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Pricing tranches of Collateralize Debt Obligation (CDO) using the one factor Gaussian Copula model, structural model and conditional survival model

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PRICING TRANCHES OF COLLATERALIZE DEBT OBLIGATION (CDO)

USING THE ONE FACTOR GAUSSIAN COPULA MODEL, STRUCTURAL MODEL AND CONDITIONAL SURVIVAL MODEL.

by

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Abstract

In this thesis we focus on the pricing of tranches of a synthetic collateralized Debt Obligation (synthetic CDO) which is a vehicle for trading portfolio of credit risk. Our purpose is not to create any new concept but we explore three different models to price the tranches of a synthetic CDO. These three models include the one factor Gaussian copula model, structural model and the conditional survival model

To this end, we provide a step by step description of the one factor Gaussian Copula model as proposed by Li, structural model as by Hull Predecu and White and conditional survival model by Peng and Kou. This thesis implement all the three models using the pricing procedure discussed in Peng and Kou paper[33]. For practical purpose, we use MATLAB to calculate a synthetic CDO tranche price based on the computation of a non-homogeneous portfolio of three reference entities under the one factor Gaussian copula model, structural model and conditional survival model. We calibrate the structural model to three cooperate bonds data to generate marginal probability of default key to all the three models.

The pricing result of the three models are very close for the risky tranches whiles that of the less risky are a little different which is attribute to the fact that the three models are affected by other parameters such as correlation parameter and loading factor. Comparisons are then made between the one factor Gaussian Copula and the structural model and the result tally with the observation Hull, Predescu and White made concerning Gaussian copula model and structural model.

Keywords : Synthetic CDO, Default Correlation,Copula, Structural model and Conditional survival model

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Table of Contents

List of Tables

List of Figures and Illustrations

List of Symbols, Abbreviations and Nomenclature

Symbol	Definition
U of C	University of Calgary

Chapter 1

Introduction To Collateralize Debt Obligation

Credit risk is the risk that a debt instrument will decline in value as a result of the borrower's inability (real or perceived) to satisfy the contractual terms of his/her borrowing arrangement. In the case of corporate debt obligations, credit risk includes default risk and credit spread, where we define :

- Default risk as exposure to loss due to non-payment by a borrower of a financial obligation when it becomes payable,
- Credit spread risk as the inability of a borrower to pay the additional interest for having a lower than satisfactory credit rating.

The most obvious way for a financial institution to transfer the credit risk of a loan or bond is by selling the loan/bond to another party. Another form of credit risk transfer is securitization, which has been used since 1980s (Fabozzi and Kothari 2007). In securitization, a financial institution, such as a bank, pulls together the loans it generates and sells them to a special purpose vehicle (SPV). The SPV obtains funds to acquire the pool of loans by issuing securities popularly called tranches. The SPV pays interest and principle to the tranche owners using the cash flow it receives from the pool of loans. While the bank retains some risk (that is, if the bank seeks a 60% protection on the loans they issue to reference entity, the bank retains 40% of the risk), majority of the risk is transferred to the investors of the tranches issued by the SPV.

The most recent developments for transferring credit risk are credit derivatives such as collateralized debt obligations (CDOs), credit default swap, total return swap etc. For financial institutions, credit derivatives such as credit default swap allow the transfer of credit risk to another party without the sale of the loan. An example of a CDO created without the sale of the loan is a synthetic CDO which seeks to generate income from swap contracts and other non-cash derivatives rather than straightforward debt instruments such as bonds, student loans, or mortgages. Here the SPV generates its income from its assets which are the credit default swaps and the tranches they issue for investors to invest. Holders of the tranche receive payments which are derived from the cash flow associated with the comprised credit swaps in the form of insurance contract premiums.

In general a CDO is a structured financial product that pulls together cash flow generating assets (mortgage, loans, bonds and credit default swaps) and repackage these assets into discrete tranches that can be sold to investors. A CDO can be initiated by one or more of the following: banks and non-bank financial institutions, also referred to as sponsor or SPV. The SPV distributes the cash flow from its asset portfolio to the holders of its various tranches in prescribed ways that take into account the relative seniority of the tranches. Any CDO can be well described by focusing on its three important attributes:

- Assets
- Liabilities
- Purpose

1.1 Assets

Like any company, a CDO has assets and its assets can be described in two main types of CDO market:

- Cash flow CDO
- Synthetic CDO

These types of CDO market only differ by their underlying portfolio.

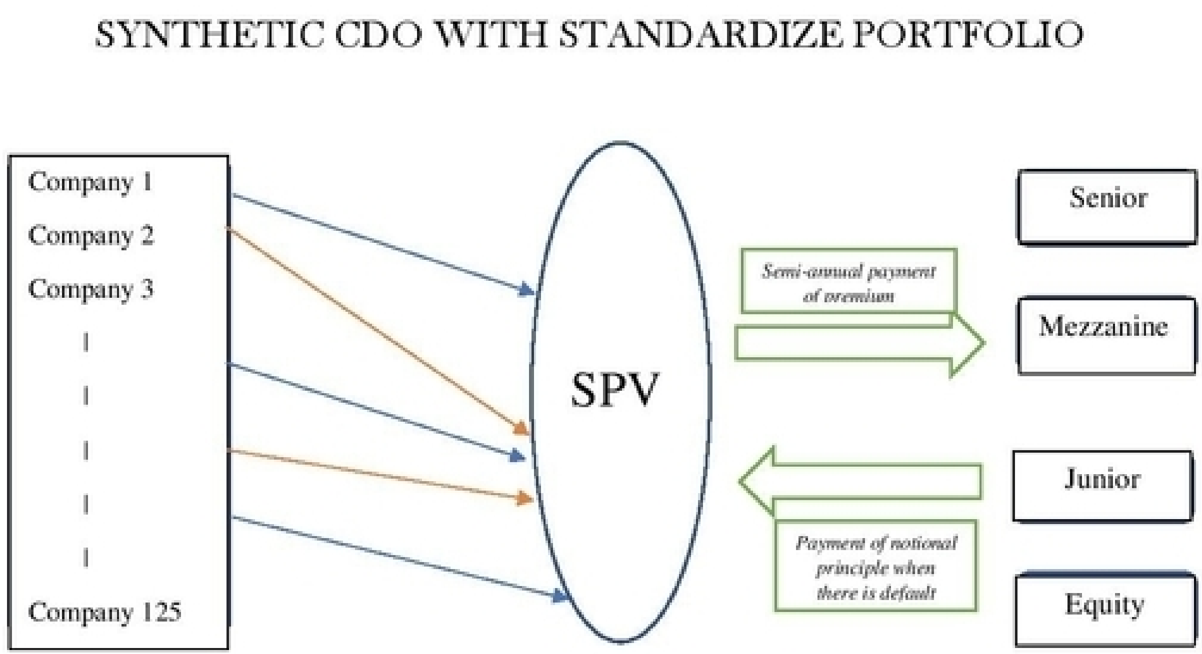
Cash flow CDOs are structured vehicles that issue different tranches of liabilities and use the income it generate from the tranches to purchase the pool of loans (CDO's assets) in the reference portfolio from a loan issuer(example bank). The cash flows generated by the assets (i.e the loan payment by reference entities) are used to pay investors interest and notional principle when there is default. Whenever there is a default event, payment of notional principles are made in a sequential order from the senior investor to the equity investor who bears the first-loss risk.

A synthetic CDO on the other hand does not actually purchase or own the portfolio of assets on which it bears credit risk. Instead, the creator of synthetic CDO gains credit exposure by selling protection through CDS. In turn, the synthetic CDO buys protection from investors through the tranches it issues. These tranches are responsible for credit loses in the reference portfolio that rises above a particular open point. Also, each tranches liability ends at a particular close point or exhaustion point.

This thesis is interested in CDOs created using credit derivatives such as credit default swaps (CDS) which are referred to as synthetic CDOs. There are two standardized CDS portfolios which are mostly used to create synthetic CDO in the market. They are the CDX NA IG and iTraxx standard portfolio. The CDX NA IG portfolio is made of 125 most liquid North American entities with investment grade credit rating as published by Markit from time to time. Its price is quoted in basis points. The iTraxx portfolio contains companies from the rest of the world and managed by the international index company also owned by Markit. It was formed by merger in 2004 and they are owned, managed and compiled by Markit. The synthetic CDO technology is applied to these portfolio into standardized tranches with different risk levels to satisfy investors with different risk favourites. Both CDX NA IG and iTraxx Europe are sliced into five tranches: equity tranche, junior mezzanine tranche, senior mezzanine tranche, junior senior tranche, and super senior tranche.The standard tranche structure of CDX NA IG is 0-3%, 3-7%, 7-10%, 10-15%, and 15-30% while that of iTraxx Europe is 0-3%, 3-6%, 6-9%, 9-12%, and 12-22%. For both portfolio, the swap premium of the equity tranche is paid differently from the non-equity tranches. For example, if the market quote of the equity is 20% then it means the SPV (the protection buyer) pays the equity tranche holder(protection seller) 20% of the principal upfront. In addition to the upfront payment, the SPV also pays the equity tranche holder a fixed 500 basis points (5%) premium per annual on the outstanding principal.

For all the non equity tranches, the market quotes are the premium in basis points, paid annually or quarterly. A structure of how these standardized portfolio are used to form synthetic CDO is the figure 1.1 below:

Figure 1.1: Synthetic CDO



Remark: The payments are not made to the companies, but to the holder of the CDS contracts. The companies represent the reference entities in the CDS contracts.

To better explain how a synthetic CDO works we define some terms.

1.1.1 Credit Default Swap

In a credit default swap the protection buyer agrees to pay a continuing premium or fee to the protection seller. Hence in case of a specified event, such as default or failure to pay by reference entity, the protection buyer can be compensated. This derivative contract permits market participants to transfer credit risk for individual credits. Credit default swaps are classified as follows:

- Single-name swaps
- Basket swaps

1.1.2 Single-name swaps

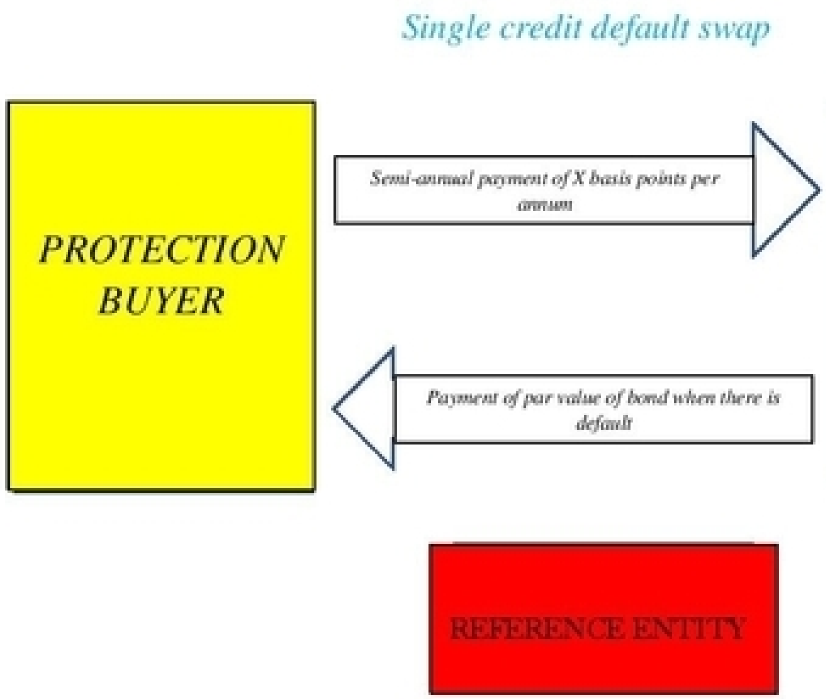
Single-name default swap involves two parties: a protection seller and a protection buyer. The protection buyer pays the protection seller a swap premium on a specified amount of the face value on the bond (the principal) from an individual company. In return, when the reference entity defaults, the protection seller pays the protection buyer who is the holder of the CDS contract its notional principle.

In the documentation of a CDS contract, a credit event is defined. The list of credit events in a CDS contract may include one or more of the following: bankruptcy, failure to pay an amount above a specified threshold over a specified period, and debt restructuring of the reference entity. The swap premium is paid on a series of dates, usually quarterly , based on the actual/360 date count convention. In the absence of a credit event, the protection buyer makes a quarterly swap premium payment until the expiration of a CDS contract. If a credit default occurs, two things happen. First, the protection buyer pays whatever premium accrued between the last payment date and the time of the credit default event (on a days fraction basis). After that payment, the protection buyer pays no further premium. Second, the protection seller makes a payment to the protection buyer. See figure 1.2 describing a single default swap.

