2014-05-08

Modeling of Currency Trading Markets and Pricing Their Derivatives in a Markov Modulated Environment

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Modeling of Currency Trading Markets and Pricing Their Derivatives in a Markov Modulated Environment

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE

DEGREE OF MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS AND STATISTICS

CALGARY, ALBERTA

May, 2014

Abstract

Using a Lévy process we generalize formulas in Bo *et al.* (2010) to the Esscher transform parameters for the log-normal distribution which ensures the martingale condition holds for the discounted foreign exchange rate. We also derive similar results, but in the case when the dynamics of the FX rate is driven by a general Merton jump-diffusion process.

Using these values of the parameters we find a risk-neural measure and provide new formulas for the distribution of jumps, the mean jump size, and the Poisson process intensity with respect to this measure. The formulas for a European call foreign exchange option are also derived.

We apply these formulas to the case of the log-double exponential and exponential distribution of jumps. We provide numerical simulations for the European call foreign exchange option prices with different parameters.

Acknowledgements

I am happy to take this opportunity to express my gratitude to my research supervisors Professor Anatoliy Swishchuk and Professor Robert Elliott. They introduced me to the new (for me) area of Mathematical Finance, gave many valuable remarks, and made suggestions for improvements. I am grateful for their attention to my thesis preparation and technical comments.

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List of Symbols, Abbreviations and Nomenclature

Symbol	Definition
Ch.	Chapter
Def.	Definition
FX rate	foreign exchange rate
Prop.	Proposition
U of C	University of Calgary
SDE	stochastic differential equation
w.r.t.	with respect to

Chapter 1

Introduction

The structure of the thesis is as follows:

Chapter 1 gives a brief introduction to the problem of FX currency option pricing with regime-switching parameters.

Chapter 2 provides an overview of results in the area under consideration. In particular, we give details of models for pricing foreign currency options incorporating stochastic interest rates, based on Merton's (1976, [30]) stochastic interest rate model for pricing equity options. We discuss pricing contingent claims on foreign currencies under stochastic interest rates using the Heath et al. (1987, [17]) model of term structure (Amin et al., 1991, [3]). A model for cross-currency derivatives, such as PRDC (Power-Reverse Dual-Currency) swaps with calibration to currency options, developed by Piterbarg (2005, [35]), is also described in our overview. We give details of an approximation formula for the valuation of currency options under a jump-diffusion stochastic volatility processes for spot exchange rates in a stochastic interest rate environment as proposed by Takahashi et al. (2006, [44]). A continuous time Markov chain which determines the values of parameters in a modified Cox-Ingersoll-Ross model was proposed by Goutte et al. (2011, [11]) to study the dynamics of foreign exchange rates. In our research we use a three state Markov chain. We present a two-factor Markov modulated stochastic volatility model with the first stochastic volatility component driven by a log-normal diffusion process and the second independent stochastic volatility component driven by a continuous-time Markov chain, as proposed by Siu et al. (2008, [41]). Finally, a Markov modulated jump-diffusion model (including a compound Poisson process) for currency option pricing as described in Bo et al. (2010, [8]). We should also mention that option pricing is discussed in the Masters Thesis "Hedging Canadian Oil.

An Application of Currency Translated Options" by Cliff Kitchen (see [26], 2010). At the end of Chapter 2 we describe a problem which will be solved in our thesis: generalize results in [8] to the case when the dynamics of the FX rate are driven by a general Lévy process ([34]).

In Chapter 3 we state all theorems and definitions necessary for our research. In §3.1 we give definitions for a general Lévy process and the most common cases of such processes: Brownian motion and the compound Poisson process. In §3.2 we give some formulas for characteristic functions of particular cases of Lévy processes that will be used to derive a main result. In the next two sections §3.3, 3.4 we give details of basic computational theorems used in the thesis: Ito's formula with jumps, Girsanov theorem with jumps.

In Chapter 4 we provide an overview of currency derivatives based on Bjork (1998, [6]). In §4.1 different types of currency derivatives with several examples are proposed. Basic foreign currency option pricing formulas are given in §4.2.

In Chapter 5 we formulate main results of our research:

1) In Section §5.1 we generalize formulas in [8] for the Esscher transform parameters which ensure that the martingale condition for the discounted foreign exchange rate is satisfied for a general Lévy process (see (5.30)). Using these values of the parameters, (see (5.39), (5.40)), we proceed to a risk-neutral measure and provide new formulas for the distribution of jumps, the mean jump size, (see (5.20)), and the Poisson process intensity with respect to this measure, (see (5.19)). At the end of §5.1 pricing formulas for the European call foreign exchange option are given (It is similar to those in [8], but the mean jump size and the Poisson process intensity with respect to the new risk-neutral measure are different).

2) In section §5.2 we apply the formulas (5.19), (5.20), (5.39), (5.40) to the case of log-double exponential processes (see (5.51)) for jumps (see (5.58)-(5.63)).

3) In Section §5.4 we derive results similar to §5.1 when the amplitude of the jumps is driven by a Merton jump-diffusion process, (see (5.88)-(5.90), (5.108)-(5.109)).

4) In Section §5.5 we apply formulas (5.88)-(5.90), (5.108)-(5.109) to a particular case of an exponential distribution, (see (5.110)) of jumps (see (5.113)-(5.115)).

In Chapter 6 we provide numerical simulations of the European call foreign exchange option prices for different parameters, (in case of log-double exponential and exponential distributions of jumps): S/K, where S is an initial spot FX rate, K is a strike FX rate for a maturity time T; parameters θ_1, θ_2 arise in the log-double exponential distribution etc.

The results of Chapters 5-6 are accepted for publication in the Journal Insurance: Mathematics and Economics (see [42]) and in the Journal of Mathematical Finance (see [43]).

In the Appendix codes for Matlab functions used in numerical simulations of option prices are provided.

Chapter 2

Literature review

Until the early 1990s the existing academic literature on the pricing of foreign currency options could be divided into two categories. In the first, both domestic and foreign interest rates are assumed to be constant whereas the spot exchange rate is assumed to be stochastic. See, e.g., Jarrow et al. (1981, [3]). The second class of models for pricing foreign currency options incorporated stochastic interest rates, and were based on Merton's 1973, [31]) stochastic interest rate model for pricing equity options. Let us mention the main points of the model in [31]. Let the stock price S_t(or spot FX rate in our case) follow the dynamics described by the following equation:

$$\frac{dS_t}{S_t} = (\alpha - \lambda k)dt + \sigma dW_t + (Z_{t_-} - 1)dN_t. \tag{2.1}$$

Here W_t is a Brownian motion, $\alpha - \lambda k$ is a drift, $\int_0^t (Z_{s_-} - 1)dN_s$ is a compound Poisson process(see Def. 3.3 in Ch. 3 for detail); k is a mean jump size if a jump occurs. In other words:

$$\frac{dS_t}{S_t} = (\alpha - \lambda k)dt + \sigma dW_t, \tag{2.2}$$

if the Poisson event does not occur, and

$$\frac{dS_t}{S_t} = (\alpha - \lambda k)dt + \sigma dW_t + Z_{t_-} - 1, \tag{2.3}$$

if a jump occurs. The solution path for spot FX rate S_t will be continuous most of the time with jumps of different signs and amplitudes occurring at discrete points in time.

The option price $\Pi(S, t)$ can be written as a twice continuously differentiable function of the stock price and time: namely, $\Pi_t = F(S, t)$. If the stock price yields the dynamics

defined in (2.1)-(2.3), then the option return dynamics can be written in a similar form as

$$\frac{d\Pi_t}{\Pi_t} = (\alpha^\pi - \lambda k^\pi)dt + \sigma^\pi dW_t + (Z^\pi - 1)dN_t, \tag{2.4}$$

where α^π is the instantaneous expected return on the option Π_t; $(\sigma^\pi)^2$ is the instantaneous variance of the return, conditional on the Poisson event not occurring; $(Z^\pi - 1)dN_t$ is an independent compound Poisson process with parameter λ; $k^\pi = \mathbb{E}(Z^\pi - 1)$.

Using Itô's theorem (see Ch. 3, § 3.4, for details) we obtain:

$$\alpha^\pi = \frac{1/2\sigma^2 S^2 F_{ss}(S,t) + (\alpha - \lambda k)SF_s(S,t) + F_t + \lambda\mathbb{E}(F(SZ,t) - F(S,t))}{F(S,t)}, \tag{2.5}$$

$$\sigma^\pi = \frac{F_s(S,t)\sigma S}{F(S,t)}, \tag{2.6}$$

where $F_{ss}(S,t), F_s(S,t), F_t(S,t)$ are partial derivatives.

Let $g(S,T)$ be the instantaneous expected rate of return on the option when the current stock price is S_t and the option expiry date is T. Then the option price $F(S,t)$ yields the following equation (see (2.5) and [30] for more details):

$$\begin{aligned} &1/2\sigma^2 S^2 F_{ss}(S,T) + (\alpha - \lambda k)SF_s(S,T) - \\ &F_t(S,T) - g(S,T)F + \lambda\mathbb{E}(F(SZ,T) - F(S,T)) = 0, \end{aligned} \tag{2.7}$$

with the following boundary conditions:

$$F(0,T) = 0, F(S,0) = \max(0, S - K), \tag{2.8}$$

where K is a strike price. We shall consider a portfolio strategy which consists of the stock(FX rate in our case), the option, and the riskless asset with return r. Assume, that we have zero-beta portfolio. From [55]: "A Zero-beta portfolio is a portfolio constructed to have zero systematic risk or, in other words, a beta of zero. A zero-beta portfolio would have the same expected return as the risk-free rate. Such a portfolio would have zero correlation with market movements, given that its expected return equals the risk-free rate, a low rate

of return". In other words the jump component of the stocks return will represent "non-systematic" risk (see again [30]). Under this assumption and using (2.5)-(2.7) we obtain the following equation for currency option price:

$$1/2\sigma^2 S^2 F_{ss}(S,T) + (r - \lambda k)SF_s(S,T) -$$
$$F_t(S,T) - rF + \lambda \mathbb{E}(F(SZ,T) - F(S,T)) = 0 \tag{2.9}$$

with the same boundary conditions (2.8). Pricing equation (2.9) is similar to (2.7), but (2.9) does not depend on either α or $g(S,T)$. Instead, as in the standard Black-Scholes formula, the dependence on the interest rate, r appears. Note, that the jump component influences the equilibrium option price in spite of the fact that it represents non-systematic risk.

The solution to (2.9) for the option price when the current stock price is S and the expiry time is T can be calculated using the formula (see again [30]):

$$F(S,T) = \sum_{n=0}^{\infty} \frac{e^{-\lambda T}(\lambda T)^n}{n!} \mathbb{E}\left[B(SZe^{-\lambda kT}, T; K, \sigma^2, r)\right]. \tag{2.10}$$

Here $B(SZe^{-\lambda kT}, T; K, \sigma^2, r)$ is the Black-Scholes option pricing formula for the no-jump case (see [7] and Ch. 4, § 4.2 for detail); $\mathbb{E}$ is an expectation operator over the distribution of Z, Z is the distribution of jumps (see (2.1), (2.3).

A similar model can be found in Grabbe (1983, [23]), Adams et al. (1987, [1]). Unfortunately, this pricing approach did not include a full-fledged term structure model into the valuation framework. Moreover, the Forex market is more complicated can not be adequately described by the model with the assumption of a zero-beta portfolio.

Amin et al. (1991, [3]), to our best knowledge, were the first to start discussing and building a general framework to price contingent claims on foreign currencies under stochastic interest rates using the Heath et al. (1987, [17]) model of term structure. We shall introduce the assumptions underlying the economy as in [3]. The authors specify the evolution of forward domestic and foreign interest rates, (see the details of forward agreements in

Ch.4, §4.1.1) in the following way: they consider four sources of uncertainty across the two economies, (domestic and foreign), defined by four independent standard Brownian motions $(W_t^1, W_t^2, W_t^3, W_t^4;\ t \in [0, \tau])$ on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The domestic forward interest rate $r_d(t, T)$, (the country's forward interest rate contracted at time t for instantaneous borrowing and lending at time T with $0 \leq t \leq T \leq \tau$) changes over time according to the following stochastic differential equation (see Assumption 1, [3]):

$$dr_d(t,T) = \alpha_d(t,T,\omega)dt + \sum_{i=1}^{2} \sigma_{di}(t,T,r_d(t,t))dW_t^i \tag{2.11}$$

for all $\omega \in \Omega$, $t \leq T \leq \tau$.

Note, that the same two independent Brownian motions W_t^1, W_t^2 drive the domestic forward interest rate. These random shocks can be explained as a short-run and long-run factor changing different maturity ranges of the term structure.

The domestic bond price in units of the domestic currency can be written (see [3]):

$$B_d(t,T) = \exp\left\{-\int_t^T r_d(t,u)du\right\}. \tag{2.12}$$

The dynamics of the domestic bond price process is given by the equation:

$$dB_d(t,T) = [r_d(t,t) + b_d(t,T)]B_d(t,T)dt + \sum_{i=1}^{2} \alpha_{di}(t,T)B_d(t,T)dW_t^i, \tag{2.13}$$

where

$$\alpha_{di} = -\int_t^T \sigma_{di}(t,u,r_d(t,u))du,\ i = 1,2, \tag{2.14}$$

$$b_d(t,T) = -\int_t^T \alpha_d(t,u,\omega)du + \frac{1}{2}\sum_{i=1}^{2}\left[\int_t^T \sigma_{di}(t,u,r_d(t,u))du\right]^2. \tag{2.15}$$

A similar assumption holds for the foreign forward interest rate dynamics (see Assumption 2, [3]):

$$dr_f(t,T) = \alpha_f(t,T,\omega)dt + \sum_{i=1}^{2} \sigma_{fi}(t,T,r_f(t,t))dW_t^i \tag{2.16}$$

for all $\omega \in \Omega$, $t \leq T \leq \tau$; $r_f(t,T)$ is a forward interest rate contracted at time t for instantaneous borrowing and lending at time T with $0 \leq t \leq T \leq \tau$.

The connection between the two markets is the spot exchange rate. The spot rate of exchange (in units of the domestic currency per foreign currency) is governed by the following SDE (see see Assumption 3, [3]):

$$dS_t = \mu_t S_t dt + \sum_{i=1}^{4} \delta_{di} S_t dW_t^i. \tag{2.17}$$

Here μ_t is a drift part, $\delta_{di}, i = 1, \cdots, 4$ are the volatility coefficients.

The spot exchange rate S_t depends on the same three Brownian motions W_t^1, W_t^2, W_t^3 running the domestic and foreign forward interest rates. As a result, there exist correlations between the spot exchange rate S_t, the domestic $r_d(t,T)$, and the foreign interest rates $r_f(t,T)$. One independent Brownian motion W_t^4 is added to govern the exchange rate dynamics.

To price options from the domestic perspective, we need to convert all of the securities to the domestic currency. We assume that a domestic investor has his holdings of foreign currency only in the form of foreign bonds B_f, or units of the foreign money market account $\widetilde{B_f}$. Both holdings are converted to domestic currency according to the formulas:

$$B_f^* = B_f S_t, \tag{2.18}$$

$$\widetilde{B_f^*} = \widetilde{B_f} S_t. \tag{2.19}$$

The stochastic processes describing these securities, denominated in the domestic currency, are given by (see Appendix, [3]):

$$d\widetilde{B_f^*} = \widetilde{B_f^*} \left[(\mu_d(t) + r_f(t,t))dt + \sum_{i=1}^{4} \delta_{di}(t) dW_t^i \right] \tag{2.20}$$

$$dB_f^* = B_f^* \left[\mu_f^*(t)dt + \sum_{i=1}^{4} (\alpha_{fi}(t,T) + \delta_{di}(t)) dW_t^i \right] \tag{2.21}$$

where

$$\mu_f^*(t) = r_f(t,t) + b_f(t,T) + \mu_d(t) + \sum_{i=1}^{4} \delta_{di}(t)\alpha_{fi}(t,T). \tag{2.22}$$

The pricing formula for the European Call option, with exercise price K and maturity time T, for this model is also given in [3](see p. 317-318):

$$C(0,T,K) = \mathbb{E}^* \max\Bigg[0, B_f(0,T)S_0 \exp\Bigg(\sum_{i=1}^{4}\int_0^T (\alpha_{fi}(t,T) + \delta_{di}(t))dW_t^{i,*} - 1/2\int_0^T (\alpha_{fi}(t,T) + \delta_{di}(t))^2 dt\Bigg) - KB_d(0,T)\times \exp\left(\sum_{i=1}^{2}\int_0^T \alpha_{di}(t,T)dW_t^{i,*} - 1/2\sum_{i=1}^{2}\int_0^T \alpha_{di}^2(t,T)dt\right)\Bigg]. \tag{2.23}$$

Here $\mathbb{E}^*$ is the mathematical expectation with respect to the risk-neutral measure P^*. (Details about risk-neutral pricing can be found in Bjork [6] and Shreve [40],V.2). The derivation of formulas for risk-neutral currency option pricing for several particular cases and general Lévy process (see Ch. 3, § 3.1) are in the main results of our research (see Ch.5).

Melino et al. (1991, [29]) examined the foreign exchange rate process, (under a deterministic interest rate), underlying the observed option prices, and Rumsey (1991, [38]) considered cross-currency options.

Mikkelsen (2001, [33]) considered cross-currency options with market models of interest rates and deterministic volatilities of spot exchange rates by simulation. The author uses a cross-currency arbitrage-free LIBOR market model, (cLMM), for pricing options, (see [33], [56]), that are evaluated from cross-market dynamics. In [33] the market consists of two economies, each of which has its own domestic bond market. An investor can invest in two fixed income markets: his home currency market and a foreign currency denominated market. To ensure arbitrage free pricing of cross-currency derivatives the spot foreign exchange rate and arbitrage free foreign exchange rate dynamics are introduced. It was proved that if LIBORs are modeled as log-normal variables all forward foreign exchange rates cannot be modeled as log-normal without breaking the arbitrage-free assumption.

Schlogl (2002, [39]) extended market models to a cross-currency framework. He did not

include stochastic volatilities into the model and focused on cross currency derivatives such as differential swaps and options on the swaps as applications. He did not consider currency options.

Piterbarg (2005, [35]) developed a model for cross-currency derivatives, such as PRDC (Power-Reverse Dual-Currency) swaps, with calibration to currency options; he used neither market models nor stochastic volatility models. An economy consists of two currencies as before (see [33]): domestic and foreign. Let P be the domestic risk-neutral measure. Let $P_i(t,T)$, $i = d, f$ be the prices, in their respective currencies, of the domestic and foreign zero-coupon discount bonds. Also let $r_i(t)$, $i = d, f$, be the short rates in the two currencies. Let S_t be the spot FX rate. The forward FX rate (the break-even rate for a forward FX transaction) $F(t,T)$ satisfies the condition(see [35]):

$$F(t,T) = \frac{P_f(t,T)}{P_d(t,T)} S_t \tag{2.24}$$

following from no-arbitrage arguments. The following model is considered in [35]:

$$\frac{dP_d(t,T)}{P_d(t,T)} = r_d(t)dt + \sigma_d(t,T)dW_t^d, \tag{2.25}$$

$$\frac{dP_f(t,T)}{P_f(t,T)} = r_f(t)dt - \rho_{fS}\sigma_f(t,T)\gamma(t,S_t)dt + \sigma_f(t,T)dW_t^f, \tag{2.26}$$

$$\frac{dS_t}{S_t} = (r_d(t) - r_f(t))dt + \gamma(t,S_t)dW_t^S, \tag{2.27}$$

where (W_t^d, W_t^f, W_t^S) is a three dimensional Brownian motion under P with the correlation matrix:

$$\begin{pmatrix} 1 & \rho_{df} & \rho_{dS} \\ \rho_{df} & 1 & \rho_{fS} \\ \rho_{dS} & \rho_{fS} & 1 \end{pmatrix}. \tag{2.28}$$

Gaussian dynamics for the rates are given by:

$$\sigma_i(t,T) = \sigma_i(t) \int_t^T \exp\left\{ -\int_t^s \chi_i(u)du \right\} ds, \ i = d, f, \tag{2.29}$$

where $\sigma_d(t), \sigma_f(t), \chi_d(t), \chi_f(t)$ are deterministic functions.

