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Finite Deformation of a Circular Inclusion with Inhomogeneous Imperfect Interface Conditions in Harmonic Materials

by

Daniel Reed McArthur

A THESIS

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ABSTRACT

The objective of this study is to determine the effect that introducing circumferential variations has on an imperfect interface in a circular inclusion in finite elasticity. The inclusion and surrounding matrix are assumed to be comprised of a particular class of materials referred to as harmonic materials and the analysis is confined to the case of plane strain deformations. This opens the door for the use of complex variable methods to derive analytic solutions for the interfacial stress under varying interface conditions.

This thesis focuses on three specific types of imperfect interface conditions, namely; the case where the degree of imperfection is the same in the coordinate directions normal and tangential to the interface, the case of a non-slip interface where the tangential degree of imperfection prevents rotation of the inclusion, and the case of a sliding interface where the normal degree of imperfection prevents separation of the interphase layer.

Using the technique of analytic continuation, analytic solutions are developed and results are contrasted to a corresponding homogeneous interface for all three of the aforementioned interface conditions. The results indicate that in general, the homogeneous interface model is not accurate in predicting the stress along the inclusion interface.

PREFACE

The bulk of this thesis is comprised of the work from three published/submitted manuscripts, they are as follows:

McArthur, D.R. and Sudak, L.J. (2016). A circular inclusion with circumferentially inhomogeneous imperfect interface in harmonic materials. *Continuum Mechanics and Thermodynamics.* 28(1-2):317-329.

McArthur,D.R. and Sudak, L.J. (2016). A circular inclusion with circumferentially inhomogeneous imperfect non-slip interface in harmonic materials. *Proceedings of the Royal Society. A, Mathematical, Physical, and Engineering Sciences*,(*DOI* : 10.1098/*rspa*.2016.0285).

McArthur, D.R. and Sudak, L.J. (2016). A circular inclusion with circumferentially inhomogeneous sliding imperfect interface in harmonic materials. *Journal of Elasticity*,(ACCEPTED).

All three of these works were completed by the present author and edited by the supervisor, Dr. Les Sudak. No other collaborators were involved in the research process. A fourth manuscript was published on the topic of arbitrarily shaped piezoelectric inclusions in linear elasticity however for the purposes of congruency it will not be included in this thesis. The citation for this work is given below

McArthur, D.R. and Sudak, L.J. (2015). An arbitrary piezoelectric inclusion with weakly and highly conducting imperfect interface. *International Journal of Solids and Structures*, 54 :215-237.

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“I was born not knowing and have only had a little time to change that here and there”

- Richard P. Feynmann

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List of Abbreviations, Symbols, and Nomeclature

Latin Symbols

x_a	components of spatial position vector
X_A	components of material position vector
F_{ab}	components of deformation gradient tensor
R_{aB}	components of rotation tensor
U_{BA}	components of right stretch tensor
v_{ab}	components of left stretch tensor
e_{ab}	components of Eularian strain tensor

E_{AB}	components of Lagrangian strain tensor
C_{AB}	components of right Cauchy Green strain tensor
b_{ab}	components of left Cauchy Green strain tensor
P_{ab}	components of first Piola stress tensor
W	strain energy function
w	deformation map
R	radius of circular inclusion

Greek Symbols

ψ_k	analytic potential function
ϕ_k	analytic potential function
χ_k	Piola stress function
μ_k	shear modulus
Γ	ratio of shear moduli

α_k	harmonic material property
β_k	harmonic material property
Ω	material constant
ω	material constant
η	material constant
ρ^*	interface imperfection parameter
λ	imperfect interface material parameter
$\boldsymbol{\pi}$	Configuration Map

Chapter 1

INTRODUCTION

As technology has evolved over the past century, composite materials have become an indispensable component of many consumer and industry products in use today. They can be found in many common electro/mechanical devices such as automobiles, computers, phones, airplanes and so on. With such widespread use, especially under the context of applications that carry the consequence of loss of human life if the material were to fail (e.g. airplane), it is paramount that the behavior of such materials is completely understood.

A composite material is typically described as a collection of two or more constituent materials (usually have varying mechanical or chemical properties) combined into a single unit. These 'composites' are manufactured in such a fashion that they tend to take advantage of specific aspects of each of the constituent materials. For example, in fiber reinforced polymers, the fiber can be oriented to increase the strength and resistance to deformation by aligning the fiber with the line of action of the applied load.

In general, the constituents of composite materials are categorized into one of two classes: *the matrix phase* is continuous and in many cases completely surrounds the secondary material, and *the reinforcing or inclusion phase*, which is dispersed throughout the matrix and can take many forms and or orientations. Between each phase of material there is an intermediary layer typically referred to as the *inter-phase layer*. Generally speaking, the inter-phase layer has a blend of the

mechanical properties of each of the constituents and more often than not contains distributed voids, micro tears and various other bonding flaws which may or may not have occurred during the material manufacturing process. Since the reinforcing phase of a composite may be comprised of either fibers oriented in a specific direction, or a distributed particulate, the inter-phase layer may be regarded as a two (in the case of a fiber with uniform plane cross section) or three dimensional region (in the case of a particulate). Regardless of the geometry of the inter-phase layer, it contains mechanical properties unique from the inclusion and matrix and therefore plays a crucial role in the transfer of loads between the two and thus it is essential to incorporate the effects of the inter-phase layer in the modeling and analysis of inclusion problems in elasticity.

To this end, a great deal of research has been conducted in the area of inclusion problems in elasticity. The field of research has rapidly progressed into the development of many complex models for inclusions which today incorporate not only a wide range of inclusion boundary geometries, but also utilize more intricate boundary conditions in the analysis. Of specific interest are inclusion problems incorporating some form of imperfect interface boundary conditions. These types of models are very important in the study of composites as, at least in most cases, failure in composite materials usually originates or is assisted by poor bonding between the matrix and reinforcing fibers. Traditionally, imperfect interface models employ a homogeneous degree of imperfection along the inclusion matrix interface, primarily to simplify the analysis. However, in recent years more research is being done in the study of inhomogeneous imperfect interface conditions, which in general give a more accurate depiction of the behavior of the interface between the inclusion and matrix. The progress however, has been somewhat limited to the case of small deformations (linear elasticity). This has created a void in the study of inclusions in finite elasticity with inhomogeneous imperfect interface conditions.

This thesis focuses on the finite deformation of inclusions with inhomogeneous imperfect interface conditions that are comprised of harmonic materials, a special rubber-like elastomer that allows for some very convenient mathematical simplifications. This work is novel in the sense that this represents the first time that an inhomogeneous imperfect interface condition has been con-

sidered in the study of harmonic materials in plane elasticity. In Chapter 2 the fundamental works of linear and finite elastic inclusion problems are reviewed and a basis for further development is established. Chapters 3 and 4 contain some of the basic prerequisites in continuum mechanics and complex analysis that are used frequently throughout the thesis. In Chapter 5 the concept of a harmonic material is defined and Chapter 6 contains the preliminary formulation for the boundary value problem of a circular inclusion with inhomogeneous imperfect interface that precedes each specific case of the imperfect interface in Chapters 7,8, and 9. In Chapter 10 conclusions are made about the results of the work and in Chapter 11 future work is discussed.

Chapter 2

BACKGROUND

2.1 Fundamental inclusion studies

Eshelby (1957) was one of the first in the field to tackle the problem of an elliptic inclusion. In his work, Eshelby used an ingenious cutting and welding technique to temporarily remove the inclusion, apply a stress free strain equivalent to the deformation of the matrix, and then apply a boundary traction comprised of a distribution of point forces along the inclusion boundary curve to 're-shape' the inclusion to fit and be 're-welded' back into the matrix. Using this technique, Eshelby was able to study several problems of ellipsoidal inclusions, and gained great insights into their behavior under mechanical loads, most notably that the stress field inside an elliptical inclusion subject to linear far field loadings is uniform.

In his second exposition, Eshelby (1959), focused more intently on the external elastic field of an ellipsoidal inclusion and was able to calculate the external elastic field at an arbitrary point in the matrix. While both are remarkable achievements in the study of inclusion problems, both of Eshelby's works of the late 1950's suffer from complex integrals that must be solved in order to calculate any interesting field quantities (i.e. stresses, displacements, etc.). As one might guess, these integrals are highly dependent on the geometry of the inclusion, and in general, become

far more complicated- to the point of being intractable- as the inclusion geometry strays from an elliptical form.

Building on the foundations of Eshelby, Jaswon et al. (1960) sought to overcome the difficulties of Eshelby's formulation by using the techniques of complex analysis for two-dimensional problems. Jaswon et al. (1960) recognized that, in the absence of body forces, the equilibrium equations on the stress furnish a homogeneous Laplace relation which may be concisely solved by postulating the stress as the real (or alternatively as the imaginary) part of a combination of analytic functions. This is a powerful approach indebted to the fact that any analytic function must satisfy the Cauchy-Riemann equations in its real and imaginary parts and hence one can show that the real and imaginary parts of an arbitrary analytic function automatically satisfy the bi-harmonic equation. The success of this method is that although Jaswon et al. (1960) employ the similar point-force technique that involves applying a fictitious stress field to "re-weld" the inclusion back into the matrix, they are able, via the techniques of Cauchy-integration, to concisely express the complicated integrals of Eshelby in a much simpler form. Not surprisingly Jaswon et al. (1960) concludes that for an elliptic inclusion the internal stresses are uniform (reconfirming Eshelby's results). Furthermore, Jaswon et al. (1960) also identifies that the equilibrium shape of the inclusion remains elliptical and identifies the mode in which the minor and major axes of the ellipse deform when exposed to plane strain deformations.

Following Jaswon et al. (1960), Bhargava et al. (1963) modified the complicated "point-force" method used in the previous studies to instead focus on the total strain energy in an elliptical inclusion and stressed matrix. In their work Bhargava et al. (1963) find the equilibrium displacements and stresses by calculating and then minimizing the total strain energy for the inclusion undergoing a deformation in absence of the matrix, and the matrix undergoing a deformation in absence of the inclusion (due to an applied far field loading on the matrix). This approach is quite powerful in that it does not require that the matrix and inclusion be made of the same material, and it also circumvents the need to evaluate the complex boundary integrals of the "point-force" method that plague the early results of Eshelby and others. In this work analysis is conducted for the cases of

plane strain without shear and for the case of plane strain with shear. In both cases the hoop stress is calculated for the inclusion and matrix where it is shown that the hoop stress contains both a uniform component and a component that varies with the angle that the normal to the inclusion/matrix boundary makes to the real axis.

Continuing the use of the complex variable formulation, Bhargava et al. (1964) study the case of a circular inclusion in an infinite elastic medium with a neighboring circular hole. In this work Bhargava et al. (1964) begins by re-stating the solution for an infinite matrix with a circular hole at the origin, exposed to a single point force located at some point outside of the boundary of the circular hole. With this solution in hand, the "point-force" method applied by Eshelby is the utilized to integrate a continuous circular curve of point forces to represent the effect of the deforming inclusion. The problem is then solved by inserting the expressions for a distribution of forces as functions of the reversed surface tractions originally proposed by Eshelby, and integrating the result. Results are then presented for the hoop stress around the inclusion and, as one can imagine, the presence of the hole does perturb the resulting stress fields in the inclusion and matrix, furthermore it is seen that the hoop stress is in fact non-uniformly discontinuous across the inclusion boundary.

The above works all utilized a interface of zero thickness and did not utilize the concept of a finite thickness interphase layer in their analysis. The incorporation of an interphase layer to model the bonding quality of an inhomogeneity to the surrounding matrix is part of the natural progression of increasing complexity for inclusion problems and in some respects served as a stepping stone for the still to come imperfect interface models.

2.2 Interphase models

Matonis (1969) represents one of the foremost discussions on interphase analysis for inclusions in composite materials. In this work, the author studies the effect of different interphase stiffnesss where, instead of modeling a radially inhomogeneous interphase stiffness, the interphase is given

by an average value of the stiffness curve. While this is a fairly significant simplification, it is well known that the solution of such problems is complicated to the point of intractability when functionally graded interface stiffnesss are incorporated into the analysis. Two cases are studied, a hard interphase with stiffness greater than the surrounding inclusion and matrix, and a soft interphase with stiffness lower than the surrounding inclusion and matrix. In both cases the shear, radial, and hoop stresses are calculated using the techniques developed by Goodier (1933). Matonis (1969) concluded that in the case of composites with hard inclusions, the introduction of an interphase layer had negligible effects however, in the case of rubber modified plastics, it was observed that interphase thickness influenced the yield stress of the composite but impaired the overall energy absorption qualities.

Broutman and Agarwal (1974) discusses a finite element model of a spherical inclusion with an interphase layer of finite thickness. In this work, Broutman and Agarwal (1974) show that by varying the strength of the interphase layer, the composite toughness and energy absorbing capability can be drastically altered. As one might expect lowering the stiffness of the interphase layer increases the interphase stress and deformation which in turn reduces the stress of the inclusion. Increasing the interphase stiffness has the converse effect. The ultimate effect of reducing the interphase modulus is an overall reduction in the strength of the composite but an increase in its energy absorbing ability. This effect is attributed to the fiber not actively contributing to the stiffness of the composite and Broutman and Agarwal (1974) analogizes this to a composite with voids where the inclusion fibers would normally be.

Agarwal and Bansal (1979) in contrast to Broutman and Agarwal (1974) modeled a cylindrical fiber but incorporated inter fiber interactions into the analysis by modeling the matrix strains as restricted by adjacent fibers. The addition of this component into the finite element analysis provides unique insights into composite conglomerates, as it was shown that reinforcing fibers can still be effective in low volume fraction situations so long as the interface stiffness is high. Furthermore it is concluded that the elastic modulus of a composite is not significantly affected by the interfacial strength whereas the toughness of the composite is highly dependent on the interfacial strength.

Papanicolaou et al. (1980) consider the anti-plane deformation of a three phase circular inclusion. The primary objective of the work was to develop a model for the variation of the modulus of elasticity throughout the interphase as a function of the radial coordinate. The form of the interphase variation relation is derived from a strain energy balance and appears as the law of mixtures for the longitudinal modulus of the cylindrical interphase region. The authors further generalize the model by supposing that the interphase region may be represented by an infinite number of cylinders and hence replace the interphase contribution to the global modulus with an integral term containing the supposed form of the elastic modulus of the interphase. This formulation allows for a great deal of analysis in the discussion of how the elastic modulus changes throughout the interphase depending on the bonding conditions between each constituent although it is limited to the case of anti-plane deformations.

Jasiuk and Kouider (1993) studied anti-plane deformations of a three phase composite with a radially inhomogeneous interphase region. In their work, Jasiuk and Kouider (1993) postulate the radial variation of the elastic modulus of the interphase region to be of either a linear or exponential form and proceed to solve the Airy stress problem for both cases. In their work they assume that the elastic modulus must be zeroth order continuous at the boundaries of the inclusion and matrix. The results suggest that variations of the elastic modulus of the interphase have a significant impact on the resulting effective properties of the composite. Specifically it was determined that the homogeneous model over-estimates the strength of the composite.

In the work of Lagache et al. (1994), the authors discuss several homogenization schemes for fibers in composites and compare the predictions to various experimental results. Through the testing and finite element simulation it was discovered that in general, composites with low fiber volume ratios have good agreement between the theory and experiments whereas composites with high fiber volume ratios do not agree with experimental data. The authors additionally tested the aforementioned homogenization techniques with inclusions containing interphase layers. They observed that the transverse (in-plane) composite properties are much more sensitive than the anti-plane properties, to variations in the interphase strength and thickness.

Lutz and Zimmerman (1996) explored the effect of an interphase in a spherical inclusion exposed to remote hydrostatic stresses. In the analysis a power law variation was applied to the elastic properties of the interphase layer which allowed for a jump between the inclusion and interphase but maintained a smooth continuous transition into the matrix. A closed form solution was obtained which indicated that the interphase zone has a significant impact on the stress in and around the inclusion. Furthermore, it was shown that a weak interphase lowers the effective moduli of a composite whereas a stiff interphase increases the effective moduli.

The work of Hashin and Monteiro (2002) is especially relevant to this thesis. In their work Hashin and Monteiro (2002) derive analytic solutions to the general inclusion problem with an imperfect interface and compare these results to the corresponding three phase model. Interestingly, it is shown that the collapsing of the interphase layer into a curve of zero thickness produces highly accurate results when contrasted to the three phase model. It is further stated that this type of imperfect interface model is highly suited to a complaint interphase region, that is, one that displays significant jumps in the displacements and negligible jumps in the traction vectors. These conclusions are important not only because this thesis utilizes the collapsible interphase model, but also because on a physical argument the interphase of a harmonic (rubber-like) material with harmonic inclusion will likely display highly compliant deformation characteristics.

The work of Wang and Zhong (2003) discusses the case of a circular inclusion with an interphase layer of non-uniform thickness in anti-plane shear. Through the generally well established complex variable approach, Wang and Zhong (2003) captured analytic solutions for the general case of a uniform remote loading, perturbation from a nearby dislocation, and nearby crack (along the real axis). A very interesting aspect of the results of said analysis is that Wang and Zhong (2003) concluded that the internal stress field of the inclusion was necessarily nonuniform as a result of the non-uniformity of the interphase layer thickness, even when subjected to uniform remote loadings. Furthermore, in the case of an adjacent screw dislocation, it was noticed that there may coexist two equilibrium positions, one stable, and another unstable.

2.3 Inclusions with an imperfect interface

An important development in the analysis of the elastic fields interior and exterior to an inclusion is the modeling of the interfacial region as a collapsible interphase. Arguably one of the most prominent models for a flexibly bonded interface was proposed by Jones and Whittier (1967). In their work, Jones and Whittier consider a composite joined by an inertia-less interface layer of a given thickness t. This assumption yields two sets of elastic boundary conditions; firstly, from the assumption that the interface is inertia-less, the boundary tractions are constrained to be equal across the interface. Secondly, the elasticity of the bond is captured through imperfect shear and normal stress conditions where in both cases the stresses are proportional to the corresponding displacements via the ratio of the shear and Youngs moduli to the interface thickness t, respectively.

The works of Benveniste and Aboudi (1983),Benveniste (1985a),Benveniste (1985b), and Aboudi (1987) all study models for the effects of imperfect bonding on the global effective properties of a composite material. Generally employing volume averaged formulations for a given unit cell configuration, these works provide much needed validation of the predictive capability of imperfect interface boundary conditions and, particularly in the work of Aboudi (1987), good agreement with experimental results is shown for a variety of different interface conditions.

Hashin (1991) shows that in the case of a spherical inclusion with homogeneous imperfect interface, the internal stresses to the inclusion are non-uniform (recall that in the perfect bonding cases modeled by Eshelby, the internal stresses were uniform). Hashin further describes the difficulties that different loading conditions may have on the modeling of an imperfect interface. Specifically, the case of compressive loads suggests interpenetration of the two parent materials. Hashin suggests that this could be overcame by either defining a different interface parameter for compression, by allowing a small negative displacement jump at most equal to the interphase thickness, or by setting the normal interface condition to be that of perfect bonding.

Gao (1995) studied a single circular inclusion in an infinite matrix with imperfect interface conditions described by a spring layer of vanishing thickness. The analysis is completed via solving the Airy stress problem and additionally the Eshelby S-tensor is calculated for the loading

cases of a uniform eigenstrain and uniform tension at the far field. Additionally, several inequality criterion are defined to prevent material interpenetration. From these discussions the authors show that no material overlap occurs for a sliding interface (where the normal interface parameter approaches infinity) and provide conditions on the interface parameters and material strengths to ensure no overlap for the cases of equal interface strength in both coordinate directions and a non slip interface (where the tangential interface parameter approaches infinity).

Bigoni, D. , Serkov, S.K. , Valentini, M., Movchan (1998) gives a particularly elegant derivation of the general imperfect interface conditions using the first order terms of the Taylor series expansions of the interface stress-displacement conditions. Bigoni et al then proceeds to give a solution to the problem of a circular inclusion with imperfect interface conditions using series expansions of several Cauchy integrals to develop relations for the unknown coefficients of the unknown potential functions from Muskhelishvili's complex variable formulation. The work concludes with an enticing section on the computation of the effective moduli of a composite containing a dilute distribution of inclusions. In the formulation of the stiffness matrix for the composite, Bigoni et al employ the Polya-Szego material matrix which attempts to incorporate the effects of the inclusion and interface properties into the global stiffness. The most unique part of this later work is that Bigoni et al provide derivations for the perturbations that the components of the Polya-Szego matrix and in general the imperfect interface properties, have on the global stiffness of the composite, giving a great deal of insight into the effects that different types of inclusions and interface conditions have on dilute composites. As an example, Bigoni et al concluded that in the particular case of a stiff inclusion, the imperfect interfacial conditions have a remarkable effect on the global properties of a composite.

2.4 Inclusions with inhomogeneous imperfect interface

In modeling an inclusion with an imperfect interface, it is not always reasonable to assume that the degree of imperfection is homogeneous throughout the interfacial curve. The extension of the

imperfect interface model into an inhomogeneous imperfect interface model was first discussed by Ru (1998) and Schiavone, P., Ru (1997), who studied a circular inclusion with a circumferentially inhomogeneous imperfect interface in plane and anti-plane elasticity, respectively. In the work by Ru (1998), the case of a circular inclusion in plane elasticity with sliding interface conditions was studied. The classification of a sliding interface is described by a very smooth interphase contact region between the inclusion and matrix that admits relative tangential displacements but maintains continuity across the interface in the radial direction. Through the use of analytic continuation the problem is simplified to a single ordinary differential equation for one analytic potential function. Upon selecting a specific form of the inhomogeneous interface function, Ru (1998) showed that an inhomogeneous imperfect sliding interface has a moderate effect on the prediction of the mean stress at a point. When contrasted to the homogeneous analog, a prediction error of up to 60 percent was observed.

Later,Sudak et al. (1999) studied a more general case of plane deformation of a circular inclusion with an inhomogeneous imperfect interface. This work focused on the interfacial description where the same degree of interface imperfection is realized in both coordinate directions normal and tangential to the interface curve. Through analytic continuation the work is reduced to two first order ordinary differential equations with variable coefficients from which results are presented for a specific case of inhomogeneous interface function. Interestingly, the authors concluded that the introduction of an inhomogeneous imperfect interface yielded a stress magnification in the average mean stress, average deviatoric, and average shear of up to 300 percent when compared to the homogeneous analog.

While many exhaustive studies exist for the study of inclusion problems in small deformations, there is a notable lack of progress in finite deformation problems. The fundamental models of inclusion problems in finite elasticity were developed in the sixties and seventies however the theories did not gain much traction until several years later.

2.5 Problems in finite elasticity

In the general class of mechanics problems in finite elasticity it is highly difficult to find exact unique solutions for problems where the deformation field is not prescribed apriori (in fact in only some very specific scenarios can one find a solution to the corresponding non-linear displacement problem). When a deformation is prescribed to a problem then one can find a solution and hence verify its validity through equilibrium conditions. These types of problems were first described by Rivlin (1948a). In the general context however, one must resort to approximate techniques to solve the non linear differential equation on the displacements where the deformation is not known. One of the more prominent methods of formulating a solution to the aforementioned problem was proposed in the work of Rivlin (1953) which is commonly referred to as "Second Order Elasticity".

Modern usage of this theory employs a version of the so called "Second Order Elasticity" whereby the non linear displacement functions are written in a power series of a non dimensional parameter ε up to second order, where the coefficients of the two terms in this expansion represent the first and second order parts of the displacement function. The general problem is then reformulated in terms of the first and second order parts of the displacement and solved via traditional techniques for partial differential equations (see for example Selvadurai (1973b),Selvadurai (1973a),Selvadurai (2016) and references therein).

While the aforementioned techniques provide approximate solutions for a wide class of problems in finite elasticity the constitutive equations employed in the theory are not conducive to the use of complex variable techniques which, for a specific class of strain energy functions can provide exact solutions to plane problems without prior knowledge of the deformation.

2.6 Inclusions in finite elasticity with harmonic materials

The study of inclusion problems in finite elasticity was popularized by the advent of John's (John (1960)) harmonic materials . In his work, Fritz John defines a harmonic material as a material where pseudo-irrotationality of the deformation (at some point in time) implies the acceleration

field (of the deformation) is also pseudo-irrotational (in the absence of a body force). This condition, with the assistance of some modified strain invariants, yields a specific form of the strain energy function which is described by John as " *necessary and sufficient*" for a material to be classified as harmonic. Utilizing the definition of harmonic materials laid out by F. John, Knowles, J.K., Sternberg (1975) studied the behavior of harmonic elastic materials near boundary conditions that, in the corresponding linear theory, caused oscillations in the deformations and stresses. In their work, Knowles and Sternberg provided a complex variable formulation for finite deformations and demonstrated that the oscillatory behavior seen in the linear analog was absent for harmonic materials.

In addition to the above, Ogden and Isherwood (1977) presented a formulation for the finite deformation of compressible, isotropic elastic materials that built on the foundations of Knowles and Sternberg providing a more streamlined formulation with complex variables. This newer formulation uses slightly modified invariants of the deformation gradient to simplify the form of the Airy stress equation coming from the equations of equilibrium. The Airy stress equation is then solved by introduction of a system of complex coordinates and strain energy function for a material of harmonic type. This allows the problem to be reduced to a separated set of differential equations on the Airy stress functions which may be solved by simple integration. Concurrently, the authors also solve for the current displacement in a similar way. The integrals are somewhat tedious in appearance however in the ensuing examples, Ogden et al show that they may be evaluated with relative ease for a variety of different problems.

Following Knowles et al and Ogden et al, Varley and Cumberbatch (1980) published an extensive exposition on the "*Finite deformation of elastic materials surrounding cylindrical holes*". In their work, Varley and Cumberbatch provide a complex variable formulation for finite deformations and discuss four special types of harmonic materials (deemed types 1 through 4) and their application to specific contexts in elasticity (for example, type 1 harmonic materials are perfectly elastic). They then proceed to several problems of cylindrical holes and provide results for a variety of loadings.

While elegant and powerful, the derivations of Knowles et al., Ogden et al. and Varley et al. did not gain a great deal of popularity in the mechanics community. Several years later, Ru (2002) presented a concise framework for the fundamental equations of harmonic materials in a complex variable formulation. In this work, Ru develops the solution to the general equilibrium equation by noting that it may be expressed in terms of analytic potential functions. In this way, Ru is able to solve a general non-homogeneous ordinary differential equation for the displacement function. Then, comparing to the work of Varley and Cumberbatch (1980), Ru developed the particular solution to the differential equation and ultimately gave the deformation and Piola stress functions for a specific class of harmonic materials. Ru then proceeds to solve the problem of an interfacial crack on the real axis of a bi-material to illustrate the power of the method.

The work of Ru (2002), would initiate several other works on the study of harmonic materials in finite deformations including the study of elliptical inclusions with uniform stress fields by Ru et al. (2005), partially de-bonded circular inclusions by Wang and Pan (2009), three phase elliptical inclusions with internal uniform hydro-static stresses by Wang and Gao (2011), three phase inclusions of arbitrary shape with internal uniform hydrostatic stresses by Wang and Schiavone (2016), a circular inclusion with imperfect interface in finite plane elastostatics by Wang (2012), and harmonic three phase circular inclusions in finite elasticity by Wang and Schiavone (2015).

Of specific interest is the work of Wang (2012) where the case of a circular inclusion with homogeneous imperfect interface (with the additional case of a rate dependent interface) is studied. The problem is solved via comparing coefficients of powers of the complex variable z in the boundary condition expressions for continuity of tractions and jump in displacements. As one might expect, Wang discovered that the stresses are non-uniform inside the inclusion for an imperfect interface in finite elasticity and provided expressions for the potentials to derive the general stress vectors. Although Wang has discussed the case of an homogeneous imperfect interface in finite deformations, the case of a inhomogeneous imperfect interface has not yet been studied in the finite deformation regime.

2.7 Experimental studies

Concurrent with the development of the theory of hyperelastic materials, many experimental studies have been conducted to determine both the micro and macroscopic properties of composites. Of particular interest are studies involving rubber composites in which properties of the interphase (or interface) were tested experimentally. Unfortunately, there very few presently available studies in rubber-rubber composites to suggest the existence of an interphase however, there are several studies of fiber reinforced rubber composites that have identified the existence of an interphase and it is supposed that this may indirectly support the existence of an interphase in rubber-rubber composites.

In the work of Nardin et al. (1993) the authors tested specimen of carbon fibre-styrene butadiene rubber (**SBR**) under uni axial tension and then compared the interfacial shear strength determined through the experimental testing to predicted values from theory. Interestingly they found that the experimental results were up to 40 times larger than then predicted theory. Among several explanations that were offered by the authors to account for this discrepancy, of particular interest is the suggestion that "*the existence of an interfacial layer, which exhibits physical and mechanical properties completely different from the bulk of **SBR**, should be assumed. In particular, an interphase constituted of elastomeric chains of reduced mobility near the fiber surface can be invoked*". Furthermore, the authors study the problem of determining the existence of an interphase layer and, through a creep analysis of **SBR** under a tensile load where an increase in the elastic modulus is observed over time it is suggested that this may be indicative of reduced mobility of macromolecular chains owing to the existence of an interfacial region near the fiber.

The studies conducted by Baranov et al. (2003) show the formation of an interphase layer in Isotactic Polypropylene (**IPP**)/Ethelene-Propylene Rubber (**EPR**) blends and the authors introduce oil into the mixing process to influence the resulting mechanical properties. The analysis was conducted by scanning electron microscopy to determine the morphology of the samples and mechanical measurements were taken from a uni axial testing machine. It was concluded that the

oil may have dissolved some of the IPP molecules with low molecular weights and more imperfect structures (shapes).

In the work by Ryu and Lee (2006), the authors study the mechanical properties of short-fiber reinforced polychloroprene rubber in terms of the interphase condition, among other things. By modifying the interphase conditions the authors were able to adjust the spring constant (change in load per unit length) of the rubber and additionally were able to increase the bursting pressure by up to 8.73 times for fiber reinforced rubbers.

It has also been suggested in the literature of composite elastomers (although little if any testing has been done), that by controlling deposition of the vulcanizing agent one may design specific interphase conditions in what may be an otherwise completely non-uniform material. This is especially relevant to the work in this thesis because the concepts of a sliding, non slip, and equal axial imperfect interface could all be explained from the perspective of vulcanizing agent deposition.

