Numerical Study Using Explicit Multistep Galerkin Finite Element Method for the MRLW Equation

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In this article, an explicit multistep Galerkin finite element method for the modified regularized long wave equation is studied. The discretization of this equation in space is by linear finite elements, and the time discretization is based on explicit multistep schemes. Stability analysis and error estimates of our numerical scheme are derived. Numerical experiments indicate the validation of the scheme by L_2 - and L_∞ -error norms and three invariants of motion.4 © 2015 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 31: 1875–1889, 2015

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I. INTRODUCTION

Nonlinear partial differential equations are useful in describing a variety of phenomena across a range of disciplines. Most of these equations do not have an analytical solution, or it is extremely difficult and expensive to compute their analytical solutions. Hence the numerical study of these nonlinear partial differential equations is important in practice. The regularized long wave equation is a class of nonlinear evolution equations, which was originally introduced by Peregrine [1] in describing the behavior of the undular bore, and later by Benjamin et al. [2]. This equation plays an important role in describing physical phenomena in various disciplines, such as the nonlinear transverse waves in shallow water, ion-acoustic waves in plasma, magnetohydrodynamics waves in plasma, longitudinal dispersive waves in elastic rods, and pressure waves in liquids gas bubbles. Many numerical methods for the RLW equation have been proposed, which include finite difference methods [3, 4], the Galerkin finite element method [5–8], least square method [9–11],

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various collocation methods with quadratic B-splines [12], cubic B-splines [13], septic splines [14], and an explicit multistep method [15].

The RLW equation is a special case of the generalized regularized long wave equation

$$u_t + u_x + \delta u^p u_x - \mu u_{xxt} = 0.$$

where δ and μ are positive constants and p is a positive integer. Some numerical methods [16–22] for the GRLW equation have also been presented, such as a finite difference method [16], a decomposition method [18], and a sinc-collation method [21]. Another special case of the GRLW equation is called the modified regularized long wave (MRLW) equation in which $p = 2$. Some authors have studied the MRLW equation applying various numerical methods, for example: Gardner et al. [23] using B-spline finite elements, Khalifa et al. [24] using a finite difference method, Khalifa et al. [25] using a collocation method with cubic B-splines, Raslan [26] using the collocation method with quadratic B-spline finite elements and used the central finite difference method for time derivative and Khalifa et al. [27] with the Adomian decomposition method.

In this article, we study the MRLW equation using the linear finite element method in space and explicit multistep method in time. We discuss the stability analysis and error estimates of this method and compare its accuracy with previous studies. Numerical experiments are also provided to verify the validation of the scheme.

The outline of this article is as follows. In the next section, the governing equation, its analytical solution and three invariants are given. The numerical methods including space semidiscretization and fully time space discretization are presented in Section III. Section IV describes the stability analysis and the optimal error estimates are derived in Section V. In Section VI, numerical results are provided to test the theoretical result, and a brief conclusion is drawn in the last section.

II. GOVERNING EQUATION

Assuming $\Omega = (a, b)$ to be an open, bounded subset of R and $T > 0$ to be a final time, setting $\Omega_T = \Omega \times (0, T)$, $J = (0, T]$, we consider the following MRLW equation

$$\begin{aligned} u_t + u_x + 6u^2 u_x - \mu u_{xxt} &= 0, \quad \text{in } \Omega_T, \\ u &= 0, \quad \text{on } \partial\Omega \times [0, T], \\ u &= u_0, \quad \text{on } \Omega \times \{t = 0\}, \end{aligned} \quad (1)$$

where $u_0: \Omega \rightarrow R$ is the initial condition, μ is a positive constant and subscripts x and t denote differentiation in space and time, respectively. The physical boundary conditions require $u \rightarrow 0$ as $x \rightarrow \pm\infty$.

The MRLW Eq. (1) has the exact solution as [23]

$$u(x, t) = \sqrt{c} \operatorname{sech}(p(x - (c + 1)t - x_0)). \quad (2)$$

where $p = \sqrt{\frac{c}{\mu(c+1)}}$, x_0 and c are arbitrary constants. Furthermore, the MRLW Eq. (1) possesses three invariants of motion corresponding to conservation of mass, momentum and energy as [23]

$$I_1 = \int_a^b u \, dx,$$

$$\begin{aligned} I_2 &= \int_a^b (u^2 + \mu u_x^2) dx, \\ I_3 &= \int_a^b (u^4 - \mu u_x^2) dx. \end{aligned} \quad (3)$$

These invariants are used to check the conservative properties of the numerical scheme, especially for problems with no analytical solution and during the collision of solitons.