1.1.3 Basket Default swaps

A basket default swap is a credit derivative on a portfolio of reference entities. The simplest basket default swaps are first-to-default swaps, second-to-default swaps, and nth-to-default swaps. With respect to a basket of reference entities (the number of people to whom the loan or bonds has been issued), a first-to-default swap provides insurance for only the first default, a second-to-default swap provides insurance for only the second default, and an nth-to-default swap provides

Figure 1.2: Single Default Swap

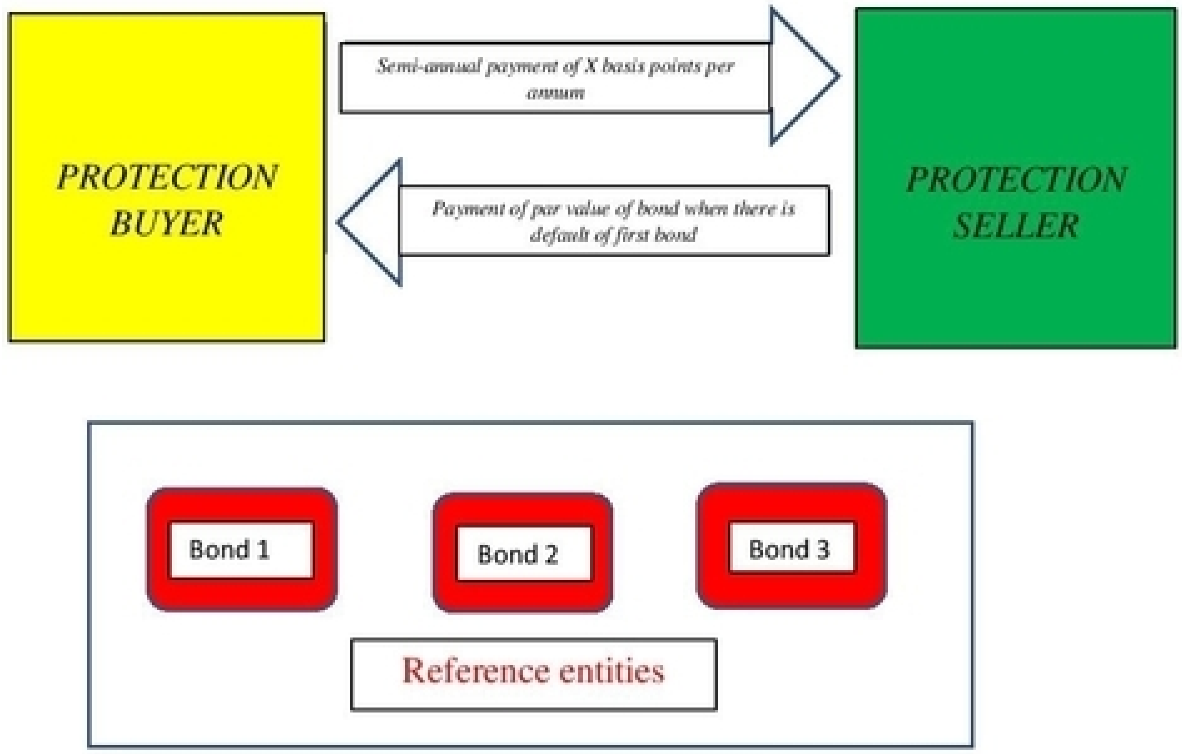


insurance for only the nth default. For example, in an nth-to-default swap, the protection seller defines and makes a payment for the nth defaulted reference entity. Once a payment is made upon the nth defaulted reference entity, the swap terminates. See figure 1.2.

1.2 Liability

Any company that has assets also has liabilities. In the case of a CDO, the liabilities of the SPV are its tranches which have a detailed and strict ranking of seniority going up the CDO's capital structure as equity, junior tranche, mezzanine and senior. These tranches are commonly labelled Class A, Class B, Class C and so forth going from top to bottom of the capital structure. They range from the most secured AAA rated tranche with the greatest amount of subordination beneath it, to the most riskier tranche called equity. In addition, the senior tranches are relatively safer because they have first priority on the collateral in the event of default. As a result, the senior tranches of a

Figure 1.3: Basket Default Swap



CDO generally has a higher credit rating and offers a lower coupon rate than the junior tranches, which offers a higher coupon rate to compensate for their higher risk level of default.

The rating of both the assets (reference entities) and liabilities (tranches) help investors to make a decision on what CDO to invest. A high rating of reference entities generally implies a lower risk for the investor investing in the tranches and therefore a lower premiums for the issuer(SPV). Rating agencies such as Moody and S & P adopt different methodologies for rating tranches. They can rate senior tranche of different CDOs differently and therefore one senior tranche may be a safer investment than another senior tranche.

1.3 Purposes

CDOs are normally created for one of the following three purposes. Depending on any of these purposes, CDOs can be created as a cash flow CDO or a synthetic CDO. The three purpose for creating CDO's are :

- To shrink a firm's balance sheet and reduce its regulatory costs since most firms that hold a large number of loans may have to pay high regulatory and funding costs.
- For asset managers to provide their management services to CDO investors.
- To increase the value of the firms asset by seeking protection.

A detailed purpose for creating CDO can be read in [26] by interested readers. We proceed with an example explaining the cash flow of the synthetic CDO. We first define a few terms.

1.3.1 Tranche Spread

A tranche is simply a percentage of a structured debt product (CDO) that seeks to distribute the interest rate risk across various credit rating securities. Examples of tranches in Collateralized Debt Obligations are Senior, Mezzanine, Junior and equity. Tranche spread simply means the price of the tranche (basically the percentage for calculating the premium) which the SPV pays to the tranche holder.

Tranche loss window

Tranche loss window is the loss that each tranche structure can absorb. As the name indicates, it has an open end and closing end. When the cumulative percentage loss of the portfolio of bonds reach the open end, investors begins to make payment to SPV, and when the cumulative percentage loss reaches the closing end, investors in the tranche make full payment of their notional principal and no further loss can occur to them. The open end is also the level of subordination that a particular tranche has beneath it. For example suppose the tranche structure is 0- 5% equity, 5%-15% junior, 15%-40% mezzanine and 40% - 100% senior debt. The equity has no subordination, the junior tranche has 5% subordination and attaches at the 5% loss level. Similarly the mezzanine tranche has 15% subordination and attaches at the 15% loss level and senior tranche has 40% subordination and attaches at 40%.

1.3.2 Types of Tranches

- Senior Tranche : A senior tranche is the highest tranche of a security, that is one deemed less risky. Any losses on the value of the security are only experienced in the senior tranche once all other tranches have lost all their value. For this safety, the senior tranche pays the lowest rate of interest. Investors who invest in senior tranche always have a credit rate of AAA since they pay huge amount of notional principle to the SPV to cover for its losses when there is a massive default event.

- Junior Tranche : A junior tranche is the lowest tranche of a security, i.e. the one deemed most risky. Any losses on the value of the security are absorbed by the junior tranche before any other tranche, but for accepting this risk the junior tranche pays the highest rate of interest. The junior tranche can also be the equity tranche.

1.4 Example explaining the dynamics of the Synthetic CDO

Let's assume there are 100 reference entities in the CDO pool with each entity given a bond with face value \$500 (hence the CDO pool has total of \$50000 notional) at 5 years maturity with zero recovery. Now suppose the issuer of these bonds seek protection on each of the bond from the SPV via CDS contract. Then the SPV protect the loan issuer against a total loss of \$50000. The SPV then creates a CDO contract by pulling together the CDS contract with the following coupon rates at the time of issuance as shown below in table 1.1

Tranches	Tranche window	Notional principle of Tranches	Tranche spread (coupon)
Senior	40%-100%	30000	100 basis point
Mezzanine	15%-40%	12500	250 basis point
Junior	5%-15%	5000	400 basis point
Equity	0-5%	2500	60%
Total		50000	

Table 1.1: Synthetic CDO table

From the table 1.1, the senior tranche is responsible for protecting the SPV against losses from \$20000 to \$50000 (which is a total of 60% of \$50000) whiles the SPV pays the senior tranche holder \$300 every three months. Likewise, the Mezzanine protects SPV against loss from 7500 to 20000 (which is total of 25% of \$50000) and the SPV pays Mezzanine \$312.5 (2.5% of \$12500) quarterly as well. The junior tranche too protects SPV against loss from \$2500 to \$7500 and SPV pays junior tranche \$200 every three months. However, the equity tranche holder protect the SPV against loss from \$0 to \$2500 whiles the SPV has to pay 60% of \$2500 (\$1500) as upfront fee to the equity tranche holder before the commence of the contract and an additional 500 basis points (5%) of \$2500 every year until the contract matures or until default. Note that the investors investing in the synthetic CDO do not pay the notional principle at the initial stage of the contract and their notional principles are determined by the total losses they protect.

Assuming 10 entities default in the CDO pool at the end of the 2nd year, then two things happen:

- Firstly, the CDO suffers a loss of \$5000 and so the holder of the equity tranche provides payment of \$2500 to SPV. Similarly the junior tranche holder provides \$2500 payment to the SPV. The equity tranche holder receives total payment of \$1750(\$1500 fee plus \$250 which is 2 years premium payment) and the junior tranche holder receives \$1600 (which is 4% of 5000 times 8 payments period) as premiums from the SPV.

- Secondly, the equity tranche holder loses all his notional principle and does not receive any premium again from the SPV. The junior tranche on the other hand loses half of its notional principle and receives premium on just the half of its notional principle ($2500) left.

This mechanism continues until the CDO matures. Now suppose the CDO does not incur any more losses after the two years and matures, then

- The junior tranche holder makes a profit of $300 that is

$$\left(\frac{400}{10000} * 2500\right) * 12 + 1600 - 2500 = \$300$$

- Similarly the mezzanine and senior tranche holders make profit of $6250 and $6000 respectively without suffering any losses. The calculation of mezzanine profit is

$$\left(\frac{250}{10000} * 12500\right) * 20 = \$6250$$

Generally the issuer of CDO (SPV) retains the equity tranche since it is too risky for investors. This means if there is no default at the end of the contract,the CDO issuer makes the most profit since the equity tranche has the highest interest. The interest rate the tranches pay depends on the risk of default of the firms in the reference portfolio. Because the risk of default of the firms are affected by the default dependency of the firms, default dependency should be taken into account in the pricing of tranches of the CDO. In particular, one needs to specify the joint distribution of the default arrival time of firms given their marginal distributions. In this vein, the structural model (developed by Merton (1974) and further extended by Zhou (2001)) and intensity-based models (see Lando, 1998, Duffie and Singleton, 1999) have been developed to capture the spirit of default correlation. However, the mostly used pricing model in the financial industry that captures default correlation is the Normal Gaussian Copula model as discussed in Li (2000) and further developed by others such as Gregory and Laurent (2003). Hull, Predescu and White (2005) incorporates default correlation in a structural model. Some other models focus on capturing the cumulative default intensities. For example, Kou and Peng in [33] propose modelling of CDO prices using the idea of default clustering where they aim to capture the prices of CDOs under the 2008 financial turmoil.

In this thesis, we examine three main models. These are :

- Gaussian Copula Model,
- Structural Model,
- Conditional Survival Model.

The Gaussian copula model is a type of a factor model developed by Li(2000). Its modelling can be understood as a combination of the copula approach and the firm value approach of Merton (1974). In the Gaussian Copula model framework, a firm defaults when its "asset value" modelled as a certain stochastic process X, falls below a barrier. Joint distribution of the firms values are defined in terms of the Gaussian copula. From this, one derive probability distribution of the number of defaults by time T and the cumulative loss distribution, which are directly related to the tranche interest rates when the companies have equal weight in the reference portfolio and recovery rates are assumed to be constant.

Hull, Predescu and White (2005) proposed a different structural model to price the spread of the tranches of a CDO. It models each firm value to has the common market risk and idiosyncratic risk. To implement the model it uses Monte Carlo simulation. The simulation was carried out by

- Drawing a set of zero mean unit variance normally distributed random variables of the common market risk and the idiosyncratic risk.
- These set of variables were then used to sample four sets of paths for the asset value X_i.

- Company i was assumed to default at the time t if the value of X_i falls below the barrier for the first time.

- The simulation was repeated 5,000 times generating 20,000 sets of paths. For each set of paths the number of defaults, d_k was determined and used to price CDO spread with a specified formula.

Conditional Survival (CS) model is a cumulative default intensity model and it is an extension to Duffie et al. (2001). This model was proposed by Peng and Kou in 2009. It models cumulative default intensity as a sum of market factors and idiosyncratic factors. These market factors are modelled using Polya processes and discrete integral CIR processes. The model was implemented to price CDO using exact simulation and pricing without simulation. The exact simulation was conducted by generating sample path from the market factors which are used to calculate the conditional survival probabilities given the market factors, which are then used to generate conditionally independent Bernoulli random variables which corresponds to default events. With these random variables the cumulative loss distribution is generated, which is used in pricing the CDO tranches.

This thesis intend to price the tranches of a synthetic CDO using these three models implemented under the exact simulation method discussed in Peng and Kou (2009).

The remaining part of the thesis is structured as follows: the second chapter provides the key formulations of the three methods. A detailed synthetic CDO tranche pricing process is described in the third chapter. Chapter 4 presents the data used and calibration performed on each model.

Chapter 5 concludes this thesis with remarks, criticism of the models and recommendations for future work.

Chapter 2

Pricing Models of Collateralized Debt Obligation (CDO)

In the rest of this thesis, we assume that there are n names in the CDO portfolio and use $\tau(i)$ to denote the default time of the i^{th} name, $i = 1, \ldots, n$. There are two types of credit risk modelling approach used in the financial world. They are the bottom-up approach, which builds models for the joint distribution of the default times of individual names in the portfolio, and the top-down approach, which builds models for the cumulative loss of the whole portfolio without referring to the underlying single names.

2.1 Bottom-up Models

In the bottom-up approach the default event of each reference entity is modelled. Then one computes the expected loss incorporating the correlation structure from which the prices of tranche spreads are obtained. Bottom up models specify the joint distribution of default times directly, usually through copula structures. A popular bottom up model is the Gaussian copula model proposed by Li (2000), which is the predominant model for the industry to price CDOs (see also Duffie et al., 2003; Andersen et al., 2003). An important advantage of the bottom up model is its ability to produce the joint distribution of default times which can cover a wider range of correlation product such as the first-to-default swap, the nth-to-default swap etc than top down model. For example, if we consider a CDO contract whose payoff depends on the underlying entity of the CDO (example vanilla bespoke CDO) then the reference entity is important and bottom up approach can be used to price it. A known disadvantage of bottom up model is its inability to fit market data well since it under prices CDO tranches. For example the one factor Gaussian copula does not appear to fit market data well (see Hull and white (2004) and Kalemanova et. al. (2005)).