Chapter 3

CONTINUUM MECHANICS PRELIMINARIES

3.1 Introduction

In this chapter we introduce some of the basic concepts of continuum mechanics of deformable bodies that are relevant to the work of this thesis. Although continuum mechanics is a very broad subject that extends from the theory of fluids to continuum thermodynamics and all the way back to deformable solids, this thesis is concerned with only the developments in the area of deformable solids. In the development of the material in continuum mechanics there are two distinct styles available in the available literature. There is the highly mathematical approach of those such as Marsden and Hughes (1983) and there is the so- called "classical" approach of Truesdell and Noll (1965), Green and Zerna (1968), Ogden (1997), Holzapfel (2000), and Chadwick (1976) among others. While the application of concepts in differential geometry in the highly mathematical works does allow for great insights into many of the developments in continuum mechanics, the concepts are difficult to wield without the prerequisite mathematical background. In contrast, the works following a more "classical" based approach offer strong insights and are much more accessible. Thus for the purpose of this thesis, which only utilizes the basic concepts of continuum

theory before departing into the world of complex analysis, we shal reference much of the material from the texts by Ogden (1997), and Holzapfel (2000). In doing so we confine the discussion to a Cartesian frame work and hence we do not differentiate between covariant and contravariant tensor components.

3.2 Material and spatial coordinates

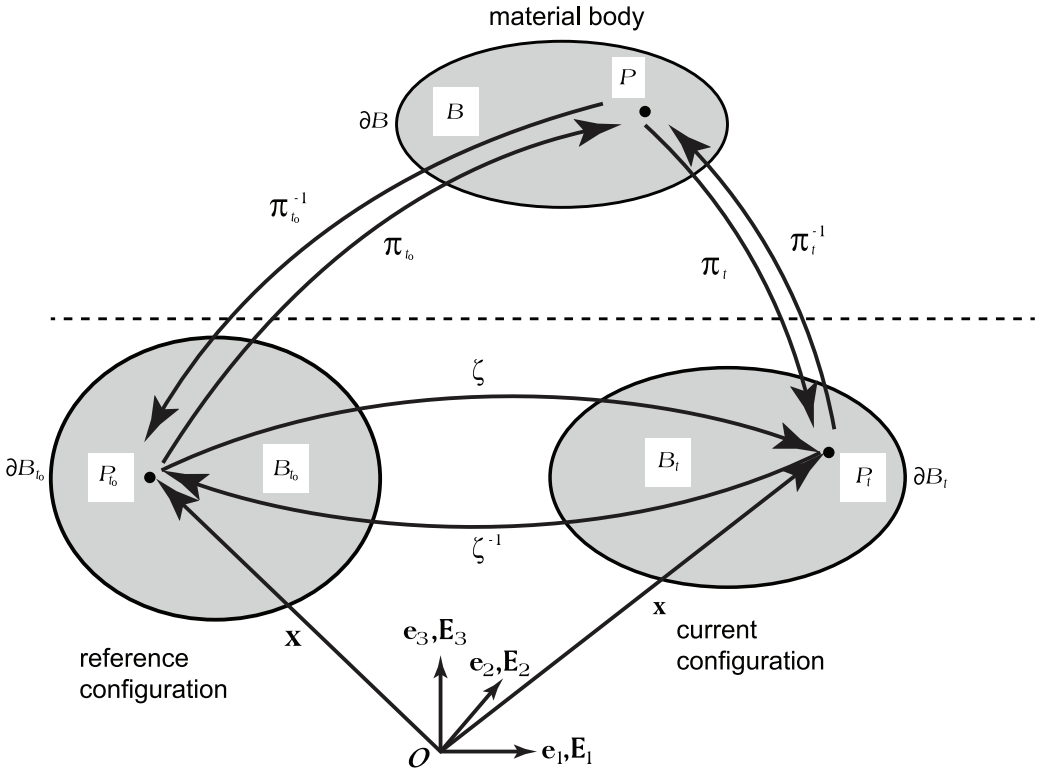

Figure 3.1: Configurations of an elastic body $\overline{B}$

Let us consider a general deformable body B characterized by some arbitrary continuous boundary ∂B. We identify each particle in the body B with a point P in a Euclidean point space. Two sets of bijective and invertible maps are then defined as π_{t_0} and π_t which map the points of the body B into a configuration of B. Let each configuration of $\overline{B}$ (where $\overline{B} = B \cup \partial B$) be embedded in an 3-dimensional Euclidean space at some instant in time t. For convenience we shall prescribe an initial undeformed configuration of $\overline{B}$ at time $t = 0$ and will refer to this as the reference configuration of $\overline{B}$ and denote it by $\overline{B_0}$. In order to describe the position of any particle in

the reference configuration, a set of orthonormal basis vectors $\mathbf{E_A},\ \ A = 1,2,3$ are introduced such that the position vector $\mathbf{X}$ may be given by

$$\mathbf{X} = X_A \mathbf{E_A},\ \ A = 1,2,3, \tag{3.1}$$

where $X_A,\ \ A = 1,2,3$ are denoted as the material coordinates of the vector $\mathbf{X}$ corresponding to an arbitrary particle P. Let us assume that the body then moves (or is deformed) into a new region $\overline{B}_t$ at some time $t > 0$. We refer to this new configuration as the current configuration of $\overline{B}$. Hence we consider a new orthonormal basis denoted by $\mathbf{e_a},\ \ a = 1,2,3$ (note that $\mathbf{e_a}$ may or may not coincide with $\mathbf{E_A}$ based on problem specific geometry and deformation criteria) from which we may express the position of a particle P as

$$\mathbf{x} = x_a \mathbf{e_a},\ \ a = 1,2,3, \tag{3.2}$$

where $x_a,\ \ a = 1,2,3$ are the so called spatial coordinates of $\mathbf{x}$. We may then define a configuration map, or motion, of the body as the convolution of the $\boldsymbol{\pi}$ mappings as follows

$$\mathbf{x} = \boldsymbol{\pi}_{\boldsymbol{t}}(\boldsymbol{\pi}_{\boldsymbol{t_0}}(\mathbf{X},t)) = \boldsymbol{\zeta}(\mathbf{X},t),\ \ or,\ \ x_a = \zeta_a(\mathbf{X},t), \forall \mathbf{X} \in \overline{B_0},\ \ \boldsymbol{\pi}_{\boldsymbol{t_0|t}} : \overline{B} \to \overline{B_{0|t}}, \tag{3.3}$$

where it is assumed that $\boldsymbol{\zeta}$ has continuous derivatives with respect to position and time and has a unique inverse given by

$$\mathbf{X} = \boldsymbol{\zeta}^{-1}(\mathbf{x},t),\ \ or,\ \ X_A = \zeta_A^{-1}(\mathbf{x},t), \forall \mathbf{x} \in \overline{B}. \tag{3.4}$$

Note that $\boldsymbol{\zeta}$ is a vector field and may be referred to as a *motion* of the body $\overline{B}$. In the context of a *deformation* the time dependence of (3.3) and (3.4) is usually dropped.

3.3 The deformation gradient

To understand how curves and tangent vectors will deform between the reference and current configurations we define the following two parametrizations for arbitrary curves in the reference and current configurations, respectively

$$\mathbf{X} = \boldsymbol{\gamma}(\xi), \;\; X_A = \gamma_A(\xi), \mathbf{X} \in \overline{B}_0, \tag{3.5}$$

$$\mathbf{x} = \boldsymbol{\zeta}(\boldsymbol{\gamma}(\xi), t), \;\; \mathbf{x} \in \overline{B}, \tag{3.6}$$

where the time dependency has been dropped in (3.5) because we have taken the reference configuration to occur at time $t = 0$. Differentiation of (3.6) with respect to ξ and application of the chain rule leads to the following relation between tangent vectors $d\mathbf{x}$ and $d\mathbf{X}$

$$d\mathbf{x} = \mathbf{F}(\mathbf{x}, t) d\mathbf{X} \;\; or \;\; dx_a = F_{aA} dX_a, \tag{3.7}$$

where

$$\mathbf{F}(\mathbf{X}, t) = \frac{\partial \boldsymbol{\zeta}(\mathbf{X}, t)}{\partial X} \;\; or \;\; F_{aA} = \frac{\partial \zeta_a(\mathbf{X}, t)}{\partial X_A}, \;\; \mathbf{F}(\mathbf{X}, t) : T_X(\overline{B_0}) \rightarrow T_x(\overline{B}), \tag{3.8}$$

is referred to as the deformation gradient, which is a two-point second order tensor meaning it maps material tangent vectors in the tangent space $T_X(\overline{B_0})$ into spatial tangent vectors in the tangent space $T_x(\overline{B})$. Note that the determinant of the deformation gradient $\mathbf{F}$ is defined as

$$J = det[\mathbf{F}], \tag{3.9}$$

where J is the volume ratio of the current and reference configurations, respectively. Naturally it is assumed that an inverse deformation gradient exists and is given by

$$\mathbf{F}^{-1}(\mathbf{x},t) = \frac{\partial \boldsymbol{\zeta}^{-1}(\mathbf{x},t)}{\partial \mathbf{x}} \;\; or \;\; F^{-1}_{Aa} = \frac{\partial \zeta^{-1}_{A}(\mathbf{x},t)}{x_a}. \tag{3.10}$$

In addition to the above, we may define the displacement field as follows

$$\mathbf{U}(\mathbf{X},t) = \mathbf{x}(\mathbf{X},t) - \mathbf{X}. \tag{3.11}$$

Equation (3.11) characterizes the material description of the displacement field. Accordingly, we may also define a spacial description of the displacement field as follows

$$\mathbf{u}(\mathbf{x},t) = \mathbf{x} - \mathbf{X}(\mathbf{x},t). \tag{3.12}$$

3.4 Polar decomposition of the deformation gradient

A fundamental theorem in continuum mechanics is that the deformation gradient may be decomposed into the single contraction of two second order tensors. In this way the deformation gradient

is re ordered as a stretch in the reference configuration followed by a two point rotation or a two point rotation followed by a stretch in the current configuration, respectively, as follows

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR} \;\; or \;\; F_{aA} = R_{aB}U_{BA} = V_{ab}R_{bA}, \tag{3.13}$$

where $\mathbf{R}$ is the rotation tensor comprised of the direction cosines between the bases in the current and reference configurations and is proper orthogonal while $\mathbf{U}, \mathbf{V}$ are symmetric and referred to as the right and left stretch tensors, respectively.

3.5 Strain Measures

Since strain is a secondary concept in the analysis of deformation, several different strain measures have been proposed over the development of the field of continuum mechanics. We introduce the concept of a strain as a change in the length of the tangent vectors $d\mathbf{x}, d\mathbf{X}$ as follows

$$|d\mathbf{x}|^2 - |d\mathbf{X}|^2 = (\mathbf{F}d\mathbf{X}) \,.\, (\mathbf{F}d\mathbf{X}) - d\mathbf{X}.d\mathbf{X} = d\mathbf{X}.(\mathbf{F}^T\mathbf{F})d\mathbf{X} - d\mathbf{X}.d\mathbf{X} = d\mathbf{X}.(\mathbf{F}^T\mathbf{F} - \mathbf{I})d\mathbf{X} \;\; . \tag{3.14}$$

The tensor $\mathbf{F}^T\mathbf{F} - \mathbf{I}$ is regarded as the Lagrangean strain tensor

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) \;\; or \;\; E_{AB} = \frac{1}{2}((F_{aA})^T F_{aB} - I_{AB}) \;\; , \tag{3.15}$$

where the factor $\frac{1}{2}$ is a normalization factor and the secondary strain measure $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ is referred to as the right Cauchy-Green deformation tensor. Note that both $\mathbf{E}$ and $\mathbf{C}$ are purely referential quantities. We may also define strain in the current configuration as

$$|d\mathbf{x}|^2 - |d\mathbf{X}|^2 = d\mathbf{x}.d\mathbf{x} - (\mathbf{F}^{-1}d\mathbf{x}).(\mathbf{F}^{-1}d\mathbf{x}) = d\mathbf{x}.d\mathbf{x} - d\mathbf{x}.(\mathbf{F}^{-T}\mathbf{F}^{-1})d\mathbf{x} = d\mathbf{x}.(\mathbf{I} - \mathbf{F}^{-T}\mathbf{F}^{-1})d\mathbf{x} \ . \tag{3.16}$$

The tensor $\mathbf{I} - \mathbf{F}^{-T}\mathbf{F}^{-1}$ is regarded as the Eularian (or Euler-Almansi) strain tensor and is defined as

$$\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{F}^{-T}\mathbf{F}^{-1}) \ \ or \ \ e_{ab} = \frac{1}{2}(I_{ab} - (F_{aA})^{-T}(F_{Ab})^{-1}) \ \ , \tag{3.17}$$

where $\frac{1}{2}$ is a normalization factor and the secondary strain measure $\mathbf{b}^{-1} = \mathbf{F}^{-T}\mathbf{F}^{-1}$ is the inverse of the left Cauchy-Green deformation tensor $\mathbf{b} = \mathbf{F}\mathbf{F}^T$. While many other alternative strain measures exist in the literature, the present work only uses the right Cauchy-Green deformation tensor and hence we shal not introduce any of the alternative strain measures.

3.6 Stress Measures

Cauchy's stress theorem simply states that there must exist a second order tensor field $\boldsymbol{\sigma}$ such that the traction vector acting on an arbitrary surface element may be represented by

$$\mathbf{t}(\mathbf{x}, t, \mathbf{n}) = \boldsymbol{\sigma}(\mathbf{x}, t)\mathbf{n} \ \ or \ \ t_a = \sigma_{ab}n_b, \tag{3.18}$$

or alternatively

$$\mathbf{T}(\mathbf{X},t,\mathbf{N}) = \mathbf{P}(\mathbf{X},t)\mathbf{N} \;\; or \;\; T_a = P_{aA}N_A, \tag{3.19}$$

where $\mathbf{T}$ represents the current force acting on a referential area and $\mathbf{t}$ represents the current force acting on the current area of a deformed body. The second order tensor fields $\mathbf{P}$ and $\boldsymbol{\sigma}$ are called the First Piola Stress and Cauchy stress, respectively. Note that we may move back and forth from the First Piola and Cauchy stress tensors via use of the so called Piola transformation

$$\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-T} \;\; or \;\; P_{aA} = J\sigma_{ab}F^{-1}_{aA}. \tag{3.20}$$

It should be noted that, much like the case of strain measures, stress measures are a secondary concept and as such a variety of different stress measures have been introduced in the literature. For the purposes of this thesis however, we shal restrict the discussion to the first Piola stress tensor shown above.

3.7 Plane strain continuum mechanics

If we assume that no deformation takes place in the anti plane direction (e_3 and E_3), then the resulting deformation which is confined to the $(1,2)$-plane is referred to as plane strain. The deformation gradient is then simplified considerably as (in the case of zero rigid body displacement) we may consider the Eulerian and Lagrangian bases (e_a and E_A) to be coincident and hence we may write

$$\boldsymbol{F} = F_{ab}e_a \otimes e_b + e_3 \otimes e_3. \tag{3.21}$$

However, for two dimensional analysis we may in fact drop the identity component of the deformation gradient which leaves (note that this does not in any way imply that the anti-plane stresses are zero, instead, it is merely that we confine the present study to only the in-plane components of the deformation)

$$\boldsymbol{F} = F_{ab} e_a \otimes e_b \ . \tag{3.22}$$

Then from the polar decomposition of the deformation gradient we have that

$$\boldsymbol{U} = \lambda_1 e_1 \otimes e_1 + \lambda_2 e_2 \otimes e_2, \tag{3.23}$$

$$\boldsymbol{R} = cos(\theta) e_1 \otimes e_1 + sin(\theta) e_1 \otimes e_2 - sin(\theta) e_2 \otimes e_1 + cos(\theta) e_2 \otimes e_2. \tag{3.24}$$

where the $\lambda_k, k = 1,2$ in (3.23) are the principal stretches and the eigenvalues of the deformation gradient.

3.8 Hyperelastic materials

A hyperelastic material is a material that is characterized by the existence of a strain energy function W. For the case of a perfectly elastic material, the constitutive relation is derived from the Clausius-Planck form of the second law of thermodynamics which states

$$\mathscr{D}_{int} = \left(\mathbf{P} - \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} \right) : \dot{\mathbf{F}} = 0, \tag{3.25}$$

where $\mathscr{D}_{int}$ is the internal energy dissipation and $\dot{\mathbf{F}}$ is the material time derivative of the deformation gradient. Hence we see that

$$\mathbf{P} = \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}}, \tag{3.26}$$

since $\dot{\mathbf{F}}$ in (3.25) is arbitrary. If we assume that the strain energy function W is objective (i.e. invariant under a rigid body translation or rotation) then we must have that

$$W(\mathbf{F}) = W(\mathbf{QF}), \forall \mathbf{F}, \mathbf{Q}, \tag{3.27}$$

where $\mathbf{Q}$ is an arbitrary orthogonal tensor representing a rotation of the basis in the current configuration. If (3.27) is satisfied, then we may choose $\mathbf{Q}$ to be $\mathbf{R}^T$ from the polar decomposition of $\mathbf{F}$ and hence

$$W(\mathbf{F}) = W(\mathbf{R}^T\mathbf{RU}) = W(\mathbf{U}) = W(\mathbf{C}), \tag{3.28}$$

since $\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{UR}^T\mathbf{RU} = \mathbf{U}^2$. For an isotropic hyperelastic solid the strain energy function W must additionally obey

$$W(\mathbf{F}) = W(\mathbf{FQ}^T), \tag{3.29}$$

where $\mathbf{Q}$ represents a rotation in the reference configuration. Equation (3.29) is equivalent to

$$W(\mathbf{C}) = W(\mathbf{QCQ}^T). \tag{3.30}$$

When (3.30) is satisfied, the strain energy function W may be rewritten in terms of the scalar invariants I_1, I_2, I_3 of $\mathbf{C}$

$$W = W(I_1, I_2, I_3). \tag{3.31}$$

The First Piola stress is then calculated by differentiating (3.31) with respect to the components of the deformation gradient such that

$$\mathbf{P} = \frac{\partial W(I_1, I_2, I_3)}{\partial \mathbf{F}} \quad or \quad P_{ab} = \frac{\partial W(I_1, I_2, I_3)}{F_{ab}}. \tag{3.32}$$

In the development of the field of finite elasticity there have been several models proposed for the form of the strain energy function $W(I_1, I_2, I_3)$. Recalling that

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = tr(\mathbf{C}), \tag{3.33}$$

$$I_2 = \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2 = \frac{1}{2}[tr(\mathbf{C})^2 - tr(\mathbf{C}^2)],$$

$$I_3 = \lambda_1^2\lambda_2^2\lambda_3^2 = det(\mathbf{C}),$$

are the principal invariants of $\mathbf{C}$, and that we may write W in the general three dimentional case as a function of the principal invariants $W = W(I_1, I_2, I_3)$, we may expand W in a power series in terms of $I_1 - 3, I_2 - 3, I_3 - 1$ as

$$W(I_1, I_2, I_3) = \sum_{l,m,n=0}^{\infty} c_{lmn}(I_1 - 3)^l (I_2 - 3)^m (I_3 - 1)^n \quad , \tag{3.34}$$

where l,m,n are natural numbers, c_{lmn} is independent of the type of deformation, and the subtraction of 3 and 1 in the series product terms is to recover zero strain energy in the undeformed state. Since in practice many rubber like materials behave in an approximate incompressible sense ($I_3 = 1$), (3.34) is generally reduced to

$$W(I_1,I_2) = \sum_{l,m=0}^{\infty} c_{lm}(I_1-3)^l(I_2-3)^m \quad . \tag{3.35}$$

From the form of W in (3.35) the following special cases may be derived

$$W(I_1,I_2) = c_{10}(I_1-3) + c_{01}(I_2-3) \quad , \tag{3.36}$$

$$W(I_1,I_2) = c_{10}(I_1-3) \quad , \tag{3.37}$$

where (3.36) is the Mooney-Rivlin strain energy function (see Rivlin (1948b)) and (3.37) is the neo-Hookean strain energy function, which may be thought of as a special case of (3.36) where $c_{01} = 0$. A further development from the general form of (3.35) was given by Ogden (see Ogden (1982)) as

$$W(\lambda_1,\lambda_2,\lambda_3) = \sum_{p=1}^{N} \frac{\mu_p}{\alpha_p}\left(\lambda_1^{\alpha_p} + \lambda_2^{\alpha_p} + \lambda_3^{\alpha_p} - 3\right) \quad , \tag{3.38}$$

where $2\mu = \sum_{p=1}^{\infty} \mu_p\alpha_p > 0$ denotes the shear modulus of the material and N is a natural number and α_p is real. While the Ogden, Mooney-Rivlin, and neo-Hookean models (among others) for finite deformations of incompressible rubber materials have each had success in describing the

response of said materials, in the present work we shal study the deformation of compressible rubber materials. In the literature there have been several propositions for models for compressible rubber materials which employ a decomposition of the strain energy function into a volumetric (volume changing) and isochoric (volume preserving) part as

$$W(\mathbf{C}) = W_{vol}(J) + W_{iso}(\overline{\mathbf{C}}), \tag{3.39}$$

where $\mathbf{C} = J^{2/3}\overline{\mathbf{C}}$ is a modified isochoric version of the right Cauchy Green deformation tensor. Naturally many of the proposed incompressible models may be rewritten in the compressible decomposition. As an example the compressible Mooney Rivlin model given in Holzapfel (2000) is

$$W(J,I_1,I_2) = c(J-1)^2 - dln(J) + c_{10}(I_1-3) + c_{01}(I_2-3), \tag{3.40}$$

where c is a material constant and $d = 2(c_{10}+c_{01})$. The compressible neo-Hookean model also given in Holzapfel (2000)is

$$W(I_1,J) = \frac{\mu}{2b}(J^{-2b}-1) + c_{10}(I_1-3), \;\; b = \frac{\nu}{1-2\nu}. \tag{3.41}$$

Unfortunately, many of the compressible models available in the literature are not conducive to application of complex variable methods and hence we shal employ the hyperelastic model proposed by John (1960) which covers a class of isotropic compressible rubber like materials under finite deformation and specifically lends itself to use of the complex variable method for plane strain problems. The detailed derivations of the harmonic material model will be deferred to chapter 5 in favor of discussing the fundamentals of complex variables first in chapter 4.

Chapter 4

COMPLEX VARIABLES

4.1 Introduction

The fundamental concepts of complex analysis are used frequently in this thesis and hence for clarity we shall recite the most relevant theories and concepts here. The content of this chapter is relayed primarily from two texts in the subject area, namely the books of England (2003) and Brown and Churchill (2004). For further details and proofs of the concepts and theorems the reader is recommended to review these texts.

4.2 Complex function preliminaries

We begin by defining the general complex coordinate $z = x_1 + ix_2$ where x_1 is the real component and x_2 is the imaginary component of z. The complex conjugate of z, denoted by $\bar{z}$, is given by $\bar{z} = x_1 - ix_2$. A complex function $\phi(z)$, which assigns a complex number $\phi(z) = u(x_1,x_2) + iv(x_1,x_2)$ to each point within some arbitrary domain, is said to be analytic or complex differentiable if it satisfies the Cauchy-Riemann equations

$$u_{,x_1} = v_{,x_2} \qquad u_{,x_2} = -v_{,x_1}. \tag{4.1}$$

A complex function $\theta(z)$ is said to be holomorphic if for some region $\mathscr{D}$ it is single valued within $\mathscr{D}$ and complex differentiable at all points of $\mathscr{D}$. Note that (4.1) may be equivalently expressed as the so called complex form of the Cauchy-Riemann equations which reads

$$\frac{d\phi(z)}{d\overline{z}} = 0. \tag{4.2}$$

4.3 The reflection principle

An important aspect of holomorphic functions is that one may use the definition of a holomorphic function, say $\phi(z)$ defined in some circular domain $\mathscr{D}$, to find an associated complex function which is holomorphic in the exterior of $\mathscr{D}$. To show this, let us define a circular simply connected domain by D_1 and denote the exterior of the closure of D_1 by D_2. Now, for $z \in D_2$, $R^2/\overline{z}$ lies in D_1 and vice versa. Let the function $\phi(z)$ be holomorphic in D_1 then $\phi(R^2/\overline{z})$ is well defined for all $z \in D_2$. Taking the complex conjugate, we may then show that

$$\frac{d}{dz}\overline{\phi(R^2/\overline{z})} = \frac{d\overline{\phi(R^2/\overline{z})}}{d\overline{R^2/\overline{z}}}\frac{d\overline{R^2/\overline{z}}}{dz} = \overline{\phi'(R^2/\overline{z})}, \tag{4.3}$$

hence, using the complex form of the Cauchy Riemann equations it is seen that

$$\frac{\partial}{\partial\overline{z}}\overline{\phi(R^2/\overline{z})} = \frac{d\overline{\phi(R^2/\overline{z})}}{d\overline{R^2/\overline{z}}}\frac{d\overline{R^2/\overline{z}}}{d\overline{z}} = \overline{\phi'(R^2/\overline{z})}\left(\frac{dz}{d\overline{z}}\right) = \overline{\phi'(R^2/\overline{z})}\left(\frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)(x_1 + ix_2)\right) = 0. \tag{4.4}$$

Thus from (4.4) $\overline{\phi}(R^2/z)$ is holomorphic for $z \in D_2$.

4.4 Analytic continuation

The technique of analytic continuation is extremely powerful in the study of problems in complex analysis. Let us suppose that two analytic functions $\phi_1(z)$ and $\phi_2(z)$ are defined in two domains D_1 and D_2, respectively. Assuming that D_1 and D_2 intersect along some curve ∂D_1 in which

$$\phi_1(z) = \phi_2(z), \quad z \in D_1 \cap D_2 = \partial D_1, \tag{4.5}$$

then we may define a continuation function

$$\theta(z) = \begin{cases} \phi_1(z) \ \ (z \in D_1), \\ \phi_2(z) \ \ (z \in D_2), \end{cases} \tag{4.6}$$

which is holomorphic in $D_1 \cup D_2 \cup \partial D_1$. From (4.6) $\phi_1(z)$ is the analytic continuation of $\phi_2(z)$ into D_1 and $\phi_2(z)$ is the analytic continuation of $\phi_1(z)$ into D_2. In practice analytic continuation also involves the process of subtracting a singularity function comprised of the singular behavior of $\phi_1(z)|_{z\to 0}$ and the singular and asymptotic behavior $\phi_2(z)|_{z\to\infty}$, from both parts of $\theta(z)$ given in (4.6). Defining this function by $L(z)$ we may then rewrite (4.6) as

$$\theta(z) = \begin{cases} \phi_1(z) - L(z) \ \ (z \in D_1), \\ \phi_2(z) - L(z) \ \ (z \in D_2). \end{cases} \tag{4.7}$$

Defined in this way, the representation of $\theta(z)$ in (4.7) is ideal for the application of Liouville's theorem.

4.5 Liouville's theorem

Before we state Liouville's theorem we must define the concept of an entire function. Briefly, an entire function is a function that is analytic at every point in the entire finite plane, including the point at infinity. With this understanding of an entire function, Liouville's theorem states

Theorem 1. If a function f is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.

This is of paramount importance to the analysis in the following chapters because we shall frequently construct functions that are analytic in the entire plane using the analytic continuation theorem and, through subtraction of some asymptotic and singular terms given by $L(z)$ as described in the previous section, will leverage Liouville's theorem to develop two differential equations for the analytic functions inside and outside of a circular domain.

4.6 Taylor series

Another important concept in complex analysis is Taylor's theorem. It is stated, without proof, as follows

Theorem 2. Suppose that a function f is analytic throughout a disk $|z-z_0| < R_0$, then $f(z)$ has the power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad |z-z_0| < R_0, \tag{4.8}$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!} \quad (n = 0, 1, 2, ...). \tag{4.9}$$

Taylor's theorem is relevant to the work in this thesis because we use it as a consistency condition to derive the expressions for the coefficients of the power series expansions of the potential functions that make up the Piola stress and deformation map.

4.7 Multivalued functions

The idea of a multivalued function, specifically the logarithmic function is important in complex analysis and especially relevant here as it will arise in the work of this thesis. To begin, we define the general form of the logarithm as follows

$$log(z) = ln(r) + i\theta, \;\; r > 0, \gamma < \theta < \gamma + 2\pi, \tag{4.10}$$

where γ is an arbitrary real number and we have used the polar form of the complex coordinate $z = re^{i\theta}$. Equation (4.10) is generally referred to as the multivalued form of the logarithmic function. However, for each value of γ equation (4.10) represents only a single branch of the multiple valued function (graphically this corresponds to one region on the Riemann surface for the general form of the logarithmic function). One particular value of γ, namely $\gamma = -\pi$, defines the so-called principal branch

$$Log(z) = ln(r) + i\Theta, \;\; r > 0, -\pi < \Theta < \pi. \tag{4.11}$$

The ray $\Theta = \pi$ in (4.11) defines a branch cut; a line or curve that portions the domain of $log(z)$ into a single region. Note that all points on the branch cut are singular points of $Log(z)$ and the point from which the branch cut emanates is referred to as a branch point. Branch cuts are generally taken along the real axis and there may be multiple branch cuts for a single branch of a multivalued function.

Chapter 5

HARMONIC MATERIALS

5.1 Introduction

The inclusion problems of this thesis are based on the use of the constitutive model for Harmonic materials. This model is highly relevant to the use of complex analysis to solve boundary value problems in finite elasticity because the equations of equilibrium for the first Piola stress are analytic. This fact allows us to leverage many of the convenient properties of analytic functions to formulate concise expressions for the displacement and first Piola stress in terms of complex potential functions.