III. NUMERICAL METHODS

Set $V = H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$. Applying the Green's formula to the problem (1) and using the boundary condition in the definition of V , we derive the variational form of problem (1):

$$\begin{cases} \text{Find } u(x, t) : J \rightarrow V, \text{ such that} \\ \left(\frac{\partial u}{\partial t}, v \right) + B(u, v) + C(u, v) + D(u, v) = 0, & \forall v \in V, t \in J, \\ (u(x, 0), v) = (u_0(x), v), & \forall v \in V, x \in \Omega, \end{cases} \quad (4)$$

where

$$B(u, v) = \int_a^b \mu u_{xt} \cdot v_x dx, \quad C(u, v) = \int_a^b u_x \cdot v dx, \quad D(u, v) = \int_a^b 6u^2 u_x \cdot v dx.$$

A. Semidiscretization in Space

Considering a uniform 1- D mesh with mesh size $h = x_{i+1} - x_i, i = 0, 1, \dots, N-1$, which consists of $N+1$ points $a = x_0 < x_1 < \dots < x_{N-1} < x_N = b$, we obtain

$$\begin{aligned} \sum_{i=0}^{N-1} & \left(\int_{x_i}^{x_{i+1}} \frac{\partial u}{\partial t} \cdot v dx + \int_{x_i}^{x_{i+1}} \mu u_{xt} \cdot v_x dx \right. \\ & \left. + \int_{x_i}^{x_{i+1}} u_x \cdot v dx + \int_{x_i}^{x_{i+1}} 6u^2 u_x \cdot v dx \right) = 0, \end{aligned} \quad (5)$$

Using the transformation $x = x_i + \eta h, 0 \leq \eta \leq 1, i = 0, 1, \dots, N-1$, we derive the following contribution

$$\int_0^1 \frac{\partial u}{\partial t} \cdot v d\eta + \frac{\mu}{h^2} \int_0^1 u_{\eta t} \cdot v_{\eta} d\eta + \frac{1}{h} \int_0^1 u_{\eta} \cdot v d\eta + \frac{6}{h} \int_0^1 u^2 u_{\eta} \cdot v d\eta = 0. \quad (6)$$

Define the finite dimension subspace $V_h \subset V, V_h = \text{span}\{L_1, L_2\}$, where

$$L_1 = \eta, \quad L_2 = 1 - \eta$$

are linear basis functions on each element. Then the semidiscrete scheme for problem (1) is formulated as follows:

$$\begin{cases} \text{Find } u_h : J \rightarrow V_h, \text{ such that} \\ \left(\frac{\partial u_h}{\partial t}, v \right) + B(u_h, v) + C(u_h, v) + D(u_h, v) = 0, & \forall v \in V_h, t \in J, \\ (u_h(x, 0), v) = (u_0^h(x), v), & \forall v \in V_h, x \in \Omega. \end{cases} \quad (7)$$

The variation of u over the element $[x_i, x_{i+1}]$, $i = 0, 1, \dots, N-1$, is expressed as

$$u^e = \sum_{j=1}^2 L_j(x) u_j(t). \quad (8)$$

For $j = 1, 2$, by choosing $v = L_j$ in problem (7), an element's contribution is obtained in the form of

$$\begin{aligned} \sum_{i=1}^2 \left\{ \left(\int_0^1 L_i \cdot L_j dx + \frac{\mu}{h^2} \int_0^1 L_i' \cdot L_j' dx \right) \frac{\partial u_i}{\partial t} \right. \\ \left. + \frac{1}{h} \left(\int_0^1 L_i' \cdot L_j dx \right) u_i + \left(\frac{6}{h} \int_0^1 \left(\sum_{i=1}^2 u_i L_i \right)^2 L_i' \cdot L_j dx \right) u_i \right\} = 0, \end{aligned} \quad (9)$$

where the symbol $(\cdot)$ denotes differentiation with respect to η , which, in matrix form, is given by

$$(A^e + B^e) \frac{\partial u^e}{\partial t} + C^e u^e + D^e (u^e) u^e = 0, \quad (10)$$

where $u^e = (u_1, u_2)^T$ are relevant nodal parameters. The element matrices are

$$\begin{aligned} A_{jk}^e &= \int_0^1 L_j L_k d\eta, \\ B_{jk}^e &= \frac{\mu}{h^2} \int_0^1 L_j' L_k' d\eta, \\ C_{jk}^e &= \frac{1}{h} \int_0^1 L_j' L_k d\eta, \\ D_{jk}^e &= \frac{6}{h} \int_0^1 \left(\sum_{m=1}^2 u_m L_m \right)^2 L_j' L_k d\eta. \end{aligned}$$

