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Motivic Classification of Regular Equivalued Orbits in the Exceptional Group $G(2)$

by

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Abstract

We exhibit a motivic parameterization of conjugacy classes of equivalued regular semisimple elements in the Lie algebra of the exceptional group $G(2)$ over local fields with residual characteristic at least 5.

Acknowledgements

Many lessons are learned and much personal growth occurs during a PhD program, but two insights seem appropriate to share here. First, completing a PhD is not only a result of intellectual prowess. What carries a student through to the end are traits common to those who successfully complete any challenging endeavour: perseverance, self-discipline, courage, fortitude, stamina, faith – an ‘unbending intent’ to succeed. Second, and most importantly, is the realization that a PhD is not an individual effort, but a group venture merely spearheaded by the recipient of the degree. It is from this perspective that I would like to specifically acknowledge a few of the most important members of my “group”, knowing there are many more whose contributions, big or small, have been essential. To these I offer my heartfelt gratitude.

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Dedication

To my parents

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$\check{X}$	cocharacter lattice	1
$<,>$	pairing $\check{\Lambda} \times \Lambda \to \mathbb{Z}$ given by $<f_i, \epsilon_j> = \delta_{i,j}$	1
R	root system of type G_2	2
Δ	basis $\{\alpha_1, \alpha_2\}$ of the root system R	2
$\{e_1, e_2, e_3\}$	basis of X, image of $\{\epsilon_1, \epsilon_2, \epsilon_3\} \in \Lambda$	2
R_{short}	short roots $\{\alpha_1, \alpha_3, \alpha_5\}$ in R with $\alpha_1 := -e_1$, $\alpha_3 := \alpha_1 + \alpha_2 = -e_2$, $\alpha_5 := 2\alpha_1 + \alpha_2 = e_3$	2
R_{long}	long roots $\{\alpha_2, \alpha_4, \alpha_6\}$ in R with $\alpha_2 := e_1 - e_2$, $\alpha_4 := 3\alpha_1 + \alpha_2 = e_3 - e_1$, $\alpha_6 := 3\alpha_1 + 2\alpha_2 = e_3 - e_2$	2
$\widetilde{\alpha}$	'longest' root with respect to the basis $\Delta = \{\alpha_1, \alpha_2\}$	2
A_n	type of family of root systems for $n \in \mathbb{Z}$	2
$\check{R}$	dual root system of R	2
$\{\check{\alpha}_1, \check{\alpha}_3, \check{\alpha}_5\}$	short roots in $\check{R}$ with $\check{\alpha}_1 := -2f_1 + f_2 + f_3$, $\check{\alpha}_3 := \check{\alpha}_1 + 3\check{\alpha}_2 = f_1 - 2f_2 + f_3$, $\check{\alpha}_5 := 2\check{\alpha}_1 + 3\check{\alpha}_2 = -f_1 - f_2 + 2f_3$	3
$\{\check{\alpha}_2, \check{\alpha}_4, \check{\alpha}_6\}$	long roots in $\check{R}$ with $\check{\alpha}_2 := f_1 - f_2$, $\check{\alpha}_4 := \check{\alpha}_1 + \check{\alpha}_2 = f_3 - f_1$, $\check{\alpha}_5 := \check{\alpha}_1 + 2\check{\alpha}_2 = f_3 - f_2$	3
$\check{\Delta}$	the basis $\{\check{\alpha}_1, \check{\alpha}_2\}$ of the dual root system $\check{R}$	3
$(X, R, \check{X}, \check{R})$	semisimple root datum of type G_2	3
$Q(R)$	root lattice of (lattice generated by) R	4
$Q(\check{R})$	root lattice of $\check{R}$	4
$\langle w_1, w_2 \rangle$	generated by w_1 and w_2	4
W	Weyl group $\langle w_1, w_2 \rangle$ for R	4
s_α	reflection across the hyperplane of root α	4
C_n	cyclic group of order n	4
D_n	dihedral group of order $2n$	4
S_n	symmetric group on n elements	4
V_4	Klein 4-group	4
$\mathfrak{g}^{\text{reg}}$	regular semisimple elements of $\mathfrak{g}$	4
w_1	equal to $s_{\check{\alpha}_1}$	5
w_2	equal to $s_{\check{\alpha}_2}$	5
X_α	Chevalley basis element corresponding to root α	6
$N_{\alpha,\beta}$	structure coefficient corresponding to the ordered pair of roots (α, β)	6
$\{\alpha, \beta\}$	ordered pair of roots in $R \times R$	7
R^+	positive roots in R	7
$\alpha \prec \beta$	if $\beta - \alpha \in R^+$ for roots $\alpha,\ \beta \in R$	7

Symbol	*Definition*	*Page*
$\mathbb{Z}[\mathfrak{g}]$	coordinate ring of $\mathfrak{g}$	9
$\mathfrak{t}_X(K)$	the Cartan subalgebra containing $X \in \mathfrak{g}^{\mathrm{reg}}(K)$	9
good of slope r	$X \in \mathfrak{g}^{\mathrm{reg}}(K)$ is if $\mathrm{ord}_K(\alpha(X)) = r$ for each root of $\mathfrak{g}(K)$ relative to $\mathfrak{t}_X(K)$	9
depth	of X is r if X is good of slope r; see [CCGS11, Def 2.1]	9
$\mathfrak{g}(r, K)$	the set of good elements in $\mathfrak{g}^{\mathrm{reg}}(K)$ of depth r	9
$\mathcal{O}(X)$	orbit of $X \in \mathfrak{g}^{\mathrm{reg}}(K)$	10
$\mathcal{O}_r(X)$	thickened orbit of $X \in \mathfrak{g}^{\mathrm{reg}}(K)$ where the depth of X is r	10
$\mathfrak{t}_X(K)_{r^+}$	the elements of depth strictly greater than r in $\mathfrak{t}_X(K)$	10
$\mathcal{O}_K$	the ring of integers of K	11
K^{int}	elements $x \in K$ with $\mathrm{ord}_K(x) \in \mathbb{Z}$	11
Def_f	the category of definable subassignments over fields containing a field f	12
μ_r	family of maps of definable subassignments	13
$\mu_{r/K}$	specialization of μ_r determined by K	13
S_r	classifies r-reductions of characteristic polynomials of regular equivalued elements $X \in \mathfrak{g}(K)$ of depth r, for each $r \in \frac{1}{6}\mathbb{Z}$	13
$\mathcal{O}_r^{\mathrm{st}}(X)$	thickened stable orbit of $X \in \mathfrak{g}^{\mathrm{reg}}(K)$	13
Spec	spectrum (set of prime ideals) of a ring	14
$\mathfrak{t} := \mathrm{Spec}(\mathbb{Z}[\check{X}])$	Cartan subalgebra $\mathrm{Spec}(\mathbb{Z}[\check{X}])$	14
$\mathbb{Z}[\mathfrak{t}]^W_{\lvert W\rvert}$	elements of $\mathbb{Z}[\mathfrak{t}]$ invariant under the action of W and localized at $\lvert W\rvert$	14
$\mathfrak{t}/W$	another notation for the Steinberg (adjoint) quotient S	14
$Q(\lambda)$	root polynomial $\prod_{\alpha\in R} = (\lambda - \alpha)$	14
$P(\lambda)$	short root polynomial $\prod_{\alpha\in R_{\mathrm{short}}} (\lambda - \alpha)$	15
s_1	coefficient of $P(\lambda)$ with $s_1 = e_1^2 e_2^2 e_3^3$	15
s_2	coefficient of $P(\lambda)$ with $s_2 = e_1e_2 + e_2e_3 + e_3e_1$	15
$P'(\lambda)$	long root polynomial $\prod_{\alpha\in R_{\mathrm{long}}} (\lambda - \alpha)$	15
$P_X(\lambda) := P_s(\lambda)$	if $(s_1, s_2) = s = \mu(X)$	16
s'_1	coefficient of $P'(\lambda)$ with $s'_1 = (e_1 - e_2)^2(e_2 - e_3)^2(e_3 - e_1)^2 = -(27s_1 + 4s_2^3)$	16
s'_2	coefficient of $P'(\lambda)$ with $s'_2 = 3(e_1e_2 + e_2e_3 + e_3e_1) = 3s_2$	16

Symbol	*Definition*	*Page*
d	discriminant of $P(\lambda)$ and $\mathfrak{t}$	16
$\mathfrak{g}/G$	another notation for the Steinberg (adjoint) quotient $S \cong \mathfrak{t}/W$	17
D	any pre-image of $d \in \mathbb{Z}[\mathfrak{t}]$ under $\mathbb{Z}[\mathfrak{g}] \to \mathbb{Z}[\mathfrak{t}]$	17
$\mu : \mathfrak{g}^{\mathrm{reg}} \to S$	restriction of $\mathfrak{g} \to \mathfrak{t}/W$ to $\mathfrak{g}^{\mathrm{reg}}$	17
$\bar{K}$	separable closure of the field K	17
$\mathcal{O}_s(K)$	stable orbit in $\mathfrak{g}(K)$ related to $s \in S$	17
stable orbit	adjoint conjugacy class over $\bar{K}$	17
r-reduction	process introduced in [CH04, §3.1] taking $P(\lambda) \in K[\lambda]$ to $R(\lambda) \in k[\lambda]$ such that $K[\lambda]/(P(\lambda))$ depends only on R	17
$\lfloor r \rfloor$	integer part of $r \in \mathbb{Q}$	18
$\{r\}$	fractional part of $r \in \mathbb{Q}$	18
fractional depth	another name for the fractional part of $r \in \mathbb{Q}$	18
$R_r(\lambda)$	r-reduction of $P_X(\lambda)$ for each $X \in \mathfrak{g}(r, K)$	18
Φ_r	the set of 'roots' of $Q_r(\lambda)$	18
$\mathfrak{t}_r$	defined by $\mathbb{Z}[\mathfrak{t}_r]_{\|W\|} = \mathbb{Z}[\Phi_r]$ over $\mathbb{Z}_{\|W\|}$	19
$\mathfrak{t}_r^{\mathrm{reg}}$	defined by $\mathbb{Z}[\mathfrak{t}_r^{\mathrm{reg}}] = \mathbb{Z}[\Phi_r]_d$	19
W_r	quotient of W for which $\mathbb{Z}[\Phi_r]^W = \mathbb{Z}[\Phi_r]^{W_r}$	19
W^r	$\{w \in W \mid w(f) = f, \forall f \in \Phi_r\}$	19
S_r	$\mathfrak{t}_r^{\mathrm{reg}}/W_r$	19
$\mu_{r/K}$	$\mu_{r/K}(X) \in S_r(k)$ is the r-reduction of $P_X(\lambda)$ for $X \in \mathfrak{g}(r, K)$	19
indexed root data	sextuple for tori $(X, \emptyset, \check{X}, \emptyset, \emptyset, \rho)$ where $\rho \in Z^1(K, W)$; see [Spr09, §16.2]	20
$Z^1(K, W)$	set of 1-cocycles of $\bar{K}/K$ over W	20
K^{tr}	a tamely ramified closure of K	20
$H^1_{\mathrm{tr}}(K, W)$	1-cohomology set of K^{tr}/K over W	20
$Q_s(\lambda)$	equals $\prod_{\alpha \in R} (\lambda - \alpha(X')) \in K[\lambda]$ where $X' \in \mathfrak{t}^{\mathrm{reg}}(\bar{K})$ and $s = \mu(X')$	20
K_s	splitting extension of $Q_s(\lambda)$ for each $s \in S_r(k)$	21
ρ_s	tame Galois representation from $\mathrm{Gal}(\bar{K}/K)$ to W for each $s \in S_r^w(k)$	21
$R_s(\lambda)$	r-reduction of $P_s(\lambda)$	21
g	$\deg(R_s)$	23
$R_{s,i}(\lambda)$	irreducible factor of $R_s(\lambda)$ in $k[\lambda]$	23
I_s	index set of irreducible factors in $R_s(\lambda)$	23

Symbol	*Definition*	*Page*
$\dot{R}_{s,i}(\lambda)$	any lift of $R_{s,i}(\lambda)$ under r-reduction	23
g_i	equals $\deg R_{s,i}$ for each $i \in I_s$	23
$K^{(n)}$	unique unramified extension of K of degree $n \in \mathbb{Z}$	23
ζ_i	root of $R_{s,i}(\lambda)$	23
$\dot{\zeta}_i$	lift in $K^{(g_i)}$ of ζ_i	23
I_w	index set of partitions into $\langle w \rangle$-orbits	24
R_i	$\langle w \rangle$-orbit in R for $i \in I_w$	24
S_w	factorization of $\mathfrak{t}^{\mathrm{reg}} \to S$	24
$S_{r,w}$	factorization of $\mathfrak{t}_r^{\mathrm{reg}} \to S_r$	25
$\mu_{r,w} : S_{r,w} \to S_r$	scheme morphism with $w \in W_r$	25
$w \leq w'$	partial order on W_r that implies the existence of a canonical map $S_{r,w'} \to S_{r,w}$ over S_r	25
S_r^w	definable subset given by $S_r^w := \mu_{r,w}(S_{r,w}) \setminus \cup_{w<w'} \mu_{r,w'}(S_{r,w'})$	25
$\rho_s : \mathrm{Gal}(\bar{K}/K) \to W$	tame Galois representation for each $s \in S_r^w(k)$	26
Fr	Frobenius automorphism	26
$\hat{\mathbb{Z}}$	Prüfer ring, equals $\varprojlim \mathbb{Z}/n\mathbb{Z} \cong \prod_p \mathbb{Z}_p$	26
σ	lift of Frobenius	26
$\mathbb{Z}_p$	ring of p-adic integers	28
D_r	depth r 'discriminant' in $\mathbb{Z}[y_1, y_2, y_3]$	31
d_r	equals $\mu_{r,w}(D_r)$	31
$\mu_{r,w}^{\#} : S_r \to S_{r,w}$	pre-image map of $\mu_{r,w}$	34
$D_{r,w}$	equals $\mu_{r,w}^{\#}(d_r)$ for $w \in W_r$	33
(w_2)	conjugacy class of w_2 in W_r	38
ζ_n	primitive nth root-of-unity	53
q	cardinality of the residue class field	54
W_s	equals $\rho_s(\mathrm{Gal}(\bar{K}/K))$	56
$\check{X}^{\mathrm{tr}_{W_s}=0}$	trace 0 elements of $\check{X}$ under the W_s action	61
$\check{X}_{W_s}$	subgroup of $\check{X}^{\mathrm{tr}_{W_s}=0}$ generated by elements $w(y) - y$ for $y \in \check{X}$ and $w \in W_s$	61
$\kappa : S \to \mathfrak{g}^{\mathrm{reg}}$	Kostant section of the Steinberg map $\mu : \mathfrak{g}^{\mathrm{reg}} \to S$	76
X_+	equals $X_{\alpha_1} + X_{\alpha_2}$	76
X_-	equals $X_{-\alpha_1} + X_{-\alpha_2}$	77
$\mathfrak{t}_2$	$\mathfrak{t}$ after base change to $\mathbb{Z}[2^{-1}]$	77

Symbol	*Definition*	*Page*
$\#H^1(K,T_s)$	cardinality of $H^1(K,T_s)$	78
A_n	the group corresponding to each $H^1(K,T_s)$, interpreted as a definable set	78
$h_r : W_r \to \mathbb{N}$	$h_r(w) = \#H^1(K,T_s)$ for T_s corresponding to each $w \in W_r$ and $r \in \frac{1}{6}\mathbb{Z}$	78
$\mathfrak{g}(r,w)$	the fibre of $S_r^w \hookrightarrow S_r$ under the map of definable subassignments $\mu_r : \mathfrak{g}(r) \to S_r$	79
$\mu_r^w : \mathfrak{g}(r,w) \to S_r^w$	maps of definable subassignments for every $r \in \frac{1}{6}\mathbb{Z}$ and $w \in W_r$	79
$\delta_{r/K}^w$	function $\delta_{r/K}^w : \mathfrak{g}(r,w,K) \to A_{h_r(w)}$ for every $r \in \frac{1}{6}\mathbb{Z}$ and $w \in W_r$	79
B_r^w	definable set $B_r^w := S_r^w \times A_{h_r(w)}$	80
$\nu_{r/K}^w : \mathfrak{g}(r,K) \to B_r^w(k)$	defined as $\nu_{r/K}^w := \mu_{r/K}^w \times \delta_{r/K}^w : \mathfrak{g}(r,K) \to B_r^w(k)$	80
$\mathcal{O}(x,a)$	equals $\mathcal{O}_r(X)$ if $\mu_{r/K}^w(X) = (x,a) \in B_r^w(k) = S_r^w(k) \times A_{h_r(w)}$	81
B_r	definable set $\coprod_{w\in W_r} \left(S_r^w \times A_{h_r(w)}\right)$	81
$\nu_r : \mathfrak{g}(r) \to B_r^w$	map of definable subassignments $\mathfrak{g}(r,K) \to \coprod_{w\in W_r} \mathfrak{g}(r,w,K) \overset{\nu_r^w}{\to} \coprod_{w\in W_r} B_r^w \to B_r$	81
S^r	definable subassignment given by $S^r(K) = \{(s_1,s_2) \in S(K) \mid \mathrm{ord}_K(s_1) = 6r \text{ and } \mathrm{ord}_K(s_2) \geq \lceil 2r \rceil\}$	82
res_r	restriction map of definable subassignments	82

Epigraph

> Anything is possible if one wants it with unbending intent...

Carlos Castaneda, The Second Ring of Power [Cas77, p. 123]

Introduction

The major step in the theory of conjugacy classes of semisimple algebraic groups was taken by Steinberg in 1965 [Ste65, §6] when he introduced a parameterization of the conjugacy classes by a variety commonly called the *adjoint quotient* but which we will refer to as the *Steinberg quotient*. This was done over an algebraically closed field, though, and this is a limitation for our purposes.

Langlands [Lan79] then parameterized the rational conjugacy classes within a stable conjugacy class and, over local and non-Archimedean fields, showed how to calculate the parameterizing object. His motivation was the desire to calculate the Arthur-Selberg trace formula, in particular "...an analysis of local orbital integrals to which the sum over a global stable conjugacy class is not directly amenable."([Lan79, p. 701]) As the name suggests, *orbital integrals* are integrals over the (semisimple) conjugacy classes – orbits – of elements in a connected semisimple group.

His method of resolving this problem was to look at calculating the error terms left over when the orbital inegrals are calculated over stable conjugacy classes. However, another solution appeared when T.C. Hales showed that p-adic *rational* orbital integrals are motivic [Hal04] and may be computed using the technique of motivic integration introduced by M. Kontsevich in 1995.

Orbital integrals were clarified in classical groups in 2004 by C. Cunningham and T.C. Hales [CH04], and in 2011 by Cluckers, Cunningham, Gordon and Spice [CCGS11]. One of their last contributions was to outline some steps towards writing a computer program to produce their results, one of the benefits of the motivic approach. Building on results from [CH04] on *good* (read equivalued) orbital integrals, Step 1 in this plan is a motivic parameterization of *thickened* good adjoint orbits in the Lie algebra of the p-adic group. However, that paper was limited to symplectic and special orthogonal groups because it relied on the classification of regular semisimple adjoint orbits given in [Wal01], which, while eminently motivic in nature, only treats classical groups.

Here we have given a recipe which could also be automated, but may be used for any linear algebraic group – we have only used the information given in the Dynkin diagram. As a demonstration we perform the calculations for the Chevalley group scheme G of type G_2: a motivic parameterization of *thickened* good adjoint orbits in the Lie algebra of G. This result should be viewed as a basic part of the infrastructure needed to compute regular semisimple orbital integrals on this Lie algebra over local fields K.

In broad strokes, our approach to this problem is familiar: we use the Steinberg quotient S over K to parameterize stable orbit varieties $\mathcal{O}_s$, with $s \in S(K)$, of regular semisimple elements; we find a stable conjugacy class of maximal tori $T_s \subset G$ attached to $s \in S(K)$; we compute $H^1(K, T_s)$ to detect how many adjoint orbits appear in the stable orbit $\mathcal{O}_s(K)$; and we use the Kostant section for $\mathfrak{g} = \operatorname{Lie} G$ over K to put a group structure on the torsor $\mathcal{O}_s(K)/G(K)$ of adjoint orbits in stable orbits.

A novelty of the approach in this thesis, however, is that all this is done in a way which is independent of the local field K, except that its residual characteristic must be at least 5, and without making use of any representation of the group G, relying instead only on the root datum for G. In particular, in this thesis we make no use of arcane knowledge of the exceptional group $G(2)$, no use of the representation of $G(2)$ in $SO(8)$, and no use of Bruhat-Tits theory; everything in this thesis is derived *directly* from the root datum of type G_2. This is all made possible by the use of r-reduction, as developed in [CH04], which in turn rests on Krasner's lemma, to show that $H^1(K, T_s)$ does not change under p-adically small perturbations of $s \in S(K)$ and show further that what 'p-adically small' means here can be expressed in the language of Pas. Making this precise leads to *thickened orbits*, a notion appearing first in [CH04] and then clarified in [CCGS11]. Putting all these pieces together proves the main result of the thesis, Theorem 1.1, giving the motivic parameterization of thickened good regular semisimple adjoint orbits in $\mathfrak{g}(K)$.

The result is a simple motivic gadget – a map of definable subassignments – which is built directly from the Chevalley group scheme G as determined by its root datum and a Chevalley basis, which is independent of any representation of G and any local field but which, after the choice of a local field K with residual characteristic of at least 5, parameterizes all thickened good regular semisimple adjoint orbits in $\mathfrak{g}(K)$. The promised map of definable subassignments is exhibited in Theorem 1.1 and described informally here, where $\mathfrak{g}$ is a Chevalley Lie algebra scheme of type G_2: *We find a family of maps of definable subassignments*

$$\forall r \in \mathbb{Q}, \qquad \nu_r : \mathfrak{g}(r) \to B_r$$

such that if K is a local field and 6 *is invertible in its residue field k then each ν_r specializes to a surjective function $\nu_{r/K} : \mathfrak{g}(r, K) \to B_r(k)$ for which the fibres are thickened orbits of good elements in $\mathfrak{g}(K)$ and all such orbits arise in this way.* The point of this thesis is not just to promise the existence of $\nu_r : \mathfrak{g}(r) \to B_r$ but to actually exhibit it.

While this thesis only considers the Chevalley group scheme G of type G_2, the strategy used here adapts to any Chevalley group scheme. It is hoped that this strategy will be implemented in the near future.

Chapter 1

Statement of the main result

We begin with a brief review of basic facts about the Chevalley group scheme G of type G_2 and its Lie algebra. We then state the main result of the thesis, the proof of which will occupy Chapters 2 and 6.

Throughout the thesis we write K for a non-Archimedean local field, k for its residue field, and π for a uniformizer of K, when we need to introduce one. Let ord_K be a valuation on K so that $\mathrm{ord}_K(K^\times) = \mathbb{Z}$.

1.1 Root datum of type G_2

Consider the lattices

$$\Lambda = \{x_1\epsilon_1 + x_2\epsilon_2 + x_3\epsilon_3 \mid x_1, x_2, x_3 \in \mathbb{Z}\}$$

and

$$\check{\Lambda} = \{y_1 f_1 + y_2 f_2 + y_3 f_3 \mid y_1, y_2, y_3 \in \mathbb{Z}\}$$

with pairing $\check{\Lambda} \times \Lambda \to \mathbb{Z}$ given by $< f_i, \epsilon_j > = \delta_{i,j}$. Now, set $\epsilon = \epsilon_1 + \epsilon_2 + \epsilon_3 \in \Lambda$ and consider the quotient lattice $X = \Lambda/\{x\epsilon \mid x \in \mathbb{Z}\}$ and the sub-lattice $\check{X} = \{y \in$

$\check{\Lambda} \mid \langle y, \epsilon \rangle = 0\}$ with pairing $\check{X} \times X \to \mathbb{Z}$ given by

$$\langle y_1 f_1 + y_2 f_2 + y_3 f_3, x_1\epsilon_1 + x_2\epsilon_2 + x_3\epsilon_3 \rangle \mapsto y_1x_1 + y_2x_2 + y_3x_3.$$

Consider the root system $R \subset X$ given by

$$R = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2), \pm(2\alpha_1+\alpha_2), \pm(3\alpha_1+\alpha_2), \pm(3\alpha_1+2\alpha_2)\}$$

where, writing e_i for the image of ϵ_i in Λ,

$$\alpha_1 := -e_1 \qquad \alpha_2 := e_1 - e_2$$

$$\alpha_3 := \alpha_1+\alpha_2 = -e_2 \qquad \alpha_4 := 3\alpha_1+\alpha_2 = -e_1+e_3$$

$$\alpha_5 := 2\alpha_1+\alpha_2 = e_3 \qquad \tilde{\alpha} = \alpha_6 := 3\alpha_1+2\alpha_2 = -e_2+e_3.$$

This is a root system of type G_2. Note that the short roots satisfy the identities

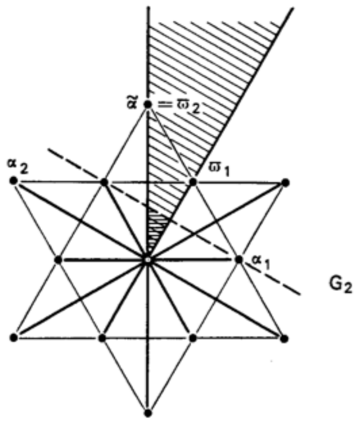
$$\begin{aligned} \alpha_1 &= \frac{1}{3}(-2e_1+e_2+e_3) \\ \alpha_1+\alpha_2 &= \frac{1}{3}(e_1-2e_2+e_3) \\ 2\alpha_1+\alpha_2 &= \frac{1}{3}(-e_1-e_2+2e_3) \end{aligned}$$

in $X \otimes \mathbb{Z}[3^{-1}]$.

The 'longest' root with respect to the basis $\Delta = \{\alpha_1, \alpha_2\}$ is $\widetilde{\alpha} = 3\alpha_1 + 2\alpha_2$ and the fundamental weights are $\varpi_1 = 2\alpha_1+\alpha_2$ and $\varpi_2 = \widetilde{\alpha}$. The short roots and long roots in R,

$$R_{\text{short}} = \{\pm\alpha_1, \pm\alpha_3, \pm\alpha_5\} \qquad R_{\text{long}} = \{\pm\alpha_2, \pm\alpha_4, \pm\alpha_6\},$$

Figure 1.1: Root system and fundamental weights of type G_2, from [Bou68, p. 276].



each form root systems of type A_2.

The dual root system $\check{R} \subset \check{X}$ is given by

$$\check{R} \quad = \quad \{\pm\check{\alpha}_1, \pm\check{\alpha}_2, \pm(\check{\alpha}_1 + \check{\alpha}_2), \pm(\check{\alpha}_1 + 2\check{\alpha}_2), \pm(\check{\alpha}_1 + 3\check{\alpha}_2), \pm(2\check{\alpha}_1 + 3\check{\alpha}_2)\}$$

with

$$\begin{aligned} \check{\alpha}_1 &:= -2f_1 + f_2 + f_3 & \check{\alpha}_2 &:= f_1 - f_2 \\ \check{\alpha}_1 + 3\check{\alpha}_2 &= f_1 - 2f_2 + f_3 & \check{\alpha}_1 + \check{\alpha}_2 &= -f_1 + f_3 \\ 2\check{\alpha}_1 + 3\check{\alpha}_2 &= -f_1 - f_2 + 2f_3 & \check{\alpha}_1 + 2\check{\alpha}_2 &= -f_2 + f_3. \end{aligned}$$

The Cartan matrix for $(R, \check{R})$ with reference to the pair of bases $\Delta = \{\alpha_1, \alpha_2\}$ and $\check{\Delta} = \{\check{\alpha}_1, \check{\alpha}_2\}$ is the matrix

$$\begin{bmatrix} <\check{\alpha}_1, \alpha_1> & <\check{\alpha}_1, \alpha_2> \\ <\check{\alpha}_2, \alpha_1> & <\check{\alpha}_2, \alpha_2> \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}.$$

The quartuple $(X, R, \check{X}, \check{R})$ is a semisimple root datum of type G_2; compare with

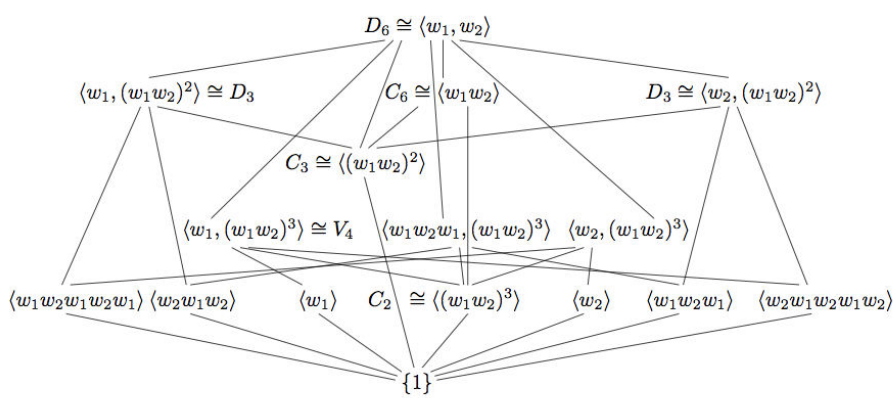
[Bou68, p. 220].

