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# Theoretical and Computational Analysis and Comparison of Stochastic Models of Energy and Interest Rate Markets

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# Theoretical and Computational Analysis and Comparison of Stochastic Models of Energy and Interest Rate Markets

by

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# Abstract

The present work summarizes information about Interest Rate Market and Energy Market. A mathematical framework is the main aspect considered here.

Different kinds of stochastic models are described for both markets. In addition, an introduction is given to a very interesting approach developed by Hinz et al. in 2005. The idea is in an application of interest rate market techniques to the energy market. Comparing this approach to the standard stochastic model we obtain the connection between two different sets of parameters. This significantly extends the possibilities of estimating the crucial parameters of the model.

In the final chapter we explore two different ways of using Heath-Jarrow-Morton framework. Firstly, we estimate parameters of the model using a specific form of volatility function and sweet crude oil forward prices. Finally, we examine the general form of volatility function of natural gas forward prices. Then using Principal Component Analysis we obtain the main principal components which allow us to reproduce prices of forward contracts.

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# Table of Contents

# List of Tables

# List of Figures and Illustrations

# List of Symbols, Abbreviations and Nomenclature

| Symbol | Definition |
| --- | --- |
| U of C | University of Calgary |

# Chapter 1

## *Interest Rate Market Models*

Interest rate market is an influential component of a general financial market.

By most people a term '*interest rate*' could be associated with savings account in bank only. In reality, however, this concept is much wider. Besides money market rates, it can refer to U.S.Treasury yields, zero-coupon yields, forward rates, swap rates. [25]

Let us see what this market consists of:

- **Money Market**

  Money market is a place for short-term (no more than 13 months) borrowing and landing. Financial instruments which are traded are called 'papers'. They have a very high liquidity and could be easily transformed into cash.

  Money market could be also associated with a savings account for institution. We need to mention however that in USA a term 'money market account' refers to a kind of savings account for individuals but with some special conditions. The difference is in compounding, which for individuals is not continuous. So under 'money market' in fixed-income modelling we assume a savings account for institution with *continuous* compounding [25].

  Here is a list of common instruments of money market:

  - *Certificate of deposit* - deposit for a fixed time period at a corresponding (to a term of deposit) rate of interest, which is called 'a term rate'.
  - *Treasury bills (also called T-bills)* - Short-term debt obligations with specific maturities (four weeks, three months, or six months),

issued by the government of United States . They have no interest rate, but are sold on a discounted basis, which for a holder of T-bill means to pay less and get more at the maturity.

– *Repurchase agreements (**repos**)* - A contract between an owner of securities and another company to sell this securities and buy them back later at a higher price [15].

– *Commercial paper* - Short-term corporate debt with maturity varying from 30 days to 270 days [22].

– *Interbank loans* - loans from one bank to another even across international boarders. The most popular rates for this type of loans are (See e.g. [17]):

  * London Interbank Offered Rate (LIBOR) - an average rate at which leading banks in UK lend money to each other for overnight;

  * Federal funds rate - a rate at which banks in United States with excess reserves lend to banks with temporarily insufficient reserves;

  * EURIBOR - a rate at which important European banks give to each other money denominated in Euros.

– *Eurodollar deposits* - Deposits in U.S. dollars placed at a bank outside of the United States (or in U.S. International Banking Facilities). Interest rate for this kind of money market instrument is LIBOR.

– *Federal agency short-term securities* - Short-term securities provided by eight major agencies, sponsored by U.S. government.

- **Bond Market**

  **A bond** is the main instrument of a **bond market**. One of the meanings of the word *bond* is a *guarantee*, which fully describes a principle by which this instrument works. A bond is a financial contract which guarantees to its buyer to receive a predefined amount of money at an expiry of this contract. The amount of money is called a ***principal***. In some contracts besides a principal buyer periodically receives some interest payments, which are called ***coupons***. In cases when there are no coupons paid we deal with ***zero-coupon bonds***. There are several different types of bonds, but in this work we deal mainly with the latter type.

  There are five individual bond markets, defined by SIFMA (the Securities Industry and Financial Markets Association):

  - **Corporate**
  - **Government and agency**
  - **Municipal**
  - **Mortgage backed, asset backed, and collateralized debt obligation**
  - **Funding**

  Bond market's players are:

  - Institutional investors
  - Governments
  - Traders
  - Individuals

- **Stock Market**

  Stock market (or equity market) is a public network of financial transactions for the trading of company stocks (shares) and derivatives at an agreed price; these are securities listed on a stock exchange as well as those only traded privately. Participants of this market are individual retail investors, institutional investors (for example: mutual funds, banks, insurance companies and hedge funds) and also publicly traded corporations trading in their own shares.

- **Currency Market** (Foreign currency exchange, forex, FX)

  Currency market is a global market for currency exchange. Market participants are: commercial companies, central banks, foreign exchange fixing, hedge funds, investment management firms, retail foreign exchange traders, non-bank foreign exchange companies and money transfer/remittance companies.

- **Retail Financial Institutions and banks**

All of the above mentioned markets and their instruments are based on a key concept of ***'interest rate'***. This implies that knowledge about the behavior of interest rates can give a lot to a person who wants to borrow/lend money.

Interest rate models are used to predict future interest rate dynamics. Interest rate models are mathematical models, which were defined in [26] as an analogy to physical models (for example, plane or train models):

> Mathematical models perform much the same function. They represent, in mathematical equation form, some activity or object. We use them because we can apply some general theory to learn more about the activity or object described by the equations, or to predict the behaviour of the activity or object.

We will consider particular types of Interest Rate Models later on. Every model is characterized by specific parameters, which are needed to be found. Here, historical data can be used to calibrate model's parameters. Historical data can be obtained from many sources.

There are some regulations and definiteness in financial market. For example, LIBOR rates are calculated only by an official LIBOR rates provider - "Thomson Reuters Corporation" - on behalf of the British Banker's Association (that is why rates are called **bbalibor**). The following information was taken from BBA's website bbalibor.com:

"BBA LIBOR is a benchmark giving an indication of the average rate at which a leading bank can obtain unsecured funding in the London interbank market for a given period, in a given currency".

There are 10 currencies with 15 maturities for each. The following diagram represents graphs of historical BBA LIBOR rates. Rates were taken monthly for a period of time starting from September 1989 and ending in January 2012 for Canadian dollars. Each curve corresponds to different terms of borrowing/landing (1 month, 2 months, 3 months, 6 months,12 months).

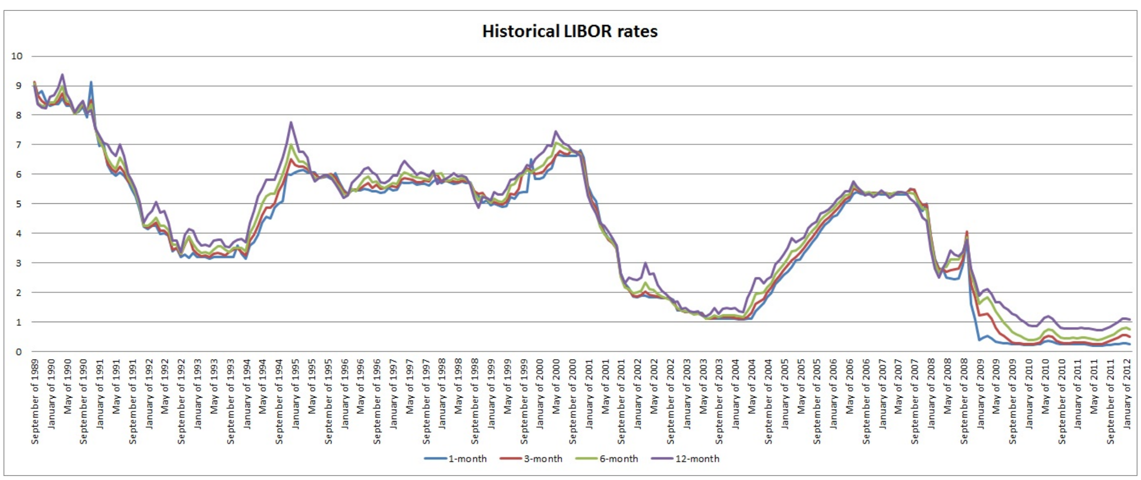



Figure 1.1: Historical LIBOR rates for different maturities

It appears that in a long term LIBOR rates follow a stochastic process.

LIBOR rates are affected by macroeconomic factors. We can observe that through a prism of Bank of England rates, since these latter rates reflect a state of global economics. We see on Figure 1.2 that LIBOR rates are co-vibrating with official Bank of England rates.

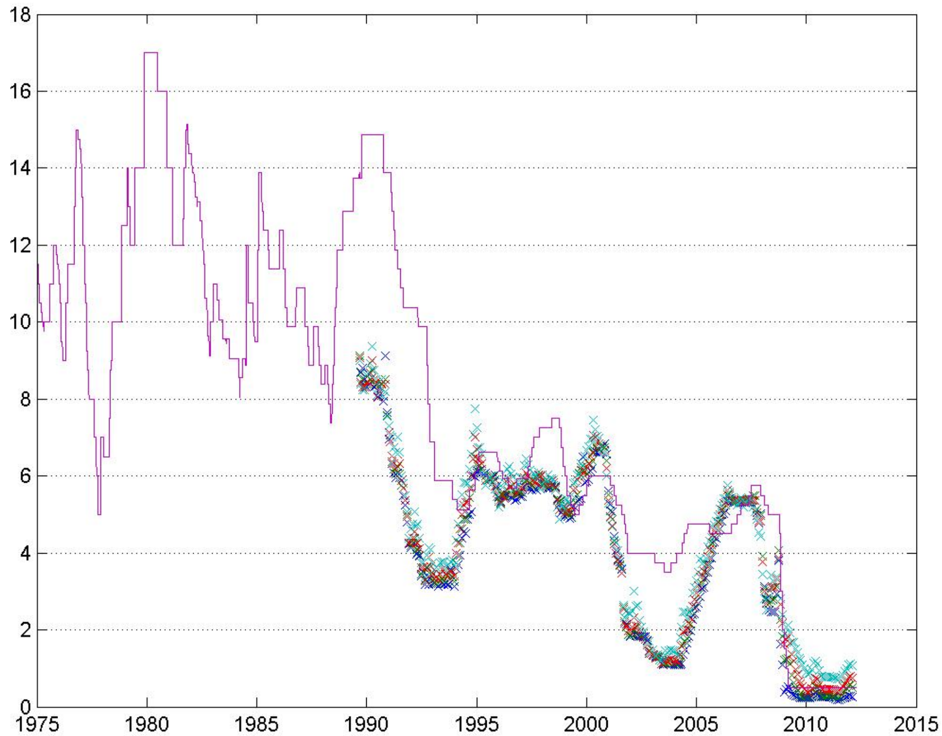



Figure 1.2: Historical Bank of England rates and LIBOR rates for different maturities

As was said in [17] national banks can manage a state of the economy by selling or purchasing government debt. It has been happened until recently. And nowadays central banks of many economically developed countries operate by the use of repurchase agreements in such cases. Another way to help banking system is a direct lending to financial institutions by national banks. Rates are predetermined and as was seen from 1.2 they have 2 interesting peculiarities: national bank's rates change not so frequently and are less attractive than rates available in money market.

The following demonstration will expose another interesting property of interest rate process. Let's see how BBA LIBOR rates behave during some short term.

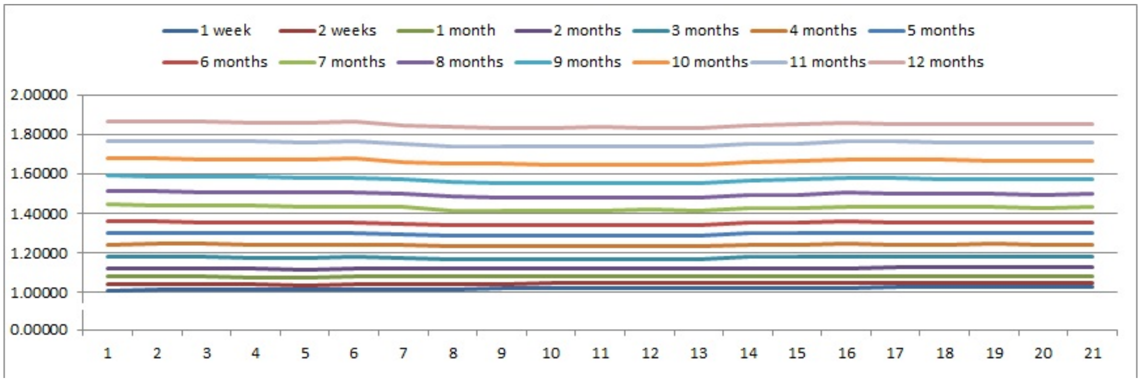



Figure 1.3: Historical LIBOR rates for July 2011

On Figure 1.3 there are graphs of BBA LIBOR rates for every day of July of 2011 for different maturities.

Here we see that on such a short term of observation BBA LIBOR remain *almost* constant. It has a very small variation around some certain value, which demonstrates that the process of LIBOR rates is a mean-reverting stochastic process.[1]. At first glance it seems that interest rate dynamics behaves chaotically. Indeed, it could be modeled better or worse by a variety of stochastic models. Another problem is how to choose (or to create) the best model. To determine that we have to take into account features and a habitat of the rate.

## 1.1 Short Rate Models

In this section we will consider types of models which describe short rate dynamics. As was said in [25]:

> Short-rate models hold a special place in fixed-income modelling: they are the first generation of interest-rate models, and some of them still play active roles in today's applications.

The short rate, usually written $r_t$, is the (annualized) interest rate at which an entity can borrow money for an infinitesimally short period of time from time $t$. So the process of

[1] rates are low at this time of observation, so volatility is low too. It would be interesting to see Year 2007 for example, but BBA LIBOR daily rates are given only for the last 6 months

interest rates can be considered as a continuous-time stochastic process.

Let us define a *filtered* probability space:

$$(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathbb{P}), \tag{1.1}$$

where $\Omega$ - a space of possible outcomes (applying to financial market it is some market condition), $\mathcal{F}$ - $\sigma$-algebra of subsets of $\Omega$ (group of events, observed on the market), $\mathbb{P}$ - probability measure on $\mathcal{F}$, $\mathbb{F} = (\mathcal{F}_n)_{n\geq 0}$ - filtration (a stream of $\sigma$-algebras such that:

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset ... \subset \mathcal{F}_n \subset ... \subset \mathcal{F}, \tag{1.2}$$

$\mathcal{F}_n$ - all possible outcomes, observable until the moment $n$ inclusively, in other words $\mathcal{F}_n$ is an available information on the market until the moment n).

Consider a *Brownian motion* $W_t$ defined on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ such that:

- $W_0 = 0$, $W_t$ is a continuous function of time $t$,
- increments $(W_t - W_s)$ are independent and normally distributed random variables with mean 0 and variance $(t - s)$,
- $W_t$ is a stochastic driver for the process of interest rates.

Many types of models driving short rates are defined using the process of Brownian motion. They could be divided into two groups :

1. Short rate is described by diffusion equations:

$$dr_t = a(t, r_t)dt + b(t, r_t)dW_t. \tag{1.3}$$

2. Short rate is described by diffusion equations with jumps:

$$dr_t = a(t, r_t)dt + b(t, r_t)dW_t + \int c(t, r_{t-}, x)(\mu(dt, dx) - \nu(dt, dx)). \tag{1.4}$$

Where the following functions are used:

- $a(t, r_t)$ and $b(t, r_t)$ are both measurable functions of $t$ and $r_t$,
- $\mu(dt, dx)$ - a Poisson random measure,
- $\nu(dt, dx)$ - a compensator of $\mu(dt, dx)$.

### 1.1.1 *Single factor models*

Even though most players in the interest rate market will agree that the model ought to have **at least** two drivers (a short-term rate and a long-term mean rate), they are still implementing one-factor models assuming that the long-term mean rate remains fixed over time. The reason is that such models take much less time to get a result, so the cost of implementation dominates benefits [19]. That is why we will consider some single factor models first.

#### ***Diffusion models***

This type of models corresponds to the short rate dynamics, described by diffusion equation (1.3). The short rate $r_t$ here is modelled under the risk-neutral probability measure, and a Brownian motion $W_t$ is the risk-neutral Brownian motion. The dimension of $W_t$ is one since we are talking about single-factor short-rate models, i.e. the models with only one source of risk.

Vasicek (1977)[23] The first representative of a class of diffusion models is a Vasicek model, which was introduced by Oldrich Vasicek in 1977. This model is a classical case of one-factor models for interest rate dynamics. It is described by stochastic differential equation:

$$dr_t = \alpha(\beta - r_t)dt + \sigma dW_t \tag{1.5}$$

One can see that (1.5) is a special case of diffusion equation (1.3) with $a(t, r_t) = \alpha(\beta - r_t)$ and $b(t, r_t) = \sigma$, where $\alpha, \beta, \sigma$ are positive constants. These constants denote the following:

- $\beta$ - the level which interest rates revert to. It is called level of *mean reversion*.
- $\alpha$ - the *velocity* at which paths of interest rate dynamics regroup around the level of mean reversion.
- $\sigma$ - *instantaneous volatility*. It reflects an amount of randomness.

The solution of the equation (1.5) is:

$$r_t = e^{-\alpha t} r_0 + (1 - e^{-\alpha t})\beta + \int_0^t e^{-\alpha(t-s)} \sigma dW_s \tag{1.6}$$

One of the shortcomings of the above model is that the interest rates can become negative. This can not happen under regular market conditions (but in some situations it is allowed for rates to go below zero, for instance, when inflation takes place).That is why Vasicek model had to be improved. This improvement was made by John C. Cox, Jonothan E. Ingersoll, Stephen A. Ross in 1985.

Cox-Ingersoll-Ross(1985)[10] In 1985 John C. Cox, Jonathan E. Ingersoll and Stephen A. Ross extended the previous model. The Cox-Ingersoll-Ross (CIR) model claims that instantaneous interest rate dynamics is described by *CIR process*, CIR namely, a stochastic process defined by the following differential equation:

$$dr_t = \alpha(\beta - r_t)dt + \sigma\sqrt{r_t}dW_t. \tag{1.7}$$

The drift factor, $\alpha(\beta - r_t)$, is exactly the same as in the Vasicek model, but the standard deviation factor now takes the form $\sigma\sqrt{r_t}$, excludes the possibility of negative interest rates.

