Turbulent wake structure and dynamics for the thin flat plate normal to a uniform flow: a study of two dynamically stable solutions

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Turbulent wake structure and dynamics for the thin flat plate normal to a uniform flow: a study of two dynamically stable solutions

by

Eric Anthony Braun

A THESIS SUBMITTED TO THE FACULTY OF GRADUATE STUDIES IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE

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Abstract

This thesis presents a comparative experimental study of the differences in the structure and dynamics of two nominally two-dimensional turbulent wakes behind a thin flat plate placed normal to a uniform flow. The flows are differentiated by their end conditions: with and without end plates. Both cases are characterized by Karman-like vortex shedding with broadband low frequency unsteadiness. Both wakes evidence a low frequency flapping motion, associated with a flow normal oscillation of the shear layers, as well as a slowly drifting baseflow that is common to cylinder wakes. However, significant differences in the mean velocity fields, back pressures, shedding frequencies, turbulence levels, and Reynolds stress magnitudes/spatial distributions indicate the existence of two dynamically stable solutions. Thus, the flat plate distinguishes itself from standard bluff body flows featuring unique solutions.

Low-order representations of the flow fields are used to reconstruct the wake dynamics. The results show that the underlying dynamics differ in terms of the energy distribution and content of otherwise similar modes of coherent motion. For the open end case, a greater cycle-to-cycle variation of the shedding process is associated with a comparatively stronger slow-varying mode in the base region. In contrast, for the closed end case, a shear layer flapping mode is more strongly expressed, which may account for greater variations in the trajectories of the shed vortices. These differences are then related to the structure and intensity of the Reynolds stress fields.

A better understanding of the vortex formation process is developed to account for differences in the vortex streets. A careful accounting of the vorticity transport in the wake is conducted and the contributions of different mechanisms are assessed. The work further contributes a new model for estimating the circulation associated with shed vortices which accounts for vorticity not captured in the core region. Despite the strength of the shed vortices being similar between the two cases, differences in the formation regions, such as the rates of vorticity decay, suggest that the concentration of vorticity within the separated
shear layers and forming vortices are important to the wake dynamics.
Preface

This dissertation is formatted as a hybrid traditional/manuscript thesis consisting of published work and original chapters:

- Chapters 1, 4, 5, and 6 are original work by the author, Eric A. Braun.

- Chapters 2 and 3 consist of work published in the International Journal of Heat and Fluid Flow (Braun et al., 2020):

Statement of contributions

This dissertation reflects original work by the author, Eric A. Braun, as well as collaborative work undertaken alongside Kaden B. Agrey. All work was conducted under the supervision of Dr.Ing Robert J. Martinuzzi.

Data collection was performed by Matthew G. Kindree and Maryam Shahroodi for the flat plate with closed ends. For the flat plate with open ends, data was collected by Dr. Meraj Mohebi.

Preliminary data processing was completed by the author and Kaden B. Agrey. Additionally, the author collaborated with Kaden B. Agrey in preparing the manuscript titled “End effects of nominally two-dimensional thin flat plates”, published in the International Journal of Heat and Fluid Flow (Braun et al., 2020). This shared work is presented in Chapters 2 and 3, as well as Appendix B. The author would like to emphasize the importance of Kaden B. Agrey’s contributions to these sections, which were the result of equal work and extensive discussions. The shared work reflects contributions involving different areas of expertise and analysis essential to the successful outcome. The remainder of this dissertation reflects the original work of the author.
Acknowledgments

First, I would thank my supervisor, Dr.Ing Robert Martinuzzi for his guidance, insight, and support. Even as an undergraduate, I could always count on you to take whatever time was necessary to properly answer a question. Your willingness to patiently share knowledge and insight with those driven to learn never went unnoticed, and I endeavor to emulate this intellectual generosity in my own life. Perhaps the most invaluable lesson that you taught me was the importance of asking the right questions; through the lens of the right question, even the most complicated of problems become clearer. Thank you for your unwavering commitment to my intellectual and professional development, I am forever grateful.

I would also like to thank my lab mate, co-author, and friend Kaden Agrey, who worked alongside me throughout this degree. I recall many a day where I would explain my codes to you, in desperate search of elusive bugs. Sometimes you would spot the bugs, and other times the bugs would live to see another day. Regardless of outcome, your willingness to lend a second pair of eyes was a kindness that was never overlooked, and will not be forgotten. I fondly remember the upbeat attitude that you maintained during the publication of our shared paper, despite all the trials, tribulations, and long nights. Your intellectual rigour and genuine delight for learning will take you very far, and deservedly so.

To all my lab mates from room 269, the 4th of EEEL, and the MEB basement, thank you for your insights, support, and motivation. Our lively conversations and debates deepened my understanding of fluid dynamics. But most importantly, I would like to thank you all for making our lab a place of laughter and comradery.
Dan Forre, thank you for your kindness, support, and the great conversations we had over coffees. Also, thank you for the fantastic work you did fixing, maintaining, and coordinating the mechanical engineering laboratories. Myself and my peers sincerely appreciated your efforts.

I would also like to voice my thanks to the Natural Sciences and Engineering Research Council of Canada, Government of Alberta, and University of Calgary for their financial support.

Finally, I would like to thank my amazing family, my loving and wonderful girlfriend Erin, and my best friend Miura for their unwavering support through the ups and downs of this degree.
Table of contents

Abstract ii
Preface iv
Statement of contributions v
Acknowledgments vi
Table of contents viii
List of figures and illustrations xi
List of symbols, abbreviations, and nomenclature xvi

1 Introduction 1
   1.1 Background and motivation .............................................. 1
   1.1.1 Objectives .......................................................... 4
   1.1.2 Thesis structure ..................................................... 5

2 Methodology 7

3 End effects of nominally two-dimensional thin flat plates 14
   3.1 Introduction ............................................................. 14
   3.2 Results ................................................................. 17
      3.2.1 Two dimensionality and phase uniformity ......................... 18
      3.2.2 Mean wake ......................................................... 18
      3.2.3 Global quantities ................................................... 22
      3.2.4 POD analysis ....................................................... 25
      3.2.5 Reynolds stresses and wake structure ............................ 32
   3.3 Concluding remarks .................................................... 35

4 Vortex identification 37
   4.1 On the identification of vortex cores: Q-criterion .......... 39
      4.1.1 Rankine vortex .................................................... 41
      4.1.2 Lamb-Oseen vortex ................................................. 42
      4.1.3 Burgers vortex ..................................................... 43
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.2 A matter of accounting: vortex circulation</td>
<td>45</td>
</tr>
<tr>
<td>4.2.1 An analytical accounting of the Burgers vortex using the Q-criterion</td>
<td>46</td>
</tr>
<tr>
<td>4.2.2 An analytical accounting of the Lamb-Oseen vortex using the Q-criterion</td>
<td>49</td>
</tr>
<tr>
<td>4.3 Burgers and Lamb-Oseen vortex models applied to flat plate flows</td>
<td>50</td>
</tr>
<tr>
<td>4.3.1 A manual accounting of vorticity</td>
<td>50</td>
</tr>
<tr>
<td>4.3.2 Results</td>
<td>51</td>
</tr>
<tr>
<td>4.4 A brief comparison of different methods for educing vortex centers</td>
<td>53</td>
</tr>
<tr>
<td>4.4.1 Vortex centroid method</td>
<td>54</td>
</tr>
<tr>
<td>4.4.2 Pressure minima method</td>
<td>58</td>
</tr>
<tr>
<td>4.4.3 Q-maxima method</td>
<td>59</td>
</tr>
<tr>
<td>4.4.4 Comparison of the centroid, pressure minima, and Q-maxima methods for determining vortex centers</td>
<td>59</td>
</tr>
<tr>
<td>4.4.5 Comparison of the centroid, pressure minima, and Q-maxima methods for determining vortex convection speeds</td>
<td>59</td>
</tr>
<tr>
<td>4.5 Concluding remarks</td>
<td>62</td>
</tr>
<tr>
<td>5 An accounting of circulation transport in the flat plate wake</td>
<td>63</td>
</tr>
<tr>
<td>5.1 Vortex shedding models</td>
<td>63</td>
</tr>
<tr>
<td>5.1.1 Gerrard’s vortex shedding model</td>
<td>64</td>
</tr>
<tr>
<td>5.1.2 Roshko’s bluff body wake model</td>
<td>66</td>
</tr>
<tr>
<td>5.1.3 Ahlborn’s bluff body wake model</td>
<td>68</td>
</tr>
<tr>
<td>5.1.4 The flat plate topology</td>
<td>70</td>
</tr>
<tr>
<td>5.2 Vorticity cancellation and annihilation</td>
<td>73</td>
</tr>
<tr>
<td>5.2.1 Vorticity cancellation (type-1)</td>
<td>73</td>
</tr>
<tr>
<td>5.2.2 Vorticity cancellation (type-2)</td>
<td>73</td>
</tr>
<tr>
<td>5.2.3 Vorticity annihilation</td>
<td>74</td>
</tr>
<tr>
<td>5.3 Vorticity transport in the flat plate wakes</td>
<td>76</td>
</tr>
<tr>
<td>5.3.1 Method 1: Semi-infinite control volume</td>
<td>76</td>
</tr>
<tr>
<td>5.3.2 Method 2 - Infinite control volume</td>
<td>80</td>
</tr>
<tr>
<td>5.3.3 Method 3 - Vortex circulation</td>
<td>84</td>
</tr>
<tr>
<td>5.3.4 Methods for estimating circulation transport rates - comparisons and results</td>
<td>85</td>
</tr>
<tr>
<td>5.4 Indicators of increased turbulence in the closed end wake</td>
<td>88</td>
</tr>
<tr>
<td>5.4.1 Centrifugal instabilities in the flat plate wakes</td>
<td>89</td>
</tr>
<tr>
<td>5.4.2 Streamline trajectories as an indicator of mixing</td>
<td>91</td>
</tr>
<tr>
<td>5.5 Concluding remarks</td>
<td>93</td>
</tr>
<tr>
<td>6 Synthesis</td>
<td>94</td>
</tr>
<tr>
<td>6.1 Conclusions</td>
<td>95</td>
</tr>
<tr>
<td>6.2 Recommendations for future work</td>
<td>98</td>
</tr>
<tr>
<td>Bibliography</td>
<td>100</td>
</tr>
<tr>
<td>A POD representations of the flat plate flows</td>
<td>107</td>
</tr>
<tr>
<td>Section</td>
<td>Page</td>
</tr>
<tr>
<td>-------------------------------</td>
<td>------</td>
</tr>
<tr>
<td><strong>B</strong> Transition modes</td>
<td>115</td>
</tr>
<tr>
<td><strong>C</strong> Q-criterion applied to Lamb-Oseen vortex</td>
<td>117</td>
</tr>
<tr>
<td><strong>D</strong> Pressure field estimation</td>
<td>119</td>
</tr>
<tr>
<td><strong>E</strong> Vortex selection code</td>
<td>121</td>
</tr>
<tr>
<td>E.1 GUI - Front End</td>
<td>122</td>
</tr>
<tr>
<td>E.2 GUI - Back End</td>
<td>127</td>
</tr>
<tr>
<td>E.3 GUI - Vortex Labeling</td>
<td>143</td>
</tr>
<tr>
<td>E.4 Subfunction - Vortex Filtering</td>
<td>146</td>
</tr>
<tr>
<td>E.5 Subfunction - Q-Criterion</td>
<td>147</td>
</tr>
</tbody>
</table>
List of figures and illustrations

2.1 Schematic representation of the geometry and nomenclature, showing the PIV field of view (FoV): a) side view b) top view. .......................................................... 8

2.2 Mean velocity and Reynolds stress distributions at different Reynolds numbers as reported in Mohebi (2016); y-profiles at x/c = 4, centerline profiles at y = 0. Reynolds number and symbol: 3000, Mean velocity and Reynolds stress profiles at x/c = 5.6, z/c = 0. Reynolds number and symbol: 3000, 4500, 6600, 8000, 9600, 10800, 12000. .......................................................... 11

3.1 Top: z-distributions of U and V for CE (left) and OE (middle) and C_Pb (right). C_Pb is shown for Re = 10000 and 20000 (circles and triangles, respectively, CE closed symbols and OE open symbols). Bottom: y-distributions of U, V, u'^2, and u'v' for CE (left two) and OE (right two) for x = 3c and z = 0c, 0.5c, and 1c. Black lines and closed symbols are for CE; green lines and open symbols for OE. ........................................................................................... 19

3.2 Cross-correlation R_{p'p'}(\tau) of pressure signals for taps at z = 0c and z = 10c. CE in black, OE in green. R_{p'p'}(\tau) = \frac{1}{T} \int_0^T p'(0,t)p'(10,t+\tau)dt. .................................................. 20

3.3 Reynolds stress contours with overlaid mean streamlines. Top half-planes with closed ends, bottom half with open ends. The separating streamline is denoted by a thicker line terminating at the saddle point at the end of the mean recirculation (x = \ell_R, y = 0). Filled symbols are for closed ends, open symbols for open ends. Yellow circles and triangles indicate points of maximum u'^2 and v'^2, respectively. The first row shows the Reynolds averaged field; second the coherent field; third the residual field. Colormap levels change across the horizontal line. Final two columns show profiles at x-location of maximum v'^2 and u'^2; ••• u'^2 as indicated in the first row. ••• v'^2, ••• u'^2. ........................................... 21

3.4 a) Power spectral density (PSD) function of u', v' at max u'^2 (left), and v'^2 (right). Successive spectra are offset by 10^5 and a reference line at (10^{-4}, absolute) is added to each for clarity. b) Auto-correlations of u' (top), v' (bottom) at max v'^2. u'^2_max is at (x, y) = (1.5c, 0.5c) (CE) and (3.4c, -0.8c) (OE) and v'^2_max is at (x, y) = (2.1c, 0c) (CE) and (4.9c, 0c) (OE). CE (green); OE (black). ............................................................................................... 22

3.5 Mean rate of vorticity transport \frac{d\Gamma}{dt} for closed end (CE) and open end (OE) conditions and their ratio (CE/OE). To allow comparison over similar domains, data from the extended open ends field were included (OE_{ext}). ........................................ 23
3.6 Vorticity thickness $\delta_w/c$ as a function of downstream distance $x/\ell_R$ for closed (CE) and open end (OE) cases. To ensure that the comparison domain was the same, data from the extended open ends field was also used (OE$_{ext}$).