The character lattice X coincides with the root lattice $Q(R)$ (the lattice generated by R) and the cocharacter lattice $\check{X}$ coincides with the coroot lattice $Q(\check{R})$ (the lattice generated by $\check{R}$). In this way we see that the root datum $(X, R, \check{X}, \check{R})$ is simultaneously of adjoint type and simply connected.

1.2 Weyl group

The Weyl group W for R may be apprehended through the action of $W = \langle s_\alpha \mid \alpha \in R \rangle$ on X given by $s_\alpha(x) = x - <\check{\alpha}, x>\alpha$. Using the Cartan matrix above, we see that $s_{\alpha_1}(\alpha_1) = -\alpha_1$ and $s_{\alpha_1}(\alpha_2) = 3\alpha_1 + \alpha_2$ while $s_{\alpha_2}(\alpha_1) = \alpha_1 + \alpha_2$ and $s_{\alpha_2}(\alpha_2) = -\alpha_2$. Henceforth we will adopt the notation $w_1 := s_{\alpha_1}$ and $w_2 := s_{\alpha_2}$.

Figure 1.2: Subgroup lattice of the Weyl group



We remark that $W = \langle w_1, w_2 \rangle$ is the dihedral group D_6 of order 12, generated by w_2 and w_2w_1, for example. The element w_2w_1 is Coxeter, and only all non-trivial powers of w_2w_1 are regular and regular elliptic elements of W. Thus, the regular and regular elliptic numbers for W are 2, 3 and 6, the latter being the Coxeter number of W.

With reference to Figure 1.2, The normal proper subgroups of W are:

1. $\langle w_1w_2\rangle \cong C_6$, $\langle w_1, (w_1w_2)^2)\rangle \cong S_3$, $\langle w_2, (w_1w_2)^2)\rangle \cong S_3$;
2. $\langle (w_1w_2)^2\rangle \cong C_3$; and
3. $\langle (w_1w_2)^3\rangle \cong C_2$.

Of the three normal subgroups of index 2, no two are conjugate.

Table 1.1: Action of Weyl group W on $\check{X} \subset \check{\Lambda}$

$w \in W$	$w(y_1, y_2, y_3)$
w_2w_1	$(-y_3, -y_1, -y_2)$
$(w_2w_1)^5 = w_1w_2$	$(-y_2, -y_3, -y_1)$
$(w_2w_1)^2$	(y_2, y_3, y_1)
$(w_2w_1)^4 = (w_1w_2)^2$	(y_3, y_1, y_2)
$(w_2w_1)^3 = (w_1w_2)^3$	$(-y_1, -y_2, -y_3)$
$w_2 = s_{\alpha_2}$	(y_2, y_1, y_3)
$w_1w_2w_1 = s_{\alpha_4}$	(y_3, y_2, y_1)
$w_2w_1w_2w_1w_2 = s_{\alpha_6}$	(y_1, y_3, y_2)
$w_1 = s_{\alpha_1}$	$(-y_1, -y_3, -y_2)$
$w_2w_1w_2 = s_{\alpha_3}$	$(-y_3, -y_2, -y_1)$
$w_1w_2w_1w_2w_1 = s_{\alpha_5}$	$(-y_2, -y_1, -y_3)$
1	(y_1, y_2, y_3)

The Weyl group may also be apprehended through the action of $\langle s_{\check{\alpha}} \mid \check{\alpha} \in \check{\Delta} \rangle$ on $\check{X}$ given by $s_{\check{\alpha}}(y) = y - <y, \alpha> \check{\alpha}$. Using the description above of $\check{X}$ as a sub-lattice of $\check{\Lambda}$, we have $s_{\check{\alpha}_1} : y_1 f_1 + y_2 f_2 + y_3 f_3 \mapsto -y_1 f_1 - y_3 f_2 - y_2 f_3$ and $s_{\check{\alpha}_2} : y_1 f_1 + y_2 f_2 + y_3 f_3 \mapsto y_2 f_1 + y_1 f_2 + y_3 f_3$. For use below, we record the action of W on $\check{X}$ in Table 1.1, in which elements of W are separated by conjugacy classes and where we use the notation $w_1 := s_{\check{\alpha}_1}$ and $w_2 := s_{\check{\alpha}_2}$; context makes this notation unambiguous.

1.3 Chevalley group scheme

Let G be a Chevalley group scheme over $\mathbb{Z}$ determined by the root datum $(X, R, \check{X}, \check{R})$; see [Che61] and [Gro96]. We remark that $G \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Q}_p)$ is a split, connected reductive algebraic group with root datum $(X, R, \check{X}, \check{R})$ for *every* prime p. Let $\mathfrak{g}$ be the Lie algebra scheme of G [CR10].

1.4 Chevalley bases and Structure Coefficients

A Chevalley basis [Che55] for $\mathfrak{g}$ is a function $R \to \mathfrak{g}$, $\alpha \mapsto X_\alpha$, with the following properties: for every $\alpha \in R$, the triple $(X_\alpha, [X_\alpha, X_{-\alpha}], X_{-\alpha})$ is an $\mathfrak{sl}_2$-triple over $\mathbb{Z}$; the union $\{X_\alpha \mid \alpha \in R\} \cup \{[X_\alpha, X_{-\alpha}] \mid \alpha \in \Delta\}$ is a basis for $\mathfrak{g}$; $[X_\alpha, X_\beta] = 0$ unless $\alpha + \beta = 0$ or $\alpha + \beta \in R$; if $\alpha + \beta \in R$ then $[X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta}$ for integers $N_{\alpha,\beta}$ called the *structure coefficients* of the Chevalley basis.

The structure coefficients satisfy the following relations:

(i) $N_{\alpha,\beta} = -N_{\beta,\alpha} \qquad \alpha, \beta \in R.$

(ii) $\frac{N_{\alpha,\beta}}{(\gamma,\gamma)} = \frac{N_{\beta,\gamma}}{(\alpha,\alpha)} = \frac{N_{\gamma,\alpha}}{(\beta,\beta)}$

if $\alpha, \beta, \gamma \in R$ satisfy $\alpha + \beta + \gamma = 0$.

(iii) $N_{\alpha,\beta}N_{-\alpha,-\beta} = -(p+1)^2, \qquad \alpha,\beta \in R$

(iv) $\frac{N_{\alpha.\beta}N_{\gamma,\delta}}{(\alpha+\beta,\alpha+\beta)} + \frac{N_{\beta,\gamma}N_{\alpha,\delta}}{(\beta+\gamma,\beta+\gamma)} + \frac{N_{\gamma,\alpha}N_{\beta,\delta}}{(\gamma+\alpha,\gamma+\alpha)} = 0$

if $\alpha, \beta, \gamma, \delta \in R$ satisfy $\alpha + \beta + \gamma + \delta$ and if no pair are opposite.

Here p is the greatest integer such that $\beta - p\alpha \in R$, and (,) is the standard inner product on R. Moreover $N_{\alpha,\beta} = \pm(p+1)$ so finding the structure coefficients amounts to determining the sign.

To calculate a Chevalley basis for $\mathfrak{g}$, we follow [Car72, §§4.1-4.2], from which we recall the following notions:

(s) a *special* (s) ordered pair of roots $\{\alpha, \beta\} \in R \times R$ is one in which $\alpha + \beta \in R$ and $0 \prec \alpha \prec \beta$, where $\alpha \prec \beta \Leftrightarrow \beta - \alpha \in R^+ = \{x\alpha_1 + y\alpha_2 \in R \mid x > 0 \text{ or } x = 0 \Rightarrow y > 0\}$;[1] and

(es) an *extraspecial* (es) pair of roots $\{\alpha, \beta\} \in R \times R$ is a special pair such that, for all special pairs $\{\gamma, \delta\}$ with $\alpha + \beta = \gamma + \delta$, $\alpha \preceq \gamma$.

Choosing the sign of the structure coefficients for the extraspecial pairs of roots $\{\alpha, \beta\}$ uniquely determines the structure coefficients $N_{\alpha,\beta}$ for all pairs [Car72, Prop. 4.2.2]; we set $\text{sign}(N_{\alpha,\beta}) = +1$ for all extraspecial pairs $\{\alpha, \beta\}$.

Then the calculation of the Chevalley basis is algorithmic:

(1) Calculate the special ordered pairs of roots; in our case the pairs $\{\alpha_1, \alpha_1 + \alpha_2\}$, $\{\alpha_1, 2\alpha_1 + \alpha_2\}$, $\{\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}$.

(2) Determine which special pairs are extraspecial; in our case all of the pairs $\{\alpha_1, \alpha_1 + \alpha_2\}$, $\{\alpha_1, 2\alpha_1 + \alpha_2\}$, $\{\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}$.

(3) Set $\text{sign}(N_{\alpha,\beta}) = +1$ for all extraspecial pairs $\{\alpha, \beta\}$.

---

[1] In fact, any total ordering on R will give an order relation $\prec$ that will work. The relation we have chosen is convenient as we already have the basis Δ of R.

Table 1.2: Structure Coefficients for $\mathfrak{g}$

	$-3\alpha_1 - 2\alpha_2$	$-3\alpha_1 - \alpha_2$	$-2\alpha_1 - \alpha_2$	$-\alpha_1 - \alpha_2$	$-\alpha_2$	$-\alpha_1$
$-3\alpha_1 - 2\alpha_2$						
$-3\alpha_1 - \alpha_2$					+1	
$-2\alpha_1 - \alpha_2$				+3		+3
$-\alpha_1 - \alpha_2$			-3			+2
$-\alpha_2$		-1				-1
$-\alpha_1$			-3	-2	+1	
α_1		-1	-2	+3		
α_2	-1			-1		
$\alpha_1 + \alpha_2$	-1		+2		-1	+3
$2\alpha_1 + \alpha_2$	+1	+1		+2		-2
$3\alpha_1 + \alpha_2$	+1		+1			-1
$3\alpha_1 + 2\alpha_2$		+1	+1	-1	-1	

	α_1	α_2	$\alpha_1 + \alpha_2$	$2\alpha_1 + \alpha_2$	$3\alpha_1 + \alpha_2$	$3\alpha_1 + 2\alpha_2$
$-3\alpha_1 - 2\alpha_2$		+1	+1	-1	-1	
$-3\alpha_1 - \alpha_2$	+1			-1		-1
$-2\alpha_1 - \alpha_2$	+2		-2		-1	-1
$-\alpha_1 - \alpha_2$	-3	+1		-2		+1
$-\alpha_2$			+1			+1
$-\alpha_1$			-3	+2	+1	
α_1		-1	+2	+3		
α_2	+1				+1	
$\alpha_1 + \alpha_2$	-2			+3		
$2\alpha_1 + \alpha_2$	-3		-3			
$3\alpha_1 + \alpha_2$		-1				
$3\alpha_1 + 2\alpha_2$						

(4) Apply condition (i) to the extraspecial pairs.

(5) Apply condition (iii) to the extraspecial pairs.

(6) Apply condition (iii) to the pairs produced in (4).

(7) Apply condition (ii) to the pairs $\{\alpha, \beta\}$ produced in (4), (5) and (6) with $\gamma = -\alpha - \beta$.

(8) Apply condition (iv) to calculate the sign of a pair $\{\alpha, \beta\}$ outside of those produced in (4), (5), (6) and (7).

(9) Apply conditions (i), (ii) and (iii) to any new pairs produced by (8).

(10) Repeat steps (8) and (9) until signs are calculated for all pairs.

Using this algorithm, and making the choice indicated above, the complete list of structure coefficients in our case is given in Table 1.2.

1.5 Equivalued/Good elements

Let $\mathfrak{g}^{\text{reg}} \hookrightarrow \mathfrak{g}$ be the open subscheme of regular semisimple elements obtained by localizing $\mathfrak{g}$ at the discriminant $D \in \mathbb{Z}[\mathfrak{g}]$ (the coordinate ring of $\mathfrak{g}$) which will be computed in Sections 2.1 and 2.2.

Let K be a local field. An element $X \in \mathfrak{g}^{\text{reg}}(K)$ is called *good of slope* r if it is equivalued in the following sense: $\text{ord}_K(\alpha(X)) = r$ for each root of $\mathfrak{g}(K)$ relative to $\mathfrak{t}_X(K)$, the Cartan subalgebra containing X. In this case, the *depth* of X is r; see [CCGS11, Def 2.1] for more detail. As they amount to the same thing, we will only use the term 'depth' henceforth. Because we are only interested in good elements which are also regular semisimple, we henceforth shorten 'good and regular semisimple' to 'good'. We write $\mathfrak{g}(r, K)$ for the set of good elements in $\mathfrak{g}^{\text{reg}}(K)$ of depth r.

1.6 Thickened orbits

Suppose $X \in \mathfrak{g}^{\text{reg}}(K)$ and let $r = \text{depth}(X)$. The *thickened orbit* of $X \in \mathfrak{g}^{\text{reg}}(K)$ is the set

$$\mathcal{O}_r(X) := \bigcup_{Y \in \mathfrak{t}_X(K)_{r+}} \mathcal{O}(X+Y),$$

where $\mathcal{O}(X+Y)$ is the $G(K)$-adjoint-orbit of $X+Y$ in $\mathfrak{g}(K)$ [CCGS11, Def. 2.5].

1.7 Definable Subassignments

There is one more definition we must make – that of a *definable subassignment*; we refer to [GY09, §§5.2.1-4]. Given the categories Field_f of fields containing a field f and *Set* of sets, define a functor $h[m,n,r] = h_{\mathbb{A}_{f((t))^m} \times \mathbb{A}_f^n \times \mathbb{Z}^r} : \mathit{Field}_f \to \mathit{Set}$ by

$$h[m,n,r](F) = h_{\mathbb{A}_{f((t))^m} \times \mathbb{A}_f^n \times \mathbb{Z}^r}(F) := F((t)) \times F^n \times \mathbb{Z}^r$$

for some field F containing f, and where $\mathbb{A}_f^n$ is *affine n-space* over f.

In general, a *subassignment* h of the functor $\mathcal{F} : \mathfrak{C} \to \mathit{Set}$ between any category $\mathfrak{C}$ and *Set* is a collection of subsets $h(C) \subset \mathcal{F}(C)$ for each $C \in \mathfrak{C}$. To define a *definable* subassignment, we need the following.

A *formal language* $\mathcal{L}$ is a set of strings made up of certain symbols. The formal languages we are interested in here are the *Language of Rings*, *Presburger's Language*, and the *Language of Denef-Pas*.

The *Language of Rings* is made up from the following symbols: countably many symbols for variables $x_1, x_2, \ldots, x_n, \ldots$, '0', '1', '$\times$', '$+$', '$=$', and parentheses '(' and ')', the existential quantifier '$\exists$', and the logical operations '$\wedge$', '$\neq$', and '$\vee$'.

Presburger's Language is made up from the following symbols: countably many symbols for variables over $\mathbb{Z}$ $x_1, x_2, \ldots, x_n, \ldots$, '0', '1', '+', '$\leq$', and for each $d = 2, 3, \ldots$ a symbol '$\equiv_d$' denoting $x \equiv y \mod d$, and the same symbols for quantifiers, logical operations and parentheses as in the Language of Rings.

The Language of Denef-Pas is an extension of the first two languages for valued fields. It has three sorts of variables: variables over the residue field whose accompanying symbols are those of the Language of Rings with symbols for every rational number (so formulas can have coefficients in $\mathbb{Q}$), variables over the value group whose accompanying symbols are those of Presburger's language along with the symbol '∞', and finally variables over the valued field itself whose accompanying symbols are those of the Language of Rings plus the symbols ord and ac, defined below.

Denote the ring of integers of K by $\mathcal{O}_K$ and a fixed uniformizer by π. Let res : $\mathcal{O}_K \to k$ be the residue map, and let $K^{int} = \{x \in K \mid \mathrm{ord}_K(x) \in \mathbb{Z}\}$. The *angular component* is a function ac : $K^{int} \to \bar{k}^{\times}$ given by $\mathrm{ac}(0) = 0$ and $\mathrm{ac}(x) = \mathrm{res}(x/\pi^{\mathrm{ord} x})$.

Finally, since we are concerned here only with elements of Field_f, we also add a symbol for each element of $f((t))$, a case which we note with the phrase "formulas with coefficients on $f((t))$".

A subassignment h of $h[m, n, r]$ is a *definable subassignment* if there is a formula ϕ in the Language of Denef-Pas with coefficients in $f((t))$ where m, n, r are the numbers of free variables of the valued field, the residue field, and the value sort, respectively, such that for every $F \in \mathrm{Field}_f$, $h(F)$ is the set of all points in $F((t))^m \times F^n \times \mathbb{Z}^r$ satisfying ϕ. Then a *morphism of definable subassignments* from h_1 to h_2 to be a definable subassignment d such that $d(C)$ is the *graph* of a function from $h_1(C)$ to $h_2(C)$ for each object C in $\mathfrak{C}$. The *category of definable subassignments* is denoted by Def_f.

Following [CH04, Lemma 5.1] we see that, for every $r \in \mathbb{Q}$, there is a formula ϕ_r in the language of Denef-Pas such that for every $F \in \mathrm{Field}_f$, $\mathfrak{g}(r, F)$ is the set of all points in $F((t))^m \times F^n \times \mathbb{Z}^r$ satisfying ϕ_r. Let $\mathfrak{g}(r)$ be the definable subassignment of equivalued regular semisimple elements of $\mathfrak{g}$ of depth r.

1.8 Statement of the main result

Theorem 1.1. *Let G be a Chevalley group scheme of type G_2 and let $\mathfrak{g}$ be its Lie algebra. Every Chevalley basis for $\mathfrak{g}$ determines a family of maps of definable subassignments*

$$\forall r \in \mathbb{Q}, \qquad \nu_r : \mathfrak{g}(r) \to B_r$$

such that if K is a local field and 6 is invertible in the residue field k of K then $\nu_{r/K}^{-1}(\nu_{r/K}(X))$ is a thickened orbit in $\mathfrak{g}(r, K)$, where $\nu_{r/K}$ is the specialization determined by K, and every thickened orbit of regular equivalued elements in $\mathfrak{g}(r, K)$ arises in this way.

Chapter 2

Motivic classification of thickened stable good orbits

As a step toward proving Theorem 1.1, in this chapter we classify thickened stable good orbits in $\mathfrak{g}(K)$. Suppose $X \in \mathfrak{g}^{\mathrm{reg}}(K)$ and let $r = \mathrm{depth}(X)$. We are now able to state the main result of this chapter.

Proposition 2.1. *Let G be a Chevalley group scheme of type G_2 and let $\mathfrak{g}$ be its Lie algebra. For every $r \in \mathbb{Q}$ there is a map of definable subassignments*

$$\mu_r : \mathfrak{g}(r) \to S_r$$

with the following property: if K is a local field and 6 is invertible in its residue field k, then the specialization $\mu_{r/K} : \mathfrak{g}(r, K) \to S_r(k)$ is surjective; and

$$\forall X \in \mathfrak{g}(r, K), \qquad \mathcal{O}^{\mathrm{st}}_r(X) = \mu^{-1}_{r/K}(\mu_{r/K}(X));$$

moreover, every thickened stable orbit in $\mathfrak{g}(r, K)$ arises in this way.

Note that Proposition 2.1 does not require the choice of a Chevalley basis for $\mathfrak{g}$, in contrast to Theorem 1.1.

2.1 Polynomials from R

Consider $\mathfrak{t} := \mathrm{Spec}(\mathbb{Z}[\check{X}])$. Since we have introduced a pair of lattices $(X, \check{X})$ through the pair of lattices $(\Lambda, \check{\Lambda})$, it is natural to use the basis for $\check{\Lambda}$ introduced above to determine a set of generators for the coordinate ring of $\mathfrak{t}$:

$$\mathbb{Z}[\mathfrak{t}] \cong \mathbb{Z}[y_1, y_2, y_2]/(y_1 + y_2 + y_3).$$

With reference to the action of the Weyl group W on $\check{X}$ in Section 1.2, invariant theory gives

$$\mathbb{Z}[\mathfrak{t}]^W_{|W|} \cong \mathbb{Z}[s_1, s_2]_6,$$

where $s_1 = y_1^2 y_2^2 y_3^3$ and $s_2 = y_1 y_2 + y_2 y_2 + y_3 y_1$.

However, because $\check{X} = Q(\check{R})$, (since $G(2)$ is adjoint), it is more natural to use the basis $\{\check{\alpha}_1, \check{\alpha}_2\}$ for $\check{R}$, as introduced above, to determine generators for the coordinate ring: $\mathbb{Z}[\mathfrak{t}] \cong \mathbb{Z}[z_1, z_2]$, where $\mathbb{Z}[y_1, y_2, y_2]/(y_1 + y_2 + y_3) \cong \mathbb{Z}[z_1, z_2]$ is determined by $z_1\check{\alpha}_1 + z_2\check{\alpha}_2 = y_1 f_1 + y_2 f_2 + y_3 f_3$.

Consider the polynomial $Q(\lambda)$ over $\mathbb{Z}[\mathfrak{t}]$ defined by

$$Q(\lambda) := \prod_{\alpha \in R} (\lambda - \alpha) .$$

Here we view each $\alpha \in R$ as an element of $\mathbb{Z}[\mathfrak{t}] = \mathbb{Z}[z_1, z_2]$ according to the identification $\alpha = \alpha(z_1\check{\alpha}_1 + z_2\check{\alpha}_2) = z_1 {<}\check{\alpha}_1, \alpha{>} + z_2 {<}\check{\alpha}_2, \alpha{>}$. Note that, with this notation, $\mathbb{Z}[\mathfrak{t}]_2 = \mathbb{Z}[R]_2$.

Since W stabilizes R, we see that the coefficients of $Q(\lambda)$ lie in $\mathbb{Z}[\mathfrak{t}]^W$ so, in fact, $Q(\lambda)$ lies in $\mathbb{Z}[\mathfrak{t}]^W[\lambda]$. We will find the coefficients of $Q(\lambda)$. Since W stabilizes R_{short}, it follows that the polynomial over $\mathbb{Z}[\mathfrak{t}]$ defined by

$$P(\lambda) := \prod_{\alpha \in R_{\text{short}}} (\lambda - \alpha)$$

also lies in $\mathbb{Z}[\mathfrak{t}]^W[\lambda]$. A simple calculation shows

$$P(\lambda) = \lambda^6 + 2s_2\lambda^4 + s_2^2\lambda^2 - s_1$$

where

$$s_2 := \sum_{\alpha \neq \beta \in \{-\alpha_1, -\alpha_3, \alpha_5\} \subset R_{\text{short}}} \alpha\beta = \alpha_1\alpha_3 - \alpha_3\alpha_5 - \alpha_5\alpha_1 = e_1e_2 + e_2e_3 + e_3e_1$$

and

$$s_1 := \prod_{\alpha \in \{-\alpha_1, -\alpha_3, \alpha_5\} \subset R_{\text{short}}} \alpha^2 = \prod_{\alpha \in R_{\text{short}}} \alpha = e_1^2e_2^2e_3^3.$$

We will sometimes use the notation $P_{s_1,s_2}(\lambda) := \lambda^6 + 2s_2\lambda^4 + s_2^2\lambda^2 - s_1$.

Likewise, since W stabilizes R_{long}, it follows that the polynomial over $\mathbb{Z}[\mathfrak{t}]$ defined by

$$P'(\lambda) := \prod_{\alpha \in R_{\text{long}}} (\lambda - \alpha)$$

also lies in $\mathbb{Z}[\mathfrak{t}]^W[\lambda]$. A simple calculation shows that

$$P'(\lambda) := \lambda^6 + 2s_2'\lambda^4 + (s_2')^2\lambda^2 - s_1',$$

so $P'_{s_1,s_2}(\lambda) = P_{s'_1,s'_2}(\lambda)$, where

$$s'_2 = \sum_{\alpha\neq\beta\in\{\alpha_2,\alpha_4,-\alpha_6\}\subset R_{\text{long}}} \alpha\beta = \alpha_2\alpha_4 - \alpha_4\alpha_6 - \alpha_6\alpha_2 = 3(e_1e_2 + e_2e_3 + e_3e_1) = 3s_2$$

and

$$s'_1 := \prod_{\alpha\in\{\alpha_2,\alpha_4,-\alpha_6\}\subset R_{\text{long}}} \alpha^2 = \prod_{\alpha\in R_{\text{long}}} \alpha = (e_1 - e_2)^2(e_2 - e_3)^2(e_3 - e_1)^2 = -(27s_1 + 4s_2^3).$$

Returning to $Q(\lambda) \in \mathbb{Z}[\mathfrak{t}]^W[\lambda]$, note that the constant term of $Q(\lambda)$ is

$$s_1s'_1 = \prod_{\alpha\in R_{\text{short}}} \alpha^2 \prod_{\alpha\in R_{\text{long}}} \alpha^2 = \prod_{\alpha\in R} \alpha^2 = e_1^2e_2^2e_3^2(e_1 - e_2)^2(e_2 - e_3)^2(e_3 - e_1)^2,$$

which is precisely the discriminant of $P(\lambda)$ and thus of $\mathfrak{t}$; we set $d := s_1s'_1 = -27s_1^2 - 4s_1s_2^3 \in \mathbb{Z}[\mathfrak{t}]^W$. We now see that $\mathbb{Z}[\mathfrak{t}^{\text{reg}}]^W \to \mathbb{Z}[\mathfrak{t}^{\text{reg}}]$ is given by

$$\mathbb{Z}[\mathfrak{t}]^W_d \to \mathbb{Z}[\mathfrak{t}]^W_d[\lambda]/(Q(\lambda)) \cong \mathbb{Z}[\mathfrak{t}]^W_d[\lambda]/(P(\lambda)) \oplus \mathbb{Z}[\mathfrak{t}]^W_d[\lambda]/(P'(\lambda)).$$

This defines the map $\mathfrak{t}^{\text{reg}} \to \mathfrak{t}^{\text{reg}}/W$, denoted by $\mu : \mathfrak{t}^{\text{reg}} \to S$ henceforth.

If we pick a K-rational point X on $\mathfrak{t}$ and replace (s_1, s_2) with $s = \mu(X)$ then we may write $P_X(\lambda) := P_s(\lambda)$ and $P'_X(\lambda) := P'_s(\lambda)$. We remark that, in this context,

$$\lambda^2 P_X(\lambda) = \det(\lambda - \text{ad}_{\mathfrak{g}}(X)),$$

is the characteristic polynomial of $X \in \mathfrak{g}(K)$.

2.2 Steinberg quotient

Let $\mathfrak{g} \to \mathfrak{g}/G \cong \mathfrak{t}/W$ be the Steinberg quotient for $\mathfrak{g}$ [CR10]. Let $D \in \mathbb{Z}[\mathfrak{g}]$ be any pre-image of $d \in \mathbb{Z}[\mathfrak{t}]$ under the quotient $\mathbb{Z}[\mathfrak{g}] \to \mathbb{Z}[\mathfrak{t}]$. Then $\mathfrak{g}^{\mathrm{reg}} := \mathrm{Spec}(\mathbb{Z}[\mathfrak{g}]_D)$, which is independent of the choice for D, is the open subscheme of regular semisimple elements in $\mathfrak{g}$. We write $\mu : \mathfrak{g}^{\mathrm{reg}} \to S$ for the restriction of $\mathfrak{g} \to \mathfrak{t}/W$ to $\mathfrak{g}^{\mathrm{reg}}$.

2.3 Parameterization of stable good orbits

One begins by working over a separable closure $\bar{K}$ and recalling the classical result [Ste65] that adjoint orbits in $\mathfrak{g}^{\mathrm{reg}}$ over $\bar{K}$ are classified by the regular part S of the Steinberg quotient over $\bar{K}$. The fibres of the Steinberg map $\mu : \mathfrak{g}^{\mathrm{reg}} \to S$ define subvarieties $\mathcal{O}_s \subset \mathfrak{g}$, for $s \in S$. Then one observes that S is in fact defined over K and if $s \in S(K)$ then $\mathcal{O}_s$ is also defined over K. The K-variety $\mathcal{O}_s$ may be apprehended as the quotient of G by the maximal torus $T_X \subseteq G$ containing X, for any $X \in \mathfrak{g}^{\mathrm{reg}}(K)$ with $\mu(X) = s$. The set $\mathcal{O}_s(K)$ is commonly called a *stable orbit* in $\mathfrak{g}(K)$.