A price of a call option on the FX rate with strike K and maturity T is

$$C(T,K) = \mathbb{E}\left(\exp\left\{-\int_0^T r_d(s)ds\right\}\max(0, S_T - K)\right). \tag{2.30}$$

The forward FX rate $F(t,T)$ has the following dynamics (see [35], Prop. 5.1):

$$\frac{dF(t,T)}{F(t,T)} = \sigma_f(t,T)dW_f^T(t) - \sigma_d(t,T)dW_d^T(t) + \gamma(t,F(t,T)D(t,T))dW_S^T(t). \tag{2.31}$$

Here $\left(W_d^T(t), W_f^T(t), W_S^T(t)\right)$ is a three-dimensional Brownian motion under the domestic T-forward measure. Moreover, there exists a Brownian motion $dW_F(t)$ under the domestic T-forward measure P^T , such that:

$$\frac{dF(t,T)}{F(t,T)} = \Lambda(t,F(t,T)D(t,T))dW_F(t), \tag{2.32}$$

where

$$\begin{aligned}
&\Lambda(t,x) = (a(t) + b(t)\gamma(t,x) + \gamma^2(t,x))^{1/2},\\
&a(t) = (\sigma_f(t,T))^2 + (\sigma_d(t,T))2 - 2\rho_{dS}\sigma_f(t,T)\sigma_d(t,T),\\
&b(t) = 2\rho_{fS}\sigma_f(t,T) - 2\rho_{dS}\sigma_d(t,T),\\
&D(t,T) = \frac{P_d(t,T)}{P_f(t,T)}.
\end{aligned} \tag{2.33}$$

In [35] a Markovian representation for the foreign FX rate is used. Define:

$$c(t,T,K) = P_d(0,t)\mathbb{E}^T\left(F(t,T) - K\right)^+, \tag{2.34}$$

where: $(x)^+ = \max(0,x)$. $T > 0$ is a the fixed settlement date of the FX forward rate. We wish to find a function $\tilde{\Lambda}(t,x)$ such that, in the model defined by the equation:

$$\frac{dF(t,T)}{F(t,T)} = \tilde{\Lambda}(t,F(t,T)D(t,T))dW_F(t), \tag{2.35}$$

the values of European options $\{c(t,T,K)\}$ for all $t, K, 0 < t \le T,\ 0 \le K < \infty$ are equal to the values of the same options in the initial model (2.25)- (2.28) (or (2.31)-(2.34)). The answer is given in [35] (see Theorem 6.1).

The local volatility function $\tilde{\Lambda}(t,x)$, for which the values of all European options $\{c(t,T,K)\}_{t,K}$ in the model (2.35) are the same as in the model (2.2), is given by

$$\tilde{\Lambda}(t,x) = \mathbb{E}^T \left(\Lambda^2(t, F(t,T)D(t,T)) | F(t,T) = x \right). \tag{2.36}$$

Here $\mathbb{E}^T$ is a conditional mathematical expectation with respect to measure P^T(see (2.31). In the case of European option pricing, the dynamics of the forward FX rate $F(,T)$ under the measure P^T are given in Corollary 6.2([35]):

$$\frac{dF(t,T)}{F(t,T)} = \tilde{\Lambda}(t, F(t,T)D(t,T))dW_F(t), \tag{2.37}$$

where

$$\tilde{\Lambda}(t,x) = (a(t) + b(t)\tilde{\gamma}(t,x) + \tilde{\gamma}^2(t,x))^{1/2}. \tag{2.38}$$

The formulas for $\tilde{\gamma}(t,x), r(t)$ can also be found in this corollary.

V. Piterbarg also derives approximation formulas for the value of options on the FX rate for a given expiry T and strike K, (see [35], Theorem 7.2).

To value options on the FX rate with maturity T, the forward FX rate can be approximated by the solution of the following stochastic differential equation:

$$dF(t,T) = \tilde{\Lambda}(t, F(0,T))(\delta_F F(t,T) + (1-\delta_F)F(0,T))dW_F(t). \tag{2.39}$$

Here

$$\delta_F = 1 + \int_0^T w(t) \frac{b(t)\eta(t) + 2\tilde{\gamma}(t, F(0,T))\eta(t)}{2\tilde{\Lambda}^2(t, F(0,T))} dt, \tag{2.40}$$

$$\eta(t) = \tilde{\gamma}(t, F(0,T))(1 + r(t))(\beta(t) - 1), \tag{2.41}$$

$$w(t) = \frac{u(t)}{\int_0^T u(t)dt}, \tag{2.42}$$

$$u(t) = \tilde{\Lambda}^2(t, F(0,T)) \int_0^t \tilde{\Lambda}^2(s, F(0,T))ds. \tag{2.43}$$

The value $c(T,K)$ of the European call option on the FX rate with maturity T and strike K is equal to:

$$c(T,K) = P_d(0,T)B(\frac{F(0,T)}{\delta_F}, k + \frac{1-\delta_F}{\delta_F}F(0,T), \sigma_F\delta_F, T), \tag{2.44}$$

$$\delta_F = \left(\frac{1}{T}\int_0^T \tilde{\Lambda}^2(t, F(0,T)dt\right)^{1/2}. \tag{2.45}$$

Here $B(F,K,\sigma,T)$ is the Black-Schloes formula for a call option with forward price F, strike K, (see [6], chapter 17), volatility σ, and time to maturity T.

A similar model and approximations for pricing FX options are given in [26] by Lech Grzelak and Kees Oosterlee (2010).

In Garman et al. (1983, [6]) and Grabbe (1983, [23]), foreign exchange option valuation formulas are derived under the assumption that the exchange rate follows a diffusion process with continuous sample paths.

Takahashi et al. (2006, [44]) proposed a new approximation formula for the valuation of currency options under jump-diffusion stochastic volatility processes for spot exchange rates in a stochastic interest rates environment. In particular, they applied market models developed by Brace et al (1998,[9]), Jamshidian (1997, [20]) and Miltersen et al (1997, [32]) to model the term structure of interest rates. As an application, Takahashi et al. applied the derived formula to the calibration of volatility smiles in the JPY/USD currency option market. In [44] the authors suppose, that the variance process of the spot FX rate is governed by the following dynamics:

$$dV_t = \xi(\eta - V_t)dt + \theta\sqrt{V_t}\vec{v}\cdot d\vec{W}_t^*. \tag{2.46}$$

Here $\vec{W}_t^*$ is a d-dimensional Brownian motion under the domestic risk neutral measure, ξ, η and θ are positive scalar parameters and $\vec{v}$ is a d-dimensional constant vector with $\|v\| = 1$ to represent the correlations between the variance and other factors. The condition $2\xi\eta > \theta^2$ ensures that V_t remains positive if $V_0 = 0$. This stochastic volatility model is introduced by

Heston in [18](1993). Let $F(t,T)$ denote a forward exchange rate with maturity T at time t. Suppose the dynamics of the process $F(t,T)$ are defined by the equation:

$$\frac{dF(t,T)}{F(t,T)} = \vec{\sigma_t^F} \cdot d\vec{W}_t^d, \tag{2.47}$$

where $\vec{\sigma_t^F}$ is a deterministic d-dimensional vector-function, and W_t^d is a d-dimensional Brownian motion, (see [44], formulas 4-6). In the case of Mertons jump-diffusion ([30], 1976) equation (2.47) can be presented in the following form:

$$\frac{d(t,T)}{(t,T)} = \vec{\sigma_t^F} \cdot d\vec{W}_t^d + Z_{t_-} dN_t - \lambda k dt, \tag{2.48}$$

where $Z_{t_-} dN_t$ is a compound Poisson process with intensity λ and jump size Z_{t_-}. (See again Ch. 3, Def.3.3 or [34] for details). A Fourier transform method for currency call option pricing is used in [44]:

$$\frac{c(S,K,T)}{B_d(0,T)} \approx \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iku} \frac{\Phi_{X^0}(u-i) - \Phi_0(u-i)}{iu(1+iu)} du + \frac{c_0(S,K,T)}{B_d(0,T)}. \tag{2.49}$$

Here $c(S,K,T)$ is an option price with strike price K, maturity time T, and S is an initial spot exchange rate, $B_d(t,T)$ is a price of zero coupon bonds with maturity T in domestic currency;

$$\Phi_0(u) = \exp\left\{ -\frac{\sigma_0^2 T}{2}(u^2 + iu) \right\}. \tag{2.50}$$

$\Phi_{X^0}(u)$ is a first order approximation of the characteristic function of the process (see [44], formulas 9-15):

$$X_t = \log \frac{F(t,T)}{F(0,T)}; \tag{2.51}$$

$$c_0(S,K,T) = B_d(0,T)(FN(d_+) - KN(d_-)), \tag{2.52}$$

$$d_\pm = \frac{k \pm \frac{1}{2}\sigma_0^2 T}{\sigma_0 \sqrt{T}}, \tag{2.53}$$

σ_0 - is a constant depending on T, S_0, domestic and foreign forward interest rates (see [44], § 3); $N(x)$ is a standard normal cumulative distribution function. The first term on the

right-hand side of the formula (2.49) is the difference between the call price of this model and the Black-Scholes call price(see [6], Chapter 17).

Also, Ahn et al. (2007, [2]) derived explicit formulas for European foreign exchange call and put options values when the exchange rate dynamics are governed by jump-diffusion processes.

Hamilton (1988) was the first to investigate the term structure of interest rates by rational expectations econometric analysis of changes in regime. Goutte et al. (2011, [11]) studied foreign exchange rate using a modified Cox-Ingersoll-Ross model under a Hamilton type Markov regime switching framework, where all parameters depend on the value of a continuous time Markov chain.

Definition 2.1 ([11], Def. 2.1) Let $(X_t)_{t\in[0;T]}$ be a continuous time Markov chain on finite space $\mathcal{S} := \{1, \cdots, K\}$. Write $\mathcal{F}_t^X := \{\sigma(X_s); 0 \leq s \leq t\}$ for the natural filtration generated by the continuous time Markov chain X. The generator matrix of X will be denoted by Π^X and given by

$$\Pi_{ij}^X \geq 0,\ i \neq j;\ \ i, j \in \mathcal{S} \tag{2.54}$$

$$\Pi_{ii}^X = -\sum_{j\neq i} \Pi_{ij}^X \tag{2.55}$$

otherwise. In our numerical simulations we shall use a slightly different matrix Π^X:

$$\Pi_{ij}^X \geq 0,\ i, j \in \mathcal{S},\ \sum_{j\in\mathcal{S}} \Pi_{ij}^X = 1 \ \text{ for any } i \in \mathcal{S}. \tag{2.56}$$

Note, that Π_{ij}^X represents probability of the jump from state i to state j. The definition of a CIR (Cox-Ingersoll-Ross) process with parameters values depending on the value of a continuous time Markov chain is given in [11]:

Definition 2.2 ([11], Def. 2.2) Let, for all $t \in [0; T]$, $(X)_t$ be a continuous time Markov chain on a finite space $\mathcal{S} := \{e_1, e_2, \cdots, e_K\}$ defined as in Def. 2.1. We will call a Regime switching CIR (in short, RS-CIR) the process (r_t) which is the solution of the SDE given by

$$dr_t = (\alpha(X_t) - \beta(X_t)r_t)dt + \sigma(X_t)\sqrt{r_t}dW_t. \tag{2.57}$$

Here for all $t \in [0;T]$, the functions $\alpha(X_t)$, $\beta(X_t)$, and $\sigma(X_t)$ are functions such that for $1 \leq i \leq K$

$$\alpha(i) \in \{\alpha(1), \cdots, \alpha(K)\},\ \beta(i) \in \{\beta(1), \cdots, \beta(K)\},$$

and

$$\sigma(i) \in \{\sigma(1), \cdots, \sigma(K)\}.$$

For all $j \in \{1, \cdots, K\}$: $\alpha(j) > 0,\ 2\alpha(j) \geq \sigma(j)^2$.

In [11] Goutte et al. investigated three state Markov switching regimes. One regime represents "normal" economic dynamics, the second one is for "crisis" and the last one means "good" economy. They also study whether more regimes capture better the economic and financial dynamics or not. The advantages and disadvantages of models with more than three regimes are discussed and supported by numerical simulations. Three state regime switching models are important in our research, because they adequately describe movements of the foreign currency exchange rate in the Forex market. In Forex we have three distinctive trends: up, down, sideways.

Zhou et al. (2012, [45]) considered an accessible implementation of interest rate models with regime-switching. Siu et al. (2008, [41]) considered pricing currency options under a two-factor Markov modulated stochastic volatility model with the first stochastic volatility component driven by a log-normal diffusion process and the second independent stochastic volatility component driven by a continuous-time Markov chain:

$$\begin{aligned}
&\frac{dS_t}{S_t} = \mu_t dt + \sqrt{V_t}dW_t^1 + \sigma_t dW_t^2, \\
&\frac{dV_t}{V_t} = \alpha_t dt + \beta dW_t^{\upsilon}, \\
&\mathrm{Cov}(dW_t^{\upsilon}, dW_t^1) = \rho dt.
\end{aligned} \tag{2.58}$$

Here S_t is a spot FX rate; $W^1 = \{W_t^1\}_{t\in\mathcal{T}},\ W^2 = \{W_t^2\}_{t\in\mathcal{T}}$ are two independent Brownian motions on $(\Omega, \mathcal{F}, P)$; $(\Omega, \mathcal{F}, P)$ is a complete probability space; $\mathcal{T} \subset [0, \infty)$ is a time index set; $W^v = \{W_t^v\}_{t\in\mathcal{T}}$ is a standard Brownian motion on $(\Omega, \mathcal{F}, P)$ correlated with W^1 with coefficient of correlation ρ. In [41] the authors define a continuous-time, finite-state Markov chain $X = \{X_t\}_{t\in\mathcal{T}}$ on $(\Omega, \mathcal{F}, P)$ with state space $\mathcal{S} = \{e_1, e_2, \cdots, e_n\}$ (similar to [11], Def. 2.1, 2.2). They take the state space $\mathcal{S}$ for X to be the set of unit vectors $(e_1, \cdots, e_n) \in \mathbb{R}^n$ with probability matrix Π^X similarly to (2.54)-(2.56). The parameters σ_t, μ_t are modelled using this finite state Markov chain:

$$\begin{aligned} \mu_t &:=< \mu, X_t >,\ \mu \in \mathbb{R}^n; \\ \sigma_t &:=< \sigma, X_t >,\ \sigma \in \mathbb{R}^n. \end{aligned} \tag{2.59}$$

Instantaneous market interest rates $\{r_t^d\}_{t\in\mathcal{T}},\ \{r_t^f\}_{t\in\mathcal{T}}$ of the domestic and foreign money market accounts are also modeled using the finite state Markov chain in [41] by:

$$\begin{aligned} r_t^d &:=< r^d, X_t >,\ r^d \in \mathbb{R}^n; \\ r_t^f &:=< r^f, X_t >,\ r^f \in \mathbb{R}^n. \end{aligned} \tag{2.60}$$

Let $B^d := \{B_t^d\}_{t\in\mathcal{T}},\ B^f := \{B_t^f\}_{t\in\mathcal{T}}$ denote the domestic and foreign money market accounts. Then their dynamics are given by the following equations (see [41]):

$$\begin{aligned} B^d &= \exp\left\{\int_0^t r_u^d du\right\}, \\ B^f &= \exp\left\{\int_0^t r_u^f du\right\}. \end{aligned} \tag{2.61}$$

Siu et al. consider the European-style and American-style currency option pricing (see [41], § 4, 5) and provide numerical simulations for options of both types (see [41], § 6).

Bo et al. (2010, [8]) deals with a Markov-modulated jump-diffusion (modeled by compound Poisson process) for currency option pricing. They use the same finite-state Markov

chain as in [41] to model coefficients in the equation for FX rate dynamics:

$$\frac{dS_t}{S_t} = (\alpha - \lambda k)dt + \sigma dW_t + (e^{Z_{t_-}} - 1)dN_t. \tag{2.62}$$

Here W_t is a Brownian motion, $\alpha - \lambda k$ is a drift, $(e^Z_{t_-} - 1)dN_t$ is a compound Poisson process (see Def. 3.3 in Ch. 3 for detail); k is a mean jump size if a jump occurs. The amplitude of the jumps Z_{t_-} is log-normally distributed. The European call currency option pricing formulas are also derived (see [8], formulas § 2: 2.18, 2.19; § 3: 3.10-3.16). A random Esscher transform (see [8], § 2, formula 2.5; [13], [15] for the detailed theory of the Esscher transform) is used to determine a risk-neutral measure, to find option pricing formulas and to evaluate the Esscher transform parameters which ensure that the discounted spot FX rate is a martingale. (See [8], § 2, Theorem 2.1).

Heston (1993, [18]) found a closed-form solution for options having stochastic volatility with applications to the valuation of currency options. Bo et al. (2010, [8]) used this model of stochastic volatility and considered the following model of a FX market:

$$dS_t = S_{t_-}(\alpha_t - k\lambda_t)dt + S_{t_-}\sqrt{V_t}dW_t + S_{t_-}(e^{Z_{t_-}} - 1)dN_t, \tag{2.63}$$

$$dV_t = (\gamma - \beta V_t)dt + \sigma_\upsilon \sqrt{V_t}W_t^1 \tag{2.64}$$

$$W_t^1 = \rho W_t^1 + \sqrt{1-\rho^2}W_t^2. \tag{2.65}$$

Here $\gamma, \sigma_\upsilon, \beta > 0, |\rho| < 1$; W_t and W_t^2 are two independent Brownian motions; S_t is a spot FX rate, $\alpha_t - k\lambda_t$ is a drift; $(e^{Z_{t_-}} - 1)dN_t$ is a compound Poisson process; V_t is a stochastic volatility. They also provided numerical simulations for the model in [18].

We note that currency derivatives for the domestic and foreign equity market and for the exchange rate between the domestic currency and a fixed foreign currency with constant interest rates are discussed in Bjork (1998, [6]). We also mention that currency conversion

for forward and swap prices with constant domestic and foreign interest rates is discussed in Benth et al. (2008, [5]).

The main goal of our research is to generalize the results in [8] to the case when the dynamics of the FX rate are driven by a general Lévy process (See Ch. 3, § 3.1 for the basic definitions of Lévy processes, or [34] for a complete overview). In other words, jumps in our model are not necessarily log-normally distributed. In particular, we shall investigate log-double exponential and exponential distributions of jumps and provide numerical simulations for European-call options which depend on a wide range of parameters. (See Ch. 5, 6).

Chapter 3

Lévy Processes

3.1 Lévy processes. Basic definitions and theorems

We shall give a definition of a general Lévy Process. Let $(\Omega, \mathcal{F}, P)$ be a filtered probability space with $\mathcal{F} = \mathcal{F}_{\mathcal{T}}$, where $(\mathcal{F}_t)_{t\in[0,T]}$ is the filtration on this space.

Definition 3.1. ([34]) An adapted, real valued stochastic process $L = (L_t)_{0\leq t\leq T}$ with $L_0 = 0$ is called a Lévy process if the following conditions are satisfied:

(L1): L has independent increments, i.e. $L_t - L_s$ is independent of $\mathcal{F}_s$ for any $0 < s < t < T$.

(L2): L has stationary increments, i.e. for any $0 < s, t < T$ the distribution of $L_{t+s} - L_t$ does not depend on t.

(L3): L is stochastically continuous, i.e. for every $0 \leq t \leq T$ and $\epsilon > 0$:

$$\lim_{t\to s} P(|L_t - L_s| > \epsilon) = 0.$$

Examples of Lévy process arising in Mathematical Finance are (linear) drift, a deterministic process, Brownian motion, non-deterministic process with continuous sample paths, and the Poisson and compound Poisson processes.

Definition 3.2. ([34]) Let (Ω, F, P) be a probability space. For each $\omega \in \Omega$, suppose there is a continuous function W_t $(t > 0)$ that satisfies $W_0 = 0$ and that depends on ω. Then W_t $(t \geq 0)$ is a Brownian motion if for all $0 = t_0 < t_1 < \cdots < t_m$ the increments $W_{t_1} = W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, ..., W_{t_m} - W_{t_{m-1}}$ are independent and each of these increments is normally distributed with

$$\mathbb{E}[W_{t_{i+1}} - W_{t_i}] = 0,$$

$$\mathrm{Var}[W_{t_{i+1}} - W_{t_i}] = t_{i+1} - t_i.$$

A Brownian motion model requires an assumption of perfectly divisible assets and a frictionless market, (i.e. there are no taxes and no transaction costs occur either for buying or selling).However, asset prices in a real market can have jumps which can be described by a compound Poisson process.

Definition 3.3. ([34]) A compound Poisson process is a continuous-time (random) stochastic process with jumps. The jumps arrive randomly according to a Poisson process and the size of the jumps is also random, with a specified probability distribution ν. A compound Poisson process, defined by a rate $\lambda_t > 0$ and jump size distribution ν, is a process

$$P_t = \sum_{i=1}^{N_t} Z_i, \tag{3.1}$$

where Z_i is an identically distributed random variable, $(N_t)_{0\leq t\leq T}$ is Poisson process with a rate λ_t.

Definition 3.4. ([34]) A compound compensated Poisson process has the form

$$P_t = \sum_{i=1}^{N_t} Z_i - \mathbb{E}[Z] \int_0^t \lambda_s ds. \tag{3.2}$$

Note, that a compound compensated Poisson process is a martingale (see [34]).

The sum of a (linear or non-linear, depending on time t) drift, a Brownian motion and a compound Poisson process is again a Lévy process. It is often called a Lévy "jump-diffusion" process. Note there exist jump-diffusion processes which are not Lévy processes. This holds, when the distribution of jumps gives non converging to zero probability of jumps with a big amplitude, and then the condition **(L3)** does not hold. We shall give several examples of these.

We now determine the characteristic functions of all three types of Lévy processes mentioned above.