3.7 Flooded iso-contours of the phase-averaged vorticity field $\langle \omega_z \rangle$ are shown with velocity streamlines; both fields are shown at the same phase. Within the fields, $|\langle \omega_z \rangle|_{max} = 6.15$ (CE), and $|\langle \omega_z \rangle|_{max} = 3.94$ (OE). To better show OE $\langle \omega_z \rangle$ detail, the color bar limits were set to $\pm 4.6$. Closed green lines indicate $Q = 0$ and enclose region $A_v$. Purple lines indicate the trajectory of the vorticity centroids, $(x_c, y_c) = \frac{1}{\Gamma_v} \int\int (x, y) \langle \omega_z \rangle \, dA_v$. Streamlines are in a convective reference frame.

3.8 Flooded isocontours and pseudo-streamlines for spatial functions together with spectra of temporal coefficients for the slow-drift, $a_\Delta$ (top), and flapping, $a_f$ (bottom), modes. CE is on the left, OE on the right. Note that, for the OE field, $a_f$ is taken from the extended domain. Spatial modes are orthonormal: $(\Phi_n, \Phi_m) = \delta_{nm}$. Right most: relative contribution to the TKE. $a_1, a_2$ fundamental harmonic pair. $a_3, a_4$ second harmonic.

3.9 Phase portraits for CE (left) and OE (right) for: a) harmonic pair $a_2$ vs. $a_1$; b) $a_2$ vs. $a_\Delta$, ($a_1 \approx 0$); and c) $a_2$ vs. $a_f$, ($a_1 \approx 0$). Note: $a_f$ is taken from the extended OE field. Solid red lines indicate a regression in b) and in a) the limit cycle ($a_\Delta = 0$). Dashed red lines in b) show bin averages for high, limit, and low cycles. The modal coefficients are scaled by $\sqrt{\lambda_n}$.

3.10 Conditionally phase averaged vorticity centroid trajectories (top) with corresponding strengths $\Gamma_v$ (bottom). Conditional phase averaging was preformed on the flapping mode. Trajectories and strengths are shown for high (red), low (blue), and average (black) cycles. CE on left, OE on the right. OE points on $x/\ell_R < 1.2$ are from near FoV, $x/\ell_R > 1.2$ from extended FoV.

3.11 High cycle left, limit cycle middle, and low cycle right. Phase averaged isocontours of vorticity at the phase when a vortex is about to shed from the negative shear layer. Overlaid are Lagrangian streamlines, the back flow (black dashed line), absolute maxima for $\langle u_c \rangle^2$ and $\langle v_c \rangle^2$ (red circles and triangles, respectively), the first local maximum $\langle u_c \rangle^2$ on the centerline (also red circles), and the vortex centroid path over the cycle (orange circles). Top, CE and bottom, OE. Note: gap in OE contours is the transition between PIV FoVs and is chosen to lie between the vortex trajectories from each FoV for visual clarity.

3.12 Left of legends; Reynolds averaged stress (top), coherent and residual production along $y = 0$ for open (OE) and closed (CE) ends as functions of $x/\ell_R$ (bottom). Right of legends; coherent and residual production (left) and coherent to residual transfer (right), both scaled by maximum production and as functions of $x/\ell_R$. CE, top and OE, bottom.

4.1 Rankine vortex. Left: the quotient of the local vorticity, and the angular velocity about the origin; $\omega_z/\Omega_0 = 2$ indicates perfect solid-body rotation. Right: the vorticity field with overlaid streamlines. $Q = 0$ is indicated by dashed green lines.
4.2 Lamb-Oseen vortex. Left: the quotient of the local vorticity and angular velocity for different $\sqrt{\Pi_2} = r/r_0$; $\omega_z/\Omega_0 = 2$ indicates perfect solid-body rotation. Right: the vorticity field for a typical Lamb-Oseen vortex at an unspecified time. $Q = 0$ is indicated by dashed green lines.

4.3 Burgers vortex. Top left: the quotient of the local vorticity, and the angular velocity about the origin; $\omega_z/\Omega_0 = 2$ indicates perfect solid-body rotation. Profiles are shown for two different $\Pi_3 = \Gamma_{\text{tot}}/\nu$ cases. The two profiles can be collapsed onto a single curve by plotting against a non-dimensionalized variable: $\Pi_2 = r^2/r_0^2$. By not collapsing the curve, it can be seen that when $Q = 0$, $\omega_z$ and $\Omega_0$ need not be equal. Top right: relationship between $\omega_z/\Omega_0$ and $\Pi_3$ when $Q = 0$. Bottom left: the vorticity field with overlaid streamlines. $Q = 0$ is indicated by dashed green lines. The red line indicates the $\omega_z/\Omega$ profile (top left) associated with the vorticity field. Bottom right: the relationship between the $u_r-u_\theta$ quotient and $\Pi_3$ when $Q = 0$.

4.4 Flooded isocontours of $\Pi_1$, with $\Pi_1 = 0$ denoted in green.

4.5 Prototypical GUI snapshot for a phase-averaged vorticity field with overlaid Lagrangian streamlines (reference frame moving with the vortex center). User selected region is marked by solid green circles along the border, and open green circles within the selected area. Magenta lines indicate the Q-identified region, while solid blue and red lines indicate isobars equal to 4% of the maximum negative and positive vorticity, respectively.

4.6 $\Gamma_v/\Gamma_{\text{tot}}^*$ as a function of $Q/Q_{\text{max}}$; $\triangle$ CE, $\circ$ OE, with analytical model as the red line.

4.7 Closed ends: $\Gamma_{\text{tot}}^*/(cU_\infty)$ for different vorticity iso-contour levels; linear regression in red.

4.8 Phase averaged vorticity fields with overlaid Lagrangian streamlines (reference frame moving with the vortex center) for the closed end case at three different phase averaged phase angles. Red circles: vorticity centroids; green squares: pressure minima; black triangles: Q-maxima.

4.9 Phase averaged vortex paths for the closed end case. Red circles: vorticity centroids; green squares: pressure minima; black triangles: Q-maxima.

4.10 Vortex convection speeds estimated from phase averaged vortex paths for the closed end case. Centroid method; pressure minima method; Q-maxima method.

5.1 Vortex shedding model, closely recreated from Williamson (1996), for flow about a circular cylinder. Saddlepoints are denoted by red circles.

5.2 Control volume for the Ahlborn et al. (1998) model in red.

5.3 Vortex shedding model for flow about a flat plate cylinder. Saddlepoints are denoted by red circles.

5.4 Schematic showing the different types of vorticity cancellation and annihilation.
5.5 Schematic showing a theoretical semi-infinite control volume. The red plane indicates the PIV domain, the arrows denote the oncoming free stream flow, the black plane represents a thin flat plate, and the blue plane illustrates an arbitrary integration plane through which the rate of vorticity transport can be calculated.

5.6 Rate of (absolute) circulation transport for the top and bottom half planes (averaged) of the PIV measurement domain for CE (left), and OE (right). Solid profiles were calculated using the averaged vorticity transport equation (5.25) and the dashed profiles were similarly calculated, less the turbulent diffusion and turbulent tilting terms \( \left( \frac{1}{2U_\infty^2} \left( \frac{u''_z^2 + u''_z''}{y_{\text{max}}} \right) - \right) \) .

5.7 Rate of circulation transport calculated along the PIV measurement domain using method-2; CE (left), and OE (right). Solid profiles were calculated from the total field, and the dashed profiles were similarly calculated, less any contributions from the incoherent field.

5.8 Estimation of the circulation transport rate; CE (left), OE (right). Circles correspond to estimates calculated using the combined mean and coherent fields; total field, triangles.

5.9 Comparison of the different methods for calculating the rate of circulation transport. Dashed lines: method-1; solid lines: method-2; circles: method-3. CE (left) and OE (right). Mean and coherent field (top); total field (bottom).

5.10 Profiles at different \( x/\ell_R \) locations for the vorticity transport integrands from method 2 i.e., \( \frac{c}{U_\infty} \left( \pm \omega_z u_x \right) \), using the total field. CE (top) and OE (bottom). Profiles taken at different \( x/\ell_R \) locations have been offset for visual clarity.

5.11 Extrapolation (red) of rates of vorticity transfer for method-1 (dashed lines) and method-2 (solid lines), using the total field; CE left, OE right. Dotted lines indicate the rate of vorticity generation \( \left( \frac{1-C_P^2}{2} \right) \) at the flat plate boundary \( (x = 0) \), as predicted by Roshko (1954b) and Ahlborn et al. (1998).

5.12 Left: the recirculation streamline \( \int_0 U/U_\infty d(y/c) = 0 \) (green) with profiles of velocity tangent to the streamline curvature. Middle and Right: the \( U/U_\infty \)-velocity fields. Top, CE; bottom, OE.

5.13 Cycle averaged streamlines colored based on i) local rates of cycle averaged vorticity transport, \( \frac{1}{U_\infty} \int_0^t \frac{\partial}{\partial t} \), (left), cycle averaged \( a_\Delta \) (top-middle, bottom-right), and cycle averaged \( a_f \) (top-left). CE (top), OE (bottom).

A.1 Closed ends (symmetric): Flooded isocontours of spatial modes \( \Phi_n^x, \Phi_m^y, \) and \( \Phi_n^w, \) as well as pseudo-streamlines and PSD of the modal temporal coefficients, \( a_n \). Spatial modes are orthonormal \( \langle \Phi_n, \Phi_m \rangle = \delta_{nm} \), and \( \int PSD(a_n)df = \overline{a_n^2} \).

A.2 Closed ends (asymmetric): Flooded isocontours of spatial modes \( \Phi_n^x, \Phi_m^y, \) and \( \Phi_n^w, \) as well as pseudo-streamlines and PSD of the modal temporal coefficients, \( a_n \). Spatial modes are orthonormal \( \langle \Phi_n, \Phi_m \rangle = \delta_{nm} \), and \( \int PSD(a_n)df = \overline{a_n^2} \).
A.3 Open ends (symmetric): Flooded isocontours of spatial modes $\Phi^u_n$, $\Phi^v_n$, and $\Phi^w_n$, as well as pseudo-streamlines and PSD of the modal temporal coefficients, $a_n$. Spatial modes are orthonormal ($\langle \Phi_n, \Phi_m \rangle = \delta_{nm}$), and $\int PSD(a_n)df = a_n^2$.  

A.4 Open ends (asymmetric): Flooded isocontours of spatial modes $\Phi^u_n$, $\Phi^v_n$, and $\Phi^w_n$, as well as pseudo-streamlines and PSD of the modal temporal coefficients, $a_n$. Spatial modes are orthonormal ($\langle \Phi_n, \Phi_m \rangle = \delta_{nm}$), and $\int PSD(a_n)df = a_n^2$.  

A.5 Open ends extended field (symmetric): Flooded isocontours of spatial modes $\Phi^u_n$, $\Phi^v_n$, and $\Phi^w_n$, as well as pseudo-streamlines and PSD of the modal temporal coefficients, $a_n$. Spatial modes are orthonormal ($\langle \Phi_n, \Phi_m \rangle = \delta_{nm}$), and $\int PSD(a_n)df = a_n^2$.  

A.6 Open ends extended field (asymmetric): Flooded isocontours of spatial modes $\Phi^u_n$, $\Phi^v_n$, and $\Phi^w_n$, as well as pseudo-streamlines and PSD of the modal temporal coefficients, $a_n$. Spatial modes are orthonormal ($\langle \Phi_n, \Phi_m \rangle = \delta_{nm}$), and $\int PSD(a_n)df = a_n^2$.  