Assembling contributions from all elements leads to the following matrix equation of the coupled nonlinear ordinary differential equation

$$(A + B) \frac{\partial u}{\partial t} + C u + D(u) u = 0, \quad (11)$$

where $u = (u_0, u_1, \dots, u_N)^T$ contains all the nodal parameters. The four assembled matrices are tridiagonal. The general row for each matrix has the following form

$$A : \left(\frac{1}{6}, \frac{2}{3}, \frac{1}{6} \right),$$

$$\begin{aligned}
 B &: \frac{\mu}{h^2} (-1, 2, -1), \\
 C &: \frac{1}{2h} (-1, 0, 1), \\
 D &: \frac{6}{h} \left(-\frac{1}{12} (u_{m-1}^2 + 2u_{m-1}u_m + u_m^2), \right. \\
 &\quad \left. \frac{1}{12} (u_{m-1}^2 - u_{m+1}^2) + \frac{1}{6} u_m (u_{m-1} - u_{m+1}), \frac{1}{12} (3u_m^2 + 2u_mu_{m+1} + u_{m+1}^2) \right).
 \end{aligned}$$

B. Full Discretization

Let $0 = t^0 < t^1 < \dots < t^M = T$ be a uniform partition of J into subintervals $J^n = (t^n, t^{n+1})$, $n = 0, 1, \dots, M-1$ with length $k = t^{n+1} - t^n = \frac{T}{M}$. For a generic function v of time, set $v^n = v(t^n)$, we formulate the following linear explicit multistep method

$$(A + B) \sum_{j=0}^r \alpha_j u^{n+j} = k \sum_{j=0}^{r-1} \gamma_j (-C - D(u^{n+j})) u^{n+j}, \quad (12)$$

where parameter variables α_j and γ_j denote the coefficients of the term ζ^j of the polynomials α and γ ,

$$\alpha(\zeta) = \sum_{j=1}^r \frac{1}{j} \zeta^{r-j} (\zeta - 1)^j, \quad \gamma(\zeta) = \zeta^r - (\zeta - 1)^r. \quad (13)$$

Here, we consider the explicit two-step method for the MRLW. For $r = 2$, we obtain

$$\alpha_0 = \frac{1}{2}, \alpha_1 = -2, \alpha_2 = \frac{3}{2}, \gamma_0 = -1, \gamma_1 = 2, \quad (14)$$

then we achieve the full discretization scheme for (12)

$$(A + B) \left(\frac{3}{2} u^{n+2} - 2u^{n+1} + \frac{1}{2} u^n \right) = k ((C + D(u^n))u^n - 2(C + D(u^{n+1}))u^{n+1}). \quad (15)$$

We can use the Thomas algorithm to solve the linear algebraic system.

IV. STABILITY ANALYSIS

In this section, we analyze the linear stability of our scheme to the linearized MRLW equation

$$u_t + \beta u_x - \mu u_{xx} = 0. \quad (16)$$

where $\beta = 1 + 6 \max |u^2|$. As in previous works [28], the stability results are obtained by examining the location of the roots of its characteristic polynomial. We use the theory of Schur and von Neumann polynomials and outline the relevant definitions and theorems.

Definition 1. The polynomial ϕ is a Schur polynomial if all its roots, z_j satisfy $|z_j| < 1$.

Definition 2. The polynomial ϕ is a simple von Neumann polynomial if all its roots, z_j satisfy $|z_j| \leq 1$ and its roots on unit circle are simple roots.

Definition 3. A numerical scheme is stable if and only if its characteristic polynomial $\phi(z)$ is a simple von Neumann polynomial.

Definition 4. Given the polynomial $\phi(z) = \sum_{j=0}^n a_j z^j$ of degree N with $a_0, a_N \neq 0$, we define the polynomial

$$\phi_1(z) = \frac{\phi^*(0)\phi(z) - \phi(0)\phi^*(z)}{z}, \quad (17)$$

where $\phi^*(z) = \sum_{j=0}^n a_{n-j}^* z^j$ and a^* denotes the complex conjugate of a .

Theorem 1. $\phi(z)$ is a simple polynomial if and only if either $|\phi(0)| < |\phi^*(0)|$ and $\phi_1(z)$ is a simple von Neumann polynomial or $\phi_1(z)$ is identically zero and $\frac{d\phi}{dz}$ is a Schur polynomial.