3.2 Characteristic functions of Lévy processes

The characteristic function of Brownian motion is given at (see for Example [34] for the derivation of all the results in this chapter):

$$\mathbb{E}\left[e^{u\int_0^t \sigma_s dW_s}\right] = \exp\left\{\frac{1}{2}u^2\int_0^t \sigma_s^2 ds\right\}, \tag{3.3}$$

where σ_t is the volatility of a market. The characteristic function of compound Poisson process, defined by a rate $\lambda_t > 0$ and jump size distribution ν, is:

$$\mathbb{E}\left[e^{u\sum_{k=1}^{N_t} Z_k}\right] = \exp\left\{\int_0^t \lambda_s \int_{\mathbb{R}} (e^{ux}-1)\nu(dx)ds\right\}. \tag{3.4}$$

The characteristic function of compound compensated Poisson process, defined by a rate $\lambda_t > 0$ and jump size distribution ν:

$$\mathbb{E}\left[e^{u\left(\sum_{k=1}^{N_t} Z_k - \mathbb{E}[Z]\int_0^t \lambda_s ds\right)}\right] = \exp\left\{\int_0^t \lambda_s \int_{\mathbb{R}} (e^{ux}-1-ux)\nu(dx)ds\right\}. \tag{3.5}$$

We shall assume that the process $L = (L_t)_{0\le t\le T}$ is a Lévy jump-diffusion, i.e. a Brownian motion plus a compensated compound Poisson process. The paths of this process can be described by

$$L_t = \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s + \sum_{k=1}^{N_t} Z_k - \mathbb{E}[Z]\int_0^t \lambda_s ds. \tag{3.6}$$

Note that since Brownian motion and compound compensated Poisson processes are martingales, a Lévy process $L_y t$ is martingale if and only if $\mu_t = 0$. Using expressions for characteristic functions (3.2)-(3.5) we obtain the characteristic function of the Lévy process L_t:

$$\mathbb{E}\left[e^{uL_t}\right] = \exp\left\{u\int_0^s \mu_s ds + \frac{1}{2}u^2\int_0^t \sigma_s^2 ds + \int_0^t \lambda_s \int_{\mathbb{R}} (e^{ux}-1-ux)\nu(dx)ds\right\}. \tag{3.7}$$

In the case of compound Poisson Process(not compensated):

$$\mathbb{E}\left[e^{uL_t}\right] = \exp\left\{u\int_0^s \mu_s ds + \frac{1}{2}u^2\int_0^t \sigma_s^2 ds + \int_0^t \lambda_s \int_{\mathbb{R}} (e^{ux}-1)\nu(dx)ds\right\}. \tag{3.8}$$

In the sequel, we shall consider jumps driven by a compound Poisson Process to describe the spot FX rate movements of currency markets. We also consider sufficient conditions for Lévy process of such a type to be a martingale.

The following decomposition of the characteristic function of general Lévy process is true (see [34], p. 12):

Theorem 3.1. *Consider a triplet (μ_t, σ_t, ν) where $\mu_t \in \mathbb{R}$, $\sigma_t \in \mathbb{R} > 0$ and ν is a measure satisfying $\nu(0) = 0$ and $\int_{\mathbb{R}}(1 \wedge |x|^2)\nu(dx) < \infty$. Then, there exists a probability space (Ω, F, P) on which four independent Lévy processes exist, L^1, L^2, L^3, L^4 where L^1 is a drift, L^2 is a Brownian motion, L^3 is a compound Poisson process and L^4 is a square integrable (pure jump) martingale with an a.s. countable number of jumps of magnitude less than 1 on each finite time interval. Taking $L_t = L^1 + L^2 + L^3 + L^4$, we have that there exists a probability space on which a Lévy process $L = (L_t)_{0 \le t \le T}$ with characteristic function*

$$\mathbb{E}\left[e^{uL_t}\right] = \exp\left\{u\int_0^s \mu_s ds + \frac{1}{2}u^2\int_0^t \sigma_s^2 ds + \int_0^t \lambda_s \int_{\mathbb{R}}(e^{ux} - 1 - ux\mathbf{1}_{|x|\le 1})\nu(dx)ds\right\} \quad (3.9)$$

for all $u \in \mathbb{R}$, is defined.

To prove the Theorem 3.1 we can use Itô's formula with jumps. (see the next chapter).

3.3 Itô's formula with jumps and its applications

In our research shall assume, that the asset price is described by the following stochastic differential equation:

$$dS_t = S_{t_-}\left(\mu_t dt + \sigma_t dW_t + (e^{Z_{t_-}} - 1)dN_t\right). \quad (3.10)$$

Here μ_t is a drift, σ_t is a volatility, $(e^{Z_{t_-}} - 1)dN_t$ is the compound Poisson process (N_t is the Poisson process with intensity $\lambda_t > 0$, $e^{Z_{t_-}} - 1$ are the jump sizes, Z_{t_-} has arbitrary distribution ν). The parameters σ_t, λ_t are modulated by a finite state Markov chain $(\xi_t)_{0 \le t \le T}$ in the following way

$$\sigma_t = \langle \sigma, \xi_t \rangle, \sigma \in \mathbb{R}^n_+,$$

$$\lambda_t = \langle \lambda, \xi_t \rangle, \lambda \in \mathbb{R}^n_+.$$

Using Itô's formula with jumps, solve (3.10). We state this theorem first, because it will be used several times in the sequel.

Theorem 3.2. ([36], p. 364-365, [46]). *Let $a(x,t), b(x,t)$ be adapted stochastic processes and f(t,x) be a function for which partial derivatives f_t, f_x, f_{xx} are defined and continuous. Let also*

$$dx = a(x_-, t)dt + b(x_-, t)dW_t + Y_t dN_t. \tag{3.11}$$

Then the following holds

$$df(x,t) = [f_t + a(x_-,t)f_x + \frac{1}{2}b^2(x_-,t)f_{xx}]dt + b(x_-,t)f_x dW_t \tag{3.12}$$

$$+[f(x_- + Y_t, t) - f(x_-, t)]dN_t.$$

The solution for (3.10) has the following form $S_t = S_0 e^{L_t}$, (S_0 is a spot FX rate at time $t = 0$), where L_t is given by the formula:

$$L_t = \int_0^t (\mu_s - 1/2\sigma_s^2)ds + \int_0^t \sigma_s dW_s + \sum_{i=1}^{N_t} Z_i. \tag{3.13}$$

We can consider jumps in (3.10) of a different form $Z_{t_-} - 1$ instead of $e^{Z_{t_-}} - 1$. Then (3.10) takes the form:

$$dS_t = S_{t_-}\left(\mu_t dt + \sigma_t dW_t + (Z_{t_-} - 1)dN_t\right). \tag{3.14}$$

In this case, to apply Itô's formula we must require $Z_{t_-} : Z_{t_-} > 0$. Then we obtain the following solution for (3.14) $S_t = S_0 e^{L_t^*}$, where L_t is given by the formula:

$$L_t^* = \int_0^t (\mu_s - 1/2\sigma_s^2)ds + \int_0^t \sigma_s dW_s + \sum_{i=1}^{N_t} \log Z_i. \tag{3.15}$$

Note that for most well-known distributions, (normal , double exponential distribution of Z_{t_-}, etc), L_t^* is not a Lévy process, since $\log Z_i \to -\infty$ for small Z_{t_-}. However, the probability of jumps with even 0 size is a positive constant, depending on the type of distribution. We shall consider the solution of the differential equation (3.14) in a separate section later. (see Ch. §5.4).

3.4 Girsanov's theorem with jumps

The primary goal of our research is to derive formulas for currency option pricing. For this purpose we need to transfer from the initial probability measure with Poisson process intensity λ_t and density function of jumps ν to a new risk neutral measure with new λ_t^*, ν^*. Note that this measure is not unique. To calculate these new quantities we shall use Girsanov's theorem with jumps (see [36], p. 375-376; [40], p. 502-503).

Theorem 3.3. *Let $W_t, 0 \leq t \leq T$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ and let $\mathcal{F}_t, 0 \leq t \leq T$ be a filtration for this Brownian motion. Let $\Theta_t, 0 \leq t \leq T$ be an adapted process; λ_t^*, ν^* are new Poisson process intensity and density functions for jumps respectively. Define*

$$Z_t^{(1)} = \exp\left\{-\int_0^t \Theta_s dW_s - 1/2 \int_0^t \Theta_s^2 ds\right\}, \tag{3.16}$$

$$Z_t^{(2)} = \exp\left\{\int_0^t (\lambda_s - \lambda_s^*) ds\right\} \int_0^t \prod_{i=1}^{N_t} \frac{\lambda_s^* \nu^*(Z_i)}{\lambda_s \nu(Z_i)} ds \tag{3.17}$$

$$Z_t = Z_t^{(1)} * Z_t^{(2)}$$

$$W_t^* = W_t + \int_0^t \Theta_s ds + Z_t - \mathbb{E}[Z] \int_0^t \lambda_s^* ds \tag{3.18}$$

and assume that $E \int_0^T \Theta_s^2 Z_s^2 ds < \infty$. Set $Z = Z(T)$. Then $EZ = 1$ and under the probability measure P^* given by $P^*(A) = \int_A Z_\omega dP_\omega, forall A \in \mathcal{F}_t$, the process $W_t^*, 0 \leq t \leq T$ is a martingale. Moreover, $W_t + \int_0^t \Theta_s ds$ – is a standard Brownian motion, $Z_t - \mathbb{E}[Z] \int_0^t \lambda_s^* ds$ is martingale with respect to new measure P^*.

When $\lambda_s = \lambda_s^*$, Theorem 3.3 coincides with the usual Girsanov theorem for the Brownian motion.

Chapter 4

Currency derivatives

4.1 Types of currency derivatives

A foreign exchange derivative is a financial derivative whose payoff depends on the foreign exchange rate(s) of two (or more) currencies. In this section we shall consider three main types of foreign exchange derivatives: foreign exchange forwards, currency futures, currency options (foreign exchange options).

4.1.1 Forward contracts

Following Bjork, (see [6], p. 102) a forward contract on X, made at t, is an agreement which stipulates that the holder of the contract pays the deterministic amount K at the delivery date T, and receives the stochastic amount $\mathcal{X}$ at T. Nothing is paid or received at the time t, when the contract is made. Note that forward price K is determined at time t.

Forward contracts are agreements to buy or sell at a certain time in the future, the maturity time, an asset at a price stipulated in advance. Forwards are a common instrument used to hedge currency risk, when the investor is expecting to receive or pay a certain amount of money expressed in foreign currency in the near future. Forward contracts are binding contracts, (contrary to options) and, therefore, both parties are obliged to exercise the contract conditions and buy (sell) the asset at agreed price. There are three types of prices associated with a forward contract. The first is a forward price F_0, that is the price of one unit of the underlying asset to be delivered at a specific time in the future, $(t = T)$. The second K is a delivery price, fixed in advance in a contract.The third is the value of a forward contract $f(t), 0 \leq t \leq T$. The value of F_0 is chosen in such a way that $f(0) = 0$ ($K = F_0$ at the initial moment). There is no initial price to enter such agreements, except

for bid-ask spread, (see [51]). After the initial time the value of $f(t)$ may vary depending on variations of the spot price of the underlying asset, the prevailing interest rates, etc (see [19], Chapter 5,6; [27], Chapter 10 for details of types of currency derivatives). The forward price $F_0, t = 0$ can be calculated from the formula([19], Chapter 5, p. 107):

$$F_0 = S_0 e^{(r-q)T}. \tag{4.1}$$

Here S is the spot price of the underlying asset, q is a rate of return of investment, r is the risk free rate, and T is maturity time. At each moment of time, until maturity, the value of a long forward contract $f(t), 0 \leq t \leq T$ (for buy) is calculated by the following formula ([19], Chapter 5, p. 108):

$$f(t) = (F_0 - K)e^{-(r-q)T}. \tag{4.2}$$

Here F_0, K are the current forward price and delivery price (at time T) respectively (see [27], p. 273). Because F_0 changes with time, the price of a forward contract may have both positive and negative values, (see (4.2)). It is important for banks, and the finance industry in general, to value the agreement each day.

In case of forward contracts on foreign currencies the underlying asset is the exchange rate, or a certain number of units of a foreign currency. Let us write S_0 for the spot exchange rate, (price of 1 unit of foreign currency (EURO for example) in domestic currency, (dollars for example)), and F_0 as the forward price, both expressed in units of the base currency per unit of foreign currency. Holding the currency gives the investor an interest rates profit at the risk-free rate prevailing in the respective country. Denote by r_d as the domestic risk-free rate and r_f as the risk-free rate in the foreign country. The forward price is then similar to (4.1), (see [19], Chapter 5, p. 113):

$$F_0 = S_0 e^{(r_d - r_f)T}. \tag{4.3}$$

Relation (4.3) is the well-known interest rate parity relationship from international finance. If condition (4.3) does not hold, there are arbitrage opportunities. If $F_0 > S_0 e^{(r_d - r_f)T}$, a profit could be obtained by:

a) borrowing $F_0 = S_0 e^{-r_f T}$ in domestic currency at rate r_d for time T;

b) buying $e^{-r_f T}$ of the foreign currency and invest this at the rate r_f;

c) short selling a forward contract on one unit of the foreign currency.

At time T, the arbitrageur will get one unit of foreign currency from the deposit, which he sells at the forward price F_0. From this result, he is able to repay the loan $S_0 e^{(r_d - r_f)T}$ and still obtain a net profit of $F_0 - S_0 e^{(r_d - r_f)T}$ (see [51]).

We now list the forward trading steps for the Forex market (see [54]):

1) Choose the buy currency and the sell currency. The exchange rate appears automatically and is called the Spot Rate.

2) Choose the forward date. The Forward Points and the Forward Rate appears automatically.

3) Choose the amount of the contract and the amount you wish to risk. The Stop-Loss rate appears.

4) Read the "Response Message". This tells you if you have enough in your account to make the deal

5) Finish the deal by pressing the "Accept" button. Then your deal is open and running.

4.1.2 Currency futures

Following Bjork, (see [6], p. 103) a futures contract is very close to the corresponding forward contract in the sense that it is still a contract for the delivery of X at T. The difference is that all the payments, from the holder of the contract to the underwriter, are no longer made at T. Let us denote the futures price by $F(t; T, X)$; the payments are delivered continuously over time, such that the holder of the contract over the time interval $[s, t]$ receives the amount $F(t; T, X) - F(s; T, X)$ from the underwriter. Finally the holder will receive X, and pay $F(T; T, X)$, at the delivery date T. By definition, the (spot) price (at any time) of the entire futures contract equals zero. Thus the cost of entering or leaving a futures contract is zero, and the only contractual obligation is the payment stream described above.

Futures are similar to forward agreements in terms of the end result. A future contract represents a binding obligation to buy or sell a particular asset at a specified price at a stipulated date. Futures have, however, specific features. For instance their standardization and their payoff procedure distinguish them from forwards. For a new contract several terms need to be specified by the exchange house prior to any transaction, (see [51]):

1) the underlying asset: whether it is a commodity such as corn, or a financial instrument such as a foreign exchange rate;

2) the size of the contract: the amount of the asset which will be delivered;

3) the delivery date and delivery arrangements;

4) the quoted price.

At the maturity time the price of the futures contract converges to the spot price of the underlying asset. If this does not hold, arbitrage opportunities would arise, thus forcing, from the action of agents in the market, the price of the futures to go up or down.

We now explain the main differences between Forward and Futures interest rates contracts for the EURO/USD currency pair. The most popular interest rate futures contract in the United States is the 3 months Eurodollar interest rate futures contract, (see [19], [52]) traded on the Chicago Mercantile Exchange. The Eurodollar currency futures contract is similar to a forward rate agreement because it locks in an interest rate for a future period. For short maturities (less than 1 year) the two contracts seem to be the same and the Eurodollar futures interest rate can be treated the same as the corresponding forward interest rate. For longer maturities the differences between contracts are very important. Let us compare a Eurodollar futures contract on an interest rate for the period of time $[T_1, T_2]$ with a Forward agreement for the same period. The Eurodollar futures contract is settled daily, (the difference in the prior agreed-upon price and the daily futures price is settled daily). The final settlement is at time T_1 and reflects the realized interest rate between times T_1, T_2. On the contrary, the forward agreement is not settled daily and the final settlement, reflecting the changes in

interest rates between T_1, T_2 is made at time T_2. So, the main difference would be on the cash-flows processes associated with forwards, (which occur only at delivery date) and with futures, which occur every trading day (see [19], Chapter 4, Section, 4.7; Chapter 6, Section 6.3 for detail). This is called "mark to market". There are several main differences between forward agreements and futures contracts (see [19]):

Forwards	Futures
Private contract	Traded on an exchange
Not standardized	Standardized
Limited range of delivery dates	Wide range of delivery dates
Settled at maturity date	Settled daily
Final cash settlement	Closed out before mature

4.1.3 Currency options

In our research our main emphasis is on currency options and pricing them. In contrast to forwards, (futures) contracts, an option is the right, but not the obligation, to buy (or sell) an asset under specified terms (see [27], Chapter 12 for details). In the currency market (the Forex market for example) it means the right to buy (or sell) some amount of foreign currency at a fixed moment of time in future at a previously stipulated exchange rate.

A foreign exchange option, (commonly shortened to just FX option), is a derivative where the owner has the right but not the obligation to exchange money denominated in one currency into another currency at a pre-agreed exchange rate on a specified date (see [53]).

The option that gives you a right to purchase something is called call option, to sell something is a put option. There are two main types of options: European options and American options. A holder of American option can exercise the option at any time before the expiration date. An owner of European option can exercise only at the expiration time. Options have an advantage over forward (futures) contracts: while protecting against a downside risk (if the trend goes against you, the investor does not exercise the option), they

do not stop the investor from profiting from unexpected upward movements of the foreign exchange rates. In other words, the investor who buys a call option anticipates that asset price will increase, for example, in the Forex market that the EURO/USD exchange rate will go up, so that at the expiration date he/she can buy the asset at the strike price K and sell it at the spot price S. His/her gain is then max(0,S-K). If the spot price decreases to less than the strike price K, the option is not exercised.

The person who buys a put option expects that asset price will decrease (or for example in Forex market EURO/USD exchange rate goes down), so that at the expiration date he/she can sell the asset at the strike price K and buy it at the spot price S. His/her gain is then max(0,K-S). If the spot price increases to more than the strike price K, the option is not exercised.

4.2 Pure currency contracts

Suppose there are two currencies: the domestic currency (say US dollars), and the foreign currency (EUROs). Denote the spot exchange rate at time t by S_t. By definition S_t is quoted as:

$$\text{Spot exchange rate } S_t = \frac{\text{Units of domestic currency}}{\text{Unit of foreign currency}}.$$

We assume now that the domestic and foreign interest rates r_d, r_f are deterministic constants; B_t^d, B_t^f are the riskless asset prices in domestic and foreign currencies respectively. We model the spot exchange rateusing geometric Brownian motion. Then we have the following dynamics for S_t:

$$dS_t = S_t \alpha_S dt + S_t \sigma_S dW_t. \tag{4.4}$$

Here W_t is a Brownian motion under the objective probability measure P; α_S, σ_S are deter-

ministic constants. The dynamics of B_t^d, B_t^f are given by

$$dB_t^d = r_d B_t^d dt, \tag{4.5}$$

$$dB_t^f = r_f B_t^f dt. \tag{4.6}$$

A T-claim (a financial derivative) is any stochastic variable $Z = \Phi(S(T))$, where Φ is a some given deterministic function.

A European call option is an example of T-claim and allows the holder to exercise the option, (i.e., to buy), only on the option expiration date T. It has the following value at time T:

$$Z = \max(S_T - K, 0). \tag{4.7}$$

The foreign currency plays the same role as a domestic stock with a continuous dividend. We shall prove this below.

We shall use the following assumption: all markets are frictionless and liquid. All holdings of the foreign currency are invested in the foreign riskless asset, i.e. they will evolve according to the dynamics (4.6), (see [6], Chapter 17).

Using the standard theory of derivatives we have the following risk neutral valuation formula

$$\Pi(t, Z) = e^{-r_d(T-t)} E_{t,S}^Q[\Phi(S(T))], \tag{4.8}$$

where Q is a risk-neutral martingale measure.

The solution to the SDE (4.4) is

$$S_t = S_0 e^{(\alpha_S - 1/2\sigma_S^2)t + \sigma_S W_t}. \tag{4.9}$$

The possibility of buying the foreign currency and investing it at the foreign short rate of interest, is equivalent to the possibility of investing in a domestic asset with a price process

$\widetilde{B}_t^f = B_t^f S_t$, where the dynamics of $\widetilde{B}_t^f$ are given by

$$d\widetilde{B}_t^f = \widetilde{B}_t^f(\alpha_t + r_t^f)dt + \widetilde{B}_t^f \sigma_t dW_t. \tag{4.10}$$

Using the Girsanov theorem with

$$\theta_t = \frac{\alpha_S + r_f - r_d}{\sigma_S}$$

the risk neutral Q-dynamics of $\widetilde{B}_t^f$ are given by

$$d\widetilde{B}_t^f = r_d \widetilde{B}_f dt + \widetilde{B}_f \sigma_S dW_t^*, \tag{4.11}$$

where W^* is a is a Q-Wiener process.