B.1 Flooded isocontours for spatial function $u$ and $v$ components together with phase portraits and spectra of temporal coefficients for the transition mode, $a_t$. Phase portraits show $a_t$ vs. $a_2$ ($a_1 \approx 0$) with modal coefficients scaled by $\sqrt{\lambda_n}$. Red lines indicate a regression. CE (top), OE (middle), OE extended (bottom). Spatial modes are orthonormal: $\langle \Phi_n, \Phi_m \rangle = \delta_{nm}$.  

\[115\]
## List of symbols, abbreviations, and nomenclature

### Latin symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_n)</td>
<td>POD temporal coefficients</td>
</tr>
<tr>
<td>(A)</td>
<td>radius of the shedding-shift mode paraboloid</td>
</tr>
<tr>
<td>(A_v)</td>
<td>area enclosed by (Q) iso-contour ((Q = 0), typically)</td>
</tr>
<tr>
<td>(b)</td>
<td>thickness of the flat plate in the streamwise direction</td>
</tr>
<tr>
<td>(c)</td>
<td>characteristic length of the flat plate (flow-normal direction)</td>
</tr>
<tr>
<td>(c_x)</td>
<td>convective speed of vortex</td>
</tr>
<tr>
<td>(C_P)</td>
<td>pressure coefficient</td>
</tr>
<tr>
<td>(C_{P_b})</td>
<td>base pressure coefficient</td>
</tr>
<tr>
<td>(e)</td>
<td>unit vector</td>
</tr>
<tr>
<td>(f)</td>
<td>frequency</td>
</tr>
<tr>
<td>(f_{sh})</td>
<td>shedding frequency</td>
</tr>
<tr>
<td>(G_k)</td>
<td>production term from TKE transport equation</td>
</tr>
<tr>
<td>(I)</td>
<td>identity matrix</td>
</tr>
<tr>
<td>(\ell_R)</td>
<td>recirculation length</td>
</tr>
<tr>
<td>(m)</td>
<td>mass</td>
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<tr>
<td>(p)</td>
<td>pressure</td>
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<tr>
<td>(p')</td>
<td>fluctuating pressure</td>
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<tr>
<td>(P)</td>
<td>mean pressure</td>
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<tr>
<td>(P_b)</td>
<td>base pressure</td>
</tr>
<tr>
<td>(P_s)</td>
<td>free streamline pressure</td>
</tr>
<tr>
<td>(P_0)</td>
<td>stagnation pressure</td>
</tr>
<tr>
<td>(P_\infty)</td>
<td>free-stream pressure</td>
</tr>
<tr>
<td>(Q)</td>
<td>(Q)-criterion and values</td>
</tr>
<tr>
<td>(r)</td>
<td>radius</td>
</tr>
<tr>
<td>(r_0)</td>
<td>“broadness” of Lamb-Oseen and Burgers vortices</td>
</tr>
<tr>
<td>(R)</td>
<td>rate of rotation tensor; streamline radius of curvature; radius at which flow transitions from solid-body to irrotational rotation for a Rankine vortex</td>
</tr>
<tr>
<td>(Re)</td>
<td>Reynolds number</td>
</tr>
<tr>
<td>(s)</td>
<td>span of flat plate</td>
</tr>
<tr>
<td>(S)</td>
<td>rate of strain tensor</td>
</tr>
</tbody>
</table>
\[ S_t \] Strouhal number

\[ t \] time

\[ \mathbf{u} = (u, v, w; (u_r, u_\theta, u_z)) \] velocity vector with components (streamwise, flow-normal, spanwise); cylindrical polar components (radius, azimuth, height)

\[ \mathbf{u}' = (u', v', w') \] fluctuating velocity vector with components (streamwise, flow-normal, spanwise)

\[ \mathbf{u}'' = (u'', v'', w'') \] incoherent part of the fluctuating velocity vector with components (streamwise, flow-normal, spanwise)

\[ \mathbf{u}_B \] base flow region

\[ \mathbf{u}_c = (u_c, v_c, w_c) \] coherent part of the fluctuating velocity vector with components (streamwise, flow-normal, spanwise)

\[ \mathbf{U}^* = (U_x^*, U_y^*, U_z^*) \] combined mean and phase averaged velocity \((\mathbf{U} + \langle \mathbf{u} \rangle)\) (streamwise, flow-normal, spanwise); combined mean and coherent velocity \((\mathbf{U} + \mathbf{u}_c)\) (streamwise, flow-normal, spanwise)

\[ \mathbf{U} = (U, V, W) \] mean velocity vector with components (streamwise, flow-normal, spanwise)

\[ U_s \] free streamline velocity

\[ U_\infty \] free-stream velocity

\[ \mathbf{x} = (x, y, z) \] cartesian coordinate vector with components (streamwise, flow-normal, spanwise)

\[ (x_n, y_n) \] location of recirculation node

\[ (x_s, y_s) \] location of maximum outward deviation of separation streamlines

**Greek symbols**

\[ \gamma \] extensional strain parameter of Burgers vortex

\[ \Gamma \] circulation

\[ \Gamma_v \] vortex circulation contained within \(A_v\)

\[ \Gamma_{tot} \] total circulation associated with a vortex

\[ \Gamma_{tot}^* \] estimation of \(\Gamma_{tot}\)

\[ \delta \] kronecker delta symbol

\[ \delta_w \] vorticity thickness

\[ \epsilon \] Levi-Civita symbol

\[ \lambda \] eigenvalue

\[ \lambda_n \] empirical POD eigenvalues

\[ \lambda_2 \] \(\lambda_2\)-criterion

\[ \Lambda \] PIV observation domain

\[ \mu \] dynamic viscosity

\[ \nu \] kinematic viscosity

\[ \omega = (\omega_x, \omega_y, \omega_z) \] vorticity vector with components (streamwise, flow-normal, spanwise)

\[ \omega' = (\omega'_x, \omega'_y, \omega'_z) \] fluctuating vorticity vector with components (streamwise, flow-normal, spanwise)

\[ \omega'' = (\omega''_x, \omega''_y, \omega''_z) \] incoherent part of the fluctuating vorticity vector with components (streamwise, flow-normal, spanwise)
vorticity at center of Burgers vortex
\( \mathbf{\Omega} = (\Omega_x, \Omega_y, \Omega_z) \) mean vorticity vector with components (streamwise, flow-normal, spanwise)
\( \mathbf{\Omega}^* = (\Omega^*_x, \Omega^*_y, \Omega^*_z) \) combined mean and phase averaged vorticity (\( \mathbf{\Omega} + \langle \mathbf{\omega} \rangle \)) (streamwise, flow-normal, spanwise); combined mean and coherent vorticity (\( \mathbf{\Omega} + \mathbf{\omega}_c \)) (streamwise, flow-normal, spanwise)
\( \mathbf{\Omega}^\dagger = (\Omega^\dagger_x, \Omega^\dagger_y, \Omega^\dagger_z) \) angular velocity vector with components (streamwise, flow-normal, spanwise)
\( \Omega_0 \) angular velocity of flow about the origin
\( \rho \) density
\( \theta \) angle
\( \phi \) phase
\( \Phi_n = (\Phi^u_n, \Phi^v_n, \Phi^w_n) \) POD spatial modes with components (streamwise, flow-normal, spanwise)
\( \tau \) stress tensor; shedding period
\( \zeta \) average vorticity found within a vortex, accumulated over a half shedding cycle using Ahlborn’s shedding model

**Abbreviations**

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td>CE</td>
<td>flat plate with closed end conditions</td>
</tr>
<tr>
<td>CV</td>
<td>control volume</td>
</tr>
<tr>
<td>det</td>
<td>determinant</td>
</tr>
<tr>
<td>DNS</td>
<td>direct Navier-Stokes</td>
</tr>
<tr>
<td>FoV</td>
<td>field of view</td>
</tr>
<tr>
<td>GUI</td>
<td>graphical user interface</td>
</tr>
<tr>
<td>IJHFF</td>
<td>International Journal of Heat and Fluid Flow</td>
</tr>
<tr>
<td>LES</td>
<td>large-eddy simulation</td>
</tr>
<tr>
<td>Nd:YLF</td>
<td>neodymium-doped yttrium lithium fluoride</td>
</tr>
<tr>
<td>OE</td>
<td>flat plate with open end conditions</td>
</tr>
<tr>
<td>PDE</td>
<td>partial differential equation</td>
</tr>
<tr>
<td>PIV</td>
<td>particle image velocimetry</td>
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<tr>
<td>POD</td>
<td>proper orthogonal decomposition</td>
</tr>
<tr>
<td>PSD</td>
<td>power spectral density function</td>
</tr>
<tr>
<td>RANS</td>
<td>Reynolds averaged Navier-Stokes</td>
</tr>
<tr>
<td>TKE</td>
<td>turbulent kinetic energy</td>
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<tr>
<td>2D</td>
<td>two dimensional</td>
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</table>

**Other notation**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td>( \cdot )</td>
<td>time average</td>
</tr>
<tr>
<td>( \langle \cdot \rangle )</td>
<td>inner product using L2-norm</td>
</tr>
<tr>
<td>( \langle \cdot \rangle )</td>
<td>phase average</td>
</tr>
<tr>
<td>( \dot{\cdot} )</td>
<td>temporal derivative</td>
</tr>
<tr>
<td>( \nabla )</td>
<td>gradient operator</td>
</tr>
<tr>
<td>( R_{xy} )</td>
<td>cross-correlation of signals ( x ) and ( y )</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

This thesis documents an analysis for the experimental study of the turbulent wakes of a thin flat plate, subject to two different end conditions. Notably, the mean field properties, spectral signatures, and variance maximizing motions of the flat plate wakes are investigated. Special attention is focused on the identification of vortical structures and their associated circulation and advection. This chapter provides an overview of the motivation and objectives of the study.

1.1 Background and motivation

Typical to many expressions of engineering design is uniqueness: the idea that for a given input, only one possible solution exists. It is common practice for engineers to simplify, linearizing non-linear expressions through approximations. In many applications, such approximations are appropriate and justified. However, simplifying approximations can be particularly costly in cases where the assumption of uniqueness is incorrect, since the alternative solution(s) can be vastly different from the anticipated solution. Fluid dynamics is an area of engineering interest that is no stranger to non-uniqueness. Indeed, the inviscid Burgers equation (Burgers, 1948), Rayleigh-Bénard convection (Koschmieder, 1993), expanding channel flows (Kline et al., 1959), and Taylor-Couette flows (Coles, 1965), are all classical
examples known for their non-uniqueness. For flows with multiple solutions, there is a fundamental interest in understanding the dynamic characteristics of the solutions, and how the solutions differ. Typically, multiple solutions arise when non-linear couplings of dynamical motions are present, such as those that exist between the boundary layers, separating free shear layers, and wakes of bluff body flows (Williamson, 1996). In this thesis, the flow about a thin flat plate, immersed normal to a uniform stream, will be investigated for two end conditions. It will be shown that two different dynamically stable wake flow solutions exist for this geometry, which satisfy the conditions for two-dimensional flow.

For many bluff body flows, the existence of two solutions is generally not a concern. Indeed, canonical examples such as the mean two-dimensional wakes of square and circular cylinders only appear to have one stable solution. As an example, flow about a very long circular cylinder with a wake that is two-dimensional in the mean will yield similar solutions to those that are obtained using a shorter cylinder with end plates (where the end plates have been designed to maintain mean two-dimensionality) (Fox and West, 1990; Hammache and Gharib, 1991; Prasad and Williamson, 1997; Williamson, 1989; Williamson and Roshko, 1990). However, unstable solutions also exist for these flows (and many other bluff body flows), including the square and circular cylinders, as well as the flat plate. From the works of Noack et al. (2003), the unsteady solution to a cylinder flow is detailed. Despite the fact that the unsteady solutions are not realized in nature, they are nonetheless critical to understanding the wake dynamics of their respective flows, as these are related to energetic exchanges between the stable (limit cycle) and unsteady solutions. Despite the continuous exchange of energy between the two states, the tendency is always towards dynamic stability; this is why stable solutions are sometimes referred to as attractors.

Considering the rectangular cylinder (which can be thought of as a more general version of the square cylinder), it has been observed that when the thickness to chord ratio drops below \( b/c \approx 0.3 - 0.4 \) (Knisely (1990); Mohebi et al. (2017); Norberg (1993)), large discrepancies exist in the wake structure and dynamics reported from different investigations. For
nominally thin flat plates \( b/c < 0.2 \) with large spans (such that the end conditions are not expected to impact the vortex formation process) there is still no consensus in the experimental literature regarding what the dynamical solution is. Moreover, comparing the steady and unsteady Reynolds-Averaged Navier-Stokes (RANS) to Large-Eddy Simulation (LES) and Direct Navier-Stokes (DNS) simulations also reveal different results (Dahlqvist, 2016; Hemmati et al., 2016, 2018; Najjar and Balachandar, 1998; Narasimhamurthy and Andersson, 2009; Tian et al., 2013). Based on these findings, it appears as though the solution type is highly sensitive to the end condition. Experiments appear to align with this notion; placing end-plates (parallel to the flow plane) on very long flat plates results in a different dynamically stable solution, as compared to the free-end condition. Hence, the characterization of these two dynamically stable solutions is of fundamental, and practical interest. Critical to the characterization of the solutions are the mean fields, and mean turbulent wake structures (Reynolds stresses). In addition, how the dynamic behavior is altered can be further elucidated by paying careful attention to the vortex formation and shedding cycles, as well as the shed vortex characteristics.

Towards a better understanding of the dynamics and characteristics of the flat plate vortex shedding, a proper accounting of the generated circulation that enters the wake, the circulation that is advected downstream (be it through the motion of the shed vortices, or some other mechanism), and the rates at which these processes occur is necessary. From the literature, methods for estimating these quantities have various weaknesses such that an investigation into the objective quantification of these quantities is warranted.

Insights and learnings included in this manuscript extend the body of knowledge concerning the use of end plates for achieving different aerodynamics purposes, thereby contributing to a broader spectrum of related applications, such as winglets on airplanes and end plates on race car wings.
1.1.1 Objectives

This manuscript will investigate the wakes of a nominally two-dimensional thin flat plate immersed normal to a uniform stream, with and without end plates. For the end plate condition, thin metal plates were positioned at the top and bottom (spanwise) edges of the flat plate, and were carefully aligned to guarantee mean two-dimensionality in the wake. Free ends were used for the second case, and the flow was also shown to be two-dimensional in the mean. In both cases, the Reynolds number is sufficiently high so that the wakes are turbulent and characterized by the nearly periodic vortex shedding. The objectives of this study can be briefly stated as:

1. Compare and characterize the open and closed end flat plate flows. This objective will provide insight into the two dynamically stable solutions that appear in thin flat plate wakes, thereby reconciling differences found in the literature.

2. Investigate the characteristics of the vortex street for the two solutions including vortex strength (circulation), trajectories, and convective speeds. With the most notable coherent motions in bluff body wakes being those related to vortex shedding, this study contributes to the understanding of how the dynamics of coherent structures influence the wake structure and global properties of engineering interest such as unsteady loading.

3. Investigate and compare methods for estimating rates of vorticity transport to complete a hermetic accounting of vorticity transport in the flat plate wakes. The accounting of vorticity captured by shed vortices is not complete without an investigation into the rates at which vorticity is generated, and the rates at which vorticity accumulates within shed vortices. By better understanding these interrelationships, more accurate assessments of mechanisms related to vorticity diffusion, cancellation, and annihilation can be achieved. Consequently, this objective contributes towards a better understanding of the processes that determine shed vortex strengths.
1.1.2 Thesis structure

This thesis will be structured as a hybrid traditional/manuscript based thesis, i.e., it will include unpublished work, as well as work that has been published in the International Journal of Heat and Fluid Flow. Literature and background information required to contextualize/frame the new developments, understandings, and contributions, will be provided at the beginning of chapters 3, 4, and 5. This choice was a conscious one, chosen to improve the manuscript’s readability by keeping relevant and specialized literature/historical methods nearby the sections where they are most pertinent.

