Rayleigh-Taylor Instability in Homogenous/Heterogenous Porous Media Under Time-Dependent Displacements

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Rayleigh-Taylor Instability in Homogenous/Heterogenous Porous Media Under Time-Dependent Displacements

by

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Abstract

The density mismatch between fluids in porous media during displacement processes leads to the formation of flow instabilities manifested as finger-like intrusions. These buoyancy-driven instabilities are thoroughly studied in this dissertation to address different aspects of the flow instability in miscible displacements. The research is conducted in two phases. In the first phase, the problem is studied under time-dependent flow displacements in homogeneous porous media. One must note that a new, more efficient approach to determine instability characteristics has been developed for the stability analysis of these time-dependent flow displacements. The effects of the cycle period and the velocity amplitude on the flow instability have been examined. The results showed that the smaller the period, the larger the growth rate but the stronger the subsequent decay over time. Finally, the first phase revealed that it is possible to control the buoyancy-driven instability through a proper choice of the period and the amplitude. In particular, the choice of the period determines the nature of the finger patterns as well as other important flow characteristics.

The buoyancy-driven instabilities are analyzed in the second phase of this study for horizontally layered heterogeneous porous media. The role of the frequency of layers and variance of the permeability distribution are examined under different scenarios of density mismatches. Qualitative analysis of the concentration fields revealed that heterogeneity induces undulated more diffuse finger structures compared to the homogeneous case. Moreover, it is found that the onset time of the instability increases with increasing number of layers and decreases with increasing variance of the permeability distribution. Finally, the study shows that for the case of shifted permeability distribution, an unstable flow in a homogeneous medium can be fully stabilized when a small number of layers is used in the heterogeneous case.
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Chapter 1

Introduction

Flow instabilities develop at the interface between fluids during displacement processes when a low viscosity fluid displaces a high viscosity fluid in porous media and are known as viscous fingering (VF) [1]. The instability may also develop as a result of densities mismatch between the fluids, in which case it is referred to as the Rayleigh–Taylor instability (RT) [2]. These hydrodynamic instabilities play an imperative role in understanding and optimizing displacement processes encountered in numerous industrial and environmental applications such as enhanced oil recovery (EOR) [3], CO₂ sequestration [4], and chromatographic separation [5]. The fingering phenomenon can be favorable and desirable in certain applications such as CO₂ sequestration in deep saline aquifers as the buoyancy-driven instability results in convective mixing [6]. This significantly enhances the CO₂ dissolution rate into brine and reduces the risk of CO₂ leakage into the environment. Contrarily, fingering can be very problematic in EOR processes in the oil industry resulting in an ineffective sweeping action that will bypass considerable amounts of recoverable oil [7].

Numerous studies have been devoted to examining the fingering instability due to its complexity and significance. In the case of miscible displacements, the effects of several parameters including the viscosity ratio [8], gravity [9-11], chemical reaction [12, 13], heat transfer [14, 15], diffusivity [16], anisotropic and velocity dependent dispersion [17, 18], inertial effects [19-21], and additives such as nanoparticles [22, 23] have been extensively analyzed. The vast majority of existing studies on the RT instability in porous media are limited to displacements
involving a constant injection rate. However, in many practical processes the injection rate is in fact time-dependent. These practical processes include some of EOR methods such as cyclic steam stimulation (CSS), and CO₂ huff-and-puff. Another aspect of the RT instability that has received very little attention is the nature of the porous medium, notably its degree of heterogeneity and how it affects the instability. Most of the existing studies have focused on the instability in homogeneous porous media. However, naturally occurring porous media are in fact heterogeneous as the permeability varies from one location to the other as a result of variations in the mediums’ microstructure.

To the best knowledge of the author, no research has been conducted on the effects of time-dependent injection rates on the RT instability. In addition, no previous studies examined the non-linear effects of layered permeability heterogeneity on the dynamics of the RT instability. The significance of the problem in addition to the lack of studies on two important aspects of the RT instability, namely the time-dependent nature of the displacement and the heterogeneity of the medium, were the main motives to conduct this study. Thus, the present study is carried out in two stages, the first analyzes the effects of time-dependent rates on buoyancy-driven instabilities developing in homogeneous porous media using both linear stability analysis (LSA) and non-linear simulations (NLS). In particular, the role of the two parameters that characterize sinusoidal time-dependent displacements namely the cycle period and velocity amplitude are analyzed. The effects of the viscosity ratio and density difference on the dynamics of the fingering instability are investigated. Moreover, the effectiveness of the time-dependent injection scheme in controlling or enhancing the instability is investigated in comparison to its constant injection counterpart. While the second stage focuses on examining the effects of layered permeability heterogeneity on the growth and development of the RT instability. Furthermore, the role of the frequency of layers and
variance of the permeability distribution under different scenarios of density mismatches are analyzed. The different flow regimes and onset time of the instability are analyzed for the heterogeneous cases. In addition, the effects of the spatial distribution are investigated. These two stages of the present study are presented in detail in Chapters 4 and 5, respectively.

It should be mentioned that this is a manuscript-based thesis consisting of published or under review journal papers. In Chapter 2, a thorough review of the existing literature on VF and RT instability is presented. In Chapter 3, a detailed description of the model under consideration is presented including the physical problem, governing equations, and numerical method used for the LSA and NLS. Chapter 4 is based on the manuscript entitled “Rayleigh–Taylor instability in porous media under sinusoidal time-dependent flow displacements” in which LSA and NLS have been carried out to analyze the effects of time-dependent injection rates on the RT instability. Following that, Chapter 5 is devoted to the NLS of the RT instability in heterogeneous porous media where layered permeability heterogeneity significantly affects the dynamics of the instability in miscible displacements. This chapter is based on a paper entitled “Dynamics of buoyancy driven miscible iso-viscous flows in heterogeneous layered porous media”. Finally, Chapter 6 summarizes the main findings of this study and provides recommendations for future works. As this is a manuscript-based thesis, it should be noted that parts of the introduction and mathematical models are repeated throughout the chapters. In particular, the introduction sections in Chapters 4 (Section 4.1) and 5 (Section 5.1) are similar with some parts of the literature review (Chapter 2) and can be skipped by the reader. Also, the sections describing the model in Chapters 4 (Section 4.2) and 5 (Section 5.2) can be skipped by the reader as Chapter 3 provides a detailed description of the model as well as the numerical method used in this study.
Chapter 2

Literature Review

The review of pertinent studies will be presented in this chapter, which will be divided into two major sections. The first dealing with the viscous fingering instability while the second one focusing on the Rayleigh-Taylor instability. Each of these sections will be divided into subsections examining different aspects of the instability, with a particular focus on the time-dependent nature of the displacement and the heterogeneity of the medium, the two aspects that will be analyzed in this study.

2.1 Viscous Fingering Instability

In this section, instabilities driven by mobility ratio differences will be reviewed. First, the review will focus on flows under constant injections and then time-dependent displacements will be discussed. For each of these flow scenarios, different aspects that affect the instability will be also summarized.

2.1.1 Constant injection velocity

First, some of the important studies about the viscous fingering (VF) instability will be reviewed. Then, this review will include a subsection examining the effects of the heterogeneity of the medium on miscible VF under constant injections.

The first scientific study on VF was carried by Hill in 1952 [24], which he referred to as channeling during his experiments on the displacement of sugar liquors by water in a uniformly packed column. The author defined a critical velocity based on the viscosities and densities of the fluids, and he concluded that the interfacial instability develops when the flow velocity exceeds
such critical velocity. The next remarkable progress regarding VF occurred in the late 1950s, when Chuoke et al. [25] carried the first exhaustive linear-stability analysis of VF in a one-dimensional immiscible displacement including surface tension effects. They determined the stability condition and the critical wavelength for the development of the fingering instability. To validate their linear-stability analysis results, they conducted experiments in a Hele-Shaw cell which confirmed their theoretical predictions. In the same year, Saffman and Taylor [1] carried essentially a similar stability analysis of immiscible displacement, however without the inclusion of surface tension. They described the physical dynamics of VF in an immiscible displacement of oil by air and water, and they proposed a mathematical model for the development of VF. Later in their study, they examined experimentally the development and the shape of the single dominant finger. Since then, numerous experimental and theoretical studies have been devoted to examining the fingering instability due to its complexity and significance.

As opposed to immiscible flows, diffusion and dispersion significantly affect the dynamics of VF in miscible displacements. In 1959, Handy [26] conducted one of the earliest studies on the effect of molecular diffusion in miscible displacements on a microscopic scale. He reported that molecular diffusion is not significant and plays no important role in the mixing of the displaced and displacing fluids. Even in cases when the fingering instability developed, he observed that molecular diffusion is still not a significant factor. He also reported that the transverse diffusion rate was insufficient to alter the shape of fingers and their geometry.

Blackwell et al. [27] experimentally examined the factors that determine the efficiency of miscible displacements. They studied the mixing of miscible fluids at the microscopic level, and they observed that molecular diffusion is significant at reservoir conditions of slow rates, length and pore sizes. On the other hand, for the macroscopic scale, they observed channeling and
bypassing of the oil due to VF, permeability variation, and gravity segregation. For displacements with unfavorable mobility ratios, diffusion wasn’t effective in preventing the development and growth of viscous fingers.

Brigham et al. [28] conducted experiments on mixing during miscible displacements in porous media of both favorable and unfavorable mobility ratio. In their experiments using sandstone cores and glass beads pack, they studied the effects of velocity, length of travel, viscosity ratio, pack diameter, and bead size on the variation of the mixing length. The authors found that the mixing rate was changed by a factor of 5.7 when they increased the viscosity ratio from 0.175 to 0.998. They concluded that the mixing rate is strongly dependent on the viscosity ratio, and the length of the mixed zone is strongly affected by the injection velocity. Also, they observed that the dispersion coefficient is directly proportional to Fick’s diffusion coefficient at slow flow rates, however at higher flow rates, the dispersion coefficient is directly proportional to $u^{1.2}$ and the rate of mixing was higher in natural cores than in glass beads pack and this is due to the heterogeneity of the reservoir rock as opposed to the glass beads pack.

Using the X-ray technique, Slobod and Thomas [29] studied the effect of diffusion in miscible displacements with unfavourable mobility ratios. They directly observed the shape of fingers in a porous medium under different injection rates. Numerous narrow fingers were observed at high flow rates, however at lower flow rates of magnitude of $1 \text{ ft}/D$, they reported a single bulging finger. The authors concluded that transverse diffusion is amply rapid to modify the shape of fingers at slow flow rates, which contradicts Handy’s conclusions [26]. On the other hand, longitudinal diffusion is not significant and plays no role in mixing because it is very slow relative to the injection rate.
Detailed literature reviews on pre-1990s pertinent studies have been presented by Homsy [30] and McCloud and Maher [31].

The first non-linear simulations of VF in miscible displacements was successfully conducted by Peaceman and Rachford in 1962 [32]. They used the finite difference method (FDM) to model and analyze the fingering phenomenon. Their observed fingers were both qualitatively and quantitively in good agreement with those observed in experiments. However, the authors could not capture the early development of the finger structures due to limitations of the FDM. Another problem, associated with using the FDM in their simulations, is that the numerical dispersion in their system was larger than the physical dispersion and this damped out the fingers. Other researchers reported some of the limitations of the FDM; in particular Ewing [33] observed severe grid orientation impacts and smear sharp distribution of the concentration in simulations of a quarter five-spot pattern, Christie and Bond [34] reported that the FDM requires large computation time and storage size therefore limiting the number of grid points and Peclet number ($Pe$), while Tan and Homsy [35] stated that the FDM has difficulties in treating the non-linear terms. Due to the several limitations of the FDM, researchers used alternative more efficient numerical techniques such as the spectral methods [36]. The spectral methods have many advantages over the FDM; the spatial derivatives are determined with very high accuracy, require less grid points and computation effort, and resolve the problem of false large numerical dispersion [37].

In a distinguished work, Tan and Homsy [8] carried a linear stability analysis of a miscible rectilinear displacement. The authors were the first to solve the problem analytically using the quasi steady-state approximation (QSSA), which assumes that the growth rate of the perturbations is much faster than the rate of change of the base state. In addition, they solved the problem numerically using the initial value (IV) calculations, which approximates the exact solution of the
partial differential equations. The authors reported that their QSSA solutions indicate that the rectilinear miscible displacement is unstable for unfavorable mobility ratios, that is $R = > 0$, and that the maximum growth rate of the perturbations occurs at $t = 0$ then it decays with time due to dispersion effects. The initial value calculations, except for short times, led essentially to the same results as the QSSA. Moreover, the authors reported good agreement between their theoretical results and the experimental observations of Slobod and Thomas [29] and Perkins et al. [38], indicating that their theory provides an accurate prediction of the most dangerous wavelength of the viscous fingers. A year later, Tan and Homsy [39] carried another linear stability analysis to investigate the influence of Peclet number on miscible radial displacements in porous media. They reported that two key parameters, Peclet number and mobility ratio, determine the stability of the displacement. They showed that there exists a critical Peclet number above which the radial displacement becomes unstable. They have found that the growth rate increases with the $Pe$, and both the most dangerous wave number and cut off wave number increase with $Pe$ as well. Finally, they concluded that increasing the mobility ratio makes the flow more unstable.

In a subsequent study, Tan and Homsy [35] utilized the Fourier spectral method to conduct non-linear simulations of miscible displacements to study the non-linear behaviour of VF in a rectilinear Hele-Shaw cell. They stated that the stability characteristics is determined by two main parameters; mobility ratio and Peclet number. The authors reported that the fingers wavelength and growth rate were essentially similar to those predicted by their previous linear stability analysis. They observed that at later times, the fingers become wider in size and their number decreases, and this phenomenon is known as spreading. Also, they have observed a shielding phenomenon, in which some of the fingers shield the growth of adjacent fingers and collapse together to form a larger finger. The authors attributed the occurrence of these two phenomena to
a spanwise secondary instability which is fostered by transverse dispersion. In addition, they observed tip-splitting phenomenon as a result of the steep concentration gradient at the front of the finger due to stretching resulting from cross flow. To characterize the fingering instability, they used the mixing length (ML), defined as the length of the channel in which the transversely averaged concentration is in the range of $0.01 < C_{ave} < 0.99$ averaged over the transverse direction. They found that for small initial disturbances, the ML initially grows as $t^{1/2}$ due to dispersion, then the ML grows linearly with time due to non-linear fingering. However, the ML always grows linearly with time for large initial disturbances.

Up to the year 1992, all non-linear simulations of miscible VF have adopted either 1-D or 2-D models of the dynamics. Zimmerman and Homsy [40] in 1992 conducted the first reliable three-dimensional (3-D) simulations of miscible VF at large Peclet numbers using the Hartley spectral method. Using a white noise initial conditions, their 3-D results demonstrated the same mechanisms of non-linear interactions of fingers observed in 2-D simulations. They observed spreading, shielding, and coalescence of viscous fingers at all Peclet numbers. However, they reported tip-splitting at Peclet numbers above a critical Peclet number comparable to that of 2-D miscible fingering. In addition, the advancement rate of the fingers remained unchanged from 2-D simulations, indicating that 2-D simulations are sufficient to capture the most important features of miscible non-linear VF. Most experiments on VF were conducted in 2-D, however Bacri et al. [41] studied the growth of VF in 3-D porous media utilizing an acoustic technique to determine the concentration profile. They performed experiments in three different porous media at various injection rates and viscosity ratios. They observed a diffusive regime ($\sim \sqrt{t}$ growth) at the early times, followed by a convective regime ($\sim t$ growth). The authors reported that their experimental results were in good agreement with those of earlier theoretical studies.
Pramanik and Mishra [42] studied the effect of $Pe$ on the stability of miscible displacements in a rectilinear Hele-Shaw cell. They carried out a linear stability analysis that confirmed that the growth rate increases with an increase in $Pe$ for rectilinear miscible displacements. The authors determined a critical Peclet number ($Pe_c$) above which the miscible displacement becomes unstable. In contrast with the radial displacement, they found that the $Pe_c$ varies with the initial time ($t_0$). In addition, they concluded that both the most dangerous wavenumber and cut-off wave number increase with $Pe$. Their conclusions for the rectilinear displacement was the same as those of Tan and Homsy [39] for the radial miscible displacement.

In a recent comprehensive study on miscible displacements, Nijjer et al. [43] examined the dynamics of VF from onset to shutdown using linear stability theory and numerical simulations. The authors reported that the displacement is unstable when the log-mobility ratio, $R > 0$ if the Peclet number exceeds a critical value. As $R$ increases, the critical Peclet number decreases. They found that the dynamics can be differentiated by three different regimes: 1) at early times, the flow is linearly unstable; 2) at intermediate times, non-linear interactions of the fingers dominate the flow; 3) at late times, a single pair of thin and long fingers counter-propagate and decay. In the first regime, the number of fingers is directly proportional to both Peclet number and log-mobility ratio, $n \sim R Pe$ and the mixing length increases as $\sim \left( \frac{t}{Pe} \right)^{\frac{1}{2}}$. In the intermediate regime, non-linear coalescence of fingers dominates, and the number of fingers decreases with time, $n \sim \left( \frac{t}{Pe} \right)^{-\frac{1}{2}}$ while the mixing length grows as $Rt$. In the late time regime, the number of fingers reduces to one and the mixing length remains constant and scales as $ML \sim R Pe$.

2.1.1.1 Permeability heterogeneity

Most of existing studies on VF have focused on the instability in homogeneous porous media. However, underground formations and naturally occurring porous media are in fact heterogeneous
as the permeability varies from one location to another due to variations in the microstructure of the medium. A review of the studies focusing on miscible displacements in heterogeneous porous media will be presented in this subsection.

The first experimental study of the displacement in heterogeneous media is attributed to Blackwell et al. [27]. The authors investigated the factors that determine the efficiency of miscible displacements in stratified porous media. They observed channeling and bypassing of the oil due to VF, permeability variation, and gravity segregation. They have found that permeability stratification reduces the oil recovery in comparison to the recovery from homogeneous sand. In addition, the oil recovery at breakthrough decreases with an increase in the mobility ratio and the amount of solvent required to completely recover the oil increases with the mobility ratio. A couple of years later, Peaceman and Rachford [32] conducted the first non-linear simulations of miscible VF in heterogeneous media. They reported that their simulated fingers structures are in good qualitative agreement with the experimental results of Blackwell et al. [27]. However, due to limitations of the finite difference method, the authors could not capture the early development of the fingers.

Tan and Homsy [44] studied the effects of random permeability variations with Gaussian distributions. They observed that the growth rate of the mixing zone increases with the variance of the permeability but exhibits a non-monotonic behaviour with the variation of the correlation length of the heterogeneity. They explained this as resonance which occurs when the length scale of the heterogeneity and that of viscous fingering are commensurate. The authors also reported particular paths taken by the fingers that correspond to regions with locally high permeability. Finally, the authors concluded that although the random permeability distribution affects the
specific paths followed by the fingers, the nonlinear dynamics and the growth rate of the fingers are still governed by the pressure fields dictated by the mobility contrast.

