$ of the equation (3.3.14). That is, ΔV is positive definite in the neighborhood of origin $y^* = 0$.

In addition, let us define the sequence $\{a_i\}$ as follows: $a_i = \frac{1}{i}$, where $i \in \mathbb{N}$. Then, $V(a_i) > 0$ for all $i \in \mathbb{N}$.

Hence, by Theorem D, the equilibrium point $y^* = 0$ of the equation (3.3.14) is unstable. That is, for any solution $\{y(n)\}$ for the equation (3.3.14) with $y(0) \neq 0$, there exists $\epsilon_0 > 0$ and $N(\epsilon_0, y(0)) \in \mathbb{N}_0$ such that for all $n \geq N$, $|y(n)| \geq \epsilon_0$.

Now, let us choose the arbitrary solution $\{x(n)\}$ for the equation (3.0.1) with the conditions (3.0.2) such that $x(0) \in \mathbb{N}_0 \setminus \{\tilde{x}(0), \tilde{x}(1), \dots, \tilde{x}(p-1)\}$. Since $x(0) \notin \{\tilde{x}(0), \tilde{x}(1), \dots, \tilde{x}(p-1)\}$, there exists the corresponding solution for the equation (3.3.14) with $y(0) \neq 0$, such that $x(n) = \tilde{x}(n) \exp(y(n))$. Then,

$$|x(n) - \tilde{x}(n)| = |\tilde{x}(n) \exp(y(n)) - \tilde{x}(n)| = |\tilde{x}(n)| |\exp(y(n)) - 1|.$$

Moreover, there exists

$$\epsilon_1 = |\exp(\epsilon_0) - 1| \min_{n \in \{0, 1, \dots, p-1\}} |\tilde{x}(n)| > 0$$

such that for all $n \geq N$, $|x(n) - \tilde{x}(n)| \geq \epsilon_1$.

Since the solution $\{x(n)\}$ is arbitrary, the latter statement is true for any solution for the equation (3.0.1) with the conditions (3.0.2) and $x(0) \notin \{\tilde{x}(0), \tilde{x}(1), \dots, \tilde{x}(p-1)\}$. Therefore, the p-periodic solution $\{\tilde{x}(0), \tilde{x}(1), \dots, \tilde{x}(p-1)\}$ is unstable. ■

Remark 1: In the proof of the Theorem 3.6, the mutual numeration of the sequences $\{\tilde{x}(n)\}$ and $\{x(n)\}$ can be easily changed by expressing $x(n)$ through the arbitrary point $\tilde{x}(m)$ of the p-periodic cycle, which generates the new sequence $\{y(n)\}$. Nevertheless, the reasoning of the Theorem 3.6 holds for any choice of the mutual numeration.

Remark 2: Theorem 3.5 claims sufficient conditions for the p-periodic cycle of the equation

(3.0.1) with the conditions (3.0.2) to remain globally asymptotically stable. The hypotheses of this theorem include the inequality for the value of parameter β_* , namely, $\beta_* > 0$.

On the other hand, Theorem 3.6 claims the sufficient conditions for the same solution to remain unstable. The hypothesis of this theorem restricts the values of the parameter β^* as follows: $\beta^* < 0$.

Hence, the whole spectrum of the cases remains with the combination of positive and negative values of $\beta(n)$, $n \in \mathbb{N}_0$, with $\beta_* < 0$ and $\beta^* > 0$.

Example 3.4. *Let us consider again the 2-periodic solutions for the equation (3.1.2) in Example 3.1 and equation (3.3.22) in Example 3.3.*

Calculation shows that for the equation (3.1.2), $\beta_ = -3.8750$ and $\beta^* = 0.9938$.*

Similarly, for the equation (3.3.22), $\beta_ = -0.0926$ and $\beta^* = 0.9456$.*

Thus, the parameters of both equations do not satisfy the hypotheses either for Theorem 3.5 or for Theorem 3.6. Despite this similarity, the 2-periodic solution for the equation (3.1.2) appears to be unstable, and the 2-periodic solution for the equation (3.3.22) appears to be globally asymptotically stable. This confirms that the hypotheses for both theorems claim the sufficient conditions.

Chapter 4

Strong Allee effect

In this chapter, we explore the existence and the properties of the p -periodic solutions for the equation (3.0.1) in the case when this equation describes the dynamics of the population at the presence of the strong Allee effect and periodic carrying capacity.

That is, the subject of the research in this chapter is the equation

$$x(n+1) = x(n) \exp \left(r \left(1 - \frac{x(n)}{K(n)} \right) \left(1 - \frac{A+C}{x(n)+C} \right) \right), \quad n \in \mathbb{N}_0 \quad (4.0.1)$$

where $\{K(n)\}$ is a strictly positive periodic sequence and the parameters of the equation satisfy the following conditions

$$r > 0, \quad C > 0, \quad \text{and} \quad A > 0. \quad (4.0.2)$$

The change in the range of the parameter A in the conditions (4.0.2) in comparison to the conditions (3.0.2) significantly alters the dynamics of the equation. The new dynamics requires the new definition for the domain of the function on the right-hand side of the equation (4.0.1).

4.1 New domain. New properties

The change of the range for the parameter A generates the changes in the properties of the function on the right-hand side of the equation (4.0.1). In this section, we explore which properties of that function remains and which new properties appears.

4.1.1 Change of domain for $x(n)$

The study of the p -periodic solutions for the equation (3.0.1) with the conditions (3.0.2) was conducted on the domain $x(n) \in \mathbb{R}^+$ for all $n \in \mathbb{N}_0$. The reason for this choice is that, in the case of the weak Allee effect defined by the equation (3.0.1), the point $x = 0$ is an unstable equilibrium for any choice of the parameters of the equation and any strictly positive sequence $\{K(n)\}$. Hence, no positive solutions converges to that equilibrium, and the point $x = 0$ does not affect the behaviour of other solutions. In the case of the strong Allee effect, the stability of the equilibrium point $x = 0$ changes. The following Lemma 4.1 provides the mathematical reasoning for that change.

Lemma 4.1. *The equilibrium point $x^* = 0$ of the equation*

$$x(n+1) = x(n) \exp \left(r \left(1 - \frac{x(n)}{K} \right) \left(1 - \frac{A+C}{x(n)+C} \right) \right), \quad n \in \mathbb{N}_0, \quad x(n) \in \mathbb{R}_0^+, \quad (4.1.1)$$

where $K > 0$, is unstable for the conditions (3.0.2) and stable for the conditions (4.0.2).