Chapter 2 will briefly detail the experimental setup and methodology.

Chapter 3 will present a comprehensive overview of the two flat plate flows. The introduction will detail the impact of end conditions on bluff body cylinders and establish an interest for an investigation into the impacts of end conditions on thin flat plates. Subsequently, the flow’s mean two-dimensionality will be established, and a comparative overview of the wake characteristics provided. Key coherent motions are identified and related to the wake structure.

Chapter 4 presents an analytical method for estimating the vorticity associated with a shed vortex, and also evaluates the abilities of three different methods to estimate vortex centers and convection speeds. The chapter begins with a review of several different vortex identification methods and subsequently presents the need for a new way to estimate the total vorticity associated with a shed vortex. It will be shown analytically that the traditional methods capture at most 71% of the circulation contained in Burgers and Lamb-Oseen vortices. The limiting ratio is then validated from experimental results.

Chapter 5 presents three different methods for estimating rates of vorticity transport in mean 2D wakes. The chapter begins by reviewing several heuristic vortex shedding models; from two of these models, estimates of vorticity generation at the flat plate edges are made possible. Next, the concepts of vorticity cancellation and annihilation are introduced for interpreting the evolution of the forming vortices. Subsequently, three methods for calculating
vorticity transport rates are detailed, tested, and compared. Finally, evidence for increased turbulence and mixing in the flat plate wake with end conditions is presented.

Chapter 6 will detail the major conclusions as well as areas for future work and research.

The Appendix is composed of five subparts. Appendix A provides an overview of supplementary proper orthogonal decomposition modes, not shown in Chapter 3. Appendix B details two modes that appear to be related to the vortex shedding modes, and an additional antisymmetric mode, referred to as the flapping mode. Appendix C shows an application of the Q-criterion being applied to a Lamb-Oseen vortex. Appendix D outlines the pressure field estimation method that was employed in this study. And in Appendix E, code for a graphical user interface that was designed for the manual selection of vortex boundaries is provided.
Chapter 2

Methodology

The following chapter was taken from a subsection of a paper published in the International Journal of Heat and Fluid Flow (IJHFF) and has been lightly modified. This paper was co-authored by Kaden B. Agrey, and Robert J. Martinuzzi (Braun et al., 2020).

The near-wake flows behind high aspect ratio thin flat plates normal to a uniform stream of velocity, \( U_\infty \), are considered for two conditions: open ends (OE), without end plates; and closed ends (CE), with end plates. Experiments were conducted in an open jet suction-type wind tunnel. The tunnel inlet had a diameter of 3 m. The conditioning section consisted of a honey-comb and 7 sieves. The air flowed through a gradual 36:1 area-ratio contraction. The working section of the wind tunnel was a 0.5 m-diameter jet in which the test-plate was mounted. The free stream velocity was monitored with a \( \frac{1}{4} \) in diameter Pitot-static tube mounted at the working section inlet penetrating the jet about 5 cm. The differential pressure transducer was a Dwyer Instruments Inc. Magnehelic 2000-OC. Based on the manufacturer’s calibration data, the Pitot-static tube accuracy is 0.25%.

The flow geometry, measurement window, and nomenclature are schematically shown in Fig. 2.1. The Cartesian coordinates in the streamwise, flow-normal, and spanwise directions are denoted \((x,y,z)\) with their corresponding velocity components \((u,v,w)\), respectively.

The thin test-plates consisted of machine-flat steel. For the open end case, the test-
Figure 2.1: Schematic representation of the geometry and nomenclature, showing the PIV field of view (FoV): a) side view b) top view.

The test plate is from a previous study (Mohebi, 2016) and had a chord of $c = 13.0\text{mm}$, a span of $s = 560\text{mm}$, such that it extended out of the flow, and a uniform thickness of $0.05c$. The wetted span-to-chord ratio was 38 and the blockage ratio, based on the frontal area, was 0.034. For closed ends, the test-plate had a chord of $c = 22.4\text{mm}$, a wetted span of $s = 508\text{mm}$, and a mid-point thickness of $0.2c$ with beveled edges. The wetted span-to-chord ratio was 23 and the blockage ratio was 0.057. The square rigid end plates were made of 1.9mm thick steel and were mounted normal to the test plate at the edge of the working jet at $z = \pm 11.5c$. For this end plate spacing, results from Fox and West (1990) indicate mean 2D flow over $-4.5 < z < 4.5$ for a circular cylinder. The plates extended to $x = -c$ upstream, $y = \pm 3c$ and $x = 5c$ downstream. The spanwise base pressure distribution was found to be sensitive to end plate orientation. The end plates were adjusted to an expanding angle of $\sim 2^\circ$ relative to the free stream to minimize spanwise gradients of the mean base pressure.

Surface pressure measurements were made at 5 locations on the leeward face of the larger plate ($z/c = 0, \pm 4, \pm 10; y/c = 0, \pm 0.1, \pm 0.2$). The pressure taps were 0.5 mm diameter holes drilled into 2 mm inner-diameter copper tubes embedded into the plate. The copper tube
ends were connected with a Tygon tube to the high-pressure side of All Sensor Corporation 5-INCH-D1-4V MINI pressure transducers. The low-pressure side of the differential transducers connected via a manifold to the static pressure of the wind tunnel’s Pitot-static tube to ensure a common reference. The pressure was sampled at a rate of 10 kHz with a 24-bit National Instruments NI9234 interface using an in-house LabView code. The pressure signals were digitally filtered using an 8th-order Butterworth filter applied forwards and backwards to eliminate the phase lag.

Time-resolved, three-component planar velocity fields were acquired using stereo particle image velocimetry (PIV) in dual-frame mode. The LaVision PIV system consisted of a Photonics Industries 20 mJ Nd:YLF laser (wavelength 532 nm), producing a ≈ 2 mm laser sheet, and a pair of Photron SA4 Fastcams (1024 × 1024 pixels). Cameras were mounted at angles of ±45° relative to the y = 0 plane. The flow was seeded with olive oil particles generated with six Laskin nozzles and introduced upstream of the tunnel inlet. The particles number-mean diameter was approximately 1µm. The effective field of view (FoV) was approximately 5.5c × 5.5c. For the closed end case, a singleFoV was used extending from x ≈ 0.5c. For the open end case, the recirculation length was significantly longer and two FoV were used. The first extended from x ≈ 0.5c and a second, overlapping FoV extended from x ≈ 3.4c. The two FoVs are schematically shown in Fig. 2.1.

For each measurement, data were acquired for three independent trials of 2728 snapshots (image-pairs). The snapshots were sampled at a rate of 1 kHz allowing 10 to 11 points per shedding cycle. Each acquisition window (one trial) spanned about 230 shedding cycles, or about 700 cycles for three trials.

PIV images were processed with DaVis Flow Master 8.3 software. A two-pass frame-straddled processing was performed. The final interrogation window was 32 × 32 pixels, with 50% overlap for a vector spacing of ∼ 0.08c. Following the procedure of (Raffel et al., 2013), statistical convergence required two trials and the estimated overall uncertainty was approximately 1% and 6%, based on the maximum values, for the mean velocity and Reynolds
Stresses, respectively, within a 95% confidence level.

Experiments are reported at $U_\infty = 14.8$ m/s (CE) and $8.7$ m/s (OE), giving a $Re = U_\infty c/\nu$, with $\nu$ the kinematic viscosity, of 20000 (CE) and 6600 (OE). Nominal ambient conditions were 20°C and 90 kPa. For both cases, the on-coming flow was uniform to within 0.2% over 97% of the test-stream and the free stream turbulence was less than 0.5%. The shedding frequency, $f_{sh}$, corresponded to $St = f_{sh}c/U_\infty = 0.142 \pm 0.002$ (CE) and $0.119 \pm 0.002$ (OE). The mean base pressures were $C_{P_b} = \frac{P_b - P_\infty}{\frac{1}{2}\rho U_\infty^2} = -1.22 \pm 0.03$ (CE) and $-0.93 \pm 0.03$ (OE), with $\rho$ the density. $St$ and $C_{P_b}$ for closed ends match earlier studies (Fage and Johansen, 1927; Leder, 1977). In Chapter 3, velocity, Reynolds stress, circulation, vorticity, and area are presented non-dimensionalized by $U_\infty$, $\rho U_\infty^2$, $(U_\infty c)^{-1}$, $U_\infty/c$, and $c^2$, respectively.

For open ends, Mohebi (2016) found that the mean velocity and Reynolds stress profiles were insensitive for $Re \geq 6600$. Representative data are shown in Figure 2.2. This figure shows $y$-profiles of non-vanishing mean quantities ($U/U_\infty$, $V/U_\infty$, $\overline{u'^2}/U_\infty^2$, $\overline{v'^2}/U_\infty^2$, $\overline{u'v'}/U_\infty^2$) at an arbitrarily chosen downstream location, and $U/U_\infty$ distributions along $y = 0$. For $Re$ of 6600 and above, the measured distributions collapse onto a single curve within the experimental uncertainty. It is seen that the distributions measured for 3000 and 4500 deviate from those at higher $Re$.

The velocity field was analyzed subject to a triple decomposition into mean, coherent, and residual contributions:

$$\mathbf{u}(x, t) = \mathbf{U}(x) + \mathbf{u_c}(x, t) + \mathbf{u''}(x, t)$$

where $\mathbf{u'}$ represents the total fluctuations. Upright bold symbols indicate vectors. The argument $\mathbf{x}$ is the position vector and $t$ time. The coherent contribution is determined from

\footnote{Chapter 3 refers to work published in the IJHFF, and thus the notation used in the publication is retained.}
Figure 2.2: Mean velocity and Reynolds stress distributions at different Reynolds numbers as reported in Mohebi (2016); $y$-profiles at $x/c = 4$, centerline profiles at $y = 0$. Reynolds number and symbol: 3000, Mean velocity and Reynolds stress profiles at $x/c = 5.6$, $z/c = 0$. Reynolds number and symbol: 3000, $-$; 4500, $-$; 6600, $¥$; 8000, $¥$; 9600, $¥$; 10800, $¥$; 12000, $¥$.

A proper orthogonal decomposition (POD) (Holmes et al., 2012) of the flow field:

$$u_c(x, t) = \sum_{n=1}^{N} a_n(t) \Phi_n(x),$$  \hspace{1cm} (2.2)

where $a_n$ and $\Phi_n$ are temporal and spatial modal functions. In subsequent chapters, the arguments $x$ and $t$ are implied for compactness.

The spatial functions are orthonormal, satisfying the inner product:

$$(\Phi_n, \Phi_m) = \int \int_{\Lambda} (\Phi_n \Phi_m) dA = \delta_{nm},$$  \hspace{1cm} (2.3)

where $\Lambda$ is the PIV observation domain. The total kinetic energy of the fluctuations is thus given by:

$$\text{TKE} = \frac{1}{2} \sum_{n} \lambda_n , \quad \lambda_n = a_n^2 ,$$

where $\lambda_n$ are the empirical eigenvalues of the fluctuating velocity’s covariance matrix.
The low-order (coherent) representation consists of \( N = 8 \) highly energetic modes. The coherent and residual contributions to the Reynolds stress tensor are defined as \( \overline{u_i u_j} \) and \( \overline{u''_i u''_j} \), respectively. By construction, their summation yields the total (Reynolds averaged) Reynolds stress field:

\[
\overline{u'_i u'_j} = \overline{u_i u_j} + \overline{u''_i u''_j}.
\]

The \( N \) modes were determined based on the following considerations. First is the contribution to the TKE (a measure of the contribution to the Reynolds normal stresses). Second is the contribution to the Reynolds shear stresses. Third is the existence of inter-modal relationships seen in phase-space as discussed in Chapter 3, Section 3.2.4. Finally, the mode statistical significance was established by comparing the co-linearity of the modes between different trials (inner product, Eq. (2.3) > 0.95). Note that modes not satisfying the latter criterion had contributions below the measurement uncertainty. For the final analysis, the modes were calculated from the three combined trials. Raw data and low-order representations were compared for randomly selected time intervals to verify that the vortex centroid locations and trajectories were well approximated. In the remainder of the text, “significant” identifies contributions that are statistically significant and result in observable changes in the \( \overline{u_i u_j} \) distributions.

Variables for conditionally averaged shedding cycles are presented as ensemble phase averages:

\[
\langle \Psi \rangle(x, \phi) = \frac{1}{N_p} \sum_{n=0}^{N_p-1} \Psi(x, \left( \frac{\phi}{2\pi} + n \right) \tau)
\]

where \( \Psi \) is the flow variable, \( N_p \) is the number of samples in each of 36 phase bins, and \( \tau \) is the shedding period. The phase \( \phi \) is determined using \( a_1 \) and \( a_2 \), the two most energetic POD temporal coefficients containing the fundamental shedding frequency, according to

\[
\phi = \arctan \left( \frac{a_2}{a_1} \right).
\]
The circulation of the shed vortex cores was estimated according to

\[ \Gamma_v = \iint_{A_v} \langle \omega_z \rangle \, dA, \]  

(2.4)

where

\[ \langle \omega_z \rangle = \frac{\partial \langle v \rangle}{\partial (x/c)} - \frac{\partial \langle u \rangle}{\partial (y/c)} \]

is the phase-averaged vorticity. For experimental measurements, the cores are first identified using the Q-criterion (Hunt et al., 1988). Letting \( A_v \) be the area enclosed by a Q iso-contour (set to \( Q = 0 \) unless otherwise stated), the estimated core strength is computed from (2.4). For shed vortices, it is noted that approximately 70% of the circulation associated with the vortex is captured when using \( Q = 0 \) to define \( A_v \), i.e., \( \frac{\Gamma_v}{\Gamma_{tot}} \approx 0.7 \) (where \( \Gamma_{tot} \) is the total circulation associated with a vortex). This result is consistent with the Burgers vortex model, as described in Chapter 4, Section 4.2.1. It is noted that results using the \( \lambda_2 \)-criterion (Jeong and Hussain, 1995) are indistinguishable from those obtained with the Q-criterion.