If the interest rate $r_t$ becomes 0, then diffusion factor $\sigma\sqrt{r_t}$ is 0, the drift $\alpha(\beta - r_t)$ becomes equal to $\alpha\beta$ (which is positive constant), so the process $r_t$ can only increase. Since the process defined by (1.7) is not Gaussian, an explicit solution is not available.

Ho-Lee (1986) This model was discovered in 1986 by Thomas Ho and Sang Bin Lee and is described by the following equation:

$$dr_t = \alpha_t dt + \sigma dW_t. \tag{1.8}$$

An explicit solution of (1.8) can be represented in the following form:

$$r_t = \int_0^t \alpha_s ds + \sigma W_t. \tag{1.9}$$

Black-Derman-Toy (1980-1990) This model was introduced by Fischer Black, Emanuel Derman, and Bill Toy in 1990. However, such kind of models were earlier developed for internal use by "Goldman Sachs" in the 1980s.

$$dr_t = \alpha_t r_t dt + \sigma_t r_t dW_t. \tag{1.10}$$

An explicit solution of (1.10) is:

$$r_t = r_0 e^{\int_0^t (-\alpha_s + \frac{\sigma_s^2}{2}) ds + \int_0^t \sigma_s dW_s}. \tag{1.11}$$

Hull-White (1990) John C. Hull and Alan White developed this model in 1990.

$$dr_t = (\beta_t - \alpha r_t) dt + \sigma_t dW_t. \tag{1.12}$$

An explicit solution of (1.12) could be represented in the following form:

$$r_t = e^{-\int_0^t \alpha_s ds} r_0 + \int_0^t e^{-\int_s^t \alpha_s ds} \beta_s ds + \int_0^t e^{-\int_s^t \alpha_s ds} \sigma_s dW_s. \tag{1.13}$$

***Jump-diffusion models***

Jump-diffusion models are models, where interest rates follow a jump-diffusion process. Robert C. Merton was the first who introduced this type of models.

Jump-diffusion processes are nicely described in a book of Rama Cont and Peter Tankov 'Financial Modelling with Jump processes' [9]. The following definitions are taken from there.

A stochastic process of the form:

$$X_t = \gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i \tag{1.14}$$

is called a *jump-diffusion process*. Here

- $(N_t)_{t\geq 0}$ is a Poisson process counting the jumps of $X_t$,
- $Y_i$ - the size of the jump, variables $Y_i$ are independent and identically distributed.

As we see this process consists of a diffusion process and jumps, which occur at random intervals. Depending on distribution of jumps there is a choice of models to use.

Merton model   Jumps in the process (1.14) have a Gaussian distribution: $Y_i \sim N(\mu, \delta^2)$. Then probability density function of $X_t$ is:

$$p_t(x) = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k \exp - \frac{(x-\gamma t - k\mu)^2}{2(\sigma^2 t + k\sigma^2)}}{k! \sqrt{2\pi(\sigma^2 t + k\sigma^2)}}. \tag{1.15}$$

Kou model   Jump sizes in this model are distributed with a density:

$$\nu_0(dx) = [p\lambda_+ e^{-\lambda_+ x} I_{x>0} + (1-p)\lambda_- e^{-\lambda_- |x|} I_{x<0}]dx, \tag{1.16}$$

where

- $\lambda_+$ and $\lambda_-$ corresponds to the decay of the tails for the distribution of positive and negative jump sizes accordingly.
- $p \in [0, 1]$ - probability of an upward jump.

### 1.1.2 *Multi factor models*

As was mentioned at the beginning of the previous section - ideally it is not enough for the model to have just one factor. There was a necessity to develop a model with more than one factor.

Some of one-factor models and multifactor models were developed in parallel. Since the presence of multifactor models does not cancel disadvantages of using models with just one factor.

Brennan and Schwartz(1982)

A model of Brennan and Schwartz has **two factors**. The model describes evolution of *'short'*, $r$, and *'long'*, $l$, interest rates. Stochastic differential equations of this evolution are the following:

$$dr = (\alpha_r + \beta_r(l - r))dt + r\sigma_r dW_t^r, \tag{1.17}$$

$$dl = l(\alpha_l + \beta_l r + \gamma_l l)dt + l\sigma_l dW_t^l, \tag{1.18}$$

where $W_t^r$ and $W_t^l$ are two standard Brownian motions with a correlation coefficient $\rho$ (i.e. $\mathbb{E}[dW_t^r dW_t^l] = \rho dt$).

Longstaff and Schwartz (1992)

This model is also a **two factor model**. Factors are a short term interest rate, $r$, and a volatility of interest rates, $\sigma$, defined by the equations:

$$r = ax + by, \tag{1.19}$$

$$\sigma = a^2 x + b^2 y, \tag{1.20}$$

where the processes $x$ and $y$, in-turn, are defined by stochastic differential equations:

$$dx = (\alpha_x - \beta_x x)dt + \sqrt{x} dW_t^x, \tag{1.21}$$

$$dy = (\alpha_y - \beta_y y)dt + \sqrt{y} dW_t^y. \tag{1.22}$$

Chen (1996)

Chen model is a representative of **three-factors models**. The following stochastic differential equations define evolution of short interest rate $r$, its mean $\mu$, and its volatility $\sigma$:

$$dr_t = (\alpha_t - \mu_t)dt + \sqrt{r_t}\sigma_t dW_t, \tag{1.23}$$

$$d\mu_t = (\beta_t - \mu_t)dt + \sqrt{\mu_t}\sigma dW_t, \tag{1.24}$$

$$d\sigma_t = (\gamma_t - \sigma_t)dt + \sqrt{\sigma_t}\theta_t dW_t. \tag{1.25}$$

Cox Ingersoll Ross

This model has **N factors**. Interest rate $r$ is represented as a sum of those factors $x_i$.

$$r = \sum_{i=0}^{N} x_i. \tag{1.26}$$

Each factor is described by a stochastic differential equation of regular 'CIR' form (1.7):

$$dx_i = \alpha_i(\beta_i - x_i)dt + \sigma_i\sqrt{x_i}dW_t^i, \tag{1.27}$$

for $i = 1, ..., N$

Interest rate models, which were described in this section are called **equilibrium** models, since they are based on macro economical factors.

Their disadvantage is that they can not automatically price bonds(unless calibration procedure was done [25]). There is a direct association of a short rate $r_t$ with a savings account in a bank. It is the most popular investment into the money market among general public [25]. Savings account $B_t$ represents a time $t$ value of a unit of cash, invested at time 0. A term *'money market account'* was developed to separate notions of savings account for individuals and savings account for institutions. From this moment on we will use the term money market account only in the latter sense, since we not deal with individuals in a present work (but keeping in mind that in the US this term is used for individuals accounts as well but with some restrictions [25]).

With a help of short interest rate $r_t$, money market account $B_t$ can be represented as

$$B_t = exp\left(\int_0^t r(s)ds\right), \tag{1.28}$$

The rate $r_t$ corresponds to a borrowing for an infinitesimal short period of time. The interest rate depends, however, on the borrowing period, therefore, the necessity of using a new characteristic arises.

## 1.2 Forward Rate Models

A forward rate $f(t;T,\Delta T)$ is an interest rate, predetermined at time $t$ for a future lending/borrowing at time $T$ for a short period of time $\Delta T$. Such future 'lending/borrowing' contract is called a *forward rate agreement*. No initial investments participate in such agreement. That is why the lender applies the following strategy ([25]):

- To short $\frac{P(t,T)}{P(t,T+\Delta T)}$ units of zero-coupon bond with maturity $(T+\Delta T)$.

  The earning therefore is $\frac{P(t,T)}{P(t,T+\Delta T)}P(t,T+\Delta T) = P(t,T)$, which will be used at the next step.

- To long 1 unit of zero-coupon bond with maturity $T$ (its price at the current moment $t$ is $P(t,T)$).

At time $T$ the lender gets 1 unit of cash from having a zero-coupon bond, maturing at time $T$, then immediately lends it out for a time-period $\Delta T$ under an interest rate $f(t;T,\Delta T)$. At the time $T+\Delta T$ gets this loan back, receiving $(1+f(t;T,\Delta T)\Delta T)$.

To avoid arbitrage on the market a very important condition must be satisfied : a net cash-flow is equal to 0.

The net cash flow of this strategy is:

$$1 + f(t;T,\Delta T)\Delta T - \frac{P(t,T)}{P(t,T+\Delta T)}. \tag{1.29}$$

Since it is zero, an expression for $f(t;T,\Delta T)$ is:

$$f(t;T,\Delta T) = \frac{1}{\Delta T}\left(\frac{P(t,T)}{P(t,T+\Delta T)} - 1\right). \tag{1.30}$$

After transition to a limiting case $\Delta T \to 0$ we get:

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T}. \tag{1.31}$$

$f(t,T)$ is an instantaneous forward rate.

Then the price of a zero-coupon bond with the maturity $T$ can be represented in the following form:

$$P(t,T) = e^{-\int_t^T f(t,s)ds}. \tag{1.32}$$

Therefore, a pricing model for any zero-coupon bond is completely defined by the model for an instantaneous forward rate.

At maturity price of zero-coupon bond by definition and as it also follows from (1.32) is equal to 1, $P(T,T) = 1$.

Also it is easy to see the relation between $r(t)$ and $f(t,T)$:

$$r(t) = \lim_{T \to t} f(t,T) = f(t,t). \tag{1.33}$$

This equation is also known as a **condition of consistency** of short rates and forward rates.

### 1.2.1 Heath-Jarrow-Morton framework

A new generation of financial models was developed by Heath, Jarrow and Morton in 1992, and were named in honor of them. The models describe evolution of forward rates.

It is more general approach. Short rate models can now be seen as special cases of models of HJM type. We will consider some cases later in this chapter.

HJM models are also called **arbitrage-free** models.

From now on a chain of concepts, which were taken from [25] will be used here.

Prices of zero-coupon bond are assumed to be log-normally distributed under some statistical probability measure $\mathbb{P}$. Then a process of those prices $P(t,T)$ can be defined by a stochastic differential equation of the following form:

$$\frac{dP(t,T)}{P(t,T)} = \mu(t,T)dt + \Sigma^T(t,T)dW_t, \tag{1.34}$$

where

- $\mu(t,T)$ is a scalar function of two variables $t$ and $T$,

- $\Sigma(t,T) = (\Sigma_1(t,T), \Sigma_2(t,T), ..., \Sigma_n(t,T))^T$ is an $(n \times 1)$ matrix of some functions $\Sigma_i$ of $t$ and $T$ $(i = 1, ..., n)$,

- $W_t$ is an $n$-dimensional Brownian motion process under the measure $\mathbb{P}$.

The parameters $\mu(t,T)$ and $\Sigma(t,T)$ of (1.34) can be estimated from market data. However, those parameters may imply arbitrage opportunities on the market. To avoid arbitrage there is a need to find a measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, such that discounted zero-coupon bond prices will be martingales with respect measure $\mathbb{Q}$. The market model defined by $(\Omega, \mathcal{F}, \mathcal{F}_{\sqcup}, \mathbb{Q})$ will be arbitrage-free according to the Fundamental Theorem of Asset Pricing .

This measure $\mathbb{Q}$ can be described by the following equation [25]:

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = \exp\left(\int_0^t (-\gamma_s^T dW_s - \frac{1}{2}\|\gamma_s\|^2 ds)\right), \tag{1.35}$$

where $\gamma_t$ is a $\mathcal{F}_t$-adaptive process and satisfies an equation:

$$\Sigma^T(t,T)\gamma_t = \mu(t,T) - r_t I. \tag{1.36}$$

According to the Cameron-Martin-Girsanov theorem a Brownian motion, corresponding to a new measure $\mathbb{Q}$, is given by the formula: $\tilde{W}_t = W_t + \int_0^t \gamma ds$. Then substituting corresponding expressions for $\mu(t,T)$ and $\tilde{W}_t$ into the original equation (1.34) we get:

$$\frac{dP(t,T)}{P(t,T)} = r_t dt + \Sigma^T(t,T)d\tilde{W}_t. \tag{1.37}$$

Thus we obtained a model for zero-coupon bond prices under risk-neutral measure $\mathbb{Q}$.

Now the goal is to make a transition to forward rates. By differentiating both sides of (1.31) $df(t,T)$ can be expressed through $d\ln P(t,T)$:

$$df(t,T) = -d\left(\frac{\partial \ln P(t,T)}{\partial T}\right) = -\frac{\partial(d\ln P(t,T))}{\partial T}. \tag{1.38}$$

Next, by applying Itô's lemma $d\ln P(t,T)$ and using (1.37) we obtain the following expression for $d\ln P(t,T)$:

$$d\ln P(t,T) = (r_t - \frac{1}{2}\Sigma^T(t,T)\Sigma(t,T))dt + \Sigma^T(t,T)d\tilde{W}_t. \tag{1.39}$$

As a result, from (1.37) and (1.39) we obtain:

$$\frac{\partial(d\ln P(t,T))}{\partial T} = \frac{\partial r_t}{\partial T} - \frac{1}{2}[\frac{\partial \Sigma^T(t,T)}{\partial T}\Sigma(t,T) + \Sigma^T(t,T)\frac{\partial \Sigma^T(t,T)}{\partial T}]dt + \frac{\partial \Sigma^T(t,T)}{\partial T}d\tilde{W}_t = -\frac{\partial \Sigma^T(t,T)}{\partial T}\Sigma(t,T)dt+ \tag{1.40}$$

To summarize all calculations fix an equation:

$$df(t,T) = \frac{\partial \Sigma^T(t,T)}{\partial T}\Sigma(t,T)dt - \frac{\partial \Sigma^T(t,T)}{\partial T}d\tilde{W}_t. \tag{1.41}$$

A volatility function of forward rates $\sigma(t,T)$ has the following form:

$$\sigma(t,T) = -\frac{\partial \Sigma^T(t,T)}{\partial T}. \tag{1.42}$$

After integrating (1.42) and using a fact that $\Sigma(t,t) = 0$ we get a representation for $\Sigma(t,T)$:

$$\Sigma(t,T) = -\int_t^T \sigma(t,s)ds. \tag{1.43}$$

Equation (1.43) interconnects volatilities of zero-coupon bonds and forward rates.

Now (1.41) can be rewritten as:

$$df(t,T) = \sigma^T(t,T)\int_t^T \sigma(t,s)ds + \sigma^T(t,T)d\tilde{W}_t. \tag{1.44}$$

This is a famous **HJM equation**, describing dynamics of forward rates.

The equation (1.44) perfectly shows that a drift term $\mu(t,T)$ of the forward rate process $f(t,T)$ is expressed in terms of forward rate volatility $\sigma(t,T)$. This equation for the drift term of forward rates is called **a drift condition**:

$$\mu(t,T) = \sigma^T(t,T)\int_t^T \sigma(t,s)ds. \tag{1.45}$$

There is another condition for a model to be of HJM type: **distribution of forward rates is Gaussian**. From this statement it follows that $f(t,T)$ can take negative values. It brings some difficulties in drawing analogy with some short-rate models, guarantying positive interest rates. In general, however, models, describing evolution of short rates, could be considered as a special cases of HJM models.

According to an equation (1.33) the short rate could be defined by a forward rate. It follows from (1.44) that by putting $T = t$, $r(t)$ can be expressed as:

$$r_t = f(t,t) = f(0,t) + \int_0^t \frac{1}{2}\frac{\partial \left\|\Sigma(s,t)\right\|^2}{\partial t}ds - \frac{\Sigma^T(s,t)}{\partial t}d\tilde{W}_s. \tag{1.46}$$

Zero-coupon bond prices in HJM framework can be calculated by the formula:

$$P(t,T) = P(0,T)e^{\int_0^t (r_s - \frac{1}{2}\Sigma^T(s,T)\Sigma(s,T))ds + \Sigma^T(s,T)d\tilde{W}_t}. \tag{1.47}$$

A disadvantage of equilibrium models is that they not reproduce market prices of basic instruments, for example, prices of zero-coupon bonds. This lack of ability is compensated by arbitrage-free models. They treat the prices of basic instruments as inputs, while the equilibrium models reproduce those prices as outputs.

Since arbitrage-free models, by their nature, do not create arbitrage opportunities on the market, basic market instruments and their derivatives can be priced consistently.

Short-rate models as sub cases of HJM framework

The following two short-rate models were considered in [25] as special cases of HJM model by specification of forward-rate volatility function:

1. **Ho-Lee Model**

2. **Hull-White Model**

Depending on the form of volatility function $\sigma(t,T)$ the model of HJM type transforms into one of the mentioned above models.

1. $\sigma(t,T) = \sigma$, where $\sigma$ is a constant.

   If volatility of forward rates is equal to some constant $\sigma$, a general HJM stochastic differential equation (1.44) transforms into:

$$df(t,T) = \sigma^2(T-t)dt + \sigma d\tilde{W}_t. \tag{1.48}$$

Integrating (1.48) over an interval from 0 to $t$ and plugging $t$ instead of $T$ we get an equation for $f(t,t)$, which is $r_t$.

Thus in HJM framework with $\sigma(t,T) = \sigma$ a short-rate dynamics is described by the equation:

$$r_t = f(0,t) + \frac{1}{2}\sigma^2 t^2 + \sigma \tilde{W}_t. \tag{1.49}$$

Denoting $(\frac{\partial f(0,t)}{\partial t} + \sigma^2 t)$ by $\alpha_t$, we get an absolute equivalence between (1.49) and (1.9).

The corresponding zero-coupon bond price is:

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left( -(\sigma \tilde{W}_t(T-t) + \frac{\sigma^2}{2} tT(T-t)) \right). \tag{1.50}$$

2. $\sigma(t,T) = \sigma \exp(-\kappa(T-t))$, where $\kappa > 0$.