Proof. Let us define the function $f : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ as follows:

$$f(x) = x \exp \left(r \left(1 - \frac{x}{K} \right) \left(1 - \frac{A+C}{x+C} \right) \right). \quad (4.1.2)$$

Since $f(0) = 0$, then, $x^* = 0$ is the equilibrium point for the equation (4.1.1).

The equation for the first derivative of the function $f(x)$ is given by the expression (2.1.3).

Then,

$$f'(0) = \exp\left(\frac{r(-A)}{C}\right).$$

For the weak Allee effect (conditions (3.0.2)), $A < 0$. Hence, $|f'(0)| > 1$, and, by Theorem A, the equilibrium point $x^* = 0$ is unstable.

Similarly, for the strong Allee effect (conditions (4.0.2)), $A > 0$. Hence, $|f'(0)| < 1$, and, by Theorem A, the equilibrium point $x^* = 0$ is stable. ■

Since the equilibrium point $x^* = 0$ is stable for the equation (4.1.1), then its basin of attraction, $\mathcal{B}(0)$, includes some interval on $\mathbb{R}^+$. Thus, in all the following reasoning in this chapter, the domain for $x(n)$ is $\mathbb{R}_0^+$, where $n \in \mathbb{N}_0$.

4.1.2 Properties of the function $f(x) = x \exp\left(r\left(1 - \frac{x}{K}\right)\left(1 - \frac{A+C}{x+C}\right)\right)$ in the case of the strong Allee effect

In Section 2.1, Lemma 2.1, the properties of the function $f(x)$, defined by the expression (4.1.2), were established for the case of the weak Allee effect. In this section, we show that the function $f(x)$ retains the same properties in the case of the strong Allee effect. Moreover, we establish the location of the maximum of that function with respect to $x = A$.

Lemma 4.2. *Let the function $f : \mathbb{R}_0^+ \mapsto \mathbb{R}_0^+$ be defined as follows:*

$$f(x) = x \exp\left(r\left(1 - \frac{x}{K}\right)\left(1 - \frac{A+C}{x+C}\right)\right), \quad (4.1.3)$$

where

$$K > 0, \quad r > 0, \quad C > 0 \quad \text{and} \quad A > 0, \quad (4.1.4)$$

and the function $g : \mathbb{R}_0^+ \mapsto \mathbb{R}_0^+$ be defined as follows:

$$g(x) = x \exp \left(r \left(1 - \frac{x}{K} \right) \right). \quad (4.1.5)$$

Then:

- (1) the functions $f(x)$ and $g(x)$ intersect exactly at two points: $x_1 = 0$ and $x_2 = K$;
- (2) $f(x) < g(x)$ on the interval $(0, K)$, and $f(x) > g(x)$ on the interval (K, ∞) .

Proof. (1). Taking into consideration the range of the parameters in conditions (4.1.4), the equation

$$\exp \left(r \left(1 - \frac{x}{K} \right) \right) = \exp \left(r \left(1 - \frac{x}{K} \right) \left(1 - \frac{A+C}{x+C} \right) \right)$$

has exactly one solution on $\mathbb{R}^+$: $x = K$. Then, the equation $f(x) = g(x)$ has exactly two solutions : $x_1 = 0$ and $x_2 = K$. Thus, the property (1) holds.

(2). Since $A > 0$, the expression $\left(1 - \frac{A+C}{x+C} \right) < 1$. At the same time, on the interval $(0, K)$, the expression $\left(1 - \frac{x}{K} \right) > 0$. Hence, $f(x) < g(x)$ on the interval $(0, K)$.

By the biological characteristics of the Allee effect (see, for example, [15]), $A < K$. Then, on the interval (K, ∞) , the expression $0 < \left(1 - \frac{A+C}{x+C} \right) < 1$, while $\left(1 - \frac{x}{K} \right) < 0$. Hence, on the interval (K, ∞) , $f(x) > g(x)$. ■

Thus, the behaviour of the function $f(x)$, defined by the expression (4.1.3), with respect to the function $g(x)$, defined by the expression (4.1.5), is similar for both weak and strong Allee effect cases.

Moreover, with Lemma 4.2 and the inequality $A < K$, all the reasoning for Lemma 2.1 holds for the case of the strong Allee effect. Hence, on $\mathbb{R}_0^+$, we can claim the following properties for the function $f(x)$ defined by the expression (4.1.3) with conditions (4.1.4):

Lemma 4.3. *Let the function $f : \mathbb{R}_0^+ \mapsto \mathbb{R}_0^+$ be defined by the expression (4.1.3) with conditions (4.1.4). Then:*

- (1) *the function $f(x)$ has the unique maximum;*
- (2) *let the x -coordinate of the maximum of the function $f(x)$ be denoted as $x_{\max}$. Then the location of the $x_{\max}$ and the maximum value of the function $f(x)$, denoted $f_{\max}$, can be described as follows:*

Case A. If $\alpha r \geq 1$, then $\frac{K}{r} < x_{\max} \leq K$ and $f_{\max} \leq \frac{K}{r} \exp(r - 1)$.

Case B. If $\alpha r \leq 1$, then $K \leq x_{\max} < \frac{K}{\alpha r}$ and $f_{\max} \leq \frac{K}{\alpha r}$.

Moreover, if $r < 1$, then $\frac{K}{r} < x_{\max} \leq \frac{K}{r\alpha}$.

Case C. If $\alpha r = 1$, then $x_{\max} = K$ and $f_{\max} = K$.

For the further reasoning in this chapter, the location of the $x_{\max}$ with respect to $x = A$ is important. The next lemma establishes the required relation.

Lemma 4.4. *Let the function $f : \mathbb{R}_0^+ \mapsto \mathbb{R}_0^+$ be defined by the expression (4.1.3) with conditions (4.1.4), and the x -coordinate of the maximum of the function $f(x)$ be denoted as $x_{\max}$. Then, $x_{\max} > A$.*

Proof. The first derivative for the function $f(x)$ is defined by the expression (2.1.3). Then, $x_{\max}$ satisfies the equation (2.1.4), which can be written as

$$1 + \frac{rx_{\max}}{K} \left(\frac{A - x_{\max}}{x_{\max} + C} + \frac{(K - x_{\max})(A + C)}{(x + C)^2} \right) = 0. \quad (4.1.6)$$

Since $A < K$, then, on the interval $[0, A]$, the left-hand side of the equation (4.1.6) is strictly positive. Hence, the equation (4.1.6) has no solutions on the interval $[0, A]$.

On the other hand, by Lemma 4.3, there exists the unique maximum for the function $f(x)$

on $\mathbb{R}_0^+$. Hence, $x_{\max} > A$.