Applying the discrete Fourier transform $u_j = e^{ij\xi h} \tilde{u}$ to Eq. (15), we obtain

$$M_1 \frac{d\tilde{u}}{dt} = -iM_2 \tilde{u}, \quad (18)$$

where $M_1 = 1 + 2\left(\frac{2\mu}{h^2} - \frac{1}{3}\right) \sin^2\left(\frac{\xi h}{2}\right)$ and $M_2 = \frac{\beta}{h} \sin(\xi h)$. Exploiting the two-step method (15) for Eq. (18) yields

$$M_1 \left(\frac{3}{2} \tilde{u}^{n+2} - 2\tilde{u}^{n+1} + \frac{1}{2} \tilde{u}^n \right) + iM_2 k (2\tilde{u}^{n+1} - \tilde{u}^n) = 0, \quad (19)$$

and the characteristic polynomial is given by

$$\phi(z) = M_1 \left(\frac{3}{2} z^2 - 2z + \frac{1}{2} \right) + iM_2 k (2z - 1), \quad (20)$$

so that $|\phi^*(0)| = |\phi(0)|$ and $\phi_1(z) \equiv 0$ hold. In view of the application of Theorem 1, we consider

$$\frac{d\phi}{dz} = 3M_1 z - 2M_1 + i2M_2 k, \quad (21)$$

and the modulus of its root is

$$|z| = \frac{2}{3} \sqrt{1 + \left(\frac{M_2}{M_1} \Delta t \right)^2} = \frac{2}{3} \sqrt{1 + \frac{\beta^2 k^2 \sin^2(\xi h)}{h^2 \left(1 + 2 \left(\frac{2\mu}{h^2} - \frac{1}{3} \right) \sin^2\left(\frac{\xi h}{2}\right) \right)^2}}, \quad (22)$$

where $\beta > 1$, then we have $|z| < 1$.

So from Theorem 1, our explicit two-step numerical scheme is stable.

V. ERROR ESTIMATES

In this section, we shall analyze the optimal error estimates for the fully discrete scheme Eq. (15) in time with explicit two-step scheme.

For $s \in \mathbb{N}$, denote H^s the Sobolev space $H^s(\Omega)$, H_0^s the Sobolev space $H_0^s(\Omega)$ and $\|\cdot\|_s$ the usual norm in H^s . Let $H = L^2(\Omega) = H^0$, $\|\cdot\|_0$ is the L_2 norm and $\|\cdot\|_\infty$ is the L_∞ norm.

Consider the explicit two-step method, we can prove that the multistep method satisfies the root condition:

1. If z_0 is a root of α , then $\|z_0\|_0 \leq 1$,
2. $C_j = 0, j = 0, 1, 2, C_3 \neq 0$, with $C_0 = \alpha(1)$, $C_i = \frac{1}{i!} \left(\sum_{j=0}^2 \alpha_j j^i - \sum_{j=0}^1 \gamma_j j^{i-1} \right)$, $i \geq 1$.

Let $A : H^2 \cap H_0^1 \rightarrow H$ and $B : H^2 \cap H_0^1 \rightarrow L^2$ defined by $Av = (I - \mu \Delta)v$ and $Bv = \nabla \cdot f(v)$, respectively, where $f(v) = -(v + 2v^3)$. Then A is a linear self-adjoint, positive definite operator and B is a nonlinear operator. Let $V := D(A^{1/2}) = H_0^1$ with the norm $\|v\| = (\|v\|_0^2 + \mu \|\Delta v\|_0^2)^{1/2}$ which is equivalent to $\|\cdot\|_1$. Identify H with its dual, let V' be the dual of V , denote by $(\cdot, \cdot)$ the duality of V' and V and by $\|\cdot\|_*$ the dual norm on V' .

For the previous space discretization, a family of finite dimensional subspaces $\{V_h\}_{0 < h \leq 1}$ satisfies the following approximation property:

$$\inf_{\chi \in V_h} (\|v - \chi\|_0 + h\|v - \chi\|_1) \leq ch^2\|v\|_2, \quad \forall v \in H^2 \cap H_0^1. \quad (23)$$

We derive the following optimal error estimates.