Using (4.5) and (4.11), Itô's formula (see Theorem 3.2) and the fact that

$$S_t = \frac{\widetilde{B}_f}{B_f}$$

we have the following dynamics for S_t under Q:

$$dS_t = S_t(r_d - r_f) + S_t \sigma S dW_t^*. \tag{4.12}$$

Theorem 4.1 (see [6], Chapter 17). *The arbitrage free price $\Pi(t, \Phi)$ for the T-claim $Z = \Phi(S_T)$ is given by $\Pi(t, \Phi) = F(t, S_t)$ where*

$$F(t, s) = e^{-r_d(T-t)} E_{t,s}^Q[\Phi(S_T)], \tag{4.13}$$

where Q-dynamics of S_t are given by (4.12) *and $S_t = s$ is the initial condition.*

This result follows directly from the fact that $S_T e^{-r_d(T-t)}$ is a martingale (see Girsanov's theorem, Theorem 3.3).

Alternatively, $F(t, s)$ can be obtained as the solution to the boundary value problem

$$\frac{\partial F}{\partial t} + s(r_d - r_f)\frac{\partial F}{\partial s} + \frac{1}{2}s^2\sigma_S^2 \frac{\partial^2 F}{\partial s^2} - r_d F = 0; \tag{4.14}$$

with the boundary condition $F(T, s) = \Phi(s)$.

As a result, foreign currency can be treated exactly as a stock with a continuous dividend. So, the price of the European call, $Z = \max[S_T - K, 0]$, on the foreign currency, is given by the modified Black-Scholes formula (see [6], Chapter 17):

$$F(t,s) = se^{-r_f(T-t)}K[d_1] - e^{-r_d(T-t)}K[d_2], \tag{4.15}$$

where

$$d_1(t,x) = \frac{1}{\sigma_S\sqrt{T-t}}\left\{\log\left(\frac{x}{K}\right) + \left(r_d - r_f + \frac{1}{2}\sigma_S^2\right)(T-t)\right\}, \tag{4.16}$$

$$d_2(t,s) = d_1(t,s) - \sigma_S\sqrt{T-t}. \tag{4.17}$$

In this section the market also includes a domestic equity with price S_t^d, and a foreign equity with a price S_t^f. We shall model the equity dynamics using geometric Brownian motion. Now, there are three risky assets in our market: S_t, S_t^f, S_t^d. As a result a three-dimensional Brownian motion is used to model the market:

The dynamic model of the entire economy, under the objective measure P, is as follows (see [6], Chapter 17):

$$dS_t = S_t\alpha_S dt + S_t\sigma_S dW_t, \tag{4.18}$$

$$dS_t^d = S_t^d\alpha_d dt + S_t^d\sigma_d dW_t, \tag{4.19}$$

$$dS_t^f = S_t^f\alpha_f dt + S_t^f\sigma_f dW_t, \tag{4.20}$$

$$dB_t^d = r_d B_t^d dt \tag{4.21}$$

$$dB_t^f = r_f B_t^f dt. \tag{4.22}$$

Here $W_t = [W_t^1 \ W_t^2 \ W_t^3]^T$ is a a three-dimensional Wiener process, consisting of three independent Brownian motions; σ is a 3×3 matrix of the following form:

$$\sigma = \begin{bmatrix} \sigma_S \\ \sigma_d \\ \sigma_f \end{bmatrix} = \begin{bmatrix} \sigma_{S1} & \sigma_{S2} & \sigma_{S3} \\ \sigma_{d1} & \sigma_{d2} & \sigma_{d3} \\ \sigma_{f1} & \sigma_{f2} & \sigma_{f3} \end{bmatrix} \tag{4.23}$$

and is invertible. There are the following types of T-contracts (see [6], Chapter 17):

1) A European foreign equity call, struck in the foreign currency, i.e. an option to buy one unit of the foreign equity at the strike price of K units of the foreign currency. The value of this claim at the date of expiration is, expressed in the foreign currency, is

$$Z^f = \max[S_T^f - K, 0]. \tag{4.24}$$

Expressed in terms of the domestic currency the value of the claim at T is:

$$Z^d = S_T \max[S_T^f - K, 0]. \tag{4.25}$$

2) A European foreign equity call, struck in domestic currency, i.e. a European option to buy one unit of the foreign equity at time T, by paying K units of the domestic currency. Expressed in domestic terms the price of this claim is given by

$$Z^d = \max[S_T S_T^f - K, 0]. \tag{4.26}$$

3) An exchange option, which gives the right to exchange one unit of the domestic equity for one unit of the foreign equity. The corresponding claim, expressed in terms of the domestic currency, is

$$Z^d = \max[S_T S_T^f - S_T^d, 0]. \tag{4.27}$$

In the general case, a T-claim has the following form

$$Z = \Phi\left(S_T, S_T^d, S_T^f\right),$$

where Z is measured in the domestic currency.

Similarly to (4.13) we shall use the following risk-neutral valuation formula:

$$F(t,s,s^d,s^f) = e^{-r_d(T-t)}E^Q_{t,s,s^d,s^f}[\Phi(S_T)], \tag{4.28}$$

where $S^d_t = s^d,\ S^f_t = s^f$ is an initial condition.

The market defined in (4.18)-(4.22) is equivalent to a market consisting of the components $S^d, \widetilde{S}^f, \widetilde{B}^f, B^d$, where

$$\widetilde{B}^f_t = B^f_t S_t, \tag{4.29}$$

$$\widetilde{S}^f_t = S^f_t S_t. \tag{4.30}$$

Using (4.29), (4.30) we can derive from (4.18)-(4.22) the following equivalent equations modeling the FX market (see [6],Chapter 17):

$$dS^d_t = S^d_t\alpha_d dt + S^d_t\sigma_d dW_t, \tag{4.31}$$

$$d\widetilde{S}^f_t = \widetilde{S}^f_t(\alpha_f + \alpha_S + \sigma_f\sigma_S^T)dt + \widetilde{S}^f_t(\sigma_f + \sigma_S)dW_t, \tag{4.32}$$

$$d\widetilde{B}^f_t = \widetilde{B}^f_t(\alpha_S + r_f)dt + \widetilde{B}^f_t\sigma_S dW_t \tag{4.33}$$

$$dB^d_t = r_d B^d_t dt. \tag{4.34}$$

S^d_t, S^f_t, B^f_t can be interpreted as the prices of domestically traded assets. Similarly to (4.10), (4.11) we can proceed to the risk-neutral measure in (4.31)-(4.34), (see [6], Chapter 17):

$$dS^d_t = S^d_t r_d dt + S^d_t\sigma_d dW^*_t, \tag{4.35}$$

$$d\widetilde{S}_t^f = \widetilde{S}_t^f r_d dt + \widetilde{S}_t^f(\sigma_f + \sigma_S)dW_t^*, \tag{4.36}$$

$$d\widetilde{B}_t^f = \widetilde{B}_t^f r_d dt + \widetilde{B}_t^f \sigma_S dW_t^*, \tag{4.37}$$

$$dS_t = S_t(r_d - r_f)dt + S_t\sigma_S dW_t^*, \tag{4.38}$$

$$dS_t^f = S_t^f(r_f - \sigma_f\sigma_s^T)dt + S_t^f\sigma_f dW_t^*. \tag{4.39}$$

Theorem 4.2 (see [6], Chapter 17). *The arbitrage free price* $\Pi(t,\Phi)$ *for the* T*-claim*

$$Z = \Phi(S_T, S_T^d, \widetilde{S_T^f})$$

is given by $\Pi(t,\Phi) = F(t,s,s^d,s^f)$ *where*

$$F(t,s,s^d,s^f) = e^{-r_d(T-t)}E^Q_{t,s,s^d,s^f}[\Phi(S_T, S_T^d, \widetilde{S_T^f})], \tag{4.40}$$

where the Q*-dynamics are given by* (4.35)-(4.39).

Alternatively $F(t,s,s^d,s^f)$ *can be obtained as a solution to the boundary value problem*

$$\begin{aligned}
&\frac{\partial F}{\partial t} + s(r_d - r_f)\frac{\partial F}{\partial s} + s^d r_d\frac{\partial F}{\partial s^d} + \widetilde{s^f} r_d\frac{\partial F}{\partial \widetilde{s^f}} + \\
&\frac{1}{2}\left\{ s^2\|\sigma_S\|^2\frac{\partial^2 F}{\partial s^2} + (s^d)^2\|\sigma_d\|^2\frac{\partial^2 F}{\partial (s^d)^2} + (\widetilde{s^f})^2\left(\|\sigma_f\|^2 + \|\sigma_S\|^2 + 2\sigma_f\sigma_S^T\right)\frac{\partial^2 F}{\partial (\widetilde{s^f})^2}\right\} + \\
&s^d s\sigma_d\sigma_S^T\frac{\partial^2 F}{\partial \widetilde{s^d}\partial s} + \widetilde{s^f} s(\sigma_f\sigma_S^T + \|\sigma_S\|^2)\frac{\partial^2 F}{\partial \widetilde{s^f}\partial s} + \\
&s^d\widetilde{s^f}(\sigma_d\sigma_f^T + \sigma_d\sigma_S^T)\frac{\partial^2 F}{\partial \widetilde{s^d}\partial \widetilde{s^f}} - r_d F = 0;
\end{aligned} \tag{4.41}$$

with a boundary condition $F(T,s,s^d,s^f) = \Phi(s,s^d,\widetilde{s^f})$.

Theorem 4.3 (see [6], Chapter 17). *The arbitrage free price* $\Pi(t,\Phi)$ *for the* T*-claim*

$$Z = \Phi(S_T, S_T^d, S_T^f)$$

is given by $\Pi(t,\Phi) = F(t,s,s^d,s^f)$ *where*

$$F(t,s,s^d,s^f) = e^{-r_d(T-t)} E^Q_{t,s,s^d,s^f}[\Phi(S_T, S^d_T, S^f_T)], \tag{4.42}$$

where the Q*-dynamics are given by* (4.35)-(4.39)*.*

Alternatively $F(t,s,s^d,s^f)$ *can be obtained as a solution of the boundary value problem*

$$\begin{aligned}
&\frac{\partial F}{\partial t} + s(r_d - r_f)\frac{\partial F}{\partial s} + s^d r_d \frac{\partial F}{\partial s^d} + s^f(r_f - \sigma_f \sigma_S^T)\frac{\partial F}{\partial s^f} + \\
&\frac{1}{2}\left\{ s^2 \|\sigma_S\|^2 \frac{\partial^2 F}{\partial s^2} + (s^d)^2 \|\sigma_d\|^2 \frac{\partial^2 F}{\partial (s^d)^2} + (s^f)^2 \frac{\partial^2 F}{\partial (s^f)^2} \right\} + \\
&s^d s \sigma_d \sigma_S^T \frac{\partial^2 F}{\partial s^d \partial s} + s^f s \sigma_f \sigma_S^T \frac{\partial^2 F}{\partial s^f \partial s} + s^d s^f \sigma_d \sigma_f^T \frac{\partial^2 F}{\partial s^d \partial s^f} - r_d F = 0;
\end{aligned} \tag{4.43}$$

with a boundary condition $F(T,s,s^d,s^f) = \Phi(s,s^d,s^f)$*.*

Chapter 5

Currency option pricing. Main results

5.1 Currency option pricing for general Lévy processes

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space with a probability measure $\mathbf{P}$. Consider a continuous-time, finite-state Markov chain $\xi = \{\xi_t\}_{0 \le t \le T}$ on $(\Omega, \mathcal{F}, \mathbf{P})$ with a state space $\mathcal{S}$, the set of unit vectors $(e_1, \cdots, e_n) \in \mathbb{R}^n$ with a rate matrix Π[1]. The dynamics of the chain are given by:

$$\xi_t = \xi_0 + \int_0^t \Pi \xi_u du + M_t \in \mathbb{R}^n, \tag{5.1}$$

where $M = \{M_t, t \ge 0\}$ is a $\mathbb{R}^n$-valued martingale with respect to $(\mathcal{F}_t^{\xi})_{0 \le t \le T}$, the $\mathbf{P}$-augmentation of the natural filtration $(\mathcal{F}_t)_{0 \le t \le T}$, generated by the Markov chain ξ. Consider a Markov-modulated Merton jump-diffusion which models the dynamics of the spot FX rate, given by the following stochastic differential equation (in the sequel SDE, see [8]):

$$dS_t = S_{t_-} \left(\mu_t dt + \sigma_t dW_t + (e^{Z_{t_-}} - 1) dN_t \right). \tag{5.2}$$

Here μ_t is drift parameter; W_t is a Brownian motion, σ_t is the volatility; N_t is a Poisson Process with intensity λ_t, $e^{Z_{t_-}} - 1$ is the amplitude of the jumps, given the jump arrival time. The distribution of Z_t has a density $\nu(x), x \in \mathbb{R}$. The parameters μ_t, σ_t, λ_t are modeled using the finite state Markov chain:

$$\begin{aligned} \mu_t &:= < \mu, \xi_t >, \ \mu \in \mathbb{R}^n_+; \\ \sigma_t &:= < \sigma, \xi_t >, \ \sigma \in \mathbb{R}^n_+; \\ \lambda_t &:= < \lambda, \xi_t >, \ \lambda \in \mathbb{R}^n_+. \end{aligned} \tag{5.3}$$

[1] In our numerical simulations we consider three-state Markov chain and calculate elements in Π using Forex market EURO/USD currency pair

The solution of (5.74) is $S_t = S_0 e^{L_t}$, (where S_0 is the spot FX rate at time $t = 0$). Here L_t is given by the formula:

$$L_t = \int_0^t (\mu_s - 1/2\sigma_s^2)ds + \int_0^t \sigma_s dW_s + \int_0^t Z_{s_-} dN_s. \tag{5.4}$$

There is more than one equivalent martingale measure for this market driven by a Markov-modulated jump-diffusion model. We shall define the regime-switching generalized Esscher transform to determine a specific equivalent martingale measure.

Using Itô's formula we can derive a stochastic differential equation for the discounted spot FX rate. To define the discounted spot FX rate we need to introduce domestic and foreign riskless interest rates for bonds in the domestic and foreign currency.

The domestic and foreign interest rates $(r_t^d)_{0\le t\le T}$, $(r_t^f)_{0\le t\le T}$ are defined using the Markov chain $(\xi_t)_{0\le t\le T}$ (see [8]):

$$r_t^d = \langle r^d, \xi_t \rangle, r^d \in \mathbb{R}_+^n,$$

$$r_t^f = \langle r^f, \xi_t \rangle, r^f \in \mathbb{R}_+^n.$$

The discounted spot FX rate is:

$$S_t^d = \exp\left(\int_0^t (r_s^d - r_s^f)ds\right) S_t, \quad 0 \le t \le T. \tag{5.5}$$

Using (5.76), the differentiation formula, see Elliott et al. (1982, [12]) and the stochastic differential equation for the spot FX rate (5.74) we find the stochastic differential equation for the discounted discounted spot FX rate:

$$dS_{t_-}^d = S_{t_-}^d (r_t^d - r_t^f + \mu_t)dt + S_{t_-}^d \sigma_t dW_t + S_{t_-}^d (e^{Z_{t_-}} - 1)dN_t. \tag{5.6}$$

To derive the main results consider the log spot FX rate

$$Y_t = \log\left(\frac{S_t}{S_0}\right).$$

Using the differentiation formula:

$$Y_t = C_t + J_t,$$

where C_t, J_t are the continuous and jump part of Y_t. They are given in (5.78), (5.79):

$$C_t = \int_0^t \left(r_s^d - r_s^f + \mu_s\right) ds + \int_0^t \sigma_s dW_s, \tag{5.7}$$

$$J_t = \int_0^t Z_{s\_} dN_s. \tag{5.8}$$

Let $(\mathcal{F}_t^Y)_{0\le t\le T}$ denote the $\mathbf{P}$-augmentation of the natural filtration $(\mathcal{F}_t)_{0\le t\le T}$, generated by Y. For each $t \in [0,T]$ set $\mathcal{H}_t = \mathcal{F}_t^Y \vee \mathcal{F}_T^\xi$. Let us also define two families of regime switching parameters

$(\theta_s^c)_{0\le t\le T}$, $(\theta_s^J)_{0\le t\le T}$: $\theta_t^m =< \theta^m, \xi_t >$, $\theta^m = (\theta_1^m, ..., \theta_n^m) \subset \mathbb{R}^n$, $m = \{c, J\}$.

Define a random Esscher transform $\mathbf{Q}^{\theta^c,\theta^J} \sim \mathbf{P}$ on $\mathcal{H}_t$ using these families of parameters $(\theta_s^c)_{0\le t\le T}$, $(\theta_s^J)_{0\le t\le T}$ (see [8], [13], [15] for details):

$$L_t^{\theta^c,\theta^J} = \left.\frac{d\mathbf{Q}^{\theta^c,\theta^J}}{d\mathbf{P}}\right|_{\mathcal{H}_t} =: \tag{5.9}$$
$$\frac{\exp\left(\int_0^t \theta_s^c dC_s + \int_0^t \theta_{s\_}^J dJ_s\right)}{\mathbb{E}\left[\exp\left(\int_0^t \theta_s^c dC_s + \int_0^t \theta_{s\_}^J dJ_s\right)\middle|\mathcal{F}_t^\xi\right]}.$$

The explicit formula for the density $L_t^{\theta^c,\theta^J}$ of the Esscher transform is given in the following theorem. A similar statement is proven for the log-normal distribution in [8]. The formula below can be obtained by another approach, considered by Elliott and Osakwe ([14]).

Theorem 5.1. *For $0 \le t \le T$ the density $L_t^{\theta^c,\theta^J}$ of Esscher transform defined in* (5.9) *is given by*

$$L_t^{\theta^c,\theta^J} = \exp\left(\int_0^t \theta_s^c \sigma_s dW_s - 1/2 \int_0^t (\theta_s^c \sigma_s)^2 ds\right) \times \tag{5.10}$$
$$\exp\left(\int_0^t \theta_{s\_}^J Z_{s\_} dN_s - \int_0^t \lambda_s \left(\int_{\mathbb{R}} e^{\theta_s^J x} \nu(dx) - 1\right) ds\right).$$

In addition,the random Esscher transform density $L_t^{\theta^c,\theta^J}$ *(see* (5.9), (5.10)*) is an exponential* $(\mathcal{H}_t)_{0\le t\le T}$ *martingale and satisfies the following SDE:*

$$\frac{dL_t^{\theta^c,\theta^J}}{L_{t_-}^{\theta^c,\theta^J}} = \theta_t^c\sigma_t dW_t + (e^{\theta_{t_-}^J Z_{t_-}} - 1)dN_t - \lambda_t\left(\int_{\mathbb{R}} e^{\theta_t^J x}\nu(dx) - 1\right)dt. \tag{5.11}$$

Proof of Theorem 5.1. The compound Poisson Process, driving the jumps $\sum_0^{N_t}(e^{Z_{s_-}} - 1)$ and the Brownian motion W_t are independent processes. Consequently:

$$\mathbb{E}\left[\exp\left(\int_0^t \theta_s^c dC_s + \int_0^t \theta_{s_-}^J dJ_s\right)\middle|\mathcal{F}_t^\xi\right] =$$

$$\mathbb{E}\left[\exp\left(\int_0^t \theta_s^c(\mu_s - 1/2\sigma_s^2)ds + \int_0^t \theta_s^c\sigma_s dW_s)\right)\middle|\mathcal{F}_t^\xi\right]\mathbb{E}\left[\exp\left(\int_0^t \theta_{s_-}^J Z_{s_-}dN_s\right)\middle|\mathcal{F}_t^\xi\right]. \tag{5.12}$$

Let us calculate:

$$\mathbb{E}\left[\exp\left(\int_0^t \theta_{s_-}^J Z_{s_-}dN_s\right)\middle|\mathcal{F}_t^\xi\right].$$

Write

$$\Gamma_t := \exp\left(\int_0^t \alpha_{s_-}dN_s\right), \alpha_s = \theta_s^J Z_s.$$

Using the differentiation rule (see [12]) we obtain the following representation of Γ_t:

$$\Gamma_t = \Gamma_0 + M_t^J + \int_{]0,t]}\Gamma_s\int_{\mathbb{R}}(e^{\alpha_s} - 1)\nu(dx)\lambda_s ds, \tag{5.13}$$

where

$$M_t^J = \int_{]0,t]}\Gamma_{s_-}(e^{\alpha_s} - 1)dNs - \int_{]0,t]}\Gamma_s\int_{\mathbb{R}}(e^{\alpha_s} - 1)\nu(dx)\lambda_s ds$$

is a martingale with respect to $\mathcal{F}_t^\xi$. Using this fact and (5.83) we obtain:

$$\mathbb{E}\left[\exp\left(\int_0^t \theta_s^J Z_{s_-}dN_s\right)\middle|\mathcal{F}_t^\xi\right] = \exp\left(\int_0^t \lambda_s\left(\int_{\mathbb{R}} e^{\theta_s^J x}\nu(dx) - 1.\right)ds\right) \tag{5.14}$$

We have from the differentiation rule:

$$\mathbb{E}\left[e^{u\int_0^t \sigma_s dW_s}\right] = \exp\left\{\frac{1}{2}u^2\int_0^t \sigma_s^2 ds\right\}, \tag{5.15}$$

where σ_t is the volatility of a market.