2.4 Steinberg by depth

One of the key tools in this thesis is *r-reduction*, as introduced in [CH04, §3.1]. Originally, r-reduction took a polynomial $P = \lambda^N + \alpha_1\lambda^{N-1} + \ldots + \alpha_{ng}$ over K, whose roots $\lambda_i \in \bar{K}$ all satisfied $\mathrm{ord}_K(\lambda_i) = r$, to a polynomial $R = \lambda^g + a_1\lambda^{g-1} + \ldots + a_g$ over k in a combinatorial manner: $r \in \mathbb{Q}$, $g, \ell, n, L, N \in \mathbb{Z}$; $N \geq 1$; $g \geq 1$; $r \geq 0$; $r = L/N$; $g = \gcd(L, N)$; $\ell = L/g$; $n = N/g$. It was then shown that the splitting field of P depended only R. Here we use it schematically: r-reducing the polynomial

$Q(\lambda) = P(\lambda)P'(\lambda)$ introduced in Section 2.1 field independently, and then specializing to a field as required.

From the form of the polynomial $Q_X(\lambda)$ it follows that $\mathfrak{g}(r, K)$ is empty unless $r \in \frac{1}{6}\mathbb{Z}$. Henceforth, we suppose $r \in \frac{1}{6}\mathbb{Z}$ and write $\lfloor r \rfloor$ to be the integer part of r and $\{r\}$ to be the fractional part, or *fractional depth*, of r, so $r = \{r\} + \lfloor r \rfloor$ and $\{r\} \in \{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}\}$. Since $Q_X(\lambda) = P_X(\lambda)P'_X(\lambda)$ in $K[\lambda]$, we may calculate the r-reduction of $P_X(\lambda)$ and $P'_X(\lambda)$ separately.

The process of r-reduction produces, for each $r \in \frac{1}{6}\mathbb{Z}$, a quotient $\mathfrak{t}_r^{\text{reg}} \to S_r$ of affine schemes which recovers the quotient $\mathfrak{t}^{\text{reg}} \to S$ when $r \in \mathbb{Z}$, as we now explain.

Table 2.1: The process of r-reduction produces $P_r(\lambda)$ and $P'_r(\lambda)$ from $P(\lambda)$ and $P'(\lambda)$ over $\mathbb{Z}[\mathfrak{t}]$.

$\{r\}$	$P_r(\lambda)$,	$P'_r(\lambda)$
0	$\lambda^6 + 2s_2\lambda^4 + s_2^2\lambda^2 - s_1 = \prod_{\alpha \in R_{\text{short}}} (\lambda - \alpha)$	$\lambda^6 + 2s'_2\lambda^4 + (s'_2)^2\lambda^2 - s'_1 = \prod_{\alpha \in R_{\text{long}}} (\lambda - \alpha)$
$\frac{1}{6}, \frac{5}{6}$	$\lambda - s_1 = \lambda - \alpha_1^2\alpha_3^2\alpha_5^2$	$\lambda - s'_1 = \lambda - \alpha_2^2\alpha_4^2\alpha_6^2$
$\frac{1}{3}, \frac{2}{3}$	$\lambda^2 - s_1 = (\lambda - \alpha_1\alpha_3\alpha_5)(\lambda + \alpha_1\alpha_3\alpha_5)$	$\lambda^2 - s'_1 = (\lambda - \alpha_2\alpha_4\alpha_6)(\lambda + \alpha_2\alpha_4\alpha_6)$
$\frac{1}{2}$	$\lambda^3 + 2s_2\lambda^2 + s_2^2\lambda - s_1 = (\lambda - \alpha_1^2)(\lambda - \alpha_3^2)(\lambda - \alpha_5^2)$	$\lambda^3 + 2s'_2\lambda^2 + (s'_2)^2\lambda - s'_1 = (\lambda - \alpha_2^2)(\lambda - \alpha_4^2)(\lambda - \alpha_6^2)$

From Table 2.1 we see how r-reduction produces from $Q(\lambda) = P(\lambda)P'(\lambda)$ a polynomial $Q_r(\lambda) = P_r(\lambda)P'_r(\lambda)$ over $\mathbb{Z}[\mathfrak{t}]$, for each $r \in \frac{1}{6}\mathbb{Z}$. Let $\Phi_r \subset \mathbb{Z}[\mathfrak{t}]$ be the set of 'roots' of $Q_r(\lambda)$ for each $r \in \frac{1}{6}\mathbb{Z}$; see Table 2.2 for a list of the sets Φ_r for each $r \in \frac{1}{6}\mathbb{Z}$.

Define $\mathfrak{t} \to \mathfrak{t}_r$ over $\mathbb{Z}_{|W|}$ by

$$\mathbb{Z}[\mathfrak{t}_r]_{|W|} = \mathbb{Z}[\Phi_r]_{|W|} \subseteq \mathbb{Z}[R]_{|W|} = \mathbb{Z}[\mathfrak{t}]_{|W|}$$

and $\mathfrak{t}^{\mathrm{reg}} \to \mathfrak{t}_r^{\mathrm{reg}}$ by $\mathbb{Z}[\mathfrak{t}_r^{\mathrm{reg}}]_{|W|} = \mathbb{Z}[\Phi_r]_{d_r} \subseteq \mathbb{Z}[R]_{|W|d} = \mathbb{Z}[\mathfrak{t}^{\mathrm{reg}}]_{|W|}$ where d_r is the restriction of d from $\mathbb{Z}[\mathfrak{t}]$ to $\mathbb{Z}[\mathfrak{t}_r]$ with the factor $|W|$ for convenience. See Chapter 3 for more detail and explicit examples. Note that the action of W on $\mathbb{Z}[\mathfrak{t}]$ descends to $\mathbb{Z}[\Phi_r]$.

The covering group of $\mathfrak{t}^{\mathrm{reg}} \to \mathfrak{t}_r^{\mathrm{reg}}$ is a quotient W_r of W for which $\mathbb{Z}[\Phi_r]^W = \mathbb{Z}[\Phi_r]^{W_r}$; the kernel of $W \to W_r$ is $W^r := \{w \in W \mid w(f) = f, \forall f \in \Phi_r\}$. In fact, in each case there is a natural section of $1 \to W^r \to W \to W_r \to 1$, as indicated in Table 2.2. Define the affine scheme

$$S_r = \mathfrak{t}_r^{\mathrm{reg}}/W_r$$

by $\mathbb{Z}[S_r] := \mathbb{Z}[\mathfrak{t}_r^{\mathrm{reg}}]^{W_r} = \mathbb{Z}[\mathfrak{t}_r]_{d_r}^{W_r}$.

Let K be a local field and suppose 6 is invertible in its residue field k. By construction, $S_r(k)$ classifies r-reductions of characteristic polynomials of regular equivalued elements $X \in \mathfrak{g}(K)$ of depth r, for each $r \in \frac{1}{6}\mathbb{Z}$. Define

$$\mu_{r/K} : \mathfrak{g}(r, K) \to S_r(k)$$

as follows: for $X \in \mathfrak{g}(r, K)$, let $\mu_{r/K}(X)$ be the element of $S_r(k)$ corresponding to the r-reduction of $P_X(\lambda)$.

Table 2.2: The coordinate ring $\mathbb{Z}[\mathfrak{t}_r] = \mathbb{Z}[\Phi_r]$ and the sets Φ_r, the groups W_r indicating a section of $1 \to W^r \to W \to W_r \to 1$, and the discriminant $d_r \in \mathbb{Z}[\mathfrak{t}_r]$ using notation from Table 2.1.

$\{r\}$	W^r	W_r	$\mathbb{Z}[\Phi_r]$	$d_r = \lvert W\rvert\, s_1 s_1'$
0	1	W	$\mathbb{Z}[\alpha_1, \alpha_3, \alpha_5] \otimes \mathbb{Z}[\alpha_2, \alpha_4, \alpha_6]$ $(\Phi_r = R)$	$\lvert W\rvert\, s_1 s_1' = -324 s_1^2 - 48 s_1 s_2^3$
$\frac{1}{6}$, $\frac{5}{6}$	W	1	$\mathbb{Z}[\alpha_1^2\alpha_3^2\alpha_5^2] \otimes \mathbb{Z}[\alpha_2^2\alpha_4^2\alpha_6^2]$ $(\Phi_r = \{\alpha_1^2\alpha_3^2\alpha_5^2, \alpha_2^2\alpha_4^2\alpha_6^2\})$	$\lvert W\rvert\, s_1 s_1' = -324 s_1^2$
$\frac{1}{3}$, $\frac{2}{3}$	$\langle w_2, (w_2 w_1)^2\rangle \cong S_3$	$\langle (w_2 w_1)^3\rangle \cong C_2$	$\mathbb{Z}[\alpha_1\alpha_3\alpha_5] \otimes \mathbb{Z}[\alpha_2\alpha_4\alpha_6]$ $(\Phi_r = \{\alpha_1\alpha_3\alpha_5, \alpha_2\alpha_4\alpha_6\})$	$\lvert W\rvert\, s_1 s_1' = -324 s_1^2$
$\frac{1}{2}$	$\langle (w_2 w_1)^3\rangle \cong C_2$	$\langle w_2, (w_2 w_1)^2\rangle \cong S_3$	$\mathbb{Z}[\alpha_1^2, \alpha_3^2, \alpha_5^2] \otimes \mathbb{Z}[\alpha_2^2, \alpha_4^2, \alpha_6^2]$ $(\Phi_r = \{\alpha_1^2, \alpha_3^2, \alpha_5^2, \alpha_2^2, \alpha_4^2, \alpha_6^2\})$	$\lvert W\rvert\, s_1 s_1' = -324 s_1^2 - 48 s_1 s_2^3$

2.5 Maximal tori

Recall that isomorphism classes of tori over K that embed into G over K as a maximal torus are classified by $H^1(K, W)$ and thus determined, up to isomorphism, by *indexed root data* of the form $(X, \emptyset, \check{X}, \emptyset, \emptyset, \rho)$ where $\rho \in Z^1(K, W) = \mathrm{Hom}(\mathrm{Gal}(\bar{K}/K), W)$; see [Spr09, §16.2].

In this thesis we are concerned only with tamely ramified maximal tori, so we will restrict our attention to $H^1_{\mathrm{tr}}(K, W) = \mathrm{Hom}(\mathrm{Gal}(K^{\mathrm{tr}}/K), W)/W$-conj. Here we see how all such data arise from elements of $\mathfrak{g}^{\mathrm{reg}}(K)$ under the hypothesis that 6 is invertible in the residue field of K.

Suppose $X \in \mathfrak{g}^{\mathrm{reg}}(K)$. Since all Cartans are conjugate over $\bar{K}$ to $\mathfrak{t}$, and since conjugation preserves depth, $X' \in \mathfrak{t}^{\mathrm{reg}}(\bar{K})$ for some conjugate X'. Let $s = \mu(X')$ and consider

$$Q_s(\lambda) = \prod_{\alpha \in R} (\lambda - \alpha(X')) \in K[\lambda];$$

this is a specialization of $Q(\lambda) \in \mathbb{Z}[\mathfrak{t}][\lambda]$, introduced in Section 2.1. Then

$$K_s := K(\alpha(X') \mid \alpha \in R)$$

is the splitting extension of $Q_s(\lambda) = P_s(\lambda)P'_s(\lambda)$. Since $\alpha(X') \in \bar{K}$, there is a natural action of $\mathrm{Gal}(\bar{K}/K)$ on the root values $\{\alpha(X') \mid \alpha \in R\}$ and since the symmetry group of $Q_s(\lambda)$ is W, there is a homomorphism $\rho_s : \mathrm{Gal}(\bar{K}/K) \to W$, unique up to W-conjugacy, so that $\sigma(\alpha(X')) = \rho_s(\sigma)(\alpha)(X')$ for each $\alpha \in R$. Note that, up to W-conjugation, the roots $\alpha(X')$ are determined by s through the splitting of the polynomial $Q_s(\lambda)$. In this way we see that every element $X \in \mathfrak{g}^{\mathrm{reg}}(K)$ determines $s \in S(K)$ and thence $[\rho_s] \in H^1(K, W)$ by way of $Q_s(\lambda) \in K[\lambda]$. In this way we define a function $S(K) \to H^1(K, W)$, from stable conjugacy classes of elements in $\mathfrak{g}^{\mathrm{reg}}(K)$ to stable conjugacy class of Cartans in $\mathfrak{g}$, by $s \mapsto [\rho_s]$. However, in order to compute the splitting extension K_s, for each $s \in S_r(k)$, we must determine the irreducible factors of $R_s(\lambda)$, the r-reduction of $P_s(\lambda)$. The next few sections explain how to do that.

2.6 Algebras attached to regular equivalued elements

In this section we prepare for a study of the function $S(K) \to H^1(K, W)$.

The coordinate ring of the fibre of $\mu : \mathfrak{t}^{\mathrm{reg}} \to S$ above a K-rational point $s \in S(K)$ is the K-algebra $K[\lambda]/(Q_s(\lambda))$. Now suppose $X \in \mathfrak{g}(r, K)$ and $s = \mu(X) \in S(K)$. By [CH04, §3.2], $K[\lambda]/(Q_s(\lambda))$ is completely determined by $\mu_{r/K}(X) \in S_r(k)$. Now [CH04, §3.2] also shows that the irreducible factors of $Q_s(\lambda)$ correspond to the irreducible factors of its r-reduction in the following way. Let $R_s(\lambda)$ be the r-reduction

Table 2.3: Factorizations of $P_r(\lambda)$ over $\mathbb{Z}[S_r]$ for $r \in \mathbb{Z}$.

$w \in W$	$P_r(\lambda) \in \mathbb{Z}[S_r][\lambda]$
w_2w_1	$\lambda^6 + 2s_2\lambda^4 + s_2^2\lambda^2 - s_1$
$(w_2w_1)^5$	$\lambda^6 + 2s_2\lambda^4 + s_2^2\lambda^2 - s_1$
$(w_2w_1)^2$	$\left(\lambda^3 + s_2\lambda + \alpha_1\alpha_3\alpha_5\right)\left(\lambda^3 + s_2\lambda - \alpha_1\alpha_3\alpha_5\right)$
$(w_2w_1)^4$	$\left(\lambda^3 + s_2\lambda + \alpha_1\alpha_3\alpha_5\right)\left(\lambda^3 + s_2\lambda - \alpha_1\alpha_3\alpha_5\right)$
$(w_2w_1)^3$	$\left(\lambda^2 - \alpha_1^2\right)\left(\lambda^2 - \alpha_3^2\right)\left(\lambda^2 - \alpha_5^2\right)$
$s_{\alpha_2} = w_2$	$\left(\lambda^2 - (\alpha_1+\alpha_3)\lambda + \alpha_1\alpha_3\right)\left(\lambda^2 + (\alpha_1+\alpha_3)\lambda + \alpha_1\alpha_3\right)(\lambda - \alpha_5)(\lambda + \alpha_5)$
$s_{\alpha_4} = w_1w_2w_1$	$\left(\lambda^2 - (\alpha_1-\alpha_5)\lambda - \alpha_1\alpha_5\right)(\lambda - \alpha_3)(\lambda + \alpha_3)\left(\lambda^2 + (\alpha_1-\alpha_5)\lambda - \alpha_1\alpha_5\right)$
$s_{\alpha_6} = w_2w_1w_2w_1w_2$	$(\lambda - \alpha_1)(\lambda + \alpha_1)\left(\lambda^2 - (\alpha_3-\alpha_5)\lambda - \alpha_3\alpha_5\right)\left(\lambda^2 + (\alpha_3-\alpha_5)\lambda - \alpha_3\alpha_5\right)$
$s_{\alpha_1} = w_1$	$\left(\lambda^2 - \alpha_1^2\right)\left(\lambda^2 - (\alpha_3+\alpha_5)\lambda + \alpha_3\alpha_5\right)\left(\lambda^2 + (\alpha_3+\alpha_5)\lambda + \alpha_3\alpha_5\right)$
$s_{\alpha_3} = w_2w_1w_2$	$\left(\lambda^2 - (\alpha_1+\alpha_5)\lambda - \alpha_1\alpha_5\right)\left(\lambda^2 - \alpha_3^2\right)\left(\lambda^2 + (\alpha_1+\alpha_5)\lambda - \alpha_1\alpha_5\right)$
$s_{\alpha_5} = w_1w_2w_1w_2w_1$	$\left(\lambda^2 - (\alpha_1+\alpha_3)\lambda - \alpha_1\alpha_3\right)\left(\lambda^2 + (\alpha_1+\alpha_3)\lambda - \alpha_1\alpha_3\right)\left(\lambda^2 - \alpha_5^2\right)$
1	$(\lambda - \alpha_1)(\lambda + \alpha_1)(\lambda - \alpha_3)(\lambda + \alpha_3)(\lambda - \alpha_5)(\lambda + \alpha_5)$

of $P_s(\lambda)$ and let $R'_s(\lambda)$ be the r-reduction of $P'_s(\lambda)$. Set $g = \deg(R_s) = \deg(R'_s)$. In the numerology $r \mapsto (g, \ell, n)$ of [CH04, §3.1], we have $ng = \deg(P)$ and $2ng = \deg(Q) = |W|$. Now let $R_s(\lambda) = \prod_{i \in I_s} R_{s,i}(\lambda)$ be the decomposition of $R_s(\lambda)$ into irreducible factors in $k[\lambda]$; likewise, $R'_s(\lambda) = \prod_{i \in I'_s} R'_{s,i}(\lambda)$. We will study the index sets I_s and I'_s, below. Let $\dot{R}_{s,i}(\lambda)$ (resp. $\dot{R}'_{s,i}(\lambda)$) be any lift of $R_{s,i}(\lambda)$ (resp. $R'_{s,i}(\lambda)$). Then [CH04, §3.2] gives

$$K[\lambda]/(Q_s(\lambda)) = \bigoplus_{i \in I_s} \frac{K[\lambda]}{(\dot{R}_{s,i}(\lambda))} \bigoplus \bigoplus_{i \in I'_s} \frac{K[\lambda]}{(\dot{R}'_{s,i}(\lambda))}.$$

For each $i \in I_s$, let $g_i = \deg R_{s,i}$ and let $K^{(g_i)}$ be the unramified extension of K of degree g_i in $\bar{K}$; likewise define $g'_i = \deg R'_{s,i}$ and $K^{(g'_i)}$. Then

$$K[\lambda]/(Q_s(\lambda)) = \bigoplus_{i \in I_s} K^{(g_i)}(\sqrt[n]{\pi^\ell \dot{\zeta}_i}) \bigoplus \bigoplus_{i \in I'_s} K^{(g'_i)}(\sqrt[n]{\pi^\ell \dot{\zeta}'_i}),$$

where π is any uniformizer for K, independent of the choice of root ζ_i of $R_{s,i}(\lambda)$ (resp. ζ'_i of $R'_{s,i}(\lambda)$) and of the lift $\dot{\zeta}_i \in K^{(g_i)}$ (resp. $\dot{\zeta}'_i \in K^{(g'_i)}$).

In order to pin down $K[\lambda]/(Q_s(\lambda))$ more precisely, we must get information about the decompositions $R_s(\lambda) = \prod_{i \in I_s} R_{s,i}(\lambda)$ and $R'_s(\lambda) = \prod_{i \in I'_s} R'_{s,i}(\lambda)$ into irreducible polynomials and their dependence on $s \in S_r(k)$. That is the topic of the next section.

2.7 Factorizations

As a sort of warm-up to the problem of finding all decompositions of the r-reduction of $Q_s(\lambda) \in K[\lambda]$, thus determining the index set I_s appearing above, in this section we find all decompositions of $Q(\lambda) \in \mathbb{Z}[S][\lambda]$. It is enough to find all decompositions of $P(\lambda) \in \mathbb{Z}[S][\lambda]$.

Each element $w \in W$ determines a partition of $R = \coprod_{i \in I_w} R_i$ into $\langle w \rangle$-orbits. The factorizations of P are listed in Table 2.3, taking $\{r\} = 0$. The composition $\mathfrak{t}^{\mathrm{reg}} \to S_w \to S$ is a factorization of $\mu : \mathfrak{t}^{\mathrm{reg}} \to S$ and all factorizations of μ arise in this manner. Each $w \in W$ thus determines a factorization $\mathfrak{t}^{\mathrm{reg}} \to S_w \to S$ of $\mathfrak{t}^{\mathrm{reg}} \to S$ corresponding to factorizations of P. We note that $S_w \cong S_{w'}$ over S if and only if if w' is W-conjugate to w.

Table 2.4: Factorizations of $P_r(\lambda)$ and $P'_r(\lambda)$ over $\mathbb{Z}[S_r]$ for $r \notin \mathbb{Z}$.

$\{r\}$	$w \in W_r$	$P_r(\lambda) \in \mathbb{Z}[S_r][\lambda]$	$P'_r(\lambda) \in \mathbb{Z}[S_r][\lambda]$
$\frac{1}{2}$	$(w_2w_1)^2$	$\lambda^3 + 2s_2\lambda^2 + s_2^2\lambda - s_1$	$\lambda^3 + 2s'_2\lambda^2 + (s'_2)^2\lambda - s'_1$
$\frac{1}{2}$	$(w_2w_1)^4$	$\lambda^3 + 2s_2\lambda^2 + s_2^2\lambda - s_1$	$\lambda^3 + 2s'_2\lambda^2 + (s'_2)^2\lambda - s'_1$
$\frac{1}{2}$	$s_{\alpha_2} = w_2$	$(\lambda^2 - (\alpha_1^2 + \alpha_3^2)\lambda + \alpha_1^2\alpha_3^2)(\lambda - \alpha_5^2)$	$(\lambda - \alpha_2^2)(\lambda^2 - (\alpha_4^2 + \alpha_6^2)\lambda + \alpha_4^2\alpha_6^2)$
$\frac{1}{2}$	$s_{\alpha_4} = w_1w_2w_1$	$(\lambda^2 - (\alpha_1^2 + \alpha_5^2)\lambda + \alpha_1^2\alpha_5^2)(\lambda - \alpha_3^2)$	$(\lambda - \alpha_4^2)(\lambda^2 - (\alpha_2^2 + \alpha_6^2)\lambda + \alpha_2^2\alpha_6^2)$
$\frac{1}{2}$	$s_{\alpha_6} = w_2w_1w_2w_1w_2$	$(\lambda^2 - (\alpha_5^2 + \alpha_3^2)\lambda + \alpha_3^2\alpha_5^2)(\lambda - \alpha_1^2)$	$(\lambda - \alpha_6^2)(\lambda^2 - (\alpha_2^2 + \alpha_4^2)\lambda + \alpha_2^2\alpha_4^2)$
$\frac{1}{2}$	1	$(\lambda - \alpha_1^2)(\lambda - \alpha_3^2)(\lambda - \alpha_5^2)$	$(\lambda - \alpha_2^2)(\lambda - \alpha_4^2)(\lambda - \alpha_6^2)$
$\frac{1}{3}, \frac{2}{3}$	$(w_2w_1)^3$	$\lambda^2 - \alpha_1^2\alpha_3^2\alpha_5^2$	$\lambda^2 - \alpha_2^2\alpha_4^2\alpha_6^2$
$\frac{1}{3}, \frac{2}{3}$	1	$(\lambda - \alpha_1\alpha_3\alpha_5)(\lambda + \alpha_1\alpha_3\alpha_5)$	$(\lambda - \alpha_2\alpha_4\alpha_6)(\lambda + \alpha_2\alpha_4\alpha_6)$
$\frac{1}{6}, \frac{5}{6}$	1	$\lambda - \alpha_1^2\alpha_3^2\alpha_5^2$	$\lambda - \alpha_2^2\alpha_4^2\alpha_6^2$

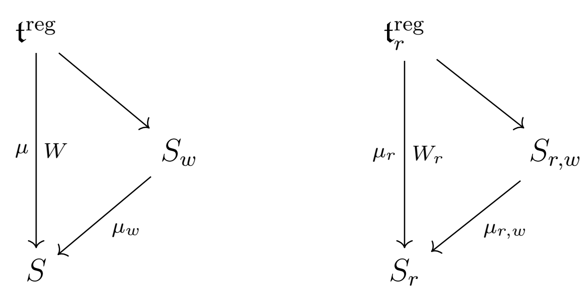
The method used above to determine all factorizations of the polynomial $P(\lambda)$ over $\mathbb{Z}[\mathfrak{t}]$ may be applied to the polynomials $P_r(\lambda)$ over $\mathbb{Z}[\mathfrak{t}_r]$, for each $r \in \frac{1}{6}\mathbb{Z}$. Chapter 3 lists the morphisms

$$\mu_r : \mathfrak{t}_r^{\mathrm{reg}} \to S_r$$

and the factors

$$\mu_{r,w} : S_{r,w} \to S_r,$$

for every $r \in \frac{1}{6}\mathbb{Z}$ and every $w \in W_r$. The results are summarized in Table 2.4 where the case $\{r\} = 0$ is omitted because that case corresponds to Table 2.3. Again arguing as above, we see that the factorizations in Tables 2.3 and 2.4 correspond to factorizations $\mathfrak{t}_r^{\mathrm{reg}} \to S_{r,w} \to S_r$ of $\mathfrak{t}_r^{\mathrm{reg}} \to S_r$, for $w \in W_r$.



Now fix $r \in \frac{1}{6}\mathbb{Z}$ and consider the family of scheme morphisms $\{\mu_{r,w} : S_{r,w} \to S_r \mid w \in W_r\}$. The partition of Φ_r into $\langle w \rangle$-orbits determines a partial order $(<)$ on W_r, corresponding to 'finer' factorizations of P_r: $w \leq w' \Leftrightarrow$ the factorization corresponding to w divides into the factorization corresponding to w'. This is not the same as the Bruhat order. Thus 1 is minimal and the Coxeter elements $w_1 w_2$ and $w_2 w_1$ are maximal. For instance, if $r \in \mathbb{Z}$ then $1 < w_2 < (w_2 w_1)^2 < w_2 w_1$. Moreover, $w \leq w'$ implies the existence of a canonical map $S_{r,w'} \to S_{r,w}$ over S_r. For each $w \in W$, let $S_r^w \subseteq S_r$ be the definable subset given by the rule

$$S_r^w := \mu_{r,w}(S_{r,w}) \setminus \cup_{w<w'} \mu_{r,w'}(S_{r,w'}).$$

The definable subsets $S_r^w \subseteq S_r$ are also recorded in Chapter 3.

The definable subsets $S_r^w \subseteq S_r$ determine the index sets I_s and I'_s appearing in $K[\lambda]/(Q_s(\lambda))$, as follows. Suppose $s \in S_r(k)$. Then $s \in S_r^w(k)$ for a unique $w \in W_r$. This w determines the factorization of $R_s(\lambda)$ and $R'_s(\lambda)$ into *irreducible* polynomials over k and thus the index sets I_s and I'_s appearing in $K[\lambda]/(Q_s(\lambda))$.

2.8 Galois representations

Having found all irreducible factors of $R_s(\lambda)$, for every $s \in S_r(k)$, we may now find the splitting extensions K_s; Table 2.5 records the results.