The choice of such volatility function was motivated by the empirical fact, which verifies that volatility of forward rates fades out as time gets closer to maturity time $T$. Substituting such volatility function $\sigma(t,T)$ into the equation (1.44), author [25] obtains specific equations for forward rates under HJM framework:

$$df(t,T) = \sigma \exp(-\kappa(T-t)) d\tilde{W}_t + \frac{\sigma^2}{\kappa}(\exp(-\kappa(T-t)) - \exp(-2\kappa(T-t)))dt. \tag{1.51}$$

Integration of 1.51 over the interval $[0,t]$ provides the formula for forward rates $f(t,T)$:

$$f(t,T) = f(0,T) + \sigma \int_0^t \exp(-\kappa(T-t))) d\tilde{W}_s + \frac{\sigma^2}{2\kappa^2}\left[ (1 - \exp(-\kappa(T-t)))^2 \right]. \tag{1.52}$$

And by making $T = t$ the formula for short interest rates is obtained:

$$r_t = f(0,t) + \sigma \int_0^t \exp(-\kappa(t-s)) d\tilde{W}_s + \frac{\sigma^2}{2\kappa^2}(1 - \exp(-\kappa t))^2. \tag{1.53}$$

For convenience the author denotes the integral in 1.53 by $X_t$:

$$X_t = \sigma \int_0^t exp(-\kappa(t-s)) d\tilde{W}_s. \tag{1.54}$$

Such process $X_t$ satisfies the following SDE:

$$dX_t = \sigma d\tilde{W}_t - \kappa X_t dt. \tag{1.55}$$

From the equation 1.53 it follows that the process $X_t$ could be represented with help of short rate process $r_t$:

$$X_t = r_t - f(0,t) - \frac{\sigma^2}{2\kappa^2}(1 - exp(-\kappa t))^2. \tag{1.56}$$

Differentiating $r_t$ 1.53 and using representation of $dX_t$ 1.55 the following expression for $dr_t$ is obtained:

$$dr_t = f_T(0,t)dt + \sigma d\tilde{W}_t - \kappa X_t dt + \frac{\sigma^2}{\kappa} \exp -\kappa t(1 - \exp -\kappa t)dt. \tag{1.57}$$

Denoting $f(0,t) + \frac{1}{\kappa} f_T(0,t) + \frac{\sigma^2}{2\kappa^2}(1 - \exp -2\kappa t)$ by $\theta_t$ author gets SDE for short rates in the following form:

$$dr_t = \kappa(\theta_t - r_t)dt + \sigma d\tilde{W}_t. \tag{1.58}$$

Which is exactly Hull White model 1.12.

Applications of HJM models

According to [16] HJM modelling technique is better applied to two kind of markets:

- A market of government debts, issued by governments of countries, which have a minimal chance of default (for example: USA, Great Britain, Germany, Japan).

- Eurodollar market, consisting of U.S. dollar accounts in European banks, which don't belong to U.S. banking system.

The frames of Heath-Jarrow-Morton modelling approach were also extended to involve other financial instruments, for example :

- risk-management instrument of mortgages,

- foreign currencies,
- pricing of equities and non interest rate commodities,
- treasury bonds, protected from inflation.

More information on that can be found in [6], [16] and references therein.

It's needed to say about the main **disadvantage** of this framework. There have not been yet discovered such specification of HJM model that it does not produce negative interest rates.

# Chapter 2

## *Energy Market Models*

Energy market is the main component of commodity market. Derivatives on crude oil, natural gas and electricity are traded on energy market.

This market consists of sub markets, for example, Electricity market, Oil market, Gas market etc. The markets, listed above, have their own peculiarities and characteristics. Let us look closely at the description of some markets and their properties.

### 2.1 Gas Market

Generally the natural gas market has a similar nature to other commodity markets. For example, natural gas prices are sensitive to a balance between supply and demand. When demand for natural gas is rising, producers must increase supply. They need to increase an exploration of natural gas resources and production potential. However, this task is not straightforward in case where mother Nature plays a central role. Putting more resources and efforts into the production does not guarantee the result. In case of natural gas it could take several years to achieve a desired goal.

***Characteristics of natural gas market*** The main features of gas market include:

- *Supply and demand*

  This characteristic of natural gas market was described above. Supply and demand is the strongest characteristic, since it affects all the other properties.

- *Production and exploration*

  The level of production and exploration has a direct influence on supply. Consequently it takes part in a price formation.

- *Storage*

  Natural gas can be stored. There are different types of gas storage.

- *Seasonality*

  Demand of gas varies throughout a year, since gas is used for heating.

## 2.2 Electricity market

Electricity is a unique commodity in a sense that unlike others it can not be stored[1]. Therefore, it is very important to have a balance between supply and demand. Even though some advanced methods of electricity production were developed (such as hydro and nuclear generation), the main source of electricity is still thermal conversion of fossil fuels such as coal, gas and oil[5]. This implies that prices on a fuel have a big impact on electricity prices. There are many other factors, however, which affect the behavior of electricity prices. Here we want to illustrate a comparison between electricity and gas prices in Alberta. Figure 2.1 represents the graph of historical prices of electricity (residential rates). Figure 2.2 shows the behavior of gas prices (for Northern customers). The gas prices were converted form \$ per GJ into \$ per kWh taking into account a relation: $1\ GJ = 278\ kWh$ and assuming that plant's efficiency is 50%. The data was taken form the **Direct Energy Regulated Services** website (http://www.directenergyregulatedservices.com). From 2.1 and 2.2 we see that jumps in electricity and gas prices appear synchronously, but the magnitude of changes in electricity prices is larger than magnitude of changes in gas prices. This means that besides the influence of gas prices there are other factors, which affect prices in electricity market.

The stochastic process of electricity prices tends to have a mean reversion. Moreover, due to seasonality the price process has sudden jumps.

[1] Actually it can be stored, but the cost of storage is very high.

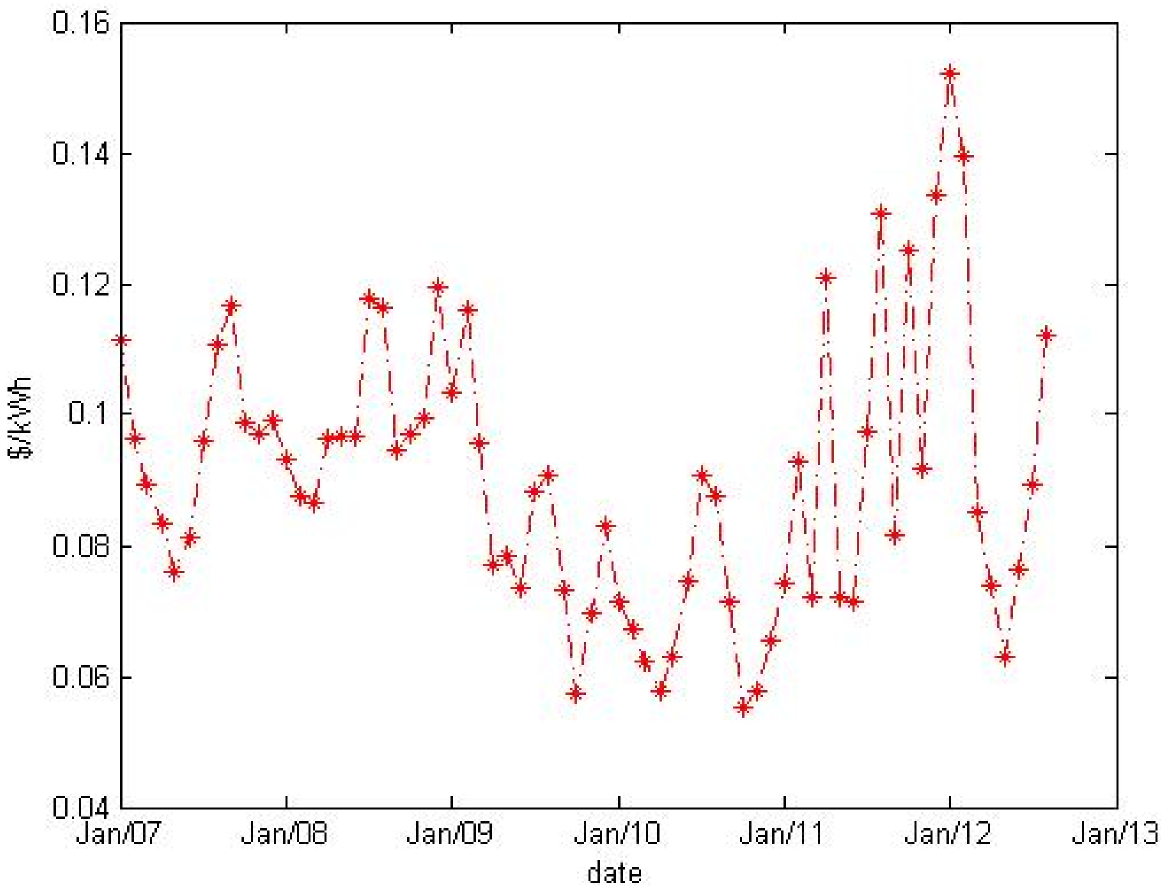



Figure 2.1: Historical electricity prices in Alberta for residential purposes

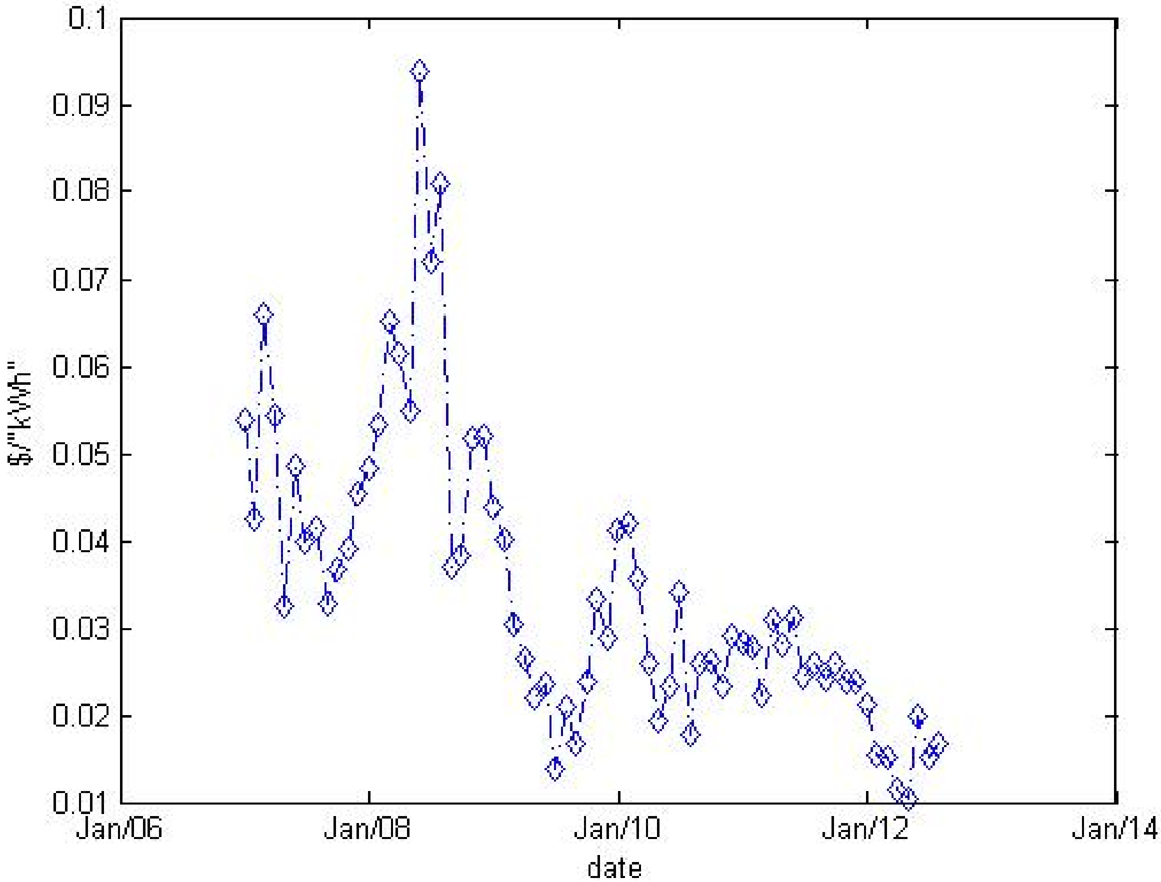



Figure 2.2: Historical gas rates for northern customers

The volatility of electricity prices is much higher than that of other commodity and financial markets.

Moreover, according to [5]: electricity markets are 'generally inefficient, high in regulatory risk and prone to persistent political interventions'.

***Characteristics of electricity market*** The electricity market has the following properties:

- *Non-storability*

  Some researchers suggest to model production processes first. To deal with the fact that electricity cannot be stored they suggest to use marginal fuel (gas, oil, etc.) prices to describe forward power prices, considering the fact that fuel is easily transformed into electrical power, provided an electricity production unit is rented for the delivery period.[14]

- *Seasonality*

  This property is well described by the following quote: "Whether deterministic or not, this seasonality does not create arbitrage opportunities, namely positions built at zero initial cost and leading to positive or strictly positive liquidation values at maturity. In this respect the example of the hedge fund Amaranth is quite instructive: this fund lost 6 billion in summer 2006, after gaining 1.5 billion in summer 2005, through the same type of calendar spread positions in natural gas futures."[2]

## 2.3 Spot Price Models

Just as a short rate is the main driver in the interest rate market, a spot price is a basic concept in energy market. Spot price, usually denoted as ***S(t)***, is a price of commodity for an immediate delivery at time $t$. A current price (at time $t$) of the forward contract for delivery at some time $T$ in the future is denoted as $F(t,T)$. And $F(t,T_1,T_2)$ represents the price of forward contract for delivery of commodity during *time interval* from $T_1$ to $T_2$. Also there is a connection between spot price $S(t)$ and forward/futures price $F(t,T)$:

$$F(t,T) = \mathbb{E}[S(T)]. \tag{2.1}$$

Main features of energy prices were listed in [21]:

- Volatility of spot prices on energy market is greater rater than volatility of interest rates and other instruments of money market.
- Energy prices are characterized by mean reversion, seasonality, spikes.

So, ideally, tools for modelling spot energy prices must be more perfect than those which are used to describe the behavior of short rates.

### 2.3.1 *Single factor models*

In single factor models we observe stochastic differential equation for description of spot price dynamics only.

#### ***Diffusion models***

Geometrical Brownian Motion   At first pricing of energy commodities was based on representation of spot price process as a ***Geometrical Brownian Motion*** process:

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{2.2}$$

where $\mu$ is a drift term, $\sigma$ is a volatility of spot prices.

The solution of this SDE could be represented in the following form:

$$S_t = S_0 exp((\mu - \frac{\sigma^2}{2})t + \sigma W_t) \tag{2.3}$$

Schwartz model This model was developed in 1997. Spot price process $S_t$ satisfies the following equation:

$$dS_t = \alpha(\mu - \ln S_t)S_t dt + \sigma S_t dW_t. \tag{2.4}$$

A new parameter $\alpha$ appears here. $\alpha$ characterizes a rate of reversion of the process $S_t$ to it's long-term level $\bar{S} = \exp \mu$. $\alpha$ is strictly positive.

Replacing $\ln S_t$ by $X_t$ and applying Itô's Lemma we will get SDE for process $X_t$:

$$dX_t = (X_t)'_{S_t} dS_t + \frac{1}{2}(X_t)''_{S_t} S_t^2 \sigma^2 dt = \frac{1}{S_t}(\alpha(\mu - \ln S_t)S_t dt + \sigma S_t dW_t) - \frac{1}{2}\frac{1}{S_t^2} S_t^2 \sigma^2 dt = \alpha(\kappa - X_t)dt + \sigma dW_t, \tag{2.5}$$

where $\kappa = \mu - \frac{\sigma^2}{2\alpha}$.

It is clear that the process $X_t$ is the OrnsteinUhlenbeck process. $X_t$ is normally distributed. And its characteristics have the following form:

$$\mathbb{E}(X_T \mid \mathcal{F}_0) = X_0 e^{-\alpha T} + (\mu - \frac{\sigma^2}{2\alpha})(1 - e^{-\alpha T}), \tag{2.6}$$

$$Var(X(T) \,|\mathcal{F}_0\,) = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha T}). \tag{2.7}$$

Now let's calculate a price of the forward (or futures) maturing at time $T$.

Since $F(0,T) = \mathbb{E}[S_T \,|\mathcal{F}_0]$ it follows that

$$F(0,T) = exp(\mathbb{E}[X_T \,|\mathcal{F}_0\,] + \frac{1}{2}Var(X_T \,|\mathcal{F}_0\,))$$

. And using 2.6 the forward price $F(0,T)$ could be represented by the expression:

$$F(0,T) = \exp[e^{-\alpha T} \ln S_0 + (1 - e^{\alpha T})(\mu - \frac{\sigma^2}{2\alpha})]. \tag{2.8}$$

With the appropriate boundary conditions, forward(or futures) prices with maturity $T$ given by:

$$F(t,T) = exp[e^{-\alpha(T-t)} \ln S + (1 - e^{-\alpha(T-t)})(\mu - \lambda - \frac{\sigma^2}{2\alpha}) + \frac{\sigma^2}{4\alpha}(1 - e^{-2\alpha(T-t)})] \tag{2.9}$$

By applying Itô's lemma to 2.9 we get:

$$\frac{dF(t,T)}{F(t,T)} = \sigma_F(t,T)dW_t, \tag{2.10}$$

where $\sigma_F(t,T)$ is the volatility function of forward prices and has the following representation:

$$\sigma_F(t,T) = \sigma e^{-\alpha(T-t)}. \tag{2.11}$$

The volatility structure under Schwartz single factor model is more realistic than the Black model but still has quite a simple shape.In particular the volatilities tend to zero for longer maturities and this happens for maturities less than one year for mean reversion rates larger than about 7. Although the market volatilities of forward prices decline with maturity they never get close to zero and so the Schwartz model has a problem for pricing options on long maturity forward contracts [8]. This model can be used for short maturity options on short maturity forwards.

***Jump-Diffusion models***

We had some discussion on that matter in previous chapter. Let's just introduced jump-diffusion type of models related to spot price process instead of interest rate process.

Merton  Merton presented this model in 1976. Under Merton's model spot price process $S_t$ satisfies the following SDE:

$$dS_t = \mu S_t dt + \sigma S_t dW_t + \kappa S_t dq, \tag{2.12}$$

where

- $\kappa$ represents the jump's size and it is normally distributed;
- $dq$ is a jump process, such that jumps appear not continuously, but at some certain points in time. $dq$ is a Poisson process with intensity $l$.

### 2.3.2 *Multi factor models of spot prices*

***Two factor model***

In [13] authors say that it's not correct to consider a constant convenience yield. According to the theory of storage there is a direct correspondence between level of inventories and the convenience yield. So besides the spot price, convenience yield must be stochastically defined. That's how one more factor arises.