Substituting (5.14) and (5.15) into (5.12) we obtain:

$$\mathbb{E}\left[\exp\left(\int_0^t \theta_s^c dC_s + \int_0^t \theta_{s_-}^J dJ_s\right)\Big|\mathcal{F}_t^\xi\right] =$$

$$\exp\left(\int_0^t \theta_s^c(\mu_s - 1/2\sigma_s^2)ds + \frac{1}{2}\int_0^t (\theta_s^c\sigma_s)^2 ds)\right)\exp\left(\int_0^t \lambda_s\left(\int_{\mathbb{R}} e^{\theta_s^J x}\nu(dx) - 1\right)ds\right). \quad (5.16)$$

Substituting (5.16) into the expression for $L_t^{\theta^c,\theta^J}$ in (5.9) we obtain:

$$L_t^{\theta^c,\theta^J} = \exp\left(\int_0^t \theta_s^c(\mu_s - 1/2\sigma_s^2)ds + \int_0^t \theta_s^c\sigma_s dW_s)\right)\exp\left(\int_0^t \theta_{s_-}^J Z_{s_-} dN_s\right)\times \quad (5.17)$$

$$\left[\exp\left(\int_0^t \theta_s^c(\mu_s - 1/2\sigma_s^2)ds + \frac{1}{2}\int_0^t (\theta_s^c\sigma_s)^2 ds)\right)\exp\left(\int_0^t \lambda_s\left(\int_{\mathbb{R}} e^{\theta_s^J x}\nu(dx) - 1\right)ds\right)\right]^{-1} =$$

$$\exp\left(\int_0^t \theta_s^c\sigma_s dW_s - 1/2\int_0^t (\theta_s^c\sigma_s)^2 ds\right)\times$$

$$\exp\left(\int_0^t \theta_{s_-}^J Z_{s_-} dN_s - \int_0^t \lambda_s\left(\int_{\mathbb{R}} e^{\theta_s^J x}\nu(dx) - 1\right)ds\right).$$

If we represent $L_t^{\theta^c,\theta^J}$ in the form $L_t^{\theta^c,\theta^J} = e^{X_t}$ (see (5.17)) and apply differentiation rule we obtain (5.11). It follows from (5.11) that $L_t^{\theta^c,\theta^J}$ is a martingale. □

We shall derive the following condition for the discounted spot FX rate ((5.76)) to be martingale. These conditions will be used to calculate the risk-neutral Esscher transform parameters $(\theta_s^{c,*})_{0\le t\le T}$, $(\theta_s^{J,*})_{0\le t\le T}$ and give to the measure $\mathbf{Q}$. Then we shall use these values to find the no-arbitrage price of European call currency derivatives.

Theorem 5.2. *Let the random Esscher transform be defined by* (5.9)*. Then the martingale condition (for S_t^d, see* (5.76)*) holds if and only if the Markov modulated parameters $(\theta_t^c, \theta_t^J, 0 \le t \le T)$ satisfy for all $0 \le t \le T$ the condition:*

$$r_t^f - r_t^d + \mu_t + \theta_t^c\sigma_t^2 + \lambda_t^{\theta,J}k_t^{\theta,J} = 0. \quad (5.18)$$

Here the random Esscher transform intensity $\lambda_t^{\theta,J}$ of the Poisson Process and the main percentage jump size $k_t^{\theta,J}$ are, respectively, given by

$$\lambda_t^{\theta,J} = \lambda_t\int_{\mathbb{R}} e^{\theta_s^J x}\nu(dx), \quad (5.19)$$

$$k_t^{\theta,J} = \frac{\int_{\mathbb{R}} e^{(\theta_t^J+1)x}\nu(dx)}{\int_{\mathbb{R}} e^{\theta_t^J x}\nu(dx)} - 1, \tag{5.20}$$

as long as $\int_{\mathbb{R}} e^{\theta_t^J x}\nu(dx) < +\infty$, $\int_{\mathbb{R}} e^{(\theta_t^J+1)x}\nu(dx) < +\infty$.

Proof of Theorem 5.2. The martingale condition for the discounted spot FX rate S_t^d is

$$\mathbb{E}^{\theta^c,\theta^J}[S_t^d|\mathcal{H}_u] = S_u^d, \quad t \geq u. \tag{5.21}$$

To derive this condition a Bayes' formula is used:

$$\mathbb{E}^{\theta^c,\theta^J}[S_t^d|\mathcal{H}_u] = \frac{\mathbb{E}[L_t^{\theta^c,\theta^J} S_t^d|\mathcal{H}_u]}{\mathbb{E}[L_t^{\theta^c,\theta^J}|\mathcal{H}_u]}, \tag{5.22}$$

taking into account that $L_t^{\theta^c,\theta^J}$ is a martingale with respect to $\mathcal{H}_u$, so:

$$\mathbb{E}\left[L_t^{\theta^c,\theta^J}\middle|\mathcal{H}_u\right] = L_u^{\theta^c,\theta^J}. \tag{5.23}$$

Using formula (5.76) for the solution of the SDE for the spot FX rate, we obtain an expression for the discounted spot FX rate in the following form:

$$S_t^d = S_u^d \exp\left(\int_u^t (r_s^f - r_s^d + \mu_s - 1/2\sigma_s^2)ds + \int_u^t \sigma_s dW_s + \int_u^t Z_{s_-} dN_s\right), \quad t \geq u. \tag{5.24}$$

Then, using (5.10), (5.24) we can rewrite(5.93) as:

$$\mathbb{E}\left[\frac{L_t^{\theta^c,\theta^J}}{L_u^{\theta^c,\theta^J}} S_t^d\middle|\mathcal{H}_u\right] = S_u^d\, \mathbb{E}\left[\exp\left(\int_u^t \theta_s^c\sigma_s dW_s - 1/2\int_0^t (\theta_s^c\sigma_s)^2 ds\right)\times\right. \tag{5.25}$$

$$\exp\left(\int_u^t \theta_{s_-}^J Z_{s_-} dN_s - \int_u^t \lambda_s\left(\int_{\mathbb{R}} e^{\theta_s^J x}\nu(dx) - 1\right) ds\right)\times$$

$$\left.\exp\left(\int_u^t (r_s^f - r_s^d + \mu_s - 1/2\sigma_s^2)ds + \int_u^t \sigma_s dW_s + \int_u^t Z_{s_-} dN_s\right)|\mathcal{H}_u\right] =$$

$$S_u^d\, \mathbb{E}\left[\exp\left(\int_u^t (\theta_s^c + 1)\sigma_s dW_s - 1/2\int_u^t ((\theta_s^c + 1)\sigma_s)^2 ds\right)\times\right. \tag{5.26}$$

$$\left.\exp\left(\int_u^t (r_s^F - r_s^D + \mu_s + \theta_s^c\sigma_s^2)ds\right)\exp\left(\int_u^t \lambda_s\Big(\int_{\mathbb{R}} e^{\theta_s^J x}\nu(dx) - 1\Big)ds\right)|\mathcal{H}_u\right]\times$$

$$\mathbb{E}\Big[\exp\Big(\int_u^t(\theta_s^c+1)Z_{s\_}dN_s\Big)\,|\mathcal{H}_u\Big].$$

Using the expression for the characteristic function of Brownian motion (see (5.15)) we obtain:

$$\mathbb{E}\Big[\exp\Big(\int_u^t(\theta_s^c+1)\sigma_s dW_s - 1/2\int_u^t((\theta_s^c+1)\sigma_s)^2 ds\Big)\,|\mathcal{H}_u\Big]=1. \tag{5.27}$$

From (5.14) we obtain:

$$\mathbb{E}\Big[\exp\Big(\int_u^t(\theta_s^c+1)Z_{s\_}dN_s\Big)\,|\mathcal{H}_u\Big]=\exp\Big(\int_0^t\lambda_s\Big(\int_{\mathbb{R}}e^{(\theta_s^J+1)x}\nu(dx)-1\Big)ds\Big). \tag{5.28}$$

Substituting (5.27), (5.28) into (5.26) we obtain finally:

$$\mathbb{E}\Big[\frac{L_t^{\theta^c,\theta^J}}{L_u^{\theta^c,\theta^J}}S_t^d\Big|\mathcal{H}_u\Big]=S_u^d\exp\Big(\int_u^t(r_s^f-r_s^d+\mu_s+\theta_s^c\sigma_s^2)ds\Big)\times \tag{5.29}$$
$$\exp\Big(-\int_u^t\lambda_s\big(\int_{\mathbb{R}}e^{\theta_s^J x}\nu(dx)-1\big)ds\Big)\exp\Big(\int_u^t\lambda_s\big(\int_{\mathbb{R}}e^{(\theta_s^J+1)x}\nu(dx)-1\big)ds\Big).$$

From (5.29) we obtain the martingale condition for discounted spot FX rate:

$$r_t^f-r_t^d+\mu_t+\theta_t^c\sigma_t^2+\lambda_t\Big[\int_{\mathbb{R}}e^{(\theta_s^J+1)x}\nu(dx)-\int_{\mathbb{R}}e^{\theta_s^J x}\nu(dx)\Big]=0. \tag{5.30}$$

We now prove, that under the Esscher transform the new Poisson process intensity and mean jump size are given by (5.19), (5.20).

Note that $L_t^J=\int_0^t Z_{s\_}dN_s$ is the jump part of Lévy process in the formula (5.75) for the solution of SDE for spot FX rate. We have:

$$\mathbb{E}_{\mathbf{Q}}\left[e^{L_t^J}\right]=\int_\Omega\exp\Big(\int_0^t Z_{s\_}dN_s\Big)\,L_t^{\theta^{c,*},\theta^{J,*}}(\omega)dP(\omega), \tag{5.31}$$

where $\mathbf{P}$ is the initial probability measure and $\mathbf{Q}$ is the new risk-neutral measure. Substituting the density of Esscher transform (5.10) into (5.31) we have (see also [14]):

$$\mathbb{E}_{\mathbf{Q}}\left[e^{L_t^J}\right]=\mathbb{E}_{\mathbf{P}}\Big[\exp\Big(\int_0^t\theta_s^c\sigma_s dW_s-1/2\int_0^t(\theta_s^c\sigma_s)^2ds\Big)- \tag{5.32}$$
$$\int_0^t\lambda_s\Big(\int_{\mathbb{R}}e^{\theta_s^J x}\nu(dx)-1\Big)ds\Big)\Big]\,\mathbb{E}_{\mathbf{P}}\Big[\exp\Big(\int_0^t(\theta_s^J+1)Z_{s\_}dN_s\Big)\Big].$$

Using (5.14) we obtain:

$$\mathbb{E}_{\mathbf{P}}\left[\exp\left(\int_0^t(\theta_s^J+1)Z_{s\_}dN_s\right)\right]=\exp\left(\int_0^t\lambda_s\left(\int_{\mathbb{R}}e^{(\theta_s^J+1)x}\nu(dx)-1\right)ds\right). \tag{5.33}$$

Substitute (5.33) into (5.32) and taking into account the characteristic function of Brownian motion (see (5.15)) we obtain:

$$\mathbb{E}_{\mathbf{Q}}\left[e^{L_t^J}\right]=\exp\left(\int_0^t\lambda_s\left(\int_{\mathbb{R}}e^{\theta_s^J x}\nu(dx)\left[\frac{\int_{\mathbb{R}}e^{(\theta_s^J+1)x}\nu(dx)}{\int_{\mathbb{R}}e^{\theta_s^J x}\nu(dx)}-1\right]\right)ds\right). \tag{5.34}$$

Returning to the initial measure $\mathbf{P}$, but with different $\lambda_t^{\theta,J}, k_t^{\theta,J}$,we have:

$$\mathbb{E}_{\tilde{\lambda},\tilde{\nu}}\left[e^{L_t^J}\right]=\exp\left(\int_0^t\lambda_s^{\theta,J}\left(\int_{\mathbb{R}}e^x\tilde{\nu}(dx)-1\right)ds\right). \tag{5.35}$$

Formula (5.19) for the new intensity $\lambda_t^{\theta,J}$ of Poisson process follows directly from (5.34), (5.35). The new density of jumps $\tilde{\nu}$ is defined from (5.35) by the following formula:

$$\frac{\int_{\mathbb{R}}e^{(\theta_t^J+1)x}\nu(dx)}{\int_{\mathbb{R}}e^{\theta_t^J x}\nu(dx)}=\int_{\mathbb{R}}e^x\tilde{\nu}(dx). \tag{5.36}$$

We now calculate the new mean jump size given the jump arrival with respect to the new measure $\mathbf{Q}$:

$$k_t^{\theta,J}=\int_\Omega(e^{Z(\omega)}-1)d\tilde{\nu}(\omega)=\int_{\mathbb{R}}(e^x-1)\tilde{\nu}(dx)=\int_{\mathbb{R}}e^x\tilde{\nu}(dx)-1=\frac{\int_{\mathbb{R}}e^{(\theta_t^J+1)x}\nu(dx)}{\int_{\mathbb{R}}e^{\theta_t^J x}\nu(dx)}-1, \tag{5.37}$$

where the new measure $\tilde{\nu}(dx)$ is defined by the formula (5.36).

We can rewrite the martingale condition (5.30) for the discounted spot FX rate in the following form:

$$r_t^f-r_t^d+\mu_t+\theta_t^c\sigma_t^2+\lambda_t^{\theta,J}k_t^{\theta,J}=0, \tag{5.38}$$

where $\lambda_t^{\theta,J}, k_t^{\theta,J}$ are given by (5.19), (5.20) respectively. □

If we put $k_t^{\theta,J}=0$, we obtain the following formulas for the regime switching Esscher transform parameters yielding the martingale condition (5.38):

$$\theta_t^{c,*}=\frac{r_t^d-r_t^f-\mu_t}{\sigma_t^2}, \tag{5.39}$$

$$\theta_t^{J,*} : \frac{\int_{\mathbb{R}} e^{(\theta_t^{J,*}+1)x}\nu(dx)}{\int_{\mathbb{R}} e^{\theta_t^{J,*}x}\nu(dx)} = 1. \tag{5.40}$$

In the next section we shall apply these formulas (5.39), (5.40) to the log-double exponential distribution of jumps.

We now proceed to the general formulas for European calls (see [8], [30]). For the European call currency options with a strike price K and the time of expiration T the price at time zero is given by:

$$\Pi_0(S,K,T,\xi) = \mathbb{E}^{\theta^{c,*},\theta^{J,*}}\left[e^{-\int_0^T (r_s^D - r_s^F)ds}(S_T - K)^+ \mid \mathcal{F}_t^{\xi}\right]. \tag{5.41}$$

Let $J_i(t,T)$ denote the occupation time of ξ in state e_i over the period $[t,T], t < T$. We introduce several new quantities that will be used in future calculations:

$$R_{t,T} = \frac{1}{T-t}\int_0^T (r_s^d - r_s^f)ds = \frac{1}{T-t}\sum_{i=1}^n (r_i^d - r_i^f)J_i(t,T), \tag{5.42}$$

where $J_i(t,T) := \int_t^T < \xi_s,\ e_i > ds$;

$$U_{t,T} = \frac{1}{T-t}\int_t^T \sigma_s^2 ds = \frac{1}{T-t}\sum_{i=1}^n \sigma_i^2 J_i(t,T); \tag{5.43}$$

$$\lambda_{t,T}^{\theta^* J} = \frac{1}{T-t}\sum_{i=1}^n \lambda_i^{\theta^* J} J_i(t,T); \tag{5.44}$$

$$\lambda_{t,T}^{\theta^*} = \frac{1}{T-t}\int_t^T (1 + k_s^{\theta^* J})\lambda_s^{\theta^* J} ds = \frac{1}{T-t}\sum_{i=1}^n (1 + k_i^{\theta^* J})\lambda_i^{\theta^* J} J_i(t,T); \tag{5.45}$$

$$V_{t,T,m}^2 = U_{t,T} + \frac{m\sigma_J^2}{T-t}, \tag{5.46}$$

where σ_J^2 is the variance of the distribution of the jumps.

$$R_{t,T,m} = R_{t,T} - \frac{1}{T-t}\int_t^T \lambda_s^{\theta^* J} k_s^{\theta^* J} ds + \frac{1}{T-t}\int_0^T \frac{\log(1 + k_s^{\theta^* J})}{T-t} ds = \tag{5.47}$$

$$R_{t,T} - \frac{1}{T-t}\sum_{i=1}^{n}\lambda_i^{\theta^* J}k_i^{\theta^* J} + \frac{m}{T-t}\sum_{i=1}^{n}\frac{\log(1+k_i^{\theta^* J})}{T-t}J_i(t,T),$$

where m is the number of jumps in the interval $[t,T]$, n is the number of states of the Markov chain ξ.

Note, that in our considerations all these general formulas (5.42)-(5.47) mentioned above are simplified by the fact that: $k_t^{\theta^* J} = 0$ with respect to the new risk-neutral measure $\mathbf{Q}$ with Esscher transform parameters given by (5.39), (5.40). From the pricing formula in Merton (1976, [30]) let us define (see [8])

$$\overline{\Pi_0}(S,K,T;R_{0,T},U_{0,T},\lambda_{0,T}^{\theta^*}) = \sum_{m=0}^{\infty}\frac{e^{-T\lambda_{0,T}^{\theta^*,J}}(T\lambda_{0,T}^{\theta^*})^m}{m!}\times \tag{5.48}$$
$$BS_0(S,K,T,V_{0,T,m}^2,R_{0,T,m})$$

where $BS_0(S,K,T,V_{0,T,m}^2,R_{0,T,m})$ is the standard Black-Scholes price formula (see [6]) with initial spot FX rate S, strike price K, risk-free rate r, volatility square σ^2 and time T to maturity.

Then, the European style call option pricing formula takes the form (see [8]):

$$\Pi_0(S,K,T) = \int_{[0,t]^n}\overline{\Pi_0}(S,K,T;R_{0,T},U_{0,T},\lambda_{0,T}^{\Theta^*,J})\times \tag{5.49}$$
$$\psi(J_1,J_2,...,J_n)dJ_1...dJ_n,$$

where $\psi(J_1,J_2,...,J_n)$ is the joint probability distribution density for the occupation time, which is determined by the following characteristic function (See [14]):

$$\mathbb{E}\left[\exp\{\langle u,J(t,T)\rangle\}\right] = \langle\exp\{(\Pi + diag(u))(T-t)\}\cdot\mathbb{E}[\xi_0],\mathbb{1}\rangle, \tag{5.50}$$

where $\mathbb{1}\in\mathbb{R}^n$ is a vector of ones, $u=(u_1,...,u_n)$ is a vector of transform variables, $J(t,T) := \{J_1(t,T),...,J_n(t,T)\}$.

5.2 Currency option pricing for log-double exponential processes

The log-double exponential distribution for Z_{t_-} ($e^{Z_{t_-}}-1$ are the jumps in (5.74)), plays a fundamental role in mathematical finance, describing the spot FX rate movements over long

period of time. It is defined by the following formula of the density function:

$$\nu(x) = p\theta_1 e^{-\theta_1 x}\Big|_{x\geq 0} + (1-p)\theta_2 e^{\theta_2 x}\Big|_{x<0}. \tag{5.51}$$

The mean value of this distribution is:

$$\text{mean}(\theta_1, \theta_2, p) = \frac{p}{\theta_1} - \frac{1-p}{\theta_2}. \tag{5.52}$$

The variance of this distribution is:

$$\text{var}(\theta_1, \theta_2, p) = \frac{2p}{\theta_1^2} + \frac{2(1-p)}{\theta_2^2} - \left(\frac{p}{\theta_1} - \frac{1-p}{\theta_2}\right)^2. \tag{5.53}$$

The double-exponential distribution was first investigated by Kou in [25]. He also gave economic reasons to use such a type of distribution in Mathematical Finance. The double exponential distribution has two interesting properties: the leptokurtic feature (see [22], §3; [25]) and the memoryless property (the probability distribution of X is memoryless if for any non-negative real numbers t and s, we have $\Pr(X > t + s|X > t) = \Pr(X > s)$, see for example [48]). The last property is inherited from the exponential distribution.

A statistical distribution has the leptokurtic feature if there is a higher peak (higher kurtosis) than in a normal distribution. This high peak and the corresponding fat tails mean the distribution is more concentrated around the mean than a normal distribution, and it has a smaller standard deviation. See details for fat-tail distributions and their applications to Mathematical Finance in [47]. A leptokurtic distribution means that small changes have less probability because historical values are centered by the mean. However, this also means that large fluctuations have greater probabilities within the fat tails.

In [28] this distribution is applied to model asset pricing under fuzzy environments. Several advantages of models including log-double exponential distributed jumps over other models described by Lévy processes (see [25]) include:

1) The model describes properly some important empirical results from stock and foreign currency markets. The double exponential jump-diffusion model is able to reflect the leptokurtic feature of the return distribution. Moreover, the empirical tests performed in [37]

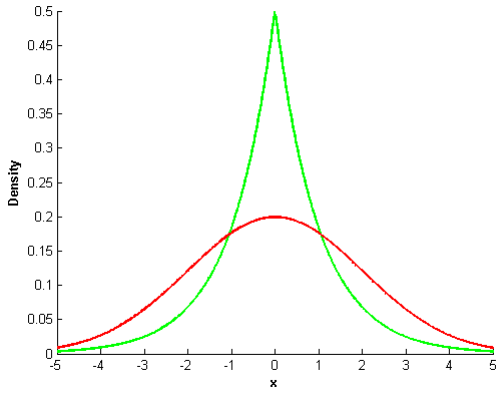


Figure 5.1: Double-exponential distribution (green) vs. normal distribution (red), mean=0, dev=2

suggest that the log-double exponential jump-diffusion model fits stock data better than the log-normal jump-diffusion model.