To remain faithful to the original IJHFF publication (presented in Chapter 3), \( \Gamma_v \) is used ubiquitously throughout Chapter 3 as a representative value for the total vortex strength (\( \Gamma_{tot} \)), since the ratio \( \frac{\Gamma_v}{\Gamma_{tot}} \approx 0.7 \) is nearly constant, and the value of Q is relatively insensitive to small perturbations due to uncertainty.
Chapter 3

End effects of nominally two-dimensional thin flat plates

The following chapter is a modified version of work published in the International Journal of Heat and Fluid Flow (IJHFF), and was co-authored by Kaden B. Agrey, and Robert J. Martinuzzi (Braun et al., 2020). In particular, certain sections that were previously covered in the IJHFF publication have since been elaborated on, and are now presented as full chapters within this thesis. In addition, the text has been modified as needed so that the terminology is consistent with the remainder of the thesis. Primarily, the term “vorticity flux” has been modified to be “the rate of vorticity transport” in this chapter.

3.1 Introduction

Experimental studies have shown that achieving mean two-dimensional (2D) wakes is subject to subtle effects, even for high aspect ratio bluff bodies. The addition of carefully aligned end plates can reduce three-dimensional effects substantially. The spanwise end condition can locally affect the separation process from the obstacle faces, thus altering the rate of vorticity transport to the wake and hence the vortex formation process. These changes can result in spanwise gradients of the mean base pressure and wake velocity. For the case of
the thin flat plate normal to a uniform stream, the effects of end conditions appear more subtle. Differences in the mean wake structure and shedding frequency are observed when comparing experimental studies, or numerical simulations, even though spanwise gradients in the wake are reported as negligible. It is thus fundamentally interesting to understand the differences in the wake dynamics when different end conditions result in different, but still 2D, mean wakes. In this work a thin flat plate normal to a uniform stream is considered for open (no end plates) and closed ends (with end plates). The aspect ratio is sufficiently large so that the flow spanwise gradients are negligible for both cases. However, differences in mean flow topology and dynamics are observed and investigated.

For 2D circular cylinders, as an example of bluff bodies without fixed separation points, the influence of end plates for reducing three-dimensionality is well understood (Fox and West, 1990; Hammache and Gharib, 1991; Prasad and Williamson, 1997; Williamson, 1989; Williamson and Roshko, 1990). The free end condition can alter the local base pressure causing the location of separation on the cylinder surface to change (Williamson and Roshko, 1990) and affect the stability of the separated shear layer (Prasad and Williamson, 1997). These changes can lead to predominantly oblique shedding of vortices. Oblique shedding is related to changes in the Strouhal number (Williamson, 1989), a small mean spanwise velocity component and pressure gradient (Hammache and Gharib, 1991). Hammache and Gharib (1991) note that when parallel shedding occurs, the mean base pressure along the span of the body is symmetric and the mean spanwise velocity component vanishes. Thus, carefully aligned end plates force the necessary boundary conditions for parallel shedding.

For rectangular cross-section, 2D bluff bodies with sharp leading edges mounted with a face normal to a uniform stream, the separation points are generally fixed at the leading edges and spanwise end conditions are not expected to affect the vortex formation process (Lee, 1991; Roshko, 1954a). Defining the obstacle spanwidth as $s$, the flow normal dimension $c$ (chord) and the streamwise thickness $b$, for cases where the mean flow does not reattach on the obstacle surfaces, $b/c \lesssim 2.3$ (Okajima, 1982), a comparison of results from experimental
studies over a wide range of $s/c > 10$ and different end conditions suggest this expectation holds for $0.4 < b/c < 2.3$ (Durao et al., 1988; Knisely, 1990; Lee, 1991; Mohebi et al., 2017; Norberg, 1993; Obasaju, 1979; Ohya, 1994). For a given $b/c$, reported (blockage corrected) values for the midspan mean base pressure, sectional drag and shedding frequency from these studies agree within experimental uncertainty and the flow about the midspan can be considered two-dimensional.

Comparing experimental results for rectangular cylinders below the critical thickness, $b/c \approx 0.3–0.4$ (Knisely, 1990; Mohebi et al., 2017; Norberg, 1993), shows large discrepancies. For nominally thin plates, $b/c < 0.2$, with large $s/c$ (Fage and Johansen, 1927; Fail et al., 1959; Kiya and Matsumura, 1988; Knisely, 1990; Leder, 1977; Lisoski, 1993; Mohebi et al., 2017), differences of up to 15% are observed in the shedding frequency or drag coefficient and 30% for the mean recirculation length. Regardless of the end conditions, separation is reported to be fixed at the plate leading edges.

The numerical prediction of infinite thin-flat plate flows using steady and unsteady Reynolds-Averaged Navier-Stokes (RANS) single-point closures has generally proven unsatisfactory (Dahlqvist, 2016; Tian et al., 2013). Simulations using Direct Navier Stokes (DNS) or Large-Eddy Simulation (LES) (Hemmati et al., 2016, 2018; Najjar and Balachandar, 1998; Narasimhamurthy and Andersson, 2009) more closely approach the experimental results of Fage and Johansen (1927) and Leder (1977), although the shedding frequency and drag are over-predicted by 10% to 15%. A comparative study (Hemmati et al., 2018) notes that simulation results are relatively insensitive to Reynolds numbers above 1000, based on the on-coming stream velocity and $c$. However, the predicted base pressure and low frequency dynamics showed great sensitivity to the implementation of boundary conditions and solution methodology.

The wake structure and dynamics for the thin plate differ significantly from those for circular or square cylinders (Balachandar et al., 1997). Differences have been attributed to the stabilizing effect of the afterbody (Balachandar et al., 1997; Mohebi et al., 2017),
the portion of the body downstream of the separation. In particular, the importance of low-frequency unsteadiness in the thin-plate wakes has been discussed extensively based on DNS studies (Hemmati et al., 2016; Najjar and Balachandar, 1998; Narasimhamurthy and Andersson, 2009). This unsteadiness, most easily observed in the recirculation region, corresponds to a broadband energy accumulation centered about 1/10 of the shedding frequency in velocity spectra. The low-frequency dynamics have not been studied in detail experimentally. Reported spectra (Fage and Johansen, 1927; Kiya and Matsumura, 1988; Leder, 1977; Lisoski, 1993; Mohebi et al., 2017) do suggest that the difference in the reported base pressure and mean recirculation length coincide with differences in the low-frequency spectral signature. These observations warrant a more detailed analysis of the low-frequency dynamics for different end conditions.

Low-frequency modulations of the shedding process have been shown to involve energy exchanges between coherent motions. For laminar, 2D cylinder wakes Noack et al. (2003) used an analysis based on proper orthogonal decomposition. They show that the dynamics resulted from an energy exchange between shed vortices and a slow-varying base flow and could be modeled as a Stuart-Landau mean-field oscillator. Bourgeois et al. (2013) extend this approach for quasi-periodic, three-dimensional turbulent wakes. Herein, this approach is used to investigate the low-frequency dynamics in the flat plate wake.

In this study, the flat plate wake is investigated for two end conditions: with and without end plates. A modal decomposition of the wake velocity fluctuations shows the existence of two low-frequency modes of motion. Their relationship to the shedding process is examined and related to differences in the Reynolds stress field and mean wake velocity field structure.

3.2 Results

The quasi-periodic, turbulent near wake of a nominally 2D thin flat plate normal to a uniform stream is investigated for open (OE), without end plates, and closed (CE), with end
plates, end conditions. The flow mean two-dimensionality is considered in §3.2.1. Differences between the mean wakes for the two end conditions are briefly identified from the mean field in §3.2.2. Differences in global quantities, such as the Strouhal number, mean base pressure, and vorticity thickness, are considered in §3.2.3. Subject to a POD modal analysis, slow-varying contributions to the wake dynamics are characterized in §3.2.4. Finally, their influence on the turbulent wake structure is discussed in §3.2.5.

3.2.1 Two dimensionality and phase uniformity

Mean two-dimensionality was verified for both flows. Figure 3.1 compactly summarizes $U$, $V$ and $C_{p_z}$ $z$-distributions and $U$, $V$, $u'^2$ and $u'v'$ $y$-distributions for different spanwise locations. It is observed that spanwise variations are negligible within experimental uncertainty. The $y$-distributions at different $z/c$ collapse onto a single curve and show the general symmetry about $y = 0$. It was verified that $W$, $u'w'$ and $v'w'$ vanish within the experimental uncertainty and are not shown for brevity.

Figure 3.2 shows a sample cross-correlation between pressure at the midspan and $z = 10c$ on the leeward face of the plate. The distributions are consistent with predominantly parallel shedding. The maximum correlation occurs at zero-lag and, within a small tolerance, correlation peaks are one period apart. By inspection, the instantaneous pressure signals are generally in phase.

3.2.2 Mean wake

Mean streamlines overlaying flooded contours of the principal Reynolds stresses ($u'^2$, $v'^2$, $u'v'$) in the central plane, $z = 0$, are shown together with their coherent and residual field contributions in Fig. 3.3. Also shown are the $u'^2$, $v'^2$, and $w'^2$ $y$-distributions at the $x$-locations of maximum $u'^2$ and $v'^2$. The mean streamlines are calculated by integration of the velocity along $y$-profiles, outward from the symmetry plane $y = 0$, and correspond to loci of constant $\dot{m} = \int U dy$ (Castro and Haque, 1987; Mohammed-Taifour and Weiss, 2006).
The separating streamline ($\dot{m} = 0$) is shown as a thicker line. The mean field shows mirror-symmetry about $y = 0$, Fig. 3.1, and hence Fig. 3.3 presents results for the closed end flow in the top half-plane and for the open end in the bottom, respectively.

While the streamline topology is qualitatively similar for both cases, the recirculation length measured along $y = 0$ to the saddle point at $x = \ell_R$, is significantly longer in the open end case, $\ell_R = 3.9c$, than for the closed end, $\ell_R = 2.4c$. Simple scaling considerations could not account for the structural differences between the recirculation regions. For example, the recirculation nodes at $(x_n, y_n) = (1.25c, \pm 0.43c)$ (CE) and $(2.53c, \pm 0.46c)$ (OE) are at different relative streamwise locations at $x_n = 0.52\ell_R$ and $x_n = 0.65\ell_R$, respectively. In contrast, the maximum outward deviation of the separating streamline from $y = 0$, at $(x_s, y_s) = (1.01c, \pm 0.77c)$ (CE) and $(1.66c, \pm 0.89c)$ (OE), correspond to similar relative streamwise locations ($x_s = 0.42\ell_R$ and $x_s = 0.43\ell_R$), such that the relative location of these two features are not the same, regardless of scale. It is also noted that $y_s/y_n \approx 1.84$ for both cases,
suggesting different scaling in the streamwise and flow-normal directions. Moreover, the curvature of the separation streamline, extending from the leading edges along the shear layer to the wake saddle point, is smaller for the open than the closed end case. Additional differences in the wake structure are discussed in relation to the Reynolds stresses below.

The magnitude of the Reynolds stresses (Reynolds averaged) are generally much larger with closed than with open ends. Downstream of the recirculation nodes, the main contribution to the Reynolds stresses in both cases is from the coherent motion, ostensibly due to vortex shedding. In this region, in contrast to the Reynolds averaged field, the three residual field normal stresses are similar in magnitude, implying that many of the differences observed in the Reynolds stress (Reynolds averaged) distributions can be addressed by considering the coherent field dynamics. Differences in the vortex formation process are suggested by the observations that: (i) the location of the maxima for $u'^2$ and $v'^2$ are located within the recirculation region for the closed end case, but outside for the open ends; (ii) in the base region upstream of the recirculation nodes, the magnitude of the Reynolds stresses are generally high for the closed, but very low for the open end cases, respectively. In the base region of both flows, very close to the plate, $u'v'$ changes sign. This inversion is associated with the coherent contribution as it is not observed in the residual field. For the open end case,
Figure 3.3: Reynolds stress contours with overlaid mean streamlines. Top half-planes with closed ends, bottom half with open ends. The separating streamline is denoted by a thicker line terminating at the saddle point at the end of the mean recirculation \((x = \ell_R, y = 0)\). Filled symbols are for closed ends, open symbols for open ends. Yellow circles and triangles indicate points of maximum \(u'^2\) and \(v'^2\), respectively. The first row shows the Reynolds averaged field; second the coherent field; third the residual field. Colormap levels change across the horizontal line. Final two columns show profiles at \(x\)-location of maximum \(v'^2\) and \(u'^2\); \(\cdots\) \(\overline{u'^2}\) as indicated in the first row. \(\cdots\) \(\overline{v'^2}\), \(\cdots\) \(\overline{w'^2}\).

the coherent contribution to the Reynolds stress strongly dominates and, upstream of the recirculation nodes, the residual field contributes negligibly to the Reynolds stresses. These observations motivated a detailed investigation of how the vortex formation and shedding process differ for the two cases.

Spectra of the \(u'\) and \(v'\) velocity components at locations of maximum \(\overline{u'^2}\) and \(\overline{v'^2}\) are shown together with their auto-correlations at the location of maximum \(\overline{v'^2}\) in Fig. 3.4. Strong periodic contributions associated with vortex shedding occur at frequencies corresponding to \(S_t = 0.119\) (OE) and 0.142 (CE). In contrast to the closed end case, the open case shows significant spectral broadening and a larger energy accumulation at low frequencies,
consistent with a more rapid decay of auto-correlations in periodic fluctuations \((R_{u'u'})\). In contrast, auto-correlations of the non-periodic fluctuations \((R_{u'u'})\) decay slower for the open end case. These results indicate greater cycle-to-cycle variation in the wake for the open than for the closed end case and will be related to stronger low-frequency coherent variations in the open end vortex shedding process.