In a later study adopting a spatially periodic permeability that varies perpendicular to the main flow direction, De Wit and Homsy [45, 46] found that the wave number at which resonance was observed depends on both the Péclet number ($Pe$) and log mobility ratio ($R$). Using both linear stability analysis and non-linear simulations, they studied two different systems, namely, the layered system ($n_x = 0$) and the checkerboard system ($n_x$ and $n_y$ are both non-zero). The authors reported that for the layered system, there exists a critical value of $\sigma_{cr}$ which depends on $Pe$ and $n_y$. For the case of $\sigma < \sigma_{cr}$, the regime is convective with the ML growing linearly with time and the fingering patterns being similar to those observed in the homogeneous case. While for the case of $\sigma > \sigma_{cr}$, the regime is dispersive in which the ML grow as $t^{1/2}$ and channeling occurs. They also observed that the channels were destroyed in the checkerboard system by the recurrence of tip splitting induced by the axial spatial variation of permeability and the system went back to the convective regime.

Interactions between viscously driven and heterogeneity induced instabilities were well depicted in the work of Sajjadi and Azaiez [47] examining periodically layered media with a heterogeneity that varies perpendicular to the main flow direction. The authors identified four regimes that govern the flow displacement; initial diffusion, channeling, lateral dispersion, and VF. In addition, generalized curves of the mixing zone length have been obtained, through a scaling of the model, for any flow scenario in which the first three regimes superpose into a single unifying curve which allowed them to clearly determine the onset of the last regime. They found that the breakthrough time (BT) varies non-monotonically with the number of layers ($q$) with a minimum BT occurring at small values of $q$ and a maximum occurring at intermediate values of
$q$. Large values of $q$ on the other hand led to the same BT as that of the homogeneous porous medium.

A year later, Norouzi and Shoghi [48] studied the effects of both anisotropic permeability and dispersion on the onset and propagation of miscible VF. Using linear stability analysis and nonlinear simulations, they found that the flow is less unstable when: 1. The ratio of the longitudinal to transverse anisotropic permeability ($\alpha_K$) increases. 2. The ratio of the longitudinal to transverse anisotropic dispersion ($\alpha_D$) decreases. The authors explained that as the permeability increases in the longitudinal flow direction opposed to the transverse direction ($\alpha_K \geq 1$), it became harder for the initial perturbations to grow into fingers which induces transport in the transverse direction. Consequently, the unfavorable conditions for the growth of the fingers led to a less unstable displacement. Similarly, a less unstable displacement occurred when the transverse dispersion was increased against the longitudinal dispersion. The stronger and greater transverse dispersion, which can smoothen the concentration gradient, resulted in a less unstable flow by hindering the development of the fingers.

In a recent study, Nijjer et al. [49] investigated the effects of permeability heterogeneities and viscosity contrast on miscible displacements in porous media using numerical simulations. They focused on large-scale permeability variations that are perpendicular to the flow direction. The authors found that the flow always evolves through three distinct regimes. At early times, the concentration evolves diffusively, independent of the viscosity ratio and permeability distribution. While at intermediate times, the interplay between the viscosity and permeability variations results in different dynamics including fingering and channeling. At late times, the flow is dominated by shear-enhanced (Taylor) dispersion, which becomes independent of the viscosity ratio and depends only on the permeability distribution.
2.1.2 Time-dependent injection schemes

Most studies on miscible displacements in porous media are limited to displacements involving a constant injection rate. However, in many practical processes the injection rate is in fact time dependent. These practical processes include some of enhanced oil recovery methods such as CO$_2$ huff-and-puff which involves cyclic injection of CO$_2$ to recover additional oil, and cyclic steam stimulation (CSS) which consists of three stages of steam injection, soaking, and oil production. A review of studies utilizing time-dependent injection velocities will be presented in this section. The review will focus on immiscible displacements and then examines miscible ones.

2.1.2.1 Immiscible displacements

Li et al. [50] suppressed VF in a circular Hele-Shaw cell utilizing a time-dependent injection rate. They investigated the dynamics of the interfacial instability experimentally and numerically under an injection rate scaled with time like $t^{-1/3}$. At an early stage, they observed substantial finger-finger interactions in which new fingers develop by tip-splitting and other fingers disappear. Later, they observed compact symmetrical shapes that grow self-similarly and depend on the injection rate.

Dias et al. conducted a series of studies employing a time varying injection rate in immiscible displacements in a radial configuration. In their first study, conducted in 2010 [51], they designed a piecewise constant injection strategy such that the average injection rate is kept constant to ensure that the same total amount of fluid is injected. During the initial stage, they utilized a relatively small injection rate before switching to a stronger injection rate during the second stage because suppressing the initial stronger instabilities needs a smaller injection rate. Their control protocol proved efficient in reducing the size of the fingers by one order of magnitude. Later in the same year, Dias and Miranda [52] adopted the same time variant injection rate employed previously by
Li et al. [50], which is proportional to $t^{-1/3}$. They were able to capture the morphology of the developing fingering patterns at the intermediate nonlinear time regime by using a second-order mode-coupling theory. The authors reported that finger competition and tip-splitting were suppressed by properly controlling the injection rate.

In a subsequent study published in 2012, Dias et al. [53] found an optimal time-dependent pumping rate that led to significant attenuation of the perturbation amplitudes and suppression of the fingers. They used an analytical variational method to obtain such optimal injection rate, which varies linearly with time. The success of their time-dependent injection strategy was due to the fact that the injection rate during the early stages was sufficiently weak to trigger strong interfacial disturbances. They concluded that the onset of the interfacial instability was delayed, and eventually when it was triggered, the disturbances grew at a reduced rate. This work was extended to a three-dimensional homogeneous porous medium in the next year by Dias [54]. Dias reported an optimal injection rate, which varies quadratically with time, so that the interfacial disturbances are indeed minimized. In order to obtain such optimal injection rate, he used the same analytical variational method proposed in his previous work [53]. He concluded that his time-dependent control strategy proved efficient even at large capillary numbers, and that the optimal injection rate does not depend on the properties of the fluids and porous medium.

Lins and Azaiez [55] analyzed the effects of time-dependent injection schemes on the growth of instabilities in immiscible radial displacements using non-linear simulations. They utilized three different time-dependent injection schemes, namely cyclic, monotonic, and controlled. The authors proposed the last injection scheme, controlled injection, in which the flow is adjusted continuously to minimize the growth of instabilities. They observed that, for cyclic injection schemes, larger amplitudes with small periods attenuate the fingering instability by decreasing the
interfacial length compared to that of the constant injection. On the other hand, larger amplitudes and longer period enhance the instability. They found that the phase shift reduces the instability of displacements involving injection stages only, however it increases the instability of displacements involving both injection and extraction stages. They reported that monotonic injection schemes, at high surface tensions and/or low mobility ratios, suppressed the fingering instability. While at high mobility ratio and low surface tension, the monotonic injection protocol couldn’t attenuate the instability and the results were similar to that of constant injection counterpart. The authors found that the optimal phase shift, leading to the strongest attenuation or enhancement of the fingering instability, depends heavily on the process amplitude and period.

A year later, Lins and Azaiez [56] observed a new physical phenomenon, they referred to as resonance, in immiscible radial displacements under sinusoidal injection strategy. They observed such resonance phenomenon at a critical value of the period, at which the interface deviates considerably from the trend observed for all other periods with the same mobility ratio and surface tension. These deviations are demonstrated in terms of distinct finger structures as well as different number of fingers. They found that the resonance period correlates with a flow characteristic time and the mobility ratio. Finally, they reported that such induced phenomenon and interfacial instability can be used to determine important physical properties such as the viscosities and surface tension of the fluids.

2.1.2.2 Miscible displacement

Chen and Meiburg [57] suppressed the growth of fingering instability in a miscible displacement in a quarter of a five-spot pattern by adopting a time-dependent injection strategy. Their strategy consisted of initiating the displacement through a smaller injection rate at a smaller Peclet number ($Pe = 400$), then at a later stage, the injection rate is increased to a larger value at
Pe = 800. They observed that the smaller injection rate during the early stages partially stabilized the flow, however increasing the injection rate at later times had negligible effects on the growth of the fingering instability. They concluded that their adopted time-dependent injection strategy enhanced the breakthrough recovery by approximately 8% compared to a constant injection rate having the same amount of injected fluid.

Inspired by the effectiveness of the time-dependent control strategies employed in immiscible displacements by Li et al. [50] and Dias and Miranda [52], Chen et al. [58] investigated a similar control strategy but for a miscible displacement in a radial Hele-Shaw geometry in which the injection rate varies as $Q(t) \sim t^{-1/3}$. Their results obtained from non-linear simulations confirm the stabilizing effects of the control strategies for miscible fluids as they found that the development of intricate fingering patterns, splitting and merging of fingers were all suppressed as compared to those of the constant injection scenario. They have also found that varying the Peclet number has almost a negligible effect on the resulting ultimate number of fingers, and the sensitivity of the control protocol is independent of the initial conditions.

Yuan and Azaiez [59] were the first to employ a time-dependent injection strategy based on a sinusoidal velocity model in miscible horizontal displacements to control viscous fingering. They observed that the sweep efficiency of their sinusoidal model was less than that of the constant injection counterpart. In addition, the breakthrough time of the sinusoidal model was shorter than that of the constant injection. They concluded that their cyclic injection model, with velocity amplitude $\Gamma = 1.5$ and frequency $\omega = 0.005$, resulted in a more unstable displacement than its constant injection counterpart due to the stronger velocities reached in the cyclic injection displacement. Also, the authors reported that inverse injection and diffusion attenuated the
hydrodynamic instability, however their effects were insufficient to make the cyclic injection displacement less unstable than its constant injection counterpart.

The same authors [60] adopted a step-size velocity profile to investigate the stability and dynamics of VF in miscible horizontal displacements in homogeneous porous media. They studied the effects of cycle period T and velocity amplitude on the fingering instability under a control strategy involving alternating stages of production and injection or of soaking and injection. Using linear stability analysis, the authors found that the disturbances growth rate follow the general trends of the time-dependent velocity yet with some differences in transition periods from extraction to injection. They have also reported that the growth rate for time-dependent velocity was smaller than that of the constant injection velocity which is equal to the minimum time-dependent velocity. In a later stage of their study, they conducted nonlinear simulations and found that the soaking-injection scenario was less unstable than the unity constant injection velocity counterpart model due to the stabilizing effects of diffusion during the soaking stage. On the other hand, they reported that the injection-soaking displacements were more unstable than the unity constant injection velocity model. For the soaking-injection displacements, they observed that the mixing area results are monotonically decreasing with the increase in cycle period T. On the contrary for injection-soaking displacements, they reported that the mixing area results are monotonically increasing with the increase in cycle period T. The authors reported the same trends for the effects of the velocity amplitude, but these were of less significance than the effects of the cycle period.
2.1.2.3 Effects of inertia under time-dependent injection velocities

Using linear stability analysis, He and Belmonte [61] investigated the effects of fluid’s inertia on the Saffman-Taylor instability under time-dependent injection rates in a circular Hele-Shaw cell. For a constant injection rate, the authors reported that small inertia has a stabilizing effect on the interface as it tends to reduce the perturbation amplitudes faster than the non-inertial case. While for the time-dependent injection rate, they found that the stabilizing effect of inertia was further enhanced by employing a sinusoidal injection scheme. Their conclusion confirms Chevalier et al. [21] experimental finding that inertia has the tendency to attenuate the instability and widen the fingers.

Yuan and Azaiez [19] extended their previous work [60] to include inertial effects in miscible time-dependent displacements. They found that the cycle period and velocity amplitude have significant effects on the stability of the displacement, and these effects are heavily influenced by considering fluid’s inertia. Regardless of the magnitude of inertia, the authors reported that the cyclic production-injection scenarios are always less unstable than the injection-production scenarios and their constant injection counterpart, irrespective of the magnitude of inertia. For production-injection processes, they reported that the flow becomes less unstable with an increase in the cycle period or velocity amplitude, and these suppression effects are greater with an increase in Reynolds number. For injection-production processes disregarding inertia, they observed that larger cycle periods or velocity amplitudes make the flow more unstable than that of its constant injection counterpart. However, the effects were non-monotonic for inertial cases. Finally, they concluded that inertia always has the tendency to attenuate the instability by slowing down the fingers’ growth during the injection stage and damping the disturbances during the production stage. Moreover, they reported that the diffusive regime was longer in time when they accounted
for inertial effects. Also, they have observed that increasing Reynolds number led to a delay in both the development of the fingers and their breakthrough time.

2.2 Rayleigh-Taylor Instability in Porous Media

In this section, instabilities triggered by a mismatch between the fluids’ densities will be reviewed. First, the review will focus on flows under constant injections in homogeneous porous media. Then, the review will be extended to discuss studies on flow instabilities in heterogeneous porous media.

2.2.1 Homogeneous porous media

The first scientific study of the Rayleigh-Taylor (RT) instability is attributed to Taylor [2], who derived a theory for the development of the instability at the interface between fluids having different densities under gravity force. He found a relationship between the rate of growth of the instability and the length of the disturbances, the densities and the acceleration. Later in the same year, Lewis [62] conducted several experiments to validate Taylor’s theory [2]. Lewis observed the fingering instability using high speed shadow photography, and his experiments confirmed Taylor’s theory.

Wooding [63] investigated experimentally the evolution of the non-linear density fingers at the interface between two fluids having the same viscosity but a linear density profile. He observed the spreading phenomenon at moderate $Pe$ numbers, and tip-splitting at large $Pe$ numbers (figure 2.1) in a Hele-Shaw cell.
Dumore [64] found that under certain conditions, vertical miscible displacements develop transition zones that are partly unstable and partly stable. The author derived a theoretical limit, based on pressure gradients, of the injection rate above which the partly unstable transition zone develops. In a later stage, he validated his theoretical stability limits using experiments on vertical downward miscible displacements. Finally, he concluded that the stability of downward miscible displacements depends not only on the endpoint values of viscosity and density but also on the details of the viscosity and density concentration relationships.
Rogerson and Meiburg [11] found that density contrasts, in vertical miscible displacement with monotonic viscosity profiles, lead to the development of potentially unstable region followed downstream by a potentially stable region and vice-versa. Their numerical work showed that density contrasts can attenuate the growth of fingers in miscible displacements with monotonic viscosity concentration relationship. The authors observed a phenomenon in which the displaced fluid fingers through the displacing fluid, and they referred to it as backward fingering.

Manickam and Homsy [65] examined the fingering instabilities, driven by both density and viscosity contrasts, in vertical downward miscible displacements using both linear stability analysis and non-linear simulations. They derived a critical velocity

$$U_c = \frac{\frac{d\rho}{dc} \bigg|_{c=0} + \frac{d\rho}{dc} \bigg|_{c=1}}{\frac{dm}{dc} \bigg|_{c=0} + \frac{dm}{dc} \bigg|_{c=1}}$$

by means of linear stability analysis, in which $c$ is the concentration of the fluid. This critical velocity determines the stability of vertical miscible displacements, and it varies with time due to dispersion of the fluids. They found that density stratification introduced zones that are locally stable, which was similarly observed in horizontal miscible displacements with non-monotonic viscosity profiles by Manickam and Homsy [66]. In the second part of their study they conducted non-linear simulations, using the Hartley spectral method, to study the evolution of the non-linear fingers. They determined the growth rates of the mixing zone for different mobility ratios and injection velocities. They found that density fingers, in the absence of viscosity contrasts, grow upwards and downwards without a preferred direction of propagation. In the presence of viscosity contrasts, they reported that locally stable regions can develop by properly adjusting the injection velocity and choosing the suitable viscosity profile. For instance, when a favourable viscosity contrast ($R < 0$) is introduced to a buoyantly unstable displacement ($\Delta \rho > 0$), the flow develops a stable upstream zone followed by an unstable downstream zone under an injection velocity. The stabilizing viscous forces coupled with the injection velocity not only
attenuated the growth of the fingers but also altered the direction of fingers propagation towards the downward direction only. The authors observed the phenomenon of reverse fingering, previously reported by Rogerson and Meiburg [11] as backward fingering. They observed it when the displaced fluid penetrated through the displacing fluid, namely in two situations: (a) viscous stabilization of density fingers \((R < 0 \text{ and } \Delta \rho > 0)\) and (b) gravity stabilization of viscous fingers \((R > 0 \text{ and } \Delta \rho < 0)\). Finally, they confirmed and validated their numerical simulation results by comparison with the experiments of Wooding [63].

![Figure 2.2: Reverse fingering in a vertical miscible displacement with unfavourable mobility ratio and stable density stratification. From Manickam and Homsy [65]](image)

Lajeunesse *et al.* [67] studied, both experimentally and theoretically, vertical miscible displacements at sufficiently high velocities and \(Pe\) numbers for diffusion effects to be negligible. They found that under gravity forces and at large \(Pe\), the interface was flattened and had a sharp shape, which is consistent with what Petitjeans and Maxworthy [68] observed. On the other hand, at low injection rates \((Pe < 1000)\), they observed significant diffusive mixing effects. They found
that under certain mobility ratio and injection rate, a two-dimensional tongue of the displacing fluid develops which is symmetric in shape. There exist thresholds in mobility ratio and injection rate above which the 2-D flow becomes unstable, leading to a 3-D fingering pattern analogous to that observed by Paterson [69].

It is well-known that RT fingering instability develops if a heavier fluid is placed on top of a lighter one in the gravity field [70], and these fingers develop upwards and downwards similarly [63]. On the other hand, when a lighter solution is on top of a heavier one, an instability can still develop if the bottom heavier solution diffuses faster than the upper one, and this instability is known as a double diffusive (DD) instability. Such instability has been investigated in numerous works in the field of oceanography by Stommel et al. [71], Stern [72], and Turner [73]. One example of DD instability is salt fingers, that develop in the ocean when warm salty water overlies cold fresh water, such instability develops as a results of heat diffusion being faster than the diffusion of the salt. Reviews of the salt fingering instability were provided by Kunze [74] and Schmitt [75].

Goyal et al. [76] studied the effects of gravity on vertical miscible displacements in homogeneous porous media using a Hele-Shaw cell. They carried out a linear stability analysis that showed that both the dominant wave number and growth rate depend weakly on the $Pe$ number, which denotes the ratio of convective to diffusive transport rates. The authors reported that both the dominant wavelength and growth rate increase with the viscosity ratio, and that the growth rate is strongly dependent on the gravitational forces represented by a parameter they defined as gravity number ($F$), which denotes the ratio of gravitational to viscous forces and a heavier fluid on top of a lighter fluid means a positive value of the gravity number. Thus, even a moderately negative value of $F$ (lighter fluid on top of a heavier fluid) can stabilize the viscously
unstable displacement. For stable density stratification, the dominant wavelength increases strongly with the gravity number, while it is independent for unstable density stratification.