■

4.2 Unimodal Allee map and its fixed points

By its definition and properties shown in Lemma 4.3 and Lemma 4.4, function $f(x)$ defined by the expression (4.1.3) with conditions (4.1.4) can be regarded as the unimodal Allee map.

Definition 4.1 ([39]). *Let $I = [0, b]$. A continuous function $f : I \rightarrow I$ is called an Allee map if the following holds:*

- (1) $f(0) = 0$, and there are positive points A and K such that
- (2) $f(x) < x$ for all $x \in (0, A) \cup (K, b)$ and $f(x) > x$ for $x \in (A, K)$.

Two remarks are required here:

- (i) In Definition 4.1, b is either finite or $b = +\infty$.
- (ii) Function $f(x)$ defined by the expression (4.1.3) with conditions (2.2.3), which describes the mechanism of the weak Allee effect, is not an Allee map by this definition, since, by the conditions (2.2.3), $A < 0$. This is a shortcoming of Definition 4.1, constructed to recognize the strong Allee effect only.

Definition 4.2 ([39]). *Let $I = [0, b]$. A continuous function $f : I \rightarrow I$ is called unimodal if*

- (1) $f(b) = 0$ when b is finite or $\lim_{x \rightarrow \infty} f(x) = 0$ otherwise.
- (2) There exists a unique critical point C of f , where $f(x)$ is strictly increasing on $[0, C)$ and strictly decreasing on (C, b) (or $(C, +\infty)$ when $b = +\infty$).

The particular cases of the unimodal Allee maps have been studied (see, for example, [39], [29], or [45]). Some conclusions can be made for all unimodal Allee maps in general.

Let us consider two types of the unimodal Allee maps that can be generated by the expression (4.1.3) with conditions (4.1.4). By Lemma 4.3, the unique maximum of this function, C , can be located between the points A and K (Figure 4.1, left panel) or on the right-hand side of the point K (Figure 4.1, right panel).

Figure 4.1: Two types of the unimodal Allee maps generated by the expression (4.1.3) with conditions (4.1.4). Parameter values: Left panel: $r = 2.5$, $K = 10$, $C = 2$, and $A = 2$. For this function, $\alpha r \approx 1.7 > 1$. Right panel: $r = 0.5$, $K = 5$, $C = 4$, and $A = 2$. For this function, $\alpha r \approx 0.17 < 1$.

In Figure 4.1, symbol $\tilde{A}$ corresponds to the pre-image of A , that is, $\tilde{A} = f^{-1}(A)$.

Here, the symbols $\tilde{A}$ and C are adopted from [39] for consistency.

Let us consider the population equation (4.1.1) with conditions (4.1.4) that describes the dynamics of the population affected by the strong Allee effect with the constant carrying capacity K , namely:

$$x(n+1) = x(n) \exp \left(r \left(1 - \frac{x(n)}{K} \right) \left(1 - \frac{A+C}{x(n)+C} \right) \right), \quad n \in \mathbb{N}_0, \quad x(n) \in \mathbb{R}_0^+, \quad (4.2.1)$$

where $K > 0$, $r > 0$, $C > 0$ and $A > 0$.

Clearly, the equation (4.2.1) has three equilibrium points: $x_1^* = 0$, $x_2^* = A$, and $x_3^* = K$.

By Lemma 4.1, the equilibrium point $x_1^* = 0$ is stable for any set of parameters of the equation (4.2.1). On the other hand, the equilibrium point $x_2^* = A$ is unstable for any set of parameters of the equation (4.2.1), which is established by the following lemma.

Lemma 4.5. *The equilibrium point $x^* = A$ of the equation (4.2.1) is unstable for any set of parameters of the equation.*

Proof. Let us denote as $f(x)$ the function on the right-hand side of the equation (4.2.1).

The equation for the first derivative of the function $f(x)$ is given by the expression (2.1.3).

Then,

$$f'(A) = 1 + \frac{rA(K-A)}{K(A+C)}.$$

Since $0 < A < K$, then, $|f'(A)| > 1$ for any set of parameters of the equation (2.1.3).

Hence, by Theorem A, the equilibrium point $x^* = A$ is unstable for any set of parameters of the equation (2.1.3). ■

To discuss stability of the third equilibrium point, $x^* = K$, it suffices to notice that the values of the parameter α for the equation with the strong Allee effect has the same range as for the equation with the weak Allee effect. That is, $0 < \alpha < 1$. Then, the reasoning of Lemma 2.2 holds for the parameters describing the strong Allee effect. Therefore, the equilibrium point $x^* = K$ of the equation (4.2.1) is asymptotically stable if $0 < \alpha r < 2$, and unstable if $\alpha r > 2$.

To complete the description of the equilibrium points, the following definition is required:

Definition 4.3 ([21]). *Let x be a point in the domain of f . If there exists a positive integer r and an equilibrium point x^* of the equation $x(n+1) = f(x(n))$ such that $f^r(x) = x^*$, $f^{r-1}(x) \neq x^*$, then x is an **eventually equilibrium** (fixed) point.*

Then, by Definition 4.3, the point $x = \tilde{A}$ is an eventually equilibrium point of the equation (4.2.1).

It has been established in [39] that two intervals on $\mathbb{R}_0^+$ are in the basin of attraction of the equilibrium point $x^* = 0$ for any unimodal Allee map. That is, $[0, A) \cup (\tilde{A}, b) \in \mathcal{B}(0)$. Therefore, for the equation (4.2.1),

$$[0, A) \cup (\tilde{A}, +\infty) \in \mathcal{B}(0). \quad (4.2.2)$$

Let us discuss the basin of attraction of the equilibrium point $x^* = K$, that is, $\mathcal{B}(K)$.

In Chapter 2, two lemmas were proven in order to construct the reasoning in Theorem 2.5.

In Lemma 2.3, the hypotheses do not include the type of the Allee effect. The reasoning in Lemma 2.4 remains correct for the strong Allee effect, since all the reasoning is conducted for $x > K$, and the distinct behaviour of the function (4.1.3) with conditions (4.1.4) on the interval $(0, A)$ does not effect the reasoning. Hence, Lemma 2.3 and Lemma 2.4 hold for the equation (4.2.1) with embedded mechanism of the strong Allee effect.

The following theorem is proved with a similar reasoning as Theorem 2.5 and describes the basin of attraction of the equilibrium point $x^* = K$.

Theorem 4.6. *If the parameters of the equation (4.2.1) satisfy the inequality*

$$\alpha r < 2, \quad (4.2.3)$$

then the interval $(A, \tilde{A}) \in \mathcal{B}(K)$, where $\tilde{A}$ is the pre-image of the point $x = A$.