Figure 3.4: a) Power spectral density (PSD) function of \(u'\), \(v'\) at max \(\overline{u'^2}\) (left), and \(\overline{v'^2}\) (right). Successive spectra are offset by \(10^{-5}\) and a reference line at \((10^{-4},\text{absolute})\) is added to each for clarity. b) Auto-correlations of \(u'\) (top), \(v'\) (bottom) at max \(\overline{v'^2}\). \(u'^2_{\text{max}}\) is at \((x,y) = (1.5c,0.5c)\) (CE) and \((3.4c,-0.8c)\) (OE) and \(v'^2_{\text{max}}\) is at \((x,y) = (2.1c,0c)\) (CE) and \((4.9c,0c)\) (OE). OE (green); CE (black).

### 3.2.3 Global quantities

The lower mean base pressure for the closed end case, \(C_{Pb} = -1.22\) (CE) vs. -0.93 (OE), is consistent with the shorter recirculation length \(\ell_R\) and the increased streamline curvature seen in Fig. 3.3. For a mean 2D flow, the separation streamline originating at the plate corners must terminate at the saddle point at \(x = \ell_R\), \(y = 0\) (Castro and Haque, 1987). This observation implies that the vorticity generated at the separation point is contained in the plane, such that a simpler 2D analysis is representative of the downstream evolution of the vorticity transport.

The local rate of change of circulation is related to the mean rate of vorticity transport according to\(^1\):

\[
\frac{d\Gamma}{dt} = - \int \overline{\omega_z u_x} \, d(y/c).
\]

\(^1\)For more information on the implementation of (3.1), the reader is directed to Chapter 5, Section 5.3.2.
Figure 3.5 shows the vorticity transport rates as a function of the streamwise location $x/\ell_R$ for the two cases, together with their ratios.

![Figure 3.5: Mean rate of vorticity transport $d\Gamma/dt$ for closed end (CE) and open end (OE) conditions and their ratio (CE/OE). To allow comparison over similar domains, data from the extended open ends field were included (OE$_{ext}$).](image)

The mean rate of vorticity generation at the separation points is expected to be proportional to $\frac{1}{2}(1-C_P_b)$ (Ahlborn et al., 1998; Roshko, 1954b), which is consistent with the higher $d\Gamma/dt$ observed immediately downstream of the plate for the closed end case. The rate of vorticity transport along the separated flow region decreases more rapidly for the closed end condition such that the (CE)/(OE) vorticity transport rate ratio approaches unity by $x \approx \ell_R$, which is close to the location of the local centerline $u'^2$ maximum, corresponding to the vortex formation length (Griffin, 1995).

The mean rate of vorticity transport downstream of the formation region is mainly attributed to the circulation $\Gamma_v$ transported by the shed vortices at an average rate of $d\Gamma/dt = \Gamma_v S_t$, within experimental uncertainty, with ratio (CE)/(OE) $\approx 0.97$. The slow decay of $d\Gamma/dt$, downstream of $\ell_R$, is due to the vorticity diffusion of the shed vortices. The difference between the two cases in the strength $\Gamma_v$ of the shed vortices and the rate of vorticity transport for $x < \ell_R$ suggests difference in the formation region dynamics.

The vorticity thickness is introduced as a measure of the shear layer spreading rates (Cas-
tro and Haque, 1987) and is defined:

\[ \delta_w = \frac{U_1 - U_2}{|\frac{\partial U}{\partial y}|_{\text{max}}} , \]

where \( U_1 \) is the maximum \( U \), at the shear layer edge, and \( U_2 \) is the minimum \( U \), at \( y = 0 \). \( |\frac{\partial U}{\partial y}|_{\text{max}} \) coincides with the \( U \) \( y \)-profile inflection point. Figure 3.6 shows \( \delta_w \) as a function of the relative streamwise location \( x/\ell_R \) allowing direct comparison of the different flow regions. In the recirculating region, the spreading rate is significantly higher for the closed end then the open end case, suggesting a strong mixing of higher-momentum fluid consistent with the observed higher drag (lower \( C_P b \)) for the closed end case. The higher spreading rate also implies a greater mean rate of vorticity transport towards \( y = 0 \) to interact with opposite-sign vorticity from the opposing shear layer, leading to a higher vorticity annihilation rate consistent with the more rapid decay rate of \( d\Gamma/dt \) seen in Fig. 3.5 for the closed end case. Downstream of the recirculation, the \( x \)-evolution of \( \delta_w \) is approximately linear for both cases, but the slopes differ significantly: \( d\delta_w/dx \approx 0.17c/\ell_R \) (CE) vs. \( 0.11c/\ell_R \) (OE). The similarity in evolution suggests similar dynamics past the formation region, which is expected as the flow far downstream must approach the classical 2D far-wake similarity solution such that the \( x \)-evolution is independent of the wake-generating body (Wygnanski et al., 1986).

Flooded iso-contours of \( \langle \omega_z \rangle \) are shown with streamlines for closed and open ends in Fig. 3.7 at the same phase in the respective averaged shedding cycles. At this phase, the vortex along the lower shear layer, \( \langle \omega_z \rangle > 0 \) (red), is in the process of shedding. The streamline saddle point upstream of the shedding vortex moves with the vortex such that there is no net vorticity transport across it. Hence, the saddle point indicates that the shedding vortex is no longer fed vorticity from the lower shear layer. The forming vortex along the upper shear layer, \( \langle \omega_z \rangle < 0 \) (blue), penetrates deeply the base region and interferes with the vorticity transport along the lower shear layer, leading to the shedding event. When compared to the closed end case, the open end formation length is much longer. The vortices
Figure 3.6: Vorticity thickness $\delta_w/c$ as a function of downstream distance $x/\ell_R$ for closed (CE) and open end (OE) cases. To ensure that the comparison domain was the same, data from the extended open ends field was also used ($\text{OE}_{\text{ext}}$).

form and shed significantly further downstream from the plate. The saddle points form at $x \approx 0.8\ell_R$ (OE, $\ell_R = 3.9c$) versus $0.6\ell_R$ (CE, $\ell_R = 2.4c$). The curvature of the feeding shear layer is also less for the open end case.

The average trajectory of the vortex cores, represented by the vorticity centroid in the region enclosed by the $Q = 0$ contour, is also shown in Fig. 3.7. The separation between the trajectories of the opposing vortices decreases inside the formation region and increases as the vortex is shed. For the closed end case, the separation increases more rapidly downstream, which is consistent with a greater spreading rate as inferred from $\delta_w$ in Fig. 3.6.

### 3.2.4 POD analysis

By construction, the phase average does not account for cycle-to-cycle variations and is thus an incomplete description of the shedding dynamics. As observed from Fig. 3.4, spectra for the open end case show greater spectral broadening and the velocity fluctuation correlations show a faster decay in periodic but a slower decay in non-periodic contributions when compared to the closed end case. These differences suggest that the cycle-to-cycle variations are important for describing the different dynamics for these two cases. Here two slow-varying coherent motions related to the wake dynamics are found for both end conditions. These
Figure 3.7: Flooded iso-contours of the phase-averaged vorticity field $\langle \omega_z \rangle$ are shown with velocity streamlines; both fields are shown at the same phase. Within the fields, $|\langle \omega_z \rangle|_{\text{max}} = 6.15$ (CE), and $|\langle \omega_z \rangle|_{\text{max}} = 3.94$ (OE). To better show OE $\langle \omega_z \rangle$ detail, the color bar limits were set to ±4.6. Closed green lines indicate $Q = 0$ and enclose region $A_v$. Purple lines indicate the trajectory of the vorticity centroids, $(x_c, y_c) = \frac{1}{\Gamma_v} \iint (x, y) \langle \omega_z \rangle \, dA_v$. Streamlines are in a convective reference frame.

motions are most strongly associated with two distinct POD modes. The first identified mode resembles the shift mode described by Noack et al. (2003), and characterizes a slow expansion and contraction of the recirculating region. Hereafter these will be referred to as the drift mode and motion. The second mode characterizes a lateral flapping of shed vortex trajectories and is hereafter called the flapping mode.

Figures 3.8 and 3.9 provide a succinct summary of key aspects of the POD analysis$^2$. The eight POD modes representing the coherent motion account for about 56% (CE) or 68% (OE) of the TKE. From the modal energy distribution in Fig. 3.8, the two most energetic modes, representing 46% (CE) and 54% (OE) of the TKE, show very strong spectral energy concentrations at the shedding frequency. Iso-contours of their $u$-spatial functions, $\Phi_1^u, \Phi_2^u$ – not shown for brevity, show the classical anti-symmetric distribution about $y = 0$ and nearly-periodic spatial patterns in the $x$-direction associated with the periodic motion of vortex shedding (Noack et al., 2003). These are a fundamental harmonic pair, with temporal coefficients $a_1$ and $a_2$, as deduced by their phase relation in Fig. 3.9a. Within any arbitrary

$^2$Appendix A presents additional POD modes of the CE and OE flows.
shedding cycle, the amplitude \( A = \sqrt{a_1^2 + a_2^2} \) is nearly constant, but \( A \) varies between cycles. This amplitude modulation is significantly greater for the open end case. In the figure, the red line closely approximates \( A \) for the average shedding cycle. A mode pair, \( a_3 \) and \( a_4 \), corresponding to the second harmonic with spectral energy concentration at \( 2f_{sh} \), is observed in both fields (not shown for brevity). Their \( u \)-spatial functions are symmetric about \( y = 0 \) and show streamwise spatial periodicity at about half the wave length observed for the fundamental pair. While the second harmonic pair contributes only about 3% of the TKE, its contribution to \( \overline{u'^2} \) is significant, especially in regions about the symmetry plane (Bourgeois et al., 2013; Hosseini et al., 2015).

Figure 3.8: Flooded isocontours and pseudo-streamlines for spatial functions together with spectra of temporal coefficients for the slow-drift, \( a_{\Delta} \) (top), and flapping, \( a_f \) (bottom), modes. CE is on the left, OE on the right. Note that, for the OE field, \( a_f \) is taken from the extended domain. Spatial modes are orthonormal: \( (\Phi_n, \Phi_m) = \delta_{nm} \). Right most: relative contribution to the TKE. \( a_1, a_2 \) fundamental harmonic pair. \( a_3, a_4 \) second harmonic.

The most energetic mode with a symmetric \( u \)-spatial modal function, \( \Phi_{\Delta}^u \) shown in the top row of Fig. 3.8, is associated with low-frequency energy concentration in spectra of \( a_{\Delta} \) in Fig. 3.8. The amplitude of this slow-varying mode, \( a_{\Delta} \), is strongly coupled with the oscillation amplitude \( A \) of the fundamental harmonic pair. From the \( a_{\Delta}, a_2 \) phase portraits of Fig. 3.9b, the relationship takes the form of a paraboloid, \( a_{\Delta} \approx c_0 + c_1A^2 \), with \( c_0 \) and
Figure 3.9: Phase portraits for CE (left) and OE (right) for: a) harmonic pair $a_2$ vs. $a_1$; b) $a_2$ vs. $a_\Delta$, ($a_1 \approx 0$); and c) $a_2$ vs. $a_f$, ($a_1 \approx 0$). Note: $a_f$ is taken from the extended OE field. Solid red lines indicate a regression in b) and in a) the limit cycle ($a_\Delta = 0$). Dashed red lines in b) show bin averages for high, limit, and low cycles. The modal coefficients are scaled by $\sqrt{\lambda_n}$.

c_1 constants. Note that the parabolic regression line is not indicative of the trajectory in the phase space since, during a typical cycle, $a_\Delta$ and $A$ vary little. This behaviour has been observed in other cylinder wakes (Bourgeois et al., 2013; Mohebi et al., 2017; Noack et al., 2003) and has been interpreted as a conservative energy exchange between shed vortices and a slow-drift of the base flow. Henceforth, the base flow is defined as: $\mathbf{u}_B = \mathbf{U} + \mathbf{u}_\Delta$, with $\mathbf{u}_\Delta = a_\Delta \Phi_\Delta$.

The relative energy content of the slow-drift mode can be related to differences in the low-frequency wake dynamics. For the closed end case, this mode represents only 2.6% of the TKE. Fluctuations along the paraboloid are concentrated near the limit (mean) shedding cycle, suggesting low modulation of the shedding amplitude consistent with the narrow spectral peak of Fig. 3.4. For the open end case, $a_\Delta$ represents 7.4% of the TKE. The
fluctuations in the phase-space range from above the limit cycle to the unstable equilibrium at $A \approx 0$ (Noack et al., 2003), suggesting strong cycle-to-cycle modulation of the shedding process with intervals of vortex shedding suppression. These observations are consistent with the spectral broadening and auto-correlation trends see in Fig. 3.4.

In both flows, the third most energetic anti-symmetric mode is characterized by the $u$-spatial modal function $\Phi_u^f$ which is most strongly expressed downstream of the mean recirculation nodes along the shear layer (bottom row of Fig. 3.8). A direct comparison of the open and closed end fields lends itself to ambiguity because these subtend dynamically different regions. Specifically, the PIV domain for the closed end case captures more of the wake dynamics downstream of the recirculating region since it extends to $2.12\ell_R$, whereas the same FoV for the open end field extends to $1.42\ell_R$, such that the dynamics downstream of the recirculation are partially captured. To maintain the same spatial resolution within the constraints of the experimental set-up, an additional PIV plane with the same FoV dimensions was considered for the open end case. This plane extends from $x = 0.87\ell_R$ to $2.13\ell_R$. The correspondence of the modes for the two FoV was ascertained by considering the similarity in the dynamics represented by the modes in the region straddled by the two FoV. It was found that $\Phi^f$ and $a^f$ appear largely unchanged and represent the same motion when the POD is computed from a truncated closed end FoV with similar limits to the open end downstream FoV. Hence, the modal function $\Phi^f_u$ and the spectrum for $a^f$ from the open end extended FoV are shown in Fig. 3.8.