Chemical reactions induced in flow displacements can significantly change the fate of the displacement process by triggering or modifying the fingering instability. Such chemical reaction could modify a physical property of the flow displacement such as density or viscosity. Reactive flows are encountered in numerous applications such as enhanced oil recovery [77], CO$_2$ sequestration [78], contaminant degradation [79] and chromatographic separation [80]. Almarcha et al. [12] investigated both numerically and experimentally the buoyancy-driven instability of a miscible system undergoing a chemical reaction. They reported that a chemical reaction as simple as $A + B \rightarrow C$ considerably impacted the stability of vertical miscible displacements. They found that such chemical reaction not only triggered instabilities in stable displacements but also broke the symmetry of the RT patterns. The authors have also observed that the displacement is unstable when: 1. The chemical product is heavier than the reactants A and B. 2. The chemical species diffusion rates is different.

Daniel and Riaz [81] studied the effect of viscosity contrast on the onset of fingering instability for buoyantly unstable displacements using two different models, namely, the fixed and moving interface. Such models, which are widely used in studying CO$_2$ sequestration in deep saline aquifers, are characterized based on whether the interface between brine and CO$_2$ is allowed to move or not. The authors found that, in the absence of injection ($U = 0$), the flow is more unstable when the viscosity decreases with depth compared to when it increases with depth. They found that both the maximum growth rate and most unstable wavenumber increase with a decrease in $R$, and the onset time increases with an increase in $R$. This increase in the instability with decreasing $R$ is in contrary with the behavior of classical VF with neutrally buoyant fluids, in
which the instability increases with $R$. On the other hand, in the presence of injection ($U > 0$), there exists a critical log-viscosity ratio $R_c$ below which the onset time increases with increasing $R$ until it reaches a maximum value at $R_c$ and the instability is dominated by buoyancy effects. While for $R > R_c$, the onset time decreases with increasing $R$, and the instability is dominated by the background flow. Finally, they observed that such critical value of $R_c$ depends on the injection velocity.

Pramanik et al. [82] investigated the effects of viscosity contrast on buoyantly unstable miscible displacements using both linear stability analysis and numerical simulations. They showed that appropriate viscosity scaling is crucial when comparing the effect of the less or more viscous fluid at the top on the dynamics of the RT instability. They used the less viscous fluid to scale the viscosity in their model. They showed that, in the absence of injection ($U = 0$), the onset of the instability happens earliest for density fingers ($R = 0$) compared to cases where the viscosities of the two fluids are different ($R > 0$ or $R < 0$). This means that the viscosity contrast delayed the onset of the instability, however, the temporal evolution of growth rates for more and less viscous upper fluid are almost indistinguishable. They reported that the dynamics of the density fingers is independent of the injection velocity, hence the onset time for the instability is the same for $U = 0$ and $U = 1$. Under an injection velocity, the fingering instability develops earlier when a less viscous fluid displaces a more viscous one in a buoyantly unstable displacement. Both the unfavourable mobility ratio and density stratification enhanced the instability. On the other hand, when a more viscous heavier fluid displaces a less viscous lighter fluid, the favourable mobility ratio ($m$) acts against the unstable density stratification and the miscible displacement becomes less unstable. They plotted a phase diagram showing three different stability regions spanned by the mobility ratio and injection velocity. Region I shows buoyancy and viscosity
dominated instabilities, II corresponds to buoyancy dominated instabilities, and III shows the stable region. For $m = 1$, the displacement is buoyantly unstable and the growth rates of the disturbances is unaffected by the injection velocity. For $m > 1$ the instability, driven by both the density and viscosity contrast, increases with the mobility ratio and injection velocity. For $m < 1$, which is the case of viscous stabilization of the gravitational fingers, the instability is attenuated when $U$ increases or $m$ decreases. Finally, the authors reported that their LSA results are contradicting with those of Daniel and Riaz [81] due to the fact that the viscosity scaling of Daniel and Riaz wasn’t appropriate to compare the displacement of a less viscous fluid by a more viscous fluid with the displacement of a more viscous fluid by a less viscous one. In addition, they stated that Manickam and Homsy [65] did not properly quantify the case of viscous stabilization of gravitational fingers ($R < 0$ and $\Delta \rho > 0$) as Manickam and Homsy observed that the mixing length increases as the mobility ratio decreases, which indicates that the instability is enhanced when the mobility ratio decreases. Pramanik et al. attributed this counter-intuitive results of Manickam and Homsy to the fact that they didn’t take the proper viscosity scaling into account.
Figure 2.3: Distinct stability regions spanned by the mobility ratio (m) and injection velocity (U) in a vertical miscible displacement. From Pramanik et al. [82]

<table>
<thead>
<tr>
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<th>$R = 0$</th>
<th>$R &lt; 0$</th>
<th>$R &gt; 0$</th>
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<tr>
<td></td>
<td>Favorable mobility ratio</td>
<td>Unfavorable mobility ratio</td>
<td></td>
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<tr>
<td>$U = 0$</td>
<td>Injection</td>
<td>$U = 0$</td>
<td>Injection</td>
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<tr>
<td>$\Delta \rho = 0$</td>
<td>Stable</td>
<td>Stable</td>
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<tr>
<td>$\Delta \rho &lt; 0$</td>
<td>Favorable density difference</td>
<td>Stable</td>
<td>Stable</td>
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<tr>
<td>$\Delta \rho &gt; 0$</td>
<td>Unfavorable density difference</td>
<td>Unstable</td>
<td>Stable if $U &gt; U_c$</td>
</tr>
</tbody>
</table>

Figure 2.4: Stability criteria for vertical miscible displacements as a function of log mobility ratio (R), density difference ($\Delta \rho$), and the injection velocity (U).
This comprehensive review for the Rayleigh-Taylor instability in homogeneous porous media leads to the following conclusions: 1. Density fingers ($R=0$) grow upwards and downwards without a preferred direction of propagation. 2. The stability of vertical miscible displacements, summarized in figure (2.4) below, depends on three main parameters: log mobility ratio ($R$), density difference ($\Delta \rho$), and the injection velocity ($U$). 3. The critical injection velocity varies with time due to dispersion between fluids. 4. The dynamics of density fingering is independent of the injection velocity. 5. In the absence of injection, the onset of the hydrodynamic instability happens earliest for density fingers ($R = 0$) compared to cases where $R > 0$ or $R < 0$. 6. In the presence of injection, the instability sets in earliest when a less viscous fluid displaces a more viscous one ($R > 0$) in a buoyantly unstable displacement ($\Delta \rho > 0$). 7. For buoyantly unstable displacements, the most unstable wavenumber increases with $R$; which is the same conclusion as that of horizontal displacement. 8. Appropriate viscosity scaling is crucial when comparing the effect of the less or more viscous fluid at the top on the dynamics of the RT instability. 9. The proper viscosity scaling is done with the less viscous fluid.

### 2.2.2 Heterogeneous porous media

Few studies have investigated the effects of permeability heterogeneity on the Rayleigh-Taylor instability. Schincariol et al. [83] adopted a random permeability field and found that longer correlation lengths and larger variance of log-normal permeability distribution promote stability. Prasad and Simmons [84] found that an increase in the standard deviation of the permeability field causes an increase in the degree of instability at earlier times but promotes stability at later times. Ranganathan et al. [85] modeled density-driven natural convection during geological CO$_2$ storage in heterogeneous formations. They represented the heterogeneity by random permeability fields. The authors reported that CO$_2$ concentration profiles show different flow patterns of convective
mixing such as channeling, gravity fingering, and dispersion based on the heterogeneity of the aquifer. In addition, they found that the variation of the mixing length with time shows three different regimes: diffusion, convection, and second diffusion.

Daniel et al. [86] adopted a permeability that varies periodically across the thickness of an aquifer to examine the stability of gravitationally unstable, transient boundary layers in heterogeneous saline aquifers. First, the authors investigated the effects of permeability variance and phase on the onset of instability using linear stability analysis. Then, using direct numerical simulations, they examined the effects of variance and wave length of the permeability oscillation on the onset time of convection. They observed that the instability is attenuated with an increase in the permeability variance when the boundary layer thickness is large relative to the permeability wavelength. Conversely, when the boundary layer thickness is smaller than the permeability wavelength, it was found that the behavior of instability as a function of the variance depends on the permeability phase.

In a later study, Ghorbani et al. [87] studied the effect of permeability heterogeneity on the stability of gravitationally unstable, transient, diffusive boundary layers in porous media. The permeability was assumed to vary periodically in the horizontal plane normal to the direction of gravity. The authors reported that thick layers enhance the instability compared to thin layers when permeability heterogeneity is increased. Contrarily, it was found that for thin layers, the instability is reduced progressively with increasing heterogeneity to the extent that it becomes less unstable than the corresponding homogeneous case.

2.3 Conclusion

The vast majority of existing studies on miscible displacements in porous media are limited to displacements involving a constant injection rate and focusing on the instability in homogeneous
porous media. However, in many practical processes the injection rate is in fact time dependent and naturally occurring porous media are in fact heterogeneous. To the best of the authors’ knowledge, no previous studies have examined the effects of time-dependent injection rates on the Rayleigh–Taylor (RT) instability. Furthermore, no research has been conducted to study the non-linear effects of layered permeability heterogeneity on the dynamics of the RT instability in miscible displacements. Therefore, the two main objectives of this present study are: 1. Analyze the effects of time-dependent injection rates on such buoyancy-driven instabilities developing in homogeneous porous media. 2. Investigate and understand the effects of layered permeability heterogeneity on the growth and development of the RT instability.
Chapter 3

Model and Methodology

In this chapter, a detailed description of the models under consideration is presented including the physical problem, governing equations, and boundary conditions. This detailed description is presented for two different models. The first discussing the physical problem and governing equations for the RT instability in homogeneous porous media under time-dependent flow displacements. While the second presents the governing equations for the RT instability in horizontally layered heterogeneous porous media. Finally, this is followed by a detailed description of the numerical method used for both the LSA and NLS.

3.1 Mathematical model for homogeneous porous medium

3.1.1 Physical problem

In this study, we examine the dynamics of buoyancy-driven instabilities in a two-dimensional porous medium under time-dependent flow displacements. The porous medium of length L and width W is assumed homogeneous with constant porosity and permeability. Fluid A of concentration $C_a$ and viscosity $\mu_a$ is injected from the top boundary (inlet) to displace fluid B of concentration $C_b$ and viscosity $\mu_b$ as shown in figure 3.1. The two fluids are incompressible, fully miscible and Newtonian. The displacement develops along the x-axis under a time-dependent displacement velocity $U(t)$, where a negative velocity corresponds to a production process, while a positive one corresponds to an injection process. Identically zero velocity results in a soaking stage.
3.1.2 Governing equations

The flow is governed by the equations for conservation of mass (continuity equation), conservation of momentum (Darcy’s law) and transport of species (convection-diffusion equation).

\[ \nabla \cdot \vec{u} = 0, \]
\[ \nabla P = -\frac{\mu}{K} \vec{u} + \rho \vec{g}, \]
\[ \phi \frac{\partial C_b}{\partial t} + \vec{u} \cdot \nabla C_b = \phi D \nabla^2 C_b, \]

where \( \vec{u} (u, v) \) is the velocity field, \( P \) the pressure, \( \mu \) the viscosity, \( K \) and \( \phi \) the porous medium permeability and porosity respectively, \( \vec{g} = (g, 0) \) the gravitational acceleration, \( C_b \) the concentration of fluid B, and \( D \) the constant diffusion coefficient. In this study, it is assumed that the mass diffusivities are isotropic and constant and the same for both species.
3.1.3 Dimensionless formulation

The governing equations are made dimensionless using diffusing scaling where $\rho_{ref}$, $D\phi/U_{ch}$ and $D\phi^2/U^2_{ch}$ are used as the reference density, length and time, respectively. The velocity is scaled using a characteristic velocity, $U_{ch} = \rho_{ref} g K/\mu_{ch}$ rather than using the injection velocity. This choice allows to include scenarios where the flow can develop in the presence of density contrast with zero injection velocity. Furthermore, the pressure and the concentration are scaled with $\mu_{ch} D\phi/K$ and $C_{b0}$ respectively, while the viscosity is scaled with $\mu_{ch} = \mu_{less}$ corresponding to the viscosity of the less viscous fluid. This is the appropriate scaling when comparing displacements of a less viscous fluid by a more viscous fluid with those of a more viscous fluid by a less viscous one [82]. Two dimensionless groups, the Péclet number $Pe = \frac{U_{ch} L}{D \phi}$ and cell aspect ratio $A_r = \frac{L}{W}$, are used to specify the domain size such that it is $\left(\frac{-Pe}{2}, \frac{Pe}{2}\right)$ in the $x$ direction and $\left(\frac{-Pe}{2A_r}, \frac{Pe}{2A_r}\right)$ in the $y$-direction. The resulting dimensionless equations, using for simplicity the same notation and expressed in a Lagrangian reference frame moving with a dimensionless injection velocity $U(t)$,

$$x' = x - \int_0^t U(\tau) d\tau$$

(3.4)

The dimensionless equations are:

$$\nabla. \tilde{u} = 0$$

(3.5)

$$\tilde{\nabla} P = -\mu [\tilde{u} + U(t) \tilde{t}] + \rho \tilde{t}$$

(3.6)

$$\frac{\partial C_b}{\partial \tau} + \tilde{u} . \tilde{\nabla} C_b = \nabla^2 C_b$$

(3.7)

Following earlier studies [8], an exponential form for the dependence of the viscosity on the concentration is adopted:

$$\mu = \begin{cases} 
\exp(R_b C_b), & \text{if } \mu_a < \mu_b \\
\exp(R_a (1 - C_b)), & \text{if } \mu_a > \mu_b 
\end{cases}$$

(3.8)
where $R_b$ and $R_a$ are the mobility ratios defined as:

$$R_b = -R_a = l \ln \left(\frac{\mu_b}{\mu_a}\right),$$  \hspace{1cm} (3.9)

while a linear dependence of the density on the concentration is adopted:

$$\rho = G_a C_a + G_b C_b,$$  \hspace{1cm} (3.10)

where $G_a$ and $G_b$ are fluid A and B density expansion coefficients, defined as:

$$G_a = \frac{\partial \rho}{\partial C_a} \quad \text{and} \quad G_b = \frac{\partial \rho}{\partial C_b}.$$  \hspace{1cm} (3.11)

### 3.1.4 Vorticity and stream function formulation

We formulate the problem in terms of the vorticity $\omega$ and stream function $\psi$.

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x},$$  \hspace{1cm} (3.12)

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$  \hspace{1cm} (3.13)

Taking the curl of Darcy’s law to eliminate the pressure, results in the following:

$$\omega = R_b \left( \frac{\partial \psi}{\partial x} \frac{\partial C_b}{\partial x} + \frac{\partial \psi}{\partial y} \frac{\partial C_b}{\partial y} + U(t) \frac{\partial C_b}{\partial y} \right) + \frac{1}{\mu} \left( \frac{\partial C_b}{\partial y} \Delta G \right).$$  \hspace{1cm} (3.14)

In the above equation $\Delta G = G_a - G_b$. $\Delta G > 0$ ($< 0$) represents a buoyantly unstable (stable) displacement where a heavier (lighter) fluid (A) is on top of a lighter (heavier) one (B).

### 3.1.5 Initial and boundary conditions

In order to complete the mathematical formulation, appropriate initial and boundary conditions need to be specified. The initial conditions for the concentration and velocity are:

$$C_b = \begin{cases} 
0, & \text{for } x < 0 \\
1, & \text{for } x > 0 
\end{cases} \quad \text{and} \quad \vec{u} = (0,0).$$  \hspace{1cm} (3.15)
Along the streamwise ($x$-) direction, the boundary conditions are:

\[(u, v, C_b) \left( \frac{-Pe}{2}, y, t \right) = (0, 0, 0), \quad (u, v, C_b) \left( \frac{Pe}{2}, y, t \right) = (0, 0, 1), \quad (3.16)\]

while in the transverse ($y$-) direction they are,

\[(u, v, C_b) \left( x, \frac{-Pe}{2A_r}, t \right) = (u, v, C_b) \left( x, \frac{Pe}{2A_r}, t \right). \quad (3.17)\]

The above boundary conditions; Eq. (3.16) state that only fluid A is present at the top boundary (inlet) while only fluid B is present at the bottom boundary (outlet), and that in the moving reference frame, the velocity field is equal to zero at the top and bottom boundaries. Periodic boundary conditions are implemented in the transverse direction (Eq. (3.17)).

In the subsequent analyses, the concentration is split as follows:

\[C_b(x, y, t) = \bar{c}_b(x, t) + c'_b(x, y, t). \quad (3.18)\]

The base-state concentration $\bar{c}_b$ is solution of the problem with zero velocity, and satisfies the streamwise boundary conditions. Consequently, $c'_b$ is zero at both top and bottom ends of the domain. This decomposition allows solving the problem in terms of $c'_b$ using spectral methods with periodic boundary conditions in the streamwise and transverse directions.

\[\frac{\partial c'_b}{\partial t} = - \left[ \frac{\partial \psi'}{\partial y} \left( \frac{\partial \bar{c}_b}{\partial x} + \frac{\partial c'_b}{\partial x} \right) - \frac{\partial \psi'}{\partial x} \frac{\partial \bar{c}_b}{\partial y} \right] + \frac{\partial^2 c'_b}{\partial x^2} + \frac{\partial^2 c'_b}{\partial y^2}, \quad (3.19)\]

\[\omega = R_b \left[ \frac{\partial \psi}{\partial x} \left( \frac{\partial \bar{c}_b}{\partial x} + \frac{\partial c'_b}{\partial x} \right) + \frac{\partial \psi}{\partial y} \frac{\partial c'_b}{\partial y} + U(t) \frac{\partial \bar{c}_b}{\partial y} \right] + \frac{1}{\mu} \left[ \frac{\partial c_b}{\partial y} \Delta G \right], \quad (3.20)\]

\[c'_b \left( \frac{-Pe}{2}, y, t \right) = c'_b \left( \frac{Pe}{2}, y, t \right) = 0; \quad c'_b \left( x, \frac{-Pe}{2A_r}, t \right) = c'_b \left( x, \frac{Pe}{2A_r}, t \right). \quad (3.21)\]
The above formulation is used to conduct LSA and NLS of the buoyancy-driven instability in porous media miscible displacements. Furthermore, the time-dependent velocity will be assumed to be cyclic, of the form

\[ U(t) = \bar{U} \left[ 1 + \Gamma \sin \left( \frac{2\pi}{T} t \right) \right], \quad (3.22) \]

where \( \Gamma \) is the amplitude and \( T \) the period. Note that when \( |\Gamma| \leq 1 \), the displacement rate will be positive and the flow will undergo simple injection while for \( |\Gamma| > 1 \) it will alternate between injection \( (U(t) > 0) \) and extraction \( (U(t) < 0) \). In the latter scenario, the displacement will be initiated by an extraction for \( \Gamma < -1 \) and by an injection when \( \Gamma > 1 \). A zero amplitude \( (\Gamma = 0) \) reduces the problem to that of a constant injection.