Theorem 2. Let $\{u_h^n\}_{n=0}^M$ satisfy the scheme Eq. (15), the solution u of Eq. (1) be sufficiently smooth, and $\|u_h^0 - u^0\|_0 + \|u_h^1 - u^1\|_0 \leq C(k^2 + h^2)$. Then, for h and k sufficiently small, there exists a constant C , such that

$$\max_{0 \leq n \leq M} \|u_h^n - u^n\|_0 \leq C(k^2 + h^2), \quad (24)$$

where the constant C is independent of h and k .

Proof. From [29], we know that we can obtain the result Eq. (24) by proving hypothetical conditions hold in Theorem 4.1 of [29]. We define the bilinear form $a(\cdot, \cdot)$ by

$$a(v, w) = (v, w) + \mu(\nabla v, \nabla w), \quad \forall v, w \in V, \quad (25)$$

and use the elliptic projection operator to $V_h, R_h: V \rightarrow V_h$, defined by

$$a(R_h v - v, \chi) = 0, \quad \forall \chi \in V_h, \quad (26)$$

this elliptic operator has the following properties:

1. There exists a constant c , such that

$$\|v - R_h v\|_0 \leq ch\|v\|, \quad \forall v \in V,$$

2. Assuming that v is sufficiently smooth, then

$$\|v - R_h v\|_0 + h\|v - R_h v\| \leq ch^2\|v\|_2, \quad \forall v \in V.$$

Denoting by T the solution operator of the problem: Given $f \in V'$, find $v \in V$, such that

$$a(v, w) = (f, w), \quad \forall w \in V. \quad (27)$$

According to the Sobolev inequality, there exists a constant C_* , such that

$$\|v\|_\infty \leq C_*\|v\|, \quad \forall v \in H_0^1, \quad (28)$$

Define the set $\bar{M}$ by $\bar{M} = \{v \in V : \exists t \in J, \|u(\cdot, t) - v\| < 1\}$. Now consider the set $\tilde{M} = \{u_h \in \mathbb{R} : \exists(x, t) \in \Omega \times J, \|u(x, t) - u_h\|_0 < C_*\}$ with C_* the constant in Eq. (28). Obviously, every real function $f \in C^1(\mathbb{R} \times J \times \mathbb{R})$ restricted to $\mathbb{R} \times J \times \tilde{M}$ satisfies a Lipschitz condition with Lipschitz constant L_f ; namely,

$$\|f(\cdot, t, v) - f(\cdot, t, w)\|_0 \leq L_f\|v - w\|_0, \quad \forall t \in J, \quad \forall v, w \in \bar{M}.$$

Thus, one can see that

$$(B(v) - B(w), \varphi) \leq \|f(v) - f(w)\|_0 \cdot \|\varphi\| \leq L_f\|v - w\|_0 \cdot \|\varphi\|, \quad \forall v, w \in \bar{M}, \varphi \in V.$$

Then, for $v, w \in \bar{M}$, we obtain

$$\begin{aligned} \|B(v) - B(w)\|_* &\leq L_1\|v - w\|, \\ \|TB(v) - TB(w)\|_0 &\leq L_2\|v - w\|_0. \end{aligned} \quad (29)$$

Define approximations $u_h^n \in V_h$ to u^n , $n = 2, \dots, M$, inductively by the scheme

$$\sum_{j=0}^2 \alpha_j \left[\left(u_h^{n+j}, \chi \right) + \mu \left(\nabla u_h^{n+j}, \nabla \chi \right) \right] = k \sum_{j=0}^1 \gamma_j \left(\nabla \cdot f(u_h^{n+j}), \chi \right), \quad \forall \chi \in V_h,$$

for $n = 0, 1, \dots, M-2$. Then, from Theorem 4.1 in [29], for h and k sufficiently small, we have

$$\max_{0 \leq n \leq M} \|u_h^n - u^n\|_0 \leq C(k^2 + h^2), \quad (30)$$

with a constant C independent of h and k . ■

VI. NUMERICAL EXPERIMENTS

In this section, we present some numerical tests to check the efficiency and accuracy of our explicit two-step scheme, which are the propagation of single soliton and collision of two solitons at different time levels. Finally, we investigate the development of the Maxwellian initial condition into solitary waves.

TABLE I. Invariants and error norms for single solitary wave, $c=1.0$, $h = 0.2$, $k = 0.025$, $x_0=40$, and $0 \leq x \leq 100$.