In both flows, the contribution of these modes, $a^f\Phi^f$ appears to describe a slow, in-tandem flow-normal oscillation, or a flapping motion, of the shear layers. The $v$-spatial modes, not shown for brevity, are symmetric about $y = 0$. The $u$-spatial modes are anti-symmetric about $y = 0$. Hence, these modes tend to increase $u$ in one shear layer, while decreasing $u$ in the opposing one, but $v$ is in the same direction for both shear layers. As shown below, this motion can be related to the trajectory of the shed vortices. The spectra of $a^f$ indicate energetic content at low frequencies, although a low-energy contribution at
$f_{sh}$ is still observed. For the closed end, the flapping mode accounts for about 3% of the TKE, which is similar to the contribution of the slow-drift. For the open end case, however, the contribution of the flapping mode is relatively weaker, accounting for about 1.5% of the TKE, compared to 7.4% for the slow-drift.

The spectral content of the flapping mode at $f_{sh}$ suggests an interaction with the pair $a_1$, $a_2$. This interaction appears to be captured in the last mode pair included in the low-order representation. Although these mode pairs contribute less than 3% of the TKE, their contribution to the Reynolds stress field in the recirculating region is not negligible and are thus retained in this model. These modes, identified as transition modes, are briefly discussed in Appendix B.

The influence of the flapping motion on the trajectories and strength, $\Gamma_v$, of shed vortices is illustrated in Fig. 3.10. The forming vortices are fed by circulation advected along the shear layers. Hence, their position is expected to follow the motion of the shear layers.

![Figure 3.10: Conditionally phase averaged vorticity centroid trajectories (top) with corresponding strengths $\Gamma_v$ (bottom). Conditional phase averaging was performed on the flapping mode. Trajectories and strengths are shown for high (red), low (blue), and average (black) cycles. CE on left, OE on the right. OE points on $x/\ell_R < 1.2$ are from near FoV, $x/\ell_R > 1.2$ from extended FoV.](image-url)
Figure 3.10 shows the conditionally phase-averaged trajectory of the vortex cores, defined by the vorticity centroid within regions enclosed by the curve \( Q = 0 \), for different states of the flapping motion. Briefly, the shedding cycles were regrouped in three equal bins based on the magnitude of \( a_f \). The bins represent the positive most, median (\( a_f \approx 0 \)), and negative most flapping mode contributions. Within each bin a phase average was then done. The influence of the flapping motion is most clearly expressed in the wake downstream of the vortex formation region. For shedding cycles within the median bin, the vortex trajectories are symmetric about \( y = 0 \). For positive most \( a_f \), the vortex trajectories are biased upward (\( y > 0 \)) and downwards (\( y < 0 \)) for the negative most \( a_f \). As can be inferred from Fig. 3.9c, the phase trajectories in the \( a_f-a_1,a_2 \) space are not correlated with \( A \) or, by extension through the paraboloid relationship, \( a_\Delta \). Hence, it is mainly the shear layer flapping motion that affects the downstream vortex trajectories.

Figure 3.10 also shows the shed vortex strength \( \Gamma_v \), conditionally phase-averaged on \( a_f \) as a function of \( x/\ell_R \). In the median bin, around \( a_f \approx 0 \) when the flapping mode contribution is least, the strength of the vortices shed from the opposing shear layers is the same. In contrast, when the shear layers are deflected upwards (positive most bin shown in red), the vortices shed along the top shear layer (\( y > 0 \)) are slightly weaker than those shed along the lower (\( y < 0 \)). The converse is true for the negative most bin (blue). This observation holds for both flows. By the Biot-Savart induction principle, the imbalance in \( \Gamma_v \) is consistent with the downstream deflection of the shear layers. It is noted that the imbalance reflects the contribution of the transitional mode pair (modes 7 and 8 of the reconstruction). It thus appears that this mode pair expresses a coupling through an energy exchange between the slow-drift, fundamental harmonic, and flapping modes, which can be inferred from the phase portraits in Appendix B.
Figure 3.11: High cycle left, limit cycle middle, and low cycle right. Phase averaged isocontours of vorticity at the phase when a vortex is about to shed from the negative shear layer. Overlaid are Lagrangian streamlines, the back flow (black dashed line), absolute maxima for $\langle u_c \rangle^2$ and $\langle v_c \rangle^2$ (red circles and triangles, respectively), the first local maximum $\langle u_c \rangle^2$ on the centerline (also red circles), and the vortex centroid path over the cycle (orange circles). Top, CE and bottom, OE. Note: gap in OE contours is the transition between PIV FoVs and is chosen to lie between the vortex trajectories from each FoV for visual clarity.

3.2.5 Reynolds stresses and wake structure

In this section, the differences in the structure of the Reynolds stress fields for the two flows, Fig. 3.3 ($u'_i u'_j$), are related to the vortex trajectories, formation lengths, and cycle-to-cycle variations. The influence of the slow-drift mode captures variations of the base flow and thus the recirculation/formation region. As observed from Fig. 3.10, the flapping mode affects little the trajectories upstream of $\ell_R$. Hence, a conditional average based on the magnitude of $a_\Delta$, reflecting the slow-varying base flow $u_B$, is used in Fig. 3.11 to investigate the formation region. The velocity data are grouped according to the $a_\Delta$ magnitude into the three levels along the paraboloid indicated in Fig. 3.9b. Each level includes 15% of the data population and is defined as high, limit ($a_\Delta \approx 0$), or low cycles. For each level, the phase average was calculated over 20 phase bins. Since $a_\Delta$ varies negligibly over a cycle, cycles remain within
the level group. A dashed black line corresponding to $u_B = 0$ delimits the backflow region for each level.

In Fig. 3.11, phase-averaged streamlines and vorticity iso-contours are shown at approximately the same shedding phase (i.e., for the same phase bin) for the three $a_\Delta$-levels. The phase corresponds to the first appearance of the saddle point in the upper shear layer ($\langle \omega_z \rangle < 0$, blue), indicating that the vortex is about to shed. Plots for the open end case show data for the two FoVs separated by white space for clarity. The two data sets were processed independently, using only the signal in the region straddled by the two FoV to synchronize the shedding phase. Note that the vortex ($\langle \omega_z \rangle > 0$, red) has already shed from the lower shear layer. The absolute maxima of the conditionally averaged Reynolds stresses, $\langle u_c \rangle^2$ and $\langle v_c \rangle^2$, are indicated together with the local maximum of $\langle u_c \rangle^2$ along $y = 0$. In each field, the loci of centroid locations, obtained over all phase bins, indicate vortex trajectories.

The limit cycle is considered first as a comparison basis. From the Reynolds stress fields, the local maximum of $\langle u_c \rangle^2$ on the centerline lies between the shed vortex and the forming vortex. This location approximately corresponds to the maximum extent of the forming vortex and is thus consistent with Griffin’s (Griffin, 1995) definition of the formation length for the circular cylinder. This definition holds for all cases and conditions considered.

The off-axis $\langle u_c \rangle^2$ maxima for closed ends are found within the core of the forming vortex, but are related to the maximum extent of the forming vortex core for open ends. It appears that the location of the off-axis maxima $\langle u_c \rangle^2$ are more related to the vorticity distribution. Griffin (1995) made a similar observations in assessing $Re$-effects. He noted that while the formation length changed little, the location of off-axis maxima $\langle u_c \rangle^2$ were sensitive to Reynolds number due to increased vorticity diffusion. $\langle v_c \rangle^2$ maxima are observed downstream of all $\langle u_c \rangle^2$ maxima, but occur further downstream for the open end case.

When considering the high and low cycles in Fig. 3.11, the location of $\langle u_c \rangle^2$ and $\langle v_c \rangle^2$ maxima relative to the forming vortices is similar to the limit-cycle case. Hence, the location and trajectories of the vortex centroids in relation to the backflow region indicated by $u_B = 0$
underlie important differences in the formation dynamics. In the closed end case, the forming core passes through the backflow region and a significant portion of the vorticity is drawn towards the flat plate. In contrast, for open ends, much of the core is located outside the backflow region. The cores move downstream during the formation leading vortices to shed farther in the wake. Correspondingly, when the vortex trajectories pass through the backflow region, the maxima of \( \langle u_c \rangle^2 \) are found upstream of the formation length, and \( \langle v_c \rangle^2 \), immediately downstream, Fig. 3.11.

Figure 3.12: Left of legends; Reynolds averaged stress (top), coherent and residual production along \( y = 0 \) for open (OE) and closed (CE) ends as functions of \( x/\ell_R \) (bottom). Right of legends; coherent and residual production (left) and coherent to residual transfer (right), both scaled by maximum production and as functions of \( x/\ell_R \). CE, top and OE, bottom.

Increased base region activity in the closed end \( u'_i u'_j \) and \( u''_i u''_j \) is related to the vorticity engulfment towards the plate and shortened formation length. Figure 3.12 shows \( u'^2 \) and \( v'^2 \) center-line distributions; coherent and residual TKE production; and the coherent production of residual fluctuations, representing a transfer of energy from the coherent to the residual field. The exact production term for a 2D-flow along \( y = 0 \), where \( \overline{u'v'} = 0 \), is given by \( G_k = (\overline{v'^2} - \overline{u'^2}) \frac{\partial u}{\partial x} \) and the coherent production of residual fluctuations is \( (\overline{u''_i u''_j} \partial u_{ci}/\partial x_j) \). Plots to the right of the legends are scaled by the maximum \( G_k \) in the field.

From the production profiles in Fig. 3.12, the closed end coherent \( G_k \) maximum is before
\( \ell_R \) so that coherent fluctuations are advected upstream where they impinge on the plate and deform against it; dispersing energy into the residual field via the coherent-residual transfer term. In a similar fashion, plate impingement draws energy from the mean field, strengthening residual fluctuations and contributing to an increase in residual TKE in the near field; this can be seen in the residual production. Correspondingly, \( \overline{u_i' u_j'} \) are much larger in the closed ends base region. The fluctuations are redirected outwards by the plate to align with the mean flow, generating a negative production, Fig. 3.12.

In contrast, with open ends, \( G_k \) and coherent Reynolds stresses are concentrated downstream of \( \ell_R \), such that fluctuations are advected downstream, resulting in relatively low coherent interaction with the plate and little transfer of coherent TKE into the residual field. Also, the mean flow gradients are lower approaching the flat plate, implying a more stable base region and resulting in negligible residual production. With a stagnant near field (Fig. 3.3), the coherent motion still realigns with the mean flow, leaving a region of negative production positioned farther downstream, suggesting an association with the larger formation length when compared to the closed ends.

### 3.3 Concluding remarks

A comparative study of the turbulent, quasi-periodic wake of a 2D thin flat plate normal to a uniform stream for open and closed end conditions was presented. While the mean flow field in both cases was 2D, important differences in the wake structure and Reynolds stress fields are observed. From a low-order representation of the flow from 8 POD modes, the two flows show similar basic underlying motions: a periodic contribution, a slow drift of the base flow related to a cycle-to-cycle variation of the backflow region, and a low-frequency flow normal non-periodic flapping of the shear layers affecting the trajectories of shed vortices. However, the relative energetic content of the motions differ between the two flows such that different dynamics in the formation regions are observed which can be related to differences
in the mean wake structure.

The open ends flow is characterized by a lower streamline curvature, greater variation of the slow drift base flow, and less engulfment of the forming vortex vorticity into the backflow region. With closed ends, the trajectory of the vortex cores are drawn inwards to the base region which causes an increase in the advection of vorticity towards the base of the plate, and results in more transfer to the residual field which translates to greater dissipation of TKE in the base region. The increased dissipation and streamline curvature would be consistent with a lower $C_{P_b}$ when comparing the closed ends to the open ends.

A plausible explanation for the different dynamics is that open ends allow for the ambient pressure to be expressed in the recirculation region, leading to an increase in the base pressure (Fox and West, 1990) and a decrease in the separation streamline curvature. A decrease in the streamline curvature has a destabilizing effect (Castro and Haque, 1987), which is consistent with the increased low-frequency energy content observed for the open end case compared to the closed end. It remains unclear why, in contrast to circular and square cylinders, two different mean 2D wakes can arise. It is clear, however, that the presence of the afterbody alters the wake dynamics (Balachandar et al., 1997). To better understand the dynamics, further investigation into these flows will then focus on how the stability of the shear layer affects the energetic exchanges between the coherent motions.
Chapter 4

Vortex identification

To this day, there exists considerable debate surrounding exact analytical definitions for the identification and characterization of vortices. Qualitatively, vortices are often pictured as regions of swirling flow whose fluid particles circle about a central point i.e., regions where the rotational motion dominates the deformation. Indeed, as early as the 14th century, Leonardo da Vinci had sketched spiraling flows in bluff body wakes, jets, and cylinder (aorta) flows (Gharib et al., 2002; Marusic and Broomhall, 2020). Today, vortices are understood to include a central “core”, where the flow rotates as though it were a solid body. Although this central core is more difficult to identify than the general swirling patterns noticed by Leonardo da Vinci, it can still be observed through even the simplest of home experiments, such as by placing a square piece of paper in a bathtub, opening the drain, and noting how the paper’s rotation about itself and the drain changes as it approaches the drain. Since Leonardo da Vinci’s time, more objective measures have been devised that do not depend on the observer’s inertial reference frame, and many of these methods involve identifying the aforementioned vortex core. Common Galilean invariant vortex detection methods include identifying i) pressure minima, ii) vorticity maxima (Hussain and Hayakawa, 1987), iii) regions wherein the eigenvalues of the velocity gradient tensor ($\nabla \mathbf{u}$) are complex (Chong and Perry, 1990), and iv) regions wherein rotational deformation is greater than straining.
deformation (commonly known as the Q-criterion) (Hunt et al., 1988). An additional method, known as the $\lambda_2$-criterion (Jeong and Hussain, 1995), has gained traction and is used extensively throughout the literature. However, for planar flows, the Q and $\lambda_2$-methods are mathematically identical (Jeong and Hussain, 1995).