2) The model gives explicit solutions for convenience of computations.

3) The model has an economical interpretation.

4) It has been suggested from extensive empirical studies that markets tend to have both overreaction and underreaction to good or bad news. The jump part of the model can be considered as the market response to outside news. In the absence of outside news the asset price (or spot FX rate) changes in time as a geometric Brownian motion. Good or bad news (outer strikes for the FX market in our case) arrive according to a Poisson process, and the spot FX rate changes in response according to the jump size distribution. Since the double exponential distribution has both a high peak and heavy tails, it can be applied to model both the overreaction (describing the heavy tails) and underreaction (describing the high peak) to outside news.

5) The log double exponential model is self-consistent. In Mathematical Finance, it means that a model is arbitrage-free.

The family of regime switching Esscher transform parameters is defined by (5.39), (5.40).

Let us define $\theta_t^{J,*}$, (the first parameter $\theta_t^{c,*}$ has the same formula as in general case) by:

$$\int_{\mathbb{R}} e^{(\theta_s^J+1)x}\left(p\theta_1 e^{-\theta_1 x}\Big|_{x\geq 0}+(1-p)\theta_2 e^{\theta_2 x}\Big|_{x<0}\right)dx = \tag{5.54}$$

$$\int_{\mathbb{R}} e^{\theta_s^J x}\left(p\theta_1 e^{-\theta_1 x}\Big|_{x\geq 0}+(1-p)\theta_2 e^{\theta_2 x}\Big|_{x<0}\right)dx.$$

We require an additional restriction for the convergence of the integrals in (5.54):

$$-\theta_2 < \theta_t^J < \theta_1. \tag{5.55}$$

Then (5.54) can be rewritten in the following form:

$$\frac{p\theta_1}{\theta_1-\theta_t^J-1}+\frac{(1-p)\theta_2}{\theta_2+\theta_t^J+1}=\frac{p\theta_1}{\theta_1-\theta_t^J}+\frac{(1-p)\theta_2}{\theta_2+\theta_t^J}. \tag{5.56}$$

Solving (5.56) we arrive at the quadratic equation:

$$(\theta_t^J)^2(p\theta_1-(1-p)\theta_2)+\theta_t^J(p\theta_1+2\theta_1\theta_2-(1-p)\theta_2)+ \tag{5.57}$$

$$p\theta_1\theta_2^2+p\theta_2\theta_1^2-\theta_2\theta_1^2+\theta_1\theta_2=0.$$

If $p\theta_1-(1-p)\theta_2\neq 0$ we have two solutions and one of them satisfies restriction (5.55):

$$\theta_t^J=-\frac{p\theta_1+2\theta_1\theta_2-(1-p)\theta_2}{2(p\theta_1-(1-p)\theta_2)}\pm \tag{5.58}$$

$$\frac{\sqrt{(p\theta_1+2\theta_1\theta_2-(1-p)\theta_2)^2-4(p\theta_1-(1-p)\theta_2)(p\theta_1\theta_2^2+p\theta_2\theta_1^2-\theta_2\theta_1^2+\theta_1\theta_2)}}{2(p\theta_1-(1-p)\theta_2)}.$$

Then the Poisson process intensity (see (5.19)) is:

$$\lambda_t^{\theta,J}=\lambda_t\left(\frac{p\theta_1}{\theta_1-\theta_t^J}+\frac{(1-p)\theta_2}{\theta_2+\theta_t^J}\right). \tag{5.59}$$

The new mean jump size (see (5.20)) is:

$$k_t^{\theta,J}=0 \tag{5.60}$$

as in the general case.

When we proceed to a new risk-neutral measure Q we have a new density of jumps ν

$$\tilde{\nu}(x) = \tilde{p}\theta_1 e^{-\theta_1 x}\Big|_{x\geq 0} + (1-\tilde{p})\theta_2 e^{\theta_2 x}\Big|_{x<0}. \tag{5.61}$$

The new probability $\tilde{p}$ can be calculated using (5.36):

$$\frac{\frac{p\theta_1}{\theta_1-\theta_t^J-1} + \frac{(1-p)\theta_2}{\theta_2+\theta_t^J+1}}{\frac{p\theta_1}{\theta_1-\theta_t^J} + \frac{(1-p)\theta_2}{\theta_2+\theta_t^J}} = \frac{\tilde{p}\theta_1}{\theta_1-1} + \frac{(1-\tilde{p}\theta_2)}{\theta_2+1}. \tag{5.62}$$

From (5.62) we obtain an explicit formula for $\tilde{p}$:

$$\tilde{p} = \frac{\frac{\frac{p\theta_1}{\theta_1-\theta_t^J-1} + \frac{(1-p)\theta_2}{\theta_2+\theta_t^J+1}}{\frac{p\theta_1}{\theta_1-\theta_t^J} + \frac{(1-p)\theta_2}{\theta_2+\theta_t^J}} - \frac{\theta_2}{\theta_2+1}}{\frac{\theta_1}{\theta_1-1} - \frac{\theta_2}{\theta_2+1}}. \tag{5.63}$$

5.3 Currency option pricing for log-normal processes

Log-normal distribution of jumps with μ_J the mean, σ_J the deviation (see [49]), and its applications to currency option pricing was investigated in [8]. More details of these distributions and other distributions applicable for the Forex market can be found in [50]. We give here a sketch of results from [8] to compare them with the case of the log-double exponential distribution of jumps discussed in this article. The main goal of our paper is a generalization of this result for arbitrary Lévy processes. The results, provided in [8] are as follows:

Theorem 5.3. *For $0 \leq t \leq T$ the density $L_t^{\theta^c,\theta^J}$ of the Esscher transform defined in* (5.9) *is given by*

$$L_t^{\theta^c,\theta^J} = \exp\left(\int_0^t \theta_s^c \sigma_s dW_s - 1/2\int_0^t (\theta_s^c\sigma_s)^2 ds\right) \times \tag{5.64}$$
$$\exp\left(\int_0^t \theta_{s_-}^J Z_{s_-} dN_s - \int_0^t \lambda_s\left(e^{\theta_s^J\mu_J + 1/2(\theta_s^J\sigma_J)^2} - 1\right) ds\right),$$

where μ_J, σ_J are the mean value and deviation of jumps, respectively. In addition, the random Esscher transform density $L_t^{\theta^c,\theta^J}$, (see (5.9), (5.10)*), is an exponential $(\mathcal{H}_t)_{0\leq t\leq T}$ martingale and satisfies the following SDE*

$$\frac{dL_t^{\theta^c,\theta^J}}{L_{t_-}^{\theta^c,\theta^J}} = \theta_t^c\sigma_t dW_t + (e^{\theta_{t_-}^J Z_{t_-}} - 1)dN_t - \lambda_t\left(e^{\theta_t^J\mu_J + 1/2(\theta_t^J\sigma_J)^2} - 1\right) dt. \tag{5.65}$$

Theorem 5.4. *Let the random Esscher transform be defined by* (5.9)*. Then the martingale condition (for S_t^d, see* (5.76)*) holds if and only if the Markov modulated parameters $(\theta_i^c, \theta_i^J, 0 \leq t \leq T)$ satisfy for all $0 \leq t \leq T$ the condition:*

$$r_i^f - r_i^d + \mu_i + \theta_i^c \sigma_i^2 + \lambda_i^{\theta,J} k_i^{\theta,J} = 0 \text{ for all } i,\ 1 \leq i \leq \Lambda \tag{5.66}$$

where the random Esscher transform intensity $\lambda_i^{\theta,J}$ of the Poisson Process and the mean percentage jump size $k_i^{\theta,J}$ are respectively given by

$$\lambda_i^{\theta,J} = \lambda_i e^{\theta_i^J \mu_J + 1/2(\theta_i^J \sigma_J)^2}, \tag{5.67}$$

$$k_i^{\theta,J} = e^{\mu_J + 1/2\sigma_J^2 + \theta_i^J \sigma_J^2} - 1 \text{ for all } i. \tag{5.68}$$

The regime switching parameters satisfying the martingale condition (5.66) *are given by the following formulas: $\theta_i^{c,*}$ is the same as in* (5.39)*,*

$$\theta_i^{J,*} = -\frac{\mu_J + 1/2\sigma_J^2}{\sigma_J^2} \text{ for all } i. \tag{5.69}$$

With such a value of a parameter $\theta_i^{J,*}$:

$$k_i^{\theta^*,J} = 0, \quad \lambda_i^{\theta^*,J} = \lambda_i \left(-\frac{\mu_J}{2\sigma_J^2} + \frac{\sigma_J^2}{8} \right) \text{ for all } i. \tag{5.70}$$

Note, that these formulas (5.67)-(5.70) follow directly from our formulas for the case of general Lévy process, (see (5.19),(5.20), (5.40)). In particular, the fact that $k_i^{\theta^*,J} = 0$ by substituting (5.40) into the expression for $k_i^{\theta,J}$ in (5.20). From (5.40) we derive:

$$\frac{\int_{\mathbb{R}} e^{(\theta_i^{J,*}+1)x} \nu(dx)}{\int_{\mathbb{R}} e^{\theta_i^{J,*} x} \nu(dx)} = 1 \tag{5.71}$$

As

$$\int_{\mathbb{R}} e^{\theta_i^{J,*} x} \nu(dx) = \frac{1}{\sqrt{2\pi\sigma_J^2}} \int_{\mathbb{R}} e^{\theta_i^{J,*} x} e^{-\frac{(x-\mu_J)^2}{2\sigma_J^2}} dx = e^{\frac{1}{2}(\sigma_J \theta_i^{J,*})^2 + \theta_i^{J,*} \mu_J} \tag{5.72}$$

we obtain from (5.71) the following equality:

$$e^{\frac{1}{2}(\sigma_J(\theta_i^{J,*}+1))^2 + (\theta_i^{J,*}+1)\mu_J} = e^{\frac{1}{2}(\sigma_J \theta_i^{J,*})^2 + \theta_i^{J,*} \mu_J}. \tag{5.73}$$

The expression for the value of the Esscher transform parameter $\theta_i^{J,*}$ in (5.69) follows immediately from (5.73). Inserting this value of $\theta_i^{J,*}$ into the expression for $\lambda_i^{\theta,J}$ in (5.19) we obtain the formula (5.70).

In the numerical simulations, we assume that the hidden Markov chain has three states: up, down, side-way, and the corresponding rate matrix is calculated using real Forex data for the thirteen-year period: from January 3, 2000 to November 2013. To calculate all probabilities we use the Matlab script (see the Appendix).

5.4 Currency option pricing for Merton jump-diffusion processes

Consider a Markov-modulated Merton jump-diffusion which models the dynamics of the spot FX rate, given by the following stochastic differential equation (in the sequel SDE, see [8]):

$$dS_t = S_{t_-}\left(\mu_t dt + \sigma_t dW_t + (Z_{t_-} - 1)dN_t\right),\ Z_t > 0. \tag{5.74}$$

Here μ_t is the drift parameter; W_t is a Brownian motion, σ_t is the volatility; N_t is a Poisson Process with intensity λ_t, $Z_{t_-} - 1$ is the amplitude of the jumps, given the jump arrival time. The distribution of Z_t has a density $\nu(x), x \in \mathbb{R}$.

The solution of (5.74) is $S_t = S_0 e^{L_t}$, (where S_0 is the spot FX rate at time $t = 0$). Here L_t is given by the formula:

$$L_t = \int_0^t (\mu_s - 1/2\sigma_s^2)ds + \int_0^t \sigma_s dW_s + \int_0^t \log Z_{s_-} dN_s. \tag{5.75}$$

Note, that for the most of well-known distributions (normal, exponential distribution of Z_t, etc) L_t is not a Lévy process (see definition of Lévy process in [34], the condition **L3**), since $\log Z_{t_-} \to -\infty$ for small Z_t, but probability of jumps with even 0 amplitude is a positive constant, depending on a type of distribution. We call the process (5.75) as Merton jump-diffusion process (see [30], Section 2, formulas 2, 3)

There is more than one equivalent martingale measure for this market driven by a Markov-modulated jump-diffusion model. We shall define the regime-switching generalized Esscher

transform to determine a specific equivalent martingale measure.

Using Ito's formula we can derive a stochastic differential equation for the discounted spot FX rate. To define the discounted spot FX rate we need to introduce domestic and foreign riskless interest rates for bonds in the domestic and foreign currency.

The discounted spot FX rate is:

$$S_t^d = \exp\left(\int_0^t (r_s^d - r_s^f)ds\right) S_t, \quad 0 \le t \le T. \tag{5.76}$$

Using (5.76), the differentiation formula, see Elliott et al. (1982, [12]) and the stochastic differential equation for the spot FX rate (5.74) we find the stochastic differential equation for the discounted discounted spot FX rate:

$$dS_{t_-}^d = S_{t_-}^d (r_t^d - r_t^f + \mu_t)dt + S_{t_-}^d \sigma_t dW_t + S_{t_-}^d (Z_{t_-} - 1)dN_t. \tag{5.77}$$

To derive the main results consider the log spot FX rate

$$Y_t = \log\left(\frac{S_t}{S_0}\right).$$

Using the differentiation formula:

$$Y_t = C_t + J_t,$$

where C_t, J_t are the continuous and diffusion part of Y_t. They are given in (5.78), (5.79):

$$C_t = \int_0^t \left(r_s^d - r_s^f + \mu_s\right) ds + \int_0^t \sigma_s dW_s, \tag{5.78}$$

$$J_t = \int_0^t \log Z_{s_-} dN_s. \tag{5.79}$$

The similar to the Theorem 5.1 statement is true in this case.

Theorem 5.5. *For $0 \le t \le T$ density $L_t^{\theta^c,\theta^J}$ of Esscher transform defined in (5.9) is given by*

$$L_t^{\theta^c,\theta^J} = \exp\left(\int_0^t \theta_s^c \sigma_s dW_s - 1/2 \int_0^t (\theta_s^c \sigma_s)^2 ds\right) \times \tag{5.80}$$

$$\exp\left(\int_0^t \theta^J_{s_-}\log Z_{s_-}dN_s - \int_0^t \lambda_s\left(\int_{\mathbb{R}_+} x^{\theta^J_s}\nu(dx)-1\right)ds\right).$$

In addition, the random Esscher transform density $L_t^{\theta^c,\theta^J}$ *(see* (5.9), (5.80)*) is an exponential* $(\mathcal{H}_t)_{0\le t\le T}$ *martingale and admits the following SDE*

$$\frac{dL_t^{\theta^c,\theta^J}}{L_{t_-}^{\theta^c,\theta^J}} = \theta^c_t\sigma_t dW_t + (Z_{t_-}^{\theta^J_{t_-}}-1)dN_t - \lambda_t\left(\int_{\mathbb{R}_+} x^{\theta^J_t}\nu(dx)-1\right)dt. \tag{5.81}$$

Proof of Theorem 5.5. The compound Poisson Process, driving jumps $\sum_0^{N_t}(Z_i-1)$, and the Brownian motion W_t are independent processes. As a result:

$$\mathbb{E}\left[\exp\left(\int_0^t \theta^c_s dC_s + \int_0^t \theta^J_{s_-}dJ_s\right)\middle|\mathcal{F}^\xi_t\right] =$$

$$\mathbb{E}\left[\exp\left(\int_0^t \theta^c_s(\mu_s-1/2\sigma_s^2)ds + \int_0^t \theta^c_s\sigma_s dW_s)\right)\middle|\mathcal{F}^\xi_t\right]\mathbb{E}\left[\exp\left(\int_0^t \theta^J_{s_-}\log Z_{s_-}dN_s\right)\middle|\mathcal{F}^\xi_t\right]. \tag{5.82}$$

Let us calculate:

$$\mathbb{E}\left[\exp\left(\int_0^t \theta^J_{s_-}\log Z_{s_-}dN_s\right)\middle|\mathcal{F}^\xi_t\right].$$

Write

$$\Gamma_t := \exp\left(\int_0^t \alpha_{s_-}dN_s\right), \alpha_s = \theta^J_s\log Z_s.$$

Using the differentiation rule (see [12]) we obtain the following representation of Γ_t:

$$\Gamma_t = \Gamma_0 + M^J_t + \int_{]0,t]}\Gamma_s\int_{\mathbb{R}}(e^{\alpha_s}-1)\nu(dx)\lambda_s ds, \tag{5.83}$$

where

$$M^J_t = \int_{]0,t]}\Gamma_{s_-}(e^{\alpha_s}-1)dNs - \int_{]0,t]}\Gamma_s\int_{\mathbb{R}}(e^{\alpha_s}-1)\nu(dx)\lambda_s ds$$

is a martingale with respect to $\mathcal{F}^\xi_t$. Using this fact and (5.83) we obtain:

$$\mathbb{E}\left[\exp\left(\int_0^t \theta^J_s\log Z_{s_-}dN_s\right)\middle|\mathcal{F}^\xi_t\right] = \exp\left(\int_0^t\lambda_s\left(\int_{\mathbb{R}}x^{\theta^J_s}\nu(dx)-1\right)ds\right). \tag{5.84}$$

We have from the differentiation rule:

$$\mathbb{E}\left[e^{u\int_0^t \sigma_s dW_s}\right] = \exp\left\{\frac{1}{2}u^2\int_0^t \sigma_s^2 ds\right\} \tag{5.85}$$

where σ_t is the volatility of a market. Substituting (5.84) and (5.85) into (5.82) we obtain:

$$\mathbb{E}\left[\exp\left(\int_0^t \theta_s^c dC_s + \int_0^t \theta_{s_-}^J dJ_s\right)\middle|\mathcal{F}_t^\xi\right] =$$

$$\exp\left(\int_0^t \theta_s^c(\mu_s - 1/2\sigma_s^2)ds + \frac{1}{2}\int_0^t (\theta_s^c\sigma_s)^2 ds)\right)\exp\left(\int_0^t \lambda_s\left(\int_{\mathbb{R}_+} x^{\theta_s^J}\nu(dx) - 1\right)ds\right) \tag{5.86}$$

Substituting (5.86) to the expression for $L_t^{\theta^c,\theta^J}$ in (5.9) we have:

$$L_t^{\theta^c,\theta^J} = \exp\left(\int_0^t \theta_s^c(\mu_s - 1/2\sigma_s^2)ds + \int_0^t \theta_s^c\sigma_s dW_s)\right)\exp\left(\int_0^t \theta_{s_-}^J \log Z_{s_-} dN_s\right)\times \tag{5.87}$$

$$\left[\exp\left(\int_0^t \theta_s^c(\mu_s - 1/2\sigma_s^2)ds + \frac{1}{2}\int_0^t (\theta_s^c\sigma_s)^2 ds)\right)\exp\left(\int_0^t \lambda_s\left(\int_{\mathbb{R}_+} x^{\theta_s^J}\nu(dx) - 1\right)ds\right)\right]^{-1} =$$

$$\exp\left(\int_0^t \theta_s^c\sigma_s dW_s - 1/2\int_0^t (\theta_s^c\sigma_s)^2 ds\right)\times$$

$$\exp\left(\int_0^t \theta_{s_-}^J \log Z_{s_-} dN_s - \int_0^t \lambda_s\left(\int_{\mathbb{R}_+} x^{\theta_s^J}\nu(dx) - 1\right)ds\right).$$

If we present $L_t^{\theta^c,\theta^J}$ in the form $L_t^{\theta^c,\theta^J} = e^{X_t}$ (see (5.87)) and and apply differentiation rule we obtain SDE (5.81). It follows from (5.81) that $L_t^{\theta^c,\theta^J}$ is a martingale. □

We shall derive the following condition for the discounted spot FX rate ((5.76)) to be martingale. These conditions will be used to calculate the risk-neutral Esscher transform parameters $(\theta_t^{c,*})_{0\le t\le T}$, $(\theta_t^{J,*})_{0\le t\le T}$ and give to the measure $\mathbf{Q}$. Then we shall use these values to find the no-arbitrage price of European call currency derivatives.

Theorem 5.6. *Let the random Esscher transform be defined by* (5.9)*. Then the martingale condition (for* S_t^d*, see* (5.76)*) holds if and only if Markov modulated parameters* $(\theta_t^c, \theta_t^J, 0 \le t \le T)$ *satisfy for all* $0 \le t \le T$ *the condition:*

$$r_t^f - r_t^d + \mu_t + \theta_t^c\sigma_t^2 + \lambda_t^{\theta,J}k_t^{\theta,J} = 0 \tag{5.88}$$

where the random Esscher transform intensity $\lambda_t^{\theta,J}$ *of the Poisson Process and the main percentage jump size* $k_t^{\theta,J}$ *are respectively given by*

$$\lambda_t^{\theta,J} = \lambda_t \int_{\mathbb{R}_+} x^{\theta_s^J} \nu(dx), \tag{5.89}$$

$$k_t^{\theta,J} = \frac{\int_{\mathbb{R}_+} x^{(\theta_t^J+1)} \nu(dx)}{\int_{\mathbb{R}_+} x^{\theta_t^J} \nu(dx)} - 1 \tag{5.90}$$

as long as $\int_{\mathbb{R}_+} x^{\theta_t^J+1} \nu(dx) < +\infty$.