### 3.1.6 Linearized perturbation equations

To carry the linear stability analysis, Eq. (3.19) and (3.20) are linearized under the assumption that the perturbation terms denoted with primes are small compared to the base state. The resulting linearized equations are:

\[
\frac{\partial c'_b}{\partial t} = - \frac{\partial \psi'}{\partial y} \left( \frac{\partial \bar{c}_b}{\partial x} \right) + \frac{\partial^2 c'_b}{\partial x^2} + \frac{\partial^2 c'_b}{\partial y^2}, \quad (3.23)
\]

\[
\omega' = R_b \left( \frac{\partial \psi'}{\partial x} \frac{\partial \bar{C}_b}{\partial x} + U(t) \frac{\partial c_b'}{\partial y} \right) + \frac{1}{\mu(\bar{C}_b)} \frac{\partial c_b'}{\partial y} \Delta G. \quad (3.24)
\]

The above equations are solved using the highly accurate Hartley pseudo-spectral method to determine the spatio-temporal evolution of \( c_b' \) and \( \psi \). This method allows us to cast the partial differential equation in time and space (Eq. (3.23)) into an ordinary differential equation in time, which is then stepped in time using a semi-implicit scheme based on Adams-Bashforth predictor.
and Adams-Moulton corrector methods. A detailed explanation of the numerical technique is provided in section 3.3.

3.2 Mathematical model for heterogeneous porous medium

3.2.1 Physical problem

In this study, we examine the dynamics of buoyancy-driven instabilities under a gravitational field \( \vec{g} = (g, 0) \) in a two-dimensional porous medium of porosity \( \phi \). The medium of length \( L \) and width \( W \) is heterogeneous with the permeability \( K \) assumed to vary in the vertical (x) direction. Fluid A of concentration \( C_a \) and viscosity \( \mu_a \) is injected from the top boundary (inlet) to displace fluid B of concentration \( C_b \) and viscosity \( \mu_b \). The two fluids are incompressible, Newtonian, and fully miscible with isotropic constant diffusion \( D \).

3.2.2 Governing equations

The flow is governed by the equations for conservation of mass (continuity equation), conservation of momentum (Darcy’s law) and transport of species (convection-diffusion equation). Following Elgahawy and Azaiez [88], the governing equations are formulated in dimensionless form using diffusing scaling, where \( C_{bo}, \rho_{ref}, D\phi / U_{ch} \) and \( D\phi^2 / U_{ch}^2 \) are used as the reference concentration, density, length and time, respectively. The permeability is scaled by the average permeability of the medium \( K_1 \). Rather than using the injection velocity, the adopted characteristic velocity is, \( U_{ch} = \rho_{ref} g K_1 / \mu_{ch} \). This general choice allows to include scenarios where the flow develops as a result of density contrast without an injection velocity. The viscosity is scaled with \( \mu_{ch} = \mu_{less} \) corresponding to the viscosity of the less viscous fluid [82].

The dimensionless equations expressed in a Lagrangian reference frame moving with a dimensionless injection velocity \( U(t) \) are:
\[ \nabla \cdot \vec{u} = 0 \quad (3.25) \]
\[ \nabla P = -\frac{\mu(C_b)}{K(x)}[\vec{u} + U(t) \vec{u}] + \rho(C_b)\vec{u} \quad (3.26) \]
\[ \frac{\partial C_b}{\partial t} + \vec{u}. \nabla C_b = \nabla^2 C_b \quad (3.27) \]

Two dimensionless groups, the Péclet number \( Pe = \frac{u_{ch}L}{\phi} \) and cell aspect ratio \( A_r = \frac{L}{w} \), are used to specify the domain size such that it is \( \left( -\frac{Pe}{2}, \frac{Pe}{2} \right) \) in the \( x \)-direction and \( \left( -\frac{Pe}{2A_r}, \frac{Pe}{2A_r} \right) \) in the \( y \)-direction.

### 3.2.3 Vorticity and stream function formulation

The problem is formulated in terms of the vorticity \( \omega \) and stream function \( \psi \).
\[
\begin{align*}
\vec{u} &= \frac{\partial \psi}{\partial y}, & v &= -\frac{\partial \psi}{\partial x}, & \omega &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\nabla^2 \psi 
\end{align*}
\] (3.28)

Eliminating the pressure term by taking the curl of Darcy’s law leads to the following equation:
\[
\omega = R_b \left[ \frac{\partial \psi}{\partial x} \frac{\partial C_b}{\partial x} + \frac{\partial \psi}{\partial y} \frac{\partial C_b}{\partial y} + U(t) \frac{\partial C_b}{\partial y} \right] + \frac{K}{\mu} \left[ \frac{\partial C_b}{\partial y} \Delta G \right] - \frac{\partial \psi}{\partial x} \frac{\partial (\ln(K))}{\partial x} 
\] (3.29)

The heterogeneity is characterized using the natural log of the permeability,
\[
F = \ln(K(x)) = s \cos \left( \frac{2 \pi x q}{Pe} \right), \quad \frac{\partial F}{\partial x} = -s \frac{2 \pi q}{Pe} \sin \left( \frac{2 \pi x q}{Pe} \right) 
\] (3.30)

Where \( s \) is the range of variation of \( F \), \( q \) is the frequency of layers in the \( x \)-vertical direction. One of the objectives of this study is to analyze the effects of these two parameters on the instability.

### 3.2.4 Initial and boundary conditions

Same initial and boundary conditions as those in section (3.1.5) are specified to complete the mathematical formulation. In the subsequent analyses, the concentration is split as follows:
\[
C_b(x, y, t) = \tilde{c}_b(x, t) + c'_b(x, y, t). 
\] (3.31)
This decomposition allows solving the problem in terms of $c'_b$ using spectral methods with periodic boundary conditions in the streamwise and transverse directions.

$$\frac{\partial c'_b}{\partial t} = - \left[ \frac{\partial \psi' (\partial \tilde{c}_b + \partial c'_b)}{\partial y} \left( \frac{\partial c'_b}{\partial x} - \frac{\partial \psi' (\partial c'_b)}{\partial y} \right) + \frac{\partial^2 c'_b}{\partial x^2} + \frac{\partial^2 c'_b}{\partial y^2} \right], \quad (3.32)$$

$$\omega = R_b \left[ \frac{\partial \psi (\partial \tilde{c}_b + \partial c'_b)}{\partial x} + \frac{\partial \psi (\partial c'_b)}{\partial y} + U(t) \frac{\partial c'_b}{\partial y} + \frac{1}{\mu} \left[ \frac{\partial \tilde{c}_b}{\partial y} \Delta \theta \right] - \frac{\partial \psi (\ln(k))}{\partial x}, \right], \quad (3.33)$$

$$c'_b \left( -\frac{P_e}{2}, y, t \right) = c'_b \left( \frac{P_e}{2}, y, t \right) = 0; \quad c'_b \left( x, -\frac{P_e}{2A_r}, t \right) = c'_b \left( x, \frac{P_e}{2A_r}, t \right). \quad (3.34)$$

### 3.3 Numerical technique

In this study, the NLS and LSA was conducted based on the Hartley pseudo-spectral method. The implemented numerical technique is briefly discussed in this section to solve Eq. (3.19) and (3.20).

The same numerical procedure is implemented to solve Eq. (3.32) and (3.33) for the heterogeneous porous media. This numerical method allows to cast the partial differential equation in time and space into an ordinary differential equation in time. The direct and inverse Hartley transforms have the following forms:

$$\hat{g}(k_x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(x) \left[ \cos(k_x x) + \sin(k_x x) \right] dx$$

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{g}(k_x) \left[ \cos(k_x x) + \sin(k_x x) \right] d(k_x) \quad (3.35)$$

Our governing equation (Eq. (3.19)) can be written as follow by introducing $J_b$:

$$\frac{\partial c'_b}{\partial t} = J_b + \frac{\partial^2 c'_b}{\partial x^2} + \frac{\partial^2 c'_b}{\partial y^2}, \quad (3.36)$$

Where
\[ J_b = - \left[ \frac{\partial \psi'}{\partial y} \left( \frac{\partial c_b'}{\partial x} + \frac{\partial c''_b}{\partial x} \right) - \frac{\partial \psi'}{\partial x} \frac{\partial c'_b}{\partial y} \right] \quad (3.37) \]

The derivatives of an arbitrary function of \( x \) and \( y \) in Hartley space are given by:

\[
H\left[ \frac{\partial}{\partial x} g(x, y, t) \right] = -k_x \hat{g}(-k_x, -k_y, t) \\
H\left[ \frac{\partial}{\partial y} g(x, y, t) \right] = -k_y \hat{g}(-k_x, -k_y, t) \\
H\left[ \frac{\partial^2}{\partial x^2} g(x, y, t) \right] = -k_x^2 \hat{g}(k_x, k_y, t) \\
H\left[ \frac{\partial^2}{\partial y^2} g(x, y, t) \right] = -k_y^2 \hat{g}(k_x, k_y, t) 
\quad (3.38)
\]

Consequently, the governing equations in Hartley transform space become:

\[
\frac{d \hat{c}_b}{dt} = \hat{J}_b - \left( k_x^2 + k_y^2 \right) \hat{c}_b \\n\hat{\psi} = \frac{\hat{\omega}}{\left( k_x^2 + k_y^2 \right)} 
\quad (3.39, 3.40)
\]

One must note that \( \hat{c}_b, \hat{J}_b, \hat{\psi}, \hat{\omega} \) are the Hartley transform functions of \( c_b, J_b, \psi, \omega \) respectively.

Eq. (3.39) is an ordinary differential equation which will be stepped in time using a semi-implicit predictor corrector scheme to find the concentration in the next step. First, we use Euler’s method to get an initial value of the concentration:

\[
C_{i,j}^{n+1} = e^{-\lambda^2 \Delta t} \left[ C_{i,j}^{n} + \Delta t J_{i,j}^{n} \right] 
\quad (3.41)
\]

and the predicted concentration value is determined using the Adams-Bashforth method:

\[
C_{i,j}^{n+1} = e^{-\lambda^2 \Delta t} \left[ C_{i,j}^{n} + \frac{\Delta t}{2} (3 J_{i,j}^{n} - J_{i,j}^{n-1} e^{-\lambda^2 \Delta t}) \right] 
\quad (3.42)
\]

The vorticity is now updated using the predicted concentration from Eq. (3.42). It is accomplished by using the Hartley transform of Eq. (3.20). The stream function is then updated using Eq. (3.40). After that, the predicted concentration is corrected using the Adams-Moulton method:
After updating the concentration, the vorticity \( \omega(\Delta t) \) and stream function \( \phi(\Delta t) \) are then corrected as well. Finally, this procedure is iterated to reach desired values of \( \tilde{c}_i(t + \Delta t) \), \( \tilde{\psi}(t + \Delta t) \), and \( \tilde{\omega}(t + \Delta t) \).
Chapter 4

Rayleigh-Taylor Instability in Porous Media under Sinusoidal Time-Dependent Flow Displacements

Abstract

1 Linear stability analysis and nonlinear simulations have been carried out to analyze the Rayleigh-Taylor instability in homogeneous porous media under time-dependent flow displacements. The flow processes consist of a sinusoidal time-dependent velocity characterized by its period (T) and amplitude (Γ) and ensure that the same amount of fluid is injected over a full flow period. A new, more efficient approach to determine instability characteristics has been developed for the stability analysis of these time-dependent injection flows, and showed a growth rate that varies in time like the displacement velocity. The effects of the period T and amplitude Γ as well as the fluids’ viscosity (R) and density differences (ΔG) have been analyzed. Consistent with constant injection displacements, larger ΔG leads to stronger instabilities. Furthermore, it is found that a larger R tends to attenuate the instability during extraction or soaking periods and to enhance it during injection. The study also revealed that for a given total injection time, the time-dependent flow can be less or more unstable than its constant injection counterpart. In particular, for Γ < −1, larger periods lead to stronger instabilities with longer more developed fingers. For Γ > 1 on the other hand, it is found that larger periods tend to attenuate the instability resulting in a smaller number of fingers and a more diffused front. Flows with unit amplitude (Γ = 1) exhibit the same qualitative trends as, but are overall more unstable than their counterparts with Γ > 1.

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¹ This chapter is reproduced from the following journal article with the permission of AIP Publishing: Y. Elgahawy and J. Azaiez, "Rayleigh–Taylor instability in porous media under sinusoidal time-dependent flow displacements," *AIP Advances*, vol. 10, no. 7, p. 075308, 2020.
4.1 Introduction

Flow instabilities develop at the interface between fluids during displacement processes in porous media when a less-viscous fluid displaces a more-viscous one or as a result of density mismatch. These hydrodynamic instabilities lead to the development of finger-like intrusions known as viscous fingering in the case of viscosities mismatch [1] or Rayleigh-Taylor instability in the case of densities mismatch between the fluids [2]. Such instabilities affect significantly the efficiency of displacement processes encountered in many applications such as enhanced oil recovery (EOR) processes [3], CO₂ sequestration [4], and chromatographic separation [89]. The fingering phenomenon is favorable and desirable in some applications such as geological storage of CO₂ in deep saline aquifers as the buoyancy-driven instability results in convective mixing [6]. This can dramatically enhance the rate of CO₂ dissolution into the brine, thus reducing the risk of CO₂ leakage into the environment. On the other hand, fingering is problematic in the oil industry as it results in an ineffective sweeping action that will bypass substantial amounts of recoverable oil.

Numerous experimental and theoretical studies have been devoted to examining the fingering instability in miscible displacements due to its complexity and importance. During the last few decades, researchers have studied the effects of several factors on the instability such as chemical reaction [12, 90], diffusivity [16], anisotropic and velocity dependent dispersion [17, 18], viscosity ratio [8], heat transfer [14, 15], gravity [9-11], injection velocity [65], inertial effects [19-21], permeability heterogeneity [44] and additives such as nanoparticles [22, 23]. Extensive reviews of this instability were given by Homsy [30] and McCloud and Maher [31]. Most of the existing studies have focused exclusively on displacements involving constant injection rates. However, in some practical processes the injection rate is in fact time-dependent. These practical processes include some of EOR methods such as CO₂ huff-and-puff which involves cyclic injection of CO₂.
to recover additional oil [91], and cyclic steam stimulation (CSS) which consists of three stages of steam injection, soaking, and oil production [92]. The soaking period allows the heat to diffuse and thus reduces the viscosity of the heavy oil enabling its production. These cycles are repeated until production rate falls to uneconomic levels. Similar to the CSS method, the Cyclic Solvent Injection (CSI) process involves solvent injection followed by a soaking period and then oil production [93]. Time-dependent injections are also encountered in other industrial applications such as trickle-bed reactors where pulsing, sinusoidal velocity profiles are utilized to enhance the mass transfer rate [94].

There are some attempts to study and understand the effects of time-dependent injection rates on miscible viscous fingering. Chen and Meiburg [57] suppressed the growth of viscous fingers in a miscible displacement using a time-dependent injection strategy. They observed that a smaller injection rate during the early stages partially stabilized the flow, however increasing the injection rate at later times had negligible effects on the growth of the fingering instability. Inspired by the effectiveness of the time-dependent control strategies employed in immiscible displacements by Li et al. [50] and Dias and Miranda [52], Chen et al. [58] investigated a similar control strategy but for a miscible displacement in which the injection rate varies as $Q(t) \sim t^{-1/3}$. Their non-linear simulations results confirmed the stabilizing effects of such control strategies, $Q(t) \sim t^{-1/3}$, and it was reported that the development of intricate fingering patterns, such as fingers’ splitting and merging were all suppressed, compared to those of the constant injection scenario. Yuan and Azaiez (2012) [59] were the first to employ a time-dependent injection strategy based on a sinusoidal velocity model in miscible horizontal displacements. They observed that the sweep efficiency of their sinusoidal model was less than that of the constant injection counterpart. In addition, the breakthrough time of the sinusoidal model was shorter than that of the constant
injection. In a subsequent study, Yuan and Azaiez [60] adopted a different control strategy involving alternating stages of production and injection or of soaking and injection. Using non-linear simulations, they found that the soaking-injection scenario attenuated the instability compared to the constant injection one, due to the stabilizing effects of diffusion during the soaking stage. On the other hand, they reported that the injection-soaking displacements enhanced the instability and the flow was more unstable than the constant injection velocity flow. This study was further expanded to analyze inertial effects [19]. It was found that the cycle period and velocity amplitude have significant effects on the stability of the displacement, and these effects are heavily influenced by considering fluid’s inertia.

To the best knowledge of the authors, no previous studies have examined the effects of time-dependent injection rates on the Rayleigh-Taylor (RT) instability. The objective of this research is therefore to investigate and understand the effects of time-dependent injection rates on such buoyancy-driven instabilities developing in porous media miscible displacements. Using Linear Stability Analysis (LSA) and non-linear simulations (NLS), the study will analyze the effects of sinusoidal time-dependent velocities on the displacement instability. The role of the cycle period and velocity amplitude under different scenarios of density and viscosity mismatches will be investigated. Furthermore, the effectiveness of the time-dependent injection scheme in attenuating or enhancing the instability relative to the constant injection scheme will be analyzed.

4.2 Mathematical model

4.2.1 Physical problem

In this study, we examine the dynamics of buoyancy-driven instabilities in a two-dimensional porous medium under time-dependent flow displacements. The porous medium of length L and width W is assumed homogeneous with constant porosity and permeability. Fluid A of
concentration $C_a$ and viscosity $\mu_a$ is injected from the top boundary (inlet) to displace fluid B of concentration $C_b$ and viscosity $\mu_b$ as shown in figure 4.1. The two fluids are incompressible, fully miscible and Newtonian. The displacement develops along the x-axis under a time-dependent displacement velocity $U(t)$, where a negative velocity corresponds to a production process, while a positive one corresponds to an injection process. Identically zero velocity results in a soaking stage.

4.2.2 Governing equations

The flow is governed by the equations for conservation of mass (continuity equation), conservation of momentum (Darcy’s law) and transport of species (convection-diffusion equation).

\[
\vec{\nabla} \cdot \vec{u} = 0, \quad (4.1)
\]

\[
\vec{\nabla} p = -\frac{\mu}{K} \vec{u} + \rho \vec{g}, \quad (4.2)
\]

\[
\varphi \frac{\partial C_b}{\partial t} + \vec{u} \cdot \vec{\nabla} C_b = \phi D \nabla^2 C_b, \quad (4.3)
\]
where \( \vec{u} (u, v) \) is the velocity field, \( P \) the pressure, \( \mu \) the viscosity, \( K \) and \( \Phi \) the porous medium permeability and porosity respectively, \( \vec{g} = (g, 0) \) the gravitational acceleration, \( C_b \) the concentration of fluid B, and \( D \) the constant diffusion coefficient. In this study, it is assumed that the mass diffusivities are isotropic and constant and the same for both species.