2.1.1 Modelling the joint distribution of default times

Default correlation measures the tendency of two companies to default at the same time. Two ways for modelling the joint distribution of default times are

- Reduced form model
- Structural model

2.1.2 Reduced Form Model

Reduced form models, focuses on modelling the probability of default rather than trying to explain default in terms of the firm's asset value. Its main assumption is that default is always a surprise, that is there is no sequence of events announcing its arrival. Reduced form model does not model firm values but rather it models the instantaneous conditional probability of default. This model was originally introduced by Jarrow and Turnbull (1992) and subsequently studied by Jarrow and Turnbull (1995) and D. Duffie and K. Singleton (1999), among others. The key postulate emphasized in reduced form model is that the modeller observes the default indicator function as a Cox process with an intensity parameter λ and a vector of state variable X_t. Some known examples of the reduced model are Poisson processes and Cox processes .

Structural Models

The structural model was first introduced by Merton (1973) and it is based on the argument that a default happens when a reference entity total asset fails to meet its own total liabilities. In another words, the total asset value of the reference entity reaches a level that is below its total liability. In the financial world, company or reference entity has two major ways of raising capital to support their business. These include issuing bonds to people or issuing equity to investors. The bond holders are mostly at risk, but they have the priority in collecting the remaining asset if the bond issuer (reference entity) go bankrupt. Merton's model is based on these basic principles. Let V be the total asset of the reference entity and S the outstanding liability of the reference entity, both time dependent. At maturity T,

- If V > S, the bond holders receive their promised amount S, while the equity holders take away the rest which is $V - S$.
- If V $\leq$S, which is the case where the reference entity would fail their obligations, the bond holders take everything that is left of the total asset, and the equity holder (stock holder) get nothing.

2.2 Top-down Model

Top-down approach refers to modelling the cumulative portfolio loss distribution ignoring what happens individually to the reference entities. This means that when using the top down approach, the arrival times of the events (default) can be observed but the defaulted reference entity cannot be recognised. One advantage of the top-down model is that it is easy to implement and calibrate. The main disadvantage is that it cannot identify the defaulted entity in the reference pool. It therefore cannot be used to price bespoke CDO's since bespoke CDO's are created specifically per the needs of a group of investors and these investors might be interested in the reference entities that form the credit portfolio which top down model cannot identify in the case of default. Some examples of papers that present the top down model are Andersen (2006), Giesecke and Goldberg (2005) and Arnsdorf and Halperin (2007).

2.3 Introduction to the Models

We investigate the following bottom up approach CDO pricing models:

- One factor Gaussian Copula Model
- Hull and White Structural Model
- Conditional Survival Model

These models are used to estimate the prices of tranches of a CDO. The main method common in our analysis of these models is to generate samples from the joint distribution of the default times of the reference entity and use the tranche spread formula discussed in [33] to price the tranches of the CDO.

2.3.1 Introduction to Copula

To understand the one factor gaussian copula, we first need to give a brief introduction to how copula function was form as explained in [7]. When modelling a credit derivative, it is intuitive to think that default rate of a reference entity tends to be higher during recession and lower when the economy is booming. That is each reference entity is subject to a common set of macroeconomic influences in addition to idiosyncratic factors. The correlation structure should take into account these common factors when evaluating the CDO. More precisely, we would like to link these macroeconomic factors to the joint distribution of default times. Generally, knowing the joint distribution of random variables allows one to derive the marginal distribution and their correlation structure among the random variables but not vice versa. To specify the joint distribution, a copula function can be used since copula function enables one to link univariate marginals to a multivariate distribution. For instance, if we consider any uniform random variables $U_1, U_2, ..., U_n$, the joint distribution function C is defined as

$$C(u_1, u_2, ..., u_n) = P[U_1 \leq u_1, U_2 \leq u_2, ..., U_n \leq u].$$

This is called a copula function and for a given collection of univariate distribution functions $F_1(x_1), F_2(x_2), ..., F_n(x_n)$, the function

$$C[F_1(x_1), F_2(x_2), ..., F_n(x_n)] = F(x_1, x_2, ..., x_n)$$

also defines a joint distribution with marginals given the F_i. This property can easily be shown as follows:

$$\begin{aligned}
&C(F_1(x_1),F_2(x_2),...,F_n(x_n))\\
&\quad = P[U_1 \leq F_1(x_1),U_2 \leq F_2(x_2),...,U_n \leq F_n(x_n)]\\
&\quad = P[F_1^{-1}(U_1) \leq x_1,F_2^{-1}(U_2) \leq x_2,...,F^{-1})(U_n) \leq x_n]\\
&\quad = P[X_1 \leq x_1,X_2 \leq x_2,...,X_n \leq x_n]\\
&\quad = F(x_1,x_2,...,x_n).
\end{aligned}$$

The marginal distribution of X_i is

$$\begin{aligned}
&F_i(x_i) = P(U_i \leq F_i(x_i)) = C(F_1(+\infty),F_2(+\infty),...,F_i(x_i),...,F_n(+\infty))\\
&\quad = P[X_1 \leq +\infty,X_2 \leq +\infty,...,X_i \leq x_i,...,X_n \leq +\infty]\\
&\quad = P[X_i \leq x_i]\\
&\quad = F_i(X_i).
\end{aligned}$$

Some common Copula functions are

- Frank Copula : The Frank copula function is a symmetric Archimedean copula which is defined as

  $$C(u,v) = -\frac{1}{\alpha} ln\Big[1+\frac{(e^{-\alpha u}-1)(e^{-\alpha v}-1)}{e^{-\alpha}-1}\Big], \qquad -\infty \leq \alpha \leq \infty$$

  .

- Bivariate or Gaussian Copula The Gaussian copula is define as

  $$C(u,v) = \Phi_2\Big(\Phi^{-1}(u),\Phi^{-1}(v),\rho\Big) \qquad -1 \leq \rho \leq 2$$

  where Φ_2 is the bivariate normal distribution function with correlation coefficient ρ, and Φ^{-1} is the inverse of a univariate normal distribution function.

The joint PDF of a standard bivariate normal distribution function with correlation coefficient ρ is given as

$$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} exp\Big\{ - \frac{1}{2(1-\rho^2)}[x^2 - 2\rho xy + y^2]\Big\} \tag{2.1}$$

2.4 The One Factor Gaussian Copula Model [8],[13]

Li (1999, 2000) introduces the one-factor Gaussian copula model for the case of two reference entities and Laurent and Gregory (2003) extend the model to the case of N companies. In this model, the individual default times are assumed to be random and after a certain transformation follow a normal distribution. Default times of each reference entity is linked to a single common factor which is accordingly assumed to be normally distributed. From this linear dependence (that is, the individual default times are linearly associated to a common factor after this transformation) a correlation structure emerges between the pairwise random default times.

2.4.1 Modelling the firm's value:

The one factor Gaussian copula model models the firm value X_i just as in CAPM (Capital Asset Pricing Model) where a certain random variable r_i (i.e. the return on an asset), can be explained both by the level of a systematic risk M (i.e. the market risk) which is said to be a non-diversifiable risk and a corresponding diversifiable one say Z_i , which depends on the firm's specific risk. From this one can write a regression as follows:

$$r_i = \rho M + \beta Z_i.$$

Consider ρ as the sensitivity of the asset return r given the level of the systematic risk M and β its corresponding sensitivity due to the diversifiable risk Z . Now, we replace the asset return by the firm's value X_i and interpret M as the common factor and Z_i as the idiosyncratic risk (i.e. firm's specific risk). ρ and β are their respective weights, measuring the sensitivity of X towards each source of risk. Note that all these variables are random and normally distributed. We can therefore write the model for the individual firm value as follows:

$$X_i = \rho M + \beta Z_i. \tag{2.2}$$

It is assumed that the correlation between pairwise idiosyncratic risks Z_i is zero. That is $Corr(Z_i Z_j) = 0$. Similarly, M is assumed to be independent of Z_i. Without loss of generality, M, Z_i and X are assumed to have 0 mean and unit variance with the restriction that $\beta = \sqrt{1-\rho^2}$. We can then write the firm's value in terms of expectation as

$$E(X_i) = \rho E(M) + \beta E(Z_i) = 0$$

and

$$Var(X_i) = \rho^2 Var(M) + 2\rho\beta Cov(M, Z_i) + \beta^2 Var(Z_i)$$

Given that $Corr(M, Z_i) = 0$, $Var(M) = 1$, $Var(Z_i) = 1$, we can then write the variance as: $Var(X_i) = \rho^2 + \beta^2 = 1$ implying that $\beta = \sqrt{1-\rho^2}$. From this, we can rewrite the firm value process as:

$$X_i = \rho_i M + \sqrt{1-\rho_i^2} Z_i, \qquad i = 1, 2, ..., n \tag{2.3}$$

This equation defines a correlation structure between the X_i dependent on a single common factor M. The factor ρ_i satisfies $-1 \leq \rho_i \leq 1$ and $corr(X_i, X_j) = \rho_i \rho_j$.

2.4.2 Conditional Default Probability Distribution Function

At this stage, we focus on an important input in pricing a pool of credit risks. In this section before setting out the probability distribution of the default time conditional on the common factor, let us first explain what we consider as default when dealing with a firm's value. A default occurs when a stock price of a firm drops below a certain level x. From this, we can say that a credit defaults when the firm's value is below the considered barrier x, that is $X_i \leq x$. The probability of default is then $Pr(X_i \leq x)$. Now let us assume a CDO includes n reference entities, $i = 1, 2, \ldots, n$ and the default times τ_i of the ith reference entity follows a general continuous distribution F_i. Hence the probability of a default occurring before the time t is

$$P(\tau_i \leq t) = F_i$$

In a one-factor copula model, it is assumed that the relationship between the default time τ_i of the ith entity and the random variable X_i is

$$P(X_i < x) = P(\tau_i < t) = F_i, \qquad i = 1, 2, ...n.$$

This relationship is satisfied if

$$X_i = \Phi^{-1}[F_i(\tau_i)], \qquad \tau_i = F_i^{-1}[\Phi(X_i)]$$

and

$$t = F_i^{-1}(\Phi_i(x)), \qquad x = \Phi_i^{-1}(F_i(t))$$

given Φ_i to be the cumulative distribution of X_i and F_i the cumulative distribution of τ_i. We now have sufficient instruments to write down the conditional default probability function. Given the common factors value, we can write the probability of default time of the individual reference entity value as

$$P(\tau_i < t \mid M) = P\left(\rho_i M + \sqrt{1-\rho_i^2} Z_i < \Phi_i^{-1}(F_i(t)) \mid M\right) = P\left[Z_i < \frac{\Phi_i^{-1}(F_i(t)) - \rho_i M}{\sqrt{1-\rho_i^2}}\right] \qquad (2.4)$$

This intuitively tells us that given a value of the common factor M a default happens when the firm's idiosyncratic risk (i.e. firms specific information) hits a certain level. The cumulative distribution function of the conditional default probability that the individual firm will default at a certain time t given the level of the common factor is:

$$P(\tau_i < t \mid M) = H\left\{\frac{\Phi_i^{-1}(F_i(t)) - \rho_i M}{\sqrt{1-\rho_i^2}}\right\} \qquad (2.5)$$

where H is the standard normal cumulative distribution function. We refer to equation 2.4 as the conditional default probability. We can simulate the default event of each firm using this model.

The advantage of the copula model is that it creates a tractable multivariate joint distribution for a set of variables that is consistent with known marginal probability distributions for the variables. It should be noted that any type of Copula functions can be used to model the default correlation in valuation of CDOs.

2.4.3 Implementation of the One factor Model

The main quantity needed to price a CDO is the expected loss in the CDO credit portfolio which is a function of the individual reference entity's default risk within the credit portfolio and the dependence between their default time. The copula method can be implemented to provide a direct way to compute these quantities. The first step that has been used to compute the expected loss in the CDO portfolio is to compute the conditional default probability for each of the reference entity. Then given the common factor, these default probabilities are independent so one can compute the unconditional default probability distribution at a time t. The unconditional default probability distribution is then given as

$$P(\tau_i < t) = \int_{-\infty}^{\infty} P(\tau_i < t \mid M)\Phi(m)dm \tag{2.6}$$

with $\Phi(m)$ being the density of the factor M. To compute this unconditional default probability several approaches have been used. These include the Fourier transform, which was used by Laurent and Gregory (2003), recursive algorithm used by Andersen, Sidenius and Basu (2003) and the Binomial and Poisson approximation used by De Prisco, Iscoe and Kreinin (2005). Note that Binomial and Poisson approximation approach for finding the unconditional distribution is best when reference entities have equal weight in the portfolio with fixed recovery rate, same pairwise correlation and same default intensity.

Now knowing the unconditional probability one can calculate the expected loss (portfolio loss distribution) and continue to derive the tranche spread of the CDO. A formula to calculate the tranche spread is derived in the next chapter. Other approaches have been used and interested readers can read [13] to learn more on how the one factor Gaussian copula has been implemented.

However, in this thesis we implement the One factor Gaussian Copula in the way Kou and Peng (2009) modelled their CDO structure, which will be discussed in detail when we talk about the Conditional Survival model.

2.5 Structural Model

The first structural model of default for a single company was introduced by Merton (1974). In this model a default occurs if the value of the assets of the company is below the face value of the bond at a particular future time. Black and Cox (1976) provided an extension to the Merton model and other extensions were provided by Ramaswamy and Sundaresan (1993), Leland (1994), Longstaff and Schwartz (1995) and Zhou (2001a). However, all these papers looked at the default probability of only one issuer. Zhou (2001b) and Hull and White (2001) were the first to incorporate default correlation between different issuers into the Black and Cox first structural model. Zhou (2001b) deduced a closed form formula for the joint default probability of two issuers, but his results cannot easily be extended to more than two issuers. However Hull and White (2001) extented the model to many issuers but their correlation model requires computationally time-consuming numerical procedures [14].