5.2 Continuum mechanics for harmonic materials using complex variables

If we let $z = x_1 + ix_2$ be the initial coordinates of a particle in the reference configuration of a domain $\mathscr{D}_0$, and let $w(z,t) = y_1(z,t) + iy_2(z,t)$ be the coordinates of the same particle in the current configuration $\mathscr{D}$. We may define the deformation gradient tensor for the case of plain strain as

follows (note that for the case of plane strain it is admissible to treat the bases of the reference and current configuration as coincident)

$$F_{ab} = \frac{\partial y_a}{\partial x_b} \ , \ a,b = 1,2. \tag{5.1}$$

From (5.1) the scalar invariants of the deformation gradient **F** and Right Cauchy Green deformation tensor **C** are given by

$$I = \lambda_1 + \lambda_2 = \sqrt{F_{ab}F_{ab} + 2J} \ , \ J = \lambda_1\lambda_2 = detF_{ab} \ , \tag{5.2}$$

where in (5.2) λ_1, λ_2 are the principal stretches. John (1966) states that a material is called Harmonic if "*pseudo-irrotationality of a deformation at a particular moment always applies that the acceleration field associated with the deformation at that moment by the equations of motion in the absence of body forces also is pseudo-irrotational*". Following the work of John (1966) we begin by defining a deformation $w_j(x_1, x_2, t)$ as pseudo-irrotational if

$$\frac{\partial y_j}{\partial x_k} - \frac{\partial y_k}{\partial x_j} = 0, \ \ j,k = 1,2. \tag{5.3}$$

Inspection of (5.3) reveals that pseudo-irrotationality of the deformation is equivalent to symmetry of the deformation gradient. The equations of motion for a perfectly elastic isotropic homogeneous material are derived from the strain energy function

$$\rho \frac{\partial^2 y_j}{\partial t^2} = \frac{\partial}{\partial x_k}\frac{\partial W}{\partial F_j k}, \tag{5.4}$$

where ρ is the constant density in the reference configuration and body forces are neglected. The two dimensional strain energy function introduced by John (1960) may be rewritten in the form

$$W(I,J) = 2\mu[H(I) - J] \ , \tag{5.5}$$

where $H(I)$ is an arbitrary scalar function of the first invariant I and J is the Jacobian. Hence we may rewrite (5.4) as

$$\rho \frac{\partial^2 y_j}{\partial t^2} = \frac{\partial}{\partial x_k} 2\mu \left[H'(I) \frac{\partial I}{\partial F_{jk}} - \frac{\partial J}{\partial F_{jk}} \right]. \tag{5.6}$$

Recalling the definition of the deformation gradient $\boldsymbol{F}$ in (5.1), the following complex representations of the first invariant I and Jacobian J may be defined

$$I = |w_{,2} + i w_{,1}| \ , \ J = -Im[w_{,1}\overline{w_{,2}}] \ . \tag{5.7}$$

Furthermore, since $F_{jk} = y_{j,k}$ we have

$$\rho \frac{\partial^2 y_j}{\partial t^2} = \frac{\partial}{\partial x_k} 2\mu \left[H'(I) \frac{\partial |y_{1,2} - y_{2,1} + i(y_{1,1} + y_{2,2})|}{\partial y_{j,k}} - \frac{\partial (-y_{1,1} - i y_{2,1})(y_{1,2} - i y_{2,2})}{\partial y_{j,k}} \right]. \tag{5.8}$$

If we then assume that at some point in time, say $t = \tau$, the deformation field y_j is pseudo-irrotational then we have that

$$y_{1,2} - y_{2,1} = 0, \ \ y_{2,1} - y_{1,2} = 0. \tag{5.9}$$

Hence (5.8) may be written as

$$\rho\frac{\partial^2 y_j}{\partial\tau^2}=\frac{\partial}{\partial x_k}2\mu\left[H'(I)\frac{\partial(y_{1,1}+y_{2,2})}{\partial y_{j,k}}-\frac{\partial(-y_{1,1}-iy_{2,1})(y_{1,2}-iy_{2,2})}{\partial y_{j,k}}\right]. \tag{5.10}$$

Expanding (5.10) we see that

$$\rho\frac{\partial^2 y_j}{\partial\tau^2}=2\mu\left[H'(I)_{,k}\frac{\partial(y_{1,1}+y_{2,2})}{\partial y_{j,k}}+H'(I)\left(\frac{\partial(y_{1,1}+y_{2,2})}{\partial y_{j,k}}\right)_{,k}-\left(\frac{\partial(-y_{1,1}-iy_{2,1})(y_{1,2}-iy_{2,2})}{\partial y_{j,k}}\right)_{,k}\right]. \tag{5.11}$$

Finally, equation (5.11) may be reduced to

$$\rho\frac{\partial^2 y_j}{\partial\tau^2}=2\mu H'(I)_{,j}, \tag{5.12}$$

which, noting that $H'(I)$ is a scalar function and the curl of the gradient of a scalar function is zero, implies that the acceleration field is also pseudo-irrotational and hence the strain energy function given by (5.5) represents a harmonic material.

Consequently, from the strain energy function the Piola stress may be written in component form as follows

$$P_{ab}=2\frac{\partial W}{\partial I}F_{ab}+\frac{\partial W}{\partial J}\varepsilon_{ar}\varepsilon_{bs}F_{rs}\ , \tag{5.13}$$

from which, inserting the definition for the strain energy function given in (5.5), yields the following

$$P_{ab} = 2\mu \left[\frac{H^{'}(I)}{I} F_{ab} + \left[\frac{H^{'}(I)}{I} - 1 \right] \varepsilon_{ar} \varepsilon_{bs} F_{rs} \right] \quad . \tag{5.14}$$

Following Knowles, J.K., Sternberg (1975), if the following auxilary functions A_1 and A_2 are defined

$$A_1 = \frac{H^{'}(I)}{I}(F_{12} - F_{21}) \ , \ A_2 = \frac{H^{'}(I)}{I}(F_{11} + F_{22}) \ , \tag{5.15}$$

we may express the Piola stress components in the following manner

$$P_{11} = 2\mu(A_2 - F_{22}) \ , \ P_{22} = 2\mu(A_2 - F_{11}) \ , \tag{5.16}$$

$$P_{12} = 2\mu(A_1 + F_{21}) \ , \ P_{21} = 2\mu(-A_1 + F_{12}) \ .$$

We may additionaly set the auxilary functions A_1, A_2 as functions of the coordinates x_1, x_2 as

$$A(x_1, x_2) = A_1(x_1, x_2) + iA_2(x_1, x_2) \ , \tag{5.17}$$

such that

$$A = \frac{H^{'}(I)}{I}(w_{,2} + iw_{,1}) \ . \tag{5.18}$$

Utilizing (5.14), the constitutive relations for the Piola stresses may now be written as

$$P_{12}+iP_{22}=2\mu\left(\frac{H^{'}(I)}{I}(w_{,2}+iw_{,1})-iw_{,1}\right)\ \ ,\ \ P_{11}+iP_{21}=2\mu i\left(w_{,2}-\frac{H^{'}(I)}{I}(w_{,2}+iw_{,1})\right)\ \ . \tag{5.19}$$

Following the complex variable formulation provided by Ru (2002) it is noted that the function

$$\frac{H^{'}(I)}{I}(w_{,2}+iw_{,1})\ \ ,$$

is analytic and may be written in terms of an arbitrary analytic function $\phi(z)$ as

$$\frac{H^{'}(I)}{I}(w_{,2}+iw_{,1})=\phi^{'}(z)\ \ . \tag{5.20}$$

From (5.20) it can be shown that $w(z)$ must satisfy the following linear non-homogeneous equation

$$w_{,2}+iw_{,1}=\phi^{'}(z)\frac{G(|\phi^{'}(z)|)}{|\phi^{'}(z)|}\ \ , \tag{5.21}$$

where $G(*)$ is the inverse function of $H^{'}(*)$. When the constitutive function $G(*)$ is of the form

$$G(|\phi^{'}(z)|)=-2|\phi^{'}(z)|\left[a+b|\phi^{'}(z)|^{2\delta}+\frac{c}{|\phi^{'}(z)|^{2}}\right]\ \ , \tag{5.22}$$

and we take the specific case of $\delta = 0$, $\alpha = 1 - b$, and $\beta = -c$ (where $1 > \alpha \geq \frac{1}{2}, \beta > 0$) , it can be shown that the deformation map $w(z)$ is then given by

$$w(z) = i\left[\alpha\phi(z) + i\overline{\psi(z)} + \frac{\beta z}{\overline{\phi'(z)}}\right] \quad , \tag{5.23}$$

where $\overline{\psi(z)}$ satisfies the homogeneous part of the differential equation given in (5.21) and α, β are derived from the ratios of the principal stretches to the principal Piola stresses given by

$$\alpha = \frac{\lambda_1 + \lambda_2}{P_1 + P_2}, \;\; \beta = \frac{\lambda_1 - \lambda_2}{P_1 - P_2}, \tag{5.24}$$

where in Varley and Cumberbatch (1980) it was determined that $\alpha = \beta = 0.5$ provides the best agreement with experimental results in uniaxial tension tests. Similarly, the complex Piola stress function is defined by

$$\chi(z) = \chi_1(z) + i\chi_2(z) \quad ,$$

$$\chi(z) = 2\mu i\left[(\alpha - 1)\phi(z) + i\overline{\psi(z)} + \frac{\beta z}{\overline{\phi'(z)}}\right] \quad , \tag{5.25}$$

where,

$$\begin{pmatrix} P_{11} & P_{21} \\ P_{12} & P_{22} \end{pmatrix} = \begin{pmatrix} \chi_{2,2} & -\chi_{1,2} \\ -\chi_{2,1} & \chi_{1,1} \end{pmatrix} \quad . \tag{5.26}$$

Equations (5.23,5.25) give rise to the following Cartesian expressions for the stress and displacement fields

$$w_k(z,\bar{z}) - z = (u_1 + iu_2)_k\,,$$
$$\chi_k(z,\bar{z})_{,1} = (P_{22} - iP_{12})_k\,,\ \ \chi_k(z,\bar{z})_{,2} = (-P_{21} + iP_{11})_k\,,\ \ k = 1,2. \tag{5.27}$$

Equation (5.27) may be transformed into polar coordinates if we allow for the assumption that either the shear components of the Cauchy stress tensor are zero or the principal stretches of the deformation gradient λ_1, λ_2 are equal, either of which implies that the Piola stress is symmetric. The primary motivation of either assumption is that symmetry of the Piola stress greatly simplifies expressions for the tractions in a polar coordinate setting. The polar form of (5.27) is then given by

$$\frac{R}{z} w_k(z,\bar{z}) - R = (u_r + iu_\theta)_k,\ \ \chi_k^{'}(z,\bar{z}) = (P_{rr} + iP_{\theta r})_k\ \ ,k = 1,2. \tag{5.28}$$

where a prime $(')$ denotes differentiation with respect to z.

5.3 Restrictions on the strain energy function

Naturally there are several restrictions on the function $W(I,J)$ given in (5.5). First we note that in the undeformed state $I = 2, J = 1$ from which the requirement that the strain energy and state of stress vanish in this configuration necessitates that

$$H(2) = 1\ \ ,\ \ H^{'}(2) = 1\ \ , \tag{5.29}$$

and

$$H''(2) = \frac{\lambda + 2\mu}{2\mu} \ , \tag{5.30}$$

is required to recover the general isotropic theory in the case of small deformations. We also require that

$$H(I) > I^2/4, \ \ \forall I > 0, I \neq 2, \tag{5.31}$$

in order to ensure that the strain energy is positive. Finally, in order to admit a normal state of uni-axial tension in the case of plane strain, Knowles, J.K., Sternberg (1975) proposed that

Theorem 3. There exists an $I_0 \in (1,2)$ such that $F'(I_0) = 0$ and

$$H''(I) > 1 \ \ (I_0 < I < \infty), \ \ \frac{H'(I)}{I} \to 1 \ \ as \ \ I \to \infty. \tag{5.32}$$

The proof of (5.32) is best stated in the work of Knowles, J.K., Sternberg (1975) and hence we shall defer an explicit proof here.

Chapter 6

A CIRCULAR INCLUSION WITH INHOMOGENEOUS IMPERFECT INTERFACE: PRELIMINARY CALCULATIONS

6.1 Introduction

The following 3 chapters discuss three separate boundary conditions for the stress-displacement jump between a circular inclusion and the surrounding matrix. The preliminary calculations and first use of the technique of analytic continuation on the continuity of tractions boundary condition, however, is identical for all three of the different stress-displacement interface conditions. Hence this chapter serves as the preliminary work that flows into each of the different stress displacement boundary conditions.

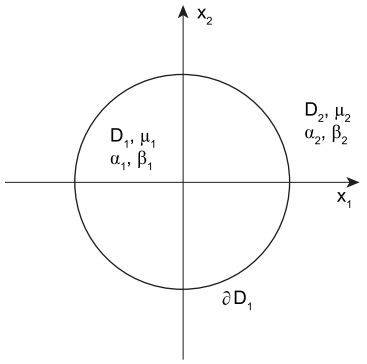

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Figure 6.1: Elastic circular inclusion(D_1) bounded by curve ∂D_1 embedded in a infinite matrix (D_2)

6.2 Formulation

Consider a single simply connected domain bounded by a continuous circular curve ∂D_1, embedded in an infinite matrix in $\mathbb{R}^2$. Let us assume that any deformation relative to the reference configuration is confined to the x_1x_2 plane (plane strain). Let $z = x_1 + ix_2$ be the Lagrangian coordinates of a particle in the reference configuration and let $w(z) = y_1(z) + iy_2(z)$ be the Eulerian coordinates of a particle in the current configuration. The inclusion is denoted by $D_1$and endowed with material properties μ_1, α_1, β_1. The matrix is denoted by domain D_2 with material properties μ_2, α_2, β_2 where $\frac{1}{2} \leq \alpha_k < 1, \beta_k > 0, k = 1,2$. The matrix and inclusion are assumed to be type 1 harmonic materials. Varley and Cumberbatch (1980) indicate that for type 1 harmonic materials the difference in the principal Piola stresses is proportional to the difference in the stretch ratios. This linear relationship not only applies for a Hookean material but it also applies outside of the Hookean regime as well. In view of this and the philosophy of the late Professor Bernard Budiansky, let us adopt the notation of Sudak et al. (1999) where the inhomogeneous imperfect interfacial boundary conditions under the linear spring model are given by

$$||P_{rr} + iP_{r\theta}|| = 0 \ , \ z \in \partial D_1 \tag{6.1}$$

$$P_{rr} = m(\theta)||u_r|| \ , \ P_{\theta r} = n(\theta)||u_\theta|| \ , \ z \in \partial D_1,$$

where $m(\theta)$ and $n(\theta)$ are two non-negative imperfect interface parameters and $||\ .\ || = (\ .\ )_2 - (\ .\ )_1$ represents the jump across ∂D_1. The concept of the linear spring model for an imperfect interface is to simulate the thin inter-phase layer separating inclusion from surrounding matrix as a 2-dimensional curve of vanishing thickness where the material properties of the inter-phase layer are represented by spring-type imperfect interface parameters. This type of boundary condition was described in detail through the work by Bigoni, D. , Serkov, S.K. , Valentini, M., Movchan (1998) however no analogous derivation currently exists for finite elasticity. Furthermore one cannot prescribe the parameters $m(\theta)$ and $n(\theta)$ a priori because it is not possible to determine the interphase properties (and hence interface properties) of an inhomogeneous imperfect interface experimentally, owing to the point wise variation of these properties throughout the interphase region.

It is assumed that the potential functions $\phi_2(z)$ and $\psi_2(z)$ exhibit the following behavior as $|z| \to \infty$

$$\phi_2(z) = Az + 0(1)\ \ ,\ \ \psi_2(z) = Bz + 0(1)\ \ ,\ \ |z| \to \infty, \tag{6.2}$$

where A and B are, in general, complex constants that reflect the far field loading and the $0(1)$ are some first order terms. Evaluating (5.28) as $|z| \to \infty$ yields

$$\frac{iP_{22}^{\infty} + P_{21}^{\infty}}{2\mu_2} = (1-\alpha_2)A - i\overline{B} - \frac{\beta_2}{\overline{A}} \tag{6.3}$$

$$\frac{-P_{21}^{\infty} + iP_{11}^{\infty}}{2\mu_2} = (1-\alpha_2)A + i\overline{B} - \frac{\beta_2}{\overline{A}} \tag{6.4}$$

Taking the real part of the sum of (6.3) and (6.4) yields the following relation

$$0 = Re[A]\left[(1-\alpha_2) - \frac{\beta_2}{|A|^2}\right] \quad , \quad |A| > 0, \tag{6.5}$$

From (6.5) it is clear that A is purely imaginary and takes the positive root of the folowing

$$A = i\left[\frac{\frac{P_{22}^{\infty}+P_{11}^{\infty}}{4\mu_2} \pm \sqrt{\left(\frac{P_{22}^{\infty}+P_{11}^{\infty}}{4\mu_2}\right)^2 + 4(1-\alpha_2)\beta_2}}{2(1-\alpha_2)}\right] \quad , \tag{6.6}$$

and B is given simply by

$$B = \frac{P_{11}^{\infty} - P_{22}^{\infty} - 2iP_{12}^{\infty}}{4\mu_2} \quad . \tag{6.7}$$

Furthermore, it is assumed that the potentials $\phi_i(z)$ and $\psi_i(z)$ for $i = 1,2$ admit the following series expansions

$$\phi_1(z) = X_0 + \sum_{k=1}^{\infty} X_k z^k \quad , \quad \psi_1(z) = Y_0 + \sum_{k=1}^{\infty} Y_k z^k \quad , \quad z \in D_1, \tag{6.8}$$

$$\phi_2(z) = Az + \sum_{k=0}^{\infty} A_k z^{-k} \quad , \quad \psi_2(z) = Bz + \sum_{k=0}^{\infty} B_k z^{-k} \quad , \quad z \in D_2.$$

To eliminate the possibility of rigid body displacement between inclusion and surrounding matrix, the coefficients $X_0$and Y_0 must be properly chosen. Since the inclusion is geometrically symmetric

about two mutually perpendicular directions it is then reasonable to assume in (6.8) that both X_0 and Y_0 are zero.

Remark 1. From (6.8) we require that $|X_1| > 0$ for $|z| \leq R$ and $|A| > 0$ for $|z| \geq R$. These assumptions guarantee that $H^{'}(I) = |\phi^{'}(z)| \neq 0$ throughout the plane.

Thus, the continuity of tractions condition from (6.1) gives the following

$$\mu_1 \left[(\alpha_1 - 1)\phi_1(z) + i\overline{\psi_1}(R^2/z) + \frac{\beta_1 z}{\overline{\phi_1^{'}}(R^2/z)} \right] = \mu_2 \left[(\alpha_2 - 1)\phi_2(z) + i\overline{\psi_2}(R^2/z) + \frac{\beta_2 z}{\overline{\phi_2^{'}}(R^2/z)} \right] \quad , \quad |z| = R. \tag{6.9}$$

Substituting $\Gamma = \frac{\mu_1}{\mu_2}$ into the above and rearranging the above in terms of regions of analyticity yields

$$\Gamma(\alpha_1 - 1)\phi_1(z) - i\overline{\psi_2}(R^2/z) - \frac{\beta_2 z}{\overline{\phi_2^{'}}(R^2/z)} = (\alpha_2 - 1)\phi_2(z) - \Gamma i\overline{\psi_1}(R^2/z) - \frac{\Gamma\beta_1 z}{\overline{\phi_1^{'}}(R^2/z)} \quad , \quad |z| = R. \tag{6.10}$$

The LHS of (6.10) is analytic for $z \in D_1$ and the RHS of (6.10) is analytic for $z \in D_2$. Using analytic continuation we now construct a function that is entire by first allowing $|z| \to 0$ in the LHS of (6.10) and capture the following singular term

$$-\frac{i\overline{B}R^2}{z} \quad , \quad z \in D_1. \tag{6.11}$$

Moving to the RHS of (6.10), we allow $|z| \to \infty$ and isolate the following singular terms

$$(\alpha_2 - 1)Az - \frac{\Gamma\beta_1 z}{\overline{X_1}} \quad , \quad z \in D_2. \tag{6.12}$$

Adding (6.11) and (6.12) yields the following function

$$S_1(z) = (\alpha_2 - 1)Az - \frac{\Gamma\beta_1 z}{\overline{X_1}} - \frac{i\overline{B}R^2}{z} \tag{6.13}$$

We now consider the function defined by subtracting (6.13) from both sides of (6.10)

$$\theta_1(z) = \begin{cases} \Gamma(\alpha_1 - 1)\phi_1(z) - i\overline{\psi_2}(R^2/z) - \frac{\beta_2 z}{\overline{\phi_2'}(R^2/z)} - S_1(z) & \text{for } z \in D_1 \\ (\alpha_2 - 1)\phi_2(z) - \Gamma i\overline{\psi_1}(R^2/z) - \frac{\Gamma\beta_1 z}{\overline{\phi_1'}(R^2/z)} - S_1(z) & \text{for } z \in D_2 \end{cases} . \tag{6.14}$$

Since (6.14) is entire, and $\theta_1(z) \to 0$ as $|z| \to \infty$, from Louisville's theorem we know that $\theta_1(z)$ is constant, in this case $\theta_1(z) \equiv 0$. This allows us to write the following relations

$$i\overline{\psi_2}(R^2/z) + \frac{\beta_2 z}{\overline{\phi_2'}(R^2/z)} = \Gamma(\alpha_1 - 1)\phi_1(z) + \left[(1 - \alpha_2)A + \frac{\Gamma\beta_1}{\overline{X_1}}\right] z + \frac{i\overline{B}R^2}{z} \quad , \quad z \in D_1 \tag{6.15}$$

$$i\overline{\psi_1}(R^2/z) + \frac{\beta_1 z}{\overline{\phi_1'}(R^2/z)} = \frac{\alpha_2 - 1}{\Gamma}\phi_2(z) + \left[\frac{1 - \alpha_2}{\Gamma}A + \frac{\beta_1}{\overline{X_1}}\right] z + \frac{i\overline{B}R^2}{\Gamma z} \quad , \quad z \in D_2.$$

The two equations in (6.15) will later be used to reduce the number of unknown analytic functions from four to two (effectively eliminating $\psi_k, k = 1, 2$ from the problem).

Chapter 7

THE CASE OF ($m(\theta) = n(\theta)$)

7.1 Introduction

In the present case we assume that the interfacial bonding is characterized by the condition $m(\theta) = n(\theta)$ (m and n are the normal and tangential imperfect interface parameters, respectively, and θ is the polar angle). This corresponds physically to the case where the directionality of the displacement jump across the interface is equal to that of the corresponding traction vector and the same degree of interface imperfection is seen in both the normal and tangential coordinate directions. As an example, if we consider any two adjacent points on opposite sides of the interface, the jump in the displacement vector between these two points post deformation is parallel to the internal force generated in the interface "spring".

7.2 Formulation

The imperfect interfacial boundary conditions in (6.1) may be combined into one more convenient expression

$$(P_{rr} + iP_{\theta r}) = \frac{m+n}{2}||u_r + iu_\theta|| + \frac{m-n}{2}||u_r - u_\theta|| \ \ , \ \ z \in \partial L. \tag{7.1}$$

Inserting (5.28) into (7.1) and multiplying both sides by i yields the following

$$i\chi_1^{'}(z)=\frac{m+n}{2}\left(\frac{R}{z}iw_2(z)-\frac{R}{z}iw_1(z)\right)+\frac{m-n}{2}\left(\frac{z}{R}i\overline{w_2}(R^2/z)-\frac{z}{R}i\overline{w_1}(R^2/z)\right)\ ,\ z\in\partial L. \quad (7.2)$$

7.2.1 Circumferentially Inhomogeneous Imperfect Interface

If we limit ourselves to the case of $m(\theta)=n(\theta)$ as discussed in the introduction, and insert (5.23,5.25) in conjunction with (6.15) into (7.2) we have the following

$$\begin{aligned}-2\mu_1\left[(\alpha_1-1)\phi_1^{'}(z)+\frac{(\alpha_2-1)}{\Gamma}\phi_2^{'}(z)-\frac{(\alpha_2-1)}{\Gamma}A+\frac{\beta_1}{\overline{X_1}}-\frac{i\overline{B}R^2}{\Gamma z^2}\right]=\\ \frac{mR}{z}\left[\alpha_2\phi_2(z)+\Gamma(\alpha_1-1)\phi_1(z)-(\alpha_2-1)Az+\frac{\Gamma\beta_1 z}{\overline{X_1}}+\frac{i\overline{B}R^2}{z}-\alpha_1\phi_1(z)-\frac{(\alpha_2-1)}{\Gamma}\phi_2(z)\right.\\ \left.+\frac{(\alpha_2-1)}{\Gamma}Az-\frac{\beta_1 z}{\overline{X_1}}-\frac{i\overline{B}R^2}{\Gamma z}\right]\ ,\ |z|=R. \quad (7.3)\end{aligned}$$

We now define a new parameter to characterize the imperfect interface

$$\delta(\theta)\equiv\frac{m(\theta)R}{2\mu_1}\ ,\ \delta(\theta)>0. \quad (7.4)$$

Since $1/\delta(\theta)$ is a non-negative periodic function on ∂L, we may redefine it as follows

$$1+f(\theta)=\frac{\delta_0}{\delta(\theta)}\ ,\ \delta_0>0\ ,\ f(\theta)>-1, \quad (7.5)$$

where δ_0 is real and $f(\theta)$ is periodic on ∂L. As $f(\theta) \to -1$, $m(\theta) \to \infty$, which is the case of a perfectly bonded interface. Given that $f(\theta)$ is periodic on ∂L, we may assume a form of $f(\theta)$ as follows

$$f(\theta) = \sum_{k=1}^{s} \left[a_k \sin(k\theta) + b_k \cos(k\theta) \right] \quad , \quad a_s^2 + b_s^2 \neq 0 \ \ unless \ \ f(\theta) \equiv 0 \ \ on \ \ \partial L. \tag{7.6}$$

In (7.6) s is a natural number and a_k, b_k are real. $f(\theta)$ may be rewritten in terms of the complex-variable z as follows

$$f(z) = \frac{1}{2} \sum_{k=1}^{s} (b_k + i a_k) \frac{R^k}{z^k} + (b_k - i a_k) \frac{z^k}{R^k} \quad , \quad z \in \partial L \quad , \quad f(z) \equiv f(\theta) \tag{7.7}$$

7.2.2 The Differential Equation for $\phi_1(z)$ and $\phi_2(z)$

To form the differential equations for ϕ_1 and ϕ_2 we again employ the technique of analytic continuation. From (7.4) and (7.5), (7.3) may be rewritten as

$$\begin{aligned} (1-\alpha_1)(1+f(z))\phi_1'(z) - \delta_0(\Gamma(\alpha_1 - 1) - \alpha_1)\frac{\phi_1(z)}{z} - \frac{(\Gamma - 1)\delta_0\beta_1}{\overline{X_1}} - (1+f(z))\frac{\beta_1}{\overline{X_1}} = \\ -\frac{(1-\alpha_2)}{\Gamma}(1+f(z))\phi_2'(z) + \delta_0\left(\alpha_2 + \frac{(1-\alpha_2)}{\Gamma}\right)\frac{\phi_2(z)}{z} + (1+f(z))\frac{(1-\alpha_2)}{\Gamma}A \\ -(1+f(z))\frac{i\overline{B}R^2}{\Gamma z^2} + \frac{\delta_0(\alpha_2 - 1)(1-\Gamma)}{\Gamma}A + \frac{\delta_0(\Gamma - 1)i\overline{B}R^2}{\Gamma z^2} \quad , \quad |z| = R. \end{aligned} \tag{7.8}$$

The LHS of (7.8) is analytic in the inclusion and the RHS is analytic in the matrix (D_1 and D_2 respectively). We first allow $|z| \to 0$ and recover the following singular terms from the LHS of (7.8)

$$(1-\alpha_1)\frac{1}{2}\sum_{k=1}^{s}(b_k+ia_k)\frac{R^k}{z^k}\sum_{r=1}^{k}rX_rz^{r-1}-\frac{\beta_1}{\overline{X_1}}\frac{1}{2}\sum_{k=1}^{s}(b_k+ia_k)\frac{R^k}{z^k} \quad , \quad z\in D_1. \tag{7.9}$$

Similarly, from the RHS of (7.8) we isolate the following singular terms as $|z| \to \infty$

$$\frac{(1-\alpha_2)}{\Gamma}\left[\frac{1}{2}\sum_{k=2}^{s}(b_k-ia_k)\frac{z^k}{R^k}\sum_{r=1}^{k-1}rA_rz^{-r-1}\right]+\delta_0A-\frac{1}{2}\frac{i\overline{B}}{\Gamma}\sum_{k=2}^{s}(b_k-ia_k)\frac{z^{k-2}}{R^{k-2}} \quad , \quad z\in D_2. \tag{7.10}$$

Adding (7.9) and (7.10) together yields the following function

$$\begin{aligned}S_2(z)=&(1-\alpha_1)\frac{1}{2}\sum_{k=1}^{s}(b_k+ia_k)\frac{R^k}{z^k}\sum_{r=1}^{k}rX_rz^{r-1}-\frac{\beta_1}{\overline{X_1}}\frac{1}{2}\sum_{k=1}^{s}(b_k+ia_k)\frac{R^k}{z^k}+\\&\frac{(1-\alpha_2)}{\Gamma}\left[\frac{1}{2}\sum_{k=2}^{s}(b_k-ia_k)\frac{z^k}{R^k}\sum_{r=1}^{k-1}rA_rz^{-r-1}\right]+\delta_0A-\frac{1}{2}\frac{i\overline{B}}{\Gamma}\sum_{k=2}^{s}(b_k-ia_k)\frac{z^{k-2}}{R^{k-2}}.\end{aligned} \tag{7.11}$$

Subtracting (7.11) from both sides of (7.8) gives a function that is entire

$$\theta_2(z)=\begin{cases}(1-\alpha_1)(1+f(z))\phi_1'(z)-\delta_0(\Gamma(\alpha_1-1)-\alpha_1)\frac{\phi_1(z)}{z}-\frac{(\Gamma-1)\delta_0\beta_1}{\overline{X_1}}-\\(1+f(z))\frac{\beta_1}{\overline{X_1}}-S_2(z) \qquad \text{for } z\ \varepsilon\ D_1\\ \\ \\ -\frac{(1-\alpha_2)}{\Gamma}(1+f(z))\phi_2'(z)+\delta_0\left(\alpha_2+\frac{(1-\alpha_2)}{\Gamma}\right)\frac{\phi_2(z)}{z}+(1+f(z))\frac{(1-\alpha_2)}{\Gamma}A-\\(1+f(z))\frac{i\overline{B}R^2}{\Gamma z^2}+\frac{\delta_0(\alpha_2-1)(1-\Gamma)}{\Gamma}A+\frac{\delta_0(\Gamma-1)i\overline{B}R^2}{\Gamma z^2}-S_2(z) \qquad \text{for } z\ \varepsilon\ D_2\end{cases}. \tag{7.12}$$