Time	I_1	I_2	I_3	L_2	L_∞
Analytical	4.44288	3.29983	1.41421		
0	4.44288	3.29981	1.41420	0	0
2	4.44288	3.29981	1.41421	2.38994×10^{-3}	1.23221×10^{-3}
4	4.44288	3.29982	1.41423	4.56503×10^{-3}	2.36119×10^{-3}
6	4.44288	3.29982	1.41421	6.49892×10^{-3}	3.42251×10^{-3}
8	4.44288	3.29983	1.41421	8.14728×10^{-3}	4.32579×10^{-3}
10	4.44288	3.29983	1.41420	9.49238×10^{-3}	5.06983×10^{-3}

To illustrate the accuracy of the present scheme, we use L_2 - and L_∞ - error norms

$$L_2 = \|u_h^n - u^n\|_0 = \sqrt{\left(h \sum_{i=1}^N |u_{hj}^n - u_j^n|^2\right)}, \quad (31)$$

$$L_\infty = \|u_h^n - u^n\|_\infty = \max_j |u_{hj}^n - u_j^n|.$$

to compare the numerical solution with the exact solution, which shows the mean and maximum differences between the numerical and the analytical solutions. The quantities I_1 , I_2 , and I_3 measure the conservation laws of our scheme during propagation.

A. Single Soliton

The analytical values of the variants are

$$I_1 = \frac{\pi\sqrt{c}}{p}, I_2 = \frac{2c}{p} + \frac{2\mu pc}{3}, I_3 = \frac{4c^2}{3p} - \frac{2\mu pc}{3}.$$

For the purpose of comparison with previous works [23] and [25], we choose the parameters $x_0 = 40$, $c = 1.0$, $h = 0.2$, $k = 0.025$, $T = 10$, and $0 \leq x \leq 100$. Then the amplitude is 1.0 and the analytical values of the three invariants are $I_1 = 4.44288$, $I_2 = 3.29983$, and $I_3 = 1.41421$. The relevant numerical results are presented in Table I, and the profiles of solitary waves at different time are given in Fig. 1. Table II presents the variants, L_∞ norm and L_2 norm at time = 10 against the cubic B-splines in [23] and [25].

To illustrate the accuracy of our algorithm, we compute the pointwise convergence rate in time and space. We define the order of accuracy of space as the formulation [7]

$$\text{order}_h = \frac{\log_{10}(|u - u_{h_j}|/|u - u_{h_{j+1}}|)}{\log_{10}(h_j/h_{j+1})}, \quad (32)$$

where $|\cdot|$ denotes L_2 or L_∞ norm, and u_{h_j} is the numerical solution with spatial step size h_j . Fixing the temporal step size $k = 0.01$, we choose the parameters as $c = 0.03$ and different spatial partition 0.2, 0.4, and 0.8. The L_2 - and L_∞ - error and the convergence order of time are presented in Tables III and IV. We can obtain L_2 - and L_∞ - error norms

$$\|u_h - u\|_0 \approx 0.036h^{2.026}, \|u_h - u\|_\infty \approx 0.01886h^{2.008}$$

by linear regression, reach $O(h^2)$ and confirm the theoretical results.

FIG. 1. Solitary wave with $c = 1.0$, $h = 0.2$, $k = 0.025$, and $x_0 = 40$, $0 \leq x \leq 100$. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

TABLE II. Invariants and error norms for single solitary wave, $c=1.0$, $h = 0.2$, $k = 0.025$, $x_0 = 40$, $0 \leq x \leq 100$, and time = 10.

Time	I_1	I_2	I_3	L_2	L_∞
Analytical	4.44288	3.29983	1.41421		
Our scheme	4.44288	3.29983	1.41420	9.49238×10^{-3}	5.06983×10^{-3}
B-splines [25]	4.44288	3.29983	1.41420	9.30196×10^{-3}	5.43718×10^{-3}
B-splines coll-CN [23]	4.442	3.299	1.413	1.639×10^{-2}	9.24×10^{-3}
B-splines coll+PA-CN [23]	4.440	3.296	1.411	2.03×10^{-2}	1.12×10^{-2}

TABLE III. The order of convergence of space for L_2 error norm with $c = 0.3$ and $k = 0.01$.

Time	$h = 0.2$	$h = 0.4$	$h = 0.8$	Order (0.2/0.4)	Order (0.4/0.8)
2	5.19056×10^{-4}	2.21847×10^{-3}	9.30613×10^{-3}	2.09559	2.06862
4	9.63368×10^{-4}	4.09489×10^{-3}	1.68473×10^{-2}	2.08767	2.04062
6	1.35709×10^{-3}	5.74562×10^{-3}	2.33937×10^{-2}	2.08194	2.02559
8	1.72854×10^{-3}	7.29675×10^{-3}	2.95259×10^{-2}	2.07770	2.01665
10	2.08976×10^{-3}	8.80161×10^{-3}	3.54682×10^{-2}	2.07443	2.01069

TABLE IV. The order of convergence of space for L_∞ error norm with $c = 0.3$ and $k = 0.01$.