Hull and White (2005) provides another way to model default correlation using the Black-Cox structural model framework. Their approach can be used to price credit derivatives and was computationally feasible when there are a large number of different companies. They assume that the correlation between the asset values of different companies is driven by one or more common factors. In their paper they consider first, a base case model, where the asset correlation and the recovery rate are both constant and then extend the base case model in three ways. The first allows the asset correlation to be stochastic and correlated with default rates. This is motivated by empirical evidence that asset correlations are stochastic and increase when default rates are high (using evidence from Servigny and Renault (2002), Das, Freed,Geng and Kapadia (2004)). They secondly assume that the recovery rate are stochastic and correlated with default rates. This was consistent with Altman et al (2005) who found that recovery rates are negatively dependent on default rates. They thirdly assume a mixture of processes for asset prices that leads to both the common factors and idiosyncratic factors having distributions with heavy tails.

2.5.1 The Firm Model

In the model, N different firms are assumed and we define $V_i(t)$ as the value of the assets of company i at time t $(1 \leq i \leq N_i)$. The value of each firm is assumed to follow a stochastic process as shown below

$$dV_i = \mu_i V_i dt + \sigma_i V_i dX_i$$

so that

$$dlnV_i = (\mu_i - \frac{\sigma_i^2}{2})dt + \sigma_i dX_i \tag{2.7}$$

where

- μ_i is the expected growth rate of the value of firm i
- σ_i is the volatility[1] of the value of firm i
- $X_i(t)$ is a Brownian motion[2].

Firm i defaults whenever the value of its assets falls below a barrier H_i. This corresponds to the Black and Cox (1976) extension of Merton (1974) model. μ_i and σ_i are assumed to be constant for the barrier to be linear.

Corresponding to H_i there is a barrier $H_i^*(t)$ such that company i defaults when X_i falls below $H_i^*(t)$ for the first time. Without loss of generality we assume that $X_i(0) = 0$. This means that

$$X_i(t) = \frac{lnV_i(t) - lnV_i(0) - (\mu_i - \frac{\sigma_i^2}{2})t}{\sigma_i}$$

[1] volatility refers to the amount of uncertainty or risk about the size of changes in the asset value of each reference entity

[2] A continuous stochastic process with stationary and independent increment with normal distribution mean 0 and standard deviation $\sqrt{t}$

and

$$H_i^*(t) = \frac{lnH_i - lnV_i(0) - (\mu_i - \frac{\sigma_i^2}{2})t}{\sigma_i}$$

where if $X_i(t) < H_i^*(t)$ firm i default. Similarly the value of each firm asset can be computed using the formula

$$V_i(t) = V_i(0)exp(\sigma X_i + (\mu - \frac{\sigma_i^2}{2})t) \tag{2.8}$$

and if $V_i(t) < H_i$ the firm i default as well. Either of the two default procedure explained above can generate the default event of each firm. Defining

$$\beta_i = \frac{lnH_i - lnV_i(0)}{\sigma_i}$$

and

$$\gamma_i = -\frac{\mu_i - \sigma^2/2}{\sigma_i}$$

then the default barrier for X_i is given as

$$H_i^*(t) = \beta_i + \gamma_i t$$

When μ_i and σ_i are constant, β_i and γ_i are constant. For this case, [16] shows that the probability of first hitting the barrier between times t and t+T is

$$P(t < \tau_i \leq t+T) = N\Big(\frac{\beta_i + \gamma_i(t+T) - X_i(t)}{\sqrt{T}}\Big) + exp[2(X_i(t) - \beta_i - \gamma_i t]N\Big(\frac{\beta_i + \gamma_i(t-T) - X_i(t)}{\sqrt{T}}\Big) \tag{2.9}$$

where N() denotes the standard normal cumulative distribution function

2.5.2 Asset Correlation

We assume that the process for modelling the state variable X_i is

$$dX_i(t) = \alpha_i(t)dF(t) + \sqrt{1-\alpha_i(t)}dU_i(t) \tag{2.10}$$

where F and U_i are Brownian motion with $F(0) = U_i(0) = 0$ and $dU_i(t)dF(t) = dU_i(t)dU_j(t) = 0, j \neq i$ The Brownian motion X_i therefore has a common component F and an idiosyncratic component U_i. The variable α_i, which may be a function of time or stochastic, defines the weighting given to the two components. F can be thought of as a common factor affecting default probability. When $F(t)$ is low there is a tendency for the X_i to be low and it causes the rate at which defaults will occur to be relatively high. Similarly, when F(t) is high there is also the tendency for the X_i to be high and the rate at which default will occur is relatively low.

The parameter α_i defines how sensitive the default probability of firm i is to the factor F. The correlation between the processes followed by the assets of firms i_1 and i_2 is $\alpha_{i_1}\alpha_{i_2}$. Note that these $X_i(t)$s are used to simulate the asset value and can also be used to simulate default events.

2.5.3 Model Implementation :

Hull and White implemented using Monte Carlo simulation. They considered points in time $t_0, t_1, ..., t_n$ (where t_0 is the initial date of the contract and typically $t_k - t_{k-1}$ is three months) and replaced the continuous barrier in the Black and Cox model with discrete barrier that was defined only at those points. The barrier was chosen using the numerical procedure so that the default probabilities in each interval (t_{k-1}, t_k) were the same as those for the continuous barrier. The simulation was carried out by

- Drawing a set of zero mean unit variance normally distributed random variables Δf_k and $\Delta u_{ik} \qquad (1 \leq k \leq n, 1 \leq i \leq N)$
- These set of variables were used to sample four sets of paths for the X_i.

$$f_k^m = f_{k-1}^m + (-1)^m \alpha_{ik} \Delta f_k \sqrt{\Delta t} \qquad m = 0,1 \qquad f_0^m = 0$$

$$u_{ik}^j = u_{i,k-1}^j + (-1)^j \sqrt{1 - \alpha_{ik}^2} \Delta u_{i,k} \sqrt{\Delta t} \qquad j = 0,1 \qquad u_{i,0}^j = 0$$

$$X_{ik}^{mj} = f_k^m + u_{ik}^j \qquad m, j = 0,1$$

where X_{ik}^{mj} is the value of X_i at time t_k for a particular combination of m and j, and α_{ik} is the value of α_i between times t_{k-1} and t_k.

The correlation parameters α_i were assumed to be constant in each interval (t_{k-1}, t_k). Company i was assumed to default at the mid point of time interval (t_{k-1}, t_k) if the value of X_i is below the barrier for the first time at time t_k. The simulation was repeated 5,000 times generating 20,000 sets of paths. For each set of paths the number of defaults, d_k, that occur in each time interval, (t_{k-1}, t_k), is determined. They then had a pricing formula which they used to estimate the prices of the tranche spread. In this thesis we implement this method but use a different pricing method to obtain the prices of the CDO tranches.

2.6 Conditional Survival (CS) Model

The main motivation behind the use of Conditional Survival model is the default clustering effect. Default clustering effect means that one default event tends to trigger more default events both across time and cross-sectionally. The most relevant empirical evidence of the default clustering effect happened in 2008 where there was a massive financial turmoil. To capture default clustering effect observed in this financial crisis was one of the motivation of the CS model (Peng and Kou 2009).

In September 20, 2007 and March 14, 2008 there was a noticable default clustering effect observed in the tranche spreads of the iTraxx Europe Series 8 5-years Index in the CDO Market [33] as shown below.

Tranche	0-3%	3%-6%	6%-9%	9%-12%	12%-22%	22% -100%
09/20/2007	1812	84	37	23	15	7
03/14/2008	5150	649	401	255	143	70

Table 2.1: iTraxx Europe Series 8 5-years Index

From the table it can be observed that on March 14, 2008 the senior tranche had an extremely high spread which was around 10 times as high as the previous year. Similarly, all the other tranches from the table had high spread on March 14 ,2008 compared to September 20, 2007. This showed that the default correlation was substantially high and the portfolio loss distribution had significantly heavy tails at the peak of the financial crisis in 2008.

Default clustering can be captured by serial correlation(where serial correlation means the relationship between a given variable and itself over various time and it is used to predict how well the past price of a security predicts the future price) but some traditional stochastic processes such as Brownian motions and Poisson processes have independent increments, which makes it hard for them to generate serial correlation. To generate serial correlation, Polya process is used since it is a counting process which has a fixed jump size 1 at each jump time. If one finds that the fixed jump size might be restrictive, one could use a compound Polya process to incorporate random jump sizes. A compound Polya process is easy to simulate, and its Laplace transform has a closed-form formula, which is derived in the Appendix of this thesis. More precisely, a Polya process $M(t)$ is a Poisson process with a random rate ξ, where ξ is a Gamma random variable with shape parameter α and scale parameter β. A gamma distribution is simply a two-parameter family of continuous probability distributions and a random variable X is said to be gamma-distributed with shape α and rate β if it is written as $X \sim \Gamma(\alpha, \beta)$.

In other words, conditional on ξ, $M(t)$ is a Poisson process with rate ξ. The marginal distribution of a Polya process $M(t)$ is given by

$$P(M(t) = i) = \binom{\alpha + i - 1}{i} \Big(\frac{1}{1+\beta t}\Big)^{\alpha} \Big(\frac{\beta t}{1+\beta t}\Big)^{i}, \qquad t > 0, i \geq 0$$

A Polya process has stationary but positively correlated increments since the covariance between their market factor at two point is greater than zero. Hence the arrival of one event tends to trigger the arrival of more events, which makes Polya processes suitable for modelling defaults that cluster in time. For more evidence of cross-sectional default clustering, one can read Peng and Kou (2009), Azizpour and Giesecke (2008) and Longstaff and Rajan (2007).

2.6.1 CS Firm modelling

Peng and Kou (2009) extended the model proposed by Duffie et al. (2001) and proposed CS model as

$$\Lambda_i = \sum_{j=1}^{J} a_{i,j} M_j(t) + X_i(t), \tag{2.11}$$

$$\tau_i = inf\{t \geq 0 : \Lambda_i(t) \geq E_i\}, \qquad i = 1, ..., n \tag{2.12}$$

where $\Lambda_i(t)$ is the cumulative default intensity of the i^{th} name. Here $M_j(t)$ represents the j^{th} market factor (here the market factors could represent both observable and unobservable macroeconomic variables that have market wide impact on the reference entity default probability) in the cumulative default intensities, which is a non-negative, increasing, and right-continuous stochastic process with $M_j(0) = 0, j = 1, ..., J$. These market factors are all independent of E_i and $X_i(t)$ but they can be correlated with each other. Also unlike the intensity based model introduced by Jarrow and Turnbull (1995), $M_j(t)$ is allowed to have jumps.

The coefficient $a_{i,j} \geq 0$ is the constant loading of the i^{th} reference entity on the j^{th} market factor, for $i = 1, ..., n; j = 1, ..., J$. $X_i(t)$ represents the idiosyncratic part of the cumulative default intensity of the i^{th} reference entity, which is a non-negative, right-continuous, and increasing process with $X_i(0) = 0$, for $i = 1, ..., n$. $X_i(t)$ are mutually independent, and also independent of the market factors $M_j(t)$. E_i are independent exponential random variables with mean 1. E_i is independent of the processes $X_i(t)$ and $M_j(t)$.

Interestingly, there is no need to specify the dynamics of the idiosyncratic default risk factors $X_i(t)$. All that we need to specify in the CS model is the dynamics of the market factors $M_j(t)$, $j = 1, ..., J$.

Conditional Survival Probabilities

We call the model conditional survival because conditional survival probabilities are the building blocks for pricing tranches of the CDO as it will be shown in chapter 3. The conditional survival probabilities in the CS model are very simple. Let $M(t) = (M_1(t), ..., M_J(t))$ be the vector of market factor processes and let

$$q_i^c(t) = P(\tau_i > t \mid M(t))$$

and

$$q_i(t) = P(\tau_i > t)$$

be the conditional survival probability and marginal survival probability of the i^{th} reference entity, respectively. Then we have this proposition as in [33]

Proposition 1 *For the i^{th} reference entity in the CS model, we have*

$$q_i^c(t) = E\left[e^{-X_i(t)}\right] e^{-\sum_{j=1}^{J} a_{i,j} M_j(t)} \tag{2.13}$$

given

$$q_i(t) = E\left[e^{-X_i(t)}\right] E\left[e^{-\sum_{j=1}^{J} a_{i,j} M_j(t)}\right] \tag{2.14}$$

And $q_i^c(t)$ can be represented by $q_i(t)$ and the market factors as

$$q_i^c(t) = q_i(t). \frac{e^{-\sum_{j=1}^{J} a_{i,j} M_j(t)}}{E\left[e^{-\sum_{j=1}^{J} a_{i,j} M_j(t)}\right]} \tag{2.15}$$

The proof of this proposition is in the appendix. This proposition shows that the conditional survival probability $q_i^c(t)$ can be computed analytically if the marginal survival probability $q_i(t)$ is known and the Laplace transforms of the market factors $M(t)$ have closed-form formulae. The marginal survival probabilities $q_i(t)$, $i = 1, ..., n$. can usually be implied from market quotes of single name derivatives or from relevant data analysis, since it seems unreasonable to include a reference entity with unknown marginal default probability in the underlying portfolio of a CDO. The conditional survival probabilities $q_i^c(t), i = 1, ..., n$ are useful because given the market factors $M(t)$, the random variables $1_{(\tau_i \le t)}, i = 1, ..., n$ are conditionally independent, and $1_{\tau_i \le t}$ has a Bernoulli $(1 - q_i^c(t))$ distribution. This enables us to compute CDO tranche spreads. To be precise, let N_i and R_i be the notional principal and recovery rate of the i^{th} reference entity in the portfolio, respectively. Then the cumulative loss process of the portfolio is given by

$$L_t = \sum_{i=1}^{n} (1 - R_i) N_i 1_{(\tau_i \le t)}, \qquad t \ge 0 \tag{2.16}$$

Therefore, conditioned on $M(t)$, L_t is equal to the linear combination of independent Bernoulli random variables, which further implies that the marginal distribution of L_t is only determined by the dynamics of the market factors $M(t)$. However, since CDO tranche spreads only depend on the marginal distribution of L_t (as shown in next chapter), it follows that CDO tranche spreads also depend only on the dynamics of the market factor $M(t)$.