Gibson and Schwartz model This model was developed by Rajna Gibson and Eduardo Schwartz in 1990 to price oil contingent claims. According to [13] the model is described by two stochastic differential equations:

$$dS_t = (\mu - \delta_t)S_t dt + \sigma_1 S_t dW_t^1, \tag{2.13}$$

$$d\delta_t = k(\alpha - \delta_t)S_t dt + \sigma_2 dW_t^2. \tag{2.14}$$

Two Brownian motion processes $W_t^1$ and $W_t^2$ have a correlation $\rho$: $dW_t^1 dW_t^2 = \rho dt$. The process 2.14 is the OrnsteinUhlenbeck process.

***Three factor model***

Schwartz model In 1997 Eduardo Schwartz extended a two-factor model 2.3.2 by adding one more stochastic factor - interest rate process $r_t$. The process $r_t$ follows mean reversion as it does in Vasicek model 1.1.1. The factors of this model are represented by the follofing stochastic differential equations:

$$dS_t = (r - \delta_t)S_t dt + \sigma_1 S_t dW_t^1, \tag{2.15}$$

$$d\delta_t = \kappa(\hat{\alpha} - \delta_t)dt + \sigma_2 dW_t^2, \tag{2.16}$$

$$dr_t = a(m - r_t)S_t dt + \sigma_3 dW_t^3. \tag{2.17}$$

And

$$dW_t^1 dW_t^2 = \rho_1 dt, dW_t^2 dW_t^3 = \rho_2 dt, dW_t^1 dW_t^3 = \rho_3 dt. \tag{2.18}$$

As was noted by [8] it is usually not necessary to use three factor model as stochastic interest rates will typically have a relatively minor impact on energy derivatives prices.

## 2.4 Heath-Jarrow-Morton type models

### 2.4.1 Clewlow and Strickland approach

There are two approaches for modelling energy market. The first one is based on a stochastic representation of spot prices and other basic variables of the market. We have described some of the models in the previous section. One of the main shortcomings is that some variables are not easily observed on the market. Even the spot price is unavailable sometimes . That is why a necessity of the second approach arose.

The second approach consists in a modeling of the forward/future curves. Clewlow and Strickland follow the second approach in their paper[7] where they assume deterministic interest rates, so that forward prices coincide with futures prices. They [7] consider an energy forward price process $F(t,T)$, defined by the following equation:

$$\frac{dF(t,T)}{F(t,T)} = \sigma e^{-\alpha(T-t)} dW_t \tag{2.19}$$

where

- $\sigma$ - the level of volatility of a spot and a forward price returns
- $\alpha > 0$ characterizes the speed of attenuation of the forward curve volatility when maturity is increasing.

As we see there is no drift term in (2.19), because forward and futures contracts do not require initial investments [8], and therefore their expected return in a risk-neutral world is zero. In case of 'real' world it would be necessary to add a drift to the model. The volatility function in this case has the following form:

$$\sigma(t,T) = \sigma e^{-\alpha(T-t)}. \tag{2.20}$$

Such choice of $\sigma(t,T)$ accounts for the fact that volatility should have negative exponential form, since short dated returns are more volatile than long dated returns. It is also possible to calibrate volatility function from observable market data. The solution of (2.19) can be represented in the following way:

$$F(t,T) = F(0,T)exp\left[-\frac{1}{2}\int_0^t \sigma(u,T)^2 du + \int_0^t \sigma(u,T)dW_u\right]. \tag{2.21}$$

We know that at maturity time the spot price is equal to the forward price, so we must have $S(t) = F(t,t)$ It follows that

$$S(t) = F(0,t)exp\left[-\frac{1}{2}\int_0^t \sigma(u,t)^2 du + \int_0^t \sigma(u,t)dW_u\right]. \tag{2.22}$$

Differentiating the logarithm of the spot price defined in (2.22) we obtain the following SDE for the spot price:

$$\frac{dS(t)}{S(t)} = \left[\frac{\partial \ln F(0,t)}{\partial t} - \int_0^t \sigma(u,t)\frac{\partial\sigma(u,t)}{\partial t}du + \int_0^t \frac{\partial\sigma(u,t)}{\partial t}dW_u\right]dt + \sigma(t,t)dW_t. \tag{2.23}$$

Now let us go back and use the particular volatility function from (2.19)

$$\sigma(t,T) = \sigma e^{-\alpha(T-t)}. \tag{2.24}$$

Its partial derivative is

$$\frac{\partial\sigma(t,T)}{\partial T} = -\alpha\sigma e^{-\alpha(T-t)}. \tag{2.25}$$

Denoting the drift term in (2.23) by $y(t)$ and using (2.25) we get:

$$y(t) = \frac{\partial \ln F(0,t)}{\partial t} + \alpha\sigma^2 \int_0^t e^{-2\alpha(t-u)}du - \alpha\int_0^t (\sigma e^{-\alpha(t-u)})dW_u. \tag{2.26}$$

The last term in representation for $y(t)$ could be found from the equation for the logarithm of the spot price (2.22) (using the volatility function (2.24)):

$$\ln S(t) = \ln F(0,t) - \frac{1}{2}\int_0^t \sigma^2 e^{-2\alpha(t-u)}du + \int_0^t \sigma e^{-\alpha(t-u)}dW_u. \tag{2.27}$$

Rearranging terms in (2.27) and multiplying by $\alpha$ we get an unknown quantity :

$$\alpha\int_0^t \sigma e^{-\alpha(t-u)}dW_u = \alpha\left[\ln S(t) - \ln F(0,t) + \frac{1}{2}\int_0^t \sigma^2 e^{-2\alpha(t-u)}du\right]. \tag{2.28}$$

Finally, the last term in the right hand side of (2.28) is:

$$\int_0^t e^{-2\alpha(t-u)}du = \frac{1}{2\alpha}(1-e^{-2\alpha t}), \tag{2.29}$$

and substituting (2.28) in (2.26) we get the SDE for the spot price process:

$$\frac{dS(t)}{S(t)} = \left[\frac{\partial F(0,t)}{\partial t} + \alpha(F(0,t) - \ln S(t)) + \frac{\sigma^2}{4}(1-e^{-2\alpha t})\right] dt + \sigma dW_t. \tag{2.30}$$

So the result is that if we have a specification of the forward price dynamics we can build a process for the spot price.

In [20] the author goes in the reverse direction: having a spot price process (2.4) we get the process for forward prices. Let's have a look at this way more closely. First of all, comparing (2.4) and (2.30), we can see that for the spot price to be consistent with the initial forward curve we need drift $\mu$ be a function of time of the following form:

$$\mu(t) = \frac{\partial F(0,t)}{\alpha \partial t} + \ln F(0,t) + \frac{\sigma^2}{4\alpha}(1-e^{-2\alpha t}). \tag{2.31}$$

Now let's rewrite the solution of (2.30):

$$F(t,T) = F(0,T)exp\left[-\frac{1}{2}\int_0^t \sigma^2 e^{-2\alpha(T-u)}du + \int_0^t \sigma e^{-\alpha(T-u)}dW_u\right]. \tag{2.32}$$

It's easy to calculate the first integral here:

$$\int_0^t \sigma^2 e^{-2\alpha(T-u)}du = \frac{\sigma^2}{2\alpha}e^{-2\alpha T}\left[e^{-2\alpha t} - 1\right]. \tag{2.33}$$

Again, using the fact that $S(t) = F(t,t)$ we obtain

$$S(t) = F(0,t)exp\left[-\frac{1}{2}\int_0^t \sigma^2 e^{-2\alpha(t-u)}du + \int_0^t \sigma e^{-\alpha(t-u)}dW_u\right]. \tag{2.34}$$

Here the first integral

$$\int_0^t \sigma^2 e^{-2\alpha(t-u)}du = \frac{\sigma^2}{2\alpha}\left[1 - e^{-2\alpha t}\right]. \tag{2.35}$$

So we can get a representation for

$$\int_0^t \sigma e^{-\alpha(t-u)} dW_u = \ln\left(\frac{S(t)}{F(0,t)}\right) + \frac{\sigma^2}{4\alpha}\left[1 - e^{-2\alpha t}\right]. \tag{2.36}$$

Now let's mention that

$$\int_0^t \sigma e^{-\alpha(T-u)} dW_u = \int_0^t \sigma e^{-\alpha T} e^{\alpha u} dW_u = e^{-\alpha T}\int_0^t \sigma e^{\alpha u} dW_u = \frac{e^{-\alpha T}}{e^{-\alpha t}}\int_0^t \sigma e^{-\alpha t} e^{\alpha u} dW_u = \frac{e^{-\alpha T}}{e^{-\alpha t}}\int_0^t \sigma e^{-\alpha(t-u)} dW_u \tag{2.37}$$

Finally using last conclusions and substituting everything into (2.32) we can get the forward price as a function of initial forward price and spot price at present moment $t$:

$$F(t,T) = F(0,T)\left(\frac{S(t)}{F(0,t)}\right)^{exp[-\alpha(T-t)]} exp\left[\frac{\sigma^2}{4\alpha}(1 - e^{-2\alpha t})(e^{-\alpha(T-t)} - 1)\right]. \tag{2.38}$$

It follows that the forward price at any time $t$ can be calculated on conditions that initial forward curve ($F(0,T)$), spot price $S(t)$ and parameters $\alpha$ and $\sigma$ are known. Fair question appears - how we can get those parameters? According to [7] these parameters can be obtained from the prices of options on the energy spot price or on forward contracts.

Also C&S suggest another way of getting $\sigma$ and $\alpha$. It consist in the best fitting to historical volatilities of forward prices.

For the first method of getting parameters of the model (2.19) we need to know what the prices of options are.

C&S are giving formulas for European options on the spot asset, options on forward contracts along with formulas for caps, floors, collars, options on swaps. Price of any contingent claim is expected value of discounted payoff under risk-neutral measure [11]. So the price of European call option is:

$$c(t, S(t), K, T) = \mathbb{E}_t[exp(-\int_t^T r(u)du)\max(0, S(t) - K)]. \tag{2.39}$$

Also (2.22) shows us that the spot price is log normally distributed and its distribution has the following view (here an equality (2.29) was used)

$$\ln S(T) \asymp N[\ln(F(0,T) - \frac{\sigma^2}{4\alpha}(1 - e^{-2\alpha T}), \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha T})]. \tag{2.40}$$

So from the fact above and an assumption that interest rates are deterministic it follows that Black-Scholes formula can be used:

$$c(t, S(t), K, T) = exp(-\int_t^T r(u)du)(F(t,T)N(h) - KN(h - \sqrt{w})], \tag{2.41}$$

where

$$h = \frac{\ln \frac{F(t,T)}{K} + \frac{w}{2}}{\sqrt{w}}, \tag{2.42}$$

$$w = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha(T-t)}). \tag{2.43}$$

Note that in a special case of (2.19), when $\sigma(t,T) = \sigma$, $w$ is putted to be equal $\sigma^2(T-t)$.

By call-put parity it is easy to get a formula for European put option.

As we will see next this is just a special case of European style option pricing on forwards (futures) (in this case maturity time of an option and maturity of a forward coincide). Since a price of European call option on a forward could be represented in the following way:

$$c(t, F(t,s), K, T, s) = \mathbb{E}_t[exp(-\int_t^T r(u)du) \max(0, F(T,s) - K)], \tag{2.44}$$

and using previous conclusions we can rewrite this price:

$$c(t, F(t,s), K, T, s) = exp(-\int_t^T r(u)du)(F(t,s)N(h) - KN(h - \sqrt{w})], \tag{2.45}$$

where

$$h = \frac{\ln \frac{F(t,s)}{K} + \frac{w}{2}}{\sqrt{w}} \tag{2.46}$$

and $w$ is given by the integral:

$$w^2(t,T,s) = \int_t^T \sigma^2 e^{-2\alpha(s-u)}du = \frac{\sigma^2}{2\alpha}(e^{-2\alpha(s-T)} - e^{-2\alpha(s-t)}). \tag{2.47}$$

Also [C&S] give formula for energy price caps in their paper. Since an energy price cap is just a set of a European call options, its price is represented as follows:

$$Cap(t, K, T, N, \Delta T) = \sum_{i=1}^{N} c(t, F(t, T + i\Delta T), K, T + i\Delta T, T + i\Delta T. \tag{2.48}$$

In the same manner floors and collars could be priced: an energy price floor is a set of a European put options, and a collar is a combination of a long position in a cap and a short position in a floor. Since the price of an option on swap is by definition:

$$Swpn(t, K, T, s, N, \Delta T) = exp(-\int_t^T r(u)du)\mathbb{E}_t[\max(0, \left\langle \frac{1}{N}\sum_{i=1}^{N} F(T, T + i\Delta T) \right\rangle - K)]. \tag{2.49}$$

authors showed that it could be represented through European call option price:

$$Swpn(t, K, T, s, N, \Delta T) = \frac{1}{N}\sum_{i=1}^{N} c(t, F(t, T + i\Delta T), K_i, T, T + i\Delta T, \tag{2.50}$$

where $K_i = F(S^*, T, T + i\Delta T)$ $F(S^*, T, s)$ is the forward price at time T for maturity s when the spot price at time T is $S^*$ . $S^*$ is can be extracted from equation:

$$\frac{1}{N}\sum_{i=1}^{N} F(S^*, T, T + i\Delta T) = K. \tag{2.51}$$

# Chapter 3

## *Comparison of two markets: Energy market vs. Interest rate market*

The nature of these two markets is so different. They have something in common though.

We start with similarities because the list of them is shorter rather then the list of differences.

### 3.1 Similarities

- *Model base*

  The base of mathematical models in interest rate market is **short interest rate**. In energy market **spot price** plays this key role.

  And the principle of these two notions is the same. Indeed, the short rate is an interest rate for immediate borrowing or landing of money for infinitely short time interval. The spot price is a price of a commodity for immediate delivery over infinitely short period of time. In Chapter 1 and Chapter 2 we considered different models, which describe dynamics of short interest rates on interest rate market and spot prices on energy market.

  In transition to forward rate models the short rate could be seen as a limiting value of forward rate for maturity of a contract approaching to a current time:

  $$r(t) = \lim_{T \to t} f(t,T) = f(t,t). \tag{3.1}$$

  There is a similar connection between forward prices and spot prices. In general: $F(t,T) = \mathbb{E}[S_t]$. And the limiting case when maturity $T = t$ gives us very helpful equation:

$$S(t) = F(t, t) \tag{3.2}$$

Despite similarity of process $r_t$ and $S_t$ the have some very important differences: the volatility of the second one is usually greater than the volatility of the first one.

- *Coincidence of models*

  Some energy market models coincide with interest rate models. For example, Schwartz model for spot prices (2.4) transforms into Vasicek model for interest rates (1.5).

  Consider (1.5):

$$dS_t = \alpha(\mu - \ln S_t)S_t dt + \sigma S_t dW_t. \tag{3.3}$$

  We have stochastic differential equation for the process $S_t$. Let us obtain SDE for the process $\ln S_t$. By Itô's lemma we get:

$$d \ln S_t = \frac{1}{S_t} dS_t + \frac{1}{2}(-1)\frac{1}{S_t^2} S_t^2 \sigma^2 dt = \alpha(\mu - \ln S_t)dt + \sigma dW_t - \frac{1}{2}\sigma^2 dt = \alpha(\mu - \frac{1}{2}\sigma^2 - \ln S_t)dt + \sigma dW_t. \tag{3.4}$$

  If we denote $\ln S_t$ by $X_t$ the last equation transforms into the following:

$$dX_t = \alpha(\mu - \frac{1}{2}\sigma^2 - X_t)dt + \sigma dW_t. \tag{3.5}$$

  SDE (3.5) has exactly the same form as equation in Vasicek model for short rates.

- There are unique representatives for each market. Energy market is represented by power forward, which will be described in the following chapter. Interest rate market is represented by zero-coupon bond.

Power forward is analogue of zero-coupon bond. Both of these instruments guarantee to receive 1 unit of specific quantity in the future. Power forward guarantees to receive 1 MhW at maturity. Zero-coupon bond assures to get 1 $ at expiry date.

## 3.2 Differences

- *Stability/instability of a price process as time approaches to maturity*

  Forward prices on energy market fluctuate even near maturity. As was said in [18] the Samuelson effect takes place in the dynamics of oil futures prices. Oil futures contracts are rarely traded at the beginning, which provokes low volatility. Liquidity of such contracts highly increases as the time approaches maturity. Consequently volatility increases as well.

  This fluctuation near maturity is impossible for bonds, since the price of a bond converges to 1 near maturity.

  Te following list of differences was formed with the help of the book [19]:

- *Lifetime*

  Interest rate market is much older than energy market.

- *Fundamental drivers*

  Energy market has much more complex fundamental drivers rather than interest rate market. For example, energy contracts depend on production and storage. Of course, there are no such drivers on interest market.

- Correlation between short-term and long-term contracts on interest market is very high, whereas energy markets have it to be very low.

- *Centralization of the the market*

Interest market is centralized unlike energy market. Most of financial institutions (for example, banks) stick to **Wall Street**. For energy market we can not claim that.

- *Seasonality*

  Seasonality is not peculiar to interest rate market. For energy market seasonality plays a huge role in price formations.

- *Storage and delivery*

  An influence of storage and delivery does not exist on interest rate market ('a piece of paper' or electronic delivery is not the same as storage and delivery by pipes of gas, for example).

- *Derivatives*

  Derivative contracts on energy market are much more complex than those one on interest rate market. Contracts were pretty standardized on interest rate market. Plus, they usually easier to model. We can not say the same about derivatives on energy market. Most of derivatives on interest rate market are popular and widely known. Whereas there are different kind of "exotic" derivatives on energy market.

- *Volatility*

  The volatility of energy contracts is much higher than the volatility of interest rate contracts.Indeed, since energy market is more complex and price process of an energy contract is governed by many factors it more unpredictable that the price of an interest rate contract.

# Chapter 4

## *Application of interest rate theory to energy markets*

This chapter is based on a very interesting paper of Juri Hinz, Lutz von Grafenstein, Michel Verschuere and Martina Wilhelm "Pricing electricity risk by interest rate methods"[14]. The authors consider an approach, consisting in application of interest rate pricing techniques to electricity market with its peculiarities.

The authors created the notional financial instrument for the electricity market. It is a *power forward*. Purpose of creating such instrument is the following: power forward plays a role of numeraire. It is artificial financial instrument and it is not actually traded.