Time	$h = 0.2$	$h = 0.4$	$h = 0.8$	Order (0.2/0.4)	Order (0.4/0.8)
2	3.41517×10^{-4}	1.45341×10^{-3}	6.17747×10^{-3}	2.08941	2.08758
4	5.60435×10^{-4}	2.38594×10^{-3}	9.76806×10^{-3}	2.08994	2.03351
6	7.24409×10^{-4}	3.07040×10^{-3}	1.25052×10^{-2}	2.08355	2.02602
8	8.77790×10^{-4}	3.70389×10^{-3}	1.48963×10^{-2}	2.07709	2.00784
10	1.02762×10^{-3}	4.34071×10^{-3}	1.69394×10^{-2}	2.07862	1.96438

TABLE V. The order of convergence of time for L_2 error norm with $c = 0.3$ and $h = 0.125$.

Time	$k = 0.05$	$k = 0.1$	$k = 0.2$	Order (0.05/0.1)	Order (0.1/0.2)
2	8.10432×10^{-4}	3.76761×10^{-3}	1.49388×10^{-2}	2.21688	1.96567
4	1.52224×10^{-3}	7.25869×10^{-3}	3.07186×10^{-2}	2.25352	2.06015
6	2.11029×10^{-3}	1.02249×10^{-2}	4.48009×10^{-2}	2.27658	2.11015
8	2.62859×10^{-3}	1.29309×10^{-2}	5.84011×10^{-2}	2.29847	2.15365
10	3.10977×10^{-3}	1.55296×10^{-2}	7.22347×10^{-2}	2.32014	2.19589

TABLE VI. The order of convergence of time for L_∞ error norm with $c = 0.3$ and $h = 0.125$.

Time	$k = 0.05$	$k = 0.1$	$k = 0.2$	Order (0.05/0.1)	Order (0.1/0.2)
2	5.29688×10^{-4}	2.46747×10^{-3}	9.73878×10^{-3}	2.21982	1.96567
4	8.87783×10^{-4}	4.23136×10^{-3}	1.77624×10^{-2}	2.25284	2.06015
6	1.13236×10^{-3}	5.47313×10^{-3}	2.36326×10^{-2}	2.27304	2.11015
8	1.34030×10^{-3}	6.55669×10^{-3}	2.90663×10^{-2}	2.29042	2.15365
10	1.53374×10^{-3}	7.59687×10^{-3}	3.45147×10^{-2}	2.30835	2.19589

Similar to the definition of the spatial order accuracy, the convergence rate of time can be defined as

$$\text{order}_t = \frac{\log_{10} (|u - u_{k_j}| / |u - u_{k_{j+1}}|)}{\log_{10} (k_j / k_{j+1})}, \quad (33)$$

where u_{k_j} is the numerical solution with time step size k_j . Tables V and VI propose the L_2 and L_∞ error convergence rate of time with $c = 0.3$, the space size $h = 0.125$ and the time size $k = 0.05, 0.2, 0.4$. We have

$$\|u_h - u\|_0 \approx 1.439k^{2.164}, \quad \|u_h - u\|_\infty \approx 0.7033k^{2.127}.$$

which reach $O(k^2)$ in accordance with the theoretical analysis.

B. Collision of Two Solitons

In this section, we study the interaction of two solitary waves with initial conditions given by a linear sum of two separate solitary waves of various amplitudes

$$u(x, 0) = \sum_{i=1}^2 A_i \text{sech}(p_i(x - x_i)), \quad (34)$$

where $A_i = \sqrt{c_i}$, $p_i = \sqrt{\frac{c_i}{\mu(c_i+1)}}$, $i = 1, 2$, x_i and c_i ($i = 1, 2$) are arbitrary constants. The analytical values of the conservation laws in this case have the following form

$$I_1 = \frac{\pi \sqrt{c_1}}{p_1} + \frac{\pi \sqrt{c_2}}{p_2},$$

$$I_2 = \frac{2c_1}{p_1} + \frac{2c_2}{p_2} + \frac{2\mu c_1 p_1}{3} + \frac{2\mu p_2 c_2}{3},$$

TABLE VII. Invariants for two solitary waves with $c_1 = 1$, $c_2 = 0.5$, $x_1 = 20$, $x_2 = 30$, $h = 0.1$, $k = 0.01$, and $0 \leq x \leq 100$.