**4.2.3 Dimensionless formulation**

The governing equations are made dimensionless using diffusing scaling where \( \rho_{ref}, D\Phi/U_{ch} \) and \( D\Phi^2/U_{ch}^2 \) are used as the reference density, length and time, respectively. The velocity is scaled using a characteristic velocity, \( U_{ch} = \rho_{ref} g K/\mu_{ch} \) rather than using the injection velocity. This choice allows to include scenarios where the flow can develop in the presence of density contrast with zero injection velocity. Furthermore, the pressure and the concentration are scaled with \( \mu_{ch} D\Phi/K \) and \( C_{b0} \) respectively, while the viscosity is scaled with \( \mu_{ch} = \mu_{less} \) corresponding to the viscosity of the less viscous fluid. This is the appropriate scaling when comparing displacements of a less viscous fluid by a more viscous fluid with those of a more viscous fluid by a less viscous one [82]. Two dimensionless groups, the Péclet number \( Pe = \frac{U_{ch} L}{D \Phi} \) and cell aspect ratio \( A_r = \frac{L}{W} \), are used to specify the domain size such that it is \( \left( \frac{-Pe}{2}, \frac{Pe}{2} \right) \) in the \( x \) direction and \( \left( \frac{-Pe}{2A_r}, \frac{Pe}{2A_r} \right) \) in the \( y \) direction. The resulting dimensionless equations, using for simplicity the same notation and expressed in a Lagrangian reference frame moving with a dimensionless injection velocity \( U(t) \),

\[
\begin{align*}
\dot{x}' &= x - \int_0^t U(\tau) d\tau \\
\end{align*}
\]

The dimensionless equations are:

\[
\nabla \cdot \vec{u} = 0
\]
Following earlier studies [8], an exponential form for the dependence of the viscosity on the concentration is adopted:

\[
\mu = \begin{cases} 
\exp(R_b C_b), & \text{if } \mu_a < \mu_b \\
\exp(R_a (1 - C_b)), & \text{if } \mu_a > \mu_b 
\end{cases}
\] (4.8)

where \(R_b\) and \(R_a\) are the mobility ratios defined as:

\[
R_b = -R_a = \ln \left( \frac{\mu_b}{\mu_a} \right),
\] (4.9)

while a linear dependence of the density on the concentration is adopted:

\[
\rho = G_a C_a + G_b C_b,
\] (4.10)

where \(G_a\) and \(G_b\) are fluid A and B density expansion coefficients, defined as:

\[
G_a = \frac{\partial \rho}{\partial C_a} \quad \text{and} \quad G_b = \frac{\partial \rho}{\partial C_b}.
\] (4.11)

### 4.2.4 Vorticity and stream function formulation

We formulate the problem in terms of the vorticity \(\omega\) and stream function \(\psi\).

\[
u = \frac{\partial \psi}{\partial x}, \quad v = -\frac{\partial \psi}{\partial x},
\] (4.12)

\[
\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.
\] (4.13)

Taking the curl of Darcy’s law to eliminate the pressure, results in the following:

\[
\omega = R_b \left( \frac{\partial \psi}{\partial C_b} + \frac{U(t)}{\partial y} \frac{\partial C_b}{\partial y} + \frac{1}{\mu} \left( \frac{\partial C_b}{\partial y} \Delta G \right) \right).
\] (4.14)
In the above equation $\Delta G = G_a - G_b. \Delta G > 0 \ (< 0)$ represents a buoyantly unstable (stable) displacement where a heavier (lighter) fluid (A) is on top of a lighter (heavier) one (B).

4.2.5 Initial and boundary conditions

In order to complete the mathematical formulation, appropriate initial and boundary conditions need to be specified. The initial conditions for the concentration and velocity are:

$$C_b = \begin{cases} 
0, & \text{for } x < 0 \\
1, & \text{for } x > 0
\end{cases} \quad \text{and} \quad \bar{u} = (0, 0). \quad (4.15)$$

Along the streamwise ($x$-) direction, the boundary conditions are:

$$(u, v, C_b) \left( \frac{-Pe}{2}, y, t \right) = (0, 0, 0), \quad (u, v, C_b) \left( \frac{Pe}{2}, y, t \right) = (0, 0, 1), \quad (4.16)$$

while in the transverse ($y$-) direction they are,

$$(u, v, C_b) \left( x, \frac{-Pe}{2Ar}, t \right) = (u, v, C_b) \left( x, \frac{Pe}{2Ar}, t \right). \quad (4.17)$$

The above boundary conditions; Eq. (4.16) state that only fluid A is present at the top boundary (inlet) while only fluid B is present at the bottom boundary (outlet), and that in the moving reference frame, the velocity field is equal to zero at the top and bottom boundaries. Periodic boundary conditions are implemented in the transverse direction (Eq. (4.17)).

In the subsequent analyses, the concentration is split as follows:

$$C_b(x, y, t) = \bar{c}_b(x, t) + c'_b(x, y, t). \quad (4.18)$$

The base-state concentration $\bar{c}_b$ is solution of the problem with zero velocity, and satisfies the streamwise boundary conditions. Consequently, $c'_b$ is zero at both top and bottom ends of the
domain. This decomposition allows solving the problem in terms of $c'_b$ using spectral methods with periodic boundary conditions in the streamwise and transverse directions.

\[
\frac{\partial c'_b}{\partial t} = -\left[ \frac{\partial \psi'}{\partial y} \left( \frac{\partial c'_b}{\partial x} + \frac{\partial c'_b}{\partial x} \right) - \frac{\partial \psi'}{\partial x} \frac{\partial c'_b}{\partial y} \right] + \frac{\partial^2 c'_b}{\partial x^2} + \frac{\partial^2 c'_b}{\partial y^2}, \tag{4.19}
\]

\[
\omega = R_b \left[ \frac{\partial \psi}{\partial x} \left( \frac{\partial c'_b}{\partial x} + \frac{\partial c'_b}{\partial x} \right) + \frac{\partial \psi}{\partial y} \frac{\partial c'_b}{\partial y} + U(t) \frac{\partial c'_b}{\partial y} \right] + \frac{1}{\mu} \left[ \frac{\partial c_b}{\partial y} \Delta G \right], \tag{4.20}
\]

\[
c'_b \left( \frac{-P_e}{2}, y, t \right) = c'_b \left( \frac{P_e}{2}, y, t \right) = 0; \quad c'_b \left( x, \frac{-P_e}{2A_r}, t \right) = c'_b \left( x, \frac{P_e}{2A_r}, t \right). \tag{4.21}
\]

The above formulation is used in the next two sections where results from linear stability analysis (LSA) and full nonlinear simulations (NS) are presented. Furthermore, the time-dependent velocity will be assumed to be cyclic, of the form

\[
U(t) = \bar{U} \left[ 1 + \Gamma \sin \left( \frac{2\pi}{T} t \right) \right], \tag{4.22}
\]

where $\Gamma$ is the amplitude and $T$ the period. Note that when $|\Gamma| \leq 1$, the displacement rate will be positive and the flow will undergo simple injection while for $|\Gamma| > 1$ it will alternate between injection ($U(t) > 0$) and extraction ($U(t) < 0$). In the latter scenario, the displacement will be initiated by an extraction for $\Gamma < -1$ and by an injection when $\Gamma > 1$. A zero amplitude ($\Gamma = 0$) reduces the problem to that of a constant injection.
4.3 Linear stability analysis

4.3.1 Linearized perturbation equations

To carry the linear stability analysis, Eq. (4.19) and (4.20) are linearized under the assumption that the perturbation terms denoted with primes are small compared to the base state. The resulting linearized equations are:

\[
\begin{align*}
\frac{\partial c'_b}{\partial t} &= -\frac{\partial \psi'}{\partial y} \left( \frac{\partial \bar{c}_b}{\partial x} \right) + \frac{\partial^2 c'_b}{\partial x^2} + \frac{\partial^2 c'_b}{\partial y^2}, \\
\omega' &= R_b \left( \frac{\partial \psi'}{\partial x} \frac{\partial \bar{c}_b}{\partial x} + U(t) \frac{\partial c'_b}{\partial y} \right) + \frac{1}{\mu(\bar{c}_b)} \frac{\partial c'_b}{\partial y} \Delta G.
\end{align*}
\] (4.23) (4.24)

The above equations are solved using the highly accurate Hartley pseudo-spectral method to determine the spatio-temporal evolution of \( c'_b \) and \( \psi \). This method allows us to cast the partial differential equation in time and space (Eq. (4.23)) into an ordinary differential equation in time, which is then stepped in time using a semi-implicit scheme based on Adams-Bashforth predictor and Adams-Moulton corrector methods. A detailed explanation of the numerical technique can be found in the study by Islam and Azaiez [95].

4.3.2 Results and discussion

One of the main objectives of LSA is to determine the most dominant wave number \( k_{max} \) and the corresponding maximum growth rate \( \sigma_{max} \). These allow to predict the early evolution of the flow and can shed light on the effects of different parameters on the instability.

To the best knowledge of the authors, all previous studies on viscous fingering involving LSA determined \( \sigma_{max} \) and \( k_{max} \) from instability characteristics curves generated by scanning the whole spectrum of wave numbers. In this study, we propose a new approach that requires less
computational time and storage space, and proved to be more efficient in determining the
domestic growth rate and most dangerous wave number. To this end, an initial small perturbation
generated from a random noise instead of a single wave is used.

\[ c_b'(x, y) = \delta \text{rand}(y) \exp \left( -\frac{x^2}{\alpha^2} \right). \quad (4.25) \]

Where \( \delta \) is the magnitude of the perturbation, \( \text{rand}(y) \) is a random number between -1 and 1, and
\( \alpha \) determines the penetration of the disturbances from the front. The linearized equations are solved
with this initial perturbation to determine the evolution of the concentration and velocity field
disturbances. The dominant growth rate is determined as [96]:

\[ \sigma_{max}(t) = \frac{1}{2E} \frac{dE}{dt}, \quad (4.26) \]

where the amplification terms are determined as [96]:

\[ E(t) = E_c(t) + E_v(t), \]
\[ E_c(t) = \iint c_b'^2(x, y, t)dx\,dy, \quad E_v(t) = \iint (u'^2 + v'^2)(x, y, t)dx\,dy. \quad (4.27) \]

In order to extract the most unstable wavenumber at any given time, the Fourier transform \( F(k, t) \)
of the contour representing the interface between the two fluids \( X = f(Y) \) is determined from
the concentration. Figure 4.2 shows a typical Fourier transform spectrum at time \( t = 400 \) for a
buoyantly unstable displacement with \( \Delta G = 1, \, R = 1 \) under natural convection \( (U(t) = 0) \). It can
be seen that the amplitude exhibits few maxima at small wavenumbers and decays rapidly as \( k \)
increases. The most dangerous wave number \( k_{max} \) is determined from the Fourier spectrum as

\[ k_{max}(t) = \frac{\sum_{i=1}^{n} |F(k_i, t)|^2 \cdot k_i}{\sum_{i=1}^{n} |F(k_i, t)|^2} \quad (4.28) \]

Where \( k \) is the wave number and \( F(k, t) \) is the Fourier transform. The numerical value for \( k_{max} \)
for the spectrum depicted in figure 4.2 at \( t = 400 \) is determined using Eq. (4.28) as \( k_{max} = 0.031 \)
Figure 4.2: Fourier transform spectrum at $t = 400 \Delta G = 1$, $R = 1$ and $U(t) = 0$.

Figure 4.3: Temporal evolution of (a) $\sigma_{\text{max}}$ and (b) $k_{\text{max}}$ for $\Delta G = 1$, $R = 1$ and $U(t) = 0$.

Figure 4.3 presents the evolution of $k_{\text{max}}$ and $\sigma_{\text{max}}$ with time for the same flow parameters as in figure 4.2. These results based on the new approach show a perfect match with those reported by Pramanik et al. [82] for the same flow parameters. We have further validated the accuracy and robustness of this approach by reproducing results for other values of the parameters $\Delta G$ and $R$ under constant injections. These however are not shown here for brevity.

In what follows we investigate the effects of time-dependent displacements on the instability by analyzing the temporal evolution of the growth rates of the disturbances. Such displacements will
involve injection/soaking/extraction stages if $|\Gamma| > 1$, only injection if $|\Gamma| < 1$ and injection/soaking if $\Gamma = \pm 1$.

First, displacements that alternate between extraction and injection with an initially decreasing velocity ($\Gamma < -1$) are examined. Figure 4.4 illustrates the evolution of the growth rate and injection velocity with time for $\Delta G = 3$, $R = 1$ and $\Gamma = -2$. It is noted that, qualitatively the variations of the growth rates follow the general trend of the velocities for all cycle periods though with some minor differences. For each period, it is observed that the oscillating growth rate decays in time, with the maximum (positive) value decreasing as the flow evolves. The minimum growth rate also decreases but at a slower rate. This decrease is due to the fact that the perturbations grow exponentially early in the flow before starting to decay (Duff et al. [97]). The global maximum growth rate occurs during the first cycle when the flow switches to injection and the velocity reaches its first maximum. Furthermore, it is found that the smaller the period, the larger the growth rate but the stronger the subsequent decay over time.
The effects of the density difference ($\Delta G$) are depicted in figure 4.5 where results for two other values of $\Delta G$; $\Delta G = 2$ and $4$ and $R = 1$, $\Gamma = -2$, are presented. The same qualitative trends reported earlier are observed here. Furthermore, it is found that the larger $\Delta G$, the larger the growth rate but also the stronger the decay over time. In particular the minimum growth rates shift away from negative values towards zero.

Figure 4.5: Variation with time of the maximum growth rate for different periods $T$ and for $R = 1$, $\Gamma = -2$, (a) $\Delta G = 2$ and (b) $\Delta G = 4$.

Figure 4.6 depicts the effects of the mobility ratio ($R$) on the variation of the oscillating growth rates with $T$ for $\Delta G = 4$ and $\Gamma = -2$. It is found that during extraction periods, a larger $R$ results in smaller (negative) growth rates while during injection the larger the $R$, the larger the growth rates. In addition, the minimum (negative) growth rate remains virtually unchanged over time for a larger $R$. These trends are to be contrasted with the effects of the density difference, where a larger $\Delta G$ leads to stronger growth rates during both extraction and injection.
Figure 4.6: Variation with time of the maximum growth rate for different periods and for $\Delta G = 4, \Gamma = -2$, (a) $R = 1$ and (b) $R = 2$.

The previous results and discussion focused on time-dependent displacements that alternate between extraction and injection with an initially decreasing velocity ($\Gamma < -1$). Next we focus on displacements that involve injection and short soaking periods ($\Gamma = 1$) with an initially increasing velocity (figure (4.7-c)) and follow the same line of analysis as for displacements with $\Gamma < -1$.

The effects of the density difference ($\Delta G$) are depicted in figure 4.7(a-b) where results for two values of $\Delta G$; $\Delta G = 2$ and $\Delta G = 4$ and $R = 1, \Gamma = 1$, are presented. It should be noted that, for each $T$, the oscillating growth rates follow the same trend as the velocities but with two tangible differences. First, there is a large increase in the growth rates during the first cycle until the maximum growth rate is reached and this is due to the exponential growth of the perturbations during the early times. Second, although all these displacement scenarios did not involve any extraction stages (negative injection), the growth rates for the case of $\Delta G = 2$ exhibited negative values specifically during soaking periods and we will refer to these as minimum growth rates.
Moreover, it is observed that during both injection and soaking, a larger $\Delta G$ leads to stronger growth rates but also a much sharper decay over time. It is also found that for a large enough value of $\Delta G$, the growth rate is strictly positive throughout the flow (figure 4.7-b).

Figure 4.7: Variation with time of the maximum growth rate for different periods and for $\Gamma = 1$, $R = 1$ and (a) $\Delta G = 2$, (b) $\Delta G = 4$, (c) Corresponding displacement velocity profiles.

Figure 4.8(a-b) depicts the effects of the mobility ratio ($R$) on the variation of the oscillating growth rates with $T$ for $\Delta G = 4$ and $\Gamma = 1$. One must note that the larger the mobility ratio, the smaller the minimum growth rates and the stronger their deviation from positive values towards zero. This
decrease in the minimum growth rates is attributed to the stabilizing effects of viscous forces on buoyantly unstable displacements during soaking periods. In conclusion, it is found that during soaking periods, larger $R$ result in smaller (negative) growth rates while during injection the larger the $R$, the larger the growth rates.

Figure 4.8: Variation with time of the maximum growth rate for different periods and for $\Delta G = 4$, $\Gamma = 1$, (a) $R = 1$, (b) $R = 2$.

For displacements that alternate between extraction and injection but with an initially increasing velocity ($\Gamma > 1$), the same qualitative trends as those for $\Gamma < -1$ were observed, where the growth rates for different periods initially increase reaching a maximum then start decreasing and keep oscillating. For the sake of brevity, these results are not presented.

The oscillatory nature of the time-dependent growth rates raises the question of how to properly quantify the instability and compare the different cases with such oscillatory behaviour. Several quantifying parameters such as the global maximum growth rates, growth rates at a fixed time e.g. $t = 1200$ and different forms of time-averaged growth rate, were explored. It was found that time-averaged growth rates over the maximum of all considered periods, offer the best characterization
of the instability and result in trends that are in better qualitative agreements with those observed in the non-linear simulations.

\[
\sigma_{\text{ave}}(R, \Delta G, T, \Gamma) = \frac{1}{T_{\text{max}}} \int_0^{T_{\text{max}}} \sigma(t; R, \Delta G, T, \Gamma) dt,
\]

(4.29)

Where \( T_{\text{max}} \) is the largest considered period, \( i.e. \) 1200.

![Graphs](image)

Figure 4.9: Variation of the average growth rate with the period for different density differences and for \( R = 1 \), (a) \( \Gamma = -2 \), (b) \( \Gamma = 1 \).

The variations of \( \sigma_{\text{ave}} \) with the period for different \( \Delta G \) and for \( R = 1 \) are depicted in figure 4.9 for two amplitudes. In the case of a negative amplitude; \( \Gamma = -2 \), it is found that for a given \( \Delta G \), \( \sigma_{\text{ave}} \) varies non-monotonically with \( T \), exhibiting two local minima at \( T \approx 500 \) and \( T \approx 800 \). Opposite trends are observed in the case of a positive amplitude; \( \Gamma = 1 \), where local maxima are attained at \( T \approx 500 \) and \( T \approx 800 \). These local extrema have been observed systematically at the same values of the period \( T \), regardless of the values of the other parameters namely, \( R, \Delta G \) and \( \Gamma \). However the amplitudes of these extrema and the rates of change with the period do depend on \( R \) and \( \Gamma \) but are independent of \( \Delta G \). In particular, it was found that larger \( R \) and larger absolute values of the amplitude \( \Gamma \) systematically lead to larger values of the extrema (see figure 4.10(a-c)).
Figure 4.10: Variation of the average growth rate with the period for different density differences and for (a) $R = 3, \Gamma = -2$, (b) $R = 3, \Gamma = 1$ and (c) $R = 1, \Gamma = 3$

The occurrence of local extrema at the same values of the period $T \approx 500$ and $\approx 800$, regardless of the values of the other parameters can be traced back to the average injection velocity or equivalently the total amount of injected fluid,

$$U_{ave} = \frac{1}{T_{max}} \int_0^{T_{max}} U(t) dt.$$  

(4.30)
The variations of $U_{ave}$ with the period for both negative and positive velocity amplitudes are presented in figure 4.11.