Since (7.12) is entire and $\theta_2(z) \to 0$ as $|z| \to \infty$, from Louisville's theorem $\theta_2(z) \equiv 0$. This allows us to develop two coupled linear ordinary differential equations with variable coefficients for $\phi_1(z)$ and $\phi_2(z)$ as follows

$$\phi_1^{'}(z) + \frac{\delta_0(\alpha_1 + \Gamma(1-\alpha_1))}{1-\alpha_1} \frac{\phi_1(z)}{z(1+f(z))} = \frac{1}{1+f(z)} \left[\frac{S_2(z)}{1-\alpha_1} + \frac{(1+f(z))\beta_1}{\overline{X_1}(1-\alpha_1)} - \frac{(1-\Gamma)\delta_0\beta_1}{\overline{X_1}(1-\alpha_1)} \right] \ , \ z \in D_1. \quad (7.13)$$

$$\phi_2^{'}(z) - \frac{\delta_0(1+\alpha_2(\Gamma-1))}{1-\alpha_2} \frac{\phi_2(z)}{z(1+f(z))} = \frac{1}{1+f(z)} \left[(1+f(z))A - \frac{(1+f(z))i\overline{B}R^2}{(1-\alpha_2)z^2} - \delta_0(1-\Gamma)A + \frac{\delta_0(\Gamma-1)i\overline{B}R^2}{(1-\alpha_2)z^2} - \frac{S_2(z)\Gamma}{1-\alpha_2} \right] \ , \ z \in D_2. \quad (7.14)$$

Equations (7.13) and (7.14) may be simplified by making the following substitutions

$$\gamma = \frac{\alpha_1 + \Gamma(1-\alpha_1)}{1-\alpha_1} \ , \ \gamma > 0 \ \ for \ \ all \ \ \Gamma, \alpha_1,$$

$$\eta = \frac{1+\alpha_2(\Gamma-1)}{1-\alpha_2} \ , \eta > 0 \ \ for \ \ all \ \ \Gamma, \alpha_2, \quad (7.15)$$

$$G(z) = \frac{S_2(z)}{1-\alpha_1} + (1+f(z)) \frac{\beta_1}{\overline{X_1}(1-\alpha_1)} - \frac{(1-\Gamma)\delta_0\beta_1}{\overline{X_1}(1-\alpha_1)} \ , \ z \in D_1,$$

$$F(z) = (1+f(z))A - \frac{(1+f(z))i\overline{B}R^2}{(1-\alpha_2)z^2} - \delta_0(1-\Gamma)A + \frac{\delta_0(\Gamma-1)i\overline{B}R^2}{(1-\alpha_2)z^2} - \frac{S_2(z)\Gamma}{1-\alpha_2} \ , \ z \in D_2.$$

From (7.15), (7.13) and (7.14) become

$$\phi_1^{'}(z) + \frac{\delta_0\gamma\phi_1(z)}{z(1+f(z))} = \frac{G(z)}{1+f(z)} \ , \ z \in D_1, \tag{7.16}$$

$$\phi_2^{'}(z) - \frac{\delta_0\eta\phi_2(z)}{z(1+f(z))} = \frac{F(z)}{1+f(z)} \ , \ z \in D_2. \tag{7.17}$$

Equations (7.16) and (7.17) are two first order linear differential equations in $\phi_1(z)$ and $\phi_2(z)$ with variable coefficients, the general solution of which is given by

$$\phi_1(z) = C(z)e^{-p(z)} \ , \ p(z) = \gamma\delta_0 \int \frac{1}{z(1+f(z))}dz \ ,$$

$$C(z) = \int_{z_1}^{z} \frac{G(z)e^{p(z)}}{1+f(z)}dz + C_0 \ , \ z \in D_1, \tag{7.18}$$

$$\phi_2(z) = H(z)e^{-q(z)} \ , \ q(z) = -\eta\delta_0 \int \frac{1}{z(1+f(z))}dz \ ,$$

$$H(z) = \int_{z_2}^{z} \frac{F(z)e^{q(z)}}{1+f(z)} dz + H_0 \ \ , \ \ z \in D_2. \tag{7.19}$$

In (7.18) and (7.19) the points z_1 and z_2 lie in D_1 and D_2, respectively and C_0, H_0 are some arbitrary constants of integration. Let us now show that any admissible solution for $\phi_1(z)$ must satisfy the following consistency condition

$$X_k = \frac{\phi_1^k(0)}{k!} \ \ , \ \ k = 1,2,...,s. \tag{7.20}$$

To derive (7.20) we begin by assuming that $\phi_1(z)$ admits a regular Taylor series expansion given by

$$\phi_1(z) = \sum_{k=0}^{\infty} Q_k z^k \ \ , Q_k = \frac{\phi_1^k(0)}{k!}. \tag{7.21}$$

Our aim is to show that $Q_k = X_k$ for $k = 1,2,...,s$. This is done by substituting (7.21) into (7.13) and comparing coefficients of the negative powers of z^k. In doing so we arrive at the following

$$(1-\alpha_1)\frac{1}{2}\sum_{k=1}^{s}(b_k + ia_k)\frac{R^k}{z^k}\sum_{r=1}^{k} rQ_r z^{r-1} = (1-\alpha_1)\frac{1}{2}\sum_{k=1}^{s}(b_k + ia_k)\frac{R^k}{z^k}\sum_{r=1}^{k} rX_r z^{r-1}. \tag{7.22}$$

Hence we see that in order for (7.22) to be true, we must have that $Q_k = X_k, k = 1,2,..,s$. Due to the fact that (7.22) has nothing to say about the coefficient X_0, we stipulate that in addition to (7.22), any analytic solution of (7.18) must also satisfy the condition that

$$X_0 = \phi_1(0) = 0. \tag{7.23}$$

Unfortunately, since $\phi_2(z)$ does not admit a regular Taylor series expansion due to the far field requirments that restrict the series terms in $\phi_2(z)$ to only negative powers of z, we may not develop a consistency condition the likes of (7.20) for $\phi_2(z)$. None the less, following the definition of the potential function $\phi_2(z)$ as an analytic function expressible by series expansion (see equation6.8), we may equate coefficients of positive powers of z including z^0 to obtain $(s-1)$ equations for the undetermined constants A_k.

The presence of singular points of the multi-valued functions appearing in equations (7.18) and (7.19) means that the solutions for ϕ_1 and ϕ_2 are in general defined only in a simply connected domain obtained by cutting D_1 and D_2 along some appropriately chosen lines. This implies that ϕ_1 and ϕ_2 are, in general, not analytic in their respective uncut domains as they could be discontinuous across any branch cut(s) or unbounded at some singular point(s). In order to ensure the analyticity of ϕ_1 and ϕ_2 in their respective domains, the following must be verified

- ϕ_1 and ϕ_2 given by (7.18) and (7.19) must be bounded at all singular points within their respective domains, and
- ϕ_1 and ϕ_2 given by (7.18) and (7.19) must be continuous across all branch cuts within their respective domains.

To recap, the solutions for ϕ_1 and ϕ_2 are given by (7.18) and (7.19), respectively, where the s undetermined constants $X_k(k = 1,...,s)$ and $s-1$ undetermined constants $A_k(k = 1,...,s-1)$ are derived via the analyticity requirements of ϕ_1 and ϕ_2 as well as the additional constraint from

(7.23). Note that in general, the solutions for ϕ_1 and ϕ_2 are coupled. This coupling can easily be seen for the case of $s = 2$ where the coefficients of ϕ_1 and ϕ_2 must be obtained from a system of equations, as opposed to being solved independently.

7.3 A Class of Inhomogeneous Interface

To illustrate a particular example, we consider the following class of inhomogeneous imperfect interface (Ru, 1998),(Sudak et al., 1999)

$$f(z) = Re[f(\theta)] = b_s cos(s\theta) \equiv \frac{b_s}{2}\left(\frac{z^s}{R^s} + \frac{R^s}{z^s}\right) \quad , \quad -1 < b_s < 1. \tag{7.24}$$

To illuminate the possibility of singularities of the coefficients of ϕ_1 and ϕ_2 in (7.16) and (7.17), we consider the roots of the polynomial formed from $[1 + f(z)]$. In light of the specific interface conditions of (7.24), the following polynomial of order $2s$ is formed

$$\left(\frac{z}{R}\right)^{2s} + \frac{2}{b_s}\left(\frac{z}{R}\right)^{s} + 1 = 0 \ . \tag{7.25}$$

From Schiavone, P., Ru (1997) it has been shown that (7.25) has no roots on ∂D_1, consequently of the $2s$ roots of the polynomial, s are located inside the inclusion and s lie in the matrix. As such, we let the s roots located inside the inclusion be denoted by

$$\rho_1, \rho_2, ..., \rho_s.$$

The s roots in the matrix are then denoted by

$$\frac{1}{\rho_1}, \frac{1}{\rho_2}, ..., \frac{1}{\rho_s}.$$

The s roots may then be determined by

$$\left(\frac{z}{R}\right)^s = \rho^s_{(1,2,..,s)} = \rho^* \quad . \tag{7.26}$$

From (7.26) and (7.25) the following expression for ρ^* is obtained

$$\rho^* = \begin{cases} -\sqrt{\frac{1}{b_s^2} - 1} - \frac{1}{b_s} > 0 \ \text{ if } b_s < 0, \\ \sqrt{\frac{1}{b_s^2} - 1} - \frac{1}{b_s} < 0 \ \text{ if } b_s > 0, \end{cases} \tag{7.27}$$

where ρ^* is real and $-1 < \rho^* < 1$. As a consequence of (7.27) we note that

$$-\frac{2}{b_s} = \frac{1+\rho^{*2}}{\rho^*} \quad . \tag{7.28}$$

From (7.24) and (7.25) we may redefine the variable coefficients of (7.16) and (7.17) as follows

$$\frac{R\delta_0}{z[1+f(z)]} = \frac{-\lambda\left(\frac{z}{R}\right)^{s-1}}{\left[\left(\frac{z}{R}\right)^s - \rho^*\right]} + \frac{\lambda\left(\frac{z}{R}\right)^{s-1}}{\left[\left(\frac{z}{R}\right)^s - \frac{1}{\rho^*}\right]}, \tag{7.29}$$

$$\lambda = -\delta_0\left(\frac{1+\rho^{*2}}{1-\rho^{*2}}\right) < 0,$$

$$\frac{1}{1+f(z)} = \frac{\left(\frac{2}{b_s}\right)\left(\frac{z}{R}\right)^s}{\left[\left(\frac{z}{R}\right)^s - \rho^*\right]\left[\left(\frac{z}{R}\right)^s - \frac{1}{\rho^*}\right]}.$$

Finally, using (7.29), the general solutions of (7.16) and (7.17) take the following form

$$\phi_1(z) = \frac{2}{b_s}\left[\left(\frac{z}{R}\right)^s - \rho^*\right]^{\frac{\gamma\lambda}{s}}\left[\left(\frac{z}{R}\right)^s - \frac{1}{\rho^*}\right]^{\frac{-\gamma\lambda}{s}} \times \int_{R\rho^*}^{z}\left(\frac{t}{R}\right)^s G(t)\left[\left(\frac{t}{R}\right)^s - \rho^*\right]^{\frac{-\gamma\lambda}{s}-1}\left[\left(\frac{t}{R}\right)^s - \frac{1}{\rho^*}\right]^{\frac{\gamma\lambda}{s}-1} dt \ , \ z \in D_1, \quad (7.30)$$

$$\phi_2(z) = \frac{2}{b_s}\left[\left(\frac{z}{R}\right)^s - \rho^*\right]^{\frac{-\eta\lambda}{s}}\left[\left(\frac{z}{R}\right)^s - \frac{1}{\rho^*}\right]^{\frac{\eta\lambda}{s}} \times \int_{\frac{R}{\rho^*}}^{z}\left(\frac{t}{R}\right)^s F(t)\left[\left(\frac{t}{R}\right)^s - \rho^*\right]^{\frac{\eta\lambda}{s}-1}\left[\left(\frac{t}{R}\right)^s - \frac{1}{\rho^*}\right]^{\frac{-\eta\lambda}{s}-1} dt \ , \ z \in D_2. \quad (7.31)$$

Following the work of Sudak et al. (1999) it can be readily shown that the solutions of ϕ_1 and ϕ_2 given by (7.30) and (7.31) are well defined in their respective domains where in each case the domain is cut along horizontal lines from each of the s singular points ($z = R\rho_k, z = R/\rho_k, k = 1,...s$). The aforementioned branch cuts are done in such a way that there is no overlap and since $\eta, \gamma > 0$ and $\lambda < 0$, from Constanda (1990), the integrals in (7.30) and (7.31) converge at each of their branch points. Furthermore, since it is required that ϕ_1 and ϕ_2 remain bounded at the singular points $z = R\rho_k$ and $z = R/\rho_k$ it is clear that $C_0 = H_0 = 0$ and

$$\int_{R\rho_1}^{R\rho_k} G(t)(\frac{t}{R})^s\left[(\frac{t}{R})^s - \rho^*\right]^{-\frac{\gamma\lambda}{s}-1}\left[(\frac{t}{R})^s - \frac{1}{\rho^*}\right]^{\frac{\gamma\lambda}{s}-1} dt = 0, \quad (7.32)$$

$$\int_{R/\rho_1}^{R/\rho_k} F(t)(\frac{t}{R})^s \left[(\frac{t}{R})^s-\rho^*\right]^{\frac{\eta\lambda}{s}-1}\left[(\frac{t}{R})^s-\frac{1}{\rho^*}\right]^{-\frac{\eta\lambda}{s}-1} dt=0,\ \ k=2,3,..,s. \tag{7.33}$$

In the case of $s>1$, equations (7.32) and (7.33) are used to add additional equations to the compatibility and consistency conditions already given.

Lastly, it can be proven that ϕ_1 and ϕ_2 are continuous across each of their branch cuts. This is done by considering the difference $\phi_1(z^+)-\phi_1(z^-)$, where $\phi_1(z^{+/-})$ denotes the value of $\phi_1(z)$ as z approaches the branch cut from above and below, respectively. Through taking the integration path along the edges of said branch cuts and passing through any one of the branch points it can be shown that $\phi_1(z^+)=\phi_1(z^-)$ and hence $\phi_1(z)$ is continuous across each of the s branch cuts. A similar procedure can be applied to $\phi_2(z)$ to prove its continuity across branch cuts in the matrix.

7.4 Example

To illustrate an example we address the case of $s=1$. For this choice of s, the inclusion potential ϕ_1 can be evaluated independently from ϕ_2 as ϕ_1 depends on X_1 only. If we additionally confine ourselves to the case of $\gamma=\eta=1$ the equations for ϕ_1 and ϕ_2 are given as follows

$$\phi_1(z)=\frac{2}{b_s}\left[\left(\frac{z}{R}\right)-\rho^*\right]^{\lambda}\left[\left(\frac{z}{R}\right)-\frac{1}{\rho^*}\right]^{-\lambda} \times\int_{R\rho^*}^{z}\left(\frac{t}{R}\right)G(t)\left[\left(\frac{t}{R}\right)-\rho^*\right]^{-\lambda-1}\left[\left(\frac{t}{R}\right)-\frac{1}{\rho^*}\right]^{\lambda-1}dt\ \ ,\ \ z\in D_1, \tag{7.34}$$

$$G(t)=\frac{1}{1-\alpha_1}\left[(1-\alpha_1)\frac{b_1X_1}{2}\frac{R}{t}-\frac{\beta_1 b_1}{2\overline{X_1}}\frac{R}{t}+\delta_0 A\right]+\frac{\beta_1 b_1}{2\overline{X_1}(1-\alpha_1)}\left[\frac{R}{t}+\frac{t}{R}\right]$$
$$+\frac{\beta_1}{\overline{X_1}(1-\alpha_1)}\left[1-(1-\Gamma)\delta_0\right] \quad ,$$

and

$$\phi_2(z)=\frac{2}{b_s}\left[\left(\frac{z}{R}\right)-\rho^*\right]^{-\lambda}\left[\left(\frac{z}{R}\right)-\frac{1}{\rho^*}\right]^{\lambda}$$
$$\times\int_{\frac{R}{\rho^*}}^{z}\left(\frac{t}{R}\right)F(t)\left[\left(\frac{t}{R}\right)-\rho^*\right]^{\lambda-1}\left[\left(\frac{t}{R}\right)-\frac{1}{\rho^*}\right]^{-\lambda-1}dt \quad , \quad z\in D_2,$$

$$F(t)=\left(A-\frac{-i\overline{B}R^2}{(1-\alpha_2)t^2}\right)\left[1+\frac{b_1}{2}\left(\frac{R}{t}+\frac{t}{R}\right)\right]-\delta_0(1-\Gamma)A+\frac{\delta_0(\Gamma-1)i\overline{B}R^2}{(1-\alpha_2)t^2}$$
$$-\frac{\Gamma}{1-\alpha_2}\left[(1-\alpha_1)\frac{b_1X_1}{2}\frac{R}{t}-\frac{\beta_1 b_1}{2\overline{X_1}}\frac{R}{t}+\delta_0 A\right], \quad (7.35)$$

where in (7.34) and (7.35) λ is given by (7.29). The unknown constant X_1 is found from the consistency condition given by (7.23) as follows

$$0=\int_{R\rho^*}^{0}\left[\frac{\frac{t}{R(1-\alpha_1)}\left[(1-\alpha_1)\frac{b_1X_1}{2}\frac{R}{t}-\frac{\beta_1 b_1}{2\overline{X_1}}\frac{R}{t}+\delta_0 A\right]}{\left[\left(\frac{t}{R}\right)-\rho^*\right]^{-\lambda-1}\left[\left(\frac{t}{R}\right)-\frac{1}{\rho^*}\right]^{\lambda-1}}\right.$$
$$\left.+\frac{\frac{t}{R}\left[\frac{\beta_1 b_1}{2\overline{X_1}(1-\alpha_1)}\left[\frac{R}{t}+\frac{t}{R}\right]+\frac{\beta_1}{\overline{X_1}(1-\alpha_1)}\left[1-(1-\Gamma)\delta_0\right]\right]}{\left[\left(\frac{t}{R}\right)-\rho^*\right]^{-\lambda-1}\left[\left(\frac{t}{R}\right)-\frac{1}{\rho^*}\right]^{\lambda-1}}\right]dt. \quad (7.36)$$

If we take the case of $\lambda=-1$ the expression for X_1 is given by

$$\frac{\delta_0 A}{1-\alpha_1}\left[\frac{R\rho^{*2}}{1-\rho^{*2}}-R\ln\left(\frac{1}{1-\rho^{*2}}\right)\right]=X_1\frac{R\rho^{*4}}{1-\rho^{*4}}+\frac{\beta_1}{\overline{X_1}}\left[\left(\frac{1-(1-\Gamma)\delta_0}{1-\alpha_1}\right)\times\right.$$
$$\left(-\frac{R\rho^{*2}}{1-\rho^{*2}}+R\ln\left(\frac{1}{1-\rho^{*2}}\right)\right)-\frac{1}{1-\alpha_1}\left(\frac{-R\rho^{*2}-R\rho^{*1}(1-\rho^{*2})}{1-\rho^{*4}}+\right.$$
$$\left.\left.\frac{2R}{1+\rho^{*2}}\ln\left(\frac{1}{1-\rho^{*2}}\right)\right)\right]. \quad (7.37)$$

With X_1 determined, $\phi_1(z)$ and $\phi_2(z)$ are evaluated through (7.34) and (7.35)

$$\phi_1(z)=-\frac{\frac{z}{R}-\frac{1}{\rho^*}}{\frac{z}{R}-\frac{1}{\rho^*}}\left[-RX_1\left(\frac{1}{\frac{1}{\rho^*}-\frac{z}{R}}-\frac{\rho^*}{1-\rho^{*2}}\right)+R\frac{1+\rho^{*2}}{\rho^*}\left(\frac{\delta_0 A}{1-\alpha_1}\right.\right.$$
$$\left.+\frac{\beta_1}{\overline{X_1}(1-\alpha_1)}(1-(1-\Gamma)\delta_0)\right)\left(\frac{\frac{1}{\rho^*}}{\frac{1}{\rho^*}-\frac{z}{R}}+\ln\left(\frac{1-\rho^*\frac{z}{R}}{1-\rho^{*2}}\right)-\frac{1}{1-\rho^{*2}}\right)$$
$$\left.-\frac{\beta_1}{\overline{X_1}(1-\alpha_1)}\left(\frac{\frac{R}{\rho^{*2}}}{\frac{1}{\rho^*}-\frac{z}{R}}+\frac{2R}{\rho^*}\ln\left(\frac{1-\rho^*\frac{z}{R}}{1-\rho^{*2}}\right)-\frac{\frac{R}{\rho^*}-R\rho^*(1-\rho^{*2})}{1-\rho^{*2}}+z\right)\right] \quad , \; z\in D_1. \quad (7.38)$$

For the sake of brevity we shall only show ϕ_2 after the integration has been preformed as inserting the limits of integration adds unnecessary clutter for the purposes of this example. $\phi_2(z)$ is given by the following

$$\phi_2(z)=-\frac{\frac{z}{R}-\rho^*}{\frac{z}{R}-\frac{1}{\rho^*}}\left[\frac{1+\rho^{*2}}{\rho^*}\left(A\left(1-\delta_0(1-\Gamma)-\frac{\Gamma\delta_0}{1-\alpha_2}\right)\right)\left[\frac{\rho^*R^2}{\rho^*R-t}+R\ln(t-\rho^*R)\right]_{\frac{R}{\rho^*}}^{z}\right.$$
$$+A\left[\frac{\rho^{*2}R^2}{\rho^*R-t}+2\rho^*R\ln(t-\rho^*R)+t\right]_{\frac{R}{\rho^*}}^{z}+\left(A-\frac{i\overline{B}-\Gamma(1-\alpha_1)X_1}{1-\alpha_2}-\frac{\Gamma\beta_1}{\overline{X_1}(1-\alpha_2)}\right)\left[\frac{R^2}{\rho^*R-t}\right]_{\frac{R}{\rho^*}}^{z}$$
$$\left.-\frac{i\overline{B}(1+\rho^{*2})}{\rho^*(1-\alpha_2)}\left[\frac{R\left(\frac{1}{\rho^*R-t}-\frac{1}{t}-2\ln(t-\rho^*R)+2\ln(t)\right)}{\rho^{*2}}\right]_{\frac{R}{\rho^*}}^{z}\right] \quad ,z\in D_2. \quad (7.39)$$

Equations (7.10) and (7.37-7.39) constitute the complete solution for the case of an homogeneous elastic inclusion embedded in an infinite matrix and surrounded by a circumferentially inhomogeneous imperfect interface characterized by the condition that $m(\theta) = n(\theta)$ and $s = \gamma = \eta = 1, \lambda = -1$. It can be shown that if we allow $\rho^* \to 0$ in (7.37), we recover the homogeneously imperfect interfacial value for X_1 given as (see appendix section 12.3 for proof)

$$2X_1 + \frac{\beta_1}{1-\alpha_1}\left(\frac{\delta_0(1-\Gamma)-1}{\overline{X_1}}\right) = \frac{\delta_0 A}{1-\alpha_1} \ . \tag{7.40}$$

Noting that in the present work the assumptions on s, γ, η, λ imply the following

$$\gamma = \frac{\alpha_1 + \Gamma(1-\alpha_1)}{1-\alpha_1} = 1 \ , \ \ \delta_0 = 1 \ \ for \ \ \lambda = -1 \ , \tag{7.41}$$

we may alternatively express the term $2X_1$ as

$$2X_1 = X_1\left(\frac{\delta_0(\alpha_1 + \Gamma(1-\alpha_1))}{1-\alpha_1} + 1\right). \tag{7.42}$$

From (7.42), (7.40) may be rewritten as

$$X_1\left(\delta_0 + \frac{1-\alpha_1}{\alpha_1 + \Gamma(1-\alpha_1)}\right) + \frac{\beta_1}{\alpha_1 + \Gamma(1-\alpha_1)}\left(\frac{\delta_0(1-\Gamma)-1}{\overline{X_1}}\right) = \frac{\delta_0 A}{\alpha_1 + \Gamma(1-\alpha_1)} \ , \tag{7.43}$$

which is in agreement with the results produced by Wang (2012).

Remark 2. In general since $\phi_1(z)$ is written in a power series we see that for the present solution where X_1 is the only non-zero coefficient, the general Piola stress vector will be non uniform when $X_1 = X_1(\rho^*)$. However, when $\rho^* = 0$ the coefficient X_1 is constant and hence the internal stress

field in the inclusion is uniform for a homogeneous imperfect interface which agrees with the results of Wang (2012).

7.5 Results

We now examine the effects of the circumferentially inhomogeneous interface on the mean stress within the inclusion. To facilitate this, we shall compare the inhomogeneous interface of the form

$$\frac{1}{n(\theta)} = \frac{R(1+b_1\cos(\theta))}{2\mu_1\delta_0}, \;\; \delta_0 = \frac{1-\rho^{*2}}{1+\rho^{*2}} \;\; , -1 < b_1 < 1 \;\; , \tag{7.44}$$

to the homogeneously imperfect interface given by

$$\frac{1}{n} = \frac{R}{2\mu_1\delta_0}, \;\; \delta_0 = \frac{1-\rho^{*2}}{1+\rho^{*2}} \;\; . \tag{7.45}$$

The exact mean stress inside the inclusion is calculated by the following

$$(P_{11}+P_{22})_1 = 4\mu_1 \mathrm{Im}\left[(1-\alpha_1)X_1 + \frac{\beta_1}{\overline{X_1}}\right] \;\; . \tag{7.46}$$

Taking the ratio of the inhomogeneous calculation for (7.46) to the homogeneous calculation of (7.46) corresponding to the case of uni-axial extension and allowing ρ^* to vary between 0 and 1 (for $\alpha_1 = \alpha_2 = \frac{1}{2}, \beta_1 = \beta_2 = \frac{1}{2}, \Gamma = 1$) yields the trends given by Figure (7.1).

Figure (7.1) clearly shows that the circumferential inhomogeneous interface parameter ρ^* has a pronounced effect on the estimation of the mean stress inside the inclusion and at its peak, the discrepancy between the inhomogeneous mean stress and homogeneous counterpart reaches over 60 percent. In comparison, the ratio of the inhomogeneous average mean stress to the homoge-

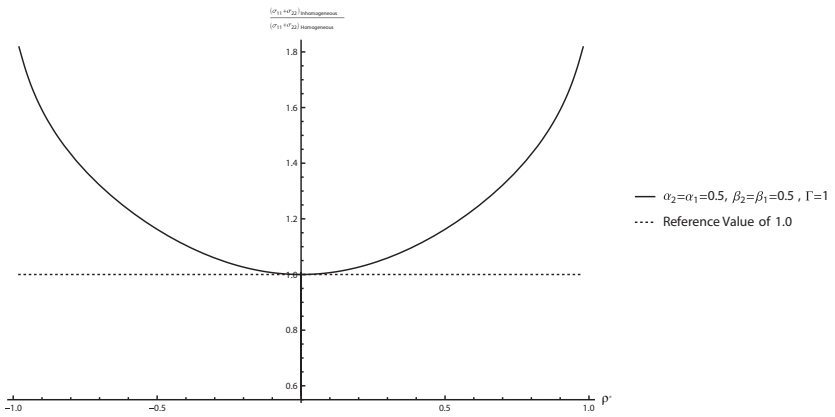

Figure 7.1: Mean stress ratio for a circular inclusion with various values of the inhomogeneous interface parameter ρ^* subject to remote loading $P_{11}^{\infty} = 10^3,\ P_{22}^{\infty} = 0\ , P_{12}^{\infty} = 0$

neous average mean stress in the linear deformation setting reached a discrepancy approaching 200 percent. While not as dramatic as in the linear setting, these results demonstrate conclusively that, in the analysis of circular inclusions with imperfect interface conditions in finite elasticity, the homogeneous interface model may fall short in accurately predicting the stress fields where a homogeneously imperfect interface is not a realistic assumption.

Chapter 8

THE CASE OF ($m(\theta) = finite, n(\theta) \to \infty$)

8.1 Introduction

In contrast to McArthur and Sudak (2016), in the following work we assume that the interfacial bonding is characterized by $m(\theta) = finite$ and $n(\theta) \to \infty$ (where $m(\theta)$ and $n(\theta)$ are the normal and tangential imperfect interface parameters, respectively, and θ is the polar angle). Physically, this type of imperfect interface condition is thought to be a potential byproduct of manufacturing techniques where an abnormal degree of surface roughness along the inter-phase layer can occur. The presence of asperities and/or interdigitations allows for no relative shear displacement along the entire interface but a certain relative normal displacement which is linearly proportional to the corresponding traction vector is permitted across the interface (ie a mechanical interlock is formed).

8.2 Formulation

8.2.1 Solution for Homogeneous Imperfect Interface

In this section we will briefly examine the homogeneous imperfect interface where the parameters m and n appearing in (8.1) are assumed to be constant along ∂D_1. Although a similar problem

has been investigated by Wang (2012), for ease of comparison, we present a solution that is more amenable to validating the present work.