Proof. Similarly, to the reasoning in Theorem 2.5, let us define the function $V(x) : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ as $V(x) = (x - K)^2$.

This function is continuous and positive definite at $x^* = K$.

The variation of $V(x)$ with respect to equation (4.2.1) is expressed by the formula

$$\begin{aligned} \Delta V(x(n)) = (x(n))^2 & \left(\exp \left(r \left(1 - \frac{x(n)}{K} \right) \left(1 - \frac{A+C}{x(n)+C} \right) \right) - 1 \right) \\ & \left(\exp \left(r \left(1 - \frac{x(n)}{K} \right) \left(1 - \frac{A+C}{x(n)+C} \right) \right) + 1 - \frac{2K}{x(n)} \right). \end{aligned} \quad (4.2.4)$$

On the interval $(A, \tilde{A})$, the expression

$$0 < \left(1 - \frac{A+C}{x(n)+C} \right) < 1.$$

Hence, the first bracket in the expression for $\Delta V(x(n))$ is positive for $A < x(n) < K$, and is negative for $K < x(n) < \tilde{A}$.

Moreover, on the interval $A < x(n) < K$, by the hypothesis of the theorem and Lemma 2.3,

$$\begin{aligned} \exp \left(r \left(1 - \frac{x(n)}{K} \right) \left(1 - \frac{A+C}{x(n)+C} \right) \right) + 1 - \frac{2K}{x(n)} & < \\ \exp \left(\alpha r \left(1 - \frac{x(n)}{K} \right) \right) + 1 - \frac{2K}{x(n)} & < \exp \left(2 \left(1 - \frac{x(n)}{K} \right) \right) + 1 - \frac{2K}{x(n)} < 0. \end{aligned} \quad (4.2.5)$$

On the other hand, by Lemma 2.4, on the interval $K < x(n) < \tilde{A}$,

$$\exp \left(r \left(1 - \frac{x}{K} \right) \left(1 - \frac{A+C}{x+C} \right) \right) + 1 - \frac{2K}{x} > 0.$$

Therefore,

$$\Delta V(x(n)) < 0 \quad \text{for all } x \in (A, K) \cup (K, \tilde{A}).$$

Hence, by Theorem B, $(A, \tilde{A}) \in \mathcal{B}(K)$.

■

Theorem 4.6 together with the statement 4.2.2 and lemmas in this chapter describe the behaviour of the solution for the equation (4.2.1) with any $x(0) \in \mathbb{R}_0^+$.

Example 4.1. *Let us discuss the statement of Theorem 4.6 with respect to the following equation:*

$$x(n+1) = x(n) \exp \left(2.0 \left(1 - \frac{x(n)}{10} \right) \left(1 - \frac{8}{x(n)+6} \right) \right), \quad n \in \mathbb{N}_0, \quad x(n) \in \mathbb{R}_0^+. \quad (4.2.6)$$

Figure 4.2: Basin of attraction for the equilibrium points 0, A , and K for the equation with the strong Allee effect and constant carrying capacity in case of the stable equilibrium point K . Data set: $r = 2.0$, $K = 10$, $C = 6$, $A = 2$, $\tilde{A} \approx 27.1928$, $\alpha r = 1$.

The set of the parameters for the equation (4.2.6) is $r = 2.0$, $K = 10$, $C = 6$, $A = 2$. Hence, this equation describes the population dynamics with the strong Allee effect present. Since for this equation $\alpha r = 1 < 2$ and $\tilde{A} \approx 27.1928$, then, by Theorem 4.6, Lemma 4.1, Lemma 4.5, and the statement 4.2.2, we expect the following behaviour for the solutions: if $x(0) \in [0, 2) \cup (27.1928, +\infty)$, then the solution converges to an equilibrium point $x^* = 0$; if $x(0) \in (2, 27.1928)$, then the solution converges to an equilibrium point $x^* = 10$; if $x(0) \in \{2, 27.1928\}$, then the solution converges to an equilibrium point $x^* = 2$. In Figure 4.2, we observe that the solutions reveal the predicted dynamics:

solutions with $x(0) = 1$, $x(0) = 27.2$, and $x(0) = 28$ converge to an equilibrium point $x^* = 0$; solutions with $x(0) = 2.1$, $x(0) = 10$, $x(0) = 20$, and $x(0) = 27.1$ converge to an equilibrium point $x^* = 10$;

the solution with $x(0) = 2.0$, remains constant.

The precision of the interval boundaries defined in Theorem 4.6 is quite remarkable: a solution with $x(0) = 27.1$ converges to an equilibrium point $x^* = 10$, whereas a solution with $x(0) = 27.2$ converges to an equilibrium point $x^* = 0$.

4.3 Ricker equation with the strong Allee effect and periodic carrying capacity

Let us add to the population model with the strong Allee effect (4.2.1) the second biological phenomenon, namely, periodic carrying capacity. That is, in this section, we discuss the p-periodic solutions of the equation (4.0.1) with conditions (4.0.2), where $\{K(n)\}$ is a strictly positive p-periodic sequence.

We denote

$$f_n(x) := x \exp \left(r \left(1 - \frac{x}{K(n)} \right) \left(1 - \frac{A+C}{x+C} \right) \right), \quad n \in \mathbb{N}_0, \quad (4.3.1)$$

where $K(n) > 0$, $r > 0$, $C > 0$ and $A > 0$.

Several observations about the mutual behaviour of two functions, $f_1(x)$ and $f_2(x)$, with $K(1) < K(2)$, are helpful for further reasoning.

As it is shown in Figure 4.3:

- (1) $f_1(x) = f_2(x)$ for $x \in \{0, A\}$;
- (2) $f_1(x) > f_2(x)$ for $x \in (0, A)$ and $f_1(x) < f_2(x)$ for $x \in (A, \infty)$;
- (3) By the observations (1) and (2), the pre-image of the point A is greater for function $f_2(x)$, that is, $\tilde{A}_1 < \tilde{A}_2$;
- (4) By the observation (2) and Lemma 4.4, we conclude that the maximum value of the

function $f_1(x)$ is smaller than the maximum value of the function $f_2(x)$ but greater than A ;

(5) if $f_{1,\max} < \tilde{A}_1$, then $x \in [A, \tilde{A}_1]$ implies $f(x) \in [A, \tilde{A}_1]$.

Figure 4.3: Two functions with embedded strong Allee effect mechanism and different values for the carrying capacity. The mutual location of these functions and their key points depend on the values of the carrying capacity.