Proof of Theorem 5.6. The martingale condition for the discounted spot FX rate S_t^d

$$\mathbb{E}^{\theta^c,\theta^J}[S_t^d|\mathcal{H}_u] = S_u^d, \quad t \geq u. \tag{5.91}$$

To derive such a condition Bayes formula is used:

$$\mathbb{E}^{\theta^c,\theta^J}[S_t^d|\mathcal{H}_u] = \frac{\mathbb{E}[L_t^{\theta^c,\theta^J} S_t^d|\mathcal{H}_u]}{\mathbb{E}[L_t^{\theta^c,\theta^J}|\mathcal{H}_u]}, \tag{5.92}$$

taking into account that $L_t^{\theta^c,\theta^J}$ is a martingale with respect to $\mathcal{H}_u$, so:

$$\mathbb{E}\left[L_t^{\theta^c,\theta^J}\middle|\mathcal{H}_u\right] = L_u^{\theta^c,\theta^J}. \tag{5.93}$$

Using formula (5.76) for the solution of the SDE for the spot FX rate, we obtain an expression for the discounted spot FX rate in the following form:

$$S_t^d = S_u^d \exp\left(\int_u^t (r_s^f - r_s^d + \mu_s - 1/2\sigma_s^2)ds + \int_u^t \sigma_s dW_s + \int_u^t \log Z_{s_-} dN_s\right), \quad t \geq u. \tag{5.94}$$

Then, using (5.80), (5.94) we can rewrite(5.93) in the following form:

$$\begin{aligned}\mathbb{E}\left[\frac{L_t^{\theta^c,\theta^J}}{L_u^{\theta^c,\theta^J}} S_t^d\middle|\mathcal{H}_u\right] = S_u^d\, \mathbb{E}\Bigg[&\exp\left(\int_u^t \theta_s^c \sigma_s dW_s - 1/2\int_0^t (\theta_s^c\sigma_s)^2 ds\right) \times \\ &\exp\left(\int_u^t \theta_{s_-}^J \log Z_{s_-} dN_s - \int_u^t \lambda_s\left(\int_{\mathbb{R}_+} x^{\theta_s^J}\nu(dx) - 1\right) ds\right) \times \\ &\exp\left(\int_u^t (r_s^f - r_s^d + \mu_s - 1/2\sigma_s^2)ds + \int_u^t \sigma_s dW_s + \int_u^t \log Z_{s_-} dN_s\right)|\mathcal{H}_u\Bigg] =\end{aligned} \tag{5.95}$$

$$S_u^d\,\mathbb{E}\Big[\exp\Big(\int_u^t(\theta_s^c+1)\sigma_s dW_s-1/2\int_u^t((\theta_s^c+1)\sigma_s)^2ds\Big)\times \tag{5.96}$$
$$\exp\Big(\int_u^t(r_s^f-r_s^d+\mu_s+\theta_s^c\sigma_s^2)ds\Big)\,\exp\Big(\int_u^t\lambda_s\big(\int_{\mathbb{R}}e^{\theta_s^J x}\nu(dx)-1\big)ds\Big)|\mathcal{H}_u\Big]\times$$
$$\mathbb{E}\Big[\exp\Big(\int_u^t(\theta_s^c+1)\log Z_{s_-}dN_s\Big)|\mathcal{H}_u\Big].$$

Using expression for characteristic function of Brownian motion (see (5.85)) we obtain:

$$\mathbb{E}\Big[\exp\Big(\int_u^t(\theta_s^c+1)\sigma_s dW_s-1/2\int_u^t((\theta_s^c+1)\sigma_s)^2ds\Big)|\mathcal{H}_u\Big]=1. \tag{5.97}$$

Using (5.84) we have:

$$\mathbb{E}\Big[\exp\Big(\int_u^t(\theta_s^c+1)\log Z_{s_-}dN_s\Big)|\mathcal{H}_u\Big]=\exp\Big(\int_0^t\lambda_s\Big(\int_{\mathbb{R}_+}x^{(\theta_s^J+1)}\nu(dx)-1\Big)ds\Big). \tag{5.98}$$

Substituting (5.97), (5.98) into (5.96) we obtain finally:

$$\mathbb{E}\Big[\frac{L_t^{\theta^c,\theta^J}}{L_u^{\theta^c,\theta^J}}S_t^d\Big|\mathcal{H}_u\Big]=S_u^d\exp\Big(\int_u^t(r_s^f-r_s^d+\mu_s+\theta_s^c\sigma_s^2)ds\Big)\times \tag{5.99}$$
$$\exp\Big(-\int_u^t\lambda_s\big(\int_{\mathbb{R}_+}x^{\theta_s^J}\nu(dx)-1\big)ds\Big)\,\exp\Big(\int_u^t\lambda_s\big(\int_{\mathbb{R}_+}x^{(\theta_s^J+1)}\nu(dx)-1\big)ds\Big).$$

From (5.99) we get the martingale condition for the discounted spot FX rate:

$$r_t^f-r_t^d+\mu_t+\theta_t^c\sigma_t^2+\lambda_t\Big[\int_{\mathbb{R}_+}x^{(\theta_s^J+1)}\nu(dx)-\int_{\mathbb{R}_+}x^{\theta_s^J}\nu(dx)\Big]=0. \tag{5.100}$$

Prove now, that under the Esscher transform the new Poisson process intensity and the mean jump size are given by (5.89), (5.90).

Note that $L_t^J=\int_0^t\log Z_{s_-}dN_s$ is the jump part of Lévy process in the formula (5.75) for the solution of SDE for spot FX rate. We have:

$$\mathbb{E}_{\mathbf{Q}}\left[e^{L_t^J}\right]=\int_\Omega\exp\Big(\int_0^t\log Z_{s_-}dN_s\Big)\,L_t^{\theta^c,\theta^J}(\omega)d\mathbf{P}(\omega), \tag{5.101}$$

where $\mathbf{P}$ is the initial probability measure, $\mathbf{Q}$ is a new risk-neutral measure. Substituting the density of the Esscher transform (5.80) into (5.101) we have:

$$\mathbb{E}_{\mathbf{Q}}\left[e^{L_t^J}\right]=\mathbb{E}_{\mathbf{P}}\Big[\exp\Big(\int_0^t\theta_s^c\sigma_s dW_s-1/2\int_0^t(\theta_s^c\sigma_s)^2ds\Big)- \tag{5.102}$$

$$\int_0^t \lambda_s \left(\int_{\mathbb{R}_+} x^{\theta_s^J} \nu(dx) - 1 \right) ds \Bigg) \Bigg] \mathbb{E}_{\mathbf{P}} \left[\exp\left(\int_0^t (\theta_s^J + 1) \log Z_{s_-} dN_s \right) \right].$$

Using (5.84) we obtain:

$$\mathbb{E}_{\mathbf{P}} \left[\exp\left(\int_0^t (\theta_s^J + 1) \log Z_{s_-} dN_s \right) \right] = \exp\left(\int_0^t \lambda_s \left(\int_{\mathbb{R}_+} x^{(\theta_s^J + 1)} \nu(dx) - 1 \right) ds \right). \tag{5.103}$$

Putting (5.103) to (5.102) and taking into account characteristic function of Brownian motion (see (5.85)) we have:

$$\mathbb{E}_{\mathbf{Q}} \left[e^{L_t^J} \right] = \exp\left(\int_0^t \lambda_s \left(\int_{\mathbb{R}_+} x^{\theta_s^J} \nu(dx) \left[\frac{\int_{\mathbb{R}_+} x^{(\theta_s^J + 1)} \nu(dx)}{\int_{\mathbb{R}_+} x^{\theta_s^J} \nu(dx)} - 1 \right] \right) ds \right). \tag{5.104}$$

Return to the initial measure $\mathbf{P}$, but with different $\lambda_t^{\theta,J}, k_t^{\theta,J}$. We obtain:

$$\mathbb{E}_{\tilde{\lambda},\tilde{\nu}} \left[e^{L_t^J} \right] = \exp\left(\int_0^t \lambda_s^{\theta,J} \left(\int_{\mathbb{R}_+} x \tilde{\nu}(dx) - 1 \right) ds \right). \tag{5.105}$$

Formula (5.89) for the new intensity $\lambda_t^{\theta,J}$ of the Poisson process follows directly from (5.104), (5.105). The new density of jumps $\tilde{\nu}$ is defined from (5.104), (5.105) by the following formula:

$$\frac{\int_{\mathbb{R}_+} x^{(\theta_t^J + 1)} \nu(dx)}{\int_{\mathbb{R}_+} x^{\theta_t^J} \nu(dx)} = \int_{\mathbb{R}_+} x \tilde{\nu}(dx). \tag{5.106}$$

Calculate now the new mean jump size given jump arrival with respect to the new measure $\mathbf{Q}$:

$$k_t^{\theta,J} = \int_\Omega (Z(\omega) - 1) d\tilde{\nu}(\omega) = \int_{\mathbb{R}_+} (x - 1) \tilde{\nu}(dx) =$$
$$\int_+ x \tilde{\nu}(dx) - 1 = \frac{\int_{\mathbb{R}_+} x^{(\theta_t^J + 1)} \nu(dx)}{\int_{\mathbb{R}_+} x^{\theta_t^J} \nu(dx)} - 1. \tag{5.107}$$

So, we can rewrite martingale condition for the discounted spot FX rate in the form in (5.88), where $\lambda_t^{\theta,J}, k_t^{\theta,J}$ are given by (5.89), (5.90) respectively. □

Using (5.100) we have the following formulas for the families of the regime switching parameters satisfying the martingale condition (5.88):

$$\theta_t^{c,*} = \frac{K_0 + r_t^d - r_t^f - \mu_t}{\sigma_t^2}, \tag{5.108}$$

$$\theta_t^{J,*} : \int_{\mathbb{R}_+} x^{(\theta_t^{J,*}+1)}\nu(dx) - \int_{\mathbb{R}_+} x^{\theta_t^{J,*}}\nu(dx) = \frac{K_0}{\lambda_t}, \tag{5.109}$$

where K_0 is any constant. Note again, that the choice for these parameters is not unique.

In the next section we shall apply these formulas (5.108), (5.109) to the exponential distribution of jumps.

5.5 Currency option pricing for exponential processes

Because of the restriction $Z_{s_-} > 0$ we can not consider a double-exponential distribution of jumps (see [25], [28]) in $\int_0^t \log Z_{s_-} dN_s$. Let us consider exponential distribution instead. It is defined by the following formula of density function:

$$\nu(x) = \theta e^{-\theta x}\Big|_{x\geq 0}. \tag{5.110}$$

The mean value of this distribution is:

$$\text{mean}(\theta) = \frac{1}{\theta}. \tag{5.111}$$

The variance of this distribution is:

$$\text{var}(\theta) = \frac{1}{\theta^2}. \tag{5.112}$$

The exponential distribution like the double-exponential distribution has also memorylessness property.

Let us derive the martingale condition and formulas for the regime-switching Esscher transform parameters in case of jumps driven by the exponential distribution. Using the martingale condition for discounted spot FX rate (5.100) we obtain:

$$r_t^f - r_t^d + \mu_t + \theta_t^c\sigma_t^2 + \lambda_t\left[\frac{\Gamma(\theta_t^J+2)}{\theta^{\theta_t^J+1}} - \frac{\Gamma(\theta_t^J+1)}{\theta^{\theta_t^J}}\right] = 0, \tag{5.113}$$

where we have such a restriction (and in the sequel): $\theta_t^J > -1$.

Using (5.89), (5.90) the random Esscher transform intensity $\lambda_t^{\theta,J}$ of the Poisson Process and the main percentage jump size $k_t^{\theta,J}$ are respectively given by

$$\lambda_t^{\theta,J} = \lambda_t \frac{\Gamma(\theta_t^J + 1)}{\theta^{\theta_t^J}}, \tag{5.114}$$

$$k_t^{\theta,J} = \frac{\theta_t^J + 1}{\theta} - 1. \tag{5.115}$$

Using (5.109) we have the following formula for the families of regime switching Esscher transform parameters satisfying martingale condition (5.113):

$$\theta_t^{J,*} : \left[\frac{\Gamma(\theta_t^{J,*} + 2)}{\theta^{\theta_t^{J,*}+1}} - \frac{\Gamma(\theta_t^{J,*} + 1)}{\theta^{\theta_t^{J,*}}}\right] = \frac{K_0}{\lambda_t}. \tag{5.116}$$

Let us simplify (5.116):

$$\theta_t^{J,*} : \frac{\Gamma(\theta_t^{J,*} + 1)}{\theta^{\theta_t^{J,*}}}\left(\frac{\theta_t^{J,*} + 1}{\theta} - 1\right) = \frac{K_0}{\lambda_t}. \tag{5.117}$$

The formula for $\theta_t^{c,*}$ in this case is the same as in (5.108).

With respect to to such values of the regime switching Esscher transform parameters we have from (5.114), (5.115), (5.117):

$$k_t^{J,*} = K_0/\lambda_t^{J,*}. \tag{5.118}$$

When we proceed to a new risk-neutral measure $\mathbf{Q}$ we have the new $\tilde{\theta}$ in (5.110). Using (5.106) we obtain:

$$\tilde{\theta} = \frac{\theta}{\theta_t^J + 1}. \tag{5.119}$$

From (5.119) we arrive at an interesting conclusion: $\tilde{\theta}$ depends on time t. So, now the distribution of jumps changes depending on time (it was not the case before for the log double-exponential distribution, where $\tilde{\theta}$ was actually a constant, see [8]). So, the compound Poisson Process depends not only on a number of jumps, but on moments of time when they arrive in this case. The same statement is true for the mean jump size in (5.115). But the pricing formulas (5.41)-(5.49) are applicable to this case as well.

Chapter 6

Numerical Results

In the numerical simulations, we assume that the hidden Markov chain has three states: up, down, side-way, and corresponding probability matrix is calculated using real Forex data for a thirteen-year period: from January 3, 2000 to November 2013. To calculate all probabilities we use Matlab script:

```
Probab_matrix_calc1(candles_back_up,candles_back_down, delta_back_up,

delta_back_down,candles_up,candles_down, delta_up, delta_down )
```

(see Appendix for details).

6.1 Log-double exponential distribution

In the Figures 6.1-6.3 we shall provide numerical simulations for the case when the amplitude of jumps is described by a log-double exponential distribution. These three graphs show a dependence of the European-call option price against S/K, where S is the initial spot FX rate, K is the strike FX rate for a different maturity time T in years: 0.5, 1, 1.2. We use the following function in Matlab:

```
Draw( S_0,T,approx_num,steps_num, teta_1,teta_2,p,mean_normal,sigma_normal)
```

to draw these graphs[1]. The arguments of this function are: S_0 is the starting spot FX rate to define first point in S/K ratio, T is the maturity time, approx num describes the number of attempts to calculate the mean for the integral in the European call option pricing formula (see Section 2, (5.49)), steps num denotes the number of time subintervals

[1]Matlab scripts for all plots are available upon request

to calculate the integral in (5.49); teta 1, teta 2, p are θ_1, θ_2, p parameters in the log-double-exponential distribution (see Section 3, (5.51)). mean normal, sigma normal are the mean and deviation for the log-normal distribution (see Section 4). In these three graphs: $\theta_1 = 10, \theta_2 = 10, p = 0.5$; mean normal $= 0$, sigma normal $= 0.1$.

Blue line denotes the log-double exponential, green line denotes the log-normal, red-line denotes the plot without jumps. The S/K ratio ranges from 0.8 to 1.25 with a step 0.05; the option price ranges from 0 to 1 with a step 0.1. The number of time intervals: num =10.

From these three plots we conclude that it is important to incorporate jump risk into the spot FX rate models. Described by Black-Scholes equations without jumps, red line on a plot is significantly below both blue and green lines which stand for the log-double exponential and the log-normal distributions of jumps, respectively.

All three plots have the same mean value 0 and approximately equal deviations for both types of jumps: log-normal and log-double exponential. We investigate the case when it does not hold (see Figures 6.4-6.6).

As we can see, the log-double exponential curve moves up in comparison with the log-normal and without jumps option prices.

If we fix the value of the θ_2 parameter in the log-double exponential distribution with $S/K = 1$ the corresponding plot is given in Figure 6.7.

Figure 6.8 represents a plot of the dependence of the European-call option price against values of the parameters θ_1, θ_2 in a log-double exponential distribution, again $S/K = 1$.

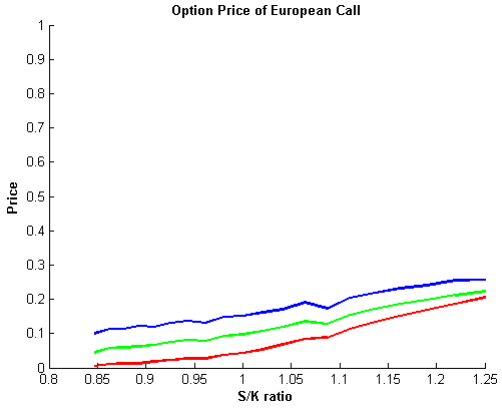


Figure 6.1: $S_0 = 1, T = 0.5, \theta_1 = 10, \theta_2 = 10, p = 0.5, \text{mean normal} = 0, \text{sigma normal} = 0.1$

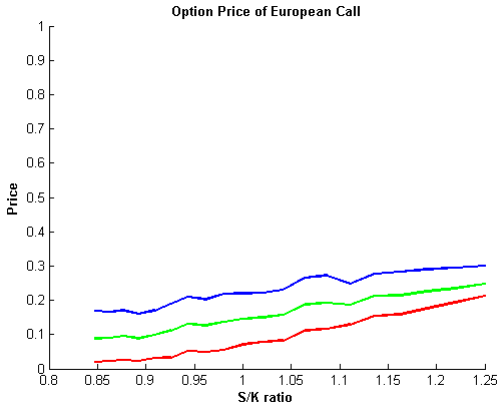


Figure 6.2: $S_0 = 1, T = 1.0, \theta_1 = 10, \theta_2 = 10, p = 0.5, \text{mean normal} = 0, \text{sigma normal} = 0.1$

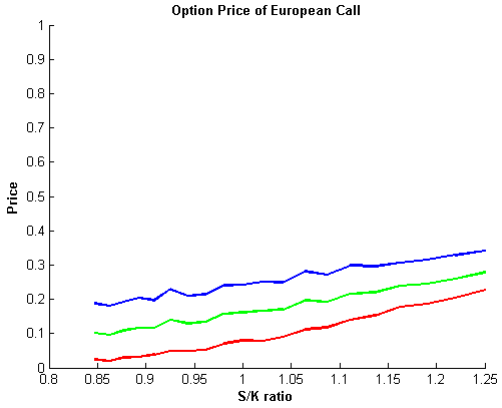


Figure 6.3: $S_0 = 1, T = 1.2, \theta_1 = 10, \theta_2 = 10, p = 0.5, \text{mean normal} = 0, \text{sigma normal} = 0.1$

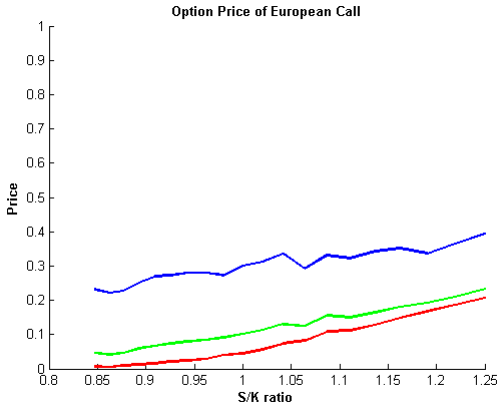


Figure 6.4: $S_0 = 1, T = 0.5, \theta_1 = 5, \theta_2 = 10, p = 0.5, \text{mean normal} = 0, \text{sigma normal} = 0.1$

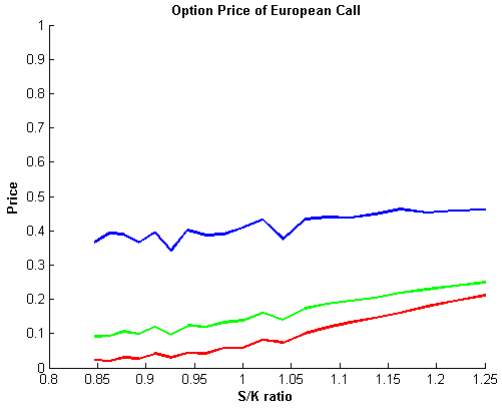


Figure 6.5: $S_0 = 1, T = 1.0, \theta_1 = 5, \theta_2 = 10, p = 0.5, \text{mean normal} = 0, \text{sigma normal} = 0.1$

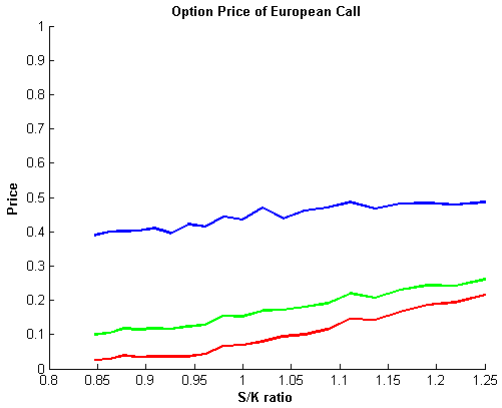


Figure 6.6: $S_0 = 1, T = 1.2, \theta_1 = 5, \theta_2 = 10, p = 0.5, \text{mean normal} = 0, \text{sigma normal} = 0.1$

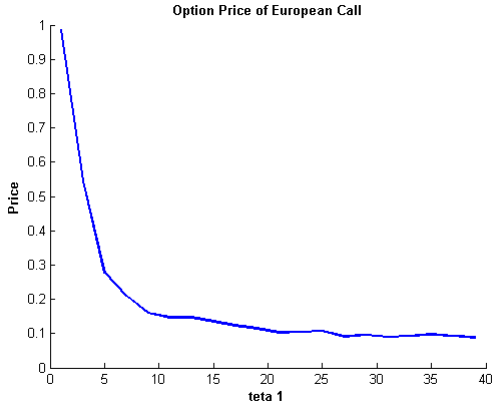


Figure 6.7: $S_0 = 1, T = 0.5, \theta_2 = 10, p = 0.5$

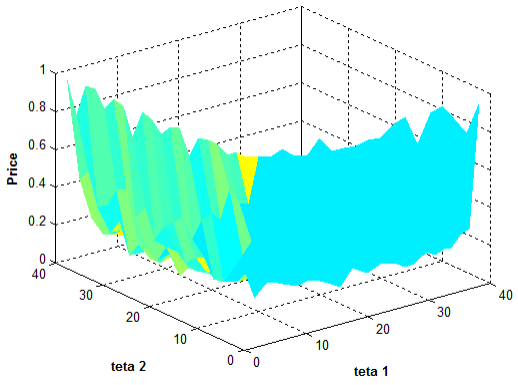
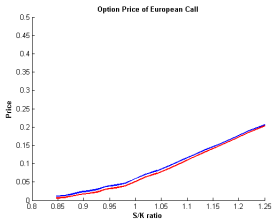


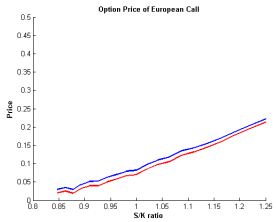
Figure 6.8: Option price of European Call: $S_0 = 1, T = 0.5, p = 0.5$

6.2 Exponential distribution

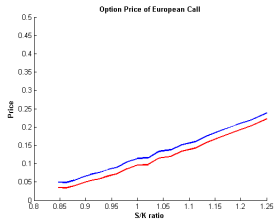
In the following figures we shall provide numerical simulations for the case when amplitude of jumps is described by the exponential distribution. These plots show the dependence of a European-call option price on S/K, where S is the initial spot FX rate ($S = 1$ in our simulations), K is a strike FX rate for various maturity times T : 0.5, 1, 1.5 in years and various values of a parameter θ : 2.5, 3.5, 5 in the exponential distribution. The blue line stands for the exponential distribution of jumps, the red-line is for the dynamics without jumps. From these plots we can make a conclusion that it is important to incorporate a jump risk into the spot FX rate models (described by the Black-Scholes equation without jumps red line on a plot is below the blue line standing for the exponential distributions of jumps).