The general form of the imperfect interface condition is given as

$$(P_{rr}+iP_{\theta r})_2 = \frac{m+n}{2}||u_r+iu_\theta|| + \frac{m-n}{2}||u_r-iu_\theta||,\ \ z\in\partial D_1, \tag{8.1}$$

Inserting the definitions of (5.28) into (8.1) yields

$$i\chi_2^{'}(z) = \frac{(m+n)R}{2z}\left(iw_2(z)-iw_1(z)\right) + \frac{(m-n)z}{2R}\left(\overline{iw_1}(R^2/z)-\overline{iw_2}(R^2/z)\right),\ \ z\in\partial D_1. \tag{8.2}$$

Next, substituting in (5.23,5.25,6.15) into (8.2) gives

$$\begin{aligned}(1-\alpha_2)\phi_2^{'}(z)+\Gamma(1-\alpha_1)\phi_1^{'}(z)+(\alpha_2-1)A-\frac{\Gamma\beta_1}{\overline{X_1}}+\frac{i\overline{B}R^2}{z^2} = \frac{(m+n)R}{4\mu_2}\left[\frac{\phi_2(z)}{z}\left(\frac{\alpha_2\Gamma-\alpha_2+1}{\Gamma}\right)\right.\\ \left.+\frac{\phi_1(z)}{z}(\Gamma(\alpha_1-1)-\alpha_1)+A(\frac{\alpha_2-1}{\Gamma}-\alpha_2+1)+\frac{\beta_1(\Gamma-1)}{\overline{X_1}}+\frac{i\overline{B}R^2}{z^2}\frac{\Gamma-1}{\Gamma}\right]+\\ \frac{m-n}{4\mu_2R}\left[z\overline{\phi_1}(R^2/z)(\alpha_1-\Gamma(\alpha_1-1))+z\overline{\phi_2}(R^2/z)(\frac{\alpha_2-1}{\Gamma}-\alpha_2)+\overline{A}R^2(\alpha_2-1-\frac{\alpha_2-1}{\Gamma})\right.\\ \left.+\frac{\beta_1R^2(1-\Gamma)}{X_1}+iBz^2\frac{(\Gamma-1)}{\Gamma}\right],\ \ z\in\partial D_1.\end{aligned} \tag{8.3}$$

Substituting the definitions of (6.8) into (8.3) and performing analytic continuation yields the following expression as a compatibility condition between the two resulting functions

$$\Gamma(1-\alpha_1)X_1 - \frac{\Gamma\beta_1}{\overline{X_1}} = \frac{mR}{4\mu_2}\left[(\Gamma(\alpha_1-1)-\alpha_1)(X_1-\overline{X_1}) + \beta_1(\Gamma-1)\left(\frac{1}{\overline{X_1}} - \frac{1}{X_1}\right) + 2A\right] + \frac{nR}{4\mu_2}\left[(\Gamma(\alpha_1-1)-\alpha_1)(X_1+\overline{X_1}) + \beta_1(\Gamma-1)\left(\frac{1}{\overline{X_1}} + \frac{1}{X_1}\right)\right], \quad (8.4)$$

As an example, it can be shown that (8.4) may be rearranged into

$$\left[\frac{R(m+n)}{4\mu_1} + \frac{1-\alpha_1}{\alpha_1+\Gamma(1-\alpha_1)}\right]X_1 + \frac{(n-m)R}{4\mu_1}\overline{X_1} + \frac{\beta_1}{\alpha_1+\Gamma(1-\alpha_1)}\left[\frac{\frac{(m+n)R}{4\mu_1}(1-\Gamma)-1}{\overline{X_1}} + \frac{\frac{(n-m)R}{4\mu_1}(1-\Gamma)}{X_1}\right] = \frac{\frac{ARm}{2\mu_1}}{\alpha_1+\Gamma(1-\alpha_1)}, \quad (8.5)$$

which is identical to the results provided by Wang, save for the insertion that A is purely imaginary, in Wang (2012).

Noting that as either (m or n) $\to \infty$ in (8.5) we recover only the displacement continuity boundary condition, we must further evaluate the stress displacement condition given by

$$\frac{(P_{rr})_2}{m} + i\frac{(P_{r\theta})_2}{n} = ||u_r + iu_\theta||, \;\; z \in \partial D_1, \quad (8.6)$$

which degenerates to

$$i\frac{\chi_2^{'}(z) + \overline{\chi_2^{'}}(R^2/z)}{2} = im\frac{R}{z}\left[w_2(z) - w_1(z)\right], \;\; z \in \partial D_1, \quad (8.7)$$

as $n \to \infty$. Substituting the definitions of $\chi_2(z)$ and $w_k(z), k = 1,2$ into (8.7) and comparing coefficients of like powers of z on the LHS and RHS gives the following

$$\Gamma(\alpha_1 - 1)(\overline{X_1} - X_1) + \Gamma\beta_1 \left(\frac{1}{X_1} - \frac{1}{\overline{X_1}} \right) = \frac{mR}{\mu_2} \left[(\Gamma(\alpha_1 - 1) - \alpha_1)X_1 + \frac{(\Gamma - 1)\beta_1}{\overline{X_1}} + A \right], (z^0), \quad (8.8)$$

$$X_2 = A_0 = 0, (z^{\pm 1}), \quad (8.9)$$

$$(1 - \alpha_2)\frac{\overline{A_1}}{R^4} - 3\Gamma(\alpha_1 - 1)X_3 + \frac{iB}{R^2} = \frac{mR}{\mu_2}(\Gamma(\alpha_1 - 1) - \alpha_1)X_3, (z^2), \quad (8.10)$$

$$(\alpha_2 - 1)A_1 + 3\Gamma(\alpha_1 - 1)\overline{X_3}R^4 + i\overline{B}R^2 = \frac{mR}{\mu_2} \left[\frac{\Gamma\alpha_2 - \alpha_2 + 1}{\Gamma} A_1 + i\overline{B}R^2 \frac{\Gamma - 1}{\Gamma} \right], (z^{-2}), \quad (8.11)$$

$$A_k = X_{k+1} = 0, \forall k \geq 2. \quad (8.12)$$

If we then input the compatibility condition from (8.4) into (8.8) for the case of $n \to \infty$ we arrive at the following expression for X_1

$$X_1\Gamma(1 - \alpha_1) + \frac{\Gamma\beta_1}{X_1} = \frac{mR}{2\mu_2} \left[(\Gamma(\alpha_1 - 1) - \alpha_1)X_1 + \frac{(1 - \Gamma)\beta_1}{X_1} + A \right]. \quad (8.13)$$

8.2.2 Circumferentially Inhomogeneous Imperfect Interface

The non-slip imperfect interface boundary conditions can be written in the following form

$$\frac{P_{rr}}{m(\theta)} = ||u_r||, \;\; ||u_\theta|| = 0, \;\; z \in \partial D_1. \tag{8.14}$$

Let us begin the analysis by considering the tangential displacement continuity condition. From (8.14) the continuity of the tangential displacement can be evaluated as follows

$$||u_\theta|| = \frac{R}{z}\left(iw_1(z,\bar{z}) - iw_2(z,\bar{z})\right) + \frac{z}{R}\left(\overline{iw_1(z,\bar{z})} - \overline{iw_2(z,\bar{z})}\right) = 0, \;\; z \in \partial D_1. \tag{8.15}$$

Inserting (6.15) in combination with (5.23) and (5.25) into (8.15) gives

$$\begin{aligned}
&[\alpha_1 - \Gamma(\alpha_1 - 1)]\frac{R}{z}\phi_1(z) + \left[\frac{\alpha_2 - 1}{\Gamma} - \alpha_2\right]\frac{z}{R}\overline{\phi_2}(R^2/z) + \left(1 - \frac{1}{\Gamma}\right)\frac{iBz^2}{R} \\
&+ \left[(1-\alpha_2)\left(1 - \frac{1}{\Gamma}\right)\left(A + \overline{A}\right) + (1-\Gamma)\beta_1\left(\frac{1}{\overline{X_1}} + \frac{1}{X_1}\right)\right]R = \\
&\left[\alpha_2 - \frac{\alpha_2 - 1}{\Gamma}\right]\frac{R}{z}\phi_2(z) + [\Gamma(\alpha_1 - 1) - \alpha_1]\frac{z}{R}\overline{\phi_1}(R^2/z) + \left(1 - \frac{1}{\Gamma}\right)\frac{i\overline{B}R^3}{z^2}, \;\; z \in \partial D_1.
\end{aligned} \tag{8.16}$$

In (8.16) the left hand side is analytic in the inclusion and the right hand side is analytic in the matrix except for possibly as $|z| \to 0$ and as $|z| \to \infty$, respectively. It is clear that the left hand side (8.16) is analytic everywhere in the inclusion including the point $z = 0$. Similarly, we observe that the right hand side of (8.16) has the following asymptotic behavior as $|z| \to \infty$

$$\left[\frac{\Gamma\alpha_2 - \alpha_2 + 1}{\Gamma}\right]AR + [\Gamma(\alpha_1 - 1) - \alpha_1]\overline{X_1}R, \;\; |z| \to \infty. \tag{8.17}$$

We now construct a piecewise continuous function $D(z)$ defined as follows (noting that $A+\overline{A}=0$)

$$D(z)=\begin{cases}[\alpha_1-\Gamma(\alpha_1-1)]\frac{R}{z}\phi_1(z)+\left[\frac{\alpha_2-1-\Gamma\alpha_2}{\Gamma}\right]\frac{z}{R}\overline{\phi_2}(R^2/z)\\ +\left(\frac{\Gamma-1}{\Gamma}\right)\frac{iBz^2}{R}+(1-\Gamma)\beta_1\left(\frac{1}{\overline{X_1}}+\frac{1}{X_1}\right)-\left[\frac{\Gamma\alpha_2-\alpha_2+1}{\Gamma}\right]AR-\\ [\Gamma(\alpha_1-1)-\alpha_1]\overline{X_1}R,\ \ z\in\ D_1,\\ \\ \left[\frac{\alpha_2\Gamma-\alpha_2+1}{\Gamma}\right]\frac{R}{z}\phi_2(z)+[\Gamma(\alpha_1-1)-\alpha_1]\frac{z}{R}\overline{\phi_1}(R^2/z)\\ +\left(\frac{\Gamma-1}{\Gamma}\right)\frac{i\overline{B}R^3}{z^2}-\left[\frac{\Gamma\alpha_2-\alpha_2+1}{\Gamma}\right]AR-[\Gamma(\alpha_1-1)-\alpha_1]\overline{X_1}R,\ \ z\in\ D_2.\end{cases}\tag{8.18}$$

Since $D(z)$ given by (8.18) is well defined and analytic in the entire plane, Louisville's theorem states that $D(z)$ must be a constant. In fact, since $|z|\to\infty$ it must be that this constant is identically equal to zero. Hence, $D(z)=0$ throughout the entire plane and we arrive at the following two equations

$$\begin{aligned}[\alpha_1-\Gamma(\alpha_1-1)]\frac{R}{z}\phi_1(z)+\left[\frac{\alpha_2-1-\Gamma\alpha_2}{\Gamma}\right]\frac{z}{R}\overline{\phi_2}(R^2/z)+\left(\frac{\Gamma-1}{\Gamma}\right)\frac{iBz^2}{R}\\ +(1-\Gamma)\beta_1\left(\frac{1}{\overline{X_1}}+\frac{1}{X_1}\right)-\left[\frac{\Gamma\alpha_2-\alpha_2+1}{\Gamma}\right]AR-[\Gamma(\alpha_1-1)-\alpha_1]\overline{X_1}R=0,\ \ z\in\ D_1,\end{aligned}\tag{8.19}$$

$$\begin{aligned}\left[\frac{\alpha_2\Gamma-\alpha_2+1}{\Gamma}\right]\frac{R}{z}\phi_2(z)+[\Gamma(\alpha_1-1)-\alpha_1]\frac{z}{R}\overline{\phi_1}(R^2/z)+\left(\frac{\Gamma-1}{\Gamma}\right)\frac{i\overline{B}R^3}{z^2}\\ -\left[\frac{\Gamma\alpha_2-\alpha_2+1}{\Gamma}\right]AR-[\Gamma(\alpha_1-1)-\alpha_1]\overline{X_1}R=0,\ \ z\in\ D_2.\end{aligned}\tag{8.20}$$

The first compatibility requirement between (8.19) and (8.20) is obtained by letting $|z| \to 0$ in (8.19) and is given as

$$[\alpha_1 - \Gamma(\alpha_1 - 1)]\left(X_1 + \overline{X_1}\right) + (1-\Gamma)\beta_1\left(\frac{1}{\overline{X_1}} + \frac{1}{X_1}\right) = 0. \tag{8.21}$$

Let us now consider the radial stress-displacement condition of (8.14) which we shall rewrite (from the matrix side) as

$$\left(\frac{P_{rr}}{m(\theta)}\right)_2 = ||u_r + iu_\theta||, \;\; z \in \partial D_1. \tag{8.22}$$

Using the definitions of (5.28), (8.22) is evaluated as follows

$$i\chi_2^{'}(z,\bar{z}) - \overline{i\chi_2^{'}(z,\bar{z})} = \frac{2m(\theta)R}{z}\left[iw_2(z,\bar{z}) - iw_1(z,\bar{z})\right], \;\; z \in \partial D_1. \tag{8.23}$$

Inserting (5.23,5.25,6.15) into (8.23) results in the following

$$\begin{aligned}
&\frac{\Gamma(1-\alpha_2-\Gamma(\alpha_1-1))}{\Gamma\alpha_2-\alpha_2+1}\phi_1^{'}(z) - \frac{\Gamma(1-\alpha_2-\Gamma(\alpha_1-1))}{\Gamma\alpha_2-\alpha_2+1}\overline{\phi_1^{'}}(R^2/z) + \\
&\frac{\Gamma(\alpha_2-1)(\alpha_1-\Gamma(\alpha_1-1))}{\Gamma\alpha_2-\alpha_2+1}\frac{2}{z}\phi_1(z) - \frac{\Gamma(\alpha_2-1)(\alpha_1-\Gamma(\alpha_1-1))}{\Gamma\alpha_2-\alpha_2+1}\frac{2z}{R^2}\overline{\phi_1}(R^2/z) + \\
&\frac{\Gamma}{\Gamma\alpha_2-\alpha_2+1}\frac{iBz^2}{R^2} + \frac{\Gamma}{\Gamma\alpha_2-\alpha_2+1}\frac{i\overline{B}R^2}{z^2} + \Gamma\beta_1\left(\frac{1}{X_1} - \frac{1}{\overline{X_1}}\right) + \\
&\frac{\Gamma(\alpha_2-1)(\alpha_1-\Gamma(\alpha_1-1))}{\Gamma\alpha_2-\alpha_2+1}\left(X_1 - \overline{X_1}\right) = \frac{m(\theta)R}{\mu_2}\Big[(\alpha_1-\Gamma(\alpha_1-1))\frac{z}{R^2}\overline{\phi_1}(R^2/z) + \\
&(\Gamma(\alpha_1-1)-\alpha_1)\frac{1}{z}\phi_1(z) + (\Gamma(\alpha_1-1)-\alpha_1)\frac{\overline{X_1}}{2} + (\alpha_1-\Gamma(\alpha_1-2))\frac{X_1}{2} \\
&\qquad + (\Gamma-1)\frac{\beta_1}{2\overline{X_1}} + (1-\Gamma)\frac{\beta_1}{2X_1} + \frac{A}{2} - \frac{\overline{A}}{2}\Big], \;\; z \in \partial D_1.
\end{aligned} \tag{8.24}$$

It is evident by the above interface condition that it remains to determine the single analytic function $\phi_1(z)$. Unlike the homogenous imperfect interface condition examined in section 3.1 where the conventional power series method led to a finite form solution, the variable parameter $m(\theta)$, in the present case, prevents the exact solution of (8.24) when the power series is adopted. To overcome this obstacle the technique of analytic continuation is employed to reduce (8.24) to a first order linear ordinary differential equation with variable coefficients for $\phi_1(z)$. In doing so the finite form of the solution for a circumferentially inhomogeneous non-slip interface can be obtained.

Firstly, let us introduce, for convenience, a new interface parameter $\delta(\theta)$ to replace $m(\theta)$ defined as follows

$$\delta(\theta) = \frac{\delta_0}{1+f(\theta)} = \frac{m(\theta)R}{\mu_2}, \ \ \{\delta_0 \in \mathbb{R} | \delta_0 > 0\}, \ \ f(\theta) > -1. \tag{8.25}$$

Here, $f(\theta)$ is a real and 2π periodic function on ∂D_1. Note that as $f(\theta) \to -1$, $\delta(\theta) \to \infty$ which corresponds to the perfect bonding condition.

Since $f(\theta)$ is periodic on ∂D_1, it admits a Fourier Series expansion as follows

$$f(\theta) = \sum_{k=1}^{s} a_k sin(k\theta) + b_k cos(k\theta), \ \ a_s^2 + b_s^2 \neq 0 \ \ \text{unless} f(\theta) = 0 \ \ \text{on} \ \ \partial D_1, \tag{8.26}$$

where s is non-negative natural number and $a_k, b_k, (k = 1, 2, ..., s)$ are given real constants. Moreover, on the interface ∂D_1, we can rewrite (8.26) as a function of the complex variable z as follows

$$f(z) = \frac{1}{2} \sum_{k=1}^{s} (b_k + ia_k) \frac{R^k}{z^k} + (b_k - ia_k) \frac{z^k}{R^k}, \ \ f(\theta) = f(z), \forall z \in \partial D_1. \tag{8.27}$$

8.2.3 The Differential Equation for $\phi_1(z)$

To derive the differential equation for $\phi_1(z)$ we rearrange and rewrite (8.24) using (8.25 and 8.27) as follows

$$
\begin{aligned}
&(1+f(z))\phi_1'(z)+\left[\frac{(\alpha_1-\Gamma(\alpha_1-1))\omega}{\Gamma}\frac{\delta_0}{z}-2\Omega\frac{(1+f(z))}{z}\right]\phi_1(z)+\\
&\frac{\omega}{\Gamma\alpha_2-\alpha_2+1}\frac{iBz^2}{R^2}(1+f(z))+(1+f(z))\left[\frac{\omega\beta_1}{X_1}+\Omega X_1\right]+\\
&\frac{\delta_0}{2}\left[\frac{(\Gamma(\alpha_1-1)\omega}{\Gamma}X_1+\frac{(\Gamma-1)\omega}{\Gamma}\frac{\beta_1}{X_1}-\frac{\omega A}{\Gamma}\right]=(1+f(z))\overline{\phi_1'}(R^2/z)+\\
&\left[\frac{(\alpha_1-\Gamma(\alpha_1-1))\omega}{\Gamma}\frac{\delta_0 z}{R^2}-2\Omega\frac{z(1+f(z))}{R^2}\right]\overline{\phi_1}(R^2/z)-\frac{\omega}{\Gamma\alpha_2-\alpha_2+1}\frac{i\overline{B}R^2}{z^2}(1+f(z))+\\
&(1+f(z))\left[\frac{\omega\beta_1}{\overline{X_1}}+\Omega\overline{X_1}\right]+\frac{\delta_0}{2}\left[\frac{(\Gamma(\alpha_1-1)\omega}{\Gamma}\overline{X_1}+\frac{(\Gamma-1)\omega}{\Gamma}\frac{\beta_1}{\overline{X_1}}-\frac{\omega\overline{A}}{\Gamma}\right],\ z\in\partial D_1, \quad (8.28)
\end{aligned}
$$

where

$$
\omega=\frac{\Gamma\alpha_2-\alpha_2+1}{1-\alpha_2-\Gamma(\alpha_1-1)}>0,\ \ \Omega=\frac{(1-\alpha_2)(\alpha_1-\Gamma(\alpha_1-1))}{1-\alpha_2-\Gamma(\alpha_1-1)}>0. \quad (8.29)
$$

In (8.28) the left-hand side is analytic in D_1 and the right-hand side is analytic in D_2 except for possibly the points $z=0$ and $|z|\to\infty$, respectively. As such we analyze the singular behavior of the left-hand side of (8.28) as $|z|\to 0$ and we obtain the following

$$
\begin{aligned}
&\sum_{j=1}^{k} jX_j z^{j-1}\sum_{k=1}^{s}\frac{b_k+ia_k}{2}\frac{R^k}{z^k}-2\Omega\sum_{j=1}^{k}X_j z^{j-1}\sum_{k=1}^{s}\frac{b_k+ia_k}{2}\frac{R^k}{z^k}+\\
&\qquad\frac{\omega}{\Gamma\alpha_2-\alpha_2+1}iB\sum_{k=3}^{s}\frac{b_k+ia_k}{2}\frac{R^{k-2}}{z^{k-2}}+\left[\frac{\omega\beta_1}{X_1}+\Omega X_1\right]\sum_{k=1}^{s}\frac{b_k+ia_k}{2}\frac{R^k}{z^k},\ \ |z|\to 0. \quad (8.30)
\end{aligned}
$$

Similarly, on the right-hand side of (8.28) we observe the following asymptotic and singular behavior as $|z| \to \infty$

$$\overline{X_1}(1-\Omega)+\frac{\omega\beta_1}{\overline{X_1}}+\sum_{k=1}^{s}(k+1)\overline{X_{k+1}}\frac{b_k+ia_k}{2}R^k+\sum_{j=1}^{k}j\overline{X_j}\left(\frac{R^2}{z}\right)^{j-1}\sum_{k=1}^{s}\frac{b_k+ia_k}{2}\frac{z^k}{R^k}+$$
$$-2\Omega\left[\sum_{k=1}^{s}\overline{X_{k+1}}\frac{b_k-ia_k}{2}R^k+\sum_{j=1}^{k}\overline{X_j}\left(\frac{R^2}{z}\right)^{j-1}\sum_{k=1}^{s}\frac{b_k-ia_k}{2}\frac{z^k}{R^k}\right]-$$
$$\frac{\omega}{\Gamma\alpha_2-\alpha_2+1}i\overline{B}\sum_{k=2}^{s}\frac{b_k-ia_k}{2}\frac{z^{k-2}}{R^{k-2}}+\frac{\delta_0}{2}\left[\frac{\alpha_1-\Gamma(\alpha_1-1))\omega}{\Gamma}\overline{X_1}+\frac{(\Gamma-1)\omega}{\Gamma}\frac{\beta_1}{\overline{X_1}}-\frac{\omega\overline{A}}{\Gamma}\right]+$$
$$\left[\frac{\omega\beta_1}{\overline{X_1}}+\Omega\overline{X_1}\right]\sum_{k=1}^{s}\frac{b_k-ia_k}{2}\frac{z^k}{R^k},\ \ |z|\to\infty. \quad (8.31)$$

We now construct a function which is analytic and well defined in the entire plane by subtracting the sum of (8.30) and (8.31) from both sides of (8.28) as follows

$$E(z)=\begin{cases}(1+f(z))\phi_1'(z)+\left[\frac{(\alpha_1-\Gamma(\alpha_1-1))\omega}{\Gamma}\frac{\delta_0}{z}-2\Omega\frac{(1+f(z))}{z}\right]\phi_1(z)+\\ \frac{\omega}{\Gamma\alpha_2-\alpha_2+1}\frac{iBz^2}{R^2}(1+f(z))+(1+f(z))\left[\frac{\omega\beta_1}{X_1}+\Omega X_1\right]+\\ \frac{\delta_0}{2}\left[\frac{(\Gamma(\alpha_1-1)\omega}{\Gamma}X_1+\frac{(\Gamma-1)\omega}{\Gamma}\frac{\beta_1}{X_1}-\frac{\omega A}{\Gamma}\right]-L(z),\ \ z\in\ D_1,\\ \\ (1+f(z))\overline{\phi_1'}(R^2/z)+\left[\frac{(\alpha_1-\Gamma(\alpha_1-1))\omega}{\Gamma}\frac{\delta_0 z}{R^2}-2\Omega\frac{z(1+f(z))}{R^2}\right]\overline{\phi_1}(R^2/z)-\\ \frac{\omega}{\Gamma\alpha_2-\alpha_2+1}\frac{i\overline{B}R^2}{z^2}(1+f(z))+(1+f(z))\left[\frac{\omega\beta_1}{\overline{X_1}}+\Omega\overline{X_1}\right]+\\ \frac{\delta_0}{2}\left[\frac{(\Gamma(\alpha_1-1)\omega}{\Gamma}\overline{X_1}+\frac{(\Gamma-1)\omega}{\Gamma}\frac{\beta_1}{\overline{X_1}}-\frac{\omega\overline{A}}{\Gamma}\right]-L(z),\ \ z\in\ D_2,\end{cases} \quad (8.32)$$

where $L(z)$ is given by

$$L(z)=\sum_{j=1}^{k} jX_j z^{j-1}\sum_{k=1}^{s}\frac{b_k+ia_k}{2}\frac{R^k}{z^k}-2\Omega\sum_{j=1}^{k}X_j z^{j-1}\sum_{k=1}^{s}\frac{b_k+ia_k}{2}\frac{R^k}{z^k}+$$
$$\frac{\omega}{\Gamma\alpha_2-\alpha_2+1}iB\sum_{k=3}^{s}\frac{b_k+ia_k}{2}\frac{R^{k-2}}{z^{k-2}}+\left[\frac{\omega\beta_1}{X_1}+\Omega X_1\right]\sum_{k=1}^{s}\frac{b_k+ia_k}{2}\frac{R^k}{z^k}+\overline{X_1}(1-\Omega)+$$
$$\frac{\omega\beta_1}{\overline{X_1}}+\sum_{k=1}^{s}(k+1)\overline{X_{k+1}}\frac{b_k+ia_k}{2}R^k+\sum_{j=1}^{k}j\overline{X_j}\left(\frac{R^2}{z}\right)^{j-1}\sum_{k=1}^{s}\frac{b_k+ia_k}{2}\frac{z^k}{R^k}+$$
$$-2\Omega\left[\sum_{k=1}^{s}\overline{X_{k+1}}\frac{b_k-ia_k}{2}R^k+\sum_{j=1}^{k}\overline{X_j}\left(\frac{R^2}{z}\right)^{j-1}\sum_{k=1}^{s}\frac{b_k-ia_k}{2}\frac{z^k}{R^k}\right]-$$
$$\frac{\omega}{\Gamma\alpha_2-\alpha_2+1}i\overline{B}\sum_{k=2}^{s}\frac{b_k-ia_k}{2}\frac{z^{k-2}}{R^{k-2}}+\frac{\delta_0}{2}\left[\frac{\alpha_1-\Gamma(\alpha_1-1))\omega}{\Gamma}\overline{X_1}+\frac{(\Gamma-1)\omega}{\Gamma}\frac{\beta_1}{\overline{X_1}}-\frac{\omega\overline{A}}{\Gamma}\right]+$$
$$\left[\frac{\omega\beta_1}{\overline{X_1}}+\Omega\overline{X_1}\right]\sum_{k=1}^{s}\frac{b_k-ia_k}{2}\frac{z^k}{R^k}. \quad (8.33)$$

Defined in this way (8.32) is analytic and single valued in both D_1 and D_2 including the point at infinity where it approaches zero. Thus, by Liouville's Theorem, we conclude that $E(z)$ is identically equal to zero. Hence, we obtain the following two equations

$$(1+f(z))\phi_1'(z)+\left[\frac{(\alpha_1-\Gamma(\alpha_1-1))\omega}{\Gamma}\frac{\delta_0}{z}-2\Omega\frac{(1+f(z))}{z}\right]\phi_1(z)+$$
$$\frac{\omega}{\Gamma\alpha_2-\alpha_2+1}\frac{iBz^2}{R^2}(1+f(z))+(1+f(z))\left[\frac{\omega\beta_1}{X_1}+\Omega X_1\right]+$$
$$\frac{\delta_0}{2}\left[\frac{\Gamma(\alpha_1-1)\omega}{\Gamma}X_1+\frac{(\Gamma-1)\omega}{\Gamma}\frac{\beta_1}{X_1}-\frac{\omega A}{\Gamma}\right]-L(z)=0,\ \ z\in\ D_1, \quad (8.34)$$

$$(1+f(z))\overline{\phi_1'}(R^2/z)+\left[\frac{(\alpha_1-\Gamma(\alpha_1-1))\omega}{\Gamma}\frac{\delta_0 z}{R^2}-2\Omega\frac{z(1+f(z))}{R^2}\right]\overline{\phi_1}(R^2/z)-$$
$$\frac{\omega}{\Gamma\alpha_2-\alpha_2+1}\frac{i\overline{B}R^2}{z^2}(1+f(z))+(1+f(z))\left[\frac{\omega\beta_1}{\overline{X_1}}+\Omega\overline{X_1}\right]+$$
$$\frac{\delta_0}{2}\left[\frac{(\Gamma(\alpha_1-1)\omega}{\Gamma}\overline{X_1}+\frac{(\Gamma-1)\omega}{\Gamma}\frac{\beta_1}{\overline{X_1}}-\frac{\omega\overline{A}}{\Gamma}\right]-L(z)=0,\ \ z\in\ D_2. \quad (8.35)$$

The function $\phi_1(z)$ determined from (8.34) must be compatible with that obtained from (8.35). It can be readily shown that by allowing $|z|\to 0$ in (8.34) we may derive the following condition which is necessary and sufficient for this required compatibility

$$L_0=\overline{L_0}, \quad (8.36)$$

where

$$L_0=X_1(1-\Omega)+\frac{\omega\beta_1}{X_1}+\frac{\delta_0}{2}\left[\frac{(\alpha_1-\Gamma(\alpha_1-1))\omega}{\Gamma}X_1+\frac{(\Gamma-1)\omega}{\Gamma}\frac{\beta_1}{X_1}-\frac{\omega A}{\Gamma}\right]+$$
$$\sum_{k=1}^{s}(k+1-2\Omega)X_{k+1}\frac{b_k+ia_k}{2}R^k+\frac{\omega}{\Gamma\alpha_2-\alpha_2+1}iB\frac{b_2+ia_2}{2}. \quad (8.37)$$

Hence, within the circular inclusion $D_1(z)$, (8.34) can be rearranged to yield

$$\phi_1'(z)+\left[\frac{(\alpha_1-\Gamma(\alpha_1-1))\omega}{\Gamma}\frac{\delta_0}{z(1+f(z))}-2\frac{\Omega}{z}\right]\phi_1(z)=P(z),\ \ z\in\ D_1. \quad (8.38)$$

where $P(z)$ is given by

$$P(z) = \frac{-\omega}{\Gamma\alpha_2 - \alpha_2 + 1}\frac{iBz^2}{R^2} - \frac{\omega\beta_1}{X_1} - \Omega X_1 - \frac{\delta_0}{1+f(z)}\left[\frac{(\Gamma(\alpha_1-1)-\alpha_1)\omega}{2\Gamma}X_1 + \frac{(\Gamma-1)\omega}{2\Gamma}\frac{\beta_1}{X_1} - \frac{\omega A}{2\Gamma}\right] + \frac{L(z)}{1+f(z)}, \quad (8.39)$$

Equation (8.38) is a first order linear ordinary differential equation with variable coefficients for $\phi_1(z)$. The general solution of which is given by

$$\phi_1(z) = e^{-T(z)}\int_{z_I}^{z} e^{T(z)}P(z)dz + C_0 e^{-T(z)}, \;\; z, \in \; D_1, \quad (8.40)$$

where

$$T(z) = \int\left(\frac{(\alpha_1 - \Gamma(\alpha_1-1))\omega}{\Gamma}\frac{\delta_0}{z(1+f(z))} - 2\frac{\Omega}{z}\right)dz. \quad (8.41)$$

Here, z_I is an arbitrary point in D_1 and C_0 is an arbitrary constant of integration.
Since the right-hand side in (8.40) contains the $(s+1)$ undetermined coefficient $X_k, (k = 1,2,...s+1)$ any admissible solution $\phi_1(z)$ of (8.40) must satisfy the following consistency condition

$$X_k = \frac{\phi_1^k(0)}{k!}, \;\; k = 1,2,...,s+1. \quad (8.42)$$

We may derive equation (8.42) by first assuming that $\phi_1(z)$ has a Taylor series expansion in D_1 given by

$$\phi_1(z) = \sum_{k=0}^{\infty} Q_k z^k \;\; , \;\; Q_k = \frac{\phi_1^k(0)}{k!}. \quad (8.43)$$

Then, by substituting (8.43) into (8.34) and comparing coefficients of negative powers of z we arrive at the following

$$\sum_{j=1}^{k}(j-2\Omega)Q_j z^{j-1}\sum_{k=1}^{s}\frac{(b_k+ia_k)}{2}\frac{R^k}{z^k}=\sum_{j=1}^{k}(j-2\Omega)X_j z^{j-1}\sum_{k=1}^{s}\frac{(b_k+ia_k)}{2}\frac{R^k}{z^k}. \tag{8.44}$$

Careful inspection of (8.44) reveals that when $\Omega \neq 1/2$ (8.42) is true for all s. However, for the case of $\Omega = 1/2$ we see that the first statement of (8.44) will be an identity, which provides no information on the form of the coefficient X_1 and implies (8.42) is not automatically satisfied for $k = 1$. Hence we must impose the additional requirement that

$$X_1 = \phi_1^{'}(0). \tag{8.45}$$

Following the work of Sudak et al. (2001), the solution for $\phi_1(z)$ captured by (8.40) will not be holomorphic in the uncut region D_1 due to the presence of multi-valued logarithmic functions resulting from the integration of $e^{T(z)}P(z)$ and isolated singular points from the zeros of the interface function $1+f(z)$. In order to ensure the holomorphicity of $\phi_1(z)$ the domain D_1 must be cut appropriately such that $\phi_1(z)$ is bounded at all isolated singular points and continuous across all branch cuts. This is accomplished, in part, by setting $C_0 = 0$ in (8.40), which is justified by the aforementioned boundedness argument. The discussion of the continuity of $\phi_1(z)$ across any and all necessary branch cuts will be delayed until a specific form of the interface function is chosen.