From equation 2.12, we clearly see that the CS model does not need the dynamics of idiosyncratic cumulative intensities $X_i(t)$. Hence no parameters for $X_i(t)$ are needed and there is no need to simulate $X_i(t)$ for pricing CDOs. But instead we assume that the functions $q_i(t)$ are specified.

Specifying Market Factors in Model

According to the previous discussion, to price CDO tranches, we only need to specify the dynamics of the market factors M(t). Peng and Kou (2009) uses Polya processes and discrete integral of CIR processes as market factors but in our simulation we just considered Polya processes as the only one market factor in the CS model.

Using a Polya process as the market factors has several advantages:

- A Polya process has stationary but positively correlated increments, which makes it suitable for modelling defaults that cluster in time.
- The jumps of the market factor Polya process cause simultaneous jumps in the cumulative intensities of all names, which produces strong cross-sectional correlation among individual names.
- A Polya process is computationally tractable. The simulation of a Polya process, which comprises the simulation of a Gamma random variable and a Poisson process, is straightforward.
- In addition, the Laplace transform of a Polya process can be calculated in closed form.

Properties of the CS Model

The CS model exhibits the following properties.

- The CS model can generate the simultaneous default of many reference entities since when the market factor $M_j(t)$ jumps, all cumulative default intensity $\Lambda_i(t)$ jump together simultaneously, which can cause many $\Lambda_i(t)$ to cross their respective barriers E_i.

- The CS model allows fast CDO pricing based on simulation because one only needs to simulate the value of market factors at coupon payment dates.

- The CS model provides automatic calibration to the underlying single-name CDS in the portfolio since it uses marginal survival probability as an input.

The implementation of this model to price CDO is discussed in chapter 3. Also the other two models, namely one factor Gaussian copula model and the structural model have been implemented and the prices of the tranches are calculated using the same procedure as in [33].

2.7 Comparison of the three models

In contrast to the one factor Gaussian copula and the structural model, the CS model does not specify the dynamics for idiosyncratic risk factors $X_i(t)$), thanks to the $q_i^c(t)$ equation that links the conditional and unconditional survival probabilities without using $X_i(t)$. To be able to compare these models, the q_i's in the CS model are extracted from the market data since that is all we need and we calibrate the parameters of the other two models to match that of the q_is. A detailed approach on how this is done is explained in chapter 4.

Now we proceed to chapter 3 and talk about the CDO pricing formula of this thesis.

Chapter 3

CDO Pricing Method

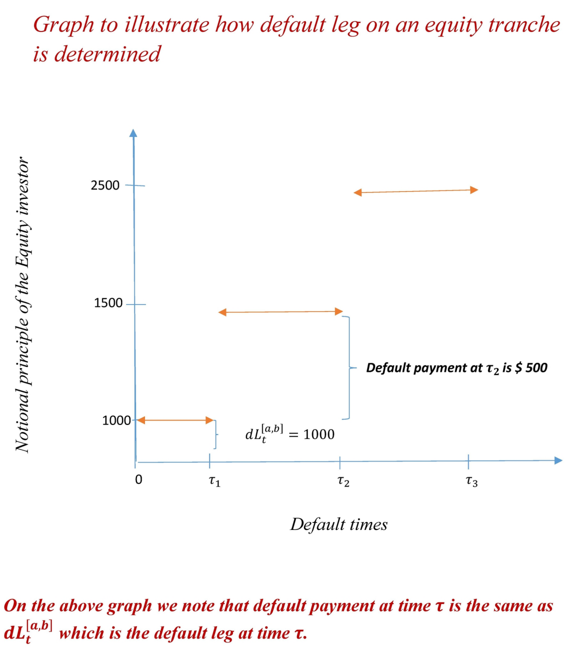
3.1 Synthetic CDO Tranche Spread Formula

Using the pricing theory explained in [33], let $T_0 = 0$ be the effective date of the synthetic CDO contract, T be the maturity date of the CDO, and $0 < T_1 < T_2 < \ldots < T_m = T$ be the coupon payment dates. Let L_t be the cumulative loss process as define in Eq. (2.15) and $[a,b]$ be the CDO tranche loss window. The tranche cumulative loss process is defined up to time t as

$$L_t^{[a,b]} = (L_t - a)^+ - (L_t - b)^+, \qquad 0 \leq t \leq T. \tag{3.1}$$

Pricing the CDO tranche involves a premium leg which is also called a fixed leg and a default or *floating* leg. To find the present value of the default leg, we already know that the tranche holder (protection seller) agrees to pay a certain amount of capital should a default occurs. Hence the loss given default that the tranche holder faces is known to be the default leg at the time of default. From an investors point of view, the payment that he could face should a default event occurs depends on the difference between the tranche loss at dates T_{i-1} and T_i which can be written as $dL_t^{[a,b]}$. Also since we are trying to set an expected amount of capital in case of default which in terms of expectation represents the tranche loss given default (default leg) we can expressed the present value of the default leg as $E[\int_0^T D(0,t)dL_t^{[a,b]}]$. The present value of default leg is just the integral for a step function mean weighted sum of jumps and $D(0,t)$ is the risk free discounter factor from time t to time 0. For simplicity, it is usually assumed that defaults only occur in the middle of coupon payment dates (see e.g. Mortensen, 2006; Andersen et al., 2003; Papageorgiou

Figure 3.1: EQUITY TRANCHE



et al.,2007). Under this simplification, the present value of the default leg is given by:

$$\begin{aligned} E\Big[\int_0^T D(0,t)dL_t^{[a,b]}\Big] &= E\Big[\sum_{i=1}^{m}\int_{T_{i-1}}^{T_i} D(0,t)dL_t^{[a,b]}\Big] \\ &= E\Big[\sum_{i=1}^{m} D(0,\frac{T_{i-1}+T_i}{2})(L_{T_i}^{[a,b]} - L_{T_{i-1}}^{[a,b]})\Big] \end{aligned} \tag{3.2}$$

Next, we compute the present value of the premium leg. The outstanding notional of the tranche at time t is $O_t^{[a,b]} = b - a - L_t^{[a,b]}$. Then the coupon payment at time T_i, $\quad i = 1,\ldots,m$ is specified as

$$S^{[a,b]} * (T_i - T_{i-1})\int_{T_{i-1}}^{T_i}\frac{O_t^{[a,b]}}{T_i - T_{i-1}}dt = S^{[a,b]}\int_{T_{i-1}}^{T_i} O_t^{[a,b]}dt \tag{3.3}$$

Assuming defaults only occur in the middle of premium periods, the present value of the premium leg is given by

$$\begin{aligned}E\Big[\sum_{i=1}^{m} D(0,T_i)S^{[a,b]}\int_{T_{i-1}}^{T_i} O_t^{[a,b]}dt\Big] &= E\Big[\sum_{i=1}^{m} D(0,T_i)S^{[a,b]}(T_i-T_{i-1})\frac{1}{2}(O_{T_{i-1}}^{[a,b]}+O_{T_i}^{[a,b]})\Big] \\ &= S^{[a,b]}E\Big[\sum_{i=1}^{m} D(0,T_i)(T_i-T_{i-1})\Big(b-a-\frac{L_{T_i}^{[a,b]}+L_{T_{i-1}}^{[a,b]}}{2}\Big)\Big]\end{aligned} \tag{3.4}$$

Note that only the T_{i-1} and T_i with $O_t^{[a,b]}$ non-zero are included in the sum, that's because there are no premium once the tranche loss reaches its end point b. By making the present value of the premium leg and that of the default leg equal, we obtain the fair spread(price) of the tranche as

$$S^{[a,b]} = \frac{E\Big[\sum_{i=1}^{m} D(0,\frac{T_{i-1}+T_i}{2})(L_{T_i}^{[a,b]}-L_{T_{i-1}}^{[a,b]})\Big]}{E\Big[\sum_{i=1}^{m} D(0,T_i)(T_i-T_{i-1})\Big(b-a-\frac{L_{T_i}^{[a,b]}+L_{T_{i-1}}^{[a,b]}}{2}\Big)\Big]}, \qquad a>0 \tag{3.5}$$

The contractual specification of cash flows for the equity tranche is different from those of the other tranches. The seller of the equity tranche pays an up-front fee at the beginning of the CDO and pays coupons at a fixed running spread of 500 basis points per year to the buyer. For example if we consider figure 3.1 at time τ_1 the equity tranche holder receives \$75 (5% of 1500) yearly from the SPV if no default occurs. However after τ_2 the equity investor receive \$ 50 as premium yearly. The equity tranche spread is defined as the ratio of the up-front fee to the notional of the equity tranche, which is given by

$$\begin{aligned}S^{[0,b]} = \frac{1}{b}\Big\{&E\Big[\sum_{i=1}^{m} D(0,\frac{T_{i-1}+T_i}{2})(L_{T_i}^{[0,b]}-L_{T_{i-1}}^{[0,b]})\Big] \\ &-0.05E\Big[\sum_{i=1}^{m} D(0,T_i)(T_i-T_{i-1})\Big(b-\frac{L_{T_i}^{[0,b]}+L_{T_{i-1}}^{[0,b]}}{2}\Big)\Big]\Big\}\end{aligned} \tag{3.6}$$

The formula for the spread of equity tranche follows as

$$b*S^{[0,b]} = E\Big[\sum_{i=1}^{m} D(0,\frac{T_{i-1}+T_i}{2})(L_{T_i}^{[0,b]}-L_{T_{i-1}}^{[0,b]})\Big]-0.05E\Big[\sum_{i=1}^{m} D(0,T_i)(T_i-T_{i-1})\Big(b-\frac{L_{T_i}^{[0,b]}+L_{T_{i-1}}^{[0,b]}}{2}\Big)\Big].$$

3.2 CDO Pricing Simulation

It is clear from the above two equations that the CDO tranche spreads are determined by the marginal distribution of the cumulative loss process L_t at coupon payment dates $T_k, k = 1, ..., m$. L_t determines $L_t^{[a,b]}$ for any tranche window hence, to price CDOs we only need to simulate L_{T_k} exactly.

3.2.1 Simulating Cumulative Loss for One factor Gaussian Copula Method

As described under the one factor Gaussian copula, to simulate the cumulative loss we

- Simulate each X_i according to the model for the firms value which is the sum of idiosyncratic component and a common component of the market as in equation (2.2).

- Compare each generated X_i with default barrier generated from the market data (explicit way of deriving the default barrier is explained under chapter 4). This comparison generates the default event of each of the firm which together forms the cumulative loss.

- We then proceed with finding the price of the tranches using equation (3.5).

3.3 Simulating Cumulative Loss for Structural Model

Under the structural model, there are two ways in which the cumulative loss can be simulated; the first approach is

- Simulate X_i from equation (2.9) which is a Brownian motion following common Weiner process M and idiosyncratic Weiner process Z_i.

- As each firm is modelled to follow X_i and the firm defaults whenever X_i falls below a default barrier H_i^* as described under the structural model [13].

- Comparing each X_i with H_i^* generates the cumulative loss of all firms and we can then proceed in finding the CDO tranche prices using equation (3.5).

The second approach involves

- Simulating X_i from equation 2.9 and using it to calculate the asset value V_i.
- Comparing each reference entities asset value V_i with default barrier H to generate the cumulative loss of all firms and then find the CDO tranche prices.

However we are interested in using the first approach.

3.3.1 Simulating Cumulative Loss for Conditional Survival Model

An advantage of using marginal survival probabilities as input to the CS model is that it renders the simulation of idiosyncratic risk factors unnecessary. One only needs to simulate the market factor M(t). Thus, it allows fast simulation for CDO pricing. The key observation is that conditional on M(t), the random variables $1_{\tau_i \le t}, i = 1, ..., n.$ are conditionally independent, and $1_{\tau_i \le t}$ has a Bernoulli $(1 - q_i^c(t))$ distribution, $i = 1, ..., n$. Suppose M(t) can be easily simulated and its Laplace transform can be calculated in closed form, then we have the following algorithm to simulate the cumulative loss $L_{T_k}, k = 1, ..., m$:

1. Generate sample path of market factors $M(T_k), k = 1, ..., m$.
2. For each $i = 1, ..., n$, calculate the conditional survival probability $q_i^c(T_k), k = 1, ..., m$, according to Eq. (2.14), using the sample of market factor generated in Step 1.
3. Generate independent Bernoulli random variables $I_{i,k} \sim$ Bernoulli $(1 - q_i^c(T_k))$ distribution, $i = 1, ..., n; k = 1, ..., m$.
4. Calculate $L_{T_k} = \sum_{i=1}^{n} (1 - R_i) N_i I_{i,k}, \qquad k = 1, ..., m$.

Using samples of L_{T_k} generated by the above algorithm, we can calculate the tranche spreads $S^{[a,b]}$ given by equation (3.5).

We move to chapter 4 and discuss the data analysis and the numerical results.

Chapter 4

Numerical Results

In this section, we show the numerical results of pricing tranches of the CDO using the three methods we discussed in Chapter 2. For simplicity of computation purposes, we choose a hypothetical non-homogeneous portfolio as reference entities for pricing of the synthetic CDO tranches. That is:

- The correlation is assumed to be non-homogeneous for both Copula model and Structural model
- The number of firms was assumed to be just 3 for easy simulation.
- The recovery rate for each firm is assumed to be non-homogeneous.
- A 5 years contract with quarterly coupon payments is assumed.