Following the strategy of Section 2.5, Table 2.6 records a tame Galois representation $\rho_s : \mathrm{Gal}(\bar{K}/K) \to W$ for each $s \in S_r^w(k)$ and thus defines a tamely ramified algebraic torus T_s for each $s \in S_r^w(k)$. Here we say a few words about the calculation of $H^1_{\mathrm{tr}}(K, W)$ above, using the inflation-restriction sequence

$$1 \to H^1(k, W) \to H^1_{\mathrm{tr}}(K, W) \to H^1_{\mathrm{tr}}(K^{\mathrm{nr}}, W)^{\mathrm{Fr}}.$$

First, we observe that

$$H^1(k, W) \cong W/W\text{-conj},$$

since $\mathrm{Gal}(\bar{k}/k) \cong \hat{\mathbb{Z}}$, and $H^1(k, W) \to H^1_{\mathrm{tr}}(K, W)$ is injective [Ser02]. Thus, the part of $H^1_{\mathrm{tr}}(K, W)$ corresponding to the case $\{r\} = 0$ is exactly the image of $H^1(k, W)$ in $H^1_{\mathrm{tr}}(K, W)$, which is $H^1(\mathrm{Gal}(K^{\mathrm{nr}}/K), W)$; clearly, this is the unramified part of $H^1_{\mathrm{tr}}(K, W)$. We fix a lift σ of Frobenius. Next, we observe that

$$H^1_{\mathrm{tr}}(K^{\mathrm{nr}}, W)^{\mathrm{Fr}} \cong W[q-1]/W\text{-conj}$$

Table 2.5: The splitting extension of lifts of $R_s(\lambda)$ for $s \in S_r^w(k)$ and $s = \mu_{r,w}(x)$

$\{r\}$	$w \in W_r$	lift of $R_s(\lambda)$	K_s
0	w_2w_1	$\lambda^6 + 2\pi^{2r}\dot{x}_2\lambda^4 + \pi^{4r}\dot{x}_2^2\lambda^2 - \pi^{6r}\dot{x}_1$ $\lambda^6 + 2x_2\lambda^4 + x_2^2\lambda^2 - x_1$	$K^{(6)} = K(\dot{\zeta})$ $\zeta^6 + 2x_2\zeta^4 + x_2^2\zeta^2 - x_1 = 0$
0	$(w_2w_1)^2$	$(\lambda^3 + \pi^{2r}\dot{x}_2\lambda + \pi^{3r}\dot{x}_1)(\lambda^3 + \pi^{2r}\dot{x}_2\lambda - \pi^{3r}\dot{x}_1)$ $(\lambda^3 + x_2\lambda + x_1)(\lambda^3 + x_2\lambda - x_1)$	$K^{(3)} = K(\dot{\zeta})$ $\zeta^3 + x_2\zeta + x_1 = 0$
0	$(w_2w_1)^3$	$(\lambda^2 - \pi^{2r}\dot{x}_1)(\lambda^2 - \pi^{2r}\dot{x}_2)(\lambda^2 - \pi^{2r}\dot{x}_3)$ $(\lambda^2 - x_1)(\lambda^2 - x_2)(\lambda^2 - x_3)$	$K^{(2)} = K(\dot{\zeta})$ $\zeta^2 - x_1 = 0$
0	w_1	$(\lambda^2 - \pi^{2r}\dot{x}_1)(\lambda^2 + \pi^{r}\dot{x}_3\lambda + \pi^{2r}\dot{x}_2)(\lambda^2 - \pi^{r}\dot{x}_3\lambda + \pi^{2r}\dot{x}_2)$ $(\lambda^2 - x_1)(\lambda^2 + x_3\lambda + x_2)(\lambda^2 - x_3\lambda + x_2)$	$K^{(2)} = K(\dot{\zeta})$ $\zeta^2 - x_1 = 0$
0	w_2	$(\lambda^2 + \pi^{r}\dot{x}_2\lambda + \pi^{2r}\dot{x}_1)(\lambda^2 - \pi^{r}\dot{x}_2\lambda + \pi^{2r}\dot{x}_1)(\lambda^2 - \pi^{2r}\dot{x}_2^2)$ $(\lambda^2 + x_2\lambda + x_1)(\lambda^2 - x_2\lambda + x_1)(\lambda^2 - x_2^2)$	$K^{(2)} = K(\dot{\zeta})$ $\zeta^2 + x_2\zeta + x_1 = 0$
0	1	$(\lambda^2 - \pi^{2r}\dot{x}_1^2)(\lambda^2 - \pi^{2r}\dot{x}_2^2)(\lambda^2 - \pi^{2r}\dot{x}_3^2)$ $(\lambda^2 - x_1^2)(\lambda^2 - x_2^2)(\lambda^2 - x_3^2)$	K
$\frac{1}{2}$	$(w_2w_1)^2$	$\lambda^6 + 2\pi^{2r}\dot{x}_2\lambda^4 + \pi^{4r}\dot{x}_2^2\lambda^2 - \pi^{6r}\dot{x}_1$ $\lambda^3 + 2x_2\lambda^2 + x_2^2\lambda - x_1$	$K^{(3)}(\sqrt{\pi\dot{\zeta}})$ $\zeta^3 + 2x_2\zeta^2 + x_2^2\zeta - x_1 = 0$
$\frac{1}{2}$	w_2	$(\lambda^2 - \pi^{2r}\dot{x}_3)(\lambda^4 - \pi^{2r}\dot{x}_2\lambda^2 + \pi^{4r}\dot{x}_1)$ $(\lambda - x_3)(\lambda^2 - x_2\lambda + x_1)$	$K^{(2)}(\sqrt{\pi\dot{\zeta}}, \sqrt{\pi\dot{x}_3})$ $\zeta^2 - x_2\zeta + x_1 = 0$
$\frac{1}{2}$	1	$(\lambda^2 - \pi^{2r}\dot{x}_1)(\lambda^2 - \pi^{2r}\dot{x}_2)(\lambda^2 - \pi^{2r}\dot{x}_3)$ $(\lambda - x_1)(\lambda - x_2)(\lambda - x_3)$	$K(\sqrt{\pi\dot{x}_1}, \sqrt{\pi\dot{x}_2}, \sqrt{\pi\dot{x}_3})$
$\frac{1}{3}$, $\frac{2}{3}$	$(w_2w_1)^3$	$\lambda^6 - \pi^{6r}\dot{x}_1$ $\lambda^2 - x_1$	$K^{(2)}(\sqrt[3]{\pi\dot{\zeta}})$, $K^{(2)}(\sqrt[3]{\pi^2\dot{\zeta}})$ $\zeta^2 - x_1 = 0$
$\frac{1}{3}$, $\frac{2}{3}$	1	$\lambda^6 - \pi^{6r}\dot{x}_1^2$ $\lambda^2 - x_1^2$	$K(\sqrt[3]{\pi\dot{x}_1}, \sqrt[3]{-\pi\dot{x}_1})$, $K(\sqrt[3]{\pi^2\dot{x}_1}, \sqrt[3]{-\pi^2\dot{x}_1})$
$\frac{1}{6}$, $\frac{5}{6}$	1	$\lambda^6 - \pi^{6r}\dot{x}_1$ $\lambda - x_1$	$K(\zeta_3, \sqrt[6]{\pi\dot{x}_1})$, $K(\zeta_3, \sqrt[6]{\pi^5\dot{x}_1})$

as pointed sets, since $\mathrm{Gal}(K^{\mathrm{tr}}/K^{\mathrm{nr}}) \cong (\hat{\mathbb{Z}}/\mathbb{Z}_p)(1)$ as a $\mathrm{Gal}(\bar{k}/k)$-module. We fix a topological generator τ for $\mathrm{Gal}(K^{\mathrm{tr}}/K^{\mathrm{nr}})$. Then, for every $\rho \in Z^1_{\mathrm{tr}}(K, W)$,

$$\rho(\sigma\tau\sigma^{-1}) = \rho(\tau)^q.$$

This makes it easy to build ρ from $\rho(\sigma)$ and $\rho(\tau)$. Case-by-case calculations are given in Chapter 4; Table 2.6 records the results.

2.9 Proof of Proposition 2.1

To see that $\mu_{r/K} : \mathfrak{g}(r, K) \to S_r(k)$ is surjective and that its fibres are thickened stable orbits, we argue as in [CH04, Thm 4.4]. Suppose $s \in S_r(k)$. Then $s \in S_r^w(k)$ for a unique $w \in W$. Then $\mathfrak{t}_s := \mathrm{Lie}\, T_s$ admits an embedding into $\mathfrak{g}$ as a Cartan subalgebra. Let $P(\lambda) \in K[\lambda]$ be any lift of $P_s(\lambda) \in k[\lambda]$. Then P determines a stable conjugacy class $\mathcal{O}_s(K) \subset \mathfrak{g}(K)$ that intersects $\mathfrak{t}_s(K)$. Any $X \in \mathfrak{t}_s(K) \cap \mathcal{O}_s(K)$ maps to s under $\mu_{r/K}$. This shows that $\mu_{r/K} : \mathfrak{g}(r, K) \to S_r(k)$ is surjective. It is clear that $Z \in \mathcal{O}_r^{\mathrm{st}}(X)$ implies $\mu_{r/K}(Z) = \mu_{r/K}(X)$. To see that $\mu_{r/K}^{-1}(\mu_{r/K}(X))$ is a thickened stable orbit we suppose $\mu_{r/K}(X) = \mu_{r/K}(Y)$. Then, up to stable conjugacy, $X, Y \in \mathfrak{t}_s(K)$ and $P_X(\lambda)$ and $P_Y(\lambda)$ have the same r-reduction, so $X - Y \in \mathfrak{t}_s(K)_{r^+}$, by [CH04, Cor 3.11], so $Y \in \mathcal{O}^{\mathrm{st}}(X)$.

To see that the collection of functions $\mu_{r/K} : \mathfrak{g}(r, K) \to S_r(k)$, for K and k as above, define a map of definable subassignments $\mu_r : \mathfrak{g}(r) \to S_r$, it is sufficient to observe that $\mathcal{O}_r^{\mathrm{st}}(s) := \mu_r^{-1}(s)$ is definable and depends on $s \in S_r$ in a definable way. Both statements are clear.

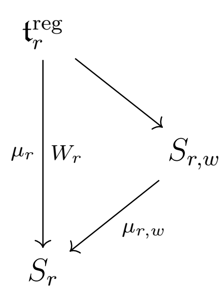
Table 2.6: Representatives $\rho_s \in Z^1_{\mathrm{tr}}(K,W)$ for $H^1_{\mathrm{tr}}(K,W)$, for all $s \in S^w_r(k)$, $s = \mu_{r,w}(x)$.

$\{r\}$ $r \in \frac{1}{6}\mathbb{Z}$	w $w \in W_r$	K_s/K $s \in S^w_r(k)$	$\rho_s(\tau) \in W$ $\tau \in \mathrm{Gal}(K^{\mathrm{tr}}/K^{\mathrm{nr}})$	$\rho_s(\sigma) \in W$ $\sigma \mapsto \mathrm{Fr}$	$\mathrm{Gal}(K_s/K)$ iso type
0	w_2w_1	$K^{(6)}$	1	w_2w_1	C_6
0	$(w_2w_1)^2$	$K^{(3)}$	1	$(w_2w_1)^2$	C_3
0	$(w_2w_1)^3$	$K^{(2)}$	1	$(w_2w_1)^3$	C_2
0	w_1	$K^{(2)}$	1	w_1	C_2
0	w_2	$K^{(2)}$	1	w_2	C_2
0	1	K	1	1	1
$\frac{1}{2}$	$(w_2w_1)^2$	$K^{(3)}(\sqrt{\pi\dot{\zeta}})$ $\zeta^3 + 2x_2\zeta^2 + x_2^2\zeta - x_1 = 0$	$(w_2w_1)^3$	$(w_2w_1)^2$	C_6
$\frac{1}{2}$	w_2	$K^{(2)}(\sqrt{\pi\dot{\zeta}}, \sqrt{\pi\dot{x}_1})$ $\zeta^2 - x_2\zeta + x_3 = 0$	$(w_2w_1)^3$	w_2	V_4
$\frac{1}{2}$	1	$K(\sqrt{\pi\dot{x}_1}, \sqrt{\pi\dot{x}_2}, \sqrt{\pi\dot{x}_3})$	$(w_2w_1)^3$	1	C_2
$\frac{1}{3},\ \frac{2}{3}$	$(w_2w_1)^3$	$K^{(2)}(\sqrt[3]{\pi\dot{\zeta}}),\ K^{(2)}(\sqrt[3]{\pi^2\dot{\zeta}})$ $\zeta^2 - x_1 = 0$	$(w_2w_1)^2$	$1, q \equiv 1(3)$ $w_2, q \equiv 2(3)$	$C_3, q \equiv 1(3)$ $S_3, q \equiv 2(3)$
$\frac{1}{3},\ \frac{2}{3}$	1	$K(\zeta_3, \sqrt[3]{\pi\dot{x}_1}),\ K(\zeta_3, \sqrt[3]{\pi^2\dot{x}_1})$	$(w_2w_1)^2$	$1, q \equiv 1(3)$ $w_2, q \equiv 2(3)$	$C_3, q \equiv 1(3)$ $S_3, q \equiv 2(3)$
$\frac{1}{6},\ \frac{5}{6}$	1	$K(\zeta_3, \sqrt[6]{\pi\dot{x}_1}),\ K(\zeta_3, \sqrt[6]{\pi^5\dot{x}_1})$	w_2w_1	$1, q \equiv 1(3)$ $w_2, q \equiv 2(3)$	$C_6, q \equiv 1(3)$ $D_6, q \equiv 2(3)$

Chapter 3

Factorizations of coverings and definable subsets

In this chapter we calculate the coverings $S_{r,w} \to S_r$ of schemes over $\mathbb{Z}[6^{-1}]$ that appeared in Section 2.7, then use these morphisms to give explicit descriptions of the definable subsets $S_r^w \hookrightarrow S_r$, for every $r \in \frac{1}{6}\mathbb{Z}$ and every $w \in W_r$.



3.1 Fractional depth 0

If the fractional depth of r is 0 (so r is an integer) then

$$P_r(\lambda) = \lambda^6 + 2s_2\lambda^4 + s_2^2\lambda^2 - s_1 \qquad \text{and} \qquad P'_r(\lambda) = \lambda^6 + 2s'_2\lambda^4 + (s'_2)^2\lambda^2 - s'_1.$$

Thus, $\Phi_r = R$ and $W_r = W$. Thus, $\mathbb{Z}[6^{-1}][\mathfrak{t}_r] = \mathbb{Z}[\Phi_r]_{|W_r|} = \mathbb{Z}[R]_6$.

Observe that $\mathbb{Z}[R]_6 \cong \mathbb{Z}[\alpha_1, \alpha_3, \alpha_5]_6 \cong \mathbb{Z}[y_1, y_2, y_3]_6/(y_1 + y_2 + y_3)$ under $y_1 = -\alpha_1$,

$y_2 = -\alpha_3$ and $y_3 = \alpha_5$. Consequently,

$$\mathfrak{t}_r^{\text{reg}} = \operatorname{Spec}(\mathbb{Z}[y_1, y_2, y_3]_{D_r}/(y_1 + y_2 + y_3)) \qquad \text{and} \qquad S_r = \operatorname{Spec}(\mathbb{Z}[s_1, s_2]_{d_r})$$

where $D_r = 6y_1^2y_2^2y_3^2(y_1 - y_2)^2(y_2 - y_3)^2(y_3 - y_1)^2$ and $d_r = -12s_1(27s_1 + 4s_2^3)$. Using this notation, the morphism $\mu_r : \mathfrak{t}_r^{\text{reg}} \to S_r$ is given by

$$\begin{aligned} \mu_r : \mathfrak{t}_r^{\text{reg}} &\to S_r \\ (y_1, y_2, y_3) &\mapsto (y_1^2y_2^2y_3^2, y_1y_2 + y_2y_3 + y_3y_1). \end{aligned}$$

Of course, it is also true that $\mathbb{Z}[R]_6 \cong \mathbb{Z}[\alpha_2, \alpha_4, \alpha_6]_6 \cong \mathbb{Z}[y_1', y_2', y_3']_6/(y_1' + y_2' + y_3')$ under $y_1' = \alpha_2$, $y_2' = \alpha_4$ and $y_3' = -\alpha_6$. Moreover, $s_1 \mapsto s_1'$ and $s_2 \mapsto s_2'$ defines an *isomorphism* $S' := \operatorname{Spec}(\mathbb{Z}[s_1', s_2']_{d_r'}) \to \operatorname{Spec}(\mathbb{Z}[s_1, s_2]_{d_r}) = S$, with d_r' defined in the obvious way; indeed, the inverse to $s_1 \mapsto s_1'$ and $s_2 \mapsto s_2'$ is given by $s_1'' = 3^6 s_1$ and $s_2'' = 3^3 s_2$. Set

$$\mathfrak{t}_r'^{\text{reg}} := \operatorname{Spec}(\mathbb{Z}[y_1', y_2', y_3']_{D_r'}/(y_1' + y_2' + y_3')) \qquad \text{and} \qquad S_r' := \operatorname{Spec}(\mathbb{Z}[s_1', s_2']_{d_r'})$$

Then the inclusion $\mathbb{Z}[R_{\text{short}}] \hookrightarrow \mathbb{Z}[R]$ induces isomorphisms $\mathfrak{t}^{\text{reg}} \to \mathfrak{t}'^{\text{reg}}$ and $S \to S'$ compatible with the map $\mu : \mathfrak{t}^{\text{reg}} \to S$. We choose to work with the short roots exclusively, for the remainder of this section, dealing with the case $\{r\} = 0$.

3.1.1 Case: $w = 1$

Since all orbits in R under the action of $\langle w \rangle = 1$ are singletons, the element $w = 1$ determines the factorization

$$P_r(\lambda) = \lambda^6 + 2s_2\lambda^4 + s_2^2\lambda^2 - s_1 = (\lambda^2 - \alpha_1^2)(\lambda^2 - \alpha_3^2)(\lambda^2 - \alpha_5^2).$$

Thus, $S_{r,w} = \mathfrak{t}_r^{\text{reg}} = \mathfrak{t}^{\text{reg}}$ and $S_{r,w} \to S_r$ is $\mu_{r,w} = \mu_r = \mu : \mathfrak{t}^{\text{reg}} \to S$ which, with reference to the notation above, is given by

$$\mathbb{Z}[s_1, s_2]_{d_r} \to \mathbb{Z}[y_1, y_2, y_3]_{D_r}/(y_1 + y_2 + y_3)$$

with $s_1 \mapsto y_1^2 y_2^2 y_3^2$ and $s_2 \mapsto y_1y_2 + y_2y_3 + y_3y_1$.

Aside: In this case, the cover $\mathfrak{t}_r^{\text{reg}} \to S_{r,w}$ is the identity on $\mathfrak{t}_r^{\text{reg}}$.

The definable subset $S_r^w \subset S_r$ attached to the morphism of affine schemes $\mu_r : \mathfrak{t}_r^{\text{reg}} \to S_r$ is $\mu_r(\mathfrak{t}_r^{\text{reg}}) \subset S_r$, which is to say,

$$S_r^1 = \{(s_1, s_2) \in S \mid \exists(y_1, y_2, y_3),\ y_1+y_2+y_3 = 0,\ (s_1, s_2) = (y_1^2y_2^2y_3^2, y_1y_2+y_2y_3+y_3y_1)\}.$$

3.1.2 Case: $w = w_1$

The action of w_1 on R determines the factorization

$$P_r(\lambda) = \lambda^6 + 2s_2\lambda^4 + s_2^2\lambda^2 - s_1 = (\lambda^2 - \alpha_1^2)(\lambda^2 - \alpha_6\lambda + \alpha_3\alpha_5)(\lambda^2 + \alpha_6\lambda + \alpha_3\alpha_5).$$

Set $x_1 = \alpha_1^2$ and $x_2 = \alpha_3\alpha_5$ and $x_3 = \alpha_3 + \alpha_5 = \alpha_6$; then $x_3^2 - 4x_2 = x_1$. Then $\mu_{r,w} : S_{r,w} \to S_r$ is given by

$$\mathbb{Z}[s_1, s_2]_{d_r} \to \mathbb{Z}[x_1, x_2, x_3]_{D_{r,w}}/(x_3^2 - 4x_2 - x_1)$$

with $s_1 \mapsto x_1x_2^2$ and $2s_2 \mapsto 2x_2 - x_3^2 - x_1$ where $D_{r,w} = \mu_{r,w}^{\#}(d_r)$.

Aside: The cover $\mathfrak{t}_r^{\text{reg}} \to S_{r,w}$ is given by $x_1 \mapsto y_1^2$ and $x_2 \mapsto -y_2y_3$ and $x_3 \mapsto y_3 - y_2$.

Since $1 < w_1$, is the only chain to w_1 in W_r, the definable subset $S_r^{w_1} \subset S_r$ attached to the morphism of affine schemes $\mu_{r,w_1} : S_{r,w_1} \to S_r$ in this case is

$$S_r^{w_1} = \mu_{r,w_1}(S_{r,w_1}) \setminus \mu_{r,1}(S_{r,1}).$$

Case: $w = w_2w_1w_2$

Since $w_2w_1w_2$ is conjugate to w_1, this case is nothing more than a re-labelling of the case $w = w_1$, above. The action of $w_2w_1w_2$ on R determines the factorization

$$P_r(\lambda) = \lambda^6 + 2s_2\lambda^4 + s_2^2\lambda^2 - s_1 = (\lambda^2 - \alpha_4\lambda + \alpha_1\alpha_5)(\lambda^2 - \alpha_3^2)(\lambda^2 + \alpha_4\lambda + \alpha_1\alpha_5).$$

Consequently, if we set $x_1 = \alpha_3^2$ and $x_2 = \alpha_1\alpha_5$ and $x_3 = \alpha_1 + \alpha_5 = \alpha_4$ then $x_3^2 - 4x_2 = x_1$, as above. Thus, $\mu_{r,w} : S_{r,w} \to S_r$ is given, again in this case, by

$$\mathbb{Z}[s_1, s_2]_{d_r} \to \mathbb{Z}[x_1, x_2, x_3]_{D_{r,w}}/(x_3^2 - 4x_2 - x_1)$$

with $s_1 \mapsto x_1x_2^2$ and $2s_2 \mapsto 2x_2 - x_3^2 - x_1$ and $D_{r,w} = \mu_{r,w}^{\#}(d_r)$, as in the case w_1, above.

Aside: However, in this case, the cover $\mathfrak{t}_r^{\mathrm{reg}} \to S_{r,w}$ is given by $x_1 \mapsto y_3^2$ and $x_2 \mapsto y_1 y_3$ and $x_3 \mapsto y_3 - y_1$.

As above, since $1 < w_2 w_1 w_2$, is the only chain to $w_2 w_1 w_2$ in W_r, the definable subset $S_r^{w_2 w_1 w_2} w \subset S_r$ attached to the morphism of affine schemes $\mu_{r,w_2 w_1 w_2} : S_{r,w_2 w_1 w_2} \to S_r$ in this case is

$$S_r^{w_2 w_1 w_2} = \mu_{r,w_2 w_1 w_2}(S_{r,w_2 w_1 w_2}) \setminus \mu_{r,1}(S_{r,1}).$$

Case: $w = w_1 w_2 w_1 w_2 w_1$

Since $w_1 w_2 w_1 w_2 w_1$ is conjugate to w_1, this case is, again, nothing more than a re-labelling of the case $w = w_1$, above. The action of $w_1 w_2 w_1 w_2 w_1$ on R determines the factorization

$$P_r(\lambda) = \lambda^6 + 2s_2\lambda^4 + s_2^2\lambda^2 - s_1 = (\lambda^2 - \alpha_5\lambda + \alpha_1\alpha_3)(\lambda^2 + \alpha_5\lambda + \alpha_1\alpha_3)(\lambda^2 - \alpha_5^2).$$

Consequently, if we set $x_1 = \alpha_5^2$ and $x_2 = \alpha_1\alpha_3$ and $x_3 = \alpha_1 + \alpha_3 = \alpha_5$ then $x_3^2 - 4x_2 = x_1$, as above. Thus, $\mu_{r,w} : S_{r,w} \to S_r$ is given in this case by

$$\mathbb{Z}[s_1, s_2]_{d_r} \to \mathbb{Z}[x_1, x_2, x_3]_{D_{r,w}}/(x_3^2 - 4x_2 - x_1)$$

with $s_1 \mapsto x_1 x_2^2$ and $2s_2 \mapsto 2x_2 - x_3^2 - x_1$ and $D_{r,w} = \mu_{r,w}^{\#}(d_r)$, as in the case w_1, above.

Aside: However, in this case, the cover $\mathfrak{t}_r^{\mathrm{reg}} \to S_{r,w}$ is given by $x_1 \mapsto y_5^2$ and $x_2 \mapsto y_1 y_3$ and $x_3 \mapsto y_3 - y_1$.

As above, since $1 < w_1 w_2 w_1 w_2 w_1$, is the only chain to $w_1 w_2 w_1 w_2 w_1$ in W_r, the definable subset $S_r^{w_1 w_2 w_1 w_2 w_1} \subset S_r$ attached to the morphism of affine schemes

$\mu_{r,w_1w_2w_1w_2w_1} : S_{r,w_1w_2w_1w_2w_1} \to S_r$ in this case is

$$S_r^{w_1w_2w_1w_2w_1} = \mu_{r,w_1w_2w_1w_2w_1}(S_{r,w_1w_2w_1w_2w_1}) \setminus \mu_{r,1}(S_{r,1}).$$

3.1.3 Case: $w = w_2$

The action of w_2 on R determines the factorization

$$P_r(\lambda) = \lambda^6 + 2s_2\lambda^4 + s_2^2\lambda^2 - s_1 = (\lambda^2 - \alpha_5\lambda + \alpha_1\alpha_3)(\lambda^2 + \alpha_5\lambda + \alpha_1\alpha_3)(\lambda - \alpha_5)(\lambda + \alpha_5).$$

Set $x_1 = \alpha_1\alpha_3$ and $x_2 = -\alpha_5$. Then $\mu_{r,w} : S_{r,w} \to S_r$ is given by

$$\mathbb{Z}[s_1, s_2]_{d_r} \to \mathbb{Z}[x_1, x_2]_{D_{r,w}}$$

with $s_1 \mapsto x_1^2 x_2^2$ and $2s_2 \mapsto 2x_1 - x_2^2$ and $D_{r,w} = \mu_{r,w}^{\#}(d_r)$.

Aside: The cover $\mathfrak{t}_r^{\mathrm{reg}} \to S_{r,w}$ is given by $x_1 \mapsto y_1y_2$ and $x_2 \mapsto -y_3$.

Since $1 < w_2$, is the only chain to w_2 in W_r, the definable subset $S_r^{w_2} \subset S_r$ attached to the morphism of affine schemes $\mu_{r,w_2} : S_{r,w_2} \to S_r$ in this case is

$$S_r^{w_2} = \mu_{r,w_2}(S_{r,w_2}) \setminus \mu_{r,1}(S_{r,1}).$$

Case: $w = w_1w_2w_1$

Since $w_1w_2w_1$ is conjugate to w_2, this case is a mere re-labelling of the case $w = w_2$, above. The action of $w_1w_2w_1$ on R determines the factorization

$$P_r(\lambda) = \left(\lambda^2 - \alpha_3\lambda - \alpha_1\alpha_5\right)(\lambda - \alpha_3)(\lambda + \alpha_3)\left(\lambda^2 + \alpha_3\lambda - \alpha_1\alpha_5\right).$$

Set $x_1 = -\alpha_1\alpha_5$ and $x_2 = -\alpha_3$. Then $\mu_{r,w} : S_{r,w} \to S_r$ is as above:

$$\mathbb{Z}[s_1, s_2]_{d_r} \to \mathbb{Z}[x_1, x_2]_{D_{r,w}}$$

with $s_1 \mapsto x_1^2 x_2^2$ and $2s_2 \mapsto 2x_1 - x_2^2$ and $D_{r,w} = \mu_{r,w}^{\#}(d_r)$.

Aside: Unlike the case above, here the cover $\mathfrak{t}_r^{\text{reg}} \to S_{r,w}$ is given by $x_1 \mapsto y_1 y_3$ and $x_2 \mapsto y_2$.

The only chain to $w_1w_2w_1$ in W_r is $1 < w_1w_2w_1$, so the definable subset $S_r^{w_1w_2w_1} \subset S_r$ attached to the morphism of affine schemes $\mu_{r,w_1w_2w_1} : S_{r,w_1w_2w_1} \to S_r$ is

$$S_r^{w_1w_2w_1} = \mu_{r,w_1w_2w_1}(S_{r,w_1w_2w_1}) \setminus \mu_{r,1}(S_{r,1}).$$

Case: $w = w_2w_1w_2w_1w_2$

Since $w_2w_1w_2w_1w_2$ is conjugate to w_2, this case is again a mere re-labelling of the case $w = w_2$, above. The action of $w_2w_1w_2w_1w_2$ on R determines the factorization

$$P_r(\lambda) = (\lambda - \alpha_1)(\lambda + \alpha_1)\left(\lambda^2 - \alpha_1\lambda - \alpha_3\alpha_5\right)\left(\lambda^2 + \alpha_1\lambda - \alpha_3\alpha_5\right).$$

Set $x_1 = -\alpha_3\alpha_5$ and $x_2 = -\alpha_1$. Then $\mu_{r,w} : S_{r,w} \to S_r$ is as above:

$$\mathbb{Z}[s_1, s_2]_{d_r} \to \mathbb{Z}[x_1, x_2]_{D_{r,w}}$$

with $s_1 \mapsto x_1^2 x_2^2$ and $2s_2 \mapsto 2x_1 - x_2^2$ and $D_{r,w} = \mu_{r,w}^{\#}(d_r)$.

Aside: Unlike the case above, here the cover $\mathfrak{t}_r^{\text{reg}} \to S_{r,w}$ is given by $x_1 \mapsto y_2 y_3$ and $x_2 \mapsto y_1$.