8.3 A Specific Class of Inhomogeneous Interface

To illustrate an example we shall consider a specific form of the interface function $\delta(\theta)$ as follows

$$\delta(\theta) = \frac{\delta_0}{1 + b_s cos(s\theta)}, \quad \delta_0 > 0, \quad -1 < b_s < 1. \tag{8.46}$$

Upon converting (8.46) into a complex variable form it is seen that there will be singularities in the interface function originating from the roots of the following polynomial of degree $2s$

$$\frac{2}{b_s}\left(\frac{z}{R}\right)^s + \left(\frac{z}{R}\right)^{2s} + 1 = 0. \tag{8.47}$$

Furthermore, it is shown in Schiavone, P., Ru (1997) that of the $2s$ roots of (8.47), s will lie inside D_1 and the remaining s will lie in D_2. Let the s roots inside D_1 be denoted by

$$\rho_1, \rho_2, \rho_3, ..., \rho_s, \tag{8.48}$$

where $\rho^s_{(1,2,...,s)} = \rho^*$ and ρ^* is real and given by

$$\rho^* = \begin{cases} \sqrt{\frac{1}{b_s^2} - 1} - \frac{1}{b_s}, < 0, b_s > 0, \\ -\sqrt{\frac{1}{b_s^2} - 1} - \frac{1}{b_s}, > 0, b_s < 0, \end{cases} \tag{8.49}$$

such that $-1 < \rho^* < 1$, and the remaining s roots in D_2 are given by $\frac{1}{\rho_1}, \frac{1}{\rho_2}, ..., \frac{1}{\rho_s}$. As a consequence of the above interface definitions we make note of the following

$$-\frac{2}{b_s}=\frac{1+\rho^{*2}}{\rho^*},$$

$$\frac{R\delta_0}{z(1+f(z))}=-\frac{\lambda(\frac{z}{R})^{s-1}}{(\frac{z}{R})^s-\rho^*}+\frac{\lambda(\frac{z}{R})^{s-1}}{(\frac{z}{R})^s-\frac{1}{\rho^*}},$$

$$\lambda=-\delta_0\left(\frac{1+\rho^{*2}}{1-\rho^{*2}}\right)<0,$$

$$\frac{1}{1+f(z)}=\frac{\frac{2}{b_s}(\frac{z}{R})^s}{\left[(\frac{z}{R})^s-\rho^*\right]\left[(\frac{z}{R})^s-\frac{1}{\rho^*}\right]}. \tag{8.50}$$

Utilizing (8.50) we may express (8.40) as follows

$$\phi_1(z)=(\frac{z}{R})^{2\Omega}\left[(\frac{z}{R})^s-\rho^*\right]^{\frac{\lambda\Omega\eta}{s}}\left[(\frac{z}{R})^s-\frac{1}{\rho^*}\right]^{\frac{-\lambda\Omega\eta}{s}}\int_{R\rho_1}^{z}(\frac{t}{R})^{-2\Omega+1}\left[(\frac{t}{R})^s-\rho^*\right]^{\frac{-\lambda\Omega\eta}{s}}$$
$$\left[(\frac{t}{R})^s-\frac{1}{\rho^*}\right]^{\frac{\lambda\Omega\eta}{s}}\frac{P(t)}{\frac{t}{R}}dt,\ \ z\in D_1, \tag{8.51}$$

$$\eta=\frac{\Gamma\alpha_2-\alpha_2+1}{\Gamma(1-\alpha_2)}>0,$$

where the integration path is taken along the edge of any branch cuts and passing through any of the s integrable branch points. In addition to having previously set $C_0=0$ we also require that

$$\int_{R\rho_1}^{R\rho_k}(\frac{t}{R})^{-2\Omega+1}\left[(\frac{t}{R})^s-\rho^*\right]^{\frac{-\lambda\Omega\eta}{s}}\left[(\frac{t}{R})-\frac{1}{\rho^*}\right]^{\frac{\lambda\Omega\eta}{s}}\frac{P(t)}{\frac{t}{R}}dz=0,\ \ k=2,3,...,s, \tag{8.52}$$

in order to maintain boundedness of $\phi_1(z)$ at any of the potential isolated singular points $R\rho_k, k = 2,3,...,s$ in D_1. Additionally, by taking the difference

$$\phi_1(z^+) - \phi_1(z^-) = 0, \tag{8.53}$$

we may prove that (8.51) is continuous across any of the s branch cuts by noting that, due to the sign change of the exponents in and outside of the integral, any increments in the multivalued logarithmic terms that will arise from inside the integral will be nullified from which (8.53) is easily confirmed. The remaining irregular point to be considered is when $z = 0$. Closer inspection of (8.51) reveals that there are three cases to be considered as $z \to 0$.

8.3.1 Case One: $\Omega > \frac{1}{2}$.

When $\Omega > \frac{1}{2}$ we see from (8.51) that $\phi_1(z) \to 0$ as $z \to 0$. However, in order to ensure the holomorphicity of $\phi_1(z)$ we must ensure that $\phi_1(z)$ is continuous across the branch cut formed from $z = R\rho^*$ along the real axis inside D_1. Closer inspection of (8.51) reveals the presence of an unintegrable singularity at $z = 0$. Hence we must define a new path of integration, L^*, to skirt around a neighborhood of $z = 0$ and set $z = z^*$,where z^* is any particular point on the branch cut from $z = 0$, to compensate for this change. In this way the continuity condition becomes

$$\int_{L^*} (\frac{z^*}{t})^{-2\Omega} \left[\frac{(\frac{z^*}{R})^s - \rho^*}{(\frac{t}{R})^s - \rho^*} \right]^{\frac{-\lambda\Omega\eta}{s}} \left[\frac{(\frac{t}{R})^s - \frac{1}{\rho^*}}{(\frac{z^*}{R})^s - \frac{1}{\rho^*}} \right]^{\frac{\lambda\Omega\eta}{s}} P(t)dt = 0 \ . \tag{8.54}$$

We may then solve for the X_{s+1} unknown coefficients using (8.36,8.54).

8.3.2 Case Two: $\Omega < \frac{1}{2}$.

For this case we shall rewrite (8.51) in the form

$$\frac{\phi_1(z)}{\frac{z}{R}} = (\frac{z}{R})^{2\Omega-1}\left[(\frac{z}{R})^s-\rho^*\right]^{\frac{\lambda\Omega\eta}{s}}\left[(\frac{z}{R})^s-\frac{1}{\rho^*}\right]^{\frac{-\lambda\Omega\eta}{s}}\int\limits_{R\rho_1}^{z}(\frac{t}{R})^{-2\Omega}$$
$$\left[(\frac{t}{R})^s-\rho^*\right]^{\frac{-\lambda\Omega\eta}{s}}\left[(\frac{t}{R})^s-\frac{1}{\rho^*}\right]^{\frac{\lambda\Omega\eta}{s}}P(t)dt,\ \ z\in D_1. \quad (8.55)$$

Given that $X_0 = 0$, the LHS of (8.55) is analytic within D_1. As a consequence, $\frac{\phi_1(z)}{\frac{z}{R}}$ must be bounded at $z = 0$ and since $\Omega < \frac{1}{2}$ this implies that

$$\int\limits_{R\rho_1}^{0}(\frac{t}{R})^{-2\Omega}\left[(\frac{t}{R})^s-\rho^*\right]^{\frac{-\lambda\Omega\eta}{s}}\left[(\frac{t}{R})^s-\frac{1}{\rho^*}\right]^{\frac{\lambda\Omega\eta}{s}}P(t)dt=0,\ \ \Omega<\frac{1}{2}. \quad (8.56)$$

Note that in (8.56) there is a singularity in the integrand owing to the term $(\frac{t}{R})^{-2\Omega}$ for $\Omega < \frac{1}{2}$. Due to the fact that the path of integration in (8.56) lies on the real axis we may treat

$$K(\rho^*,t) = (\frac{t}{R})^{-2\Omega}\left[(\frac{t}{R})^s-\rho^*\right]^{\frac{-\lambda\Omega\eta}{s}}\left[(\frac{t}{R})^s-\frac{1}{\rho^*}\right]^{\frac{\lambda\Omega\eta}{s}}, \quad (8.57)$$

as a weakly singular kernel function and is thus integrable along such a domain (see Constanda (1990)). We may then solve for the X_{s+1} unknown coefficients using (8.36,8.56).

8.3.3 Case Three: $\Omega = \frac{1}{2}$.

In this case from (8.51) we see that $z = 0$ is not a singular point of $\phi_1(z)$ owing to the fact that $P(0) = 0$ and thus the expression $P(t)/(t/R)$ is non-singular at the point $z = 0$. We may then

proceed to solve for the X_{s+1} unknown coefficients by recalling relation (8.36) and by evaluating (8.42) as

$$RX_1 = \left[-\rho^*\right]^{\frac{\lambda\eta}{2s}} \left[-\frac{1}{\rho^*}\right]^{-\frac{\lambda\eta}{2s}} \int_{R\rho_1}^{0} \left[(\frac{t}{R})^s - \rho^*\right]^{\frac{-\lambda\eta}{2s}} \left[(\frac{t}{R})^s - \frac{1}{\rho^*}\right]^{\frac{\lambda\eta}{2s}} \frac{P(t)}{\frac{t}{R}} dt = 0. \tag{8.58}$$

These two conditions provide the $s+1$ necessary equations to solve for the unknown coefficients.

In summary, we have shown that for the case of an inhomogeneous imperfect non-slip interface described by (8.15), the stress inside the inclusion can be determined through (8.4) and (8.51) whereby the unknown coefficients X_{s+1} are determined through (8.36) and one of (8.42,8.56,8.57).

8.4 Example

For ease of analysis in illustrating the method we shall assume that $\Omega = \frac{1}{2}, \lambda = -1, \eta = 2$ and we shall confine ourselves to the case $s = 1$. From these preliminaries we may evaluate (8.51) as

$$\phi_1(z) = \frac{z}{R} \frac{\left(\frac{z}{R} - \frac{1}{\rho^*}\right)}{\left(\frac{z}{R} - \rho^*\right)} \left[-\frac{i\omega B}{\Gamma\alpha_2 - \alpha_2 + 1} I_1(z) - \left(X_1 + \frac{\omega\beta_1}{X_1}\right) I_2(z) + \left(X_2 R + \frac{2}{b_1}\frac{\delta_0}{2} X_1\right) I_3(z)\right], \quad z \in D_1, \tag{8.59}$$

where

$$I_1(z) = \int_{R\rho^*}^{z} \frac{\frac{t}{R}\left(\frac{t}{R} - \rho^*\right)}{\frac{t}{R} - \frac{1}{\rho^*}} dt = \left[\frac{t^2}{2R} + \left(\frac{1-\rho^{*2}}{\rho_1}\right) t + R\left(\frac{1-\rho^{*2}}{\rho^{*2}}\right) Log\left(\frac{t}{R} - \frac{1}{\rho^*}\right)\right]_{R\rho^*}^{z}, \tag{8.60}$$

$$I_2(z) = \int_{R\rho^*}^{z} \frac{\frac{t}{R}}{\left(\frac{t}{R} - \frac{1}{\rho^*}\right)^2} dt = \left[RLog\left(\frac{t}{R} - \frac{1}{\rho^*}\right) - \frac{t}{\frac{t}{R} - \frac{1}{\rho^*}} \right]_{R\rho^*}^{z},$$

$$I_3(z) = \int_{R\rho^*}^{z} \frac{1}{\left(\frac{t}{R} - \frac{1}{\rho^*}\right)^2} dt = \left[-\frac{R}{\frac{t}{R} - \frac{1}{\rho^*}} \right]_{R\rho^*}^{z}, \quad z \in D_1.$$

The coefficients $X_1, \overline{X_1}, X_2, \overline{X_2}$ are then computed from the following two equations (which are derived from (8.36) and (8.57))

$$X_1 \left[1 + \frac{\delta_0(\alpha_1 - \Gamma(\alpha_1 - 1))\omega}{\Gamma} \right] + \frac{1}{X_1} \left[2\omega\beta_1 + \frac{(\Gamma - 1)\omega\beta_1\delta_0}{\Gamma} \right] + \frac{b_1 R}{2}(X_2 - \overline{X_2}) = \frac{\delta_0 \omega A}{\Gamma}, \quad (8.61)$$

$$X_2 \frac{R^2\rho^{*3}}{\rho^{*2} - 1} = \frac{i\omega B}{\Gamma\alpha_2 - \alpha_2 + 1} \left[R\left(\frac{1 - \rho^{*2}}{\rho^{*2}}\right) ln\left(\frac{1}{1 - \rho^{*2}}\right) + \frac{R\rho^{*2}}{2} - R \right] + \left(X_1 + \frac{\omega\beta_1}{X_1} \right) \left[RLog\left(\frac{1}{1 - \rho^{*2}}\right) + \frac{R\rho^{*2}}{\rho^{*2} - 1} \right], \quad -1 < \rho^* < 1. \quad (8.62)$$

Solving for X_1, X_2 in (8.61,8.62), we may then calculate the stress in the inclusion via (5.28).

We must now validate the expressions given in (8.61) and (8.62) for X_1 and X_2. This is done in part by considering the case where $\rho^* \to 0$ in (8.61,8.62) from which we recover

$$X_1\Gamma(1 - \alpha_1) + \frac{\Gamma\beta_1}{X_1} = \frac{\delta_0}{2} \left[(\Gamma(\alpha_1 - 1) - \alpha_1)X_1 + \frac{(1 - \Gamma)\beta_1}{X_1} + A \right], \quad (8.63)$$

$X_2 = 0.$

Where in (8.63) we have used the relation

$$2\Gamma\left(\frac{1}{\omega}-\frac{\Omega}{\omega}\right)=2\Gamma(1-\alpha_1). \tag{8.64}$$

In order to validate (8.63) the case of a homogeneous imperfect interface must be considered and the corresponding homogeneous coefficients X_k must be calculated. This is done in section 8.2.1 where we see that (8.63) is identical to (8.13).

Remark 3. In general since $\phi_1(z)$ is written in a power series we see that for the present solution where X_1 and X_2 are the only non-zero coefficients, the general Piola stress vector will be non uniform. However, when $\rho^*=0$ the coefficient $X_2=0$ and X_1 is constant and hence the internal stress field in the inclusion is uniform for a homogeneous imperfect interface which agrees with the results of Wang (2012).

8.5 Results

Having verified the formulation we may now proceed to compare the homogeneous imperfect interface to the inhomogeneous one. For the purpose of this example we will compare the inhomogeneous interface of the form

$$\frac{m(\theta)R}{\mu_2}=\frac{\delta_0}{1+b_1Cos(\theta)},\quad \delta_0=\frac{1-\rho^{*2}}{1+\rho^{*2}},\quad -1<b_1<1, \tag{8.65}$$

to the homogeneous imperfect interface given by

$$\frac{mR}{\mu_2}=\delta_0. \tag{8.66}$$

To facilitate this comparison we shall take the average of the mean stress on the boundary defined as follows

$$(P_{11}+P_{22})_{2,Avg} = \frac{1}{C_{\partial D_1}} \int_{\partial_{D_1}} 4\mu_2 Im\left[\Gamma(1-\alpha_1)X_1 + \frac{\Gamma\beta_1}{X_1}\right] ds, \tag{8.67}$$

where $C_{\partial D_1}$ is the circumference of the boundary ∂D_1. If we compute the ratio the inhomogeneous version of (8.67) to the homogeneous version using the corresponding definitions of X_1 for the inhomogeneous and homogeneous cases given in (8.61) and (8.63) we observe the following

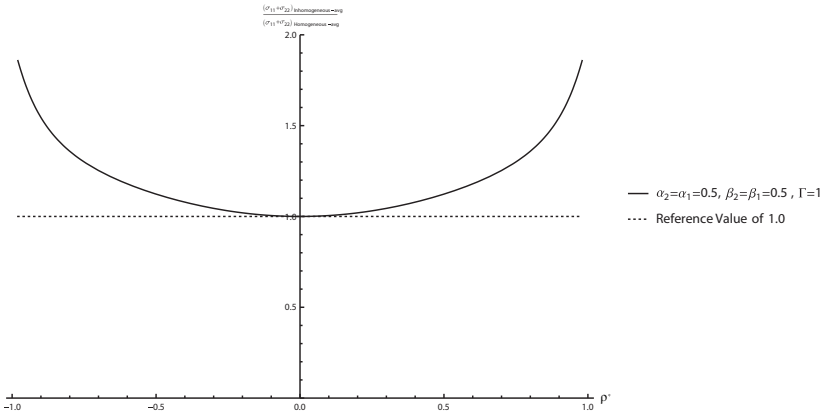

Figure 8.1: Ratio of inhomogeneous to homogeneous average mean stress on ∂D_1 for the remote loading $P_{11}^{\infty} = 0, P_{22}^{\infty} = 10^3, P_{12}^{\infty} = 0$

Figure 8.1 clearly demonstrates that the inhomogeneous interface parameter ρ^* has a significant effect on the estimation of the average mean stress on the inclusion boundary and at its peak reaches an error of 80 percent. In the linear analog Sudak et al. (1999) reported an error of up to 400 percent between the inhomogeneous and homogeneous cases. While the present work does not reach errors of a similar magnitude, these results clearly demonstrate that when analyzing a circular inclusion in finite elasticity with non-slip interfacial boundary conditions, the traditional homogeneous imperfect interface model is in general not sufficient when calculating the average mean stress on the boundary.

Chapter 9

THE CASE OF ($m(\theta) \to \infty, n(\theta) = finite$)

9.1 Introduction

In this case we assume that the interfacial bonding is characterized by $m(\theta) \to \infty$ and $n(\theta) = finite$ (where $m(\theta)$ and $n(\theta)$ are the normal and tangential imperfect interface parameters, respectively, and θ is the polar angle). This type of imperfect interface condition is thought to be a potential byproduct of manufacturing techniques where improper cooling can result in a smooth inter-phase layer. Such an interface condition allows for a relative tangential displacement but maintains continuity of radial displacements across the inter-phase layer. The physical interpretation of said boundary condition is that in the direction tangential to the boundary curve a jump in the tangential displacement across the boundary is linearly proportional to the corresponding traction vector at the same point. Conversely, in the direction normal to the boundary curve the displacement vector is continuous where the relative normal displacement between the inclusion and matrix is zero.

9.2 Formulation

9.2.1 Solution for Homogeneous Imperfect Interface

In this section we will briefly examine the homogeneous imperfect interface where the parameters m and n appearing in (6.1) are assumed to be constant along ∂D_1. Although a similar problem has been investigated by Wang (2012), for ease of comparison, we present a solution that is more amenable to validating the present work.

The general form of the imperfect interface condition is given as

$$(P_{rr}+iP_{\theta r})_2 = \frac{m+n}{2}||u_r+iu_\theta|| + \frac{m-n}{2}||u_r-iu_\theta||, \;\; z\in\partial D_1. \tag{9.1}$$

Inserting the definitions of (5.28) into (9.1) yields

$$i\chi_2^{'}(z) = \frac{(m+n)R}{2z}\left(iw_2(z)-iw_1(z)\right) + \frac{(m-n)z}{2R}\left(\overline{iw_1}(R^2/z)-\overline{iw_2}(R^2/z)\right), \;\; z\in\partial D_1. \tag{9.2}$$

Next, substituting in (5.23,5.25,6.15,5.28) into (9.2) gives

$$\begin{aligned}(1-\alpha_2)\phi_2^{'}(z)+\Gamma(1-\alpha_1)\phi_1^{'}(z)+(\alpha_2-1)A-\frac{\Gamma\beta_1}{\overline{X_1}}+\frac{i\overline{B}R^2}{z^2} = \frac{(m+n)R}{4\mu_2}\left[\frac{\phi_2(z)}{z}\left(\frac{\alpha_2\Gamma-\alpha_2+1}{\Gamma}\right)\right.\\ \left.+\frac{\phi_1(z)}{z}(\Gamma(\alpha_1-1)-\alpha_1)+A(\frac{\alpha_2-1}{\Gamma}-\alpha_2+1)+\frac{\beta_1(\Gamma-1)}{\overline{X_1}}+\frac{i\overline{B}R^2}{z^2}\frac{\Gamma-1}{\Gamma}\right]+\\ \frac{m-n}{4\mu_2R}\left[z\overline{\phi_1}(R^2/z)(\alpha_1-\Gamma(\alpha_1-1))+z\overline{\phi_2}(R^2/z)(\frac{\alpha_2-1}{\Gamma}-\alpha_2)+\overline{A}R^2(\alpha_2-1-\frac{\alpha_2-1}{\Gamma})\right.\\ \left.+\frac{\beta_1R^2(1-\Gamma)}{X_1}+iBz^2\frac{(\Gamma-1)}{\Gamma}\right], \;\; z\in\partial D_1.\end{aligned} \tag{9.3}$$

Substituting the definitions of (6.8) into (9.3) and performing analytic continuation yields the following expression as a compatibility condition between the two resulting functions

$$\Gamma(1-\alpha_1)X_1 - \frac{\Gamma\beta_1}{\overline{X_1}} = \frac{mR}{4\mu_2}\left[(\Gamma(\alpha_1-1)-\alpha_1)(X_1-\overline{X_1})+\beta_1(\Gamma-1)\left(\frac{1}{\overline{X_1}}-\frac{1}{X_1}\right)+2A\right] + \frac{nR}{4\mu_2}\left[(\Gamma(\alpha_1-1)-\alpha_1)(X_1+\overline{X_1})+\beta_1(\Gamma-1)\left(\frac{1}{\overline{X_1}}+\frac{1}{X_1}\right)\right], \quad (9.4)$$

As an example, it can be shown that (9.4) may be rearranged into

$$\left[\frac{R(m+n)}{4\mu_1}+\frac{1-\alpha_1}{\alpha_1+\Gamma(1-\alpha_1)}\right]X_1+\frac{(n-m)R}{4\mu_1}\overline{X_1}+ \frac{\beta_1}{\alpha_1+\Gamma(1-\alpha_1)}\left[\frac{\frac{(m+n)R}{4\mu_1}(1-\Gamma)-1}{\overline{X_1}}+\frac{\frac{(n-m)R}{4\mu_1}(1-\Gamma)}{X_1}\right]=\frac{\frac{ARm}{2\mu_1}}{\alpha_1+\Gamma(1-\alpha_1)}, \quad (9.5)$$

which is identical to the results provided by Wang, save for the insertion that A is purely imaginary, in Wang (2012).

Noting that as either (m or n) $\to \infty$ in (9.4) we recover only the displacement continuity boundary condition, we must further evaluate the stress displacement condition given by

$$\frac{(P_{rr})_2}{m}+i\frac{(P_{r\theta})_2}{n}=||u_r+iu_\theta||, \;\; z\in\partial D_1, \quad (9.6)$$

which degenerates to

$$\frac{\chi_2'(z)-\overline{\chi_2'}(R^2/z)}{2i}=-in\frac{R}{z}\left[w_2(z)-w_1(z)\right], \;\; z\in\partial D_1, \quad (9.7)$$

as $m \to \infty$. Substituting the definitions of $\chi_2(z)$ and $w_k(z), k = 1,2$ into (9.6) and comparing coefficients of like powers of z on the LHS and RHS gives the following

$$\Gamma(\alpha_1 - 1)(X_1 + \overline{X_1}) + \Gamma\beta_1 \left(\frac{1}{X_1} + \frac{1}{\overline{X_1}} \right) = \frac{nR}{\mu_2} \left[(\alpha_1 - \Gamma(\alpha_1 - 1))X_1 + (1 - \Gamma)\frac{\beta_1}{\overline{X_1}} - A \right], (z^0), \quad (9.8)$$

$$X_2 = A_0 = 0, (z^{\pm 1}), \quad (9.9)$$

$$3\Gamma(\alpha_1 - 1)X_3 + (1 - \alpha_2)\frac{\overline{A_1}}{R^4} + \frac{iB}{R^2} = \frac{nR}{\mu_2} \left[(\alpha_1 - \Gamma(\alpha_1 - 1))X_3 \right], (z^2), \quad (9.10)$$

$$(1 - \alpha_2)A_1 + 3\Gamma(\alpha_1 - 1)\overline{X_3}R^4 - i\overline{B}R^2 = \frac{nR}{\mu_2} \left[i\overline{B}R^2 \frac{(1 - \Gamma)}{\Gamma} + \frac{\alpha_2 - 1 - \Gamma\alpha_2}{\Gamma} A_1 \right], (z^{-2}), \quad (9.11)$$

$$A_k = X_{k+2} = 0, \forall k \geq 2. \quad (9.12)$$

If we then input the compatibility condition from (9.4) into (9.8) for the case of $m \to \infty$ we arrive at the following expression for X_1 and its conjugate

$$\Gamma(1-\alpha_1)(\overline{X_1}+X_1)-\Gamma\beta_1\left(\frac{1}{X_1}+\frac{1}{\overline{X_1}}\right)=\frac{nR}{\mu_2}\left[(\Gamma(\alpha_1-1)-\alpha_1)(X_1+\overline{X_1})+\right.$$
$$\left.(\Gamma-1)\beta_1\left(\frac{1}{\overline{X_1}}+\frac{1}{X_1}\right)\right]. \quad (9.13)$$

9.2.2 Circumferentially Inhomogeneous Imperfect Interface

We shall now consider a circular inclusion for which the inhomogeneous imperfect interface is characterized by $m(\theta)\to\infty$, $n(\theta)=finite$. For this so called sliding interface the boundary conditions take the form

$$\frac{(P_{r\theta})_2}{n(\theta)}=||u_\theta||,\ \ ||u_r||=0,\ \ z\in\partial D_1, \quad (9.14)$$

where $n(\theta)$ is non-negative and periodic along ∂D_1. The displacement continuity condition is evaluated as follows

$$||u_r||=\frac{R}{z}\left(iw_2(z)-iw_1(z)\right)+\frac{z}{R}\left(\overline{iw_1}(R^2/z)-\overline{iw_2}(R^2/z)\right),\ \ z\in\partial D_1. \quad (9.15)$$

Inserting (6.8) in combination with (5.23,5.25) into (9.15) gives

$$[\Gamma(\alpha_1-1)-\alpha_1]\frac{R}{z}\phi_1(z)+\left[\frac{\alpha_2-1-\Gamma\alpha_2}{\Gamma}\right]\frac{z}{R}\overline{\phi_2}(R^2/z)+\frac{\alpha_2-1+\Gamma(1-\alpha_2)}{\Gamma}AR$$
$$+\frac{iBz^2}{R}\frac{\Gamma-1}{\Gamma}+\frac{\beta_1}{X_1}(1-\Gamma)=[\Gamma(\alpha_1-1)-\alpha_1]\frac{z}{R}\overline{\phi_1}(R^2/z)+\left[\frac{\alpha_2-1-\Gamma\alpha_2}{\Gamma}\right]\frac{R}{z}\phi_2(z)$$
$$+\frac{\alpha_2-1+\Gamma(1-\alpha_2)}{\Gamma}\overline{A}R+\frac{i\overline{B}R^3}{z^2}\frac{1-\Gamma}{\Gamma}+\frac{\beta_1}{\overline{X_1}}(1-\Gamma),\ \ z\in\partial D_1. \quad (9.16)$$

In (9.16) the left hand side is analytic in D_1 and the right hand side is analytic in D_2 except for possibly at the point $|z| = 0$ and as $|z| \to \infty$, respectively. Employing the technique of analytic continuation, we first analyze the behavior of the left hand side of (9.16) as $|z| \to 0$ as follows

$$(\Gamma(\alpha_1 - 1) - \alpha_1)X_1 R + \frac{\alpha_2 - 1 - \Gamma\alpha_2}{\Gamma}\overline{A}R + \frac{\alpha_2 - 1 + \Gamma(1-\alpha_2)}{\Gamma}AR + \frac{\beta_1}{X_1}(1-\Gamma), \quad |z| \to 0. \tag{9.17}$$