We want to make our comparisons using relevant values of the parameters used in each of the models. To obtain the set of relevant values for the parameters, we use market data calibrated to the structural model. This allows us to estimate the q_i in CS model and find the barrier for both the one factor Gaussian copula and the structural model. To establish these we pick three firms from the market namely:

- Apple Corporate Bond
- American Credit Express Corporate Bond
- Pepsi Corporate Bond

These three bonds are used to estimate the volatility(σ) and distance to default. To estimate μ which is the risk-free interest rate we obtain it from the US treasury bill and it was 2% as of 07-02-2017

Reference Entity	Notional principle of loan given	Recovery rate
A	1113.57	40%
B	1114.22	50%
C	1000	30%
Total	3227.79	

Table 4.1: Portfolio Structure Table

Table 4.1 above highlights the firm structure data used in pricing the tranches. This data is just fictious it has nothing to do with the market data.

4.1 Description of Data for extracting Probability of default (q_i) and default barrier

We obtained Corporate coupon bond data from Bloomberg. Three bond firms were used as listed above as Pepsi Company, Apple Company and American Credit Express corporate bond. All the bonds had maturity of at most five years and these bonds were used since our simulation on pricing the tranches of CDO was executed for firms with five year maturity and the current prices for each bond was selected on 7-02-2017. A typical description of a particular coupon bond data is shown in the figure below.

From figure 4.1 the apple bond is AAPL 2.5 02/09/22 corporate bond and it has a semi-annual coupon payment of 2.5 and was Announced on 02-February-2017 with a maturity date 02-September-2022.

Figure 4.1: Apple bond.

4.1.1 Estimating Probability of Default, Distance to Default, Mean and Volatility

Using the idea explained under Bootstrap method in [27] the price of a coupon bond over a particular maturity is given as

$$C_{j,k} = \sum_i C_{j,k,i} Z_j(0, T_i) \tag{4.1}$$

where $C_{j,k}$ is the price of the kth bond in the jth category (for instance the first apple bond under the apple company), $C_{j,k,i}$ is the coupon payment of this bond at date T_i and $Z_j(0,T_i)$ is the zero coupon bonds of the firm. The pricing formula of a zero coupon bond is given by

$$Z(0,T) = e^{-rT} P(\tau_m > T) \tag{4.2}$$

where τ_m is the first passage of the barrier m and m is a positive number. Using the idea of maximum Brownian motion we know that $\tau_m > T$ if and only if the maximum of Brownian motion on $[0,T]$ denoted by $M(T)$ satisfies $M(T) \leq m$. Conversely $M(T) \leq m$ if and only if $\tau_m > T$. Then both $\tau_m > T$ and $M(T) \leq m$ are identical so we have

$$P(\tau_m > T) = P(M(T) \leq m).$$

Hence the zero coupon bond can be written as

$$Z(0,T) = e^{-rT} P(M(T) \leq m). \tag{4.3}$$

To deduce a formula for $P(M(T) \leq m)$, we find the joint distribution between Brownian motion with drift and the maximum of Brownian motion which is given as

$$f_{M(T),\tilde{X}_t}(m,w) = \frac{2(2m-w)}{T\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2}, \quad w \leq m, \quad m \geq 0$$

and is zero for other values of m and w. Then integrating

$$\int_0^m \int_{-\infty}^u f_{M(T),\tilde{X}_t}(u,w) e^{\alpha w - 1/2\alpha^2 T (2u-w)^2} dw du$$

gives

$$P(M(T) \leq m) = N(\frac{m - \alpha T}{\sqrt{T}}) - e^{2\alpha m} N(\frac{-m - \alpha T}{\sqrt{T}}), \quad m \geq 0. \tag{4.4}$$

where α is the drift and m is the barrier of the Brownian motion $\tilde{X}_t$. Looking at the structural model, the asset value follows a geometric Brownian motion

$$X_t = rX_t dt + \sigma X_t d\tilde{X}_t \tag{4.5}$$

and applying Ito's formula[31] we obtain

$$ln\frac{X_t}{X_0} = (\mu - \frac{\sigma^2}{2})t + \sigma\tilde{X}_t \tag{4.6}$$

and taking the exponential and multiplying both side by X_0 gives the solution

$$X_t = X_0 \, e^{\left(\left(r - \frac{1}{2}\sigma^2\right)t + \sigma\tilde{X}_t\right)} = X_0 e^{\sigma\hat{X}_t}$$

where

$$\hat{X}_t = \hat{\alpha} t + \tilde{X}_t \tag{4.7}$$

and

$$\alpha = \frac{r}{\sigma} - \frac{1}{2}\sigma.$$

Then the first passage of distance to default (d) by X_t is the same as the first passage by $\hat{X}_t$ of

$$\frac{1}{\sigma}log\Big(\frac{d}{X_0}\Big),$$

which is negative. Similarly the first passage of distance to default (d) is the same as the first passage by $-\hat{X}_t$ of

$$-\frac{1}{\sigma}log\Big(\frac{d}{X_0}\Big),$$

which is positive. So the default time is τ_m, the first passage by $-\hat{X}_t$ of the barrier $m = -\frac{1}{\sigma}log\Big(\frac{d}{X_0}\Big)$ given

$$\hat{X}_t = -\hat{}\alpha t - \tilde{X}_t$$

with $\alpha = -\hat{\alpha}$. Therefore using formula 4.4 we can rewrite the zero coupon bond pricing formula as

$$Z(0,T) = e^{-rT}\Big[N\Big(\frac{-m+\alpha T}{\sqrt{T}}\Big) - e^{2\alpha m}N\Big(\frac{m+\alpha T}{\sqrt{T}}\Big)\Big]$$

and $r =$ risk-less rate which is the treasury bill rate obtained from Bloomberg. We then rewrite the bond pricing formula as

$$C_{j,k} = \sum_i C_{j,k,i}e^{-rT_i}\Big[N\Big(\frac{-m+\alpha T_i}{\sqrt{T_i}}\Big) - e^{2\alpha m}N\Big(\frac{m+\alpha T}{\sqrt{T}}\Big)\Big] \tag{4.8}$$

4.1.2 Simulating Distance to default and Volatility

Now using the new derived formula for the coupon bond price we can calibrate distance to default and volatility using optimization by minimizing the error between our observed price and model structural price by constructing the minimization equation as

$$Minimize \sum_J \Big(C_{j,k} - \sum C_{j,k,i}e^{-rT_i}\Big[N\Big(\frac{-m+\alpha T_i}{\sqrt{T_i}}\Big) - e^{2\alpha m}N\Big(\frac{m+\alpha T_i}{\sqrt{T_i}}\Big)\Big]\Big)^2 \tag{4.9}$$

subject to the condition that $\alpha < 0, \quad m > 0$. This optimization problem can be solved using non-linear least-squares solver in MATLAB which solves non-linear least-squares curve fitting problems of the form as discussed above.

4.1.3 Implementation of the Method

The optimization method is implemented as follows. We consider all bonds to have a maturity at most five years. For example the Apple Firm data was compiled as shown below

Apple Data from Bloomberg

Current Price	Coupon	Maturity
100.084	1.55	2
100.492	1.9	3
100.067	1.23817	3
100.328	2.5	5
100.247	1.53817	5

Figure 4.2: Apple Excel data.

From the above data sheet we can construct the first bond summation for the optimization problem as

$$\begin{aligned}
100.08 = {} & 1.55e^{-0.02*0.5}\left[N\left(\frac{m-\alpha*0.5}{\sqrt{0.5}}\right) - e^{2\alpha m}N\left(\frac{-m-\alpha*0.5}{\sqrt{0.5}}\right)\right] \\
& + 1.55e^{-0.02*1}\left[N\left(\frac{m-\alpha*1}{\sqrt{1}}\right) - e^{2\alpha m}N\left(\frac{-m-\alpha*1}{\sqrt{1}}\right)\right] \\
& + 1.55e^{-0.02*1.5}\left[N\left(\frac{m-\alpha*1.5}{\sqrt{1.5}}\right) - e^{2\alpha m}N\left(\frac{-m-\alpha*1.5}{\sqrt{1.5}}\right)\right] \\
& + 101.55e^{-0.02*2}\left[N\left(\frac{m-\alpha*2}{\sqrt{2}}\right) - e^{2\alpha m}N\left(\frac{-m-\alpha*2}{\sqrt{2}}\right)\right]
\end{aligned}$$

Which is coded in Matlab as

```
  % Coupon data for the five bond issued under apple bond
xdata = [1.55 1.55 1.55 101.55 0 0 0 0 0 0 ];
x1data = [1.9 1.9 1.9 1.9 1.9 101.9 0 0 0 0 ];
x2data = [1.24 1.24 1.24 1.24 1.24 101.24 0 0 0 0];
x3data = [2.5 2.5 2.5 2.5 2.5 2.5 2.5 2.5 2.5 102.5];
x4data = [1.54 1.54 1.54 1.54 1.54 1.54 1.54 1.54 1.54 101.54] ;
% Coupon interest payment time
 t = [0.5 1 1.5 2 2.5 3 3.5 4 4.5 5];
%Simulating the Cummulative distribution in the equation
r = zeros(1,10);r1 = zeros(1,10);k =zeros(1,10); g=zeros(1,10);g1=zeros(1,1
%x(1)=-2.2983
%x(2)= 0.4922
for i =1:10
r(i) = (-x(1)+x(2)*t(i))/sqrt(t(i));
end
for i =1:10
g(i) = cdf('Normal',r(i),0,1);
end
for i =1:10
r1(i) = (x(1)+x(2)*t(i))/sqrt(t(i));
end
for i = 1:10
g1(i) = cdf('Normal',r1(i),0,1);
end
for i =1:10
    w(i) = (exp(-0.02*t(i)));
```

```
end
% Generating the function under the Lsqnonlin
fun= @(x)(((ones(1,5))*[(times(w,(g-exp(2*x(1)*x(2))*g1))*xdata'-100.084) (
x0 = [-2.3,0.5];
options.MaxIter = 1000;
options.TolFun = 1e-38;
options.TolX = 1e-100;
options.Algorithm = 'levenberg-marquardt';
[x,output,exitflag] = lsqnonlin(fun,x0,[],[],options)
```

Running this code with r = 0.02 gives the best α and m to be

$$\alpha = 0.4922 \qquad m = -2.2983$$

solving for σ from this α gives

$$\sigma = 0.039082$$

and using this σ and m gives

$$\frac{d}{X_0} = 0.914094$$

which is very close to 1, it seems like default is imminent which does not sound realistic. Yet these results are acceptable since we are marching market return with a structural model which is well known to underestimate the rate of return of defaultable asset if σ is chosen to match historical default rates. The results of volatility, m and α were estimated as shown in the table below for the three firms.

Table 4.2: Estimated Parameters

Estimated values			
Firms	Apple	American Credit	Pepsi
Estimate m	-2.2983	-2.2754	-2.2993
Estimate Volatility	0.0391	0.0474	0.0375
Estimated α	0.4922	0.3947	0.4966
$\frac{d}{X_0}$	0.91409	0.89776	0.917389

Now we can find the probability of default for Apple bond using the default probability formula explained here which coincide with the probability of default of the Structural model and this probability default becomes the marginal probability $q_i(t)$ for the Conditional Survival model and probability of default under the copula model. For the default barrier under the structural model we use

$$H^*(t) = -m_i - \alpha_i(t) \qquad i = 1, 2, 3. \tag{4.10}$$

which similar to the default barrier $H^*(t)$ deduce for structural model in [13]. Coding all these models under Matlab and simulating it 10,000 times, the prices of the tranches under each model were determined to be as shown in the table below:

Table 4.3: Tranche Prices

Model Prices				
Tranches	0%-5%	5%-15%	15%-40%	40%-100%
CS model	0.3996	8.0489	8.0038	6.5368
Copula model	0.3957	8.0288	7.6966	3.0587
Structural Model	0.3485	7.7723	2.2647	0.3383

The Matlab Code for pricing tranches under the three main models are attached at the appendix.

4.2 Interpretation of Results

From the table 4.3 0% to 5% tranche (equity) is quoted as the percentage upfront payment required on the assumption that subsequent payments will be 500 basis points per year. The other tranches are also quoted as basis points per every three months. As observed from the prices table above, it is clearly seen that all the three models give very close prices for the equity tranche. Similarly the premium coupons of the junior tranche for all the three models are also very close. However the Mezzanine tranche spread for the structural model is quite small as compared to the one factor copula model and the CS model. For the senior tranche spread the prices differ for the three models. The CS model gives a high price compared to the other two models. Also the one factor copula has a higher senior tranche than the structural model which produced a small price for the senior tranche. The equity tranche price simply determines the upfront fee to be paid to the equity investors. For example in our pricing problem the CS model value the equity tranche as 39.96% which means the equity tranche holder receives $38.5 (39.96% of 96.2626) as an upfront fee. The tranche prices of the other tranches are the fractions of the notional principle on each tranche (premium) paid quarterly.

As shown from the table, the CS model and the one factor copula model are almost giving the same value to the tranches which means they seem to be priced well under the cumulative loss method explained above. Now we continue to experiment how changes in correlation factor for the one factor Gaussian copula model and the structural model will affect each price.

4.2.1 Effect of Correlation on Valuation of CDO Tranches

From the description of each of the model, one factor Gaussian Copula model and the Structural model are affected by the correlation factor. We then experiment how changes in the correlation factor will affect the tranche prices. We experiment when the coefficient of the market factor for the three firms increases from 0.1,0.2,0.3 to 0.4,0.5,0.6 and 0.7,0.8,0.9 respectively. This experiment is the same as increasing correlation factor from 0.02,0.06,0.03 to 0.2,0.3,0.24 and 0.56,0.72,0.63

respectively. Noting that the correlation factor between reference entities X_1, X_2, X_3 are define as $corr(X_1X_2) = 0.02$, $corr(X_2X_3) = 0.06$ and $corr(X_1X_3) = 0.03$.