Two local minima (maxima) at $T \approx 500$ and $T \approx 800$ are observed in the case of a negative (positive) amplitude. These local extrema coincide with those observed in the variations of $\sigma_{ave}$ reported earlier, explain the sudden decrease or increase in $\sigma_{ave}$ at those periods. The exact values of the periods ($T^*$) at which local extrema occur can be determined from the extrema of:

$$
Z = \frac{U_{ave}}{U}(T) = 1 + \frac{\Gamma}{2\pi} \left( \frac{T}{T_{max}} \right) \left[ 1 - \cos \left( 2\pi \frac{T_{max}}{T} \right) \right]
$$

(4.31)

Which are obtained as solution of the algebraic equation:

$$
1 - \cos (\beta) - \beta \sin (\beta) = 0 \Leftrightarrow \sin (\beta) \left[ \tan \left( \frac{\beta}{2} \right) - \beta \right] = 0 \text{ where } \beta = 2\pi \frac{T_{max}}{T^*}
$$

(4.32)
The sinusoidal time-dependent velocities are such that their average over the time (t = 1200) is unity; \( \overline{U(t)} = 1 \) except for the two cases where the period is about 500 or 800. For these two periods, the cumulative volume of injected fluid into the porous medium is either lower (for \( \Gamma < 0 \)) or larger (for \( \Gamma > 0 \)) than that of the other periods.

The \( \sigma_{ave} \) results presented in figures 4.9 and 4.10 follow the same trends for different values of \( \Delta G \) and seem self-similar. This implies that it may be possible to present all results through a single master curve. This was indeed possible through the following shifted expression of the average growth rate:

\[
\sigma_{ave}^{*} = \sigma_{ave} - f(R)(\Delta G - 1), \tag{4.33}
\]

where \( f(R) \) is a factor that depends on the mobility ratio and that was determined as 0.00590 for \( R = 1 \), 0.00395 for \( R = 2 \), and 0.00275 for \( R = 3 \). With this shift, the curves for different values of \( \Delta G \) shown in figures 4.9 and 4.10 collapse into a single curve depicted in figure 4.12.
Figure 4.12: Variation of the average growth rate with period for different density differences and for (a) $R = 1, \Gamma = -2$, (b) $R = 1, \Gamma = 1$, (c) $R = 3, \Gamma = -2$ and (d) $R = 3, \Gamma = 1$. 
4.4 Non-linear simulations

4.4.1 Numerical technique

Numerical solutions of Eq. (4.19) and (4.20) were obtained to capture the non-linear effects of fingers development that were previously ignored in the LSA. The numerical code convergence was ascertained by varying the spatial and temporal step sizes. It was also validated by comparison with the results of a number of earlier studies. In particular, its predictions in the case of vertical miscible displacements under a constant injection velocity were compared with corresponding ones published in the literature. As an illustration, the concentration contours for a viscously and buoyantly unstable displacement with the same flow parameters reported by Pramanik et al. [82] is shown in figure 4.13. This figure shows a good qualitative agreement with Pramanik et al. [82] results; both having similar number of fingers and structures.

Figure 4.13: Concentration contour for $\Delta G = 1$, $R = 1$ and for $U(t) = 1$. 
4.4.2 Results and discussion

In all what follows, the fingering instability is qualitatively characterized using concentration fields of fluid B. In addition, a quantitative analysis is conducted to assess the degree of instability and capture the development of the fingers. The mixing length ($ML$), mixing quality ($MQ$), and interfacial length ($IL$) are used as robust tools to quantify the hydrodynamic instability. The $ML$ is defined as the ratio of the length of the channel in which the transversely averaged concentration is in the range of $0.01 < C_{b,ave} < 0.99$ to the length of the whole channel, where

$$C_{b,ave}(x,t) = \frac{A_r}{Pe} \int_{0}^{A_r} C_b(x,y,t) \, dy.$$  

(4.34)

The $MQ$, which is based on the variance of the concentration field, quantifies the degree of mixing and is defined by,

$$MQ (t) = 1 - \frac{\sigma^2}{\sigma_{max}^2}$$ 

(4.35)

Where $\sigma^2 = \langle C_b^2 \rangle - \langle C_b \rangle^2$ and $\langle \cdot \rangle$ denotes spatial averaging over the domain. $\sigma_{max}^2$ is the variance of a perfectly segregated state which occurs at the initial time. Note that, $\sigma^2 = 0$ leading to $MQ = 1$ corresponds to the perfectly mixed case.

The interfacial length ($IL$), commonly referred to as the relative contact area, measures the temporal evolution of the stream-wise and transverse gradients of concentration and accurately determines the total length of the diffusive region between fluids $A$ and $B$. It is virtually constant during the early diffusive regime, and increases rapidly during the convective regime when fingers start developing.

Figure 4.14, displaying the variations with time of $ML$, $MQ$ and $IL$, shows the strong capability of these three quantitative tools to capture the temporal evolution of the fingering instability driven
by both density and viscosity differences. The three parameters increase with time as the instability develops with the progression of the flow.

![Graph showing normalized quantities vs. time]

Figure 4.14: Comparison of different quantitative tools for $R = 1$, $\Delta G = 1$, and $U(t) = 1$.

The effects of the density difference on the instability for the constant injection case ($U(t) = 1$) is illustrated in figure 4.15 depicting concentration fields for different $\Delta G$ and $R = 1$. Please note that in all concentration fields, the time is indicated at the top left corner. It can be seen that a larger $\Delta G$ leads to a stronger instability with more developed complex finger structures due to the non-linear interactions between the fingers. These qualitative trends are confirmed with the analysis of the $IL$, shown in figure 4.16(a), where a larger density difference results in larger magnitudes of $IL$. The temporal variations of $IL$ depicted in figure 4.16 shows that for each $\Delta G$ or $R$, $IL$ is the same and constant during the early time before undergoing a rapid increase. It is also clear that the early plateau stage corresponding to the diffusive regime lasts for a shorter duration for larger $\Delta G$ or $R$. Furthermore, larger mobility ratios, which intuitively trigger a stronger instability, result in larger magnitudes of $IL$. Both $ML$ and $MQ$ exhibited the same qualitative behaviour. However, for brevity, in all that follows only results based on $IL$ will be presented.
In what follows we study the effects of time-dependent displacements on the instability qualitatively and quantitively by examining both the temporal evolution of the $IL$ and
concentration fields of fluid B. Following the same line of analysis adopted in the LSA section, first displacements that alternate between extraction and injection with an initially decreasing velocity ($\Gamma < -1$) are analyzed. It should be noted that throughout this study, the $IL$ results of different cases will be compared and examined at $t = 1200$.

![Graph](image)

Figure 4.17: Variation with time of the interfacial length for different periods $T$. $R = 1, \Delta G = 3$ and $\Gamma = -2$.

Figure 4.17 illustrates the evolution of the interfacial length with time for different periods, in the case $\Delta G = 3$, $R = 1$ and $\Gamma = -2$. The corresponding velocity profiles are depicted in figure 4.4. It is observed that the early diffusive regime extends longer in time compared to the constant injection case (see case of $\Delta G = 3$ in figure 4.16(a)) and lasts longer for larger periods. As a result, larger periods lead to a later onset of the convective regime but also to larger values of $IL$ once the flow is in the convective regime. Subsequently, $IL$ undergoes an oscillatory regime with an amplitude that increases with time.

The effects of $\Delta G$ on the time-dependent displacement are further investigated using the temporal evolution of $IL$ for $R = 1$ and two values of $\Delta G$; $\Delta G = 2$ and 4 (see figure 4.18(a-b)). The qualitative trends reported earlier in figure 4.17 are also observed here. It is found that the early diffusive regime, for each period, becomes shorter in time and convection starts earlier at larger
values of $\Delta G$. Furthermore, under a large $\Delta G$, the flow undergoes a stronger oscillatory regime in which the $IL$ increases prominently during injection stages and decreases rapidly during extraction periods. These trends are to be contrasted with those of the constant injection case in which $IL$ keeps increasing after the onset of convection (see figure 4.16(a)). Concentration fields for the period $T = 600$ depicted in figure 4.19 confirm that a larger $\Delta G$ leads to earlier onset of the instability. It is also identified that the oscillatory regime of $U(t)$ results in a fluctuation of the instability with the stabilizing effects of diffusion being well noticeable at $t = 800$ after an extraction period that started at $t = 650$. It is clear that the front at $t = 800$ is more diffused than that at $t = 650$, and stops advancing, with less intricate fingers of almost of the same length. As time evolves, the flow becomes more unstable as $U(t)$ switches to injection, enhancing convective forces that destabilize the flow as seen at $t = 1200$. In addition, it is observed that under a larger $\Delta G$, tip splitting of the fingers is observed as a result of steeper concentration gradients at the front which is due to the stretching associated with stronger cross flow. It should be noted that after the fingers split, they become narrower and longer as they become more stretched.
Figure 4.18: Variation with time of the interfacial length for different periods $T$ and for $R = 1, \Gamma = -2$, (a) $\Delta G = 2$ and (b) $\Delta G = 4$. 

(a) $\Delta G = 2$
In the following, we examine displacements that involve injection and short soaking periods with an initially increasing velocity ($\Gamma = 1$).

The temporal evolution of the $IL$ for different periods is presented in figure 4.20 for two values of $\Delta G$; ($\Delta G = 2$ and 4) and $R = 1$. Corresponding velocity profiles have already been depicted in figure 4.7. For each $\Delta G$, it is observed that a larger $T$ leads to smaller values of $IL$; hence reduced flow instability. It is also found that the diffusive regime is considerably shorter in time compared to the negative amplitude displacements ($\Gamma = -2$). Moreover, the duration of the diffusive regime is almost the same for different periods. It is also observed that during both injection and soaking, a larger $\Delta G$ leads to larger values of $IL$. Figure 4.21 depicting the concentration fields for the period $T = 600$ confirms that a larger $\Delta G$ results in a more unstable displacement. It is also clear that as the flow evolves, less number of fingers is observed; however, these fingers are more developed and longer as a result of the oscillatory regime of $U(t)$. 

Figure 4.19: Temporal evolution of $C_b$ for $T = 600$, $\Gamma = -2$, $R = 1$ and $\Delta G = 2$, (b) $\Delta G = 4$. 

(b) $\Delta G = 4$
Figure 4.20: Variation with time of the interfacial length for different periods and for $\Gamma = 1$, $R = 1$ and (a) $\Delta G = 2$, (b) $\Delta G = 4$. 
Figure 4.21: Temporal evolution of concentration contours for $T = 600$ and $\Gamma = 1$, $R = 1$, and $\Delta G = 2$, (b) $\Delta G = 4$.

Figure 4.22 depicts the effects of the period and mobility ratio on the variation of $IL$ with time for $\Delta G = 3$ and $\Gamma = 1$. It is observed that a larger period ultimately leads to a less unstable displacement with a reduced number of fingers and a more diffused front (See concentration fields). It should however be noted that fingers’ advancement is virtually unaffected by the period. It is also found that a larger $R$ leads to a sharper increase (decrease) of $IL$ during injection (soaking) periods. This sharper decrease is attributed to the stabilizing effects of viscous forces on buoyantly unstable displacements during soaking periods. Furthermore, a larger $R$ results in a shorter diffusive regime ultimately leading to an overall stronger instability.
The effects of the amplitude and period on the development of the instabilities are depicted in figure 4.23 for \( R = 1 \) and \( \Delta G = 3 \). It is observed that under a negative amplitude (\( \Gamma = -2 \)), the larger the period the more unstable the flow. In particular, larger periods lead to a stronger instability with longer more developed fingers and earlier breakthrough. On the other hand, under a positive amplitude (\( \Gamma = 2 \)), a larger period leads to a reduced flow instability with less fingers and a more diffused front. These qualitative trends are confirmed with the analysis of the IL\(_{1200}\) (figure 4.24) where a larger period results in larger interfacial length for the case of negative amplitude (\( \Gamma = -2 \)). Conversely, under a positive one (\( \Gamma = 2 \)), a larger period leads to smaller interfacial length. One must note that compared to the constant injection (\( \Gamma = 0 \)), \( \Gamma = -2 \) displacements result in stronger instabilities while \( \Gamma = 2 \) displacements tend to attenuate the instability. It is worth noting that longer periods result in a shorter breakthrough time (fingers (a) \( R = 1 \), (b) \( R = 2 \).
extend further downstream) for the negative amplitude ($\Gamma = -2$) but have virtually no effect on the breakthrough time in the case of the positive amplitude ($\Gamma = 2$).

Figure 4.23: Concentration fields for different periods and $R = 1, \Delta G = 3$, (a) $\Gamma = -2$, (b) $\Gamma = 2$. 
Figure 4.24: $IL_{1200}$ versus density difference at $R = 1$ for different $T$. Black solid curves are for negative amplitude ($\Gamma = -2$), red curves are for positive amplitude ($\Gamma = 2$), and blue curve is for the constant injection ($\Gamma = 0$).

The effectiveness of the time-dependent injection scheme in attenuating or enhancing the instability relative to the constant injection scheme is now analyzed. Figure 4.25 depicts the variation of the relative change of the interfacial length with the period for different values of $\Delta G$ and $R = 1$. For consistency, the interfacial length of the constant injection flow is also determined at $t = 1200$.

$$IL_{relative} = \frac{IL_{Time-dependant} - IL_{Constant~injection}}{IL_{Constant~injection}} \times 100 \quad (4.36)$$

It is clear that the interfacial length exhibits the same qualitative trends as those of $\sigma_{ave}$ reported earlier in the LSA section (see figure 4.9). This good qualitative agreement between the NLS and LSA results indicates that $\sigma_{ave}$ proved efficient in characterizing and predicting the instability. It is found that in the case of a negative amplitude; $\Gamma = -2$, the strongest attenuating effect is observed at the periods $T \approx 500$ and $T \approx 800$, while the strongest enhancing effect is observed at $T = 1200$. These significant effects are well illustrated by the concentration fields at $t = 1200$ shown in figure 4.26 in the case $\Gamma = -2$ and for $\Delta G = 1, R = 1$. These qualitative trends
confirm the $IL_{relative}$ analysis (figure 4.25(a)) where the displacement with a period $T = 800$ attenuates the instability, while that with $T = 1200$ leads to the strongest increase in the instability. This can be attributed to the fact that the injection scheme; $T = 800$, results in shorter overall periods of injection, thus a less cumulative volume of injected fluid into the porous medium; hence a suppressed instability with much less developed finger structures. The time-dependent case with $T = 1200$ results in longer periods of injection thus enhancing convection forces, which strongly destabilise the flow. One must note that even though the same cumulative volume of fluid is injected for periods $T = 200, 400, 600,$ and $1200$, the structures of the fingers are different. On the other hand, opposite trends are observed in the case of a positive amplitude; $\Gamma = 1$, where the strongest *attenuating* effect is observed at $T = 1200$ and the strongest *enhancing* effect is observed at $T \approx 800$ (Figure 4.27). It is clear that a large enough value of the period ($T = 1200$) tends to attenuate the instability the most compared to the constant injection case (see figure 4.16) and other time-dependent scenarios. In particular, for $T = 1200$, the stabilizing effect of diffusion with time is more pronounced starting at $t = 600$; when $U(t)$ starts decreasing. This leads to a more diffused front with a reduced number of fingers due to diffusion that smooths out the viscosity profile at $t = 1200$. Finally, it should be mentioned that larger values of the extrema were observed for larger $R$ and larger absolute values of the amplitude $\Gamma$. However, for brevity, these results are not presented.
Figure 4.25: Variation of the relative change of the interfacial length with period for $R = 1$, different density differences and for (a) $\Gamma = -2$, (b) $\Gamma = 1$.

Figure 4.26: Concentration fields for $\Delta G = 1$, $R = 1$ and three periods: $\Gamma = -2$. 

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4.5 Conclusion

This study focuses on investigating and analyzing the effects of time-dependent, miscible displacements on the Rayleigh-Taylor instability in porous media. Both linear stability analysis (LSA) and non-linear simulations (NLS) were used to determine the effects of two parameters characterizing sinusoidal time-dependent displacements, namely, the period $T$ and the amplitude $\Gamma$, as well as the viscosity $R$ and density ratios $\Delta G$ of the fluids.

In order to determine instability characteristics, a new approach to extract the maximum growth rate has been developed. The resulting findings of the LSA revealed that that the smaller the period, the larger the growth rate but the stronger the subsequent decay over time. It is also found that for all values of the period and amplitude, the larger the $\Delta G$, the larger the growth rate but also the stronger the decay over time. It was observed that during extraction or soaking periods, larger $R$ result in smaller (negative) growth rates while during injection the larger the $R$, the larger the growth rates. One must note that time-averaged growth rates over the maximum of all considered
periods ($\sigma_{ave}$), offer the best characterization of the instability and result in trends that are in better qualitative agreements with those observed in the non-linear simulations.

The results of NLS showed that for $\Gamma < -1$, larger periods lead to stronger instabilities with longer more developed fingers. On the other hand, for $\Gamma > 1$ it is found that a larger period leads to a less unstable displacement with a reduced number of fingers and a more diffused front. These qualitative trends are confirmed with the analysis of the interfacial length ($IL$), where a larger period results in larger $IL$ for $\Gamma < -1$. and smaller $IL$ for $\Gamma > 1$. It is also noted that longer periods result in a shorter breakthrough time for $\Gamma < -1$ but have virtually no effect on the breakthrough time in the case of $\Gamma > 1$. For displacements with $\Gamma = 1$, it is revealed that, similar to the case $\Gamma > 1$, a larger period ultimately results in a reduced flow instability. It was however found that in general, displacements with unit amplitude are more unstable than their counterpart with $\Gamma > 1$.