Taking into consideration the mutual properties of the functions $f_n(x)$, we prove the persistence property of the solution for the Ricker equation with the strong Allee effect and p-periodic carrying capacity.

Theorem 4.7. *Let $\{x(n)\}$, $n \in \mathbb{N}_0$, be a solution for the equation (4.0.1) with conditions (4.0.2), where $\{K(n)\}$ is a strictly positive p-periodic sequence, $p \geq 1$.*

Let $\{f_n(x)\}$, with $n \in \{0, 1, \dots, p-1\}$ be the set of functions defined by the expression (4.3.1);

$\tilde{A}_{\min} := \min_{n \in \{0, 1, \dots, p-1\}} \{\tilde{A}(n)\}$, where $\tilde{A}(n)$ is the pre-image of the Allee threshold, A , for the function $f_n(x)$;

$f_{\max} := \max_{n \in \{0, 1, \dots, p-1\}} \{f_{n,\max}\}$, where $f_{n,\max}$ is the maximum value of the function $f_n(x)$.

If $f_{\max} \leq \tilde{A}_{\min}$, then $x(0) \in [A, \tilde{A}_{\min}]$ implies $x(n) \in [A, \tilde{A}_{\min}]$ for all $n \in \mathbb{N}_0$.

Proof. Let the initial point of the solution $\{x(n)\}$ satisfies

$$x(0) \in [A, \tilde{A}_{\min}].$$

Since $A < f_{0,\max} \leq f_{\max} \leq \tilde{A}_{\min} \leq \tilde{A}_0$, then, by observations (4) and (5),

$$f_0(x(0)) \in [A, \tilde{A}_{\min}].$$

Let $N > 0$ and

$$x(N) \in [A, \tilde{A}_{\min}].$$

The sequence $\{K(n)\}$ is p -periodic, hence, $f_N(x) \in \{f_n(x)\}$, $n \in \{0, 1, \dots, p-1\}$. Therefore, $A < f_{N,\max} \leq f_{\max} \leq \tilde{A}_{\min} \leq \tilde{A}_N$.

Then, by observations (4) and (5),

$$x(N+1) = f_N(x(N)) \in [A, \tilde{A}_{\min}].$$

Then, by the Principle of Mathematical Induction, $x(n) \in [A, \tilde{A}_{\min}]$ for all $n \in \mathbb{N}_0$. ■

Theorem 4.7 states that, under specific conditions, all the solutions for the equation (4.0.1) with conditions (4.0.2) are contained in the closed interval. Then, by the same reasoning as in Theorem 3.2, it is possible to prove the existence of the p -periodic solution for the equation (4.0.1) with conditions (4.0.2), where $\{K(n)\}$ is a strictly positive p -periodic sequence, $p \geq 1$.

Theorem 4.8. *Under the hypotheses of Theorem 4.7, there exists a p -periodic solution $\{\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_{p-1}\}$ for the equation (4.0.1) with conditions (4.0.2).*

Moreover, $\tilde{x}_n \in [A, \tilde{A}_{\min}]$ for all $n \in \{0, 1, \dots, p-1\}$.

Proof. The reasoning for the existence of the p -periodic solution is the same as in Theorem 3.2 with the only difference that the closed interval mapped into itself is now $[A, \tilde{A}_{\min}]$.

To prove that $\tilde{x}_n \in [A, \tilde{A}_{\min}]$ for all $n \in \{0, 1, \dots, p-1\}$, let us recall that, by Theorem C,

there exists $x^* \in [A, \tilde{A}_{\min}]$ such that $F(x^*) = (f_{p-1} \circ f_{p-2} \cdots \circ f_0)(x^*) = x^*$ and $\tilde{x}_0 := x^*$. Hence, the p -periodic solution $\{\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_{p-1}\}$ satisfies the hypotheses of Theorem 4.7. Therefore, by Theorem 4.7, $\tilde{x}_n \in [A, \tilde{A}_{\min}]$ for all $n \in \{0, 1, \dots, p-1\}$. ■

Let us illustrate the statements of Theorem 4.8 with the following example.

Example 4.2. Let us set the following parameters for the equation (4.0.1) with conditions (4.0.2): $\{K(n)\} = \{10, 14\}$, $r = 2$, $A = 2$, $C = 6$.

Then, $f_{0,\max} = 9.995$, $f_{\max} = f_{2,\max} = 14.2$; $\tilde{A}_{\min} = \tilde{A}_0 = 27.2$, $\tilde{A}_1 = 39.3$.

Hence, $f_{\max} = 14.2 < 27.2 = \tilde{A}_{\min}$. By Theorem 4.8, there exists a p -periodic solution contained in the interval $[2, 27.2]$.

Figure 4.4: Existence of the 2-periodic solution for the equation (4.0.1) with conditions (4.0.2) and 2-periodic sequence $\{K(n)\}$. In addition, all solutions with the initial point in the interval $[2, 27.2]$ converge to this 2-cycle.

In Figure 4.4, we observe that the 2-periodic solution $\{9.281, 12.8\}$, indeed, exists and its points are contained in the interval $[2, 27.2]$. Moreover, this cycle is an attractor for all solutions that start on this interval.

Chapter 5

Attenuant property of the periodic solutions for Ricker equation with Allee effect and periodic carrying capacity.

In this chapter, we discuss the necessary and sufficient conditions for the attenuant property of the p -periodic solution for the equation (1.0.1) where $\{K(n)\}$ is a strictly positive p -periodic sequence.

5.1 Monotonicity and concavity

The conjecture on deleterious effect of periodic changes in the carrying capacity of the environment on the population density, made by Cushing and Henson in [19], became a foundation for a new direction in the mathematical studies of population equations. Since the conjecture was formulated for the Beverton-Holt equation with the concave increasing

function on the right-hand side of the equation (1.2.1), first proofs for the conjecture were based on the properties of monotonicity and concavity of function $f(x(n), K(n))$ in equation (1.2.1).

However, in 2005, Kon in [36] proved an attenuant property of the cycle for the wider range of non-autonomous difference equations:

$$x(n+1) = g\left(\frac{x(n)}{K(n)}\right)x(n), \quad x(0) \in \mathbb{R}_+, n \in \mathbb{N}_0, \quad (5.1.1)$$

where $\{K(n)\}$ is a strictly positive p-periodic sequence, and the continuous function $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfies the properties

$$g(1) = 1; \quad g(x) > 1, \quad x \in (0, 1); \quad g(x) < 1, \quad x \in (1, \infty). \quad (5.1.2)$$

In [36], it was also proven that any periodic solution for the Ricker equation with periodic carrying capacity, namely,

$$x(n+1) = x(n) \exp\left(r\left(1 - \frac{x(n)}{K(n)}\right)\right), \quad n \in \mathbb{N}_0 \quad (5.1.3)$$

is attenuant under the hypothesis

$$\frac{K^*}{K_*} \exp(r-1) \leq 2.$$

That is, the globally asymptotic p-periodic cycle, which existence was proven under the same hypothesis in [46], is attenuant. Since the function on the right-hand side of the equation (5.1.3) is not monotone, that proves that the monotonicity is not the necessary condition for the periodic solution to be attenuant.