Table 4.4: Copula model verses structural model

Model Prices				
Correlation 0.1,0.2,0.3				
Tranches	0%-5%	5%-15%	15%-40%	40%-100%
Copula model	0.3957	8.0288	7.6966	3.0587
Structural Model	0.3485	7.7723	2.2647	0.3383
Correlation 0.4,0.5,0.6				
Tranches	0%-5%	5%-15%	15%-40%	40%-100%
Copula model	0.3737	7.9327	7.4830	4.0507
Structural Model	0.3261	7.6533	2.1347	0.3252
Correlation 0.7,0.8,0.9				
Tranches	0%-5%	5%-15%	15%-40%	40%-100%
Copula model	0.2252	7.6789	7.3704	5.6133
Structural Model	0.3124	7.5777	2.0910	0.3384

From table 4.4 it is observed that as we increase the correlation, the tranche prices for structural model decrease across all the tranches which means that the cost of protection decreases as correlation increases.

Similarly, as correlation increases we observed that the tranche spread for the one factor Gaussian copula model decreases for the equity tranche, junior tranche and mezzanine whiles the spread of the senior tranche increases. This means that the lower tranches spread (premium of cost of protection) is inversely proportional to correlation while the senior tranche premium is directly proportional to correlation.

This results tally with the observation made by Hull and White in their paper [13]. They observed that increasing correlation lowers the value for the junior tranches that bears the initial losses and increases the value for the senior tranche that bears the later losses.

4.2.2 Difference Between Structural Model and One Factor Gaussian Copula Model

Table 4.5: Difference between Copula model and structural model

Difference:Structural - Gaussian Copula				
	Tranches			
Correlation	0%-5%	5%-15%	15%-40%	40%-100%
0.1,0.2,0.3	-0.0472	-0.2565	-5.4319	-2.7204
0.4,0.5,0.6	-0.0476	-0.2794	-5.3483	-3.7255
0.7,0.8,0.9	0.028	-0.1012	-5.2794	-5.3049

Considering the table 4.2.2 above, it can be seen that the models produce quite similar results for the equity and junior tranche but a wide difference for mezzanine and senior tranche. We measure the difference between the two models by calculating the spread difference across tranches. We observe that the one factor Gaussian copula model has high spreads across tranches as compared to the structural model. The results obtained are similar to that of Hull, Predescu and White (2005) in [14] where they stated that the one factor Gaussian copula model is a good approximation to the basic structural model in a wide range of situations. Moreover considering all the three model prices we can say with certainty that the CS model gives the best prices to the CDO structure since default is imminent and investors need these high prices to be able to invest in the CDO.

Chapter 5

Conclusion and Recommendations

This thesis has been able to present the detailed pricing of synthetic CDO using one factor Gaussian copula model, structural model and conditional survival model. To achieve these goals, various correlation modelling approaches were used such as Li (2000), Hull and White (2004), Hull, Predescu and White (2005) and Peng and Kou (2009). We found out that all these models, despite the small differences tend to give comparable prices for the tranches of the portfolio of the three sample reference entities.

Considering the tranche prices observed in chapter four, the three models give a very close price for the equity tranche and junior tranche while the prices for mezzanine tranche and senior tranche differ. However this thesis does not give any explicit reason why the tranche prices for the less risky tranches are quite different since they can be attributed to several different features of the modelling technique (structural verses copula, correlation structure, etc. Also from the prices obtained from our simulation the CS model gives a higher prices for all four tranches compared to the other two models and this high price favours investors because they receive a higher cash flow from SPV. On the other hand the structural model gives the lowest tranche price in all the tranche categories which favours SPV since it makes them spend less on premium payment.

In view of the experiment made on the one factor Gaussian copula model and the structural model in chapter four, the experiment tends to give interesting results when the correlation parameter in both models are increased even for the small number of reference entities we used in our simulation. Our conclusions coincide with the conclusions made by Hull, Predescu and White (2005), in particular, we observed that as correlation increases the tranche price for the structural model and that of the one factor Gaussian copula model decrease while the price of the senior tranche increases. Moreover since we did not price these tranches under market data, we could not determine which model is best fit for the market modelling so this thesis is silent on which model is best for market pricing.

All these three models have their own advantages and drawbacks which can affect the price of the tranches one way or the other. For instance Hull, Predescu and White (2005) revealed that structural model was not a good fit to market data and also observed that the market standard model does not appear to fit market data well [13]. Where the market standard model is the one factor Gaussian copula model under the following assumptions:

- a fixed recovery rate of 40%,
- same CDS spreads for all of the underlying reference entities,
- same pairwise correlations,
- same default intensities for all the underlying reference entities.

Also the CS model is known to price tranches well when the economy is in recession as discussed by Peng and Kou (2009). Peng and Kou modelled the tranche prices based on three economical factors where Polya process was used to simulate the financial crises state in the economy and CIR model for the normal fluctuation in the economy. Their method gave a very close tranche price compared to the market prices.

Lastly a common similarity between these three models is the fact that they all incorporate in their model an economic factor or common factor and marginal default probability which makes comparison possible.

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Appendix A

First Appendix

A.1 Appendix

A.1.1 Appendix 1 Structural Model Matlab Code

```
function [default] = Structural Model
T = 5;
mu = 0.01 ; sigma = 0.3;  cor = 0.1;
N = 20; n=20;
t = zeros(N,1);
for i = 1:20;
    t(i) = (T/N)*i;
    t;
end


 % Generating Brownian motions for Common risk
M = (sigma*sqrt(T/N))*(tril(ones(N))*((normrnd(0,1,[N,1])))) + mu*t;
%plot(t,M,'blue')
%hold on;
% For firm A
x = (sigma*sqrt(T/N))*(tril(ones(N))*((normrnd(0,1,[N,1])))) + mu*t; % Cons
%plot(t,x,'red')
%hold on;
X = cor*M + sqrt(1-cor^2)*x; % Brownian motion of firm A
```

```
v = 0.8; % initial asset vale of firm A
V =(exp(sigma*X + (mu-(sigma^2)/2)*(t)))*v;
% For firm B
y = (sigma*sqrt(T/N))*(tril(ones(N))*((normrnd(0,1,[N,1])))) + mu*t;
%plot(t,M,'green')
%hold on;
Y = cor*M + sqrt(1-cor^2)*y;
s = 0.8;
S = (exp(sigma*Y + (mu-(sigma^2)/2)*(t)))*s;
%Firm C
z = (sigma*sqrt(T/N))*(tril(ones(N))*((normrnd(0,1,[N,1])))) + mu*t;
%plot(t,M,'yellow')
%hold on ;
Z = cor*M + sqrt(1-cor^2)*z;
r = 0.8;
R = (exp(sigma*Z + (mu-(sigma^2)/2)*(t)))*r;
%Firm D
%plot(t,w,'black')
%hold on;
%U =(exp(sigma*W + (mu-(sigma^2)/2)*(t)))*u;%
plot(t,V,'red')
hold on;
plot (t,R,'black')
hold on;
plot(t,S,'yellow')
hold on;
```

```
%plot(t,U,'green')
%hold on;
V
S
R
%U
%Probability of default barrier for Apple bond for firm A
a1= -2.2983 ;  b1= 0.4923; prob = zeros(20,1); c1= -2.2753; d1 =0.3945; e1=
prob1=zeros(20,1); prob2=zeros(20,1);
for i = 1:20
    prob(i) = a1 + b1*t(i);
end
prob
for i = 1:20
    prob1(i) = c1 + d1*t(i);
end
prob1
for i = 1:20
    prob2(i) = e1 + f1*t(i);
end
prob2
% Constructing barrier for asset
a = zeros(20,1); e =zeros(20,1); f= zeros(20,1); h = zeros(20,1);
%default = zeros(1,4);
for i = 1:20
 if  ( X(i) <= prob(i))
```

```
    a(i) = 1;
 elseif  X(i) > prob(i)
 a(i) = 0;
 end
end
for i = 1:20
 if Y(i) <= prob1(i)
    e(i) = 1;
elseif  Y(i) > prob1(i)
    e(i) = 0;
 end
end
for i = 1:20
 if Z(i) < prob2(i)
    f(i) = 1;
elseif Z(i) > prob2(i)
    f(i) = 0 ;
 end
end
%for i =1:20
%Aif U(i) < D(i);
    %h(i) = 1
%elseif U(i) > D(i);
   % h(i) = 0
%end
defaultmatrix = [a e f ]
```

```
% Loss on each firm
 % Given that firm 1 has a loan of 1113.57 with a 40% recovery,
 %Firm 2 loan 1114.22 and 50% and
 %Firm 3 has 1000 and 30% recovery
 %firm 1 loss
 L1 = (1-0.4)*1113.57;
 %firm 2 loss
 L2 = (1-0.5)*1114.22;
 %firm 3 loss
 L3 = (1-0.3)*1000;
 Loss = [L1;L2;L3];
 %cummulative loss
 CL = defaultmatrix*Loss
for i=1:20
        t(i)= (1/4)*i;
end
 %Discount Factor
 dt = zeros(20,1);
 Dt = zeros(20,1);
 r = 0.05;
 dt(1) = 1/((1+r)^(t(1)/2));
for i = 2:20
     dt(i) = 1/((1+r)^((t(i)+t(i-1))/2));
end
dt
Dt(1) = 1/((1+0.05)^(t(1)));% second discount factor
```

```
for i = 2: 20
    Dt(i) = 1/((1+0.05)^(t(i)));
end
Dt
delta = zeros(20,1);% change in time
delta(1) = t(1)-0;
for i = 2: length(t)
    delta(i) = t(i)-t(i-1);
end
 % Tranche Cummulative losses
%Equity    0-96.2626
%Junior    96.2626 - 288.7878
%Mezzanine 288.7878 - 770.1008
%Senior     770.1008 - 1925.252
LE = 96.2626;%*ones(length(Mt),1);
LJ = 192.5252;%*ones(length(Mt),1);
LM = 481.313;%*ones(length(Mt),1);
LS = 1155.1512;%*ones(length(Mt),1);
%%Default
Lequity = zeros(n,1);
Ljunior =zeros(n,1);
Lmezzanine = zeros(n,1);
Lsenior = zeros(n,1);
LKR = zeros(n,1);
LKT = zeros(n,1);
LKS = zeros(n,1);
```

```
for i = 1:n;
    if CL(i) > LE
        %if LK > LE
Lequity(i) = (CL(i)-0)-(CL(i)-96.2626) ;% cummulatve loss on equity
    else if (0 < CL(i))&&(CL(i) < LE)
            Lequity(i) = CL(i);
        elseif CL < 0
                Lequity(i) = 0;
         end


   end
LKR(i) = CL(i)-LE;
if   LKR(i) > LJ
    Ljunior(i) = ((LKR(i)-96.2626)-(LKR(i)-288.7878)); % cummulative loss o
elseif (0 < LKR(i)) && (LKR(i) < LJ)
       Ljunior(i) = max((min(LKR(i),288.7878)-96.2626),0);
else if LKR(i) < 0
            Ljunior(i) = 0;
    end
 end
LKT(i)= LKR(i)-LJ;
if LKT(i) > LM
Lmezzanine(i) = ((LKT(i)-288.7878)-(LKT(i)-770.1008));
elseif (0 < LKT(i)) && (LKT(i) < LM)
    Lmezzanine(i)= LKT(i);%max((min(LKT(i),770.1008)-288.7878),0);
elseif LKT(i) < 0
```

```
            Lmezzanine(i) = 0;
    end
    LKS(i) = LKT(i)- LM;
    if LKS(i) > LS
        Lsenior(i) = ((LKS(i)-770.1008)-(LKS(i)-1925.252));

     elseif (0 < LKS(i)) && (LKS(i) < LS)
       Lsenior(i) = LKS(i);%max((min(LKS(i),1925.252)-770.1008),0);

    elseif LKS < 0
            Lsenior(i) = 0;
    end
    end
    Lequity
    Ljunior
    Lmezzanine
    Lsenior
    % tranche prices
    %Equity spread
    cumequity = zeros(n,1);
    cumequity(1) = Lequity(1)-0;
     for i = 2:n;
        cumequity(i) = Lequity(i)-Lequity(i-1);
     end
     cumesum = zeros(n,1);
     cumesum(1) = (Lequity(1)+0)/2;
```

```
 for i = 2:n;
          cumesum(i) = (Lequity(i)+Lequity(i-1))/2;
 end
f = times(Dt,delta);
R = LE - cumesum;
SE = ((1/LE)*dot(dt,cumequity))-0.05*(dot(f,R));
% Junior Spread
cumjunior = zeros(n,1);
cumjunior(1) = Ljunior(1)-0;
 for i = 2:n;
     cumjunior(i) = Ljunior(i)-Ljunior(i-1);
 end
 sumjunior = zeros(n,1);
 sumjunior(1) = (Ljunior(1)+0)/2;
 for i = 2:n;
          sumjunior(i) = (Ljunior(i)+Ljunior(i-1))/2;
 end
 U = (288.7878-96.2626)-(sumjunior);
 SJ = (dot(dt,cumjunior))/(dot(f,U));
 % Mezzanine Spread
 cummezz = zeros(n,1);
cummezz(1) = Lmezzanine(1)-0;
 for i = 2:n;
     cummezz(i) = Lmezzanine(i)-Lmezzanine(i-1);
 end
 summezz = zeros(n,1);
```

```
summezz(1) = (Lmezzanine(1)+0)/2
for i = 2:n;
        summezz(i) = (Lmezzanine(i)+Lmezzanine(i-1))/2;
end
Z = (770.1008 - 288.7878)-(summezz);
SM = (dot(dt,cummezz))/(dot(f,Z));
% Senior tranche
cumsenior = zeros(n,1);
cumsenior(1) = Lsenior(1)-0;
for i = 2:n;
    cumsenior(i) = Lsenior(i)-Lsenior(i-1);
end
sumsenior = zeros(n,1);
sumsenior(1) = (Lsenior(1)+0)/2;
for i = 2:n;
        sumsenior(i) = (Lsenior(i)+Lsenior(i-1))/2;
end
O = (1925.252-770.1008)-(sumsenior);
SS = (dot(dt,cumsenior))/(dot(f,O));
SE
SJ
SM
SS
default = [SE SJ SM SS]
end
```