The only chain to $w_2w_1w_2w_1w_2$ in W_r is $1 < w_2w_1w_2w_1w_2$, so the definable subset $S_r^{w_2w_1w_2w_1w_2} \subset S_r$ attached to the morphism of affine schemes $\mu_{r,w_2w_1w_2w_1w_2} : S_{r,w_2w_1w_2w_1w_2} \to S_r$ is

$$S_r^{w_2w_1w_2w_1w_2} = \mu_{r,w_2w_1w_2w_1w_2}(S_{r,w_2w_1w_2w_1w_2}) \setminus \mu_{r,1}(S_{r,1}).$$

3.1.4 Case: $w = (w_2w_1)^3$

The action of $(w_2w_1)^3$ on R determines the factorization

$$P_r(\lambda) = \lambda^6 + 2s_2\lambda^4 + s_2^2\lambda^2 - s_1 = (\lambda^2 - \alpha_1^2)(\lambda^2 - \alpha_3^2)(\lambda^2 - \alpha_5^2).$$

Set $x_1 = \alpha_1^2$ and $x_2 = \alpha_3^2$ and $x_3 = \alpha_5^2$. Let $I_{r,w}$ be the ideal in $\mathbb{Z}[x_1, x_2, x_3]$ generated by the relation $x_1^2 + x_2^2 + x_3^2 = 2(x_1x_2 + x_2x_3 + x_3x_1)$. Then $\mu_{r,w} : S_{r,w} \to S_r$ is given by

$$\mathbb{Z}[s_1, s_2]_{d_r} \to \mathbb{Z}[x_1, x_2, x_3]_{D_{r,w}}/I_{r,w}$$

with $s_1 \mapsto x_1x_2x_3$ and $-2s_2 \mapsto x_1 + x_2 + x_3$ and $D_{r,w} = \mu_{r,w}^{\#}(d_r)$.

Aside: The cover $\mathfrak{t}_r^{\text{reg}} \to S_{r,w}$ is given by $x_1 \mapsto y_1^2$ and $x_2 \mapsto y_2^2$ and $x_3 \mapsto y_3^2$.

The only chain to $(w_2w_1)^3$ in W_r is $1 < (w_2w_1)^3$. So, the definable subset $S_r^{(w_2w_1)^3} \subset S_r$ attached to the morphism of affine schemes $\mu_{r,(w_2w_1)^3} : S_{r,(w_2w_1)^3} \to S_r$ is

$$S_r^{(w_2w_1)^3} = \mu_{r,(w_2w_1)^3}(S_{r,(w_2w_1)^3}) \setminus \mu_{r,1}(S_{r,1}).$$

3.1.5 Case: $w = (w_2w_1)^2$

The action of $(w_2w_1)^2$ on R determines the factorization

$$P_r(\lambda) = \lambda^6 + 2s_2\lambda^4 + s_2^2\lambda^2 - s_1 = (\lambda^3 + s_2\lambda + \alpha_1\alpha_3\alpha_5)(\lambda^3 + s_2\lambda - \alpha_1\alpha_3\alpha_5).$$

Set $x_1 = \alpha_1\alpha_3\alpha_5$ and $x_2 = \alpha_1\alpha_3 - \alpha_3\alpha_5 - \alpha_5\alpha_1$. Then $\mu_{r,w} : S_{r,w} \to S_r$ is given by

$$\mathbb{Z}[s_1, s_2]_d \to \mathbb{Z}[x_1, x_2]_D$$

with $s_1 \mapsto x_1^2$ and $s_2 \mapsto x_2$.

Aside: The cover $\mathfrak{t}_r^{\mathrm{reg}} \to S_{r,w}$ is given by $x_1 \mapsto y_1y_2y_3$ and $x_2 \mapsto y_1y_2 + y_2y_3 + y_3y_1$.

The complete list of elements less than $(w_2w_1)^2$ in W_r is: 1, w_2, $w_1w_2w_1$ and $w_2w_1w_2w_1w_2$. So, the definable subset $S_r^{(w_2w_1)^2} \subset S_r$ attached to the morphism of affine schemes $\mu_{r,(w_2w_1)^2} : S_{r,(w_2w_1)^2} \to S_r$ is

$$S_r^{(w_2w_1)^2} = \mu_{r,(w_2w_1)^2}\left(S_{r,(w_2w_1)^2}\right) \setminus \bigcup_{w \in (w_2)} \mu_{r,w}(S_{r,w}),$$

where (w_2) denotes the conjugacy class of w_2 in W_r.

3.1.6 Case: $w = w_2w_1$

Since w_2w_1 acts transitively on R, this element of W_r determines no factorization of $P(\lambda)$. Thus, $S_{r,w} = S_r$ in this case and $\mu_{r,w} : S_{r,w} \to S_r$ is the identity on S_r.

Aside: In this case, the cover $\mathfrak{t}_r^{\mathrm{reg}} \to S_{r,w}$ is exactly μ_r.

All elements of order less than 6 are less than w_2w_1 in W_r are, so

$$S_r^{w_2w_1} = S_r \setminus \left(\mu_{r,(w_2w_1)^2}(S_{r,(w_2w_1)^2}) \cup \mu_{r,(w_2w_1)^3}(S_{r,(w_2w_1)^3}) \bigcup_{w\in(w_1)} \mu_{r,w}(S_{r,w}) \right)$$

3.2 Fractional depth $\frac{1}{6}$ or $\frac{5}{6}$

If the fractional depth of r is $\frac{1}{6}$ or $\frac{5}{6}$ then

$$Q_r(\lambda) = P_r(\lambda)P_r'(\lambda) = (\lambda - s_1)(\lambda - s_1') = (\lambda - \alpha_1^2\alpha_3^2\alpha_5^2)(\lambda - \alpha_2^2\alpha_4^2\alpha_6^2),$$

so $\Phi_r = \{\alpha_1^2\alpha_3^2\alpha_5^2, \alpha_2^2\alpha_4^2\alpha_6^2\}$ and $W_r = 1$. Recall the notation $s_1 = \alpha_1^2\alpha_3^2\alpha_5^2$ and $s_1' = \alpha_2^2\alpha_4^2\alpha_6^2$ from Section 2.1; set $D_r = 6s_1s_1'$. Then

$$\mathfrak{t}_r^{\text{reg}} = \text{Spec}(\mathbb{Z}[s_1, s_1']_{D_r}) = S_r$$

and $\mu_r : \mathfrak{t}_r^{\text{reg}} \to S_r$ is the identity map, as are $\mu_{r,1} : S_{r,1} \to S_r$ and $\mathfrak{t}_r^{\text{reg}} \to S_{r,1}$ and $S_r^1 = S_r$.

3.3 Fractional depth $\frac{1}{3}$ or $\frac{2}{3}$

If the fractional depth of r is $\frac{1}{3}$ or $\frac{2}{3}$ then

$$Q_r(\lambda) = P_r(\lambda)P_r'(\lambda) = (\lambda^2 - \alpha_1^2\alpha_3^2\alpha_5^2)(\lambda^2 - \alpha_2^2\alpha_4^2\alpha_6^2).$$

Thus, in this case, $\Phi_r = \{\alpha_1\alpha_3\alpha_5, \alpha_2\alpha_4\alpha_6\}$ so $W_r = \langle (w_2w_1)^3 \rangle \cong C_2$. Set $y = \alpha_1\alpha_3\alpha_5$ and $y' = \alpha_2\alpha_4\alpha_6$ so $y^2 = s_1$ and $y'^2 = s_1'$. Set $D_r = 6y^2y'^2$ and $d_r = 6s_1s_1'$. Then

$$\mathfrak{t}_r^{\mathrm{reg}} = \mathrm{Spec}(\mathbb{Z}[y, y']_{D_r}) \qquad \text{and} \qquad S_r = \mathrm{Spec}(\mathbb{Z}[s_1, s_1']_{d_r}),$$

and $\mu_r : \mathfrak{t}_r^{\mathrm{reg}} \to S_r$ is given by $\mu_r^\# : s_1 \mapsto y^2$ and $\mu_r^\# : s_1' \mapsto y'^2$.

3.3.1 Case: $w = 1$

The element $1 \in W_r$ determines the factorizations

$$P_r(\lambda) = (\lambda - \alpha_1\alpha_3\alpha_5)(\lambda + \alpha_1\alpha_3\alpha_5) \qquad \text{and} \qquad P_r'(\lambda) = (\lambda - \alpha_2\alpha_4\alpha_6)(\lambda + \alpha_2\alpha_4\alpha_6).$$

Thus, $S_{r,w} = \mathfrak{t}_r^{\mathrm{reg}}$ and $S_{r,w} \to S_r$ is $\mu_{r,w} = \mu_r : \mathfrak{t}_r^{\mathrm{reg}} \to S_r$. Recall that $\mu_r^\#(s_1) = y^2$ and $\mu_r^\#(s_1') = y'^2$.

Aside: The cover $\mathfrak{t}_r^{\mathrm{reg}} \to S_{r,w}$ is the identity on $\mathfrak{t}_r^{\mathrm{reg}}$.

The definable subset $S_r^1 \subset S_r$ attached to the morphism of affine schemes $\mu_r : \mathfrak{t}_r^{\mathrm{reg}} \to S_r$ is $\mu_r(\mathfrak{t}_r^{\mathrm{reg}}) \subset S_r$, which is to say,

$$S_r^1 = \{(s_1, s_1') \in S_r \mid \exists y, y',\ y^2 = s_1, y'^2 = s_1'\}.$$

3.3.2 Case: $w = (w_2w_1)^3$

The element $(w_2w_1)^3 \in W_r$ determines the trivial factorization of $P_r(\lambda)$ and $P_r'(\lambda)$. Thus, $S_{r,w} \subset S_r$ and $\mu_{r,w} : S_{r,w} \to S_r$ is the identity on S_r.

Aside: In this case, the cover $\mathfrak{t}_r^{\mathrm{reg}} \to S_{r,w}$ is exactly μ_r.

The definable subset attached to $\mu_{r,1}$ is

$$S_r^{(w_2w_1)^3} = \mu_{r,(w_2w_1)^3}\left(S_{r,(w_2w_1)^3}\right) \setminus \mu_{r,1}(S_{r,1}),$$

which is to say,

$$S_r^{(w_2w_1)^3} = \{(s_1, s_1') \in S_r \mid \forall y, y',\ y^2 \neq s_1 \text{ or } y'^2 \neq s_1'\}.$$

3.4 Fractional depth $\frac{1}{2}$

If the fractional depth of r is $\frac{1}{2}$ then

$$Q_r(\lambda) = P_r(\lambda)P_r'(\lambda) = (\lambda^3 + 2s_2\lambda^2 + s_2^2\lambda - s_1)(\lambda^3 + 2s_2'\lambda^2 + (s_2')^2\lambda - s_1').$$

Thus, in this case, $\Phi_r = \{\alpha_1^2, \alpha_3^2, \alpha_5^2, \alpha_2^2, \alpha_4^2, \alpha_6^2\}$ and $W_r = \langle w_2, (w_2w_1)^2\rangle \cong S_3$. Set $y_1 = \alpha_1^2$, $y_2 = \alpha_3^2$, $y_3 = \alpha_5^2$; also set $y_1' = \alpha_2^2$, $y_2' = \alpha_4^2$, $y_3' = \alpha_6^2$. Then

$$y_1^2 + y_2^2 + y_3^2 = 2(y_1y_2 + y_2y_3 + y_3y_1);$$

let I_r be the ideal in $\mathbb{Z}[y_1, y_2, y_3]$ generated by this relation. Set $D_r = 6y_1y_2y_3$ and $d_r = -12s_1(27s_1 + 4s_2^3)$. Then

$$\mathfrak{t}_r^{\text{reg}} = \operatorname{Spec}(\mathbb{Z}[y_1, y_2, y_3]_{D_r}/I_r) \qquad \text{and} \qquad S_r = \operatorname{Spec}(\mathbb{Z}[s_1, s_2]_{d_r})$$

The morphism $\mu_r : \mathfrak{t}_r^{\text{reg}} \to S_r$ is given by $s_1 \mapsto y_1y_2y_3$ and $-2s_2 \mapsto y_1 + y_2 + y_3$. If $\mathfrak{t}_r'$ and S_r' are the schemes defined using y_1', y_2' and y_3' in place of y_1, y_2 and y_3 then

$\mathfrak{t}'_r \cong \mathfrak{t}_r$ and $S'_r \cong S_r$. Accordingly, we work only with $P_r(\lambda)$, $\mathfrak{t}_r$ and S_r, below.

3.4.1 Case: $w = 1$

The element $1 \in W_r$ determines the factorization

$$P_r(\lambda) = \lambda^3 + 2s_2\lambda^2 + s_2^2 - s_1 = (\lambda - \alpha_1^2)(\lambda - \alpha_3^2)(\lambda - \alpha_5^2).$$

Thus, $S_{r,w} = \mathfrak{t}_r^{\text{reg}}$ and $S_{r,w} \to S_r$ is $\mu_{r,w} = \mu_r : \mathfrak{t}_r^{\text{reg}} \to S_r$.

Aside: The cover $\mathfrak{t}_r^{\text{reg}} \to S_{r,w}$ is the identity on $\mathfrak{t}_r^{\text{reg}}$.

The definable subset $S_r^1 \subset S_r$ attached to the morphism of affine schemes $\mu_r : \mathfrak{t}_r^{\text{reg}} \to S_r$ is $\mu_r(\mathfrak{t}_r^{\text{reg}}) \subset S_r$, which is to say,

$$S_r^1 = \{(s_1, s_2) \in S_r \mid \exists (y_1, y_2, y_3),\ s_1 = y_1y_2y_3, -2s_2 = y_1 + y_2 + y_3\}.$$

3.4.2 Case: $w = w_2$

The element $w_2 \in W_r$ determines the factorization

$$P_r(\lambda) = \lambda^3 + 2s_2\lambda^2 + s_2^2\lambda - s_1 = (\lambda^2 - (\alpha_1^2 + \alpha_3^2)\lambda + \alpha_1^2\alpha_3^2)(\lambda - \alpha_5^2)$$

Set $x_1 = \alpha_1^2\alpha_3^2$ and $x_2 = \alpha_1^2 + \alpha_3^2$ and $x_3 = \alpha_5^2$. Then $(x_2 + x_3)^2 = 4(x_1 + x_2x_3)$; let $I_{r,w}$ be the ideal in $\mathbb{Z}[x_1, x_2, x_3]$ generated by this relation. Then $\mu_{r,w} : S_{r,w} \to S_r$ is given by

$$\mathbb{Z}[s_1, s_2]_{d_r} \to \mathbb{Z}[x_1, x_2, x_3]_{D_r}/I_{r,w}$$

with $s_1 \mapsto x_1x_3$ and $-2s_2 \mapsto x_2 + x_3$ and $D_r = 6x_1x_3$.

Aside: In this case, the cover $\mathfrak{t}_r^{\mathrm{reg}} \to S_{r,w}$ is $x_1 \mapsto y_1 y_2$, $x_2 \mapsto y_1 + y_2$ and $x_3 \mapsto y_3$.

The definable subset attached to μ_{r,w_2} is

$$S_r^{w_2} = \mu_{r,w_2}(S_{r,w_2}) \setminus \mu_{r,1}(S_{r,1}).$$

Case: $w = w_1 w_2 w_1$

Since w is conjugate to w_2 this case is like the case w_2, above. The element $w_1 w_2 w_1 \in W_r$ determines the factorization

$$P_r(\lambda) = \lambda^3 + 2s_2\lambda^2 + s_2^2 - s_1 = (\lambda^2 - (\alpha_1^2 + \alpha_5^2)\lambda + \alpha_1^2\alpha_5^2)(\lambda - \alpha_3^2)$$

Set $x_1 = \alpha_1^2\alpha_5^2$ and $x_2 = \alpha_1^2 + \alpha_5^2$ and $x_3 = \alpha_3^2$. Then the relation determining the ideal $I_{r,w}$ in $\mathbb{Z}[x_1, x_2, x_3]$ is the same as that in the case w_2 , above; likewise, $\mu_{r,w} : S_{r,w} \to S_r$ is again given by

$$\mathbb{Z}[s_1, s_2]_{d_r} \to \mathbb{Z}[x_1, x_2, x_3]_{D_r}/I_{r,w}$$

with $s_1 \mapsto x_1 x_3$ and $-2s_2 \mapsto x_2 + x_3$ and $D_r = 6x_1x_3$.

Aside: In this case, the cover $\mathfrak{t}_r^{\mathrm{reg}} \to S_{r,w}$ is $x_1 \mapsto y_1 y_3$, $x_2 \mapsto y_1 + y_3$ and $x_3 \mapsto y_2$.

The definable subset attached to $\mu_{r,w_1 w_2 w_1}$ is

$$S_r^{w_1 w_2 w_1} = \mu_{r,w_1 w_2 w_1}(S_{r,w_1 w_2 w_1}) \setminus \mu_{r,1}(S_{r,1}).$$

Case: $w = w_2w_1w_2w_1w_2$

Since w is conjugate to w_2 this case is also like the case w_2, above. The element $w = w_2w_1w_2w_1w_2 \in W_r$ determines the factorization

$$P_r(\lambda) = \lambda^3 + 2s_2\lambda^2 + s_2^2 - s_1 = (\lambda^2 - (\alpha_3^2 + \alpha_5^2)\lambda + \alpha_3^2\alpha_5^2)(\lambda - \alpha_1^2)$$

Set $x_1 = \alpha_3^2\alpha_5^2$ and $x_2 = \alpha_3^2 + \alpha_5^2$ and $x_3 = \alpha_1^2$. Then the relation determining the ideal $I_{r,w}$ in $\mathbb{Z}[x_1, x_2, x_3]$ is the same as that in the case w_2, above; likewise, $\mu_{r,w} : S_{r,w} \to S_r$ is again given by

$$\mathbb{Z}[s_1, s_2]_{d_r} \to \mathbb{Z}[x_1, x_2, x_3]_{D_r}/I_{r,w}$$

with $s_1 \mapsto x_1x_3$ and $-2s_2 \mapsto x_2 + x_3$ and $D_r = 6x_1x_3$.

Aside: In this case, the cover $\mathfrak{t}_r^{\mathrm{reg}} \to S_{r,w}$ is $x_1 \mapsto y_2y_3$, $x_2 \mapsto y_2 + y_3$ and $x_3 \mapsto y_1$.

The definable subset attached to $\mu_{r,w_2w_1w_2w_1w_2}$ is

$$S_r^{w_2w_1w_2w_1w_2} = \mu_{r,w_2w_1w_2w_1w_2}(S_{r,w_2w_1w_2w_1w_2}) \setminus \mu_{r,1}(S_{r,1}).$$

3.4.3 **Case:** $w = (w_2w_1)^2$

The element $(w_2w_1)^2 \in W_r$ determines the trivial factorization of $P_r(\lambda) = \lambda^3 + 2s_2\lambda^2 + s_2^2\lambda - s_1$. Thus, $S_{r,w} = S_r$ and $\mu_{r,w} : S_{r,w} \to S_r$ is the identity on S_r.

Aside: In this case, the cover $\mathfrak{t}_r^{\mathrm{reg}} \to S_{r,w}$ is exactly μ_r.

The definable subset attached to $\mu_{r,(w_2w_1)^2}$ is

$$S_r^{(w_2w_1)^2} = \mu_{r,(w_2w_1)^2}\left(S_{r,(w_2w_1)^2}\right) \setminus \bigcup_{w \in (w_2)} \mu_{r,w}(S_{r,w}),$$

where (w_2) denotes the conjugacy class of w_2 in W_r.

Chapter 4

Galois cohomology: $H^1(K, W)$

In this chapter we review the calculations summarized in Tables 2.5 and 2.6. Recall that 'fractional depth' refers to the fractional part $\{r\}$ of the depth $r \in \frac{1}{6}\mathbb{Z}$. To define ρ_s we will exploit the action of $\mathrm{Gal}(K_s/K)$ on the fibre in $\check{X} \otimes K_s$ through the second component, which determines an action of W on $\check{X} \otimes K_s$ through the first component.

4.1 Fractional depth 0

If $\{r\} = 0$ and $s = (s_1, s_2) \in S_r(K)$ then

$$Q_s(\lambda) = P_s(\lambda)P_s'(\lambda) = (\lambda^6 + 2s_2\lambda^4 + s_2^2\lambda^2 - s_1)(\lambda^6 + 2s_2'\lambda^4 + (s_2')^2\lambda^2 - s_1').$$

Recall the partition

$$S_r(k) = \coprod_{w \in W_r} S_r^w(k)$$

introduced in Section 2.7 with supporting calculations presented in Chapter 3. Then each $s \in S_r(k)$ lies in $S_r^w(k)$ for a unique $w \in W_r$. We will find the splitting extension K_s of every polynomial with r-reduction $Q_s(\lambda)$, for every $s \in S_r^w(k)$. In the cases below, we consider only $P_s(\lambda)$ since it contains all the needed information.

4.1.1 Case: $w = 1 \in W_0$

Suppose $s \in S_r^1(k)$. Then $s = \mu_{r,1}(x)$ for some $x = (x_1, x_2, x_3) \in S_{r,1}(k)$; see Section 3.1.1. Then

$$P_s(\lambda) = \lambda^6 + 2s_2\lambda^4 + s_2^2\lambda^2 - s_1 = (\lambda^2 - x_1^2)(\lambda^2 - x_2^2)(\lambda^2 - x_3^2)$$

so $P_s(\lambda)$ splits in $k[\lambda]$, and any lift

$$P(\lambda) = \lambda^6 + 2\pi^{2r}\dot{s}_2\lambda^4 + \pi^{4r}\dot{s}_2^2\lambda^2 - \pi^{6r}\dot{s}_1 = (\lambda^2 - \pi^{2r}\dot{x}_1^2)(\lambda^2 - \pi^{2r}\dot{x}_2^2)(\lambda^2 - \pi^{2r}\dot{x}_3^2).$$

splits in $K[\lambda]$.

In this case, $\rho_s : \mathrm{Gal}(\bar{K}/K) \to W$ is trivial.

4.1.2 Case: $w \in (w_1) \subset W_0$

Suppose w lies in the conjugacy class of w_1 in W_0 and $s \in S_r^w(k)$; without loss of generality, suppose $w = w_1$. Then $s = \mu_{r,w_1}(x)$ for some $x = (x_1, x_2, x_3) \in S_{r,w_1}(k)$. Then

$$P_s(\lambda) = \lambda^6 + 2s_2\lambda^4 + s_2^2\lambda^2 - s_1 = (\lambda^2 - x_1)(\lambda^2 - x_3\lambda + x_2)(\lambda^2 + x_3\lambda + x_2)$$

is the decomposition of $P_s(\lambda)$ into irreducible monic factors in $k[\lambda]$. Let ζ be a root of the irreducible factor $\lambda^2 - x_1$ of $P_s(\lambda)$. Then $[k(\zeta) : k] = 2$. Let $K^{(2)}$ be the unique unramified extension of K of degree 2. Then the factors of any lift

$$P(\lambda) = (\lambda^2 - \pi^{2r}\dot{x}_1)(\lambda^2 + \pi^r\dot{x}_3\lambda + \pi^{2r}\dot{x}_2)(\lambda^2 - \pi^r\dot{x}_3\lambda + \pi^{2r}\dot{x}_2)$$

are also irreducible and the splitting extension of this polynomial is $K^{(2)} = K(\dot{\zeta})$ where $\dot{\zeta}$ is any lift of $\zeta \in k(\zeta)$ to $K^{(2)}$.

With reference to Section 5.1.2, pick $y = (y_1, y_2, y_3) \in \check{X} \otimes K^{(2)} = \mathfrak{t}(K^{(2)})$, regular, such that its image under $\mu_{r/K^{(2)}} : \mathfrak{t}^{\text{reg}}(K^{(2)}) \to S_r(K^{(2)})$ is a lift of $s = \mu_{r,w}(x) \in S_r(k)$. Then, without loss of generality, $y_1 = \dot{\zeta}$ with $\zeta = \sqrt{x_1}$. Let $\sigma \in \text{Gal}(K^{(2)}/K)$ be the element defined by $\sigma(\dot{\zeta}) = -\dot{\zeta}$. Then, comparing the form of $P_s(\lambda)$ with $P_r(\lambda)$ from Section 5.1.2, we have

$$\sigma(y_1, y_2, y_3) = (-y_1, -y_3, -y_2) = w_1(y_1, y_2, y_3).$$

In this way we determine a homomorphism $\text{Gal}(\bar{K}/K) \to W$ with $\rho_s(\sigma) = w_1$. Since $\text{Gal}(K^{(2)}/K) = \langle \sigma \rangle$, this determines $\rho_s : \text{Gal}(\bar{K}/K) \to W$ with kernel $\text{Gal}(\bar{K}/K^{(2)})$ and image $\langle w_1 \rangle \subset W$.

4.1.3 Case: $w \in (w_2) \subset W_0$

Suppose w lies in the conjugacy class of w_2 in W_r and $s \in S_r^w(k)$; without loss of generality, suppose $w = w_2$. Then $s = \mu_{r,w_2}(x)$ for some $x = (x_1, x_2) \in S_{r,w_2}(k)$. Then

$$P_s(\lambda) = \lambda^6 + 2s_2\lambda^4 + s_2^2\lambda^2 - s_1 = (\lambda^2 + x_2\lambda + x_1)(\lambda^2 - x_2\lambda + x_1)(\lambda + x_2)(\lambda - x_2)$$

is the decomposition of $P_s(\lambda)$ into irreducible monic factors in $k[\lambda]$. Let ζ be a root of the irreducible quadratic factor $\lambda^2 + x_2\lambda + x_1$ of $P_s(\lambda)$; write $\zeta = \frac{-x_2 + \sqrt{x_2^2 - 4x_1}}{2}$. Then $[k(\zeta) : k] = 2$. Let $K^{(2)}$ be the unique unramified extension of K of degree 2. Then the factors of any lift

$$P(\lambda) = (\lambda^2 + \pi^r \dot{x}_2\lambda + \pi^{2r}\dot{x}_1)(\lambda^2 - \pi^r \dot{x}_2\lambda + \pi^{2r}\dot{x}_1)(\lambda^2 - \pi^{2r}\dot{x}_1^2)$$

are also irreducible and the splitting extension of this polynomial is $K^{(2)} = K(\dot{\zeta})$ where $\dot{\zeta}$ is any lift of $\zeta \in k(\zeta)$ to $K^{(2)}$.

With reference to Section 3.1.3, pick $y = (y_1, y_2, y_3) \in \check{X} \otimes K^{(2)} = \mathfrak{t}(K^{(2)})$, regular, such that its image under $\mu_{r/K^{(2)}} : \mathfrak{t}^{\text{reg}}(K^{(2)}) \to S_r(K^{(2)})$ is a lift of $s = \mu_{r,w}(x) \in S_r(k)$. Let $\sigma \in \text{Gal}(K^{(2)}/K)$ be non-trivial. Then $\sigma(\sqrt{x_2^2 - 4x_1}) = -\sqrt{x_2^2 - 4x_1}$. Then, comparing the form of $P_s(\lambda)$ with $P_r(\lambda)$ from Section 3.1.3, we have

$$\sigma(y_1, y_2, y_3) = (y_2, y_1, y_3) = w_2(y_1, y_2, y_3).$$

In this way we determine a homomorphism $\text{Gal}(\bar{K}/K) \to W$ with $\rho_s(\sigma) = w_2$. Since $\text{Gal}(K^{(2)}/K) = \langle\sigma\rangle$, this determines $\rho_s : \text{Gal}(\bar{K}/K) \to W$ with kernel $\text{Gal}(\bar{K}/K^{(2)})$ and image $\langle w_2\rangle \subset W$.

4.1.4 Case: $w = (w_2w_1)^3 \in W_0$

Suppose $s \in S_r^{(w_2w_1)^3}(k)$, so $s = \mu_{r,(w_2w_1)^3}(x)$ for some $x = (x_1, x_2, x_3) \in S_r^{(w_2w_1)^3}(k)$. Then

$$P_s(\lambda) = \lambda^6 + 2s_2\lambda^4 + s_2^2\lambda^2 - s_1 = (\lambda^2 - x_1)(\lambda^2 - x_2)(\lambda^2 - x_3)$$

is the decomposition of $P_s(\lambda)$ into irreducible monic factors in $k[\lambda]$. Let ζ be any root of the irreducible quadratic factor $\lambda^2 - x_1$ of $P_s(\lambda)$; write $\zeta = \sqrt{x_1}$. Then $[k(\zeta) : k] = 2$. Let $K^{(2)}$ be the unique unramified extension of K of degree 2. Then the factors of any lift

$$P(\lambda) = (\lambda^2 - \pi^{2r}\dot{x}_1)(\lambda^2 - \pi^{2r}\dot{x}_2)(\lambda^2 - \pi^{2r}\dot{x}_3)$$

are also irreducible and the splitting extension of this polynomial is $K^{(2)} = K(\dot{\zeta})$ where $\dot{\zeta}$ is any lift of $\zeta \in k(\zeta)$ to $K^{(2)}$.