Ricker equation with Allee effect (1.0.1) is not of the same type as the equation (5.1.1)

because of the factor $\left(1 - \frac{A+C}{x(n)+C}\right)$ in the exponent. Therefore, this equation does not satisfy the hypotheses of the lemmas and theorems in [36]. Nevertheless, the idea of the concavity of the multivariate function $f(x(n), K(n))$ on the specific convex set can be used to prove that the periodic solution of the equation (1.2.1) is attenuant.

The following lemma is an important tool in the reasoning for the attenuant property of the cycle.

Lemma 5.1. *Suppose $\{\tilde{x}(n)\}$ is a p -periodic solution with $p > 1$ of the equation*

$$x(n+1) = F(x(n), K(n)) \quad , \quad n \in \mathbb{N}_0, \quad (5.1.4)$$

where $F(x(n), K(n))$ is a strictly concave continuous function of two variables, and $\{K(n)\}$ is a strictly positive p -periodic sequence. Then,

$$\frac{1}{p} \sum_{n=1}^p \tilde{x}(n) < F\left(\frac{1}{p} \sum_{n=1}^p \tilde{x}(n), \frac{1}{p} \sum_{n=1}^p K(n)\right). \quad (5.1.5)$$

Proof. The proof is based on the Method of the Mathematical Induction.

Base Case: $p=2$.

Since $\tilde{x}(2) = F(\tilde{x}(1), K(1))$, $\tilde{x}(3) \equiv \tilde{x}(1) = F(\tilde{x}(2), K(2))$, and by the strict concavity of the function F , we obtain

$$\begin{aligned} \frac{1}{2} \sum_{n=1}^2 \tilde{x}(n) &= \frac{1}{2} F(\tilde{x}(1), K(1)) + \frac{1}{2} F(\tilde{x}(2), K(2)) \\ &< F\left(\frac{1}{2}(\tilde{x}(1), K(1)) + \frac{1}{2}(\tilde{x}(2), K(2))\right) = F\left(\frac{\tilde{x}(1) + \tilde{x}(2)}{2}, \frac{K(1) + K(2)}{2}\right) \\ &= F\left(\frac{1}{2} \sum_{n=1}^2 \tilde{x}(n), \frac{1}{2} \sum_{n=1}^2 K(n)\right). \end{aligned} \quad (5.1.6)$$

Hence, the base case holds for the inequality (5.1.5).

Inductive Step.

Inductive Hypothesis: Under the hypotheses of Lemma 5.1,

$$\frac{1}{t} \sum_{n=1}^t \tilde{x}(n) < F \left(\frac{1}{t} \sum_{n=1}^t \tilde{x}(n), \frac{1}{t} \sum_{n=1}^t K(n) \right).$$

Let us consider $p = t + 1$. Applying the Inductive Hypothesis, we obtain the following:

$$\begin{aligned} \frac{1}{t+1} \sum_{n=1}^{t+1} \tilde{x}(n) &= \frac{1}{t+1} \sum_{n=1}^{t+1} F(\tilde{x}(n), K(n)) \\ &= \frac{1}{t+1} F(\tilde{x}(t+1), K(t+1)) + \frac{1}{t+1} \sum_{n=1}^t F(\tilde{x}(n), K(n)) \\ &< \frac{1}{t+1} F(\tilde{x}(t+1), K(t+1)) + \frac{t}{t+1} F \left(\frac{1}{t} \sum_{n=1}^t \tilde{x}(n), \frac{1}{t} \sum_{n=1}^t K(n) \right). \end{aligned} \quad (5.1.7)$$

Now, choosing $\lambda_1 = \frac{1}{t+1}$, $\lambda_2 = \frac{t}{t+1}$, and the points $(\tilde{x}(t+1), K(t+1))$ and $\left(\frac{1}{t} \sum_{n=1}^t \tilde{x}(n), \frac{1}{t} \sum_{n=1}^t K(n) \right)$, by the concavity of the function F , we obtain from (5.1.7):

$$\begin{aligned} \frac{1}{t+1} \sum_{n=1}^{t+1} \tilde{x}(n) &< F \left(\frac{1}{t+1} \tilde{x}(t+1) + \frac{1}{t+1} \sum_{n=1}^t \tilde{x}(n), \frac{1}{t+1} K(t+1) + \frac{1}{t+1} \sum_{n=1}^t K(n) \right) \\ &= F \left(\frac{1}{t+1} \sum_{n=1}^{t+1} \tilde{x}(n), \frac{1}{t+1} \sum_{n=1}^{t+1} K(n) \right). \end{aligned} \quad (5.1.8)$$

Therefore, by the Principle of Mathematical Induction, the inequality (5.1.5) holds for any $p > 1$. ■

Corollary 5.1.1. *Suppose $\{\tilde{x}(n)\}$ is a p -periodic solution with $p > 1$ of the equation (5.1.4), where $F(x(n), K(n))$ is the strictly convex function of two variables, and $\{K(n)\}$ is a strictly positive p -periodic sequence. Then,*

$$\frac{1}{p} \sum_{n=1}^p \tilde{x}(n) > F \left(\frac{1}{p} \sum_{n=1}^p \tilde{x}(n), \frac{1}{p} \sum_{n=1}^p K(n) \right). \quad (5.1.9)$$

The inequality (5.1.5) in Lemma 5.1 shows the implicit relation between the average of the cycle, $\frac{1}{p} \sum_{n=1}^p \tilde{x}(n)$, and the average of the carrying capacity, $\frac{1}{p} \sum_{n=1}^p K(n)$. The further usual strategy for the proof of the attenuant property is to use the specific properties of the given function $f(x(n), K(n))$ to derive the explicit relation, as it was done, for example, in [36].

5.2 Concavity of the forcing function and the attenuancy property of the periodic solutions for Ricker equation with the weak Allee effect and periodic carrying capacity

In this section, we demonstrate that the common method of proving the attenuancy of the periodic solutions via the concavity of the forcing function is not applicable to the equation (1.0.1).