Since (9.17) does not contain any strictly singular terms, we may conclude that the left hand side of (9.16) is analytic in D_1 as $|z| \to 0$. Moving on to the right hand side of (9.16), we observe the following as $|z| \to \infty$

$$(\Gamma(\alpha_1 - 1) - \alpha_1)\overline{X_1}R + \frac{\alpha_2 - 1 - \Gamma\alpha_2}{\Gamma}AR + \frac{\alpha_2 - 1 + \Gamma(1-\alpha_2)}{\Gamma}\overline{A}R + \frac{\beta_1}{\overline{X_1}}(1-\Gamma), \quad |z| \to \infty. \tag{9.18}$$

Equation (9.18) represents the asymptotic behavior of the right hand side of (9.16) and , subtracting (9.18) from both sides of (9.16), we may form the following function

$$D(z)=\begin{cases}[\Gamma(\alpha_1-1)-\alpha_1]\frac{R}{z}\phi_1(z)+\left[\frac{\alpha_2-1-\Gamma\alpha_2}{\Gamma}\right]\frac{z}{R}\overline{\phi_2}(R^2/z)+\frac{\alpha_2-1+\Gamma(1-\alpha_2)}{\Gamma}AR\\ +\frac{iBz^2}{R}\frac{\Gamma-1}{\Gamma}+\frac{\beta_1}{X_1}(1-\Gamma)-(\Gamma(\alpha_1-1)-\alpha_1)\overline{X_1}R-\frac{\alpha_2-1-\Gamma\alpha_2}{\Gamma}AR-\\ \frac{\alpha_2-1+\Gamma(1-\alpha_2)}{\Gamma}\overline{A}R-\frac{\beta_1}{\overline{X_1}}(1-\Gamma),\ \ z\in D_1,\\ \\ [\Gamma(\alpha_1-1)-\alpha_1]\frac{z}{R}\overline{\phi_1}(R^2/z)+\left[\frac{\alpha_2-1-\Gamma\alpha_2}{\Gamma}\right]\frac{R}{z}\phi_2(z)\\ +\frac{\alpha_2-1+\Gamma(1-\alpha_2)}{\Gamma}\overline{A}R+\frac{i\overline{B}R^3}{z^2}\frac{1-\Gamma}{\Gamma}+\frac{\beta_1}{\overline{X_1}}(1-\Gamma)-\frac{\alpha_2-1-\Gamma\alpha_2}{\Gamma}AR-\\ \frac{\alpha_2-1+\Gamma(1-\alpha_2)}{\Gamma}\overline{A}R-\frac{\beta_1}{\overline{X_1}}(1-\Gamma),\ \ z\in D_2.\end{cases}\tag{9.19}$$

Now, since $D(z)$ is well defined and analytic in the entire plane including as $|z|\to\infty$, Louisvilles theorem states that $D(z)=constant$. This implies, through the subtraction of (9.18) on the left hand and right hand side of (9.16), that $D(z)=0$ and hence we arrive at the following two equations

$$\begin{aligned}&[\Gamma(\alpha_1-1)-\alpha_1]\frac{R}{z}\phi_1(z)+\left[\frac{\alpha_2-1-\Gamma\alpha_2}{\Gamma}\right]\frac{z}{R}\overline{\phi_2}(R^2/z)+\frac{\alpha_2-1+\Gamma(1-\alpha_2)}{\Gamma}AR\\ &+\frac{iBz^2}{R}\frac{\Gamma-1}{\Gamma}+\frac{\beta_1}{X_1}(1-\Gamma)-(\Gamma(\alpha_1-1)-\alpha_1)\overline{X_1}R-\frac{\alpha_2-1-\Gamma\alpha_2}{\Gamma}AR-\\ &\qquad\frac{\alpha_2-1+\Gamma(1-\alpha_2)}{\Gamma}\overline{A}R-\frac{\beta_1}{\overline{X_1}}(1-\Gamma)=0,\ \ z\in D_1,\end{aligned}\tag{9.20}$$

$$\begin{aligned}&[\Gamma(\alpha_1-1)-\alpha_1]\frac{z}{R}\overline{\phi_1}(R^2/z)+\left[\frac{\alpha_2-1-\Gamma\alpha_2}{\Gamma}\right]\frac{R}{z}\phi_2(z)\\ &+\frac{\alpha_2-1+\Gamma(1-\alpha_2)}{\Gamma}\overline{A}R+\frac{i\overline{B}R^3}{z^2}\frac{1-\Gamma}{\Gamma}+\frac{\beta_1}{\overline{X_1}}(1-\Gamma)-\frac{\alpha_2-1-\Gamma\alpha_2}{\Gamma}AR-\\ &\qquad\frac{\alpha_2-1+\Gamma(1-\alpha_2)}{\Gamma}\overline{A}R-\frac{\beta_1}{\overline{X_1}}(1-\Gamma)=0,\ \ z\in D_2.\end{aligned}\tag{9.21}$$

The compatability requirement between equations (9.20) and (9.21) is given by

$$(\Gamma(\alpha_1-1)-\alpha_1)(X_1-\overline{X_1})+\beta_1(1-\Gamma)\left(\frac{1}{X_1}-\frac{1}{\overline{X_1}}\right)=-2A. \tag{9.22}$$

We now consider the tangential stress-displacement interface condition which may be written as

$$\frac{(P_{r\theta})_2}{n(\theta)}=-i||u_r+iu_\theta||,\ \ z\in\partial D_1, \tag{9.23}$$

which, in terms of (5.23,5.25,5.28) and (6.8) becomes

$$\begin{aligned}
&\left[\frac{\Gamma(1-\alpha_2+\Gamma(1-\alpha_1))}{\alpha_2-1-\Gamma\alpha_2}\right]\phi_1'(z)+\left[\frac{\Gamma(1-\alpha_2+\Gamma(1-\alpha_1))}{\alpha_2-1-\Gamma\alpha_2}\right]\overline{\phi_1}'(R^2/z)+\\
&\frac{2\Gamma(\alpha_2-1)(\alpha_1-\Gamma(\alpha_1-1))}{\alpha_2-1-\Gamma\alpha_2}\frac{\phi_1(z)}{z}+\frac{2\Gamma(\alpha_2-1)(\alpha_1-\Gamma(\alpha_1-1))}{\alpha_2-1-\Gamma\alpha_2}\overline{\phi_1'}(R^2/z)\frac{z}{R^2}\\
&+\frac{i\overline{B}R^2}{z^2}\frac{\Gamma}{\alpha_2-1-\Gamma\alpha_2}-\frac{iBz^2}{R^2}\frac{\Gamma}{\alpha_2-1-\Gamma\alpha_2}+\frac{\Gamma(\alpha_2-1)(\Gamma(\alpha_1-1)-\alpha_1)}{\alpha_2-1-\Gamma\alpha_2}\overline{X_1}\\
&+\frac{\Gamma(\alpha_2-1)(\Gamma(\alpha_1-1)-\alpha_1)}{\alpha_2-1-\Gamma\alpha_2}X_1+\frac{\Gamma\beta_1}{\overline{X_1}}+\frac{\Gamma\beta_1}{X_1}=\\
&\frac{n(\theta)R}{\mu_2}\left[(\alpha_1-\Gamma(\alpha_1-1)\frac{\phi_1(z)}{z}+(\alpha_1-\Gamma(\alpha_1-1)\frac{z}{R^2}\overline{\phi_1}(R^2/z)+\right.\\
&\qquad\left.(\Gamma(\alpha_1-1)-\alpha_1)\frac{\overline{X_1}}{2}+\frac{\beta_1}{2\overline{X_1}}(1-\Gamma)+(\Gamma(\alpha_1-1)-\alpha_1)\frac{X_1}{2}+\frac{\beta_1}{2X_1}(1-\Gamma)\right],z\ \in\partial D_1.
\end{aligned} \tag{9.24}$$

The problem is now reduced to determining the single analytic function $\phi_1(z)$. Unlike the homogeneous analog, direct substitution of the power series expansion of $\phi_1(z)$ into (9.24) will result in an unsolvable system of equations for the coefficients of $\phi_1(z)$. As such, we shall seek to use analytic continuation to further reduce (9.24) into an ordinary differential equation with variable

coefficients for $\phi_1(z)$. To aid in this process we will define a new imperfect interface parameter to replace $n(\theta)$ in (9.24) as follows

$$\delta(\theta) = \frac{n(\theta)R}{\mu_2}, \;\; \delta(\theta) > 0, \tag{9.25}$$

and since $1/\delta(\theta)$ is a non-negative and periodic function on ∂D_1, we may write

$$\frac{\delta_0}{\delta(\theta)} = 1 + f(\theta), \;\; \delta_0 > 0, \;\; f(\theta) > -1, \tag{9.26}$$

where δ_0 is real, $f(\theta)$ is 2π periodic on ∂D_1, and as $f(\theta) \to -1$, $n(\theta) \to \infty$, which is the case of a perfectly bonded interface. Given $f(\theta)$ is 2π periodic on ∂D_1 we are afforded a Fourier series expansion for $(1 + f(z))$ which we may then rewrite as a function of the complex variable z as follows

$$f(z) = \frac{1}{2}\sum_{k=1}^{s}(b_k + ia_k)\frac{R^k}{z^k} + (b_k - ia_k)\frac{z^k}{R^k}, \;\; \forall z \in \partial D_1, f(\theta) = f(z). \tag{9.27}$$

9.2.3 The Differential Equation for $\phi_1(z)$

Before returning to (9.24) we introduce the following two material parameters

$$\Omega = \frac{(1-\alpha 2)(\alpha_1 - \Gamma(\alpha_1 - 1))}{1 - \alpha_2 + \Gamma(1-\alpha_1)} > 0, \;\; \omega = \frac{\Gamma\alpha_2 - \alpha_2 + 1}{1 - \alpha_2 + \Gamma(1-\alpha_1)} > 0. \tag{9.28}$$

Using (9.26,9.27,9.28) we may rewrite (9.24) as

$$(1+f(z))\phi_1'(z)+\left[\frac{\omega(\alpha_1-\Gamma(\alpha_1-1)}{\Gamma}\frac{\delta_0}{z}-\frac{2\Omega(1+f(z))}{z}\right]\phi_1(z)+$$
$$(1+f(z))\left[\Omega X_1-\frac{iBz^2}{R^2}\frac{\omega}{\Gamma\alpha_2-\alpha_2+1}-\frac{\omega\beta_1}{X_1}\right]-\frac{\omega(\alpha_1-\Gamma(\alpha_1-1))}{\Gamma}\frac{\delta_0}{2}X_1+$$
$$\frac{\beta_1}{2X_1}\frac{(1-\Gamma)\omega\delta_0}{\Gamma}=-(1+f(z))\overline{\phi_1'}(R^2/z)+$$
$$\left[\frac{2\Omega(1+f(z))z}{R^2}-\frac{\omega(\alpha_1-\Gamma(\alpha_1-1))}{\Gamma}\frac{\delta_0 z}{R^2}\right]\overline{\phi_1}(R^2/z)-$$
$$(1+f(z))\left[\Omega\overline{X_1}-\frac{\omega\beta_1}{\overline{X_1}}+\frac{i\overline{B}R^2}{z^2}\frac{\omega}{\Gamma\alpha_2-\alpha_2+1}\right]+\frac{\omega(\alpha_1-\Gamma(\alpha_1-1))}{\Gamma}\frac{\delta_0}{2}\overline{X_1}-$$
$$\frac{\beta_1}{2\overline{X_1}}\frac{(1-\Gamma)\delta_0\omega}{\Gamma},\ \ z\in\partial D_1. \quad (9.29)$$

The left hand side of (9.29) is analytic in D_1 and the right hand side is analytic in D_2. Using the technique of analytic continuation we may construct an entire function by studying the behavior of the left hand and right hand side of (9.29) as $|z|=0$, and $|z|\to\infty$, respectively. Beginning with the left hand side we allow $|z|\to 0$ and recover the following singular terms

$$\sum_{j=1}^{k} jX_j z^{j-1}\sum_{k=1}^{s}\frac{b_k+ia_k}{2}\frac{R^k}{z^k}-2\Omega\sum_{j=1}^{k}X_j z^{j-1}\sum_{k=1}^{s}\frac{b_k+ia_k}{2}\frac{R^k}{z^k}+$$
$$\left[\Omega X_1-\frac{\omega\beta_1}{X_1}\right]\sum_{k=1}^{s}\frac{b_k+ia_k}{2}\frac{R^k}{z^k}-\frac{iB\omega}{\Gamma\alpha_2-\alpha_2+1}\sum_{k=3}^{s}\frac{b_k+ia_k}{2}\frac{R^{k-2}}{z^{k-2}},\ \ |z|\to 0. \quad (9.30)$$

Proceeding to the right hand side of (9.29), we find the following asymptotic and singular behavior as $|z|\to\infty$

$$\overline{X_1}(\Omega-1)-\sum_{k=1}^{k} j\overline{X_j}\left(\frac{R^2}{z}\right)^{j-1}\sum_{k=1}^{s}\frac{b_k-ia_k}{2}\frac{z^k}{R^k}-\sum_{k=1}^{s}(k+1)\overline{X_{k+1}}\frac{b_k-ia_k}{2}R^k+$$
$$2\Omega\left[\sum_{j=1}^{k}\overline{X_j}\left(\frac{R^2}{z}\right)^{j-1}\sum_{k=1}^{s}\frac{b_k-ia_k}{2}\frac{z^k}{R^k}+\sum_{k=1}^{s}\overline{X_{k+1}}\frac{b_k-ia_k}{2}R^k\right]-\frac{\omega(\alpha_1-\Gamma(\alpha_1-1))}{\Gamma}\frac{\delta_0}{2}\overline{X_1}+$$
$$\frac{\omega\beta_1}{\overline{X_1}}-\left[\Omega\overline{X_1}-\frac{\omega\beta_1}{\overline{X_1}}\right]\sum_{k=1}^{s}\frac{b_k-ia_k}{2}\frac{z^k}{R^k}-\frac{i\overline{B}\omega}{\Gamma\alpha_2-\alpha_2+1}\sum_{k=2}^{s}\frac{b_k-ia_k}{2}\frac{z^{k-2}}{R^{k-2}}-$$
$$\frac{\beta_1}{\overline{X_1}}\frac{(1-\Gamma)\delta_0\omega}{2\Gamma},\quad |z|\to\infty. \quad (9.31)$$

The sum of (9.30) and (9.31) is defined by $L(z)$

$$L(z)=\sum_{j=1}^{k} jX_jz^{j-1}\sum_{k=1}^{s}\frac{b_k+ia_k}{2}\frac{R^k}{z^k}-2\Omega\sum_{j=1}^{k}X_jz^{j-1}\sum_{k=1}^{s}\frac{b_k+ia_k}{2}\frac{R^k}{z^k}+$$
$$\left[\Omega X_1-\frac{\omega\beta_1}{X_1}\right]\sum_{k=1}^{s}\frac{b_k+ia_k}{2}\frac{R^k}{z^k}-\frac{iB\omega}{\Gamma\alpha_2-\alpha_2+1}\sum_{k=3}^{s}\frac{b_k+ia_k}{2}\frac{R^{k-2}}{z^{k-2}}+$$
$$\overline{X_1}(\Omega-1)-\sum_{k=1}^{k} j\overline{X_j}\left(\frac{R^2}{z}\right)^{j-1}\sum_{k=1}^{s}\frac{b_k-ia_k}{2}\frac{z^k}{R^k}-\sum_{k=1}^{s}(k+1)\overline{X_{k+1}}\frac{b_k-ia_k}{2}R^k+$$
$$2\Omega\left[\sum_{j=1}^{k}\overline{X_j}\left(\frac{R^2}{z}\right)^{j-1}\sum_{k=1}^{s}\frac{b_k-ia_k}{2}\frac{z^k}{R^k}+\sum_{k=1}^{s}\overline{X_{k+1}}\frac{b_k-ia_k}{2}R^k\right]-\frac{\omega(\alpha_1-\Gamma(\alpha_1-1))}{\Gamma}\frac{\delta_0}{2}\overline{X_1}+$$
$$\frac{\omega\beta_1}{\overline{X_1}}-\left[\Omega\overline{X_1}-\frac{\omega\beta_1}{\overline{X_1}}\right]\sum_{k=1}^{s}\frac{b_k-ia_k}{2}\frac{z^k}{R^k}-\frac{i\overline{B}\omega}{\Gamma\alpha_2-\alpha_2+1}\sum_{k=2}^{s}\frac{b_k-ia_k}{2}\frac{z^{k-2}}{R^{k-2}}-$$
$$\frac{\beta_1}{\overline{X_1}}\frac{(1-\Gamma)\delta_0\omega}{2\Gamma}\quad , \quad (9.32)$$

such that by subtracting $L(z)$ from both the left hand side and right hand side of (9.29) we obtain the following entire function

$$E(z)=\begin{cases}(1+f(z))\phi_1'(z)+\left[\frac{\omega(\alpha_1-\Gamma(\alpha_1-1)}{\Gamma}\frac{\delta_0}{z}-\frac{2\Omega(1+f(z))}{z}\right]\phi_1(z)+\\ (1+f(z))\left[\Omega X_1-\frac{iBz^2}{R^2}\frac{\omega}{\Gamma\alpha_2-\alpha_2+1}-\frac{\omega\beta_1}{X_1}\right]-\frac{\omega(\alpha_1-\Gamma(\alpha_1-1))}{\Gamma}\frac{\delta_0}{2}X_1+\\ \frac{\beta_1}{2X_1}\frac{(1-\Gamma)\omega\delta_0}{\Gamma}-L(z),\ \ z\in D_1,\\ \\ -(1+f(z))\overline{\phi_1'}(R^2/z)+\\ \left[\frac{2\Omega(1+f(z))z}{R^2}-\frac{\omega(\alpha_1-\Gamma(\alpha_1-1))}{\Gamma}\frac{\delta_0 z}{R^2}\right]\overline{\phi_1}(R^2/z)-\\ (1+f(z))\left[\Omega\overline{X_1}-\frac{\omega\beta_1}{\overline{X_1}}+\frac{i\bar{B}R^2}{z^2}\frac{\omega}{\Gamma\alpha_2-\alpha_2+1}\right]+\frac{\omega(\alpha_1-\Gamma(\alpha_1-1))}{\Gamma}\frac{\delta_0}{2}\overline{X_1}-\\ \frac{\beta_1}{2\overline{X_1}}\frac{(1-\Gamma)\delta_0\omega}{\Gamma}-L(z),\ \ z\in D_2.\end{cases}\tag{9.33}$$

Once again we seek to take advantage of Louisville's theorem whereby it is realized that $E(z)=$ *constant* in (9.33) . Owing to the subtraction of $L(z)$, $E(z)=0$ and we generate the following two equations for $\phi_1(z)$

$$\begin{aligned}&(1+f(z))\phi_1'(z)+\left[\frac{\omega(\alpha_1-\Gamma(\alpha_1-1)}{\Gamma}\frac{\delta_0}{z}-\frac{2\Omega(1+f(z))}{z}\right]\phi_1(z)+\\ &(1+f(z))\left[\Omega X_1-\frac{iBz^2}{R^2}\frac{\omega}{\Gamma\alpha_2-\alpha_2+1}-\frac{\omega\beta_1}{X_1}\right]-\frac{\omega(\alpha_1-\Gamma(\alpha_1-1))}{\Gamma}\frac{\delta_0}{2}X_1+\\ &\qquad\qquad\frac{\beta_1}{2X_1}\frac{(1-\Gamma)\omega\delta_0}{\Gamma}-L(z)=0,\ \ z\in D_1,\end{aligned}\tag{9.34}$$

$$
\begin{aligned}
&-(1+f(z))\overline{\phi_1^{'}}(R^2/z)+\\
&\left[\frac{2\Omega(1+f(z))z}{R^2}-\frac{\omega(\alpha_1-\Gamma(\alpha_1-1))}{\Gamma}\frac{\delta_0 z}{R^2}\right]\overline{\phi_1}(R^2/z)-\\
&(1+f(z))\left[\Omega\overline{X_1}-\frac{\omega\beta_1}{\overline{X_1}}+\frac{i\overline{B}R^2}{z^2}\frac{\omega}{\Gamma\alpha_2-\alpha_2+1}\right]+\frac{\omega(\alpha_1-\Gamma(\alpha_1-1))}{\Gamma}\frac{\delta_0}{2}\overline{X_1}-\\
&\qquad\qquad\frac{\beta_1}{2\overline{X_1}}\frac{(1-\Gamma)\delta_0\omega}{\Gamma}-L(z)=0,\ \ z\in D_2. \quad (9.35)
\end{aligned}
$$

The compatibility requirement between (9.34) and (9.35) is given by setting $|z|\to 0$ in (9.34) and may be written as

$$
L_0=-\overline{L_0}, \tag{9.36}
$$

where

$$
\begin{aligned}
L_0=X_1(1-\Omega)-\frac{\omega\beta_1}{X_1}+\sum_{k=1}^{s}(k+1-2\Omega)X_{k+1}\frac{b_k+ia_k}{2}R^k+\\
\frac{\omega(\alpha_1-\Gamma(\alpha_1-1))}{\Gamma}\frac{\delta_0}{2}X_1+\frac{\beta_1(1-\Gamma)\omega\delta_0}{2\Gamma X_1}-\frac{iB\omega}{\Gamma\alpha_2-\alpha_2+1}\frac{b_2+ia_2}{2}. \quad (9.37)
\end{aligned}
$$

Given equation (9.36), equations (9.34) and (9.35) are equivalent and hence we may use (9.34) to define a simplified differential equation for $\phi_1(z)$ as follows

$$
\phi_1^{'}(z)+\left[\frac{\omega(\alpha_1-\Gamma(\alpha_1-1)}{\Gamma}\frac{\delta_0}{z(1+f(z))}-\frac{2\Omega}{z}\right]\phi_1(z)=P(z),\ \ z\in D_1, \tag{9.38}
$$

where

$$P(z) = \frac{\omega\beta_1}{X_1} - \Omega X_1 + \frac{iBz^2}{R^2}\frac{\omega}{\Gamma\alpha_2 - \alpha_2 + 1} - \frac{\delta_0/2}{1+f(z)}\left[\frac{\beta_1(1-\Gamma)\omega}{\Gamma X_1} - \frac{\omega(\alpha_1 - \Gamma(\alpha_1 - 1))}{\Gamma}X_1\right] + \frac{L(z)}{1+f(z)}. \quad (9.39)$$

Equation (9.38) is a first order ordinary differential equation with variable coefficients which has the following general solution

$$\phi_1(z) = e^{-T(z)}\int_{z_1}^{z} e^{T(z)}P(z)dz + C_0 e^{-T(z)}, \;\; z \in D_1, \quad (9.40)$$

where

$$T(z) = \int\left(\frac{\omega(\alpha_1 - \Gamma(\alpha_1 - 1))}{\Gamma}\frac{\delta_0}{z(1+f(z))} - \frac{2\Omega}{z}\right)dz, \quad (9.41)$$

and z_1 is any point in D_1 and C_0 is an arbitrary constant of integration. In light of the fact that $P(z)$ in (9.40) contains the X_{s+1} coefficients of the power series expansion of $\phi_1(z)$, any solution of (9.40) must satisfy the consistency condition given by

$$X_k = \frac{\phi_1^k(0)}{k!}, \;\; k = 1, 2, ..., s, s+1, \quad (9.42)$$

The derivation of (9.42) proceeds in an identical fashion to the previously derived case of a non-slip interface. We restate it here for convenience. First, we suppose that $\phi_1(z)$ has a Taylor series expansion in D_1 given by

$$\phi_1(z) = \sum_{k=0}^{\infty} Q_k z^k \ , \ Q_k = \frac{\phi_1^k(0)}{k!}. \tag{9.43}$$

Then, by substituting (9.43) into (9.35) and comparing coefficients of negative powers of z we arrive at the following

$$\sum_{j=1}^{k} (j-2\Omega) Q_j z^{j-1} \sum_{k=1}^{s} \frac{(b_k + ia_k)}{2} \frac{R^k}{z^k} = \sum_{j=1}^{k} (j-2\Omega) X_j z^{j-1} \sum_{k=1}^{s} \frac{(b_k + ia_k)}{2} \frac{R^k}{z^k}. \tag{9.44}$$

Careful inspection of (9.44) reveals that when $\Omega \neq 1/2$ (9.42) is true for all s. However, for the case of $\Omega = 1/2$ we see that the first statement of (9.44) will be an identity, which provides no information on the form of the coefficient X_1 and implies (9.42) is not automatically satisfied for $k = 1$. Hence we must impose the additional requirement that

$$X_1 = \phi_1^{'}(0). \tag{9.45}$$

Following the work of Ru (1998), the solution for $\phi_1(z)$ in (9.40) is not holomorphic in the uncut domain D_1 due to the presence of multivalued logarithmic functions from under the integral and from isolated singular points stemming from the zeros of the interface function $(1 + f(z))$. To ensure the holomprphicity of $\phi_1(z)$ the domain must be cut appropriately such that $\phi_1(z)$ both single valued and bounded at all isolated singular points. As a preliminary step towards this end we may, without loss of generality, set $C_0 = 0$ in (9.40), which is justified by the aforementioned boundedness argument. We shall, however, intentionally delay the discussion of the continuity of

$\phi_1(z)$ across the necessary branch cuts until a specific form of the interface function $(1+f(z))$ has been chosen.

9.3 A Specific Class of Inhomogeneous Interface

To illustrate an example we shall consider a specific form of the interface function $\delta(\theta)$ as follows

$$\delta(\theta) = \frac{\delta_0}{1+b_s cos(s\theta)}, \quad \delta_0 > 0, \quad -1 < b_s < 1. \tag{9.46}$$

Upon converting (9.46) into a complex variable form it is seen that there will be singularities in the interface function originating from the roots of the following polynomial of degree $2s$

$$\frac{2}{b_s}\left(\frac{z}{R}\right)^s + \left(\frac{z}{R}\right)^{2s} + 1 = 0. \tag{9.47}$$

Furthermore, it is shown in Schiavone, P., Ru (1997) that of the $2s$ roots of (9.47), s will lie inside D_1 and the remaining s will lie in D_2. Let the s roots inside D_1 be denoted by

$$\rho_1, \rho_2, \rho_3, ..., \rho_s, \tag{9.48}$$

where $\rho^s_{(1,2,...,s)} = \rho^*$ and ρ^* is real and given by

$$\rho^* = \begin{cases} \sqrt{\frac{1}{b_s^2} - 1} - \frac{1}{b_s}, < 0, b_s > 0, \\ -\sqrt{\frac{1}{b_s^2} - 1} - \frac{1}{b_s}, > 0, b_s < 0, \end{cases} \tag{9.49}$$

such that $-1<\rho^*<1$, and the remaining s roots in D_2 are given by $\frac{1}{\rho_1},\frac{1}{\rho_2},...,\frac{1}{\rho_s}$. As a consequence of the above interface definitions we make note of the following

$$-\frac{2}{b_s}=\frac{1+\rho^{*2}}{\rho^*}, \tag{9.50}$$

$$\frac{R\delta_0}{z(1+f(z))}=-\frac{\lambda(\frac{z}{R})^{s-1}}{(\frac{z}{R})^s-\rho^*}+\frac{\lambda(\frac{z}{R})^{s-1}}{(\frac{z}{R})^s-\frac{1}{\rho^*}},$$

$$\lambda=-\delta_0\left(\frac{1+\rho^{*2}}{1-\rho^{*2}}\right)<0,$$

$$\frac{1}{1+f(z)}=\frac{\frac{2}{b_s}(\frac{z}{R})^s}{\left[(\frac{z}{R})^s-\rho^*\right]\left[(\frac{z}{R})^s-\frac{1}{\rho^*}\right]}.$$

Utilizing (9.50) we may express (9.40) as follows

$$\phi_1(z)=(\frac{z}{R})^{2\Omega}\left[(\frac{z}{R})^s-\rho^*\right]^{\frac{\lambda\Omega\eta}{s}}\left[(\frac{z}{R})^s-\frac{1}{\rho^*}\right]^{\frac{-\lambda\Omega\eta}{s}}\int\limits_{R\rho^*}^{z}(\frac{t}{R})^{-2\Omega+1}\left[(\frac{t}{R})^s-\rho^*\right]^{\frac{-\lambda\Omega\eta}{s}}$$
$$\left[(\frac{t}{R})^s-\frac{1}{\rho^*}\right]^{\frac{\lambda\Omega\eta}{s}}\frac{P(t)}{\frac{t}{R}}dt,\ \ z\in D_1, \tag{9.51}$$

$$\eta=\frac{\Gamma\alpha_2-\alpha_2+1}{\Gamma(1-\alpha_2)}>0,$$

where the integration path is taken along the edge of any branch cuts and passing through any of the s integrable branch points. In addition to having previously set $C_0=0$ we also require that

$$\int\limits_{R\rho_1}^{R\rho_k}(\frac{t}{R})^{-2\Omega+1}\left[(\frac{t}{R})^s-\rho^*\right]^{\frac{-\lambda\Omega\eta}{s}}\left[(\frac{t}{R})-\frac{1}{\rho^*}\right]^{\frac{\lambda\Omega\eta}{s}}\frac{P(t)}{\frac{t}{R}}dz=0,\ \ k=2,3,...,s, \tag{9.52}$$

in order to maintain boundedness of $\phi_1(z)$ at any of the potential isolated singular points $R\rho_k, k = 2,3,...,s$ in D_1. Additionally, by taking the difference

$$\phi_1(z^+) - \phi_1(z^-) = 0, \tag{9.53}$$

we may prove that (9.51) is continuous across any of the s branch cuts by noting that, due to the sign change of the exponents in and outside of the integral, any increments in the multivalued logarithmic terms that will arise from inside the integral will be nullified from which (9.53) is easily confirmed. The remaining irregular point to be considered is when $z = 0$. Closer inspection of (9.51) reveals that there are three cases to be considered as $z \to 0$.