With reference to Section 3.1.4, pick $y = (y_1, y_2, y_3) \in \check{X} \otimes K^{(2)} = \mathfrak{t}(K^{(2)})$, regular, such that its image under $\mu_{r/K^{(2)}} : \mathfrak{t}^{\text{reg}}(K^{(2)}) \to S_r(K^{(2)})$ is a lift of $s = \mu_{r,w}(x) \in S_r(k)$. Let $\sigma \in \text{Gal}(K^{(2)}/K)$ be non-trivial; then $\sigma(\zeta) = -\zeta$. Then, comparing the form of $P_s(\lambda)$ with $P_r(\lambda)$ from Section 3.1.4, we have

$$\sigma(y_1, y_2, y_3) = (-y_1, -y_2, -y_3) = (w_2 w_1)^3 (y_1, y_2, y_3).$$

In this way we determine a homomorphism $\text{Gal}(\bar{K}/K) \to W$ with $\rho_s(\sigma) = (w_2 w_1)^3$. Since $\text{Gal}(K^{(2)}/K) = \langle \sigma \rangle$, we have now determined $\rho_s : \text{Gal}(\bar{K}/K) \to W$ with kernel $\text{Gal}(\bar{K}/K^{(2)})$ and image $\langle (w_2 w_1)^3 \rangle \subset W$.

4.1.5 Case: $w = (w_2 w_1)^2 \in W_0$

Suppose $s \in S_r^{(w_2 w_1)^2}(k)$. Then $s = \mu_{r,(w_2 w_1)^2}(x)$ for some $x = (x_1, x_2) \in S_r^{(w_2 w_1)^2}(k)$. Then

$$P_s(\lambda) = \lambda^6 + 2 s_2 \lambda^4 + s_2^2 \lambda^2 - s_1 = (\lambda^3 + x_2 \lambda + x_1)(\lambda^3 + x_2 \lambda - x_1).$$

is the decomposition of $P_s(\lambda)$ into irreducible monic factors in $k[\lambda]$. Let ζ be any root of the irreducible cubic factor $\lambda^3 + x_2 \lambda + x_1$ of $P_s(\lambda)$. Then $[k(\zeta) : k] = 3$. Let $K^{(3)}$

be the unique unramified extension of K of degree 3. Then the factors of any lift

$$P(\lambda) = (\lambda^3 + \pi^{2r}\dot{x}_2\lambda + \pi^{3r}\dot{x}_1)(\lambda^3 + \pi^{2r}\dot{x}_2\lambda - \pi^{3r}\dot{x}_1)$$

are also irreducible and the splitting extension of this polynomial is $K^{(3)} = K(\dot{\zeta})$ where $\dot{\zeta}$ is any lift of $\zeta \in k(\zeta)$ to $K^{(3)}$.

With reference to Section 3.1.5, pick $y = (y_1, y_2, y_3) \in \check{X} \otimes K^{(3)} = \mathfrak{t}(K^{(3)})$, regular, such that its image under $\mu_{r/K^{(3)}} : \mathfrak{t}^{\text{reg}}(K^{(3)}) \to S_r(K^{(3)})$ is a lift of $s = \mu_{r,w}(x) \in S_r(k)$. Let $\sigma \in \text{Gal}(K^{(3)}/K)$ be non-trivial, hence a generator. Then, comparing the form of $P_s(\lambda)$ with $P_r(\lambda)$ from Section 3.1.5, we have

$$\sigma(y_1, y_2, y_3) = (y_2, y_3, y_1) = (w_2 w_1)^2 (y_1, y_2, y_3).$$

In this way we determine a homomorphism $\text{Gal}(\bar{K}/K) \to W$ with $\rho_s(\sigma) = (w_2 w_1)^2$. Since $\text{Gal}(K^{(3)}/K) = \langle \sigma \rangle$, we have now determined $\rho_s : \text{Gal}(\bar{K}/K) \to W$ with kernel $\text{Gal}(\bar{K}/K^{(3)})$ and image $\langle (w_2 w_1)^2 \rangle \subset W$.

4.1.6 Case: $w = w_2 w_1 \in W_0$

If $s = (s_1, s_2) \in S_r^{w_2 w_1}(k)$ then

$$P_s(\lambda) = \lambda^6 + 2 s_2 \lambda^4 + s_2^2 \lambda^2 - s_1 = \lambda^6 + 2 x_2 \lambda^4 + x_2^2 \lambda^2 - x_1$$

is irreducible in $k[\lambda]$. Let ζ be any root of $P_s(\lambda)$. Then $[k(\zeta):k]=6$. Let $K^{(6)}$ be the unique unramified extension of K of degree 6. Then any lift

$$P(\lambda)=\lambda^6+2\pi^{2r}\dot{x}_2\lambda^4+\pi^{4r}\dot{x}_2^2\lambda^2-\pi^{6r}\dot{x}_1\in K[\lambda]$$

is also irreducible, where $\dot{x}_1,\dot{x}_2\in\mathcal{O}_K$ are any lifts of $x_1,x_2\in k$. The splitting extension of this polynomial in $K[\lambda]$ is $K^{(6)}=K(\dot{\zeta})$ where $\dot{\zeta}$ is any lift of $\zeta\in k(\zeta)$ to $K^{(6)}$.

The lift $P(\lambda)\in K[\lambda]$ above determines a W-conjugacy class of homomorphisms $\rho_s:\mathrm{Gal}(K_s/K)\to W$ as follows. Split $P(\lambda)\in K[\lambda]$ in $K^{(6)}$:

$$P(\lambda)=(\lambda^2-\lambda_1^2)(\lambda^2-\lambda_2^2)(\lambda^2-\lambda_3^2).$$

With reference to Section 3.1.6, pick $y=(y_1,y_2,y_3)\in\check{X}\otimes K^{(6)}=\mathfrak{t}(K^{(6)})$, regular, such that its image under $\mu_{r/K^{(6)}}:\mathfrak{t}^{\mathrm{reg}}(K^{(6)})\to S_r(K^{(6)})$ is a lift of $s=\mu_{r,w}(x)\in S_r(k)$. Let $\sigma\in\mathrm{Gal}(K^{(6)}/K)$ be a generator. Then, comparing the form of $P_s(\lambda)$ with $P_r(\lambda)$ from Section 3.1.6, we have

$$\sigma(y_1,y_2,y_3)=(-y_3,-y_1,-y_2)=w_2w_1(y_1,y_2,y_3).$$

In this way we determine a homomorphism $\mathrm{Gal}(\bar{K}/K)\to W$ with $\rho_s(\sigma)=w_2w_1$. Since $\mathrm{Gal}(K^{(6)}/K)=\langle\sigma\rangle$, we have now determined $\rho_s:\mathrm{Gal}(\bar{K}/K)\to W$ with kernel $\mathrm{Gal}(\bar{K}/K^{(6)})$ and image $\langle w_2w_1\rangle\subset W$.

4.2 Fractional depth $\frac{1}{6}$ or $\frac{5}{6}$

If $\{r\} = \frac{1}{6}$ or $\frac{5}{6}$ and $s = (s_1, s_1') \in S_r(K)$ then

$$Q_s(\lambda) = P_s(\lambda)P_s'(\lambda) = (\lambda - x_1)(\lambda - x_1')$$

for $x = (x_1, x_1') \in S_r(k)$. Since $W_r = 1$, there is only one case to consider: $S_r = S_r^1$ and $P_s(\lambda)$ and $P_s'(\lambda)$ are evidently irreducible; see Section 2.7 and 3.2. Then, for any lifts $\dot{x}_1, \dot{x}_1' \in \mathcal{O}_K$, the sextic factors of

$$Q(\lambda) = (\lambda^6 - \pi^{6r}\dot{x}_1)(\lambda^6\pi^{6r}\dot{x}_1')$$

are irreducible. The splitting extension K_s of this lift is $K(\zeta_3, \sqrt[6]{\pi\dot{x}_1}) = K(\zeta_3, \sqrt[6]{\pi\dot{x}_1'})$ if $\{r\} = \frac{1}{6}$ and is $K(\zeta_3, \sqrt[6]{\pi^5\dot{x}_1}) = K(\zeta_3, \sqrt[6]{\pi^5\dot{x}_1'})$ if $\{r\} = \frac{5}{6}$.

Next, we see how s determines a representation $\rho_s : \mathrm{Gal}(K_s/K) \to W$, unique up to W-conjugation. Split $\lambda^6 - \pi^{6r}\dot{x}_1$ in K_s:

$$\lambda^6 - \pi^{6r}\dot{x}_1 = (\lambda - \theta)(\lambda - \zeta_3\theta)(\lambda - \zeta_3^2\theta)(\lambda + \theta)(\lambda + \zeta_3\theta)(\lambda + \zeta_3^2\theta).$$

where $\theta = \pi^r\sqrt[6]{\dot{x}_1}$ if $\{r\} = \frac{1}{6}$ and $\theta = \pi^r\sqrt[6]{\pi^5\dot{x}_1}$ if $\{r\} = \frac{5}{6}$. Set $y_1 = \theta$, $y_2 = \zeta_3\theta$ and $y_3 = \zeta_3^2\theta$. With reference to the notation of Table 2.6, define $\sigma \in \mathrm{Gal}(K_s/K)$ by $\sigma(\zeta_3) = \zeta_3^2$ if $q \equiv 2 \mod 3$ and $\sigma(\zeta_3) = \zeta_3$ if $q \equiv 1 \mod 3$; then

$$\sigma(y_1, y_2, y_3) = \begin{cases} (y_1, y_3, y_2) = w_2w_1w_2w_1w_2(y_1, y_2, y_3), & q \equiv 2 \mod 3; \\ (y_1, y_2, y_3) & q \equiv 1 \mod 3. \end{cases}$$

Define $\tau \in \mathrm{Gal}(K_s/K)$ by $\tau(\theta) = \zeta_6\theta$ where $\zeta_6 := -\zeta_3^2$, a primitive sixth root-of-unity in K_s; then,

$$\tau(y_1, y_2, y_3) = (-y_3, -y_1, -y_2) = w_2w_1(y_1, y_2, y_3).$$

Since

$$\sigma\tau\sigma^{-1} = \tau^q,$$

this completely defines a homomorphism $\mathrm{Gal}(K_s/K) \to W$. We conjugate this homomorphism by w_1w_2 to define $\rho_s : \mathrm{Gal}(\bar{K}/K) \to W$ appearing in Table 2.5; note that the image W_s of ρ_s is $\langle w_2w_1\rangle$ if $q \equiv 1 \bmod 3$ while the image of ρ_s is $\langle w_2, w_2w_1\rangle = W$ if $q \equiv 2 \mod 3$.

4.3 Fractional depth $\frac{1}{3}$ or $\frac{2}{3}$

If $\{r\} = \frac{1}{3}$ or $\frac{2}{3}$ and $s = (s_1, s_1') \in S_r(K)$ then

$$Q_s(\lambda) = P_s(\lambda)P_s'(\lambda) = (\lambda^2 - s_1)(\lambda^2 - s_1').$$

In this case, $W_r = \langle(w_2w_1)^3\rangle$.

4.3.1 Case: $w = 1$

Suppose $s \in S_r^1(k)$. Then, using Section 3.3.1, $s = \mu_{r,1}(x)$ is given by $\mu_{r,1}(x_1, x_1') = (x_1^2, {x_1'}^2)$. Thus,

$$P_s(\lambda) = (\lambda - x_1)(\lambda + x_1) \qquad \text{and} \qquad P_s'(\lambda) = (\lambda - x_1')(\lambda + x_1').$$

so $P_s(\lambda)$ splits in $k[\lambda]$. Consider a lift to $K[\lambda]$:

$$P(\lambda) = (\lambda^3 - \pi^{3r}\dot{x}_1)(\lambda^3 + \pi^{3r}\dot{x}_1) \qquad \text{and} \qquad P'(\lambda) = (\lambda^3 - \pi^{3r}\dot{x}'_1)(\lambda^3 + \pi^{3r}\dot{x}'_1).$$

The splitting extension K_s of $P(\lambda)P'(\lambda)$ is $K(\zeta_3, \sqrt[3]{\pi\dot{x}_1})$ if $\{r\} = \frac{1}{3}$ and $K(\zeta_3, \sqrt[3]{\pi^2\dot{x}_1})$ if $\{r\} = \frac{2}{3}$.

We now define the representation $\rho_s : \mathrm{Gal}(K_s/K) \to W$ appearing in Table 2.5. Split $P(\lambda)$ in K_s:

$$(\lambda^3 - \pi^{3r}\dot{x}_1)(\lambda^3 + \pi^{3r}\dot{x}_1) = (\lambda - \theta)(\lambda - \zeta_3\theta)(\lambda - \zeta_3^2\theta)(\lambda + \theta)(\lambda + \zeta_3\theta)(\lambda + \zeta_3^2\theta),$$

where $\theta = \pi^r\sqrt[3]{\pi\dot{x}_1}$ if $\{r\} = \frac{1}{3}$ and $\theta = \pi^r\sqrt[3]{\pi^2\dot{x}_1}$ if $\{r\} = \frac{2}{3}$ and where ζ_3 is a primitive third root-of-unity in K_s. As above, set $y_1 = \theta$, $y_2 = \zeta_3\theta$ and $y_3 = \zeta_3^2\theta$, and define $\sigma \in \mathrm{Gal}(K_s/K)$ by $\sigma(\zeta_3) = \zeta_3^2$ if $q \equiv 2 \mod 3$ and $\sigma(\zeta_3) = \zeta_3$ if $q \equiv 1 \mod 3$; then

$$\sigma(y_1, y_2, y_3) = \begin{cases} (y_1, y_3, y_2) = w_2w_1w_2w_1w_2(y_1, y_2, y_3), & q \equiv 2 \mod 3; \\ (y_1, y_2, y_3) & q \equiv 1 \mod 3. \end{cases}$$

Define $\tau \in \mathrm{Gal}(K_s/K)$ by $\tau(\theta) = \zeta_3\theta$; then,

$$\tau(y_1, y_2, y_3) = (y_2, y_3, y_1) = (w_2w_1)^2(y_1, y_2, y_3).$$

Since

$$\sigma\tau\sigma^{-1} = \tau^q,$$

this completely defines a homomorphism $\mathrm{Gal}(K_s/K) \to W$. We conjugate this homomorphism by w_1w_2 to define $\rho_s : \mathrm{Gal}(\bar{K}/K) \to W$ appearing in Table 2.5; note that the image W_s of ρ_s is $\langle (w_2w_1)^2 \rangle$ if $q \equiv 1 \mod 3$ while the image of ρ_s is $\langle w_2, (w_2w_1)^2 \rangle$ if $q \equiv 2 \mod 3$.

4.3.2 Case: $w = (w_2w_1)^3$

Suppose $s \in S_r^{(w_2w_1)^3}(k)$. Recall from Section 3.3.2 that $\mu_{r,(w_2w_1)^3} : S_{r,w} \to S_r$ is the identity. Here,

$$Q_s(\lambda) = P_s(\lambda)P_s'(\lambda) = (\lambda^2 - s_1)(\lambda^2 - s_1'),$$

and $P_s(\lambda)$ and $P_s'(\lambda)$ are irreducible in $k[\lambda]$. Let ζ be a root of $P_s(\lambda) = \lambda^2 - s_1$; thus, $\zeta = \sqrt{s_1}$; let ζ' be a root of $P_s'(\lambda) = \lambda^2 - s_1'$; thus, $\zeta' = \sqrt{s_1'}$. Consider a lift of $P_s(\lambda)$ to $K[\lambda]$:

$$P(\lambda) = \lambda^6 - \pi^{6r}\dot{s}_1 = (\lambda^3 - \pi^{3r}\dot{\zeta})(\lambda^3 + \pi^{3r}\dot{\zeta}).$$

Then the splitting extension K_s of $P(\lambda)$ is $K^{(2)}(\sqrt[3]{\pi\dot{\zeta}})$ if $\{r\} = \frac{1}{3}$ and $K^{(2)}(\sqrt[3]{\pi^2\dot{\zeta}})$ if $\{r\} = \frac{2}{3}$. Let ζ be a root of the irreducible factor $\lambda^2 - x_1$ of $P_s(\lambda)$. Again $[k(\zeta) : k] = 2$, and $K^{(2)} = K(\dot{\zeta})$ where $\dot{\zeta}$ is any lift of $\zeta \in k(\zeta)$ to $K^{(2)}$.

We now define the representation $\rho_s : \mathrm{Gal}(K_s/K) \to W$ appearing in Table 2.5. Split $P(\lambda)$ in K_s:

$$(\lambda^3 - \pi^{3r}\dot{\zeta})(\lambda^3 + \pi^{3r}\dot{\zeta}) = (\lambda - \theta)(\lambda - \zeta_3\theta)(\lambda - \zeta_3^2\theta)(\lambda + \theta)(\lambda + \zeta_3\theta)(\lambda + \zeta_3^2\theta),$$

where $\theta = \pi^r\sqrt[3]{\pi\dot{\zeta}}$ if $\{r\} = \frac{1}{3}$ and $\theta = \pi^r\sqrt[3]{\pi^2\dot{\zeta}}$ if $\{r\} = \frac{2}{3}$ and where ζ_3 is a primitive third root-of-unity in K_s. As above, define $\sigma \in \mathrm{Gal}(K_s/K)$ by $\sigma(\zeta_3) = \zeta_3^2$ if $q \equiv 2$

mod 3 and $\sigma(\zeta_3) = \zeta_3$ if $q \equiv 1 \mod 3$; then

$$\sigma(y_1, y_2, y_3) = \begin{cases} (y_1, y_3, y_2) = w_2 w_1 w_2 w_1 w_2 (y_1, y_2, y_3), & q \equiv 2 \mod 3; \\ (y_1, y_2, y_3) & q \equiv 1 \mod 3. \end{cases}$$

Define $\tau \in \mathrm{Gal}(K_s/K)$ by $\tau(\theta) = \zeta_3 \theta$; then,

$$\tau(y_1, y_2, y_3) = (y_2, y_3, y_1) = (w_2 w_1)^2 (y_1, y_2, y_3).$$

Since

$$\sigma \tau \sigma^{-1} = \tau^q,$$

this completely defines a homomorphism $\mathrm{Gal}(K_s/K) \to W$. We conjugate this homomorphism by $w_1 w_2$ to define $\rho_s : \mathrm{Gal}(\bar{K}/K) \to W$ appearing in Table 2.5; note that the image W_s of ρ_s is $\langle (w_2 w_1)^2 \rangle$ if $q \equiv 1 \mod 3$ while the image of ρ_s is $\langle w_2, (w_2 w_1)^2 \rangle$ if $q \equiv 2 \mod 3$.

4.4 Fractional depth $\frac{1}{2}$

If $\{r\} = \frac{1}{2}$ and $s = (s_1, s_2) \in S_r(K)$ then

$$P_s(\lambda) = \lambda^3 + 2 s_2 \lambda^2 + s_2^2 \lambda - s_1$$

and $W_r = \langle w_2, (w_2 w_1)^2 \rangle$. Any lift of $P_s(\lambda)$ to $K[\lambda]$ takes the form

$$P(\lambda) = \lambda^6 + 2\pi^{2r} \dot{s}_2 \lambda^4 + \pi^{4r} \dot{s}_2^2 \lambda^2 - \pi^{6r} \dot{s}_1.$$

4.4.1 Case: $w = 1 \in W_{1/2}$

Suppose $s = (s_1, s_2) \in S_r^1(k)$. Then, with reference to Section 3.4.1, $s = \mu_{r,1}(x)$ for $x = (x_1, x_2, x_3) \in S_{r,1}(k)$ and

$$P_s(\lambda) = \lambda^3 + 2s_2\lambda^2 + s_2^2\lambda - s_1 = (\lambda - x_1)(\lambda - x_2)(\lambda - x_3),$$

so $P_s(\lambda)$ splits in $k[\lambda]$. Consider a lift:

$$P(\lambda) = (\lambda^2 - \pi^{2r}\dot{x}_1)(\lambda^2 - \pi^{2r}\dot{x}_2)(\lambda^2 - \pi^{2r}\dot{x}_3)$$

Then the splitting extension of $P(\lambda)$ is $K_s = K(\sqrt{\pi\dot{x}_1}, \sqrt{\pi\dot{x}_2}, \sqrt{\pi\dot{x}_3})$. Set $y_1 = \sqrt{\pi\dot{x}_1}$ and $y_2 = \sqrt{\pi\dot{x}_2}$ and $y_3 = \sqrt{\pi\dot{x}_3}$. From the structure of $S_r^1(k)$ we find that if $\sigma \in \mathrm{Gal}(K_s/K)$ is non-trivial, then $\sigma(y_1) = -y_1$ and $\sigma(y_2) = -y_2$ and $\sigma(y_3) = -y_3$, so $\mathrm{Gal}(K_s/K) = \langle\sigma\rangle$ and, moreover,

$$\sigma(y_1, y_2, y_3) = (-y_1, -y_2, -y_3) = (w_2w_1)^3(y_1, y_2, y_3).$$

This defines $\mathrm{Gal}(K_s/K) \to W$ by $\sigma \mapsto (w_2w_1)^3$ and thus defines $\rho_s : \mathrm{Gal}(\bar{K}/K) \to W$ for $s \in S_{1/2}^1(k)$ in Table 2.6.

4.4.2 Case: $w \in (w_2) \subset W_{1/2}$

Suppose $s = (s_1, s_2) \in S_r^w(k)$. Then, with reference to Section 3.4.2, $s = \mu_{r,w}(x)$ for $x = (x_1, x_2, x_3) \in S_{r,w}(k)$ where $(x_2 + x_3)^2 = 6(x_1 + x_2x_3)$. Then

$$P_s(\lambda) = (\lambda^2 - x_2\lambda + x_1)(\lambda - x_3).$$

Let ζ be a root of the irreducible polynomial $\lambda^2 - x_2\lambda + x_1$, so $\zeta = \frac{x_2+\sqrt{x_2^2-4x_1}}{2}$; set $\zeta' = \frac{x_2-\sqrt{x_2^2-4x_1}}{2}$. Then $k(\zeta)/k$ is a splitting extension for $P_s(\lambda)$.

Consider a lift of $P_s(\lambda)$ to $K[\lambda]$:

$$P(\lambda) = (\lambda^4 - \pi^{2r}\dot{x}_2\lambda^2 + \pi^{4r}\dot{x}_3)(\lambda^2 - \pi^{2r}\dot{x}_1).$$

The splitting extension for $P(\lambda)$ over K is $K_s = K^{(2)}(\sqrt{\pi\dot{\zeta}}, \sqrt{\pi\dot{x}_1})$. Set $y_1 = \pi^r\sqrt{\pi\dot{\zeta}}$, $y_2 = \pi^r\sqrt{\pi\dot{\zeta}'}$, $y_3 = \pi^r\sqrt{\pi\dot{x}_3}$. Then

$$\sigma(y_1, y_2, y_3) = (y_2, y_1, y_3) = w_2(y_1, y_2, y_3)$$

and

$$\tau(y_1, y_2, y_3) = (-y_1, -y_2, -y_3) = (w_2w_1)^3(y_1, y_2, y_3)$$

generate $\mathrm{Gal}(K_s/K)$. Since $\sigma\tau\sigma^{-1} = \tau^q$, this defines $\mathrm{Gal}(K_s/K) \to W$ with $\sigma \mapsto w_2$ and $\tau \mapsto (w_2w_1)^3$ with image $W_s \cong V_4$. This defines $\rho_s : \mathrm{Gal}(\bar{K}/K) \to W$ in this case, as appearing in Table 2.6.

4.4.3 Case: $w = (w_2w_1)^2 \in W_{1/2}$

Suppose $s = (s_1, s_2) \in S_r^{(w_2w_1)^2}(k)$. Then, with reference to Section 3.4.3,

$$P_s(\lambda) = \lambda^3 + 2s_2\lambda^2 + s_2^2\lambda - s_1$$

is irreducible. Let ζ_1, ζ_2, ζ_3 be roots of this polynomial; let $\dot{\zeta}_1$, $\dot{\zeta}_2$, $\dot{\zeta}_3$ be lifts to $K^{(3)}$. Then $y_1 = \pi^r\sqrt{\pi\dot{\zeta}_1}$, $y_2 = \pi^r\sqrt{\pi\dot{\zeta}_2}$, $y_3 = \pi^r\sqrt{\pi\dot{\zeta}_3}$ are roots of a lift of $P_s(\lambda)$ to $P(\lambda)$.

The splitting extension K_s of this lift is $K^{(3)}(\pi^r\sqrt{\pi\dot{\zeta}})$, where ζ is any root of $P_s(\lambda)$. The Galois group $\mathrm{Gal}(K_s/K)$ is generated by σ and τ with $\sigma\tau\sigma^{-1} = \tau^q$ where

$$\sigma(y_1, y_2, y_3) = (y_2, y_3, y_1) = (w_2w_1)^2(y_1, y_2, y_3)$$

and

$$\tau(y_1, y_2, y_3) = (-y_1, -y_2, -y_3) = (w_2w_1)^3(y_1, y_2, y_3).$$

This determines $\rho_s : \mathrm{Gal}(\bar{K}/K) \to W$ in this case, with kernel $\mathrm{Gal}(K_s/K)$ and image $W_s = \langle (w_2w_1)^2, (w_2w_1)^3 \rangle = \langle w_2w_1 \rangle \cong C_6$.

Chapter 5

Galois cohomology of maximal tori

In Chapter 3 we found the sets $S_r^w(k)$ and in Chapter 4 we found the Galois representation $\rho_s : \mathrm{Gal}(\bar{K}/K) \to W$ for every $s \in S_r^w(k)$, and therefore a torus T_s over K which embeds into G over K as a maximal torus. In this chapter we find $H^1(K, T_s)$ and therefore find the cardinality of $G(K)$-conjugacy classes of embeddings of T_s into G over K; the results of this chapter are summarized in Table 6.1 where they are used to prove Theorem 1.1.

To determine $H^1(K, T_s)$ we use Tate-Nakayama ([Lan79, p. 3] or [Ser02] more generally):

$$H^1(K, T_s) = \check{X}^{\mathrm{tr}_{W_s}=0}/\check{X}_{W_s},$$

where $W_s = \rho_s(\mathrm{Gal}(\bar{K}/K))$ (so $W_s \cong \mathrm{Gal}(K_s/K)$, since $K_s = \ker \rho_s$) and

$$\check{X}^{\mathrm{tr}_{W_s}=0} = \{y \in \check{X} \mid \sum_{w \in W_s} w(y) = 0\} \qquad \text{and} \qquad \check{X}_{W_s} = \langle w(y) - y \mid y \in \check{X}, w \in W_s \rangle.$$

If $|W_s| > 2$, we use $\vee$ (the logical 'or') to separate the non-trivial cases in the calculations.