The power forward is an agreement, which is made at time $t$ by two counter parties. The agreement guarantees a delivery of predetermined amount of an electricity during some time period $(T_1, T_2)$. The price of the power forward, denoted by $p(t)$, is set at time $t$. The price $p(t)$ is not a price in our regular perception. The difference is that unlike regular forward prices it is measured in megawatt hours (MWh). The advantage of such measure is that it protects a pricing process from errors caused by **MWh-EURO** (or **MWh-dollar**) fluctuations.

Also derivatives on power forward were developed to hedge electricity price risk. Pricing methods of European call and put options were discussed in the mentioned above paper.

The interpretation of power forwards as an electricity market instrument in terms of interest market instruments can be as follows. Power forward guarantees to get a flow of 1 MhW of electricity over some small time interval at its maturity. And a famous representative of interest rate market, a bond, guarantees to receive 1 dollar at bond's maturity. So similarity is very direct and transparent. Power forward appears in a role of a zero-coupon bond.

Taking into account the electricity production process the author shows that one justifies the applicability of efficient martingale methods to pricing arbitrary electricity contracts. It turns out that equilibrium asset prices are given by their future payoff, expected with respect to some equivalent measure. In paper [14] first of all authors build an equilibrium model for pricing energy contracts. This model is based on a finite set of tradable assets $\mathcal{E} = \mathcal{E}^{\text{pca}} \cup \mathcal{E}^{\text{fin}}$. They denoted physical assets (production capacity agreements) by $\mathcal{E}^{\text{pca}}$, and financial assets by $\mathcal{E}^{\text{fin}}$. Accordingly, $S_t = (S_t(\epsilon)$ stands for vector of prices of all assets $(\epsilon \in \mathcal{E})$. $(R_t)_{t=1}^{T}$ is a $\mathbb{R}^{\mathcal{E}}$-valued adapted process representing revenues ($R_t(\epsilon)$ - the revenue from holding the asset $\epsilon \in \mathcal{E}$ within $[t-1,t]$). Also there are I agents on the market who can share the assets. Each agent is characterized by pair $(x_i, U_i)$, where positive $x_i$ is initial endowment of i-th agent and $U_i$ its utility function such that

$$U_i \in \{U \in C^1(0,\infty) : U' > 0 \text{ is strictly decreasing function}, \lim_{z\to\infty} U_i'(z) = 0\}$$

If we have observed market data, therefore probability space is defined:

$$(\Omega, F, (F_t)_{t=0}^{T}, P)$$

$$(N_t)_{t=0}^{T}, (R)_{t=1}^{T}, (U_i, x_i)_{i=1}^{I}$$

#### 4.0.1 Notations used in this chapter:

The **domestic currency**: currency unit at time $t$ is 1 MW h constantly delivered within $[t, t+\Delta]$.

The **saving security** $(N_t)_{t=0}^{T}$:a bank account in EURO paying a constant interest rate $r > 0$ ($\Leftrightarrow e^{-rt}N_t$ is reciprocal EURO-price at time $t$ for electricity delivered within $[t, t+\Delta]$ ).

$E = E^{phys} \cup E^{fin}$ is a finite set of tradeable assets, where $E^{pca}$ denotes physical assets (production capacity agreements), $E^{fin}$ - financial assets.

$(R_t)_{t=1}^{T}$ is an $\mathbb{R}^{E}$valued process describing revenues, where $R_t(e)$ is the revenue from holding the asset $e \in E$ within $[t-1, t]$ .

$S_t = (S_t(e))_{e \in E}$ - the price vector of all physical and financial assets $e \in E$ at time $t$.

A trading strategy $((\theta_t, \vartheta_t))_{t=0}^T$ determines the number $\theta_t$ of savings security units and the part $\vartheta_t(e)$ of each asset $e \in E$ held by the agent within $(t, t+1]$.

The strategy $((\theta_t, \vartheta_t))_{t=0}^T$ is called self-financed, if $X_{t+1} = X_t + \theta_t(N_{t+1} - N_t) + \vartheta_t \circ (S_{t+1} - S_t + R_{t+1})$for all $t = 0, ..., T-1$,

where $(X_t = \theta_t N_t + \vartheta_t \circ S_t)_{t=0}^T$ denotes the wealth of this strategy.

#### 4.0.2 Assumptions:

Suppose that $I \in \mathbb{N}$ agents may share the assets. An agent $i = 1, ..., I$ is determined by $(x_i, U_i)$, where $x_i \in (0, \infty)$ denotes its initial endowment and $U_i$ is its utility function.

$\Delta > 0$ is fixed pre-specified delivery duration of forward contracts.

Electricity market with contract valuation by formula:

$$\hat{S}_t^*(e) = \mathbb{E}_Q(\sum_{u=t+1}^{T} \hat{R}_u(e)|F_t).$$

All agents are sufficiently wealthy.

Following conditions for the existence of the equilibrium are satisfied:

**Assumption 1**: The one period revenue is integrable and bounded from below: $\mathbb{E}(|\hat{R}_t(e)|) < \infty$, essinf $\hat{R}_t(e) > -\infty$ for all $e \in E^{pca}, t = 1, ..., T$.

**Assumption 2**: All contracts lose their values at the final date:

$S_T(e) = 0$ for all $e \in E$.

**Assumption 3**: All agents $(x_i, U_i)_{i=1}^I$ are equal: There exists a utility function $U$ and an initial endowment $x \in (0, \infty)$ such that $U_i = U, x_i = x$ for all $i = 1, ..., I$ .

In the symmetric equilibrium, there exists a measure Q such that the market price $p_t(\tau)$ at time $t$ for power forward maturing at $\tau$ is given by $p_t(\tau) = N_t \mathbb{E}_Q(\frac{1}{N_\tau}|F_t) t = 0, ...\tau, \tau = 0, ..., T$, (since $p_t(t) = 1$).

Hence to describe the dynamics of PF prices using interest rate theory they apply HJM formulation for *spot martingale measure*, which assumes that *the wealth of the self financing* *strategy investing entirely in just maturing bonds* is the standard numeraire security and supposes that all asset prices, expressed in units of this numeraire follow martingales with respect to the spot martingale measure. we have to choose the *wealth of the self financing strategy* investing entirely in just maturing power forwards as the new numeraire.

The **sliding MWh** $(B_t)_{t=0}^T$: $B_t = \prod_{u=1}^t p_{u-1}(u)^{-1}, t = 0, ..., T$ which mimics the wealth of the strategy.

Choosing $(B_t)_{t=0}^T$ as a numeraire we have to change from $Q$ to the spot martingale measure$\hat{Q}$ such that

$d\hat{Q} := \frac{N_0 B_T}{N_T B_0} dQ$ in order to ensure the martingalizing property:

for each process $(F_t)_{t=0}^T$ such that $(\frac{F_t}{N_t})_{t=0}^T$ is a $Q$-martingale and $(\frac{F_t}{B_t})_{t=0}^T$ is a $\hat{Q}$-martingale.

$\Rightarrow$ discounted PF prices $(\hat{p}_t(\tau) = \frac{p_t(\tau)}{B_t})_{t=0}^T$ are $\hat{Q}$-martingales for all $\tau = 0, ..., T$ moreover a discounted savings security $(\hat{N}_t = \frac{N_t}{B_t})_{t=0}^T$ is a $\hat{Q}$-martingale.

Let us describe notations in details. Suppose that our energy market is described by two types of instruments: power forward contracts and savings bank account.[3] Since authors have associated pricing process with production process it would be convenient to use power units as a currency. In other words the domestic currency unit at time $t$ is 1 MW h, constantly delivered within the interval $[t, t + \Delta]$.

$(N_t)_{t=0}^T$:a bank account in EURO paying a constant interest rate $r > 0$ ($\Leftrightarrow e^{-rt} N_t$ is reciprocal EURO-price at time $t$ for electricity delivered within $[t, t + \Delta]$).

In other words we can describe this account as following: $N_t$ is a number of units of currency (power) you can buy at time $t$ with one EURO invested at time 0.

Let us bring pricing methods from two papers [7],[14]. For better understanding first we should state that $F(t, T) = P_t(T)$, where (according to [7]) $F(t, T)$ is a price of a forward contract maturing at $T$. And $P_t(T)$ is a forward price in Euro according to [14]. Consequently:

$$F(t, T) = P_t(T). \tag{4.1}$$

$$P_t(\tau) = \frac{p_t(\tau)}{e^{-rt}N_t}. \tag{4.2}$$

where $p_t(\tau)$ is a forward price in currency units. $p_t(\tau)$ is analogous to the zero-coupon bond on the financial market[4]. Initial forward curve:

$$F(0,T) = P_0(T) = \frac{\hat{p}_0(T)}{\hat{N}_0} = \frac{p_0(T)}{N_0} = p_0(T).$$

By definition $F(t,T) = P_t(T)$ that's why

$$F(t,T) = \frac{p_t(\tau)}{e^{-rt}N_t} = \frac{\hat{p}_t(\tau)}{e^{-rt}\hat{N}_t}. \tag{4.3}$$

But

$$\frac{p_t(\tau)}{e^{-rt}N_t} = \frac{\hat{p}_t(\tau)}{e^{-rt}\hat{N}_t}. \tag{4.4}$$

Then it follows that:

$$F(t,T) = \frac{\hat{p}_t(\tau)}{e^{-rt}\hat{N}_t}. \tag{4.5}$$

In [14] $\hat{p}_t$ and $\hat{N}_t$ are defined by DEs:

$$d\hat{p}_t = \hat{p}_t s dW_t \tag{4.6}$$

and

$$d\hat{N}_t = \hat{N}_t \nu dW_t. \tag{4.7}$$

Let us differentiate $(e^{rt}\frac{\hat{p}}{\hat{N}})$. Applying Itô's formula we get:

$$\begin{aligned}
dF(t,T) = re^{rt}\frac{\hat{p}}{\hat{N}}dt + e^{rt}\frac{1}{\hat{N}}d\hat{p} + e^{rt}(-\frac{\hat{p}}{\hat{N}^2}d\hat{N}) + \\
\frac{1}{2}(r^2e^{rt}\frac{\hat{p}}{\hat{N}}(dt)^2 + re^{rt}\frac{d\hat{p}}{\hat{N}}dt + re^{rt}(-\frac{\hat{p}d\hat{N}}{\hat{N}^2}dt)) + \\
\frac{1}{2}(re^{rt}\frac{1}{\hat{N}}d\hat{p}dt + 0 * (d\hat{p})^2 + e^{rt}(-\frac{1}{\hat{N}^2}d\hat{p}d\hat{N})) + \\
\frac{1}{2}(-re^{rt}\frac{\hat{p}}{\hat{N}^2}d\hat{N}dt - e^{rt}\frac{1}{\hat{N}^2}d\hat{p}d\hat{N} - e^{rt}(-2\frac{\hat{p}}{\hat{N}^3}(d\hat{N})^2) \\
= re^{rt}\frac{\hat{p}}{\hat{N}}dt + e^{rt}\frac{1}{\hat{N}}d\hat{p} - e^{rt}\frac{\hat{p}}{\hat{N}^2}d\hat{N} - e^{rt}\frac{1}{\hat{N}^2}d\hat{p}d\hat{N} + e^{rt}\frac{\hat{p}}{\hat{N}^3}(d\hat{N})^2
\end{aligned} \tag{4.8}$$

(here we used the fact that terms with $dtd\hat{p}$, $dtd\hat{N}$, $(dt)^2$ are vanishing, because $d\hat{N}$ and $d\hat{p}$ are multiples of $dW$ and $dtdW = 0$)

Finally, using (4.6) and (4.7) and replacing $e^{rt}\frac{\hat{p}_t(\tau)}{\hat{N}_t}$ by $F(t,T)$ we get:

$$dF(t,T) = F(t,T)(r - s\nu + \nu^2)dt + F(s-\nu)dW_t \tag{4.9}$$

If we write $(r - s\nu + \nu^2) = (s-\nu)\lambda$ for some $\lambda$,then $dF(t,T) = F(t,T)(s-\nu)(dW_t + \lambda dt)$. In this case we can denote $(dW_t + \lambda dt)$ by $d\hat{W}_t$

From Girsanov's Theorem it follows that $d\hat{W}_t$ is again a Brownian motion under measure $\hat{Q}$.

So we have that $\frac{dF(t,T)}{F(t,T)} = (s-\nu)d\hat{W}_t$.

But in [7] $\frac{dF(t,T)}{F(t,T)} = \sigma e^{-\alpha(T-t)}dW_t$

It follows that under the measure $\hat{Q}$: $(s-\nu) = \sigma e^{-\alpha(T-t)}$

We can see that, if we identify $\hat{W}_t$ with $W_t$ in [7], then the model in [7] can be seen to be a special case of the model in [14] by setting $s - \nu = \sigma e^{-\alpha(T-t)}$. This means that parameters $s$ and $\nu$ of the model (4.6) and (4.7), which describe the dynamics of production process and bank account correspondingly, define parameters of the model of forward prices.

In the next chapter we performed the estimation of parameters $\sigma$ and $\alpha$ of such model for forward prices.

Consider the following ratio: $\frac{\hat{p}}{\hat{N}}$.

From (4.5) follows that:

$$\frac{\hat{p}}{\hat{N}} = F(t,T)e^{-rt}. \tag{4.10}$$

Let us differentiate $\frac{\hat{p}}{\hat{N}}$ by using Itô's formula:

$$d(\frac{\hat{p}}{\hat{N}}) = d(F(t,T)e^{-rt}) = e^{-rt}F(t,T)(-\nu(s-\nu)dt + (s-\nu)dW_t) = \\ (s-\nu)e^{-rt}F(t,T)(-\nu dt + dW_t). \tag{4.11}$$

Consider the case when $(s-\nu) = 0$.

From (4.6) and (4.7) it follows that the volatility of production process equals the volatility of money market account. Recalling that $(s-\nu) = \sigma e^{-\alpha(T-t)}$, it follows that volatility of forward price process $F(t,T)$ equals to zero and parameters of the model, defined in [7] are the following: $\sigma = 0$ and $\alpha$ could be any number.

In the case when $s - \nu = 0$ from the representation for $d(\frac{\hat{p}}{\hat{N}})$ given above follows that $\frac{\hat{p}}{\hat{N}} = c$, where $c$ - some constant. Since $F(t,T) = e^{rt}\frac{\hat{p}}{\hat{N}}$ we can use $\frac{\hat{p_0}}{\hat{N_0}}$ for $c$ for example. Recalling that $\frac{\hat{p_0}}{\hat{N_0}} = F(0,T)$ we obtain the following equation for forward rates:

$$F(t,T) = e^{rt}\frac{p_0}{N_0} = e^{rt}p_0. \tag{4.12}$$

Which means that we can get prices of forward contracts at any point in time just using initial values of power forward.

At the end of this chapter we'd like to clarify some questions on that matter - why this method is , may be, not so widely used. The best way is to refer to the authors of our source of ideas, described above : "However, the valuation of these contracts is still under discussion owing to the lack of convincing economical pricing concepts. The point here is that the electrical energy is not economically storable. Thus, power forwards with non- overlapping delivery intervals seem to have different underlying commodities (electrical energy, delivered in different periods) without any opportunity to transfer one commodity into the other which makes hedging by commodity storage impossible." [14]

# Chapter 5

# *Case study*

In this chapter we examine the model(2.19) on page 32. To calibrate the model we have to adjust its parameters $\sigma$ and $\alpha$ to real market data. Then it would be possible to use obtained values of those parameters to price forwards/futures of any maturity and at any point in time.

## 5.1 Model's setup

Recall the model (2.19) of forward price behavior:

$$\frac{dF(t,T)}{F(t,T)} = \sigma e^{-\alpha(T-t)} dW_t.$$

To calibrate the model (2.19) we need, first of all, some observed data. Suppose for some *fixed* maturity time $T$ we have forward prices $F(t,T)$ identically distributed over the time interval $[0,T]$ with the time step $\Delta t$. Now let's introduce a new function $r(t,T)$ which has the following form:

$$r(t,T) = \frac{F(t+\Delta t, T)}{F(t,T)} - 1. \tag{5.1}$$

From the definition above it follows that we have observed data $r(t_i,T)$ for each moment of time $t_i$, except the last one $t_N = T$. Here $N$ is a number of observations and $N = \frac{T}{\Delta t} + 1$. It is obvious that $t_i = i\Delta t$ and $t_{i+1} = t_i + \Delta t$. Consequently we get another representation of $r(t_i,T)$. For simplification let's denote $r(t_i,T) = r_i$ and $F(t_i,T) = F_i$:

$$r_i = \frac{F_{i+1}}{F_i} - 1. \tag{5.2}$$

Starting calibration of real market data we make a very important assumption that our returns $r_i$ are normally distributed. Such assumption is commonly used in financial pricing models. But dealing with 'real world' data we need, of course, to check the normality first of all. There are some kinds of statistical tests. In the present work we used MatLab function "jbtest" which is (according to its name) based on Jarque-Bera (JB) test of Normality. Jarque-Bera test probes the null hypothesis that $r_i$-s have the normal distribution. In the Table 5.1 we see values of skewness, kurtosis and results of Jarque-Bera test of normality.

We see negative skewness here, it turns out that returns are tend to be positive more often.

| skewness | kurtosis | acceptance of the null hypothesis | significance level |
|---|---|---|---|
| -0.100499 | 3.321838 | accepted | 0.3 |

Table 5.1: Characteristics of distribution of observed data $r_i$

The latter means that observed forward prices frequently increase from day to day.

We can conclude (using (2.19), (5.2) and the assumption about normality of $r(t,T)$) that

$$r_i = \sigma\sqrt{\Delta t}\exp\left(-\alpha(T-t_i)\right)Z_i, \tag{5.3}$$

where $Z_i \sim N(0,1)$.

Thus we have a vector of observations $r = (r_1, ..., r_N)$ and we know that the observations are independent identically distributed random variables ($r_i \sim N(0, \sigma^2\Delta t\exp\left(-2\alpha(T-t_i)\right))$).

The graph 5.1 shows us - how the distribution of our observed data is closed to the normal.

From the representation 5.3 we see dependence of $r_i's$ on parameters $\sigma$ and $\alpha$. Since we don't know the values of these parameters yet our goal will be to find them using observations $r_i$ $(i = 1, ..., N-1)$.