This study showed conclusively that it is possible to control the Rayleigh-Taylor instability through a proper choice of the two parameters characterizing the time-dependent flow, i.e. the period $T$ and the amplitude $\Gamma$. In particular, for a fixed overall time of the process, initiating the flow by an extraction versus an injection plays a critical role in the flow development and the choice of the period $T$ determines the nature of the finger structures as well as other important flow characteristics such as the breakthrough time.
Chapter 5

Dynamics of Buoyancy Driven Miscible Iso-viscous Flows in Heterogeneous Layered Porous Media

Abstract

1 Buoyancy-driven instabilities in horizontally layered heterogeneous porous media are investigated using numerical simulations. The analysis is conducted for two different permeability distributions, where the permeability attains its maximum (minimum) at the initial interface. The effects of the frequency of layers (q) and variance of the permeability distribution (s) under different scenarios of density mismatches, were analyzed and characterized both qualitatively and quantitatively. Results revealed that heterogeneity induces undulated more diffuse finger structures compared to the homogeneous case. In cases where the heterogeneity at the initial interface is maximum, it is found that the larger the q, the less unstable the flow. It is shown that the onset time of the instability increases with increasing number of layers and decreases with increasing heterogeneity variance. Moreover, it is found that flow mixing increases (decreases) with increasing heterogeneity variance before (after) a critical flow time. The trends observed are however reversed in the case of shifted permeability heterogeneity where the smallest permeability is at the initial interface. Interestingly, it was found that for the shifted permeability distribution, an unstable flow in a homogeneous medium can be fully stabilized when a small number of layers are used in the heterogeneous case.

1 This chapter is the exact reproduction of the following manuscript submitted for publication: Y. Elgahawy and J. Azaiez, "Dynamics of Buoyancy Driven Miscible Iso-viscous Flows in Heterogeneous Layered Porous Media"
5.1 Introduction

Fingering instabilities play a significant role in understanding and optimizing displacement processes encountered in serval industrial and environmental applications such as enhanced oil recovery [3], CO₂ sequestration [4], and chromatographic separation [5]. The instability may develop as a result of differences in the fluids’ viscosities and is known as viscous fingering (VF) [1] or can be triggered by a mismatch between the fluids’ densities, in which case it is referred to as the Rayleigh–Taylor instability [2]. In addition to the physical properties of the fluids namely the viscosity, density and diffusivity, which can be affected by different mechanisms such as heat and reactions, these instabilities depend on the nature of the porous medium, notably its degree of heterogeneity. This instability is favorable in some applications such as CO₂ sequestration in deep saline aquifers in which the buoyancy-driven instability results in convective mixing [6]. Consequently, enhancing the rate of CO₂ dissolution into brine and reducing the risk of CO₂ leakage into the environment. Conversely, fingering is unfavorable in the oil industry as it results in an ineffective sweeping action that will bypass substantial amounts of recoverable oil [7].

Numerous studies have been conducted to analyze the fingering instability due to its complexity and significance. In the case of miscible displacements, the effects of different parameters including the viscosity ratio [8], heat transfer [14, 15], chemical reaction [12, 13], gravity [9-11], diffusivity [16], anisotropic and velocity dependent dispersion [17, 18], inertial effects [19-21], and additives such as nanoparticles [22, 23] have been extensively analyzed. Detailed literature reviews on pre-1990s relevant studies have been presented by Homsy [30] and McCloud and Maher [31]. The vast majority of existing studies have focused on the instability in homogeneous porous media. However, underground formations and naturally occurring porous media are in fact
heterogeneous as the permeability varies from one location to another due to variations in the microstructure of the medium.

A limited number of studies have investigated the effects of permeability heterogeneity on miscible VF. In particular, Tan and Homsy [44] studied the effects of random permeability variations with Gaussian distributions. They observed that the growth rate of the mixing zone increases with the variance of the permeability but exhibits a non-monotonic behaviour with the variation of the correlation length of the heterogeneity. They explained this as resonance which occurs when the length scale of the heterogeneity and that of viscous fingering are commensurate. The authors also reported particular paths taken by the fingers that correspond to regions with locally high permeability. In a later study adopting a spatially periodic permeability that varies perpendicular to the main flow direction, De Wit and Homsy [45, 46] found that the wave number at which resonance was observed depends on both the Péclet number ($Pe$) and log mobility ratio ($R$). Interactions between viscously driven and heterogeneity induced instabilities were well depicted in the work of Sajjadi and Azaiez [47] examining periodically layered media with a heterogeneity that varies perpendicular to the main flow direction. The authors identified four regimes that govern the flow displacement: initial diffusion, channeling, lateral dispersion, and VF. They found that the breakthrough time (BT) varies non-monotonically with the number of layers ($q$) with a minimum BT occurring at small values of $q$ and a maximum occurring at intermediate values of $q$. Large values of $q$ on the other hand led to the same BT as that of the homogeneous porous medium. A year later, Norouzi and Shoghi [48] studied the effects of both anisotropic permeability and dispersion on the onset and propagation of VF. They found that the larger the ratio of longitudinal to transverse anisotropic permeability ($\alpha_K$) and dispersion ($\alpha_D$), the less and more unstable the flow becomes, respectively. The authors explained that for $\alpha_K \geq 1$, it
became harder for the initial perturbations to grow into fingers which induces transport in the transverse direction. The recent study by Nijjer et al. [49], who focused on large-scale permeability variations that are perpendicular to the flow direction found that the flow always evolves through three regimes. At early times, the concentration evolves diffusively, independent of both the viscosity ratio and permeability distribution. While at intermediate times, the interplay between the viscosity and permeability variations results in different dynamics including fingering and channeling. At late times, the flow is dominated by shear-enhanced (Taylor) dispersion, which becomes independent of the viscosity ratio and depends only on the permeability distribution.

A small number of studies have investigated the effects of permeability heterogeneity on the Rayleigh-Taylor instability. Schincariol et al. [83] adopted a random permeability field and found that longer correlation lengths and larger variance of log-normal permeability distribution promote stability. Prasad and Simmons [84] found that an increase in the standard deviation of the permeability field causes an increase in the degree of instability at earlier times but promotes stability at later times. Later in 2012, Ranganathan et al. [85] modeled density-driven natural convection during geological CO₂ storage in heterogeneous formations. They represented the heterogeneity by random permeability fields. The authors reported that CO₂ concentration profiles show different flow patterns of convective mixing such as channeling, gravity fingering, and dispersion based on the heterogeneity of the aquifer. In addition, they found that the variation of the mixing length with time shows three different regimes: diffusion, convection, and second diffusion. Daniel et al. [86] adopted a permeability that varies periodically across the thickness of an aquifer to examine the stability of gravitationally unstable, transient boundary layers in heterogeneous saline aquifers. First, the authors investigated the effects of permeability variance and phase on the onset of instability using linear stability analysis. Then, using direct numerical
simulations, they examined the effects of variance and wave length of the permeability oscillation on the onset time of convection. They observed that the instability is attenuated with an increase in the permeability variance when the boundary layer thickness is large relative to the permeability wavelength. Conversely, when the boundary layer thickness is smaller than the permeability wavelength, it was found that the behavior of instability as a function of the variance depends on the permeability phase. In a later study, Ghorbani et al. [87] studied the effect of permeability heterogeneity on the stability of gravitationally unstable, transient, diffusive boundary layers in porous media. The permeability was assumed to vary periodically in the horizontal plane normal to the direction of gravity. The authors reported that thick layers enhance the instability compared to thin layers when permeability heterogeneity is increased. Contrarily, it was found that for thin layers, the instability is reduced progressively with increasing heterogeneity to the extent that it becomes less unstable than the corresponding homogeneous case.

To the best of our knowledge, no research has been conducted to study the non-linear effects of layered permeability heterogeneity on the dynamics of the Rayleigh-Taylor instability in miscible displacements. The aim of the present study is therefore to investigate and understand the effects of layered permeability heterogeneity on the growth and development of such gravity-driven instabilities developing in porous media. The effects of a periodic permeability distribution on the non-linear development of the instability will be analyzed qualitatively through concentration fields and quantitively through the mixing length and quality, breakthrough time, and center of mass. The role of the frequency of layers \( q \) and variance of the permeability distribution \( s \) under different scenarios of density mismatches will be investigated. In addition, the effects of a phase shift of the permeability distribution will be also examined. Furthermore, the
effectiveness of the periodic permeability distribution in attenuating or enhancing the instability relative to the homogeneous case will be analyzed.

5.2 Mathematical Model

5.2.1 Physical problem

In this study, we examine the dynamics of buoyancy-driven instabilities under a gravitational field \( \bar{g} = (g, 0) \) in a two-dimensional porous medium of porosity \( \Phi \). The medium of length \( L \) and width \( W \) is heterogeneous with the permeability \( K \) assumed to vary in the vertical (x) direction. Fluid A of concentration \( C_a \) and viscosity \( \mu_a \) is injected from the top boundary (inlet) to displace fluid B of concentration \( C_b \) and viscosity \( \mu_b \) as shown in figure 5.1. The two fluids are incompressible, Newtonian, and fully miscible with isotropic constant diffusion \( D \).

![Figure 5.1: Schematic of the flow.](image)

5.2.2 Governing equations

The flow is governed by the equations for conservation of mass (continuity equation), conservation of momentum (Darcy’s law) and transport of species (convection-diffusion equation). Following
Elgahawy and Azaiez [88], the governing equations are formulated in dimensionless form using diffusing scaling, where $C_{b0}$, $\rho_{ref}$, $D\bar{\phi}/U_{ch}$ and $D\bar{\phi}^2/U_{ch}^2$ are used as the reference concentration, density, length and time, respectively. The permeability is scaled by the average permeability of the medium $K_1$. Rather than using the injection velocity, the adopted characteristic velocity is, $U_{ch} = \rho_{ref} g K_1/\mu$. This general choice allows to include scenarios where the flow develops as a result of density contrast without an injection velocity. The viscosity is scaled with $\mu_{ch} = \mu_{less}$ corresponding to the viscosity of the less viscous fluid [82].

The dimensionless equations expressed in a Lagrangian reference frame moving with a dimensionless injection velocity $U(t)$ are:

\[
\nabla \cdot \bar{u} = 0 \quad (5.1)
\]

\[
\bar{\nabla} P = -\frac{\mu(C_b)}{K(x)} [\bar{u} + U(t) \bar{l}] + \rho(C_b)\bar{r} \quad (5.2)
\]

\[
\frac{\partial C_b}{\partial t} + \bar{u} \cdot \bar{\nabla} C_b = \nabla^2 C_b \quad (5.3)
\]

Two dimensionless groups, the Péclet number $Pe = \frac{U_{ch} L}{D\bar{\phi}}$ and cell aspect ratio $A_r = \frac{L}{W}$, are used to specify the domain size such that it is $\left(\frac{-Pe}{2}, \frac{Pe}{2}\right)$ in the $x$-direction and $\left(\frac{-Pe}{2A_r}, \frac{Pe}{2A_r}\right)$ in the $y$-direction.

Following earlier studies [8], an exponential form for the dependence of the viscosity on the concentration is adopted:

\[
\mu(C_b) = \begin{cases} 
\exp(R_b C_b), & \text{if } \mu_a < \mu_b \\
\exp(R_a (1 - C_b)), & \text{if } \mu_a > \mu_b
\end{cases} \quad (5.4)
\]

where $R_b$ and $R_a$ are the mobility ratios defined as:

\[
R_b = -R_a = \ln \left( \frac{\mu_b}{\mu_a} \right) \quad (5.5)
\]
while a linear dependence of the density on the concentration is adopted:

\[ \rho(C_b) = G_a C_a + G_b C_b, \quad (5.6) \]

where \( G_a \) and \( G_b \) are fluid A and B density expansion coefficients, respectively:

\[ G_a = \frac{\partial \rho}{\partial C_a} \quad \text{and} \quad G_b = \frac{\partial \rho}{\partial C_b}. \quad (5.7) \]

### 5.2.3 Vorticity and stream function formulation

The problem is formulated in terms of the vorticity \( \omega \) and stream function \( \psi \).

\[
\begin{align*}
\mathbf{u} &= \frac{\partial \psi}{\partial y}, \\
\mathbf{v} &= -\frac{\partial \psi}{\partial x}, \\
\omega &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\nabla^2 \psi
\end{align*}
\]

(5.8)

Eliminating the pressure term by taking the curl of Darcy’s law leads to the following equation:

\[
\begin{align*}
\omega &= R_b \left[ \frac{\partial \psi}{\partial x} \frac{\partial C_b}{\partial x} + \frac{\partial \psi}{\partial y} \frac{\partial C_b}{\partial y} + U(t) \frac{\partial C_b}{\partial y} \right] + \frac{K}{\mu} \left[ \frac{\partial C_b}{\partial y} \Delta G \right] - \frac{\partial \psi}{\partial x} \frac{\partial (\ln(K))}{\partial x}
\end{align*}
\]

(5.9)

In the above equation \( \Delta G = G_a - G_b \). \( \Delta G > 0 \) \((< 0)\) represents a buoyantly unstable (stable) displacement where a heavier (lighter) fluid (A) is on top of a lighter (heavier) one (B).

The heterogeneity is characterized using the natural log of the permeability,

\[
F = \ln(K(x)) = s \cos \left( \frac{2 \pi x q}{P_e} \right), \quad \frac{\partial F}{\partial x} = -s \frac{q}{P_e} \sin \left( \frac{2 \pi x q}{P_e} \right)
\]

(5.10)

Where \( s \) is the range of variation of \( F \), \( q \) is the frequency of layers in the x-vertical direction. One of the objectives of this study is to analyze the effects of these two parameters on the instability.

### 5.2.4 Initial and boundary conditions

In order to complete the mathematical formulation, appropriate initial and boundary conditions need to be specified. The initial conditions for the concentration and velocity are:

\[ C_b = (0 \text{ for } x < 0, 1 \text{ for } x > 0), \quad \mathbf{u} = (0,0). \]

(5.11)

Along the streamwise (x-) direction, the boundary conditions are:
\[ (u, v, C_b) \left( \frac{-P_e}{2}, y, t \right) = (0, 0, 0), \quad (u, v, C_b) \left( \frac{P_e}{2}, y, t \right) = (0, 0, 1), \]  

while in the transverse (y-) direction they are,

\[ (u, v, C_b) \left( x, \frac{-P_e}{2A_r}, t \right) = (u, v, C_b) \left( x, \frac{P_e}{2A_r}, t \right). \]  

(5.13)

Periodic boundary conditions are implemented in the transverse direction (Eq. (5.13)). In the subsequent analyses, the concentration is split as follows:

\[ C_b(x, y, t) = \bar{c}_b(x, t) + c'_b(x, y, t). \]  

(5.14)

The base-state concentration \( \bar{c}_b \) is solution of the problem with zero velocity, and satisfies the streamwise boundary conditions. Consequently, \( c'_b \) is zero at both top and bottom ends of the domain. This decomposition allows solving the problem in terms of \( c'_b \) using spectral methods with periodic boundary conditions in the streamwise and transverse directions.

\[ \frac{\partial c'_b}{\partial t} = - \left[ \frac{\partial \psi'}{\partial y} \left( \frac{\partial \bar{c}_b}{\partial x} + \frac{\partial c'_b}{\partial x} \right) - \frac{\partial \psi'}{\partial x} \frac{\partial c'_b}{\partial y} \right] + \frac{\partial^2 c'_b}{\partial x^2} + \frac{\partial^2 c'_b}{\partial y^2}, \]  

(5.15)

\[ \omega = R_b \left[ \frac{\partial \psi}{\partial x} \left( \frac{\partial \bar{c}_b}{\partial x} + \frac{\partial c'_b}{\partial x} \right) + \frac{\partial \psi}{\partial y} \frac{\partial c'_b}{\partial y} + U(t) \frac{\partial c'_b}{\partial y} \right] + \frac{1}{\mu} \left[ \frac{\partial \bar{c}_b}{\partial y} \Delta G \right] - \frac{\partial \psi}{\partial x} \frac{\partial (\ln(k))}{\partial x}, \]  

(5.16)

\[ c'_b \left( \frac{-P_e}{2}, y, t \right) = c'_b \left( \frac{P_e}{2}, y, t \right) = 0; \quad c'_b \left( x, \frac{-P_e}{2A_r}, t \right) = c'_b \left( x, \frac{P_e}{2A_r}, t \right). \]  

(5.17)

### 5.3 Numerical Technique

Numerical solutions of Eq. (5.15) and (5.16) were obtained to capture the non-linear effects of fingers development. The numerical code convergence was ascertained by varying the spatial and temporal step sizes. It was also validated by comparison with the results of a number of earlier studies. In particular, its predictions in the case of vertical miscible displacements under a constant injection velocity in a homogeneous porous media were compared with corresponding ones published in the literature.
As an illustration, the concentration contours for a viscously and buoyantly unstable displacement with the same flow parameters reported by Pramanik et al. [82] is shown in figure 5.2. This figure shows a good qualitative agreement with Pramanik et al. [82] results; both having similar number of fingers and structures.

5.4 Results and Discussion

Even though the above problem formulation is general and can be used to analyze the dynamics for any time-dependent injection flow, the subsequent analysis will be limited to purely buoyancy driven instabilities with zero injection velocity; \( U(t) = R_b = 0 \). The fingering instability is qualitatively characterized using concentration fields of fluid B. Furthermore, a quantitative analysis is conducted to assess the degree of instability and capture the development of the fingers. This was achieved using the mixing length \( (ML) \), mixing quality \( (MQ) \), center of mass \( (x_m) \), and breakthrough time \( (T_{bt}) \). \( ML \) is defined as the ratio of the length of the channel in which the
transversely averaged concentration is in the range of $0.01 < C_{b,ave} < 0.99$ to the length of the whole channel, where

$$C_{b,ave}(x, t) = \frac{A_r}{Pe} \int_0^{Pe} C_b(x, y, t) \, dy.$$ (5.18)

$MQ$ which is based on the variance of the concentration field, quantifies the degree of mixing and is defined by,

$$MQ(t) = 1 - \frac{\sigma^2}{\sigma^2_{max}}$$ (5.19)

Where $\sigma^2 = <C_b^2> - <C_b>^2$ and $<>$ denotes spatial averaging over the domain. $\sigma^2_{max}$ is the variance of a perfectly segregated state which occurs at the initial time. Note that, $\sigma^2 = 0$ leading to $MQ = 1$ corresponds to the perfectly mixed case.

The center of mass, which is determined through the first moment of the transversely averaged concentration, provides information on the location of the front and can be used as a criterion for determining the time evolution of the vertical location of the region of mixing of the two species.

$$x_m(t) = \frac{1}{Pe} \int_{-Pe/2}^{Pe/2} xC_b(x, t) \, dx.$$ (5.20)

The breakthrough time is another effective tool to quantify the instability and it is commonly used in field applications in the oil industry. It is defined as the time when the leading edge of the mixing zone reaches the downstream end of the porous medium.

In what follows we investigate the effects of the permeability heterogeneity on the gravitational instability by analyzing the concentration fields. The heterogeneous porous medium is characterized by the frequency of layers ($q$) and variance of the permeability distribution ($s$). First, we focus on the effects of the frequency of layers on the flow structures and mixing. Then, the analysis is expanded to examine the effects of other parameters. Unless specified otherwise, the
following parameters are fixed as $Pe = 2000$, $\Delta G = 1$, $Ar = 2$, and $s = 1$. It should be noted that the time shown in the following results is the dimensionless time ($t$).