In Theorem 3.5, we proved the global asymptotic stability for a wide range of the p-periodic solutions under the assumption that the domain of the function $f(x(n), K(n))$ is limited to the convex set $S = [L_*, L^*] \times [K_*, K^*]$. If the function of two variables

$$f(x, K) = x \exp \left(r \left(1 - \frac{x}{K} \right) \left(1 - \frac{A + C}{x + C} \right) \right) \quad (5.2.1)$$

were strictly concave on the set S , then Lemma 5.1 would be the way to prove the attenuancy of those p-periodic solutions in Theorem 3.5. Otherwise, the strict convexity would help to prove the resonant property of the cycles.

The following lemma and example will study the question if the concavity of the function (5.2.1) could be used to determine the attenuant property of the cycles with the weak Allee

effect and periodic carrying capacity.

Lemma 5.2. *Let the function $G : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ be defined as follows:*

$$G(x, y) = x \exp \left(r \left(1 - \frac{x}{y} \right) \left(1 - \frac{A+C}{x+C} \right) \right), \quad (5.2.2)$$

where $r > 0$, $A < 0 < C$, and $|A| < C$.

Let $K_*, K^* \in \mathbb{R}_+$ with $K_* < K^*$.

Suppose

$$\beta_* > 0 \quad \text{and} \quad \frac{r\beta^* L^*}{K_*} < 2.$$

Then,

$$\frac{\partial^2 G}{\partial x^2} < 0, \quad (x, y) \in S. \quad (5.2.3)$$

Proof. The second derivative of the function $G(x, y)$ with respect to variable x can be written in the form

$$\begin{aligned} \frac{\partial^2 G}{\partial x^2} = & \left(-2 \frac{r}{y} \left(1 - \frac{(A+C)(y+C)}{(x+C)^2} \frac{C}{x+C} \right) + x \left(\frac{r}{y} \right)^2 \left(1 - \frac{(A+C)(y+C)}{(x+C)^2} \right)^2 \right) \\ & \exp \left(r \left(1 - \frac{x}{y} \right) \left(1 - \frac{A+C}{x+C} \right) \right). \end{aligned} \quad (5.2.4)$$

Hence, the inequality (5.2.3) is equivalent to the inequality

$$2 \frac{r}{y} \left(1 - \frac{(A+C)(y+C)}{(x+C)^2} \frac{C}{x+C} \right) > x \left(\frac{r}{y} \right)^2 \left(1 - \frac{(A+C)(y+C)}{(x+C)^2} \right)^2. \quad (5.2.5)$$

Since $(x, y) \in S$, then

$$L_* \leq x \leq L^*, \quad K_* \leq y \leq K^*,$$

and, therefore, by the hypothesis of the lemma,

$$0 < \beta_* \leq \left(1 - \frac{(A+C)(y+C)}{(x+C)^2}\right) \leq \beta^*.$$

Moreover, since $0 < \frac{C}{x+C} < 1$, then

$$1 - \frac{(A+C)(y+C)}{(x+C)^2} \frac{C}{x+C} > 1 - \frac{(A+C)(y+C)}{(x+C)^2}.$$

Therefore, to prove the inequality (5.2.5), it suffices to show that

$$2\frac{r}{y} \left(1 - \frac{(A+C)(y+C)}{(x+C)^2}\right) > x \left(\frac{r}{y}\right)^2 \left(1 - \frac{(A+C)(y+C)}{(x+C)^2}\right)^2,$$

or, equivalently,

$$2 > x \frac{r}{y} \left(1 - \frac{(A+C)(y+C)}{(x+C)^2}\right). \quad (5.2.6)$$

By the hypotheses of the lemma,

$$\frac{r\beta^*L^*}{K_*} < 2. \quad (5.2.7)$$

Since $x < L^*$, $y > K_*$, and $1 - \frac{(A+C)(y+C)}{(x+C)^2} < \beta^*$, then the inequality (5.2.7) implies the inequality (5.2.6), and the inequality (5.2.3) holds. ■

Example 5.1. Let us define the set of parameters for the equation (3.0.1) with the conditions (3.0.2) as follows: $\{K(n)\} = \{12, 15\}$, $r = 1.5$, $A = -2$, and $C = 6$.

For $K_* = 12$ and $K^* = 15$, we obtain: $\beta_* = 0.4912 > 0$, $\frac{r\beta^*L^*}{K_*} = 1.7675 < 2$. Hence, the hypotheses of Theorem 3.5, are satisfied. Then, by Theorem 3.5, there exists globally asymptotically stable 2-cycle. This 2-periodic solution $\{11.09, 14.96\}$ is shown in Figure 5.1. For this solution, the average on the cycle, 13.025, is smaller than average of the carrying capacities, 13.5. Hence, this cycle is attenuant.

Figure 5.1: Globally asymptotically stable 2-periodic solution $\{11.09, 14.96\}$ for the set of parameters: $\{K(n)\} = \{12, 15\}$, $r = 1.5$, $A = -2$, and $C = 6$.

For this set of parameters, $L_* = 6.8483$ and $L^* = 16.4872$. To establish the connection between the attenuancy property of this cycle and the concavity, let us define the function of two variables $G(x, y)$ as follows:

$$G(x, y) = x \exp \left(1.5 \left(1 - \frac{x}{y} \right) \left(1 - \frac{4}{x+6} \right) \right). \quad (5.2.8)$$

on the convex set $S = [L_*, L^*] \times [K_*, K^*]$. Then, by Lemma 5.2, $\frac{\partial^2 G}{\partial x^2} < 0$ on S .

Choosing the point $(x, y) = (7, 12.2) \in S$, we calculate that the determinant of the Hessian for the function $G(x, y)$ at this point

$$\det H(G)(7, 12.2) = -0.0022.$$

The matrix $H(G)(x, y)$ is negative-definite if and only if its principal minors of the odd order are negative and its principal minors of the even order are positive ([32]). For this matrix to be positive-definite, the signs of the principal minors must switch.

Therefore, matrix $H(G)(x, y)$ is neither negative-definite nor positive-definite on S .

On the other hand, by the theory of the concave (convex) multivariate functions [41], the differentiable function $G(x, y)$ is strictly (convex) if and only if its Hessian is negative-definite (positive-definite) on its entire convex domain S .

Thus, we conclude that the function $G(x, y)$ is neither concave nor convex on S . Nevertheless, the 2-cycle is attenuant.

Example 5.1 shows that the usual approach for the proof of the attenuancy of the cycles is not applicable to the equation (1.0.1).

We conclude our studies of the equation (1.0.1) with the proof of the following lemma which states quite a remarkable property of p -periodic solutions. These properties can be used for further research of the attenuant properties of the cycles.