There are 3 main statistical methods for estimating model parameters:

- Method of moments
- Least squares
- Maximum likelihood estimation

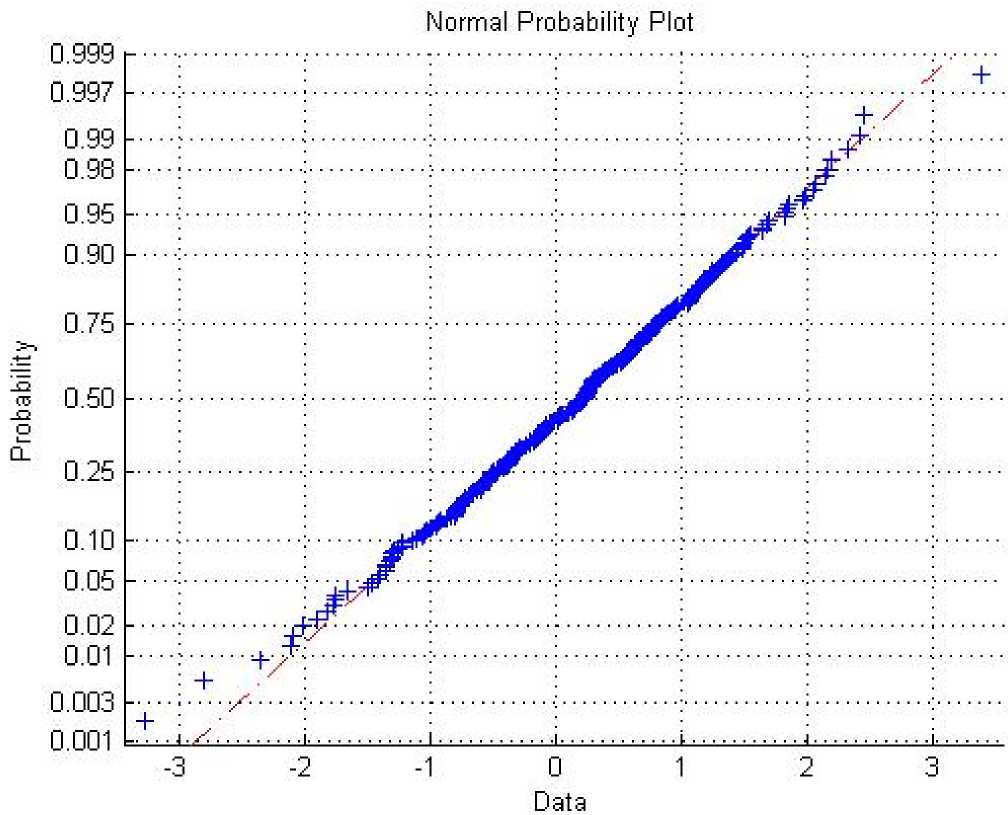



Figure 5.1: Normal plot of returns of NYMEX sweet crude oil forward prices

Because the type of a distribution is known we will use the maximum likelihood method. First of all we need to determine likelihood function of $r_i$. Likelihood function has the same form as probability distribution function. And probability density function of $r_i$ is

$$f(r_i) = \frac{\exp\left(-\frac{(r_i-0)^2}{2\sigma^2}\right)}{\sqrt{2\pi\sigma^2}} = \frac{\exp\left(-\frac{r_i^2}{2\sigma^2}\right)}{\sqrt{2\pi\sigma^2}}. \tag{5.4}$$

Then the joint density function of all $r_i$'s is just a product of $f(r_i)$'s and equal to:

$$\frac{\exp\left(-\frac{\sum_{i=1}^{N} r_i^2}{2\sigma^2}\right)}{\sqrt{(2\pi\sigma^2)^N}}. \tag{5.5}$$

As we said before likelihood function of $r_i$ looks totally the same as (5.5). But we will use its natural logarithm for simplification. It will be our **log-likelihood** function and it has the following form:

$$l(\theta|r_i) = -\frac{N}{2}\ln(\sigma^2\Delta t) - \frac{N}{2}\ln(2\pi) + \sum_{i=1}^{N}\alpha(T-t_i) + \frac{\sum_{i=1}^{N} r_i^2 \exp\left(2\alpha(T-t_i)\right)}{2\sigma^2\Delta t}. \tag{5.6}$$

Derivatives of log-likelihood function $l(\theta|r_i)$ by $\alpha$ and $\sigma$ are represented by following formulas:

$$\frac{\partial l}{\partial \alpha} = \sum_{i=1}^{N}(T-t_i) - \frac{\sum_{i=1}^{N} r_i^2(T-t_i)}{\sigma^2\Delta t}\exp\left(2\alpha(T-t_i)\right), \tag{5.7}$$

$$\frac{\partial l}{\partial \sigma} = \frac{N}{\sigma} - \frac{\sum_{i=1}^{N} r_i^2 \exp\left(2\alpha(T-t_i)\right)}{\sigma^3\Delta t}. \tag{5.8}$$

Solving $\frac{\partial l}{\partial \alpha} = 0$ and $\frac{\partial l}{\partial \sigma} = 0$ we get numerical solutions $\hat{\alpha}$ and $\hat{\sigma}$. These solutions compose a maximum likelihood estimator $\hat{\theta} = (\hat{\alpha}, \hat{\sigma})$.

Once we have $\hat{\alpha}$ and $\hat{\sigma}$ we can go backward and create a sequence of simulated returns $(r_1^{sim}, r_2^{sim}, ..., r_{N-1}^{sim})$ using (5.3). With the help of simulated $r_i^{sim}$'s we can get (solving (5.7) = 0 and (5.8) = 0) new parameters of MLE. Here is a comparison table of parameters $\hat{\alpha}$ and $\hat{\sigma}$ recovered form observed data (NYMEX sweet crude oil prices for a period of time: February 2004- July 2005), denoted as $\hat{\alpha}_{obs}$ and $\hat{\sigma}_{obs}$, and simulated data, denoted as $\hat{\alpha}_{sim}$ and $\hat{\sigma}_{sim}$:

| type of data | $\hat{\alpha}$ | $\hat{\sigma}$ |
|---|---|---|
| observations | 0.001001 | 0.152177 |
| simulations | 0.001023 | 0.157803 |

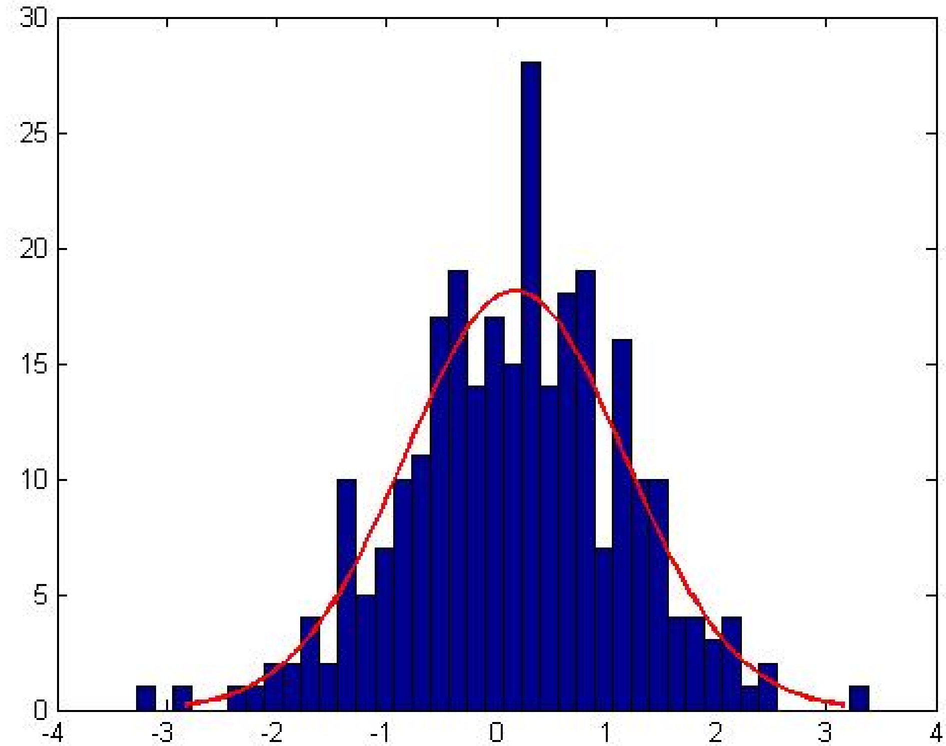



Figure 5.2: Histogram of distribution of $Z_i$, which are defined by formula (5.9)

It was shown that on the assumption that data are normally distributed we found parameters of our model using method of maximum likelihood estimation. Than a new set of data $r_{sim}$ was created using such parameters $\hat{\alpha}_{obs}$ and $\hat{\sigma}_{obs}$ and Brownian motion. After that parameters $\hat{\alpha}_{sim}$ and $\hat{\sigma}_{sim}$ were received using MLE. And the fact is that these two pairs of parameters are very similar.

To verify that parameters $\hat{\alpha}$ and $\hat{\sigma}$ are the right ones we can check normality of random variables $Z_i$:

$$Z_i = \frac{r_i}{\hat{\sigma} * exp(-(\hat{\alpha}(T-t)))\sqrt{dt}}. \tag{5.9}$$

The normality is demonstrated on the histogram 5.2.

Also we have received the following results for kurtosis and skewness 5.2 of $Z_i$.

| kurtosis | skewness |
|---|---|
| 3.34995 | -0.15532 |

Table 5.2: Characteristics of distribution of random variables $Z_i$

## 5.2 Real data analysis

As was said in [24] Heath-Jarrow-Merton model can be used in two main directions. The first one defines the volatility structure “to be sufficiently ’nice’ to make a tractable Markov model”. Another one is to figure out - what kind of form should a volatility structure take to fit observed market data. Here **Principal Component Analysis** could be really helpful. It will help to get volatility functions empirically from historical observations.

PCA is based on one of techniques of matrix decomposition. This type of matrix decomposition is called **Singular value decomposition** of matrix. The concept is the following:

Suppose we have a matrix $X$ of returns. $M$ is covariance matrix of these returns. Let $U$ be the orthogonal matrix of eigenvectors of $M$. Now we can consider matrix $C$ in the form:

$$C = XU.$$

The columns of matrix $C$ are *principal components* of $M$. So, as [1] describes Principal Components, "the linear transformation defined by $U$ transforms our original data $X$ on $n$ correlated random variables into a set of *orthogonal* random variables: That is, the columns of the matrix $C = XU$ are uncorrelated”.

Given natural gas daily forward prices from January 6, 2000 to February 7, 2011 for 12 maturities.The graphs of later are illustrated on Figure 5.3.

A market date was denoted by $t$. Hence $t$ varies from 06/01/2000 to 07/02/2011. Maturity date of a contract is $T$. And for more convenient compounding of data it’s useful to denote $\tau = T - t$ which means ’time-to-maturity’. Therefore we have just twelve values of $\tau$ ($\tau$ varies from 1 month to 12 month). According to above notations we can construct a matrix $P$, consisting of forward prices for each value of $t$ and corresponding ’time-to-maturity’ $\tau$ . The size of matrix $P$ is $n \times 12$, where $n$ - a number of different market dates.

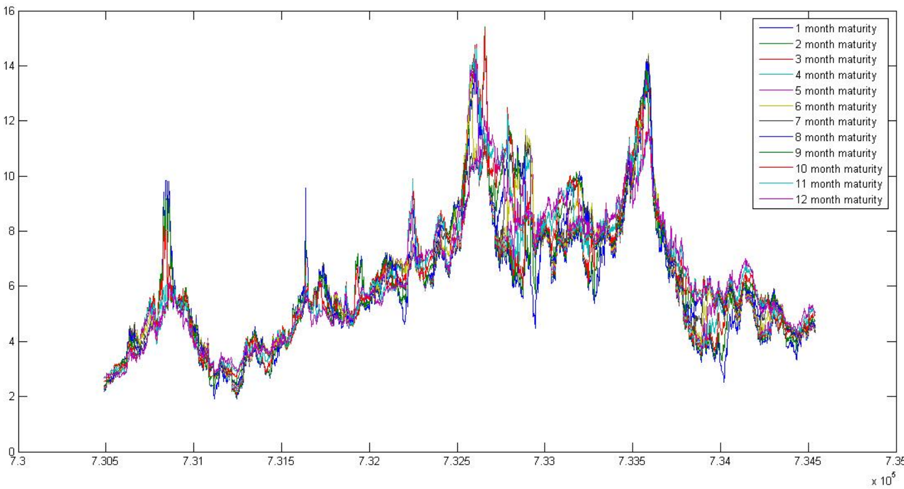



Figure 5.3: Natural Gas forward prices for different maturities

Figure 5.4 represents plots of the Henry Hub natural gas prices of forward contracts in two shapes. The difference is in one of the axis, assigned for maturity. The first plot was built using actual time of maturity ($T$). The second one - using *relative* maturity ($\tau$).$\tau$ varies form 1 to 12 months.

The same procedure was implemented by authors in [12]. They described one interesting fact, which we can easily observe from our pictures 5.4 as well. The fact is that area of the surface, corresponding to higher prices, moves toward the axis, assigned for market dates $t$, after transition to a $\tau$-parameterization of maturity. It turns out that relatively higher prices (or the highest prices) appear near $t - axis$ on the second plot of Figure 5.4. This is natural situation, since forward contracts on energy market become more expensive as the time approaches to its maturity. It would be interesting to analyze the behavior of separate forward curves. We pick 4 different dates such that they belong to different seasons: October 14,2006, January 14,2007, April 14,2007, July 14,2007. The resulting forward curves are represented on Figure 5.5. We see that November forward curve looks like a graph of increasing function. Such event is called ***contango***. It means that prices on natural gas are expected to be higher in the future. The opposite process is called ***backwardation***.

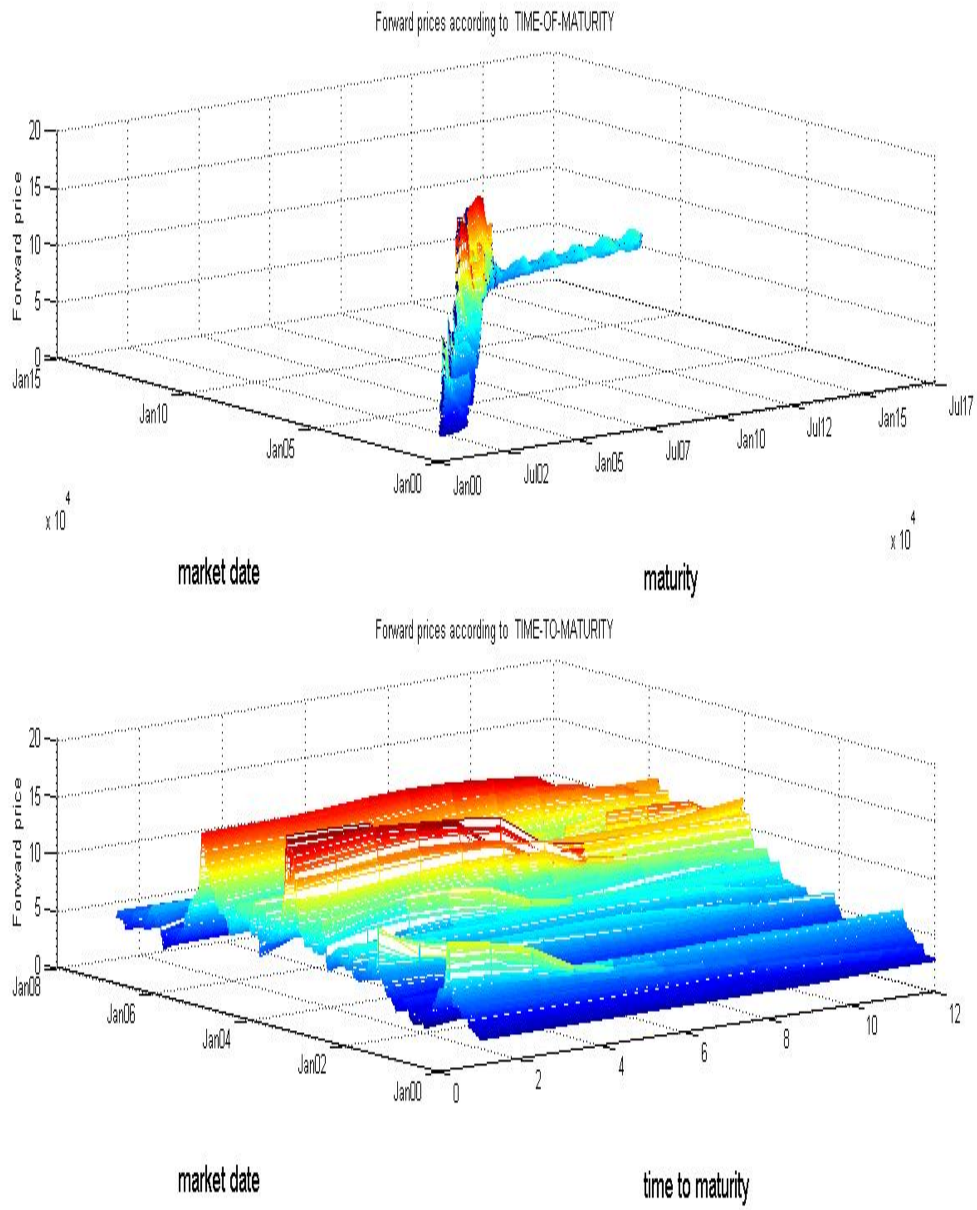



Figure 5.4: Prices of forward contracts on natural gas at different market dates for different maturities (two types of maturity parameterization)

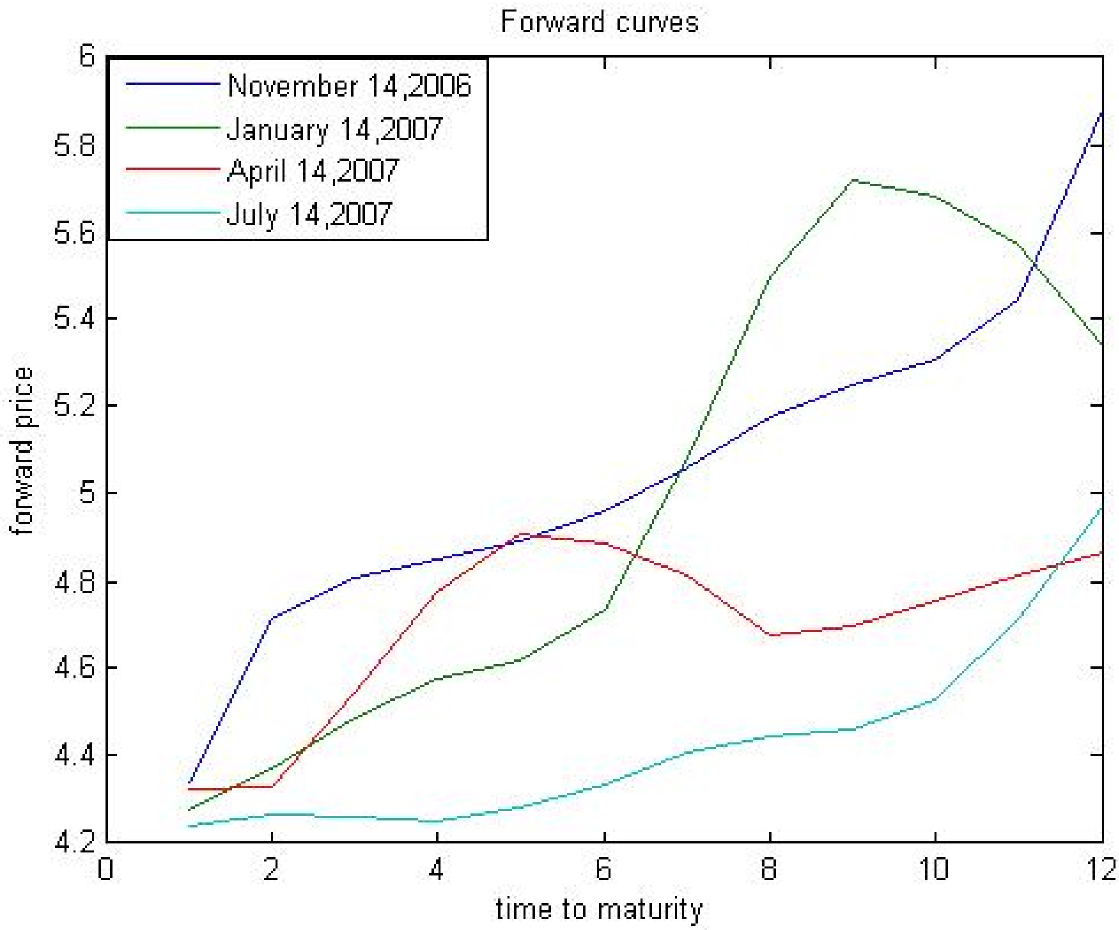



Figure 5.5: Forward curves for different market dates

But we don't observe it in this particular case. We can explain that in November natural gas prices are supposed to increase, because the cold season starts and the demand will increase. The maximum of prices of November's contract corresponds to the maximum time to maturity (12 months). The interesting point is that the picks of prices for January's and April's contracts fall on the following November as well.