$T = 0.5,\ \theta = 5$

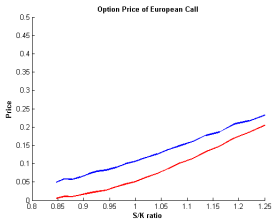


$T = 1.0,\ \theta = 5$

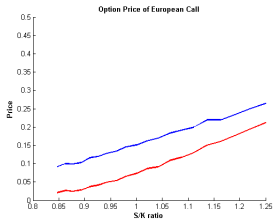


$T = 1.5,\ \theta = 5$

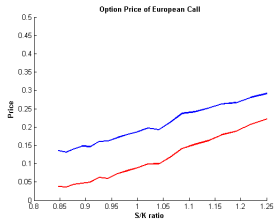
Figure 6.9: Option price of European Call: $\theta = 5$



$T = 0.5,\ \theta = 3.5$

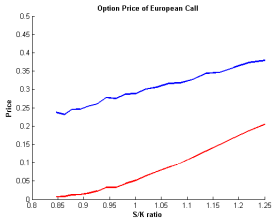


$T = 1.0,\ \theta = 3.5$

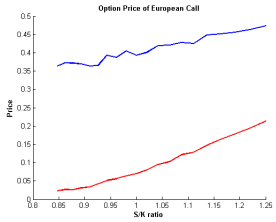


$T = 1.5,\ \theta = 3.5$

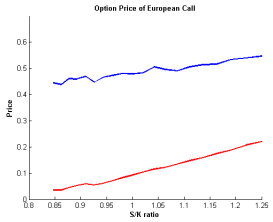
Figure 6.10: Option price of European Call: $\theta = 3.5$



$T = 0.5,\ \theta = 2.5$



$T = 1.0,\ \theta = 2.5$



$T = 1.5,\ \theta = 2.5$

Figure 6.11: Option price of European Call: $\theta = 2.5$

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Chapter 7

Apendix

Matlab functions for numerical simulations of currency option pricing:

```
Markov_chain.m
function [ alpha_m, sigma_m, lambda_m, int_rate_d_m, int_rate_f_m, time_matrix ]
 =Markov_chain( steps_num)
alpha_state=[0.05 0.02 0.03];
sigma_state=[0.2 0.6 0.2];
lambda_state=[10 20 30];
int_rate_d_state=[0.05 0.03 0.02];
int_rate_f_state= [0.04 0.02 0.01];
trans=[0.5 0.3 0.2;0.25 0.5 0.25; 0.2 0.3 0.5 ];

emis_matrix=1/3*ones(3, 3);
[seq,states] = hmmgenerate(steps_num,trans,emis_matrix);
time_matrix=zeros(3,1);
for i=1:steps_num
    time_matrix(states(i))=time_matrix(states(i))+1;
end
for i=1:3
    time_matrix(i)= time_matrix(i)/steps_num;
end
for i=1:steps_num
    alpha_m(i)=alpha_state(states(i));
```

```
    sigma_m(i)=sigma_state(states(i));
    lambda_m(i)=lambda_state(states(i));
    int_rate_d_m(i)=int_rate_d_state(states(i));
    int_rate_f_m(i)=int_rate_f_state(states(i));
end
end
```

Additional_Param_double_exp.m

```
function [V, R,Price_normal, Price,Price_no_jumps] =
Additional_Param_double_exp(step_num,  time_matrix, teta_1,
 teta_2, p,mean_normal,sigma_normal,  S, K, T )
A=p*teta_1-(1-p)*teta_2;
B=p*teta_1+2*teta_1*teta_2-(1-p)*teta_2;
C=p*teta_1*teta_2^2+p*teta_2*teta_1^2-teta_2*teta_1^2+teta_2*teta_1;

discrimin=B^2-4*A*C;
if A==0
    teta_jump=-C/B;
else
    temp=(-B-sqrt(discrimin))/(2*A);
    if (temp>-teta_2)&&(temp<teta_1);
        teta_jump=temp;
    else
        temp=(-B+sqrt(discrimin))/(2*A);
        teta_jump=temp;
```

```
    end
end
%teta_jump=-0.5;
%teta_jump=round(teta_jump*10000)/10000;

sigma_state=[0.2 0.6 0.2];
lambda_state=[10 20 30];
int_rate_d_state=[0.05 0.03 0.02];
int_rate_f_state= [0.04 0.02 0.01];

lambda_m_new=
lambda_state*(p*teta_1/(teta_1-teta_jump)+(1-p)*teta_2/(teta_2+teta_jump));
lambda_m_new_normal=
lambda_state*exp(-mean_normal^2/sigma_normal^2+sigma_normal^2/8);
time_matrix_trans=time_matrix';

R_T=sum((int_rate_d_state- int_rate_f_state).*time_matrix_trans);
U_T=sum(sigma_state.^2.*time_matrix_trans);
lambda_T=sum(lambda_m_new.*time_matrix_trans);
lambda_T_normal=sum(lambda_m_new_normal.*time_matrix_trans);
%calculate mean and variance for jump with double exponential disribution
mean_jump=p/teta_1-(1-p)/teta_2;
var_jump=2*p/teta_1^2+2*(1-p)/teta_2^2-mean_jump^2;
for i=1:200
    V(i)=U_T+(i-1)*var_jump/T;
    V_normal(i)=U_T+(i-1)*sigma_normal^2/T;
```

```
    R(i)=R_T;
end;
[Call1, Put]=blsprice(S, K, R(1), T, V(1));
Price_no_jumps=Call1;
Price=0;
Price_normal=0;
for m=0:199
    [Call, Put] = blsprice(S, K, R(m+1), T, V(m+1));
    [Call_normal, Put_normal] = blsprice(S, K, R(m+1), T, V_normal(m+1));
    Price=Price+exp(-T*lambda_T)*(T*lambda_T)^m/factorial(m)*Call;
    Price_normal=Price_normal+
    exp(-T*lambda_T_normal)*(T*lambda_T_normal)^m/factorial(m)*Call_normal;
end
end
```

Price_real_double_exponential.m

```
function [Price_Real_normal, Price_Real, Price_Real_no_jumps ] =
Price_real_double_exponential(S, K, T, approx_num,
steps_num, teta_1, teta_2, p)
Price_Real_normal=0;
Price_Real=0;
Price_Real_no_jumps=0;
trans=[0.5 0.3 0.2;0.25 0.5 0.25; 0.2 0.3 0.5 ];
for j=1:approx_num
  [ alpha_m, sigma_m, lambda_m, int_rate_d_m, int_rate_f_m, time_matrix] =
```

```
  Markov_chain(steps_num);
  [V, R, lambda_m_new,Price_normal, Price, Price_no_jumps] =
  Additional_Param_double_exp(10,  time_matrix, teta_1, teta_2, p, S,K,T );
  Price_Real_normal=Price_Real_normal+
  Price_Real=Price_Real+Price;
  Price_Real_no_jumps= Price_Real_no_jumps+Price_no_jumps;
end
Price_Real=Price_Real/approx_num;
Price_Real_no_jumps=Price_Real_no_jumps/approx_num;
end
```

Draw.m

```
function [ output_args ] =
Draw( S,T,approx_num,steps_num, teta_1,teta_2,p,mean_normal,sigma_normal)
points_num=20;
K_array=S*ones(1,points_num);
for i=1:points_num
    K_array(i)=K_array(i)*(1.2-i/50);
end;
Price_array_normal=S*ones(1,points_num);
Price_array=S*ones(1,points_num);
Price_array_no_jumps=S*ones(1,points_num);
for i=1:points_num
    [Price_Real_normal,Price_Real,  Price_Real_no_jumps]=
    Price_real_double_exponential(S, K_array(i),
```

```
     T,approx_num,steps_num, teta_1, teta_2, p,mean_normal,sigma_normal);
     Price_array_normal(i)=Price_Real_normal;
     Price_array(i)=Price_Real;
     Price_array_no_jumps(i)=Price_Real_no_jumps;
end;
inv_array=ones(1,points_num);
for i=1:points_num
    inv_array(i)=S/K_array(i);
end;

hold on;
plot1=plot(inv_array,Price_array_normal,'g-','LineWidth',2);
plot2=plot(inv_array,Price_array,'b-','LineWidth',2);
plot3=plot(inv_array,Price_array_no_jumps,'r-','LineWidth',2);
ylim([0 3]);
hold off;
title('Option Price of European Call','FontWeight','bold');
xlabel('S/K ratio','FontWeight','bold');
ylabel('Price','FontWeight','bold');
end
```

Draw_teta_1.m

```
function [ output_args ] = Draw_teta_1( S,T,approx_num,steps_num,
teta_1, teta_2,p,mean_normal,sigma_normal)
points_num=20;
step_teta_1=2;
```

```
K_array=S*ones(1,points_num);
teta_array=S*ones(1,points_num);

for i=1:points_num
    teta_array(i)=teta_1+step_teta_1*(i-1);
end
Price_array=S*ones(1,points_num);
for i=1:points_num
    [Price_Real_normal,Price_Real,  Price_Real_no_jumps]=
    Price_real_double_exponential(S, K_array(i),

    T,approx_num,steps_num, teta_1+step_teta_1*(i-1),
     teta_2, p,mean_normal,sigma_normal);
    Price_array(i)=Price_Real;
end;

plot2=plot(teta_array,Price_array,'b-','LineWidth',2);
ylim([0 1]);
title('Option Price of European Call','FontWeight','bold');
xlabel('teta_1','FontWeight','bold');
ylabel('Price','FontWeight','bold');

end
```

Draw_teta_1_3d.m

```
function [ X, Y,Price_array ] =
 Draw_teta_1_3d( S,T,approx_num,steps_num,teta_1, teta_2,p,mean_normal,sigma_normal)
points_num=20;
step_teta_1=2;
step_teta_2=2;

K_array=S*ones(1,points_num);

for i=1:points_num
    teta_1_array(i)=teta_1+step_teta_1*(i-1);
    teta_2_array(i)=teta_2+step_teta_2*(i-1);
end

Price_array=S*ones(points_num,points_num);

for i=1:points_num
    for j=1:points_num
    [Price_Real_normal,Price_Real,  Price_Real_no_jumps]=
    Price_real_double_exponential(S, K_array(i), T,
    approx_num,steps_num, teta_1+step_teta_1*(i-1),
     teta_2+step_teta_2*(j-1), p,mean_normal,sigma_normal);
     Price_array(i,j)=Price_Real;
    end
end;
[X Y]=meshgrid( teta_1_array,  teta_2_array);
plot2=surfl(X,Y,Price_array);
```

```
xlabel('teta_1','FontWeight','bold');
ylabel('teta_2','FontWeight','bold');
zlabel('Price','FontWeight','bold');
shading interp
colormap(blue);
title('Option Price of European Call','FontWeight','bold');
end
```

Draw_teta.m

```
function [ teta_array, Price_array] = Draw_teta( S,T,approx_num,steps_num, teta,K_0)
points_num=40;
K_array=S*ones(1,points_num);
Price_array=S*ones(1,points_num);
for i=1:points_num
    [Price_Real,  Price_Real_no_jumps, teta_exp_real]=
    Price_real_double_exponential_teta(S, K_array(i), T,approx_num,steps_num, teta,K_0);
    Price_array(i)=Price_Real;
    teta_array(i)=teta_exp_real;
end;
for i = 2:points_num
    key = teta_array(i);
    key1=Price_array(i);
    j = i - 1;
    while j >= 1&& teta_array(j) > key
        teta_array(j+1)= teta_array(j);
```

```
        Price_array(j+1)=Price_array(j);
        j = j - 1;
    end
    teta_array(j+1)= key;
    Price_array(j+1)=key1;

end

plot2=plot(teta_array, Price_array,'b-','LineWidth',2);

ylim([0 0.1]);

title('Option Price of European Call','FontWeight','bold');
xlabel('Mean  Esscher transform theta','FontWeight','bold');
ylabel('Price','FontWeight','bold');
end
```

Additional Functions used in this script Draw_teta.m

Price_real_double_exponential_teta.m

```
function [ Price_Real, Price_Real_no_jumps, teta_exp_real ] =

 Price_real_double_exponential_teta(S, K, T, approx_num,steps_num, teta, K_0)
Price_Real=0;
Price_Real_no_jumps=0;
```

```
teta_exp_real=0;
trans=[0.5 0.3 0.2;0.25 0.5 0.25; 0.2 0.3 0.5 ];
for j=1:approx_num
  [ alpha_m, sigma_m, lambda_m, int_rate_d_m, int_rate_f_m, time_matrix] =
  Markov_chain(steps_num);
  [V, R,  Price, Price_no_jumps, teta_exp] =
  Additional_Param_double_exp_teta(10,  time_matrix, teta, K_0, S,  K, T);
  Price_Real=Price_Real+Price;
  Price_Real_no_jumps= Price_Real_no_jumps+Price_no_jumps;
  teta_exp_real=teta_exp_real+teta_exp;
end
Price_Real=Price_Real/approx_num;
Price_Real_no_jumps=Price_Real_no_jumps/approx_num;
teta_exp_real=teta_exp_real/approx_num;
end
```

Additional_Param_double_exp_teta.m

```
function [V, R, Price,Price_no_jumps,teta_exp] =
Additional_Param_double_exp_teta(step_num,  time_matrix, teta, K_0,  S, K, T )
sigma_state=[0.2 0.6 0.2];
lambda_state=[10 20 30];
int_rate_d_state=[0.05 0.03 0.02];
int_rate_f_state= [0.04 0.02 0.01];
syms x;
for i=1:3
```

```
teta_J(i)=double(solve(gamma(x+1)/(teta^x)*((x+1)/teta-1)-K_0/lambda_state(i), x));
k_new(i)=(teta_J(i)+1)/teta-1;
lambda_m_new(i)=K_0/((teta_J(i)+1)/teta-1);
end;
time_matrix_trans=time_matrix';
teta_exp=sum(teta_J.*time_matrix_trans);
R_T=sum((int_rate_d_state- int_rate_f_state).*time_matrix_trans);
U_T=sum(sigma_state.^2.*time_matrix_trans);
lambda_T=sum(lambda_m_new.*time_matrix_trans);
lambda_T_neutral=sum(lambda_m_new.*(k_new+1).*time_matrix_trans);
%calculate mean and variance for jump with exponential disribution
mean_jump=1/teta;
var_jump=1/teta^2;
for i=1:200
    V(i)=U_T+(i-1)*var_jump/T;
    R(i)=R_T-K_0+(i-1)*sum(log(1+k_new).*time_matrix_trans);
end;
[Call1, Put]=blsprice(S, K, R(1), T, V(1));
Price_no_jumps=Call1;

Price=0;
for m=0:199
    [Call, Put] = blsprice(S, K, R(m+1), T, V(m+1));
    Price=Price+exp(-T*lambda_T_neutral)*(T*lambda_T_neutral)^m/factorial(m)*Call;
end
end
```

Matlab function to calculate probability matrix for Markov chain modeling cross rates of currency pairs in Forex market.

We assume that Markov chain has only three states: "trend up", "trend down", "trend sideway". Such choice of states is justified by numerous articles for FX market(see www.mql5.com). In a file MaxDataFile open.CSV there are open prices of EURO/ESD currency pair of japanese candles over 13 year period. This file was generated in the platform MT5 using MQL5 programming language.

```
function [ Probab_matrix ] =Probab_matrix_calc1(candles_back_up, candles_back_down,
delta_back_up, delta_back_down, candles_up,candles_down, delta_up, delta_down )
Probab_matrix=zeros(3,3);
m_open=csvread('MaxDataFile_open.CSV');
[size_open temp]=size(m_open);
m_before=zeros(1,size_open);
upper_border=size_open-max(candles_up, candles_down);
delta_up=delta_up/10000;
delta_down=delta_down/10000;
count_up=0;
count_down=0;
count_sideway=0;
beforeborder=max(candles_back_up, candles_back_down)+1;
for i=beforeborder:size_open
    if (m_open(i)-m_open(i-candles_back_up)>=delta_up)
        m_before(i)=1;
    end
     if (m_open(i-candles_back_down)-m_open(i)>=delta_down)
```

```
        m_before(i)=-1;
     end
end;
for i=1:upper_border
    if(m_before(i)==1)
        if(m_open(i+candles_up)-m_open(i)>=delta_up)
        Probab_matrix(1,1)= Probab_matrix(1,1)+1;
        else
            if(m_open(i)-m_open(i+candles_down)>=delta_down)
                Probab_matrix(1,2)= Probab_matrix(1,2)+1;
            else
                Probab_matrix(1,3)= Probab_matrix(1,3)+1;
            end
        end
    end
    if(m_before(i)==-1)
        if(m_open(i+candles_up)-m_open(i)>=delta_up)
        Probab_matrix(2,1)= Probab_matrix(2,1)+1;
        else
            if(m_open(i)-m_open(i+candles_down)>=delta_down)
                Probab_matrix(2,2)= Probab_matrix(2,2)+1;
            else
                Probab_matrix(2,3)= Probab_matrix(2,3)+1;
            end
        end
    end
```

```
    if(m_before(i)==0)
        if(m_open(i+candles_up)-m_open(i)>=delta_up)
        Probab_matrix(3,1)= Probab_matrix(3,1)+1;
        else
            if(m_open(i)-m_open(i+candles_down)>=delta_down)
                Probab_matrix(3,2)= Probab_matrix(3,2)+1;
            else
                Probab_matrix(3,3)= Probab_matrix(3,3)+1;
            end
        end
    end
end
count_up=sum(Probab_matrix(1,:));
count_down=sum(Probab_matrix(2,:));
count_sideway=sum(Probab_matrix(3,:));
for j=1:3
      Probab_matrix(1,j)= Probab_matrix(1,j)/count_up;
      Probab_matrix(2,j)= Probab_matrix(2,j)/count_down;
      Probab_matrix(3,j)= Probab_matrix(3,j)/count_sideway
end
end
```

For example run in Matlab:

```
[ Probab_matrix ] = Probab_matrix_calc1(30, 30, 10, 10, 30, 30, 10, 10);
```

Probability matrix is as follows:

$$\begin{pmatrix} \text{up} & \text{down} & \text{sideway} \\ 0.4408 & 0.4527 & 0.1065 \\ 0.4818 & 0.4149 & 0.1033 \\ 0.4820 & 0.4119 & 0.1061 \end{pmatrix}$$