Lemma 5.3. *The p -periodic solution $\{\tilde{x}(0), \tilde{x}(1), \dots, \tilde{x}(p-1)\}$ for the equation (3.2.1) with the conditions (3.2.2) and $\{K(n)\}$ a strictly positive p -periodic sequence, satisfies the property*

$$\sum_{n=0}^{p-1} \frac{\tilde{x}(n)}{K(n)} \left(1 - \frac{A+C}{\tilde{x}(n)+C}\right) = \sum_{n=0}^{p-1} \left(1 - \frac{A+C}{\tilde{x}(n)+C}\right). \quad (5.2.9)$$

Proof. Let $\{\tilde{x}(0), \tilde{x}(1), \dots, \tilde{x}(p-1)\}$ be the p -periodic solution for the equation (3.2.1) with the conditions (3.2.2). From the equation (3.2.2), we obtain:

$$\frac{\tilde{x}(n+1)}{\tilde{x}(n)} = \exp \left(r \left(1 - \frac{A+C}{\tilde{x}(n)+C}\right) \right) \exp \left(-r \frac{\tilde{x}(n)}{K(n)} \left(1 - \frac{A+C}{\tilde{x}(n)+C}\right) \right),$$

where $n \in \{0, 1, 2, \dots, (p-1)\}$.

Re-arranging the terms:

$$\frac{\tilde{x}(n+1)}{\tilde{x}(n)} \exp \left(r \frac{\tilde{x}(n)}{K(n)} \left(1 - \frac{A+C}{\tilde{x}(n)+C} \right) \right) = \exp \left(r \left(1 - \frac{A+C}{\tilde{x}(n)+C} \right) \right). \quad (5.2.10)$$

By the writing the equation (5.2.10) for each point of the solution $\{\tilde{x}(0), \tilde{x}(1), \dots, \tilde{x}(p-1)\}$ and multiplying the left-hand parts and the right-hand parts of those equations respectively, we obtain:

$$\exp \left(r \sum_{n=0}^{p-1} \frac{\tilde{x}(n)}{K(n)} \left(1 - \frac{A+C}{\tilde{x}(n)+C} \right) \right) \prod_{n=0}^{p-1} \frac{\tilde{x}(n+1)}{\tilde{x}(n)} = \exp \left(r \sum_{n=0}^{p-1} \left(1 - \frac{A+C}{\tilde{x}(n)+C} \right) \right). \quad (5.2.11)$$

Taking into consideration that the solution $\{\tilde{x}(0), \tilde{x}(1), \dots, \tilde{x}(p-1)\}$ is p-periodic, we can simplify the product in the equation (5.2.11) as follows:

$$\prod_{n=0}^{p-1} \frac{\tilde{x}(n+1)}{\tilde{x}(n)} = \frac{\tilde{x}(1)}{\tilde{x}(0)} \frac{\tilde{x}(2)}{\tilde{x}(1)} \dots \frac{\tilde{x}(p)}{\tilde{x}(p-1)} = \frac{\tilde{x}(p)}{\tilde{x}(0)} = 1.$$

Hence, the equation (5.2.11) can be written as

$$\exp \left(r \sum_{n=0}^{p-1} \frac{\tilde{x}(n)}{K(n)} \left(1 - \frac{A+C}{\tilde{x}(n)+C} \right) \right) = \exp \left(r \sum_{n=0}^{p-1} \left(1 - \frac{A+C}{\tilde{x}(n)+C} \right) \right). \quad (5.2.12)$$

After taking the logarithm of the both sides of this equation, we obtain the equation (5.2.9). ■

Chapter 6

Conclusions and further research directions

In this research, we developed the model which includes the mechanisms of two biological phenomena: the Allee effect and periodically changing environmental conditions. Our modelling allows us to describe both the weak and the strong Allee effect. Hence, when the set of the component Allee effects changes or the mutual interference of the component Allee effects changes, the same equations can be used to describe the population dynamics for the species.

The thesis has shown that just an addition of one biological phenomenon to the Ricker equation, namely, the Allee effect, significantly changes the sufficient conditions for the equilibrium point of the equation to obtain stability. This also increases the range of parameters which allows for the equilibrium solution to be globally asymptotically stable. The new parameter $\alpha = \frac{K - A}{K + C} < 1$ appears as the coefficient of that range increase.

The addition of the second mechanism, namely, the periodic carrying capacity, generates the multiple cycles with the variety of periods. It was shown that the p-periodic carrying capacity always generates the p-periodic solutions. Moreover, under certain conditions, those p-periodic solutions become globally attractive.

In addition, enhancing the model for the periodic carrying capacity with the mechanism for the weak Allee effect also increases the range of parameters for cycles to stay globally asymptotically stable. The parameter of this increase, β , not only effect the stability of the cycle but also serves as the unique criterion of instability.

The intriguing phenomenon of "entrapment" was revealed when the weak or strong Allee effect was added to the equation with the periodic carrying capacity. For the weak Allee effect, all the solutions are persistent. However, for the strong Allee effect this property is exclusive for solutions on the specific interval. The p-periodic solutions of the Ricker equation with the Allee effect and p-periodic carrying capacity were further studied with respect to the attenuant and resonant properties of the cycles. It turned out that despite the fact that the function on the right-hand side of the equation cannot be described either as concave or as convex, the cycles consistently demonstrate the attenuant properties.

Thus, it was proven that the concavity of the corresponding function is not a necessary condition for the periodic solutions of the Ricker equation with the Allee effect and p-periodic carrying cappacity to be attenuant.

The study of the Ricker equation with the Allee effect and p-periodic carrying capacity completed in this research rises the spectrum of possibilities for the further research:

1. The bifurcation procces for the four types of the equation: weak Allee effect, strong Allee effect, and both of those enhanced with the periodic carrying capacity.
2. There has been proven a theorem on stability of the periodic solutions based on the derivative of the composition function ([39]). The comparison of the stability conditions developed in this thesis with the condition of that theorem may reveal hidden relations between parameters of the equation.
3. Both parameters α and β are connected to the stability of the solutions. What are the origin of these parameters and what is the connection between them?
4. It has been shown that the concavity of the forcing function is not a necessary condition for the attenuant property of the cycles. The new approach to the studies of the attenuant

properties has to be developed.

5. One of the reasons of the complex behaviour of the solutions for the equation (1.0.1) is the interplay of the polynomials and exponents. It could be promising to explore this mutual interference further.
6. Most importantly, as for any mathematical model, is to compare the analytical result with experimental biological data to tune the model accordingly.

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