Now it is necessary to make a transition to a matrix of log-returns $r_{ij} = ln(P_{(i+1)j}/P_{ij})$ (this is an $i$-th element of $j$-th column).

Using approach of receiving volatility from historical data suggested in [12], we calculate volatility function $\sigma(t)$ in a sliding window of length 30. Here let us to have a look at the theoretical basis described in the work mentioned above.

Recall multifactor (with $n$ factors) model of forward prices:

$$\frac{dF(t,T)}{F(t,T)} = \mu(t,T)dt + \sum_{i=1}^{n} \sigma_k(t,T)dW_k(t).. \tag{5.10}$$

There was an assumption about risk-adjustment. Under the risk-neutral probability the term $\mu(t,T) = 0$. Formula (5.10) transforms into the following:

$$\frac{dF(t,T)}{F(t,T)} = \sum_{i=1}^{n} \sigma_k(t,T)dW_k(t). \tag{5.11}$$

Authors [12] represent the volatility function $\sigma_k(t,T)$ (5.11) of forward price model in the following way:

$$\sigma_k(t,T) = \sigma(t)\sigma_k(T-t) = \sigma(t)\sigma_k(\tau). \tag{5.12}$$

Since the instantaneous spot price volatility $\sigma_S(t,t)$ could be represented in the following way:

$$\sigma_S(t)^2 = \sqrt{\sum_{i=1}^{n} \sigma_k^2(t,t)}, \tag{5.13}$$

authors get a representation of a spot price volatility using a new parameterization of maturity.

$$\sigma_S(t) = \tilde{\sigma}(0)\sigma(t), \tag{5.14}$$

where $\tilde{\sigma}(\tau) = \sqrt{\sum_{k=1}^{n} \sigma_k(\tau)^2}$.

So the conclusion is that "the function $t \hookrightarrow \sigma(t)$ is, up to a constant multiplicative factor, necessarily equal to the instantaneous volatility of the spot price"( [12]). And this result makes a great contribution to a calculation of $\sigma(t)$ using market data.

In our case we use the first column of our matrix $P$ as a spot price, since a prompt-month forward price is placed for a spot price. And applying a method of volatility calculation to given observations in a sliding window we get a vector of volatilities $[\sigma_S]$. Now we try to find a general form of a volatility function $\sigma(t)$ for any time $t$. Suppose the volatility function could be represented in the following way:

$$\sigma(t) = exp(\alpha + \beta t + \sum_{m=1}^{M} (\gamma_m sin(2\pi mt) + \delta_m cos(2\pi mt))), \tag{5.15}$$

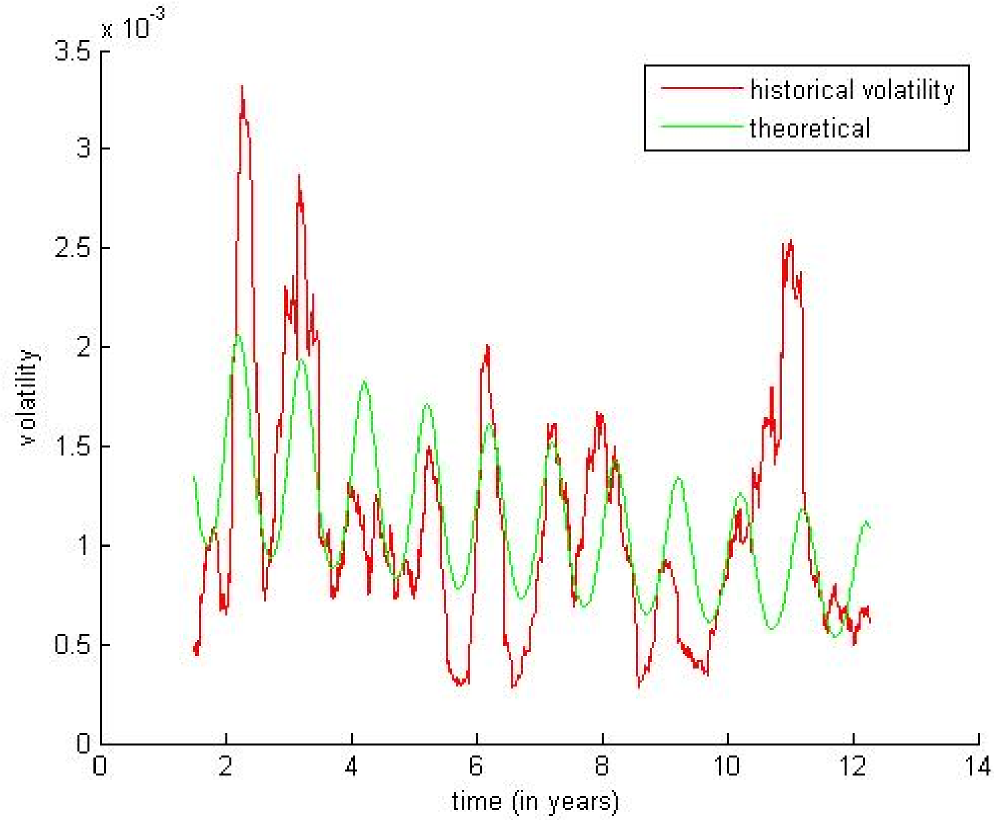
Now it's possible to find a coefficients $\alpha$, $\beta$, $\gamma_m$, $\delta_m$ having values of spot price volatility $(\sigma_S(t_i))$ of historical data for some period of time. We need to minimize a sum:

$\sum_{i=1}^{n} (\sigma_S(t_i) - \sigma(t_i))^2$

To implement this we used least-square data fitting function *lsqcurvefit* in Matlab.

Figures below represent the results, produced by plotting spot price volatility of the historical data versus theoretical volatility. Specific function "*getvols*"(see the code) was used to obtain volatility of spot prices. The length of a sliding window is equal to 30. The choice of the length of sliding window was dictated by the fact that for smaller windows we would have higher volatilities, whereas for larger windows volatilities are lower.

Graphs of the theoretical volatility function are different on these two pictures only because of a choice of number $M$ in the formula 5.15.



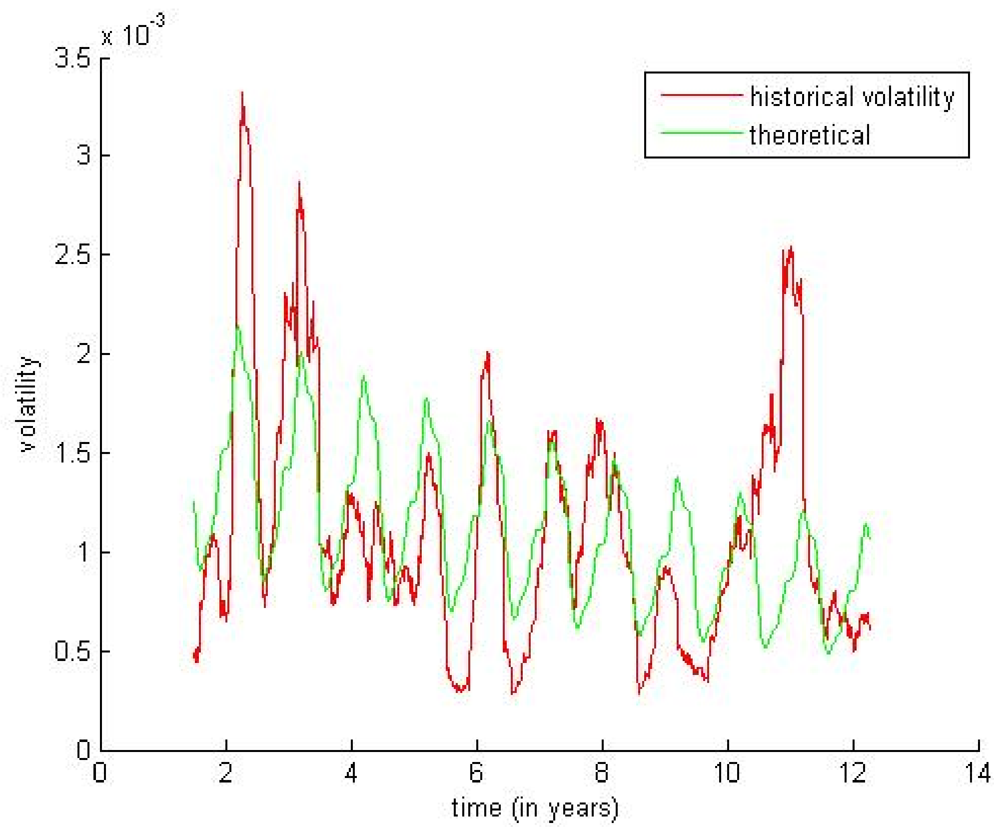




Figure 5.6: Graphs of Historical spot price volatility vs Theoretical volatility($\sigma(t)$) for M=1 and M=4 (5.15)

We chose the number M to be equal to 4. Values of the parameters of function $\sigma(t)$ are represented in the following table:

| $\alpha$ | $\beta$ | $\gamma_1$ | $\delta_1$ | $\gamma_2$ | $\delta_2$ | $\gamma_3$ | $\delta_3$ | $\gamma_4$ | $\delta_4$ |
|---|---|---|---|---|---|---|---|---|---|
| -6.42 | -0.06 | 0.37 | 0.103 | -0.11 | -0.03 | 0.03 | -0.02 | -0.06 | 0.001 |

Using volatility function $\sigma(t)$ and its parameters shown in the table, we can construct a matrix of normalized returns dividing $r_{ij}$ by $\sigma(t_i)$. Note that here $r_{ij}$ are sorted out in a correspondence with its $[\sigma(t)]$. And after that we create a matrix $M$ of covariances of$(r_{ij}/\sigma)$.

Now we have necessary inputs for **Principal Component Analysis**.

$M = U\Lambda V$ we find matrices $U$, $\Lambda$ and $V$ $(V = U')$. Matrix $\Lambda$ is a diagonal matrix of eigenvalues $\lambda_1, \lambda_2, ..., \lambda_{12}$ in such an order that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq ... \geq \lambda_{12}$ .

Matrix $U$ is called the matrix of factor loadings. It is an orthogonal matrix where $i^th$ column $[u_i]$ represents the eigenvector corresponding to $\lambda_i$.

$PU$ forms the matrix of *principal components.*

From our matrix definitions follows that it is necessary to have at least $m$ (in our case $m = 12$) principal components to describe all variations in the original data matrix $P$. At the same time we want to minimize the number of factors. So if we use only first $q < m$ eigenvalues from matrix $\Lambda$ and put the rest $\lambda_{q+1}, ..., \lambda_m$ equal to zero we can calculate a proportion of total variance given by the first $q$ factors. It is :

$$l = \frac{\sum_{i=1}^{q} \lambda_i}{\sum_{i=1}^{m} \lambda_i}.$$

In practice it's usually enough to have such number of factors so that $l$ is around 95%.

The following graphs support theoretical conclusions about the meaning of first factors in PCA. For example:

- The $1^st$ factor is assigned for a shift. By changing this factor we'll move all contracts in the same direction.

- The $2^d$ factor is assigned for a tilt. A change in this factor will move a half of our contracts in one direction and another half - in opposite direction.

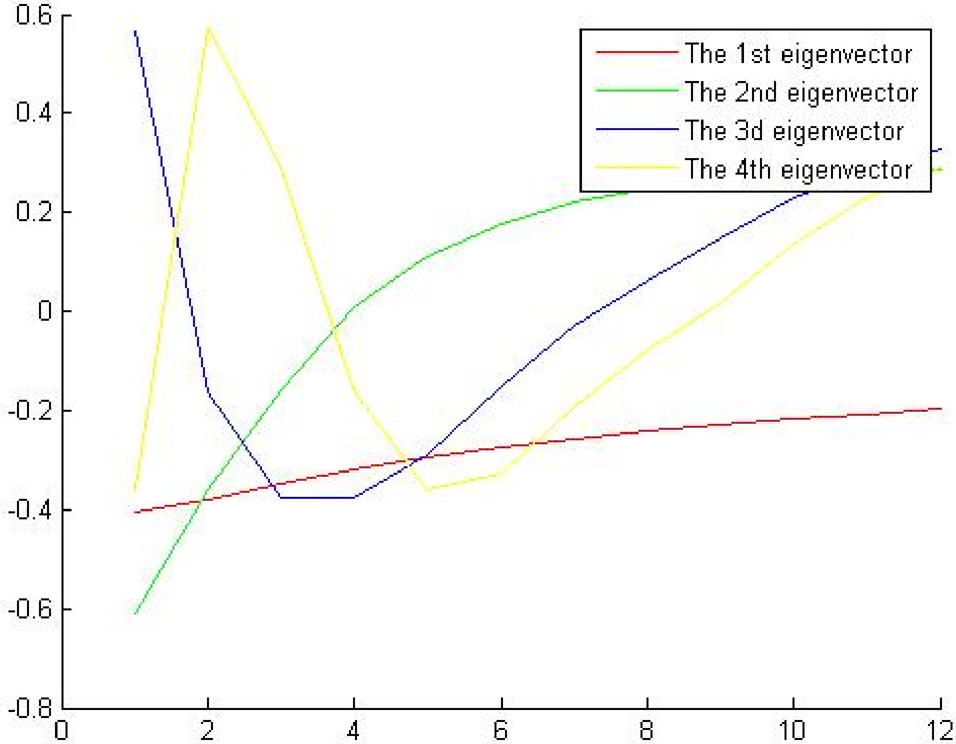



Figure 5.7: The first 4 eigenvectors of the NG daily forward prices covariance matrix

- The $3^d$ factor is not so determined as the first two factors, but it could be reasonable to define it as a bend.

Also to see the influence of number of factors involved in our estimation we can construct a matrix $M_i$ where $i$ is a number of factors in the following way: $M_i = U * \Lambda_i * U'$, where $\Lambda_i$ is a diagonal matrix, consisting of $i$ elements (the first $i$ eigenvalues of $\Lambda$). And then to assess a ratio: $(M - M_i)/M$. The elements of later will approach 0 with a growth of number $i$.

Natural gas forward prices have seasonal behavior. So it would be reasonable to extract seasonal component in forward prices. This could be represented as:

$$F(t, T) = \Lambda(T)X(t, T), \tag{5.16}$$

where $\Lambda(T)$ is a seasonal factor in natural gas forward prices representation.

Now we need simulate volatility function in the following form:

$$\sigma(t) = exp(\alpha + \beta t + \sum_{m=1}^{M} (\gamma_m sin2\pi mt + \delta_m cos2\pi mt)) \qquad (5.17)$$

We reconstructed forward prices using Monte Carlo simulations, function $\sigma(t)$ and the first 3 principal components . The Figure 5.8 represents 3D plots of original data and simulated.