Time	I_1	I_2	I_3
0	8.29053	5.29563	1.88406
2	8.29054	5.29562	1.88387
4	8.29054	5.29557	1.88364
6	8.29054	5.29550	1.88333
8	8.29055	5.29545	1.88306
10	8.29055	5.29545	1.88309

$$I_3 = \frac{4c_1^2}{3p_1} + \frac{4c_2^2}{3p_2} - \frac{2\mu p_1 c_1}{3} - \frac{2\mu p_2 c_2}{3}.$$

In our numerical scheme, we choose $h = 0.1$, $k = 0.01$, $c_1 = 1.0$, $c_2 = 0.5$, $x_1 = 20$, and $x_2 = 30$ with interval $[0, 100]$, then the amplitudes are in ratio $\sqrt{2} : 1$, where $\sqrt{2}A_2 = A_1$ and the analytical values for the conservations are $I_1 = 8.29053$, $I_2 = 5.22433$, and $I_3 = 1.79911$. The changes of the invariants are satisfactorily small, as the changes of the invariants I_1 , I_2 , and I_3 are 2.41239×10^{-6} , 3.39903×10^{-5} , and 5.14845×10^{-4} , respectively. The results are recorded in Table VII. The geometry of initial state and profiles at time $t = 0, 3, 7$, and 10 shown graphically in Fig. 2, respectively.

C. The Maxwellian Initial Condition

Now, we consider the development of the Maxwellian initial condition

$$u(x, 0) = e^{-(x-40)^2},$$

into a train of solitary waves. We choose different values $\mu = 1.0$ and 0.5 in our numerical scheme. The comparisons of the three variants I_1 , I_2 , and I_3 with the earlier result in [30] are presented in Table VIII. For $\mu = 1.0$, the change of the variants I_2 and I_3 with respect to the initial values are 6.1624×10^{-4} and 8.5756×10^{-3} , respectively, and the change of the variants I_1 from the initial variants approaches zeros. For $\mu = 0.5$, the variants are changed by 0 , 4.2646×10^{-5} and 5.9698×10^{-3} in this case. The development of the Maxwellian initial condition is shown in Fig. 3 with different parameter values, respectively. The smaller μ is, the more solitary waves will form. The simulations are done up to time = 10 in this case. From the previous work [23], the total number of solitary waves and the values μ has the approximate relation

$$N \cong \lceil \frac{1}{\sqrt[3]{\mu}} \rceil.$$

VII. CONCLUSION

In this article, the explicit multistep Galerkin finite element method is taken to investigate the propagation of nonlinear partial differential equation of the MRLW equation. The stability analysis is given, and optimal error estimate proves that the convergence of our scheme can reach $O(k^2 + h^2)$. The numerical results in Tables III–VI show that the L_2 - and L_∞ - error norms are of order $O(k^2 + h^2)$, corresponding with theoretical results. The high efficiency and accuracy of

FIG. 2. Interaction of two solitary waves. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

TABLE VIII. Values of I_1, I_2, I_3 for Maxwellian initial condition when $h = 0.1, k = 0.01$, and the space interval $[0, 100]$.

μ	Time	I_1	I_2	I_3	I_1 [30]	I_2 [30]	I_3 [30]
1	2	1.77254	2.49904	−0.363239	1.77245	2.50635	−0.366815
	4	1.77254	2.49904	−0.364212	1.77245	2.50624	−0.366697
	6	1.77254	2.49934	−0.365635	1.77245	2.50617	−0.366633
	8	1.77254	2.50037	−0.365984	1.77245	2.50612	−0.366586
	10	1.77254	2.50058	−0.366354	1.77244	2.50609	−0.366555
0.5	2	1.77254	1.87592	0.259807	1.77245	1.87989	0.259650
	4	1.77254	1.87585	0.258868	1.77245	1.87986	0.259683
	6	1.77254	1.87588	0.258465	1.77245	1.87984	0.259697
	8	1.77254	1.87585	0.258335	1.77245	1.87983	0.259705
	10	1.77254	1.87584	0.258256	1.77245	1.87983	0.259709

FIG. 3. Maxwellian initial condition with different values of μ . [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

our numerical method are demonstrated by the numerical experiments: the propagation of single solitons, collision of two and the development of Maxwellian initial condition into solitary waves. Moreover, our scheme satisfies the three conservation laws of mass, momentum and energy.

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