5.1 Fractional depth 0

5.1.1 Case: $w = 1$

If $s \in S_r^1(k)$ then $W_s = 1$; see Table 2.6, Section 4.1.1 and Section 3.1.1. In this case,

$$\begin{aligned}
\check{X}^{\mathrm{tr}_{W_s}=0}/\check{X}_{W_s} &= \frac{\{y \in \check{X} \mid \sum_{w \in W_s} w(y) = 0\}}{\langle w(y) - y \mid y \in \check{X}, w \in W_s\rangle} \\
&= \frac{\left\{(y_1, y_2, y_3) \mid \sum_{w \in \{1\}} w(y_1, y_2, y_3) = (0,0,0)\right\}}{\langle w(y_1, y_2, y_3) - (y_1, y_2, y_3) \mid y \in \check{X}, w \in \{1\}\rangle} \\
&= \frac{\{(y_1, y_2, y_3) \mid (y_1, y_2, y_3) = (0,0,0)\}}{\langle (y_1, y_2, y_3) - (y_1, y_2, y_3)\rangle} \\
&= \frac{\{(0,0,0)\}}{\langle (0,0,0)\rangle} \cong 0
\end{aligned}$$

5.1.2 Case: $w = w_1$

If $s \in S_r^{w_1}(k)$ then $W_s = \langle w_1 \rangle = \{1, w_1\}$; see Table 2.6, Section 4.1.2 and Section 5.1.2. In this case,

$$\begin{aligned}
\check{X}^{\mathrm{tr}_{W_s}=0}/\check{X}_{W_s} &= \frac{\{y \in \check{X} \mid \sum_{w \in W_s} w(y) = 0\}}{\langle w(y) - y \mid y \in \check{X}, w \in W_s\rangle} \\
&= \frac{\left\{(y_1, y_2, y_3) \mid \sum_{w \in \{1, w_1\}} w(y_1, y_2, y_3) = (0,0,0)\right\}}{\langle w(y_1, y_2, y_3) - (y_1, y_2, y_3) \mid y \in \check{X}, w \in \{1, w_1\}\rangle} \\
&= \frac{\{(y_1, y_2, y_3) \mid (y_1, y_2, y_3) + (-y_1, -y_3, -y_2) = (0,0,0)\}}{\langle (y_1, y_2, y_3) - (y_1, y_2, y_3), (-y_1, -y_3, -y_2) - (y_1, y_2, y_3)\rangle} \\
&= \frac{\{(y_1, y_2, y_3) \mid (0, y_2 - y_3, y_3 - y_2) = (0,0,0)\}}{\langle (0,0,0), (-2y_1, -y_3 - y_2, -y_2 - y_3)\rangle} \\
&= \frac{\{(y_1, y_2, y_3) \mid y_2 = y_3,\ y_1 = -y_2 - y_3 = -2y_2\}}{\langle (-2y_1, y_1, y_1)\rangle} \\
&= \frac{\{(-2y_2, y_2, y_2)\}}{\langle (-2y_1, y_1, y_1)\rangle} \cong \frac{\mathbb{Z}}{\mathbb{Z}} \cong 0
\end{aligned}$$

Case: $w = w_2 w_1 w_2$

Since $w_2w_1w_2$ is conjugate to w_1, this case is nothing more than a re-labelling of the case $w = w_1$, as in Section . If $s \in S_r^{w_2w_1w_2}(k)$ then $W_s = \langle w_2w_1w_2\rangle = \{1, w_2w_1w_2\}$; see Section 4.1.2 and Section 5.1.2. In this case,

$$\begin{aligned}
\check{X}^{\mathrm{tr}_{W_s}=0}/\check{X}_{W_s} &= \frac{\{y \in \check{X} \mid \sum_{w\in W_s} w(y) = 0\}}{\langle w(y) - y \mid y \in \check{X}, w \in W_s\rangle} \\
&= \frac{\left\{(y_1, y_2, y_3) \mid \sum_{w\in\{1,w_2w_1w_2\}} w(y_1, y_2, y_3) = (0, 0, 0)\right\}}{\langle w(y_1, y_2, y_3) - (y_1, y_2, y_3) \mid y \in \check{X}, w \in \{1, w_2w_1w_2\}\rangle} \\
&= \frac{\{(y_1, y_2, y_3) \mid (y_1, y_2, y_3) + (-y_3, -y_2, -y_1) = (0, 0, 0)\}}{\langle (y_1, y_2, y_3) - (y_1, y_2, y_3), (-y_3, -y_2, -y_1) - (y_1, y_2, y_3)\rangle} \\
&= \frac{\{(y_1, y_2, y_3) \mid (y_1 - y_3, 0, y_3 - y_1) = (0, 0, 0)\}}{\langle (0, 0, 0), (-y_3 - y_1, -2y_2, -y_1 - y_3)\rangle} \\
&= \frac{\{(y_1, y_2, y_3) \mid y_1 = y_3,\ y_2 = -y_1 - y_3 = -2y_1\}}{\langle (y_2, -2y_2, y_2)\rangle} \\
&= \frac{\{(y_1, -2y_1, y_1)\}}{\langle (y_2, -2y_2, y_2)\rangle} \cong \frac{\mathbb{Z}}{\mathbb{Z}} \cong 0
\end{aligned}$$

Case: $w = w_1 w_2 w_1 w_2 w_1$

Since $w_2w_1w_2w_1w_2$ is conjugate to w_1, this case is nothing more than a re-labelling of the case $w = w_1$, above. If $s \in S_r^{w_1w_2w_1w_2w_1}(k)$ then $W_s = \langle w_1w_2w_1w_2w_1\rangle = \{1, w_1w_2w_1w_2w_1\}$; see Section 4.1.2 and Section 5.1.2. In this case,

$$\begin{aligned}
\check{X}^{\mathrm{tr}_{W_s}=0}/\check{X}_{W_s} &= \frac{\{y \in \check{X} \mid \sum_{w\in W_s} w(y) = 0\}}{\langle w(y) - y \mid y \in \check{X}, w \in W_s\rangle} \\
&= \frac{\left\{(y_1, y_2, y_3) \mid \sum_{w\in\{1,w_1w_2w_1w_2w_1\}} w(y_1, y_2, y_3) = (0, 0, 0)\right\}}{\langle w(y_1, y_2, y_3) - (y_1, y_2, y_3) \mid y \in \check{X}, w \in \{1, w_1w_2w_1w_2w_1\}\rangle} \\
&= \frac{\{(y_1, y_2, y_3) \mid (y_1, y_2, y_3) + (-y_2, -y_1, -y_3) = (0, 0, 0)\}}{\langle (y_1, y_2, y_3) - (y_1, y_2, y_3), (-y_2, -y_1, -y_3) - (y_1, y_2, y_3)\rangle}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\{(y_1, y_2, y_3) \mid (y_1 - y_2, y_2 - y_1, 0) = (0,0,0)\}}{\langle (0,0,0), (-y_2 - y_1, -y_1 - y_2, -2y_3)\rangle} \\
&= \frac{\{(y_1, y_2, y_3) \mid y_1 = y_2,\ y_3 = -y_1 - y_2 = -2y_1\}}{\langle (y_3, y_3, -2y_3)\rangle} \\
&= \frac{\{(y_1, y_1, -2y_1)\}}{\langle (y_3, y_3, -2y_3)\rangle} \cong \frac{\mathbb{Z}}{\mathbb{Z}} \cong 0
\end{aligned}$$

5.1.3 Case: $w = w_2$

If $s \in S_r^{w_2}(k)$ then $W_s = \langle w_2 \rangle = \{1, w_2\}$; see Table 2.6, Section 4.1.3 and Section 3.1.3. In this case,

$$\begin{aligned}
\check{X}^{\mathrm{tr}_{W_s}=0}/\check{X}_{W_s} &= \frac{\{y \in \check{X} \mid \sum_{w \in W_s} w(y) = 0\}}{\langle w(y) - y \mid y \in \check{X}, w \in W_s\rangle} \\
&= \frac{\left\{(y_1, y_2, y_3) \mid \sum_{w \in \{1, w_2\}} w(y_1, y_2, y_3) = (0,0,0)\right\}}{\langle w(y_1, y_2, y_3) - (y_1, y_2, y_3) \mid y \in \check{X}, w \in \{1, w_2\}\rangle} \\
&= \frac{\{(y_1, y_2, y_3) \mid (y_1, y_2, y_3) + (y_2, y_1, y_3) = (0,0,0)\}}{\langle (y_1, y_2, y_3) - (y_1, y_2, y_3), (y_2, y_1, y_3) - (y_1, y_2, y_3)\rangle} \\
&= \frac{\{(y_1, y_2, y_3) \mid (y_1 + y_2, y_2 + y_1, 2y_3) = (0,0,0)\}}{\langle (0,0,0), (y_2 - y_1, y_1 - y_2, 0)\rangle} \\
&= \frac{\{(y_1, y_2, y_3) \mid y_2 = -y_1,\ y_3 = 0\}}{\langle (y_2 - y_1, -(y_2 - y_1), 0)\rangle} \\
&= \frac{\{(y_1, -y_1, 0)\}}{\langle (y_2 - y_1, -(y_2 - y_1), 0)\rangle} \cong \frac{\mathbb{Z}}{\mathbb{Z}} \cong 0
\end{aligned}$$

Case: $w = w_1 w_2 w_1$

Since $w_1 w_2 w_1$ is conjugate to w_2, this case is nothing more than a re-labelling of the case $w = w_2$, above. If $s \in S_r^{w_1 w_2 w_1}(k)$ then $W_s = \langle w_1 w_2 w_1 \rangle = \{1, w_1 w_2 w_1\}$; see

Table 2.6, Section 4.1.3 and Section 3.1.3. In this case,

$$\begin{aligned}
\check{X}^{\mathrm{tr}_{W_s}=0}/\check{X}_{W_s} &= \frac{\{y \in \check{X} \mid \sum_{w\in W_s} w(y) = 0\}}{\langle w(y) - y \mid y \in \check{X}, w \in W_s\rangle} \\
&= \frac{\left\{(y_1, y_2, y_3) \mid \sum_{w\in\{1,w_1w_2w_1\}} w(y_1, y_2, y_3) = (0,0,0)\right\}}{\langle w(y_1, y_2, y_3) - (y_1, y_2, y_3) \mid y \in \check{X}, w \in \{1, w_1w_2w_1\}\rangle} \\
&= \frac{\{(y_1, y_2, y_3) \mid (y_1, y_2, y_3) + (y_3, y_2, y_1) = (0,0,0)\}}{\langle (y_1, y_2, y_3) - (y_1, y_2, y_3), (y_3, y_2, y_1) - (y_1, y_2, y_3)\rangle} \\
&= \frac{\{(y_1, y_2, y_3) \mid (y_1 + y_3, 2y_2, y_3 + y_1) = (0,0,0)\}}{\langle (0,0,0), (y_3 - y_1, 0, y_1 - y_3)\rangle} \\
&= \frac{\{(y_1, y_2, y_3) \mid y_3 = -y_1,\ y_2 = 0\}}{\langle (y_3 - y_1, 0, -(y_3 - y_1))\rangle} \\
&= \frac{\{(y_1, 0, -y_1)\}}{\langle (y_3 - y_1, 0, -(y_3 - y_1))\rangle} \cong \frac{\mathbb{Z}}{\mathbb{Z}} \cong 0
\end{aligned}$$

Case: $w = w_2w_1w_2w_1w_2$

Since $w_2w_1w_2w_1w_2$ is conjugate to w_2, this case is nothing more than a re-labelling of the case $w = w_2$, above. If $s \in S_r^{w_2w_1w_2w_1w_2}(k)$ then $W_s = \langle w_2w_1w_2w_1w_2\rangle = \{1, w_2w_1w_2w_1w_2\}$; see Table 2.6, Section 4.1.3 and Section 3.1.3. In this case,

$$\begin{aligned}
\check{X}^{\mathrm{tr}_{W_s}=0}/\check{X}_{W_s} &= \frac{\{y \in \check{X} \mid \sum_{w\in W_s} w(y) = 0\}}{\langle w(y) - y \mid y \in \check{X}, w \in W_s\rangle} \\
&= \frac{\left\{(y_1, y_2, y_3) \mid \sum_{w\in\{1,w_2w_1w_2w_1w_2\}} w(y_1, y_2, y_3) = (0,0,0)\right\}}{\langle w(y_1, y_2, y_3) - (y_1, y_2, y_3) \mid y \in \check{X}, w \in \{1, w_2w_1w_2w_1w_2\}\rangle} \\
&= \frac{\{(y_1, y_2, y_3) \mid (y_1, y_2, y_3) + (y_1, y_3, y_2) = (0,0,0)\}}{\langle (y_1, y_2, y_3) - (y_1, y_2, y_3), (y_1, y_3, y_2) - (y_1, y_2, y_3)\rangle} \\
&= \frac{\{(y_1, y_2, y_3) \mid (2y_1, y_2 + y_3, y_3 + y_2) = (0,0,0)\}}{\langle (0,0,0), (0, y_3 - y_2, y_2 - y_3)\rangle} \\
&= \frac{\{(y_1, y_2, y_3) \mid y_3 = -y_2,\ y_1 = 0\}}{\langle (0, y_3 - y_2, -(y_3 - y_2))\rangle} \\
&= \frac{\{(0, y_2, -y_2)\}}{\langle (0, y_3 - y_2, -(y_3 - y_2))\rangle} \cong \frac{\mathbb{Z}}{\mathbb{Z}} \cong 0
\end{aligned}$$

5.1.4 Case: $w = (w_2w_1)^3$

If $s \in S_r^{(w_2w_1)^3}(k)$ then $W_s = \langle (w_2w_1)^3 \rangle = \{1, (w_2w_1)^3\}$; see Table 2.6, Section 4.1.4 and Section 3.1.4. In this case,

$$\begin{aligned}
\check{X}^{\mathrm{tr}_{W_s}=0}/\check{X}_{W_s} &= \frac{\{y \in \check{X} \mid \sum_{w \in W_s} w(y) = 0\}}{\langle w(y) - y \mid y \in \check{X}, w \in W_s \rangle} \\
&= \frac{\left\{(y_1, y_2, y_3) \mid \sum_{w \in \{1,(w_2w_1)^3\}} w(y_1, y_2, y_3) = (0,0,0)\right\}}{\langle w(y_1, y_2, y_3) - (y_1, y_2, y_3) \mid y \in \check{X}, w \in \{1, (w_2w_1)^3\} \rangle} \\
&= \frac{\{(y_1, y_2, y_3) \mid (y_1, y_2, y_3) + (-y_1, -y_2, -y_3) = (0,0,0)\}}{\langle (y_1, y_2, y_3) - (y_1, y_2, y_3), (-y_1, -y_2, -y_3) - (y_1, y_2, y_3) \rangle} \\
&= \frac{\{(y_1, y_2, y_3) \mid (0,0,0) = (0,0,0)\}}{\langle (0,0,0), (-2y_1, -2y_2, -2y_3) \rangle} \\
&= \frac{\{(y_1, y_2, y_3) \mid y_1, y_2 \text{ arbitrary, } y_3 = -y_1 - y_2\}}{\langle (-2y_1, -2y_2, -2y_3) \rangle} \\
&= \frac{\{(y_1, y_2, -y_1 - y_2)\}}{\langle (-2y_1, -2y_2, 2y_1 + 2y_2) \rangle} \\
&\cong \frac{\mathbb{Z} \times \mathbb{Z}}{2\mathbb{Z} \times 2\mathbb{Z}} \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}
\end{aligned}$$

5.1.5 Case: $w = (w_2w_1)^2$

If $s \in S_r^{(w_2w_1)^2}(k)$ then $W_s = \langle (w_2w_1)^2 \rangle = \{1, (w_2w_1)^2, (w_2w_1)^4\}$; see Table 2.6, Section 4.1.5 and Section 3.1.5. In this case,

$$\begin{aligned}
\check{X}^{\mathrm{tr}_{W_s}=0}/\check{X}_{W_s} &= \frac{\{y \in \check{X} \mid \sum_{w \in W_s} w(y) = 0\}}{\langle w(y) - y \mid y \in \check{X}, w \in W_s \rangle} \\
&= \frac{\left\{(y_1, y_2, y_3) \mid \sum_{w \in \{1,(w_2w_1)^2,(w_2w_1)^4\}} w(y_1, y_2, y_3) = (0,0,0)\right\}}{\langle w(y_1, y_2, y_3) - (y_1, y_2, y_3) \mid y \in \check{X}, w \in \{1, (w_2w_1)^2 . (w_2w_1)^4\} \rangle} \\
&= \frac{\{(y_1, y_2, y_3) \mid (y_1, y_2, y_3) + (y_2, y_3, y_1) + (y_3, y_1, y_2) = (0,0,0)\}}{\langle (0,0,0), (y_2 - y_1, y_3 - y_2, y_1 - y_3), (y_3 - y_1, y_1 - y_2, y_2 - y_3) \rangle}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\{(y_1,y_2,y_3) \mid y_1+y_2+y_3=0\}}{\langle (y_2-y_1,-y_1-2y_2,2y_1+y_2),(-y_2-2y_1,y_1-y_2,2y_2+y_1)\rangle} \\
&= \frac{\{(y_1,y_1+v_1,-2y_1-v_1)\}}{\langle (v,-3y_1-2v,3y_1+v)\vee(-v-3y_1,-v,2v+3y_1)\rangle} \\
&= \frac{\mathbb{Z}\times\mathbb{Z}}{\mathbb{Z}\times 3\mathbb{Z}\vee 3\mathbb{Z}\times\mathbb{Z}} \cong \frac{\mathbb{Z}}{3\mathbb{Z}}
\end{aligned}
$$

5.1.6 Case: $w = w_2w_1$

If $s \in S_r^{w_2w_1}(k)$ then $W_s = \langle w_2w_1\rangle = \{1, w_2w_1, (w_2w_1)^2, (w_2w_1)^3, (w_2w_1)^4, (w_2w_1)^5\}$; see Table 2.6, Section 4.1.6 and Section 3.1.6. In this case,

$$
\begin{aligned}
\check{X}^{\mathrm{tr}_{W_s}=0}/\check{X}_{W_s} &= \frac{\{y\in\check{X} \mid \sum_{w\in W_s} w(y)=0\}}{\langle w(y)-y \mid y\in\check{X}, w\in W_s\rangle} \\
&= \frac{\left\{(y_1,y_2,y_3) \mid \sum_{w\in\langle w_2w_1\rangle} w(y_1,y_2,y_3)=(0,0,0)\right\}}{\langle w(y_1,y_2,y_3)-(y_1,y_2,y_3) \mid y\in\check{X}, w\in\langle w_2w_1\rangle\rangle} \\
&= \frac{\left\{\begin{array}{c}(y_1,y_2,y_3)+(-y_3,-y_1,-y_2)+(y_2,y_3,y_1)\\ +(-y_1,-y_2,-y_3)+(y_3,y_1,y_2)+(-y_2,-y_3,-y_1)=(0,0,0)\end{array}\right\}}{\left\langle\left\{\begin{array}{c}(y_1,y_2,y_3)-(y_1,y_2,y_3),(-y_3,-y_1,-y_2)-(y_1,y_2,y_3),\\ (y_2,y_3,y_1)-(y_1,y_2,y_3),(-y_1,-y_2,-y_3)-(y_1,y_2,y_3),\\ (y_3,y_1,y_2)-(y_1,y_2,y_3),(-y_2,-y_3,-y_1)-(y_1,y_2,y_3)\end{array}\right\}\right\rangle} \\
&= \frac{\{(y_1,y_2,y_3) \mid (0,0,0)=(0,0,0)\}}{\left\langle\left\{\begin{array}{c}(0,0,0),(y_2,y_3,y_1),(y_2-y_1,y_3-y_2,y_1-y_3),\\ (-2y_1,-2y_2,-2y_3),(y_3-y_1,y_1-y_2,y_2-y_3),(y_3,y_1,y_2)\end{array}\right\}\right\rangle}
\end{aligned}
$$

$$= \frac{\{(y_1,y_2,y_3) \mid y_3 = -y_1 - y_2\}}{\left\langle \left\{ \begin{array}{c} (y_2,y_3,y_1) \vee (y_2-y_1, y_3-y_2, y_1-y_3) \vee (-2y_1,-2y_2,-2y_3) \\ \vee (y_3-y_1, y_1-y_2, y_2-y_3) \vee (y_3,y_1,y_2) \end{array} \right\} \right\rangle}$$

$$= \frac{\{(y_1,y_2,-y_1-y_2)\}}{\left\langle \left\{ \begin{array}{c} (y_2,-y_1-y_2,y_1) \vee (v,-3y_1-2v,3y_1-v) \\ \vee(-2y_1,-2y_2,2y_1+2y_2) \vee (-v-3y_1,-v,2v+3y_1) \\ \vee(-y_1-y_2,y_1,y_2) \end{array} \right\} \right\rangle}$$

$$= \frac{\mathbb{Z}\times\mathbb{Z}}{\mathbb{Z}\times\mathbb{Z} \vee \mathbb{Z}\times 3\mathbb{Z} \vee 2\mathbb{Z}\times 2\mathbb{Z} \vee 3\mathbb{Z}\times\mathbb{Z} \vee \mathbb{Z}\times\mathbb{Z}} \cong \frac{\mathbb{Z}\times\mathbb{Z}}{\mathbb{Z}\times\mathbb{Z}} \cong 0$$

5.2 Fractional depth $\frac{1}{6}$ or $\frac{5}{6}$

Suppose $s \in S_r(k)$. Refer to Table 2.6, Section 4.2 and Section 3.2.

If $q \equiv 1(3)$ then $W_s = \langle 1, w_2w_1\rangle = \{1, w_2w_1, (w_2w_1)^2, (w_2w_1)^3, (w_2w_1)^4(w_2w_1)^5\}$ In this case, $\check{X}^{\mathrm{tr}_{W_s}=0}/\check{X}_{W_s} = 0$ as in the $w = w_2w_1$ case of fractional depth 0.

If $q \equiv 2(3)$ then $W_s = \langle w_2, w_2w_1\rangle = \langle w_2, w_1\rangle = \{1, w_2, w_1, w_2w_1, w_1w_2, w_2w_1w_2, w_1w_2w_1, w_2w_1w_2w_1, w_1w_2w_1w_2, w_2w_1w_2w_1w_2, w_1w_2w_1w_2w_1, w_2w_1w_2w_1w_2w_1\} \cong D_6$. Thus, in this case,

$$\begin{aligned}
\check{X}^{\mathrm{tr}_{W_s}=0}/\check{X}_{W_s} &= \frac{\{y \in \check{X} \mid \sum_{w\in W_s} w(y) = 0\}}{\langle w(y) - y \mid y \in \check{X}, w \in W_s\rangle} \\
&= \frac{\left\{(y_1,y_2,y_3) \mid \sum_{w\in\{\langle w_2,w_1\rangle\}} w(y_1,y_2,y_3) = (0,0,0)\right\}}{\langle w(y_1,y_2,y_3) - (y_1,y_2,y_3) \mid y \in \check{X}, w \in \langle w_2, w_1\rangle\rangle}
\end{aligned}$$

$$= \frac{\left\{ \begin{array}{c} (y_1, y_2, y_3) + (y_2, y_1, y_3) + (-y_1, -y_3, -y_2) \\ +(-y_3, -y_1, -y_2) + (-y_2, -y_3, -y_1) + (-y_3, -y_2, -y_1) \\ +(y_3, y_2, y_1) + (y_2, y_3, y_1) + (y_3, y_1, y_2) \\ +(y_1, y_3, y_2) + (-y_2, -y_1, -y_3) + (-y_1, -y_2, -y_3) = (0,0,0) \end{array} \right\}}{\left\langle \left\{ \begin{array}{c} (y_1, y_2, y_3) - (y_1, y_2, y_3), (y_2, y_1, y_3) - (y_1, y_2, y_3), \\ (-y_1, -y_3, -y_2) - (y_1, y_2, y_3), (-y_3, -y_1, -y_2) - (y_1, y_2, y_3), \\ (-y_2, -y_3, -y_1) - (y_1, y_2, y_3), (-y_3, -y_2, -y_1) - (y_1, y_2, y_3), \\ (y_3, y_2, y_1) - (y_1, y_2, y_3), (y_2, y_3, y_1) - (y_1, y_2, y_3), \\ (y_3, y_1, y_2) - (y_1, y_2, y_3), (y_1, y_3, y_2) - (y_1, y_2, y_3) \\ (-y_2, -y_1, -y_3) - (y_1, y_2, y_3), (-y_1, -y_2, -y_3) - (y_1, y_2, y_3) \end{array} \right\} \right\rangle}$$

$$= \frac{\{(y_1, y_2, y_3) \mid (0,0,0) = (0,0,0)\}}{\left\langle \left\{ \begin{array}{c} (0,0,0), (y_2 - y_1, y_1 - y_2, 0), \\ (-2y_1, -y_3 - y_2, -y_2 - y_3), (-y_3 - y_1, -y_1 - y_2, -y_2 - y_3), \\ (-y_2 - y_1, -y_3 - y_2, -y_1 - y_3), (-y_3 - y_1, -2y_2, -y_1 - y_3), \\ (y_3 - y_1, 0, y_1 - y_3), (y_2 - y_1, y_3 - y_2, y_1 - y_3), \\ (y_3 - y_1, y_1 - y_2, y_2 - y_3), (0, y_3 - y_2, y_2 - y_3), \\ (-y_2 - y_1, -y_1 - y_2, -2y_3), (-2y_1, -2y_2, -2y_3) \end{array} \right\} \right\rangle}$$

$$= \frac{\{(y_1, y_2, y_3) \mid y_1, y_2 \text{ arbitrary, } y_3 = -y_1 - y_2\}}{\left\langle \left\{ \begin{array}{c} (0,0,0), (v_3, -v_3, 0), (-2y_1, y_1, y_1), (y_2, y_3, y_1), (y_3, y_1, y_2), \\ (y_2, -2y_2, y_2), (v_2, 0, -v_2), (y_2 - y_1, -y_1 - 2y_2, 2y_1 + y_2) \\ (-y_2 - 2y_1, y_1 - y_2, y_1 + 2y_2), (0, v_1, -v_1), (y_3, y_3, -2v_3), \\ (-2y_1, -2y_2, -2y_3) \end{array} \right\} \right\rangle}$$

$$= \frac{\{(y_1, y_2, -y_1 - y_2\}}{\left\langle \left\{ \begin{array}{c} (v_3, -v_3, 0) \vee (-2y_1, y_1, y_1) \vee (y_2, -y_1 - y_2, y_1) \\ \vee(-y_1 - y_2, y_1, y_2) \vee (y_2, -2y_2, y_2) \vee (v_2, 0, -v_2) \\ \vee(v_3, -3y_1 - 2v_3, 3y_1 + v_3) \vee (-v_3 - 3y_1, -v_3, 2v_3 + 3y_1) \\ \vee(0, v_1, -v_1) \vee (y_3, y_3, -2v_3) \vee (-2y_1, -2y_2, 2y_1 + 2y_2) \end{array} \right\} \right\rangle}$$

$$= \frac{\mathbb{Z} \times \mathbb{Z}}{\left\{ \begin{array}{c} \mathbb{Z} \times 0 \vee \mathbb{Z} \times 0 \vee \mathbb{Z} \times \mathbb{Z} \vee \mathbb{Z} \times \mathbb{Z} \vee \mathbb{Z} \times 0 \vee \mathbb{Z} \times 0 \\ \vee \mathbb{Z} \times 3\mathbb{Z} \vee 3\mathbb{Z} \times \mathbb{Z} \vee \mathbb{Z} \times 0 \vee \mathbb{Z} \times 0 \vee 2\mathbb{Z} \times 2\mathbb{Z} \end{array} \right\}} \cong \frac{\mathbb{Z} \times \mathbb{Z}}{\mathbb{Z} \times \mathbb{Z}} \cong 0$$

5.3 Fractional depth $\frac{1}{3}$ or $\frac{2}{3}$

5.3.1 Case: $w = 1$

Suppose $s \in S_r^1(k)$. The cases below refer to Table 2.6, Section 3.3.1 and Section 4.3.1.

If $q \equiv 1(3)$ then $W_s = \langle 1, (w_2w_1)^2 \rangle = \langle (w_2w_1)^2 \rangle = \{1, (w_2w_1)^2, (w_2w_1)^4\}$. In this case, $\check{X}^{\mathrm{tr}_{W_s}=0}/\check{X}_{W_s} = \dfrac{\mathbb{Z}}{3\mathbb{Z}}$ as in the $w = (w_2w_1)^2$ case of fractional depth 0.

If $q \equiv 2(3)$ then $W_s = \langle w_2, (w_2w_1)^2 \rangle \cong S_3$. In this case,

$$\begin{aligned} \check{X}^{\mathrm{tr}_{W_s}=0}/\check{X}_{W_s} &= \frac{\{y \in \check{X} \mid \sum_{w \in W_s} w(y) = 0\}}{\langle w(y) - y \mid y \in \check{X}, w \in W_s \rangle} \\ &= \frac{\left\{(y_1, y_2, y_3) \mid \sum_{w \in \{\langle w_2, (w_2w_1)^2 \rangle\}} w(y_1, y_2, y_3) = (0,0,0)\right\}}{\langle w(y_1, y_2, y_3) - (y_1, y_2, y_3) \mid y \in \check{X}, w \in \langle w_2, (w_2w_1)^2 \rangle \rangle} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\left\{\begin{array}{c}(y_1,y_2,y_3)+(y_2,y_1,y_3)+(y_3,y_2,y_1)\\ +(y_1,y_3,y_2)+(y_2,y_3,y_1)+(y_3,y_1,y_2)=(0,0,0)\end{array}\right\}}{\left\langle\left\{\begin{array}{c}(y_1,y_2,y_3)-(y_1,y_2,y_3),(y_2,y_1,y_3)-(y_1,y_2,y_3),\\ (y_3,y_2,y_1)-(y_1,y_2,y_3),(y_1,y_3,y_2)-(y_1,y_2,y_3)\\ (y_2,y_3,y_1)-(y_1,y_2,y_3),(y_3,y_1,y_2)-(y_1,y_2,y_3)\end{array}\right\}\right\rangle}\\
&= \frac{\{(y_1,y_2,y_3)\mid y_1+y_2+y_3=0\}}{\left\langle\left\{\begin{array}{c}(0,0,0),(y_2-y_1,y_1-y_2,0),\\ (y_3-y_1,0,y_1-y_3),(0,y_3-y_2,y_2-y_3)\\ (y_2-y_1,y_3-y_2,y_1-y_3),(y_3-y_1,y_1-y_2,y_2-y_3)\end{array}\right\}\right\rangle}\\
&= \frac{\{(y_1,y_2,y_3)\mid y_1,y_2 \text{ arbitrary, } y_3=-y_1-y_2\}}{\left\langle\left\{\begin{array}{c}(0,0,0),(v_3,-v_3,0),(v_2,0,-v_2),(0,v_1,-v_1),\\ (y_2-y_1,-y_1-2y_2,2y_1+y_2),(-y_2-2y_1,y_1-y_2,y_1+2y_2)\end{array}\right\}\right\rangle}\\
&= \frac{\{(y_1,y_2,-y_1-y_2\}}{\left\langle\left\{\begin{array}{c}(v_3,-v_3,0)\vee(v_2,0,-v_2)\vee(0,v_1,-v_1)\\ \vee(v_3,-3y_1-2v_3,3y_1+v_3)\vee(-v_3-3y_1,-v_3,2v_3+3y_1)\end{array}\right\}\right\rangle}\\
&= \frac{\mathbb{Z}\times\mathbb{Z}}{\mathbb{Z}\times\{0\}\vee\mathbb{Z}\times\{0\}\vee\mathbb{Z}\times\{0\}\vee\mathbb{Z}\times 3\mathbb{Z}\vee 3\mathbb{Z}\times\mathbb{Z}}\\
&\cong \frac{\mathbb{Z}\times\mathbb{Z}}{\mathbb{Z}\times 3\mathbb{Z}}\cong\frac{\mathbb{Z}}{3\mathbb{Z}}
\end{aligned}
$$

5.3.2 Case: $w = (w_2w_1)^3$

Suppose $s \in S_r^{(w_2w_1)^3}(k)$. The calculations in this case are identical to those in the $w = 1$ case for fractional depth $\frac{1}{3}$ or $\frac{2}{3}$ above.