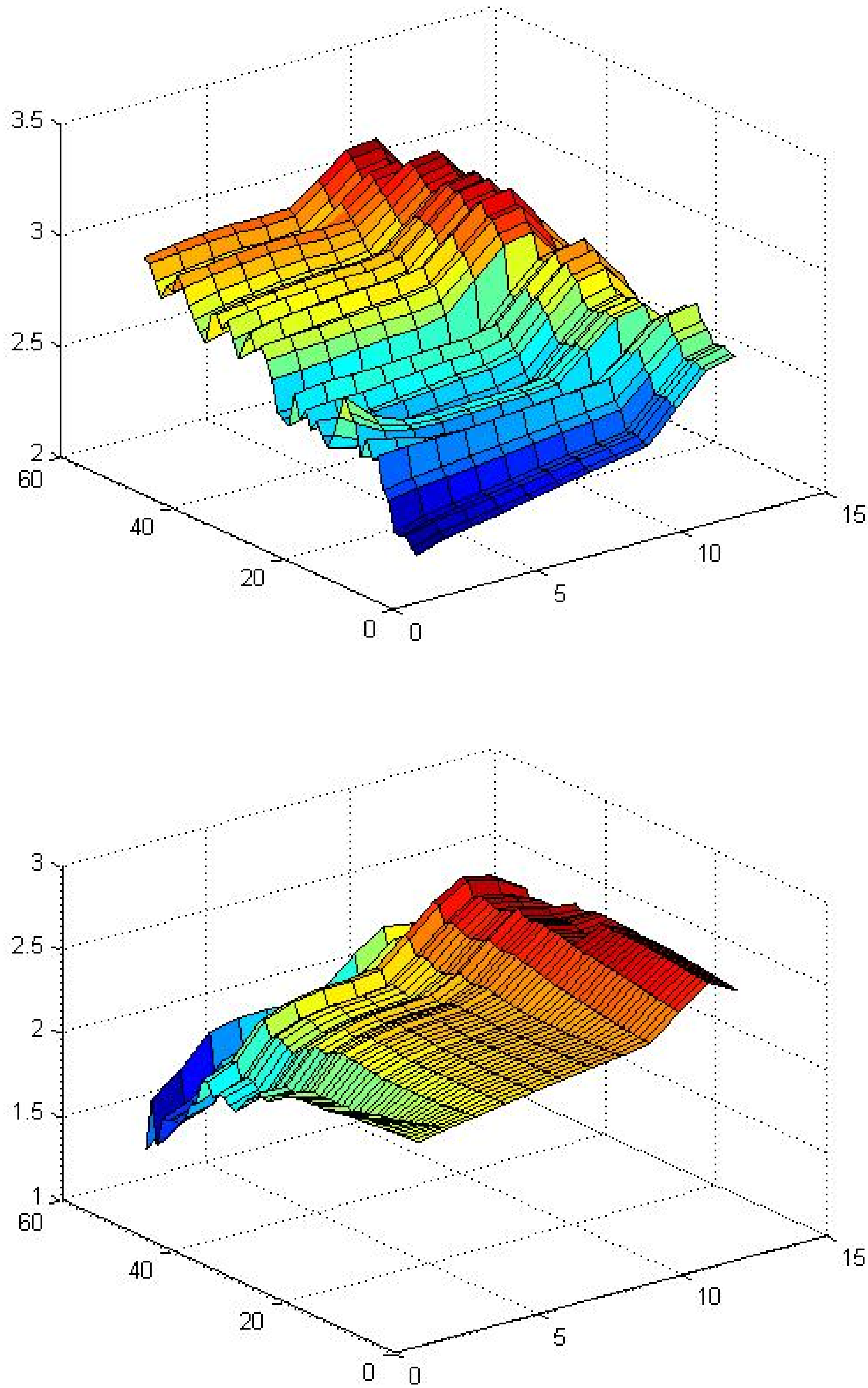



Figure 5.8: Graphs of Historical forward prices vs Simulated forward prices

# Chapter 6

# Conclusions

In the present work we compared the two most valuable components of financial market: Interest Rate Market and Energy Market. Structures and peculiarities of these two markets were considered in the first two chapters. We concentrated mostly on a mathematical framework fitted to each market.

There is a description of different kinds of models defining dynamics of short rates on Interest Rate Market. However these models do not solve the problem of pricing some of the basic market instruments such as zero-coupon bonds. Hence there was a need to introduce a new generation of financial models - Heath-Jarrow-Morton (HJM) framework. This model describes the dynamics of forward rates. This allows to represent each model described earlier as a sub-case of HJM model.

Correspondingly in Chapter 2 there were considered Spot Price Models and HJM models for Energy Market. The technique of pricing of European options as well as energy price caps and swaps was demonstrated.

Interest Rate market and Energy market have a lot of differences as well as similarities, which we describe in Chapter 3. Nevertheless there was a very interesting approach, developed in [14]. The approach applies interest rate market techniques to energy market. We combined two approaches from [7] and [14] to show how the pricing model from one approach transforms into another by change of measure.

The HJM model can be used in two different ways. We explored both of them in the last two chapters. In Chapter 4 we calibrated the model, described in Chapter 2, using prices of forward contracts on sweet crude oil and obtained parameters $\sigma$ and $\alpha$ of the model. Finally, in Chapter 5 we examine the form of volatility function of forward prices using Principal

Component Analysis on real market data.

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# Appendix A

# Appendix. MatLab codes

```
clear all; close all; clc;
load excel
%%
%building nx12 matrix P
for ind_t = 1:length(t)
x = F(ind_t,1:end);
ind = find(~isnan(x));
x = x(ind);
if isempty(x),x=nan(1,12);ind = 1:12;end
if length(x)==11, x(12)=NaN;ind = [ind ind(end)+1]; end
if sum(abs(diff(x)))==0, x(:)=NaN;end
F(ind_t,ind) = x;
S(ind_t) = x(1); %vector of spot prices (prompt month)
for j = 1:12
P(ind_t,j) = x(j);
end
end
% plot(S)
% datetick('x','12','keepticks')
%%
LR = diff(log(P));% matrix of log-returns on P
ind = find(isnan(sum(LR,2))); % find rows with NaNs in
```

```
holidays = find(diff(t)>1);
% Also we want to find rows where the prompt month shifts.
[tmp,ia,ib]=intersect(t,T);
missout=union(union(ind,holidays),ia);
keep = setdiff(1:length(t)-1,missout);
tkeep = t(keep);
P = P(keep,:);
LR = LR(keep,:);
win = 30; %window length could be changed
[vols,indv]= getvols(LR(:,1),win);% volatility "sigma"
tvol = tkeep(indv)/365-2000;
%
%matrices P nad LR without unnecessary data were constructed
%% making X, deseasonalizing of F
for i=1:12
tmp = log(F(:,i:12:end));
lambda_m(i) = mean(tmp(~isnan(tmp)));%calculating a seasonal
    component of log-prices
% lambda_m(i) = mean(tmp(~isnan(tmp)) - mean(log(F(1,1:12)),2));
for j=0:16
X(:,12*j+i) = exp(log(F(:,12*j+i))-lambda_m(i));
end
end

%% make matrix XP - analogue of P from X:
for ind_tx = 1:length(t)
```

```
xX = X(ind_tx,1:end);
xX = xX(~isnan(xX));
if isempty(xX),xX=nan(1,12);ind = 1:12;end
if length(xX)==11, xX(12)=NaN; end
% if sum(abs(diff(xX)))==0, xX(:)=NaN;end %%%there was some rows of the same numbers, exclude them
for j = 1:12
XP(ind_tx,j) = xX(j);
end
end
LRX = diff(log(XP)); %deseas.logreturns
%% excluding some data. The same procedure as was in F.
ind = find(isnan(sum(LRX,2)));
holidays = find(diff(t)>1);
[tmp,ia,ib]=intersect(t,T);
missout=union(union(ind,holidays),ia);
keep = setdiff(1:length(t)-1,missout);
tkeep = t(keep);
XP = XP(keep,:);
LRX = LRX(keep,:);
%% historical volatility of deseasonalized log-returns
win = 30;
[volsX,indX]= getvols(LRX(:,1),win);
tvolX = tkeep(indX)/365-2000;
%% fitting volatilities
x0 = [1 1 1 1 1 1 1 1 1 1];
```

```
x_fit = lsqcurvefit(@sigmaFunction11,x0,tvolX,volsX);
sigma_f = (sigmaFunction11(x_fit,tvolX)); %daily volatility
sigma_fd = sqrt(252)*sigma_f; %yearly volatility
%%
% figure(1),clf
% hold on
% plot(tvolX, volsX,'r')
% plot(tvolX,sigma_f,'g')
% xlabel('time (in years)');
% ylabel('volatility')
% legend('historical volatility','theoretical')



%% replacing vols by sigma_f
for j = 1:12
%normolized deseas. returns
LRX_norm(:,j) = LRX((win+1):(end-win),j)./sigma_f;
end
MX = cov(LRX_norm); %covarinace matrix of log_norm-s
[UX,LX,VX] = svd(MX);% SVD of covariance matrix
% tvolX = tkeep(win+1:end-win); %times according to vols
%% rebuild X using first 3 components
SDX = sqrt(LX(1:3,1:3));
VDX = LRX_norm*UX(:,1:3)*inv(SDX(1:3,1:3));%FIXED
MDX = VDX*SDX*UX(:,1:3)';
%% first,rebuild logreturnes
```

```
LRX_norm_reb = MDX;%%% deseas
%%%%
%

%%
tt=1:0.01:12;
x = lambda_m([10,11,12,1:12, 1,2]);
y = csape([10,11,12,1:12, 1,2],x,'periodic'); %defining a periodic function "lambda" by cubic spline interpolation
%%%calcul-n of trend's polynomial
trend_h = volsX - detrend(volsX); %historical
trend_t = spline(tvolX,trend_h,tt);% trend for any time 'tt'
% l_tt = lambda_tt + trend_t;
%%
%%%%%%%%%%%%OPTION PRICING%%%%%%%%%%%%%%
OEDate = 0.25;
today = 0;
N = 100; % No of MC simulations
tX = 0:1/252:OEDate;
% tX = tvolX(1:50);
dt = 1/252;
f=@(tX) ppval(y,12*mod(tX,1));
lambda_ttX = @(tX)f(tX);
trend_tX = spline(tvolX,trend_h,tX);%
sigma_fd = @(tX) sqrt(sigmaFunction11(x_fit,tX)*252); %the values of volatility function (yearly)
```

```
sim_VX = randn(length(tX),3,N)/sqrt(length(tvolX)); % MONTE CARLOOOOOO simulation of matrix VX(Nx3) ~ N(0,1)
for k=1:N
LRX_sim(:,:,k) = sim_VX(:,:,k)*SDX*UX(:,1:3)';
for j = 1:12
LRX_norm_sim(:,j,k) = LRX_sim(:,j,k)./sigma_fd(tX)'; %normolized returns
end
LRX_norm_reb_sim(:,:,k) = LRX_norm_sim(:,:,k);
for i=2:length(LRX_norm_reb_sim(:,1,k))
    XP_reb_sim(1,:,k) = XP(1,:);
 XP_reb_sim(i,:,k) =  XP_reb_sim(i-1,:,k).*exp(-(sigma_fd(i-1)^2/2)*(tX(i-1)*dt) + sigma_fd(i-1)* LRX_norm_sim(i-1,:,k));
end
for j=1:12
    for i=1:length(tX)
P_reb_sim(i,j,k) =  XP_reb_sim(i,j,k) *exp(lambda_ttX(i/252+j/12));%+trend_tX(i));
    end
end
K =  mean(P(1:50,4))-0.5; % strike price for European Call
payoff(k) = max(0,P_reb_sim(end,4,k)- K);%%%we take the last element of P_reb_sim because prices were simulated till option expiry date


```

```
C_mc(k) = exp(-0.05*(OEDate - today))*payoff(k);%%% European Call by MC (for each simulation)
end

tmpC = cov(squeeze(P_reb_sim(end,4,:)),payoff);
b_hat = tmpC(2,1)/tmpC(1,1);

C_mc_mean = mean(C_mc);%E-call price,received by MC simulations
% P_mc_mean = mean(P_reb_sim(:,4,:),3);
value_sigma = @(OEDate) sigma_fd(OEDate);

cv_estimator = (payoff(:) - b_hat * (squeeze(P_reb_sim(end,4,:)) - exp(0.05*OEDate)*squeeze(P_reb_sim(1,4,:))));
C_cv = C_mc_mean - mean(cv_estimator);
figure(2),clf
hold on
plot(UX(:,1),'r')
plot(UX(:,2),'g')
plot(UX(:,3),'b')
plot(UX(:,4),'y')
legend('The 1st eigenvector','The 2nd eigenvector','The 3d eigenvector','The 4th eigenvector')
% [mean(C_cv) - 1.96*std(C_cv)/sqrt(N),mean(C_cv) + 1.96*std(C_cv)/sqrt(N)]
```

```
function f = sigmaFunction11(x,tvol)

```

```
f = exp(x(1) + x(2)*tvol + ...
      x(3)*sin(2*pi*tvol) + x(4)*cos(2*pi*tvol)+...
      x(5)*sin(4*pi*tvol) + x(6)*cos(4*pi*tvol)+...
      x(7)*sin(6*pi*tvol) + x(8)*cos(6*pi*tvol)+...
      x(9)*sin(8*pi*tvol) + x(10)*cos(8*pi*tvol));
```

```
data = xlsread('CLData.xls');
% load data
T = data(:,5); %
t = data(:,10); %
F = data(:,9);
ud = unique(T); %
ind = find(T==ud(40));
T = ud(40);
t = t(ind);
F = F(ind);
ind = find(T>t);
t = t(ind);
F = F(ind);
N = length(F);
r = zeros(1,N-1);
for i = 1:1:N-1
      r(i)=(F(i+1)/F(i))-1;
end
t = t(1:end-1)';
weekends=0;
% eliminate zero returns
```

```
ind = find(r);
t = t(ind);
d_w = weekday(t);
r = r(ind);
if weekends==0
      disp('Eliminating weekends')
% eliminate weekends
 ind = find(1<d_w & d_w<7);
elseif weekends==1
      ind = find(1<d_w &d_w<8);
elseif weekends==1
      ind = find(d_w==1);
elseif weekends==2
      % only weekends
      ind = find(d_w==1 & d_w == 7);
end
N = length(ind);
r=r(ind);
t=t(ind);
dt=1;
alpha_obs = fzero(@(alpha) mle(alpha, T, N, t, r), 0.05);
sigma_obs = std(r.*exp(alpha_obs * (T-t)));

for i = 1:1:N
Sigma(i) = sigma_obs*exp(-(alpha_obs *(T-t(i))));
end
```

```
%Now we have parameters alpha&sigma of our model. They will help us to
%simulate r's since r*exp(alpha*(T-t)=sigma*sqrt(delta_t)*Z, where Z~N(0,1)
%or r~N(0,(sigma^2)*delta_t*exp(-2*alpha(T-t))
% T = 12;
N = 1000;
dt = 1;
ti = linspace(t(1),T,N+1);
rs = zeros(1,N);
rs = sigma_obs*sqrt(dt)*randn(1,N).*exp(-alpha_obs*(T-ti(1:end-1)));
%recover Fs :
Fs (1) = F(1); % we need initial value of simulated F
Fs = cumprod( [Fs(1) ; (rs+1)' ],1);
%now let's calculate volatilities using these simulated rs's
%%%%%%FOR different window lengths:
win1=2; %window length
[vol1,ind1,avs] = getvols(rs,win1);
win2=20; %window length
[vol2,ind2,avs] = getvols(rs,win2);
win3=200; %window length
[vol3,ind3,avs] = getvols(rs,win3);
figure(1)
plot(ti(ind1),sqrt(vol1)/sqrt(dt),'-r', ...
     ti(ind2),sqrt(vol2)/sqrt(dt),'-g', ...
```

```
        ti(ind3),sqrt(vol3)/sqrt(dt),'-b')
legend('win1','win2', 'win3', 'Location', 'EastOutside')

% we could estimate alpha using a least-square fit on this curve
% or - use the MLE approach:
alpha_new = fzero(@(alpha) mle(alpha, T, N, ti(1:end-1), rs),
    alpha_obs);
sigma_new = std(rs.*exp(alpha_new * (T-ti(1:N))));
% exp(alpha_new * (T - ti(1:end-1))));
for i = 1:1:N
Sigma_new (i) = sigma_new*exp(-(alpha_new *(T-ti(i))));
end
format long
disp([alpha_obs,alpha_new;sigma_obs sigma_new])
r_over_sigma = r ./Sigma;
rs_over_sigma = rs./Sigma_new;
% r_over_sigma = r .*sigma_new* exp(alpha_new *(T-t))/Sigma_new ;
%%%and now let's test normality of r_over_sigma
 histfit(r_over_sigma,40)
qqplot(r_over_sigma)
kurt = kurtosis(r_over_sigma)
skew = skewness(r_over_sigma)
normplot(r_over_sigma)
[h,p] = jbtest(r_over_sigma,0.01)
 histfit(rs_over_sigma,40)
qqplot(rs_over_sigma)
```

```
kurt = kurtosis(rs_over_sigma)
skew = skewness(rs_over_sigma)
normplot(rs_over_sigma)
[h,p] = jbtest(rs_over_sigma,0.01)

%  histfit(r,40)
% qqplot(r)
% kurt = kurtosis(r)
% skew = skewness(r)
% normplot(r)
% [h,p] = jbtest(r,0.01)
```

```
%This is a derivative by alfa of log-likelihood function for r_i (i=1,...N-1) in the code
%"calculateRs"
function f = mle(alpha, T, N, t, r)


f = ...
    sum(T - t) * sum(r.^2 .* exp(2 * alpha * (T - t)))    ...
    -                                                      ...
    N * sum(r.^2 .* (T - t) .* exp(2 * alpha * (T - t)))   ;
```

```
clear all; close all; clc;
load ngdata
% load excel
t = ngdata.data.Curve(2:end,1);
```

```
T = ngdata.data.Curve(1,2:end);
% T = ngdata.data.Expiry(:,2);
cT = ngdata.data.Expiry(:,1);
F = ngdata.data.Curve(2:end,2:end);
for ind_t = 1:length(t)
x = F(ind_t,1:end);
ind = find(~isnan(x));
x = x(ind);
if isempty(x),x=nan(1,12);ind = 1:12;end
if length(x)==11, x(12)=NaN;ind = [ind ind(end)+1]; end
if sum(abs(diff(x)))==0, x(:)=NaN;end
F(ind_t,ind) = x;
S(ind_t) = x(1); %vector of spot prices (prompt month)
for j = 1:12
P(ind_t,j) = x(j);
end
end
% subplot(2,1,2);
% mesh(P)
% % datetick('x',12)
% datetick('y',12)
% xlabel('time to maturity','fontsize',14)
% ylabel('market date','fontsize',14)
% zlabel('Forward price')
% title('Forward prices according to TIME-TO-MATURITY')
% subplot(2,1,1)
```

```
% mesh(T,t,F)%actual maturity
% datetick('x',12)
% datetick('y',12)
% xlabel('maturity','fontsize',14)
% ylabel('market date','fontsize',14)
% zlabel('Forward price')
% title('Forward prices according to TIME-OF-MATURITY')

x=1:12;%maturities
plot(x,P(2479,:),x,P(2571,:),x,P(2661,:),x,P(2752,:))%forward curves for October 14,2006 and March 29,2007
legend('November 14,2006','January 14,2007','April 14,2007','July 14,2007')
xlabel('time to maturity')
ylabel('forward price')
title('Forward curves')
```