5.4.1 Non-shifted permeability distribution

5.4.1.1 Effects of the number of layers

Figure 5.3 depicts concentration contours for $\Delta G = 1$ at $t = 1200$ in the homogeneous case as well as the non-homogeneous one with numbers of layers $q$ varying from 1 to 12. A plot of the permeability is superposed in white solid line. It is clear that regardless of the number of layers, the non-homogeneous case is more unstable than its homogeneous counterpart during this early stage of the flow. Furthermore, as the number of layers is increased, the instability tends to develop slower and becomes less complex. As a result, the smaller the number of layers the further the fingers extend in the upstream and downstream directions. It is noted that the extent of fingers development correlates with the spatial period of the permeability that decreases inversely with $q$. This observed trend can be explained by the fact that the region of the porous medium located at the initial interface offers an easy path for the flow and in turn for the development of the instability. Therefore, the larger this region of high permeability, the stronger the initial instability. In order to track the evolution of the instability and its later development, time sequences of the concentration for the homogeneous case and non-homogeneous one with $q = 8$ are presented in figure 5.4(a-b).
In the early stages of the flow, the instability is stronger in the non-homogeneous case (see figure 5.3 at $t = 1200$). At later times however, fingers in the homogeneous medium tend to develop and
extend further in the domain compared to those in the non-homogeneous medium (figure 5.4). Unlike the homogeneous case where the fingers extend mainly in the vertical direction with little or no interactions, those developing in the non-homogeneous medium tend to interact with each other in the early stages of the flow and to widen in the transverse direction. As a result, they extend less in the direction of the main flow and this, as shall be seen later, will result in longer breakthrough times. As the front exits the first high permeability layer, it exhibits a succession of constriction and expansion structures that develop in both the upstream and downstream directions and that tend to be more diffusive as time evolves. These structures correlate with the regions of high and low permeability leading to undulated fingers of fluid A falling down and fluid B rising up.

(a) Homogeneous case

\[ t = 1800 \quad t = 2200 \quad t = 3000 \quad t = 4000 \]
To analyze the dependence of the previous results on the density difference, concentration fields at $t = 1200$, for $\Delta G = 2$ and 3, and different values of $q$ are shown in figure 5.5. It can be seen that a larger $\Delta G$ leads to a stronger instability with more intricate finger structures extending further in both upstream and downstream directions. A similar effect of the number of layers on the instability is observed for $\Delta G = 2$ in which a smaller $q$ results in a stronger instability with fingers extending further in the upstream and downstream directions. Successive constriction and expansion structures are also observed for both $\Delta G = 2$ and 3. It is worth noting that the size of the undulations, i.e. the length (x-direction) of the constrictions and width (y-direction) of the expansions, seems to be proportional to the inverse of the number of layers. Furthermore, it is found that for the largest $\Delta G$, the structures tend to be longer with limited inter-finger interactions like in the homogeneous case, except for their undulated surface. This can be attributed to the fact that for large density differences, the fluid residence time in the high- and low-permeability layers
is shorter, reducing the development of local finger interactions. In this case the flow is able to advance faster and hence exhibits a larger number of undulations.

(a) $\Delta G = 2$

(b) $\Delta G = 3$

Figure 5.5: Contours of $C_3$ at $t = 1200$ with plots of the permeability in solid white lines for the homogeneous case and the non-homogeneous one with different values of $q$, and for (a) $\Delta G = 2$ and (b) $\Delta G = 3$. 

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To better understand the dynamics of the successive constriction and expansion structures, a plot of the average concentration ($C_{b,ave}$) superposed on the concentration contours of $C_b$ is depicted in figure 5.6(a) for $q = 12$ and $\Delta G = 3$. It is clear that local maxima of $C_{b,ave}$ occur at layers where expansion structures of fluid B are observed (see blue contour). Further analysis of $C_{b,ave}$, utilizing plots of the derivative of the permeability ($K'(x)$), shows that qualitatively, the variation of $C_{b,ave}$ follows the same trend as $K'(x)$ in the flow region where the instability develops. Moreover, it is observed that the local extrema of $C_{b,ave}$ occur at the same locations (layers) as the local extrema of $K'(x)$ (see figure 5.6(b)).

![Figure 5.6](image)

Figure 5.6: Contours of $C_b$ with plots of the average concentration in solid white lines, (b) corresponding plot of the derivative of the permeability ($q = 12$, $\Delta G = 3$, $t = 1200$).
In what follows we conduct a quantitative analysis to further characterize the evolution of the fingering instability. The previous qualitative results of figure 5.3 are confirmed with the analysis of the $ML$, where a larger number of layers results in a smaller $ML$ as demonstrated in figure 5.7. It is also observed that regardless of the number of layers, the $ML$ for the non-homogeneous case is larger than its homogeneous counterpart during the early times. However, as the flow evolves, the rate of increase of $ML$ with time for the homogeneous case becomes larger than those of the non-homogeneous cases especially those with large values of $q$. As a result, at later times, the $ML$ for the homogeneous medium exceeds those of the non-homogeneous ones with large $q$.

![Graph showing variation with time of the mixing length for different number of layers (q).](image)

**Figure 5.7:** Variation with time of the mixing length for different number of layers ($q$).

Figure 5.8 allows the identification of the different dominant flow regimes as well as the onset time of convection. In the non-homogeneous cases, the flow evolves through three different regimes; an initial diffusive regime in which $ML$ grows almost as $t^{0.5}$, followed by a convective
regime where $ML$ grows almost as $t^{1.0}$ and finally the flow goes through a diffusion dominant regime with $ML$ growing almost as $t^{0.5}$. On the other hand, the flow in the homogeneous medium evolves only through two regimes, a diffusion dominant regime followed by a convection dominant one.

The onset time of convection as a function of the number of layers is depicted in figure 5.9 (solid line for $s = 1$). It is observed that for small values of $q$, this time is almost constant. However, for larger value of $q$, the onset time of convection grows almost linearly with increasing $q$, indicating that instabilities are slower to develop as the number of layers in the heterogeneous medium increases.

Figure 5.8: Dominant flow regimes for different number of layers ($q$).
Figure 5.9: Effect of the number of layers ($q$) and variance ($s$) on the onset of convection.

The center of mass analysis depicted in figure 5.10 shows that $x_m$ varies rapidly with time for small numbers of layers but the rate of change with time decreases rapidly for large $q$. These trends are most noticeable at late times, and are less important in the early stages of the flow. For the small $q$, the smaller the number of layers, the stronger the vertical downward shift, indicating again a stronger instability where fingers extend further in the vertical direction. It should be noted that the larger the number of layers the smaller the center of mass, but this dependence on the number of layers becomes weak for large values of $q$. These results confirm the previously reported qualitative (contours) and quantitative ($ML$) trends.
The effect of the number of layers on the breakthrough time is depicted in figure 5.11. For the purpose of notations, \( q = 0 \) refers to the homogeneous case. It is found that the breakthrough time varies non-monotonically with the increase in \( q \). The earliest breakthrough occurs at the smallest value of \( q \), as predicted by the center of mass results (see figure 5.10), thus leading to the strongest instability. The breakthrough time keeps increasing reaching a maximum at \( q = 4 \) then it slightly decreases and reaches a plateau. It should be noted that for all considered values of \( \Delta G \), the heterogeneous medium with one layer leads to the earliest breakthrough time, even shorter than that of the homogeneous medium. However, all heterogeneous media with more than one layer have longer breakthrough times than that of the homogeneous medium. The breakthrough time results follow the same trend for different values of \( \Delta G \) and seem self-similar. This indicates that it may be possible to present all results by a single master curve.
5.4.1.2 Effects of the variance of the permeability

The effects of the variance of the permeability distribution \((s)\) are now analyzed, and illustrated for two values; \(q=1\) and 8. The results are presented for \(s = 1.2, 1.6\) and 2.0 and should be considered in conjunction with the corresponding previously discussed ones for \(s = 1\).

Figure 5.12 depicts the concentration fields at \(t = 1200\). In the case of \(q=1\), there is a tendency for the fingers to extend further for larger variances. It is also found that a larger variance results in more diffuse fingers. This tendency for more diffuse structures is also observed in the case \(q=8\), but it was noted that the extent of development of fingers is virtually unaffected by changes in the variance.
Figure 5.12: Contours of $C_p$ with plots of the permeability in solid white lines at $t = 1200$ for different values of variance ($s$) and for (a) $q = 1$, and (b) $q = 8$. 
The picture is however reversed at later times where it is found that the spread of the fingers in the direction of the flow is virtually unaffected by the variance in the case $q=1$, while it is suppressed with increasing variance for the case $q=8$ as seen in figure 5.13. It is therefore expected that the breakthrough time will increase with increasing $s$ for large $q$ and will be virtually unaffected by the variance for small $q$.

(a) $q = 1$

\begin{itemize}
  \item $s = 1.2$
  \item $s = 1.6$
  \item $s = 2.0$
\end{itemize}
To assess the role of variance of the permeability distribution on the degree of instability, further quantitative analysis is provided in figure 5.14 for different values of \( q \) at \( \Delta G = 1 \). The MQ increases monotonically with time, but exhibits two time regimes in which its dependence on the variance is reversed. In the first regime, the larger the intensity of the heterogeneity, the larger the MQ. This trend is reversed at a critical time, after which larger values of the variance lead to smaller MQ. The time at which the trends are reversed decreases with increasing number of layers. At early time when the front has not progressed much, the permeability tends to enhance the instability and as a result a larger intensity of permeability leads naturally to stronger mixing. However, as the flow progresses, the displacing fluid enters regions of adverse permeability where larger variances result in stronger resistance to the flow. This and the fact that stronger mixing for
large values of $s$ in the first regime attenuate the density gradient, lead to weaker instability and in turn less fluids mixing in the second regime.

Figure 5.14: Variation with time of the mixing quality for different values of variance ($s$) and for (a) $q = 1$, (b) $q = 4$, (c) $q = 8$, and (d) $q = 12$. 

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The time for the onset of convection decreases with increasing variance as can be seen in figure 5.9, indicating that the instability develops earlier for larger variances. It should be noted that for large enough values of $s$, the onset time of convection is virtually independent of the number of layers $q$. This reflects that a strong enough permeability will result in the development of the instability very early in time and when the front is within a narrow region that falls within the first layer of high permeability, at least for the cases considered where $q$ ranges from 1 to 12.

### 5.4.2 Shifted permeability distribution

The previous results and discussion focused on displacements in which the largest permeability is at the initial interface. We now turn our attention to displacements in which the permeability is shifted such that the lowest permeability is at the initial interface.

#### 5.4.2.1 Effects of the number of layers

Concentration contours for the shifted permeability heterogeneity are depicted in figure 5.15 for $\Delta G = 1$ at $t = 6000$ with numbers of layers varying from 1 to 12. It was found that as $q$ is increased, the instability develops faster and becomes stronger with more complex finger structures. These trends are to be contrasted with those of the non-shifted permeability where a larger $q$ attenuates the instability. Moreover, the larger the number of layers the further the fingers extend in the upstream and downstream directions. This can be attributed to the fact that the first layer of the porous medium located at the initial interface offers a resistive path to the flow and in turn to the development of the instability. Thus, the larger this region of low permeability (smaller $q$), the weaker the initial instability. It should be mentioned that similar qualitative results were observed for larger $\Delta G$, however for brevity, these results are not presented. It is also interesting to note that for the smaller values of $q$, the displacement is virtually stable even after a large time. This indicates that the heterogeneity can stabilize an otherwise unstable system. These qualitative
trends are confirmed with the quantitative analyses. This is illustrated in figure 5.16, where larger values of $q$ lead to a more unstable displacement with larger $MQ$. It is worth noting that the cases of smaller values of $q$ ($q = 1$ and 2) exhibit only one stable diffusive regime even for very large times ($t = 8000$) and thus the displacement is deemed to be stable.

Figure 5.15: Contours of $C_b$ with plots of the permeability in solid white lines at $t = 6000$ for the shifted permeability case for different values of $q$. 
5.4.2.2 Effects of the variance of the permeability

Figure 5.17 depicts the effects of $s$ for the shifted permeability heterogeneity case for different number of layers at $t = 7500$. It is found that larger variances tend to attenuate the instability with less developed finger structures. It is also observed that for all values of $s$, displacements with small values of $q$ are stable. The previous qualitative trends are confirmed by the quantitative analyses, as illustrated by plot of $MQ$ for different permeability variances and for two values of $q$: $q = 1$ and 12 (see figure 5.18). It is clear that flows with small $q$ exhibit only one regime; diffusive regime and these displacements are virtually stable regardless of the variance. For larger values of $q$ however, it is observed that the smaller the permeability variance, the larger the $MQ$. One must note that the permeability variance has stronger effects on the instability in the shifted permeability case compared to the non-shifted one. These stronger effects are in regard to the extent of fingers’ development and complexity of the finger structures (see figure 5.17).
Figure 5.17: Contours of $C_\theta$ with plots of the permeability in solid white lines at $t = 7500$ for the shifted permeability case at different values of variance ($s$) and for (a) $q = 1$, and (b) $q = 12$. 
5.5 Conclusion

In this study, we have analyzed the fingering instability in horizontally layered heterogeneous porous media. The analysis is conducted for two different permeability distributions, the first focuses on displacements in which the largest permeability is at the initial interface, while the second deals with distributions in which the permeability is shifted such that the lowest permeability is at the initial interface. The non-linear effects of layered permeability heterogeneity on the growth and development of the Rayleigh-Taylor instability were examined. In particular, the role of the frequency of layers and variance of the permeability distribution under different scenarios of density mismatches were analyzed using non-linear simulations (NLS).

The results of NLS revealed that heterogeneity induces undulated finger structures that are more diffuse compared to the homogeneous case. It was found that for permeability distribution with a maximum at the initial interface, larger number of layers leads to a less unstable displacement. Furthermore, the larger the variance, the less unstable the flow in the case of large number of
layers, and the more unstable for small number of layers. Results show that the onset time of the instability increases with increasing number of layers however it decreases with increasing heterogeneity variance. The flow was also characterized through the breakthrough time ($T_{bt}$) that was found to vary non-monotonically with the number of layers, reaching a minimum for the medium with one layer. It was also shown that flow mixing increases (decreases) with increasing heterogeneity variance before (after) a critical flow time.

However opposite trends are observed in the case of shifted permeability heterogeneity in which the smallest permeability is at the initial interface. It is interesting to note that for the shifted permeability distribution, an unstable flow in a homogeneous medium was fully stabilized when a small number of layers was used in the heterogeneous case.

The present study showed conclusively that layered permeability heterogeneity significantly affects the dynamics of the fingering instability in miscible displacements. One must also note that it is possible to stabilize or enhance the fingering instability through a proper choice of the two parameters characterizing the heterogeneous displacement, $i.e.$ the number of layers $q$ and the variance $s$. 
Chapter 6

Conclusion

In this study, buoyancy-driven instabilities developing in porous media have been thoroughly studied through the aid of linear stability analysis and non-linear simulations. The significance of the Rayleigh-Taylor problem in addition to the lack of studies on the time-dependent nature of the displacement and the heterogeneity of the medium were the main motives to conduct this study. Therefore, this research is conducted in two stages. In the first stage, the problem is analyzed under sinusoidal time-dependent flow displacements in homogeneous porous media. The effects of two parameters characterizing sinusoidal time-dependent displacements, namely the period $T$ and the amplitude $\Gamma$, as well as the density and viscosity ratios $\Delta G$ of the fluids have been examined. A new approach has been developed to extract the maximum growth rate in order to determine instability characteristics. The flow instability and dynamics of the sinusoidal time-dependent displacement are found to be considerably different than those of their constant injection counterpart. The LSA results show that smaller periods lead to larger growth rates and stronger subsequent decay over time. Moreover, it is found that for all values of the period and amplitude, larger $\Delta G$ result in larger growth rates and also stronger decay over time. While it is found that the larger the $R$, the smaller the growth rates during extraction or soaking periods, however during injection periods larger $R$ results in larger growth rates. It should be noted that the time-averaged growth rates over the maximum of all considered periods ($\sigma_{\text{ave}}$), offer the best characterization of the instability and result in trends that are in better qualitative agreements with those observed in the NLS. The NLS results revealed that for $\Gamma < -1$, the larger the period the stronger the instability.
with longer more developed fingers. Conversely, for displacements with $\Gamma > 1$ it is found that the larger the period the less unstable the displacement with a reduced number of fingers and a more diffused front. Similarly, for displacements with $\Gamma = 1$, a larger period ultimately leads to a reduced flow instability. However, displacements with unit amplitude are more unstable than their counterpart with $\Gamma > 1$. Finally, this study revealed that it is possible to control the buoyancy-driven instability through a proper choice of the cycle period and the velocity amplitude. Particularly, initiating the flow by an extraction as opposed to an injection plays an important role in the flow development and the choice of the period $T$ dictates the nature of the finger structures and determines important flow characteristics such as the breakthrough time.

In the second stage of this study, the RT instability is analyzed in heterogeneous porous media where layered permeability heterogeneity plays a major role in affecting the flow dynamics. The analysis is conducted for two different permeability distributions, the first dealing with displacements where the maximum permeability is at the initial interface, the second focusing on displacements in which the permeability is shifted with the minimum permeability located at the initial interface. Using numerical simulations, the effects of the frequency of layers and heterogeneity variance under various scenarios of density mismatches have been investigated. The results showed that heterogeneity induces undulated more diffuse finger structures compared to the homogeneous case. It is found that for heterogeneity with a maximum at the initial interface, the larger the number of layers the less unstable the flow. For the heterogeneous cases, regardless of the number of layers, it is identified that the flow always evolves through three different regimes; an initial diffusive regime where the $ML$ grows almost as $t^{0.5}$, followed by a convective regime in which $ML$ grows linearly with time, and finally the flow goes through a second diffusive regime. However, the flow in the homogeneous medium is characterized by two regimes only, a
diffusion dominant regime followed by a convection dominant regime. Further analysis revealed that the onset time of the instability increases with increasing number of layers and decreases with increasing heterogeneity variance. Additionally, it is found that flow mixing increases (decreases) with increasing heterogeneity variance before (after) a critical flow time. The trends observed are however reversed in the case of shifted permeability heterogeneity where the minimum permeability is at the initial interface. Finally, for the case of shifted permeability distribution, it is interesting to note that an unstable flow in a homogeneous medium can be fully stabilized when a small number of layers is used in the heterogeneous case.

The scope of the present study can be expanded for future work on the same topic. In the present study, we considered isothermal displacements in heterogeneous porous media. However, it would be interesting to analyze non-isothermal displacements in heterogeneous porous media due to its extensive industrial application. This might be a little challenging as the thermal properties of the medium are strongly dependent on the porosity and thus a correlation accounting for the variations of the porosity and permeability needs to be considered. Thus, several parameters that are assumed constant will need to be redefined. Moreover, this study can be extended to investigate the effects of a step-size permeability distribution that varies in the vertical direction as some oil reservoirs are naturally layered such that the variations of the permeability from one layer to the other is not periodic. Finally, this study can also be extended to analyze the effects of inertia on the stability of the RT instability or for displacements in heterogeneous porous media.
Bibliography