9.3.1 Case One: $\Omega > \frac{1}{2}$.

When $\Omega > \frac{1}{2}$ we see from (9.51) that $\phi_1(z) \to 0$ as $z \to 0$. However, in order to ensure the holomorphicity of $\phi_1(z)$ we must ensure that $\phi_1(z)$ is continuous across the branch cut formed from $z = R\rho^*$ along the real axis inside D_1. Closer inspection of (9.51) reveals the presence of an unintegrable singularity at $z = 0$. Hence we must define a new path of integration, L^*, to skirt around a neighborhood of $z = 0$ and set $z = z^*$,where z^* is any particular point on the branch cut from $z = 0$, to compensate for this change. In this way the continuity condition becomes

$$\int_{L^*} (\frac{z^*}{t})^{-2\Omega} \left[\frac{(\frac{z^*}{R})^s - \rho^*}{(\frac{t}{R})^s - \rho^*} \right]^{\frac{-\lambda\Omega\eta}{s}} \left[\frac{(\frac{t}{R})^s - \frac{1}{\rho^*}}{(\frac{z^*}{R})^s - \frac{1}{\rho^*}} \right]^{\frac{\lambda\Omega\eta}{s}} P(t)dt = 0 \ . \tag{9.54}$$

We may then solve for the X_{s+1} unknown coefficients using (9.36,9.54).

9.3.2 Case Two: $\Omega < \frac{1}{2}$.

For this case we shall rewrite (9.51) in the form

$$\frac{\phi_1(z)}{\frac{z}{R}} = (\frac{z}{R})^{2\Omega-1}\left[(\frac{z}{R})^s-\rho^*\right]^{\frac{\lambda\Omega\eta}{s}}\left[(\frac{z}{R})^s-\frac{1}{\rho^*}\right]^{\frac{-\lambda\Omega\eta}{s}}\int\limits_{R\rho_1}^{z}(\frac{t}{R})^{-2\Omega}$$
$$\left[(\frac{t}{R})^s-\rho^*\right]^{\frac{-\lambda\Omega\eta}{s}}\left[(\frac{t}{R})^s-\frac{1}{\rho^*}\right]^{\frac{\lambda\Omega\eta}{s}}P(t)dt,\ \ z\in D_1. \quad (9.55)$$

Given that $X_0 = 0$, the LHS of (9.55) is analytic within D_1. As a consequence, $\frac{\phi_1(z)}{\frac{z}{R}}$ must be bounded at $z = 0$ and since $\Omega < \frac{1}{2}$ this implies that

$$\int\limits_{R\rho_1}^{0}(\frac{t}{R})^{-2\Omega}\left[(\frac{t}{R})^s-\rho^*\right]^{\frac{-\lambda\Omega\eta}{s}}\left[(\frac{t}{R})^s-\frac{1}{\rho^*}\right]^{\frac{\lambda\Omega\eta}{s}}P(t)dt=0,\ \ \Omega<\frac{1}{2}. \quad (9.56)$$

Note that in (9.56) there is a singularity in the integrand owing to the term $(\frac{t}{R})^{-2\Omega}$ for $\Omega < \frac{1}{2}$. Due to the fact that the path of integration in (9.56) lies on the real axis we may treat

$$K(\rho^*,t) = (\frac{t}{R})^{-2\Omega}\left[(\frac{t}{R})^s-\rho^*\right]^{\frac{-\lambda\Omega\eta}{s}}\left[(\frac{t}{R})^s-\frac{1}{\rho^*}\right]^{\frac{\lambda\Omega\eta}{s}}, \quad (9.57)$$

as a weakly singular kernel function and is thus integrable along such a domain (see Constanda (1990)). We may then solve for the X_{s+1} unknown coefficients using (9.36,9.56).

9.3.3 Case Three: $\Omega = \frac{1}{2}$.

In this case from (9.51) we see that $z = 0$ is not a singular point of $\phi_1(z)$ owing to the fact that $P(0) = 0$ which renders the ratio $P(t)/(t/R)$ non-singular. We may then proceed to solve for the X_{s+1} unknown coefficients by recalling relation (9.36) and by evaluating (9.42) as

$$RX_1 = [-\rho^*]^{\frac{\lambda\eta}{2s}} \left[-\frac{1}{\rho^*}\right]^{-\frac{\lambda\eta}{2s}} \int_{R\rho_1}^{0} \left[(\frac{t}{R})^s - \rho^*\right]^{\frac{-\lambda\eta}{2s}} \left[(\frac{t}{R})^s - \frac{1}{\rho^*}\right]^{\frac{\lambda\eta}{2s}} \frac{P(t)}{\frac{t}{R}} dt = 0. \tag{9.58}$$

These two conditions provide the $s+1$ necessary equations to solve for the unknown coefficients.

In summary, we have shown that for the case of an inhomogeneous imperfect sliding interface described by (9.1), the stress inside the inclusion can be determined through (5.28) and (9.51) whereby the unknown coefficients X_{s+1} are determined through (9.36) and one of (9.54,9.56,9.58).

9.4 Example

For ease of analysis in illustrating the method we shall assume that $\Omega = \frac{1}{2}, \lambda = -1, \eta = 2$ and we shall confine ourselves to the case $s = 1$. From these preliminaries we may evaluate (9.51) as

$$\phi_1(z) = \frac{z}{R}\left(\frac{z/R - 1/\rho^*}{z/R - \rho^*}\right)\left[I_1(z)\left(\omega\beta_1\left(\frac{1}{X_1} + \frac{1}{\overline{X_1}}\right) - \frac{1}{2}\left(X_1 + \overline{X_1}\right)\right) + X_2 R I_2(z) + \delta_0(2/b_1)\frac{\omega\beta_1(\alpha_1 - \Gamma(\alpha_1 - 1))}{\Gamma} X_1 I_2(z) + \frac{iB\omega}{\Gamma\alpha_2 - \alpha_2 + 1} I_3(z)\right], \quad z \in D_1, \tag{9.59}$$

where

$$I_1(z) = \int_{R\rho^*}^{z} \frac{t/R}{(t/R - 1/\rho^*)^2} dt, \tag{9.60}$$

$$I_2(z) = \int_{R\rho^*}^{z} \frac{1}{(t/R - 1/\rho^*)^2} dt,$$

$$I_3(z) = \int_{R\rho^*}^{z} \frac{t/R(t/R - \rho^*)}{(t/R - 1/\rho^*)^2} dt, \;\; z \in D_1.$$

The unknown coefficients $X_1, \overline{X_1}, X_2, \overline{X_2}$ are then evaluated from (9.36,9.58) as follows

$$\frac{1}{2}\left(X_1 + \overline{X_1}\right) - \omega\beta_1 \left(\frac{1}{X_1} + \frac{1}{\overline{X_1}}\right) + \frac{b_1}{2} R(X_2 + \overline{X_2}) = \frac{\delta_0}{2}\left[\frac{\omega(\Gamma(\alpha_1 - 1) - \alpha_1)}{\Gamma}(X_1 + \overline{X_1}) + \frac{\beta_1(\Gamma - 1)\omega}{\Gamma}\left(\frac{1}{X_1} + \frac{1}{\overline{X_1}}\right)\right], \tag{9.61}$$

$$\left[\frac{R\rho^{*2}}{\rho^{*2} - 1} + Rln\left(\frac{1}{1 - \rho^{*2}}\right)\right]\left(\omega\beta_1\left(\frac{1}{X_1} + \frac{1}{\overline{X_1}}\right) - \frac{1}{2}(X_1 + \overline{X_1})\right) + X_2 R\left(\frac{R\rho^{*3}}{\rho^{*2} - 1}\right) + \frac{iB\omega}{\Gamma\alpha_2 - \alpha_2 + 1}\left[\frac{R(1 - \rho^{*2})}{\rho^{*2}} ln\left(\frac{1}{1 - \rho^{*2}}\right) + \frac{R\rho^{*2}}{2} - R\right] = 0, \quad -1 < \rho^* < 1. \tag{9.62}$$

Noting that $\frac{1}{2} = 1 - \Omega$ it can be shown that since $\frac{\delta_0}{2} = \frac{nR}{2\mu_2}$, (9.61) is in fact identical to (9.13) in the case where $\rho^* \to 0$.

Remark 4. In general since $\phi_1(z)$ is written in a power series we see that for the present solution where X_1 and X_2 are the only non-zero coefficients, the general Piola stress vector will be non uniform. However, when $\rho^* = 0$ the coefficient $X_2 = 0$ and X_1 is constant and hence the

internal stress field in the inclusion is uniform for a homogeneous imperfect interface which agrees with the results of Wang (2012).

9.5 Results

Having verified the formulation we may now proceed to compare the homogeneous imperfect interface to the inhomogeneous one. For the purpose of this example we will compare the inhomogeneous interface of the form

$$\frac{n(\theta)R}{\mu_2} = \frac{\delta_0}{1+b_1 Cos(\theta)}, \quad \delta_0 = \frac{1-\rho^{*2}}{1+\rho^{*2}}, \quad -1 < b_1 < 1, \tag{9.63}$$

to the homogeneous imperfect interface given by

$$\frac{nR}{\mu_2} = \delta_0. \tag{9.64}$$

Close inspection of the expression given by (6.7) reveals that in the cases of either a uni-axial or bi-axial remote loading, B is in fact purely real. Hence we may prove from (9.48,9.49) that X_1, X_2 must both be purely imaginary and we may solve for them using (9.36,9.48,9.49). As a point of congruency we further assume that in the homogeneous case X_1 must also be purely imaginary and hence we see that, using the average mean stress on the boundary defined by

$$(P_{11}+P_{22})_{2,Avg} = \frac{1}{C_{\partial D_1}} \int\limits_{\partial D_1} 4\mu_2 Im\left[\Gamma(1-\alpha_1)X_1 + \frac{\Gamma\beta_1}{X_1}\right] ds, \tag{9.65}$$

the ratio of the inhomogeneous to homogeneous interfaces will be one to one since X_1 is identical in both interface conditions. To further explore the results we compute the ratio of the mean stresses at $z = R$ given by the relations

$$(P_{11}+P_{22})_{2,Homogeneous} = 4\mu_2 Im\left[\Gamma(1-\alpha_1)(X_1)+\frac{\Gamma\beta_1}{X_1}\right], \tag{9.66}$$

$$(P_{11}+P_{22})_{2,Inhomogeneous} = 4\mu_2 Im\left[\Gamma(1-\alpha_1)(X_1+2X_2 z)+\frac{\Gamma\beta_1}{X_1+2X_2 z}\right], \tag{9.67}$$

from which the following results are observed

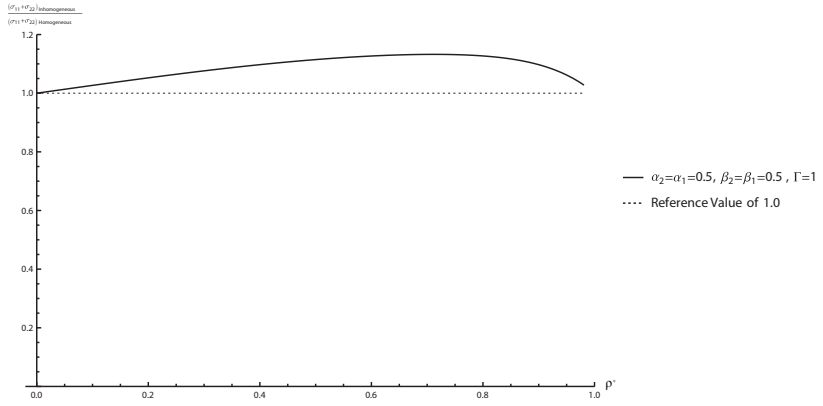

Figure 9.1: Ratio of inhomogeneous to homogeneous mean stress at $z = R$ for the remote loading $P_{11}^{\infty} = 0, P_{22}^{\infty} = 10^3, P_{12}^{\infty} = 0$

From Figure 2 we conclude that the inhomogeneous interface parameter ρ^* has a moderate effect on the mean stress at the point $z = R$ on the boundary ∂D_1, which at its peak reaches an error of 13 percent. In contrast, Ru (1998) observed a relative error of up to 50 percent in the case of a inhomogeneous sliding interface in linear elasticity. The present results indicate that for an inhomogeneous sliding interface the traditional homogeneous interface model is sufficient in predicting the average mean stress on the boundary, and is only off by a small margin when calculating

the mean stress at a point and hence, in most engineering applications the homogeneous interface model is sufficient.

Chapter 10

CONCLUSION

In this thesis rigorous general solutions have been developed for three different types of boundary conditions in the finite deformation of circular inclusions with inhomogeneous imperfect interface conditions in harmonic materials.

In the first problem, a general closed form solution was developed for the case of an imperfect interface characterized by $m(\theta) = n(\theta)$, where the degree of imperfection is the same in both normal and tangential directions, and an example was given for a specific class of inhomogeneous interface from which the corresponding homogeneous results were derived and corroborated by other works. Furthermore, results were presented for the inclusion mean stress ratio as a function of the inhomogeneous interface scaling parameter ρ^*. From these results it was observed that the introduction of an inhomogeneous imperfect interface has a profound effect on the mean stress value in comparison to the homogeneous case (up to 60 percent or more). This signifies, for the first time in finite elasticity, that in circumstances where the inclusion interface may have circumferentially non-homogeneous imperfections, the homogeneously imperfect interface model is insufficient in determining the mean stress.

Following the case of $m(\theta) = n(\theta)$, a general solution was developed for the case of an inhomogeneous imperfect interface with so-called non-slip boundary conditions which is captured by setting $m(\theta) = finite$ and $n(\theta) \to \infty$ in the normal and tangential coordinate directions, re-

spectively. The formulation was validated through studying the analogous case of a homogeneous imperfect interface and corroborating the result with other works. To simulate an example, a new measure of stress approximation on the boundary of the inclusion was introduced and results were presented for the ratio of the inhomogeneous average boundary stress to the homogeneous average boundary stress as a function of the imperfect interface parameter ρ^*. From the results it was observed that the imperfect interface parameter ρ^* has a significant impact on the average boundary stress when compared to the homogeneous case (up to 80 percent) and, when the case of a non-slip boundary is warranted, the homogeneous imperfect interface model is insufficient.

Finally, a general solution was developed for the case of an inhomogeneous imperfect sliding interface characterized by the setting the imperfect interface parameters $m(\theta), n(\theta)$ to $m(\theta) \to \infty, n(\theta) = finite$ in the normal and tangential coordinate directions, respectively. The formulation was validated by referring to the solution of a homogeneous imperfect interface under the same sliding interface boundary conditions and subsequently results were presented for the mean stress at a point along the inclusion matrix boundary curve. From these results it was observed that the average mean stress along the boundary was identical between the inhomogeneous and homogeneous interface conditions and that there was a maximum error of 13 percent when comparing the mean stress at a point between inhomogeneous and homogeneous interfaces. It is thus concluded that for most engineering design problems the homogeneous interface model is sufficient, up to an error in the mean stress of 13 percent, in predicting the behavior of a circular inclusion with imperfect interface in finite deformation.

In summation, the work of this thesis indicates that the study of inhomogeneous imperfect interface conditions provides meaningful insights into the elastic response of a harmonic inclusion and hence should be considered when building a model for composite materials containing inclusions with imperfect interfacial conditions.

Chapter 11

FUTURE WORK

While there are a myriad of different unsolved problems in the study of harmonic inclusions in finite elasticity, there are three immediate and natural extensions to the present work, namely the case of $s = 2$ for all three different boundary conditions described in chapters 7,8, and 9. These are important and interesting problems as the case of $s = 2$ describes a completely different form of the interface function $f(z)$ and further complicates the steps required to solve for the coefficients of $\phi_k(z)$.

Another natural extension to the present work could be the study of a new imperfect interface model for finite elasticity. It was pointed out in chapter 2 of this thesis that no derivation currently exists for an imperfect interface model in finite deformation of harmonic materials, and that the present work relied on the use of the linear spring model derived by Bigoni, D. , Serkov, S.K. , Valentini, M., Movchan (1998). While use of this model is warranted for harmonic materials (due to a proportionality in the principal Piola stress and corresponding principal stretch) it is believed that more accurate results could be achieved by derivation of a new imperfect interface model.

An additional avenue of further development could also be found in the study of three phase inclusions with interphase layers of non homogeneous thickness. This model has been studied in the linear analog however no derivation currently exists in finite elasticity and it is thought that this work would provide many interesting results, as it did in the linear regime where it was concluded

that the introduction of an interphase of non-uniform thickness results in a non-uniform inclusion stress.

Of general interest is also the development of a thermo-elastic model for harmonic materials in finite elasticity. Such an endeavor would involved developing a modified deformation gradient containing a purely mechanical component and a thermo-mechanical component. Such a decomposition does exist however it has never been applied to the study of harmonic material inclusion problems.

Finally, a rather ambitions, though plausible, avenue of study could be the extension of the harmonic material model into three dimensional problems. While the fundamental basis for this analysis has already been addressed by J. Fritz in terms of real variables, it is possible to extend the complex plane into a three dimensional space using what are known as quaternions. For simple inclusion geometries it may be possible to obtain relatively concise analytic solutions for the harmonic inclusion problem using a three dimensional complex variable framework.

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Chapter 12

Appendix

12.1 Derivation of mean stress at a point

Beginning with the definitions

$$\sigma_{11} + i\sigma_{21} = -i\chi_{,2}(z), \tag{12.1}$$

$$\sigma_{22} - i\sigma_{12} = \chi_{,1}(z), \tag{12.2}$$

we may show that

$$\sigma_{11} + \sigma_{22} = \frac{d\chi(z)}{dz} + \overline{\frac{d\chi(z)}{dz}}. \tag{12.3}$$

Recalling that

$$\chi(z) = 2i\mu \left[(\alpha - 1)\phi(z) + i\overline{\psi(z)} + \frac{\beta z}{\overline{\phi'(z)}} \right], \tag{12.4}$$

we may rewrite (12.3) as

$$\sigma_{11} + \sigma_{22} = 2i\mu \left[(\alpha - 1)\phi'(z) + \frac{\beta}{\overline{\phi'(z)}} \right] - 2i\mu \left[(\alpha - 1)\overline{\phi'(z)} + \frac{\beta}{\phi'(z)} \right]. \tag{12.5}$$

Noting that

$$2iIm[\phi'(z)] = \phi'(z) - \overline{\phi'(z)}, \;\; 2iIm[\frac{1}{\phi'(z)}] = \frac{1}{\phi'(z)} - \frac{1}{\overline{\phi'(z)}}, \tag{12.6}$$

equation (12.5) is rewritten as

$$\sigma_{11}+\sigma_{22}=4\mu Im\left[(1-\alpha)\phi^{'}(z)+\frac{\beta}{\phi^{'}(z)}\right]. \tag{12.7}$$

12.2 Derivation of average mean stress on the boundary

Directly integrating (12.7) results in some less than ideal integrals and hence we seek to work around this obstacle by looking back to (12.3) and inserting the relations given through the continuity of tractions boundary condition in (6.15). This yields the following expression for the mean stress

$$\begin{aligned}(\sigma_{11}+\sigma_{22})_2 &= 2i\mu_2\left[(\alpha_2-1)\phi_2^{'}(z)+\Gamma(\alpha_1-1)\phi_1^{'}(z)+(1-\alpha_2)A+\frac{\Gamma\beta_1}{\overline{X_1}}-\frac{i\overline{B}R^2}{z^2}\right]\\ &-2i\mu_2\left[(\alpha_2-1)\overline{\phi_2^{'}(z)}+\Gamma(\alpha_1-1)\overline{\phi_1^{'}(z)}+(1-\alpha_2)\overline{A}+\frac{\Gamma\beta_1}{X_1}+\frac{iBz^2}{R^2}\right],\ |z|=R. \end{aligned}\tag{12.8}$$

Inserting the power series definitions for $\phi_1(z)$ and $\phi_2(z)$ yields

$$\begin{aligned}(\sigma_{11}+\sigma_{22})_2 &= 2i\mu_2\left[(\alpha_2-1)(A-\sum_{k=1}^{\infty}A_kz^{-k-1})+\Gamma(\alpha_1-1)(\sum_{k=1}^{\infty}kX_kz^{k-1})\right.\\ &\left.+(1-\alpha_2)A+\frac{\Gamma\beta_1}{\overline{X_1}}-\frac{i\overline{B}R^2}{z^2}\right]-2i\mu_2\left[(\alpha_2-1)(\overline{A}-\sum_{k=1}^{\infty}\overline{A_k}(R^2/z)^{-k-1})+\right.\\ &\left.\Gamma(\alpha_1-1)(\sum_{k=1}^{\infty}k\overline{X_k}(R^2/z)^{k-1})+(1-\alpha_2)\overline{A}+\frac{\Gamma\beta_1}{X_1}+\frac{iBz^2}{R^2}\right],\ |z|=R. \end{aligned}\tag{12.9}$$

Canceling out the remote loading terms in the above leaves us with

$$\begin{aligned}(\sigma_{11}+\sigma_{22})_2 &= 2i\mu_2\left[(\alpha_2-1)(-\sum_{k=1}^{\infty}A_kz^{-k-1})+\Gamma(\alpha_1-1)(\sum_{k=1}^{\infty}kX_kz^{k-1})\right.\\ &\left.+\frac{\Gamma\beta_1}{\overline{X_1}}-\frac{i\overline{B}R^2}{z^2}\right]-2i\mu_2\left[(\alpha_2-1)(-\sum_{k=1}^{\infty}\overline{A_k}(R^2/z)^{-k-1})+\right.\\ &\left.\Gamma(\alpha_1-1)(\sum_{k=1}^{\infty}k\overline{X_k}(R^2/z)^{k-1})+\frac{\Gamma\beta_1}{X_1}+\frac{iBz^2}{R^2}\right],\ |z|=R. \end{aligned}\tag{12.10}$$

Equation (12.10) may then be re-stated as the imaginary part of the RHS as follows

$$(\sigma_{11}+\sigma_{22})_2 = 4\mu_2 Im\left[(1-\alpha_2)(\sum_{k=1}^{\infty} A_k z^{-k-1}) + \Gamma(1-\alpha_1)(\sum_{k=1}^{\infty} kX_k z^{k-1}) + \frac{\Gamma\beta_1}{X_1} + \frac{iBz^2}{R^2}\right], \quad |z| = R. \quad (12.11)$$

Integrating (12.11) with respect to the polar angle θ and dividing by the inclusion circumference yields

$$(\sigma_{11}+\sigma_{22})_2 = \frac{1}{C_{\partial D_1}} \int_{\partial D_1} 4\mu_2 Im\left[\Gamma(1-\alpha_1)X_1 + \frac{\Gamma\beta_1}{X_1}\right] ds, \quad |z| = R. \quad (12.12)$$

12.3 Homogeneous coefficient calculation

Here we provide an example calculation for the derivation of the homogeneous imperfect interface coefficient when the imperfect interface parameter $\rho^* \to 0$ for the case of $m(\theta) = n(\theta)$. We begin at equation (7.37) which we restate here for convenience

$$\frac{\delta_0 A}{1-\alpha_1}\left[\frac{R\rho^{*2}}{1-\rho^{*2}} - R\ln\left(\frac{1}{1-\rho^{*2}}\right)\right] = X_1\frac{R\rho^{*4}}{1-\rho^{*4}} + \frac{\beta_1}{\overline{X_1}}\left[\left(\frac{1-(1-\Gamma)\delta_0}{1-\alpha_1}\right)\times \left(-\frac{R\rho^{*2}}{1-\rho^{*2}} + R\ln\left(\frac{1}{1-\rho^{*2}}\right)\right) - \frac{1}{1-\alpha_1}\left(\frac{-R\rho^{*2} - R\rho^{*1}(1-\rho^{*2})}{1-\rho^{*4}} + \frac{2R}{1+\rho^{*2}}\ln\left(\frac{1}{1-\rho^{*2}}\right)\right)\right]. \quad (12.13)$$

Conveniently, the logarithmic terms in (12.13) may be expanded in the following manner

$$ln\left(\frac{1}{1-\rho^{*2}}\right) = ln(1) - ln(1-\rho^{*2}) = \sum_{k=1}^{\infty}\frac{(-1)^k(-\rho^{*2})^k}{K} = \rho^{*2} + \rho^{*4}/2 + \rho^{*6}/3.., (-1 < \rho^* < 1). \quad (12.14)$$

Substituting the above into (12.13) gives

$$\frac{\delta_0 A}{1-\alpha_1}\left[\frac{R\rho^{*2}}{1-\rho^{*2}} - R(\rho^{*2}+\rho^{*4}/2+\rho^{*6}/3...)\right] = X_1\frac{R\rho^{*4}}{1-\rho^{*4}} + \frac{\beta_1}{\overline{X_1}}\left[\left(\frac{1-(1-\Gamma)\delta_0}{1-\alpha_1}\right)\times \left(-\frac{R\rho^{*2}}{1-\rho^{*2}} + R(\rho^{*2}+\rho^{*4}/2+\rho^{*6}/3...)\right) - \frac{1}{1-\alpha_1}\left(\frac{-R\rho^{*2} - R\rho^{*1}(1-\rho^{*2})}{1-\rho^{*4}} + \frac{2R}{1+\rho^{*2}}(\rho^{*2}+\rho^{*4}/2+\rho^{*6}/3...)\right)\right]. \quad (12.15)$$

Next, multiplying both sides of (12.16) by $(1+\rho^{*2})(1-\rho^{*2})$ yields

$$\frac{\delta_0 A}{1-\alpha_1}\left[R\rho^{*2}(1+\rho^{*2}) - R(1-\rho^{*4})(\rho^{*2}+\rho^{*4}/2+\rho^{*6}/3...)\right] = X_1 R\rho^{*4} + \frac{\beta_1}{\overline{X_1}}\left[\left(\frac{1-(1-\Gamma)\delta_0}{1-\alpha_1}\right)\left(-R\rho^{*2}(1+\rho^{*2}) + R(\rho^{*2}+\rho^{*4}/2+\rho^{*6}/3...)\right) - \frac{1}{1-\alpha_1}\left(-R\rho^{*2} - R\rho^{*1}(1-\rho^{*2}) + 2R(1-\rho^{*2})(\rho^{*2}+\rho^{*4}/2+\rho^{*6}/3...)\right)\right]. \quad (12.16)$$

Canceling like terms we may then set $\rho^* \to 0$ from which we are left with

$$2X_1 + \frac{\beta_1}{1-\alpha_1}\left[\frac{\delta_0(1-\Gamma)}{\overline{X_1}} - \frac{1}{\overline{\overline{X_1}}}\right] = \frac{\delta_0 A}{1-\alpha_1}, \quad (12.17)$$

which is identical to (7.40).

12.4 Interface parameter quotient calculation

In this section we show the derivation of relation (7.29), which is subsequently used in all three boundary conditions (chapters 7,8,9). Beginning with

$$\frac{R\delta_0}{z[1+f(z)]}, \quad (12.18)$$

we expand the denominator $1+f(z)$ as follows

$$\frac{R\delta_0}{z[1+f(z)]} = \frac{(R/z)\delta_0}{1+\frac{b_s}{2}(\frac{z}{R})^s + \frac{b_s}{2}(\frac{R}{z})^s}. \quad (12.19)$$

Next, we multiply the numerator and denominator by $(z/R)^s(2/b_s)$ and factor the resulting polynomial in the denominator

$$\frac{R\delta_0}{z[1+f(z)]} = \frac{(z/R)^{s-1}(\frac{2}{b_s})\delta_0}{\left[(\frac{z}{R})^s - \rho^*\right]\left[(\frac{z}{R})^s - \frac{1}{\rho^*}\right]}, \tag{12.20}$$

and we may expand the result via partial fractions as follows

$$\frac{(z/R)^{s-1}(\frac{2}{b_s})\delta_0}{\left[(\frac{z}{R})^s - \rho^*\right]\left[(\frac{z}{R})^s - \frac{1}{\rho^*}\right]} = \frac{D}{\left[(\frac{z}{R})^s - \rho^*\right]} + \frac{M}{\left[(\frac{z}{R})^s - \frac{1}{\rho^*}\right]}. \tag{12.21}$$

Simple analysis of (12.21) reveals that A and B are given by

$$D = (z/R)^{s-1}\delta_0\frac{(1+\rho^{*2})}{(1-\rho^{*2})}, \tag{12.22}$$

$$M = -(z/R)^{s-1}\delta_0\frac{(1+\rho^{*2})}{(1-\rho^{*2})}. \tag{12.23}$$

Hence, if we define $\lambda = -\delta_0\left(\frac{1+\rho^{*2}}{1-\rho^{*2}}\right)$ we arrive at

$$\frac{R\delta_0}{z[1+f(z)]} = \frac{-\lambda\left(\frac{z}{R}\right)^{s-1}}{\left[\left(\frac{z}{R}\right)^s - \rho^*\right]} + \frac{\lambda\left(\frac{z}{R}\right)^{s-1}}{\left[\left(\frac{z}{R}\right)^s - \frac{1}{\rho^*}\right]}. \tag{12.24}$$

Similarly, we may rewrite $1/(1+f(z))$ by multiplying the numerator and denominator by $(z/R)^s(2/b_s)$ which yields

$$\frac{1}{1+f(z)} = \frac{\left(\frac{2}{b_s}\right)\left(\frac{z}{R}\right)^s}{\left[\left(\frac{z}{R}\right)^s - \rho^*\right]\left[\left(\frac{z}{R}\right)^s - \frac{1}{\rho^*}\right]}. \tag{12.25}$$

12.5 Proof that $Re[X_1] = 0$ in the sliding interface case

Taking the real part of (9.62), in the case where B is purely real (biaxial loading only), yields the following equation

$$(X_2+\overline{X_2})R\left(\frac{R\rho^{*3}}{\rho^{*2}-1}\right)=2\left[\frac{R\rho^{*2}}{\rho^{*2}-1}+Rln\left(\frac{1}{1-\rho^{*2}}\right)\right]\left(\omega\beta_1\left(\frac{1}{X_1}+\frac{1}{\overline{X_1}}\right)-\frac{1}{2}(X_1+\overline{X_1})\right). \tag{12.26}$$

Then, we may insert (12.26) into (9.61) and , noting that

$$\frac{1}{X_1}+\frac{1}{\overline{X_1}}=\frac{X_1+\overline{X_1}}{X_1\overline{X_1}},\ \ |X_1|^2\neq 0, \tag{12.27}$$

we may then rewrite (9.61) in terms of only a non-zero coefficient multiplied by $X_1+\overline{X_1}$. Hence we may show that $X_1+\overline{X_1}=0$ and, in virtue of (12.26) we see that $X_2+\overline{X_2}=0$. Finally, using (9.61) we may solve for X_2 and then X_1 from (9.22).

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