5.4 Fractional depth $\frac{1}{2}$

5.4.1 Case: $w = 1$

Suppose $s \in S_r^1(k)$. Then $W_s = \langle 1, (w_2w_1)^3 \rangle = \{1, (w_2w_1)^3\}$; see Table 2.6, Section 3.4.1 and Section 4.4.1. In this case, $\check{X}^{\mathrm{tr}_{W_s}=0}/\check{X}_{W_s} = \dfrac{\mathbb{Z}}{2\mathbb{Z}} \times \dfrac{\mathbb{Z}}{2\mathbb{Z}}$ as in the $w = (w_2w_1)^3$ case of fractional depth 0.

5.4.2 Case: $w = w_2$

Suppose $s \in S_r^{w_2}(k)$. Then $W_s = \langle w_2, (w_2w_1)^3 \rangle = \{1, w_2, w_1w_2w_1w_2w_1, w_2w_1w_2w_1w_2w_1\} \cong V_4$; see Table 2.6, Section 3.4.2 and Section 4.4.2. In this case,

$$
\begin{aligned}
\check{X}^{\mathrm{tr}_{W_s}=0}/\check{X}_{W_s} &= \frac{\{y \in \check{X} \mid \sum_{w \in W_s} w(y) = 0\}}{\langle w(y) - y \mid y \in \check{X}, w \in W_s \rangle} \\
&= \frac{\left\{(y_1, y_2, y_3) \mid \sum_{w \in \{w_2, (w_2w_1)^3\}} w(y_1, y_2, y_3) = (0,0,0)\right\}}{\langle w(y_1, y_2, y_3) - (y_1, y_2, y_3) \mid y \in \check{X}, w \in \{w_2, (w_2w_1)^3\} \rangle} \\
&= \frac{\left\{ (y_1, y_2, y_3) \;\middle|\; \begin{array}{c} (y_1, y_2, y_3) + (y_2, y_1, y_3) \\ +(-y_2, -y_1, -y_3) + (-y_1, -y_2, -y_3) = (0,0,0) \end{array} \right\}}{\left\langle \left\{ \begin{array}{c} (y_1, y_2, y_3) - (y_1, y_2, y_3), (y_2, y_1, y_3) - (y_1, y_2, y_3), \\ (-y_2, -y_1, -y_3) - (y_1, y_2, y_3), (-y_1, -y_2, -y_3) - (y_1, y_2, y_3) \end{array} \right\} \right\rangle} \\
&= \frac{\{(y_1, y_2, y_3) \mid (0,0,0) = (0,0,0)\}}{\left\langle \left\{ \begin{array}{c} (0,0,0), (y_2 - y_1, y_1 - y_2, 0), \\ (-y_2 - y_1, -y_1 - y_2, -2y_3), (-2y_1, -2y_2, -2y_3) \end{array} \right\} \right\rangle}
\end{aligned}
$$

$$= \frac{\{(y_1, y_2, y_3) \mid y_1, y_2 \text{ arbitrary}, \ y_3 = -y_1 - y_2\}}{\left\langle \left\{ \begin{array}{c} (0,0,0), (v_3, -v_3, 0), \\ (y_3, y_3, -2y_3), (-2y_1, -2y_2, -2y_3) \end{array} \right\} \right\rangle}$$

$$= \frac{\{(y_1, y_2, -y_1 - y_2\}}{\langle (v_3, -v_3, 0), (y_3, y_3, -2y_3), (-2y_1, -2y_2, 2y_1 + 2y_2) \rangle}$$

$$= \frac{\mathbb{Z} \times \mathbb{Z}}{\mathbb{Z} \times \{0\} \vee \mathbb{Z} \times \{0\} \vee 2\mathbb{Z} \times 2\mathbb{Z}} \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$$

Case: $w = w_1 w_2 w_1$

Since $w_1 w_2 w_1$ is conjugate to w_2, this case is nothing more than a re-labelling of the case $w = w_2$, above. Suppose $s \in S_r^{w_1 w_2 w_1}(k)$. Then $W_s = \langle w_1 w_2 w_1, (w_2 w_1)^3 \rangle = \{1, w_1 w_2 w_1, w_2 w_1 w_2, w_2 w_1 w_2 w_1 w_2 w_1\} \cong V_4$; see Table 2.6, Section 3.4.2 and Section 4.4.2. In this case,

$$\check{X}^{\mathrm{tr}_{W_s}=0} / \check{X}_{W_s} = \frac{\{y \in \check{X} \mid \sum_{w \in W_s} w(y) = 0\}}{\langle w(y) - y \mid y \in \check{X}, w \in W_s \rangle}$$

$$= \frac{\left\{(y_1, y_2, y_3) \mid \sum_{w \in \{w_1 w_2 w_1, (w_2 w_1)^3\}} w(y_1, y_2, y_3) = (0,0,0)\right\}}{\langle w(y_1, y_2, y_3) - (y_1, y_2, y_3) \mid y \in \check{X}, w \in \{w_1 w_2 w_1, (w_2 w_1)^3\} \rangle}$$

$$= \frac{\left\{ (y_1, y_2, y_3) \;\middle|\; \begin{array}{c} (y_1, y_2, y_3) + (y_3, y_2, y_1) \\ +(-y_3, -y_2, -y_1) + (-y_1, -y_2, -y_3) = (0,0,0) \end{array} \right\}}{\left\langle \left\{ \begin{array}{c} (y_1, y_2, y_3) - (y_1, y_2, y_3), (y_3, y_2, y_1) - (y_1, y_2, y_3), \\ (-y_3, -y_2, -y_1) - (y_1, y_2, y_3), (-y_1, -y_2, -y_3) - (y_1, y_2, y_3) \end{array} \right\} \right\rangle}$$

$$= \frac{\{(y_1, y_2, y_3) \mid (0,0,0) = (0,0,0)\}}{\left\langle \left\{ \begin{array}{c} (0,0,0), (y_3 - y_1, 0, y_1 - y_3), \\ (-y_3 - y_1, -2y_2, -y_1 - y_3), (-2y_1, -2y_2, -2y_3) \end{array} \right\} \right\rangle}$$

$$= \frac{\{(y_1,y_2,y_3) \mid y_1,y_2 \text{ arbitrary, } y_3 = -y_1-y_2\}}{\left\langle \left\{ \begin{array}{c} (0,0,0),(v_2,0,-v_2), \\ (y_2,-2y_2,y_2),(-2y_1,-2y_2,-2y_3) \end{array} \right\} \right\rangle}$$

$$= \frac{\{(y_1,y_2,-y_1-y_2\}}{\langle (v_2,0,-v_2),(y_2,-2y_2,y_2),(-2y_1,-2y_2,2y_1+2y_2)\rangle}$$

$$= \frac{\mathbb{Z}\times\mathbb{Z}}{\mathbb{Z}\times\{0\}\vee\mathbb{Z}\times\{0\}\vee 2\mathbb{Z}\times 2\mathbb{Z}} \cong \frac{\mathbb{Z}}{2\mathbb{Z}}\times\frac{\mathbb{Z}}{2\mathbb{Z}}$$

Case: $w = w_2w_1w_2w_1w_2$

Since $w_2w_1w_2w_1w_2$ is conjugate to w_2, this case is nothing more than a re-labelling of the case $w = w_2$, above. Suppose $s \in S_r^{w_2w_1w_2w_1w_2}(k)$. Then $W_s = \langle w_2w_1w_2w_1w_2, (w_2w_1)^3\rangle = \{1, w_2w_1w_2w_1w_2, w_1, w_2w_1w_2w_1w_2w_1\} \cong V_4$; see Table 2.6, Section 3.4.2 and Section 4.4.2. In this case,

$$\check{X}^{\mathrm{tr}_{W_s}=0}/\check{X}_{W_s} = \frac{\{y\in\check{X} \mid \sum_{w\in W_s} w(y)=0\}}{\langle w(y)-y \mid y\in\check{X}, w\in W_s\rangle}$$

$$= \frac{\left\{(y_1,y_2,y_3) \mid \sum_{w\in\{w_2w_1w_2w_1w_2,(w_2w_1)^3\}} w(y_1,y_2,y_3)=(0,0,0)\right\}}{\langle w(y_1,y_2,y_3)-(y_1,y_2,y_3) \mid y\in\check{X}, w\in\{w_2w_1w_2w_1w_2,(w_2w_1)^3\}\rangle}$$

$$= \frac{\left\{ (y_1,y_2,y_3) \;\middle|\; \begin{array}{c} (y_1,y_2,y_3)+(y_1,y_3,y_2) \\ +(-y_1,-y_3,-y_2)+(-y_1,-y_2,-y_3)=(0,0,0) \end{array} \right\}}{\left\langle \left\{ \begin{array}{c} (y_1,y_2,y_3)-(y_1,y_2,y_3),(y_1,y_3,y_2)-(y_1,y_2,y_3), \\ (-y_1,-y_3,-y_2)-(y_1,y_2,y_3),(-y_1,-y_2,-y_3)-(y_1,y_2,y_3) \end{array} \right\} \right\rangle}$$

$$= \frac{\{(y_1,y_2,y_3) \mid (0,0,0)=(0,0,0)\}}{\left\langle \left\{ \begin{array}{c} (0,0,0),(0,y_3-y_2,y_2-y_3), \\ (-2y_1,-y_3-y_2,-y_2-y_3),(-2y_1,-2y_2,-2y_3) \end{array} \right\} \right\rangle}$$

$$
\begin{aligned}
&= \frac{\{(y_1, y_2, y_3) \mid y_1, y_2 \text{ arbitrary, } y_3 = -y_1 - y_2\}}{\left\langle \left\{ \begin{matrix} (0,0,0), (0, v_1, -v_1), \\ (-2y_1, y_1, y_1), (-2y_1, -2y_2, -2y_3) \end{matrix} \right\} \right\rangle} \\
&= \frac{\{(y_1, y_2, -y_1 - y_2\}}{\langle (0, v_1, -v_1), (-2y_1, y_1, y_1), (-2y_1, -2y_2, 2y_1 + 2y_2) \rangle} \\
&= \frac{\mathbb{Z} \times \mathbb{Z}}{\mathbb{Z} \times \{0\} \vee \mathbb{Z} \times \{0\} \vee 2\mathbb{Z} \times 2\mathbb{Z}} \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}
\end{aligned}
$$

5.4.3 Case: $w = (w_2 w_1)^2$

Suppose $s \in S_r^{(w_2w_1)^2}(k)$. Then $W_s = \langle (w_2w_1)^2, (w_2w_1)^3 \rangle = \langle w_2w_1 \rangle$; see Table 2.6, Section 3.4.3 and Section 4.4.3. In this case, $\check{X}^{\mathrm{tr}_{W_s}=0}/\check{X}_{W_s} = 0$ as in the $w = w_2w_1$ case of fractional depth 0.

Chapter 6

Proof of the main result

In this chapter we prove the main result in this thesis, stated again here for convenience.

Theorem 1.1. *Let G be a Chevalley group scheme of type G_2 and let $\mathfrak{g}$ be its Lie algebra. Every Chevalley basis for $\mathfrak{g}$ determines a family of maps of definable subassignments*

$$\forall r \in \mathbb{Q}, \qquad \nu_r : \mathfrak{g}(r) \to B_r$$

such that if K is a local field and 6 is invertible in the residue field k of K then the specialization $\nu_{r/K}$ determined by K is surjective and

$$\mathcal{O}_r(X) = \nu_{r/K}^{-1}(\nu_{r/K}(X)).$$

6.1 Kostant section

In this section we recall the Kostant section $\kappa : S \to \mathfrak{g}^{\text{reg}}$ of the Steinberg map $\mu : \mathfrak{g}^{\text{reg}} \to S$.

Following [Kos63] (and a nice précis in [Kot99, §2.4]), set $X_+ = X_{\alpha_1} + X_{\alpha_2}$ and

$X_- = X_{-\alpha_1} + X_{-\alpha_2}$. Using the structure coefficients of Table 1.2, the centralizer $\ker \mathrm{ad}(X_-)$ of X_- in $\mathfrak{g}$ is found to be the linear span of

$$\{X_{-\alpha_1} - X_{-\alpha_2}, X_{-\tilde{\alpha}}\}.$$

Passing from $\mathbb{Z}$ to $\mathbb{Z}[2^{-1}]$ and using [Kos63, Prop 19], we find that the restriction of $\mathfrak{g} \to \mathfrak{t}/W$ to $X_+ + \ker \mathrm{ad}(X_-)$ is an isomorphism of $\mathbb{Z}[2^{-1}]$-schemes. In this way we find that the Kostant section $\mathfrak{t}_2/W \to \mathfrak{g}_2$, with image $X_+ + \ker \mathrm{ad}(X_-)$, is given by

$$(s_1, s_2) \quad \mapsto \quad X_{\alpha_1} + X_{\alpha_2} + \frac{s_1}{4} X_{-\tilde{\alpha}} - \frac{s_2}{2} \left(X_{-\alpha_1} - X_{-\alpha_2}\right).$$

The restriction of μ to $X_+ + \ker \mathrm{ad}(X_-)$ is not an isomorphism of schemes over $\mathbb{Z}$, but is after base change to $\mathbb{Z}[2^{-1}]$. This recipe for the section $\mathfrak{t}_2/W \to \mathfrak{g}_2$ of the base change of $\mathfrak{g} \to \mathfrak{t}/W$ to $\mathbb{Z}[2^{-1}]$ depends only on the basis $\Delta = \{\alpha_1, \alpha_2\}$ for R and the Chevalley basis $\{X_\alpha \mid \alpha \in R\}$. We write $\kappa : S_2 \to \mathfrak{g}_2^{\mathrm{reg}}$ for the restriction of the Kostant section to S_2.

6.2 Tate-Nakayama

In Chapter 2 we attached a torus T_s to every $s \in S_r(k)$ using the partition $S_r = \coprod_{w \in W_r} S_r^w$ of definable sets of Section 2.7. The torus T_s was determined by the cocycle (just a homomorphism, actually) $\rho_s : \mathrm{Gal}(\bar{K}/K) \to W$ defined in Section 2.8. This made it easy to calculate $H^1(K, T_s)$ using Tate-Nakayama, and the calculation of $\check{X}^{\mathrm{tr}_{W_s}=0}/\check{X}_{W_s}$, for every $r \in \frac{1}{6}\mathbb{Z}$ and $w \in W_r$ and $s \in S_r^w(k)$, was carried out in Chapter 5. The groups $H^1(K, T_s)$ are listed in Table 6.1, from which we see that

$H^1(K, T_s)$ depends only on the conjugacy class of $w \in W_r$ for which $s \in S_r^w(k)$. Recall that $H^1(K, T_s)$ classifies $G(K)$-conjugacy classes of subtori of G of type T_s.

Table 6.1: $H^1(K, T_s)$ for tori T_s determined by $s \in S_r^w(k)$.

$\{r\}$ $r \in \frac{1}{6}\mathbb{Z}$	w $w \in W_r$	$H^1(K, T_s)$	$h_r(w)$ $= \#H^1(K, T_s)$
0	w_2w_1	0	1
0	$(w_2w_1)^2$	$\mathbb{Z}/3\mathbb{Z}$	3
0	$(w_2w_1)^3$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	4
0	w_1	0	1
0	w_2	0	1
0	1	0	1
$\frac{1}{2}$	$(w_2w_1)^2$	0	1
$\frac{1}{2}$	w_2	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	4
$\frac{1}{2}$	1	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	4
$\frac{1}{3}, \frac{2}{3}$	$(w_2w_1)^3$	$\mathbb{Z}/3\mathbb{Z}$	3
$\frac{1}{3}, \frac{2}{3}$	1	$\mathbb{Z}/3\mathbb{Z}$	3
$\frac{1}{6}, \frac{5}{6}$	1	0	1

From Table 6.1 we note that the group $H^1(K, T_s)$ is determined by its cardinality: if $\#H^1(K, T_s) = 1$ then $H^1(K, T_s)$ is trivial; if $\#H^1(K, T_s) = 3$ then $H^1(K, T_s) \cong \mathbb{Z}/3\mathbb{Z}$; and if $\#H^1(K, T_s) = 4$ then $H^1(K, T_s) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. For each $n \in \{1, 3, 4\}$, let A_n be the corresponding group, interpreted as a definable set. Using these facts we define $h_r : W_r \to \mathbb{N}$, for each $r \in \frac{1}{6}\mathbb{Z}$, by the data appearing in the final column of Table 6.1.

6.3 Proof of the main result

For each $w \in W_r$, let $\mathfrak{g}(r,w) \hookrightarrow \mathfrak{g}(r)$ be the fibre of $S_r^w \hookrightarrow S_r$ under the map of definable subassignments $\mu_r : \mathfrak{g}(r) \to S_r$ from Proposition 2.1. By pull-back, the partition $S_r = \coprod_{w\in W_r} S_r^w$ defines a partition

$$\mathfrak{g}(r) = \coprod_{w\in W_r} \mathfrak{g}(r,w)$$

and maps of definable subassignments

$$\mu_r^w : \mathfrak{g}(r,w) \to S_r^w.$$

We now define a function

$$\delta_{r/K}^w : \mathfrak{g}(r,w,K) \to A_{h_r(w)}$$

for every $r \in \frac{1}{6}\mathbb{Z}$ and $w \in W_r$. Suppose $X \in \mathfrak{g}(r,w,K)$; set $s = \mu(X)$. Using the Kostant section, set $X_0 := \kappa(\mu(X))$ and note that $X_0 \in \mathfrak{g}(r,K)$ and X is stably conjugate to X_0. The relationship between the stable orbit $\mathcal{O}_s(K)$ and the $G(K)$-orbit $\mathcal{O}(X_0)$ of X_0 is found by computing the connecting homomorphism of the long exact sequence in Galois cohomology

$$1 \longrightarrow T_{X_0}(K) \longrightarrow G(K) \longrightarrow \mathcal{O}_s(K) \xrightarrow{\delta_{X_0}} H^1(K,T_{X_0}) \longrightarrow H^1(K,G)$$

derived from the short exact sequence of K-varieties

$$1 \longrightarrow T_{X_0} \longrightarrow G \longrightarrow G/T_{X_0} \longrightarrow 1.$$

Since $H^1(K,G)=0$, the Galois cohomology of T_{X_0} measures how many $G(K)$-orbits lie in $\mathcal{O}_s(K)$: with the choice of $X_0 \in \mathcal{O}_s(K)$ as a base point, the torsor $\mathcal{O}_s(K)/G(K)$ becomes a group isomorphic to $H^1(K,T_{X_0})$. By Tate-Nakayama, $H^1(K,T_{X_0})$ may be calculated directly from the action of $\mathrm{Gal}(\bar{K}/K)$ on the cocharacter lattice $X_*(T_{X_0})$. Indeed, since $T_{X_0}=T_s$, we have already determined the group $H^1(K,T_{X_0})$, above. In particular, from Table 6.1 we see

$$H^1(K,T_{X_0}) \cong A_{h_r(w)}.$$

Since X is stably conjugate to X_0, we have $X \in \mathcal{O}_s(X_0)$, so the connecting homomorphism $\delta_{X_0} : \mathcal{O}_s(X_0) \to H^1(K,T_{X_0})$ sends X to an element of $H^1(K,T_{X_0})$. In this way we have defined the function

$$\delta^w_{r/K} : \mathfrak{g}(r,w,K) \to A_{h_r(w)}.$$

Note that $\delta^w_{r/K}$ is clearly surjective.

Set

$$B^w_r := S^w_r \times A_{h_r(w)};$$

note that this is a definable set. The argument above shows that

$$\nu^w_{r/K} := \mu^w_{r/K} \times \delta^w_{r/K} : \mathfrak{g}(r,K) \to B^w_r(k)$$

is surjective.

Moreover, arguing as in the proof of Proposition 2.1, we see that the fibre of $\nu^w_{r/K}$ above $\nu^w_{r/K}(X) \in B^w_r(k)$, for $X \in \mathfrak{g}(r,w,K)$, is precisely the thickened orbit of X in $\mathfrak{g}(K)$:

$$\mathcal{O}_r(X) = (\nu^w_{r/K})^{-1}(\nu^w_{r/K}(X)).$$

This justifies the notation $\mathcal{O}(x,a)$ for $\mathcal{O}_r(X)$ if $\mu^w_{r/K}(X) = (x,a) \in B^w_r(k) = S^w_r(k) \times A_{h_r(w)}$. Since thickened orbits are definable and since the dependence of $\mathcal{O}(x,a)$ in $(x,a) \in S^w_r(k) \times A_{h_r(w)}$ is definable, the functions $\nu^w_{r/K} : \mathfrak{g}(r,K) \to B^w_r(k)$ define a map of definable subassignments,

$$\nu^w_r : \mathfrak{g}(r) \to B^w_r.$$

Set

$$B_r := \coprod_{w \in W_r} \left(S^w_r \times A_{h_r(w)} \right);$$

note that this too is a definable set. Let

$$\nu_r : \mathfrak{g}(r) \to B_r$$

be the map of definable subassignments defined by composing the isomorphism of definable subassignments $\mathfrak{g}(r,K) \to \coprod_{w \in W_r} \mathfrak{g}(r,w,K)$ with the coproduct of the maps $\nu^w_r : \mathfrak{g}(r,w,K) \to B^w_r$ and the isomorphism of definable subassignments $\coprod_{w \in W_r} B^w_r \to B_r$. Then $\nu_r : \mathfrak{g}(r) \to B_r$ is a map of definable subassignments and if 6 is invertible in the residue field of K then the specialization $\nu_{r/K} : \mathfrak{g}(r,K) \to B_r(k)$ is surjective, and

$$\mathcal{O}_r(X) = \nu_{r/K}^{-1}(\nu_{r/K}(X))$$

for every $X \in \mathfrak{g}(r, K)$. This completes the proof of Theorem 1.1.

6.4 Application to stable orbit representatives

We conclude by explaining how to use this thesis to enumerate representatives for stable conjugacy classes of good (equivalued) elements in $\operatorname{Lie} G(2)$ over K, assuming only that the residual characteristic of K is at least 5.

For each $r \in \frac{1}{6}\mathbb{Z}$, consider the definable subassignment $S^r \subset S$ given by the specializations

$$S^r(K) = \{(s_1, s_2) \in S(K) \mid \operatorname{ord}_K(s_1) = 6r \quad \text{and} \quad \operatorname{ord}_K(s_2) \geq \lceil 2r \rceil\}.$$

Recall the definition of S_r from Section 2.4 and the map of definable subassignments $\mu_r : \mathfrak{g}(r) \to S_r$ appearing in Proposition 2.1. Let

$$\operatorname{res}_r : S^r \to S_r$$

be the map of definable subassignments given by the surjective specializations

$$\operatorname{res}_{r/K} : S^r(K) \to S_r(k)$$

where

$$\operatorname{res}_{r/K}(s_1, s_2) := \begin{cases} (\operatorname{res}_{6r}(s_1), \operatorname{res}_{\lceil 2r \rceil}(s_2)) & \{r\} = 0, \frac{1}{2} \\ (\operatorname{res}_{6r}(s_1), \operatorname{res}_{6r}(-3^3 s_1)) & \{r\} = \frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \frac{5}{6}. \end{cases}$$

Then the image of $\mathfrak{g}(r, K)$ under the Steinberg quotient $\mu_K : \mathfrak{g}^{\text{reg}}(K) \to S(K)$ is precisely $S^r(K)$ and $\mu_r : \mathfrak{g}(r) \to S_r$ factors through res_r:

$$\begin{array}{ccccc} \mathfrak{g}^{\text{reg}}(K) & \xrightarrow{\mu_K} & S(K) & & \\ \uparrow{\scriptstyle \text{def'ble}} & & \uparrow{\scriptstyle \text{def'ble}} & & \\ \mathfrak{g}(r, K) & \xrightarrow{\mu_K|_{\mathfrak{g}(r,K)}} & S^r(K) & \xrightarrow{\operatorname{res}_{r/K}} & S_r(k) \end{array}$$

$$\mu_r : \mathfrak{g}(r, K) \to S_r(k)$$

Now, suppose $s \in S_r(k)$. Then $s \in S_r^w(k) \subseteq S_r(k)$ for a unique $w \in W_r$. This parameterizes the components of s by $s = \mu_{r,w}(x)$ for $x \in S_{r,w}(k)$. Let $\dot{s}$ be any lift of $s \in S_r(k)$ to $S^r(K)$; thus, $\operatorname{res}_{r/K}(\dot{s}) = s$. Using Section 6.1, we see that

$$\kappa(\dot{s}) = X_{\alpha_1} + X_{\alpha_2} + \frac{\dot{s}_1}{4} X_{-\widetilde{\alpha}} - \frac{\dot{s}_2}{2} \left(X_{-\alpha_1} - X_{-\alpha_2} \right)$$

lies in $\mathfrak{g}(r, K)$. Letting s range over $S_r(k)$, the set

$$\{\kappa(\dot{s}) \in \mathfrak{g}(r, K) \mid \dot{s} \in \operatorname{res}_{r/K}^{-1}(s),\ s \in S_r(k)\}$$

is a set of representatives for the stable orbits in $\mathfrak{g}(r, K)$.

6.5 Future work

The techniques presented in this thesis may also be used to produce a complete list of Cartan subalgebras of $\mathfrak{g}(K)$. We leave that for another day.

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Appendix A

Representation of $G(2)$ in $SO(8)$

Traditionally, getting a handle on $G(2)$ has been difficult. Élie Cartan noted in 1914 [Car14] that $G(2)$ was the automorphism group of the octonions. Springer [Spr09, §17.4] realizes this using a direct sum of 2×2-matrices over a field.

A standard, but even more involved, way of saying this has been through the use of Cayley algebras: $G(2)$ is the automorphism group of a Cayley algebra where the Cayley algebra is built, for example, from 2×2-matrices this time over a 3-dimensional cross product algebra over $\mathbb{Q}$.

Using this Cayley algebra, Bump and Joyner [BJ87, §1] define a group we call $G(2)$ of automorphisms of type G_2 and its Lie algebra $\mathfrak{g}(2) = \operatorname{Lie} G(2)$. They show $G(2)$ embeds in $SO(8)$ the special orthogonal group, so $\mathfrak{g}(2)$ embeds in $\mathfrak{so}(8) = \operatorname{Lie} SO(8)$ the special orthogonal Lie algebra.

Then they offer a Chevalley basis for $\mathfrak{g}(2)$ in $\mathfrak{so}(8)$ which we used for doing explicit calculations. It is, in our notation:

$$X_{\alpha_1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_{\alpha_2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$X_{\alpha_1+\alpha_2} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_{2\alpha_1+\alpha_2} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$X_{3\alpha_1+\alpha_2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ and } X_{3\alpha_1+2\alpha_2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with the Chevalley basis elements for the negative roots being simply the transpose of those for the positive roots. So $[X_{\alpha_1}, X_{-\alpha_1}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$ and

$$[X_{\alpha_2}, X_{-\alpha_2}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$