A.1.2 Appendix 2 Conditional Survival Model Matlab Code

```
function [default] = Nnewcsmodel
alpha = 3; beta = 2;
lambda = gamrnd(alpha,beta);
%lambda =0.5;
maturity = 20;
t = 0
tau = -log(rand)/lambda;
    T= t + tau;
    k =0 ;
    index = 1;
    while (T < maturity)
     k(index) = k(end) + 10*betarnd(0.5,0.5);
     T(index) = T(end) + tau;
     index = index + 1;
    end


    k
    T


     stairs(T,k,'red')
     t = zeros(20,1); at = zeros(20,1); Nt = zeros(20,1);Mt = zeros (20,1);
   for i=1:20
        t(i)= (1/4)*i;
        at(i)=lambda*t(i);
        Nt(i) = poissrnd(at(i));
        Mt(i) = sum(rand(Nt(i),1));
```

```
        %sum(-log(rand(Nt(i),1))/lambda);
        %sum(rand(Nt(i),1))

    end
    t
    at
    Nt
    Mt
    % Firm 1
    %marginal probability of each firm
   a=-2.2983; q =zeros(20,1);q11=zeros(20,1); q3 =zeros(20,1);
b= 0.4923;
for i = 1: 20
    q(i)= normcdf((-a+b*t(i))/sqrt(t(i)))- exp(2*a*b)*normcdf((a+b*t(i))/sqr
end
q % marginal probability of firm 1
c = -2.2753;
d = 0.3945;
for i =1:20
     q11(i)= normcdf((-c+d*t(i))/sqrt(t(i)))- exp(2*c*d)*normcdf((c+d*t(i))/
end
q11 % marginal survival probability of firm 2
e = -2.2993;
f = 0.4966;
for i = 1:20
  q3(i)= normcdf((-e+f*t(i))/sqrt(t(i)))- exp(2*e*f)*normcdf((e+f*t(i))/sqrt
```

```
end
q3
%q = gampdf(rand(length(Mt),1),1,1);% pdf from gamma function
% generating the expected value in q(i) equation
 a1= 0.18;
g1=a1*Mt;
e1 = -g1;
k1 = exp(1).^(e1);
%expectation for poisson
p = 1/(1+beta*maturity);% 20 is the time frame ie T = 20
expectedpoisson1 = ((p)/(1-((1-p)*exp(1)^(0.5*a1^2))))^alpha;% moment gener

 % finding the conditional survival probability for firm 1
 w1= ones(1,length(Mt))*((1/(expectedpoisson1))*k1);
 qx = q*w1
 % Firm 2
% generating the expected value in q(i) equation
 a2= 0.17;
g2=a2*Mt;
e2 = -g2;
k2 = exp(1).^(e2);
%expectation for poisson
p = 1/(1+beta*maturity);
expectedpoisson11 = ((p)/(1-((1-p)*exp(1)^(0.5*a2^2))))^alpha;%revisit the r
 % finding the conditional survival probability for firm 1
 w11= ones(1,length(Mt))*((1/(expectedpoisson11))*k2);
```

```
 qs = q11*w11
 %firm 3
% generating the expected value in q(i) equation
 a3= 0.16;
g3=a3*Mt;
e3 = -g3;
k3 = exp(1).^(e3);
%expectation for poisson
p = 1/(1+beta*maturity);
expectedpoisson13 = ((p)/(1-((1-p)*exp(1)^(0.5*a3^2))))^alpha;%revisit the 
 % finding the conditional survival probability for firm 3
 w13= ones(1,length(t))*(1/(expectedpoisson13)*k3);
 qc = q3*w13
 %Bernouli random variable
 h=zeros(length(t),1);h1 = zeros(length(t),1); h2 = zeros(length(t),1);
 for i = 1:20
     if rand(1)< qx(i)
     h(i) =1;
     elseif rand(1)> qx(i)
         h(i) = 0;
    end
 end
 for i = 1:20
     if rand(1)< qs(i)
     h1(i) =1;
     elseif rand(1)> qs(i)
```

```
            h1(i) = 0;
        end
    end
    for i = 1:20
         if rand(1)< qc(i)
         h2(i) =1;
         elseif rand(1)> qc(i)
            h2(i) = 0;
        end
    end
    h
    h1
    h2
    % Loss on each firm
    % Given that firm 1 has a loan of 1113.57 with a 40% recovery,
    %Firm 2 loan 1114.22 and 50% and
    %Firm 3 has 1000 and 30% recovery
    %firm 1 loss
    L1 = (1-0.4)*1113.57;
    %firm 2 loss
    L2 = (1-0.5)*1114.22;
    %firm 3 loss
    L3 = (1-0.3)*1000;
    Loss = [L1;L2;L3]
    %cummulative loss
    CL = [h, h1, h2]*Loss
```

```
%Discount Factor
dt = zeros(length(t),1);
Dt = zeros(length(t),1);
r = 0.05;
dt(1) = 1/((1+r)^(t(1)/2));
for i = 2:length(t)
    dt(i) = 1/((1+r)^((t(i)+t(i-1))/2));
end
dt
Dt(1) = 1/((1+0.05)^(t(1)));% second discount factor
for i = 2: length(t)
    Dt(i) = 1/((1+0.05)^(t(i)));
end
Dt
delta = zeros(length(t),1);% change in time
delta(1) = t(1)-0;
for i = 2: length(Mt)
    delta(i) = t(i)-t(i-1);
end
% Tranche Cummulative losses
%Equity    0-96.2626
%Junior    96.2626 - 288.7878
%Mezzanine 288.7878 - 770.1008
%Senior     770.1008 - 1925.252
LE = 96.2626;%*ones(length(Mt),1);
LJ = 192.5252;%*ones(length(Mt),1);
```

```
LM = 481.313;%*ones(length(Mt),1);
LS = 1155.1512;%*ones(length(Mt),1);
%%Default
 Lequity = zeros(length(t),1);
 Ljunior =zeros(length(t),1);
 Lmezzanine = zeros(length(t),1);
 Lsenior = zeros(length(t),1);
LKR = zeros(length(t),1);
LKT = zeros(length(t),1);
LKS = zeros(length(t),1);
 for i = 1:length(t)
      if CL(i) > LE
          %if LK > LE
 Lequity(i) = (CL(i)-0)-(CL(i)-96.2626) ;% cummulatve loss on equity
      else if (0 < CL(i))&&(CL(i) < LE)
               Lequity(i) = CL(i);
          elseif CL < 0
                    Lequity(i) = 0;
          end


    end
LKR(i) = CL(i)-LE;
 if    LKR(i) > LJ
      Ljunior(i) = ((LKR(i)-96.2626)-(LKR(i)-288.7878)); % cummulative loss o
 elseif (0 < LKR(i)) && (LKR(i) < LJ)
         Ljunior(i) = max((min(LKR(i),288.7878)-96.2626),0);
```

```
else if LKR(i) < 0
            Ljunior(i) = 0;
    end
 end
LKT(i)= LKR(i)-LJ;
if LKT(i) > LM
Lmezzanine(i) = ((LKT(i)-288.7878)-(LKT(i)-770.1008));
elseif (0 < LKT(i)) && (LKT(i) < LM)
    Lmezzanine(i)= LKT(i);%max((min(LKT(i),770.1008)-288.7878),0);
elseif LKT < 0
            Lmezzanine = 0;
end
LKS(i) = LKT(i)- LM;
if LKS(i) > LS
    Lsenior(i) = ((LKS(i)-770.1008)-(LKS(i)-1925.252));

  elseif  (0 < LKS(i)) && (LKS(i) < LS)
   Lsenior(i) = LKS(i);%max((min(LKS(i),1925.252)-770.1008),0);

elseif LKS < 0
            Lsenior(i) = 0;
end
end

Lequity
Ljunior
```

```
Lmezzanine
Lsenior
 % tranche prices
%Equity spread
cumequity = zeros(length(t),1);
cumequity(1) = Lequity(1)-0;
 for i = 2:length(t)
      cumequity(i) = Lequity(i)-Lequity(i-1);
 end
 cumesum = zeros(length(t),1);
 cumesum(1) = (Lequity(1)+0)/2;
 for i = 2:length(t)
          cumesum(i) = (Lequity(i)+Lequity(i-1))/2;
 end
f = times(Dt,delta);
R = LE - cumesum;
SE = ((1/LE)*dot(dt,cumequity))-0.05*(dot(f,R));
% Junior Spread
cumjunior = zeros(length(t),1);
cumjunior(1) = Ljunior(1)-0;
 for i = 2:length(t);
      cumjunior(i) = Ljunior(i)-Ljunior(i-1);
 end
 sumjunior = zeros(length(t),1);
 sumjunior(1) = (Ljunior(1)+0)/2;
 for i = 2:length(t)
```

```
        sumjunior(i) = (Ljunior(i)+Ljunior(i-1))/2;
 end
 U = (288.7878-96.2626)-(sumjunior);
 SJ = (dot(dt,cumjunior))/(dot(f,U));
 % Mezzanine Spread
cummezz = zeros(length(t),1);
cummezz(1) = Lmezzanine(1)-0;
 for i = 2:length(t)
     cummezz(i) = Lmezzanine(i)-Lmezzanine(i-1);
 end
 summezz = zeros(length(t),1);
 summezz(1) = (Lmezzanine(1)+0)/2;
 for i = 2:length(t)
        summezz(i) = (Lmezzanine(i)+Lmezzanine(i-1))/2;
 end
 Z = (770.1008 - 288.7878)-(summezz);
 SM = (dot(dt,cummezz))/(dot(f,Z));
 % Senior tranche
cumsenior = zeros(length(t),1);
cumsenior(1) = Lsenior(1)-0;
 for i = 2:length(t)
     cumsenior(i) = Lsenior(i)-Lsenior(i-1);
 end
 sumsenior = zeros(length(t),1);
 sumsenior(1) = (Lsenior(1)+0)/2;
 for i = 2:length(t)
```

```
        sumsenior(i) = (Lsenior(i)+Lsenior(i-1))/2;
    end
    O = (1925.252-770.1008)-(sumsenior);
    SS = (dot(dt,cumsenior))/(dot(f,O));
    %default = SE
   default= [SE SJ SM SS]
   end
```

A.1.3 Appendix 3 Proof of Proposition

$$
\begin{aligned}
q_i^c(t) &= E\left[1_{\sum_{j=1}^{J} a_{i,j}M_j(t)+X_i(t)<E_i} \mid M(t)\right] \\
&= E\left[E\left[1_{\sum_{j=1}^{J} a_{i,j}M_j(t)+X_i(t)<E_i} \mid X_i(t), M(t)\right] \mid M(t)\right] \\
&= E\left[e^{-X_i(t)-\sum_{j=1}^{J} a_{i,j}M_j(t)} \mid M(t)\right] \\
&= E\left[e^{-X_i(t)} \mid M(t)\right] e^{-\sum_{j=1}^{J} a_{i,j}M_j(t)} \\
&= E\left[e^{-X_i(t)})\right] e^{-\sum_{j=1}^{J} a_{i,j}M_j(t)}
\end{aligned}
\tag{A.1}
$$

where the last equality follows from independence of $X_i(t)$ and $M(t)$. We then take expectation of both side of the last equation and it gives equation 2.14. Dividing Eq. (2.13) by (2.14), we obtain Eq. (2.15).

A.1.4 Appendix 4 Laplace Transform of a Compound Polya Process

Let $P(t)$ be a polya process with parameters α and β. Then the Laplace transform of $P(t)$ is given by

$$
E\left[e^{-\mu P(t)}\right] = \left(\frac{p}{1-(1-p)e^{-\mu}}\right)^{\alpha}, \qquad P = \frac{1}{1+\beta t}. \tag{A.2}
$$

Let $Y_1, Y_2, \ldots$ be i.i.d random variables with Laplace transform $\Phi_Y(\mu) = E\left[e^{-\mu Y_i}\right]$ that are independent of $P(t)$. Then the Laplace transform of the compound Polya process $M(t) = \sum_{i=1}^{P(t)} Y_i$ is given by

$$
E\left[e^{-\mu M(t)}\right] = \left(\frac{P}{1-(1-P)\Phi_Y(u)}\right)^{\alpha}, \qquad \frac{1}{1+\beta t}. \tag{A.3}
$$

Indeed, we can show this by noting that

$$
\begin{aligned}
E\left[e^{-uM(t)}\right] &= E\left[E\left[e^{-uM(t)} \mid P(t)\right]\right] \\
&= E\left[\left(E[e^{-uY_i}]\right)^{P(t)}\right] \\
&= E\left[e^{-log(\frac{1}{\Phi_Y(u)})P(t)}\right] \\
&= E\left[e^{-vP(t)}\right] \big|_{v=\log} \left(\frac{1}{\Phi_Y(u)}\right) \\
&= E\left[e^{-\mu P(t)}\right] = \left(\frac{p}{1-(1-p)e^{-\mu}}\right)^{\alpha}, \qquad P = \frac{1}{1+\beta t}
\end{aligned}
\tag{A.4}
$$

where the last equality follows from Eq. (A.2).