Regrettably, the aforementioned techniques are somewhat arbitrary in their placement of vortex boundaries. These techniques generally define the vortex core and therefore the exact boundary of the core will depend on the criterion used. Moreover, some vorticity attached to the vortex extends beyond its core, such that considering only the core region under estimates the circulation of the vortex. As an example, the Q-criterion will identify regions where the vorticity magnitude is greater than the rate of strain magnitude, placing the vortex boundary along iso-surfaces where the two are equal. Consequently, the Q-criterion biases the identified regions towards areas where flow undergoes solid body rotation. With these considerations in mind, the Q-criterion will perfectly discover the vorticity boundary for a Rankine vortex (where a sharp cutoff between solid-body rotational flow and irrotational flow exists), but will exclude vorticity from real flows where the transition between vortex cores and irrotational\(^1\) flow is not abrupt.

Throughout the investigation into the impact of end conditions on the wakes of thin flat plates, the need arose for a more accurate accounting of the shed vortex strengths, quantified through the circulation. It was observed that not all the vorticity that rolled up and convected downstream was being included during vortex strength calculations. In the following sections, the ability of the Q-criterion to identify regions of flow that rotate approximately as solid bodies will be assessed for three different analytical vortex models. Subsequently, analytical models will be derived that allow for the estimation of “missing” vorticity for Burgers and Lamb-Oseen vortices. The models will then be applied to the two flat-plate flows, and it will be shown that the models are not only robust, but well suited for representing the shed vortices in the flat plate wakes. Finally, three different methods for

\(^1\)Irrotational refers to the fact that the fluid element does not rotate about itself, i.e., there is no vorticity.
spatially tracking vortices will be assessed.

4.1 On the identification of vortex cores: Q-criterion

Simple visual analysis of fluid flows will often reveal spirals, saddle points, sources, and sinks. With such features, it is unsurprising that fluid flows have often been assessed using critical point theory (Strogatz, 1994). Indeed, observing fluid motion in a reference frame that moves with the flow, nearby fluid motion can be expressed as:

\[ \dot{x} = (\nabla u)x \] (4.1)

Solving the characteristic equation (\( \det[\nabla u - \lambda I] = 0 \)) to this system of partial differential equations (PDE) yields three invariants (Chong and Perry, 1990). From the works of Hunt et al. (1988), the second invariant of \( \nabla u \) will be greater than some thresholding value when the vorticity is large compared to the irrotational straining; this is known as the Q-criterion. The second invariant of \( \nabla u \) can be written as:

\[ Q = -u_{i,j}u_{j,i}. \] (4.2)

Letting \( S_{ij} \) be the rate of strain tensor \( \frac{1}{2}(u_{i,j} + u_{j,i}) \), and \( \omega_i \) the vorticity \( \epsilon_{ijk}u_{k,j} \), the Q-criterion can be rewritten to demonstrate its dependence on the vorticity and the rate of strain. Indeed,

\[
S_{ij}S_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \cdot \frac{1}{2}(u_{i,j} + u_{j,i}) \\
= \frac{1}{4}(u_{i,j}u_{i,j} + u_{i,j}u_{j,i} + u_{i,j}u_{j,i} + u_{j,i}u_{j,i}) \\
= \frac{1}{2}(u_{i,j}u_{i,j} + u_{i,j}u_{j,j}) \\
= S_{ij}^2 \] (4.3)
and

\[ \omega_i \omega_i = \epsilon_{ijk} u_{k,j} \cdot \epsilon_{mn} u_{n,m} \]
\[ = (\delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km})(u_{k,j} u_{n,m}) \]
\[ = \delta_{jm} \delta_{kn} u_{k,j} u_{n,m} - \delta_{jn} \delta_{km} u_{k,j} u_{n,m} \]
\[ = u_{k,j} u_{k,j} - u_{k,j} u_{j,k} \]
\[ = u_{i,j} u_{i,j} - u_{i,j} u_{j,i} \]
\[ = \omega_i^2. \]  

(4.4)

Thus, (4.2) can be rewritten as

\[ Q = \frac{1}{2} \omega_i^2 - S_{ij}^2. \]  

(4.5)

Finally, for Newtonian fluids, \( \tau_{ij} = 2\mu S_{ij} \), and therefore:

\[ Q = \frac{1}{2} \omega_i^2 - \frac{1}{4\mu^2} \tau_{ij}^2. \]  

(4.6)

Typically, the thresholding value is set to zero, such that regions where the vorticity magnitude is greater than the rate of strain are identified. Thus, this criterion is very effective at identifying regions of solid body rotation, since a fluid rotating as a solid body will not experience viscous stresses (i.e., the fluid elements do not undergo internal deformation). Consequently, the Q-criterion can be used to define vortex cores, where the solid-body rotation dominates the internal deformation. To illustrate the Q-criterion’s ability to identify the cores of vortices, the Q-criterion was applied to three idealized vortex models, i) the Rankine vortex, ii) the Lamb-Oseen vortex, and iii) the Burgers vortex. To objectively determine how “closely” the flows rotate compared to a solid-body in rotation, the following measure was developed:

\[ \frac{\omega_z}{\Omega_0} = \frac{1}{\tau} \frac{\partial}{\partial \tau} (ru_{\theta}) - \frac{1}{\tau} \frac{\partial u_{\theta}}{\partial \theta}, \]  

(4.7)

\[ R_{ij} R_{ij} = \frac{1}{2} \omega_i^2 \therefore Q = R_{ij} R_{ij} - S_{ij} S_{ij} = R_{ij}^2 - S_{ij}^2, \]  

where \( R \) is the rate of rotation tensor:

\[ R_{ij} = \frac{1}{2} (u_{i,j} - u_{j,i}) \]
where $\Omega_0$ is the angular velocity of the flow about the origin (not to be confused with the traditional definition of angular velocity, $\Omega^\dagger$, which describes the rotation of a fluid particle about itself, and is equal to one-half the vorticity $\Omega^\dagger = \omega/2$). In the limit, as $r \to 0$, $\Omega_0 \to \Omega_z^\dagger$. For a flow to behave as though it is rotating as a solid body, $u_\theta \propto r$. Thus, for the chosen vortex models, (4.7) can be expressed as:

\[
\frac{\omega_z}{\Omega_0} = \frac{1}{r} \frac{\partial}{\partial r} \left( r u_\theta \right) - \frac{1}{r} \frac{\partial}{\partial \theta} \frac{u_\theta}{r} = 1 + \frac{r}{u_\theta} \frac{\partial u_\theta}{\partial r} = 1 + \frac{\partial r}{\partial u_\theta} \frac{\partial u_\theta}{\partial r} = 1 + 1 = 2 \tag{4.8}
\]

when the flow is rotating as a solid body.

### 4.1.1 Rankine vortex

Apart from the potential point vortex, the Rankine vortex is the simplest theoretical vortex model. The Rankine vortex is a steady, 2D vortex with perfectly circular streamlines. With these conditions, there are three possible solutions that satisfy continuity and angular momentum: i) $u_\theta = cr$, ii) $u_\theta = c/r$, and iii) $u_\theta = cr + c/r$ (with $c$ being a constant). From i) the model describes a vortex that rotates as a solid body, and ii) describes a potential vortex. Thus, combining the rotational and irrotational solutions yields a vortex whose core undergoes solid-body rotation before transitioning to irrotational rotation. The velocity profiles that describe this vortex are:

\[
\begin{align*}
    u_\theta &= \frac{\Gamma_{\text{tot}}}{2\pi R^2} r \quad \text{for} \quad r \leq R \\
    u_\theta &= \frac{\Gamma_{\text{tot}}}{2\pi r} \quad \text{for} \quad r > R.
\end{align*}
\tag{4.9}
\]

Where $\Gamma_{\text{tot}}$ is the total circulation, and $R$ denotes the radius at which the flow transitions from solid-body to irrotational rotation. From Fig. 4.1, it can be seen that the Q-criterion perfectly identifies the transition between solid-body, and irrotational flow for a Rankine
vortex, thus capturing all of the vorticity associated with the idealized vortex. Indeed,

\[
Q = R_{ij}R_{ij} - S_{ij}S_{ij} = \frac{\Gamma_{\text{tot}}^2}{2\pi^2 R^4} \quad \text{for} \quad r \leq R
\]

\[
Q = R_{ij}R_{ij} - S_{ij}S_{ij} = -\frac{\Gamma_{\text{tot}}^2}{2\pi^2 r^4} \quad \text{for} \quad r > R.
\]  

(4.10)

Figure 4.1: Rankine vortex. Left: the quotient of the local vorticity, and the angular velocity about the origin; \(\omega_z/\Omega_0 = 2\) indicates perfect solid-body rotation. Right: the vorticity field with overlaid streamlines. \(Q = 0\) is indicated by dashed green lines.

4.1.2 Lamb-Oseen vortex

The Lamb-Oseen vortex is a 2D vortex subject to viscous effects (Lamb, 1945). Because the vortex is 2-dimensional, vortex stretching is not present and cannot counterbalance the viscous diffusion; thus the Lamb-Oseen vortex is non-steady. In this model, the vortex is a function of radius and time, and has the following boundary conditions:

\[
\begin{align*}
u_{\theta}(r = 0, t) &= 0 \\
u_{\theta}(r \to \infty, t) &= \frac{\Gamma_{\text{tot}}}{2\pi r} \\
u_{\theta}(r, t = 0) &= \frac{\Gamma_{\text{tot}}}{2\pi r}
\end{align*}
\]

(4.11)
The velocity can be shown to be:

\[ u_\theta = \frac{\Gamma_{\text{tot}}}{2\pi r} \left[ 1 - \exp \left( -\frac{r^2}{4r_0^2} \right) \right]. \quad (4.12) \]

And from the velocity field, the vorticity is:

\[ \omega_z = \frac{1}{r} \frac{\partial}{\partial r} \left( ru_\theta \right) - \frac{1}{r'} \frac{\partial}{\partial \theta} \frac{\partial r}{\partial r} = \frac{\Gamma_{\text{tot}}}{4\pi \nu t} \exp \left( -\frac{r^2}{4r_0^2} \right). \quad (4.13) \]

From (4.12, 4.13), \( r_0 = \sqrt{\nu t} \) is a measure of the “broadness” of the vortex as a function of time (\( \sqrt{2r_0} \) is the standard deviation of the Gaussian vorticity profile). Figure 4.2 shows that the Q-criterion always identifies regions where the vorticity is greater than the angular velocity about the center of rotation i.e., \( Q = 0 \) when \( \omega_z = \Omega_0 \); an analytical proof of this result is presented in Appendix C.

![Figure 4.2: Lamb-Oseen vortex. Left: the quotient of the local vorticity and angular velocity for different \( \sqrt{\Pi_2} = r/r_0 \); \( \omega_z/\Omega_0 = 2 \) indicates perfect solid-body rotation. Right: the vorticity field for a typical Lamb-Oseen vortex at an unspecified time. \( Q = 0 \) is indicated by dashed green lines.](image)

### 4.1.3 Burgers vortex

Contrasting the Lamb-Oseen vortex is the Burgers vortex, which describes a homogeneous, incompressible, and time invariant vortex whose viscous diffusion is balanced by an exten-
sional strain i.e., vortex stretching (Burgers, 1948). Denoting $\Gamma$ as the circulation contained within an arbitrary radius $r$ from the centroid, $\Gamma_{\text{tot}}$ the total circulation associated with the vortex, $\gamma$ as the extensional strain parameter, and $\nu$ as the kinematic viscosity, the Burgers vortex velocities are:

\begin{align}
  u_z &= \gamma z \\
  u_r &= -\frac{1}{2} \gamma r \\
  u_\theta &= \frac{\Gamma_{\text{tot}}}{2\pi r} \left[ 1 - \exp \left( -\frac{r^2}{4r_0^2} \right) \right].
\end{align}

(4.14)

where $r_0$ expresses the competing action between the viscous diffusion and the vortex stretching,

\[ r_0 = \sqrt{\frac{\nu}{\gamma}}. \]

(4.15)

Consistent with the aforementioned velocities, the vorticity profile is:

\begin{equation}
  \omega_z = \frac{1}{r} \frac{\partial}{\partial r} (ru_\theta) - \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial}{\partial r} \left( \omega_0 \exp \left( -\frac{r^2}{4r_0^2} \right) \right)
\end{equation}

(4.16)

where $\omega_0$ is the vorticity at $r = 0$. Analogous to the Lamb-Oseen vortex, the “broadness” of the Burgers vortex is often defined by $r_0$. Letting $r \to \infty$, the total circulation in the field is:

\[ \Gamma_{\text{tot}} = 4\pi \omega_0 r_0^2, \]

(4.17)

and the fraction of vorticity captured up to an arbitrary radius can be expressed as:

\[ \frac{\Gamma(r)}{\Gamma_{\text{tot}}} = 1 - \exp \left( -\frac{r^2}{4r_0^2} \right). \]

(4.18)

Similar to the Lamb-Oseen model, the Q-criterion identifies regions where the vorticity is greater than the angular velocity about the vortex center, as seen in Fig. 4.3. However, unlike the Lamb-Oseen vortex, when $Q = 0$, $\omega_z/\Omega_0$ is not constant and ranges from $1 \leq \omega_z/\Omega_0 \leq 2$, ...
depending on the value of $\Pi_3 = \Gamma_{tot}/\nu$. For large $\nu$, i.e., when the viscous diffusion of vorticity is very strong, $\omega_z/\Omega_0 \rightarrow 2$, and the Q-criterion becomes increasingly biased towards regions that behave as though they are undergoing solid-body rotation. From Fig. 4.3, it can also be seen that at locations where $Q = 0$, the $|u_r/u_\theta|$ ratio becomes greatest when $\Pi_3$ is small. In other words, as the viscous diffusion increases, so too does the strain rate ($\gamma$), leading to increases in the radial velocity towards the vortex center ($u_r = \frac{1}{2} \gamma r$). Consequently, a greater $|u_r/u_\theta|$ ratio at the $Q = 0$ boundary indicates stronger spiraling motions of the fluid particles and streamlines. These observations lend further insight into the behavior of the Q-criterion. In particular, a vortex can be very “broad” in its distribution of vorticity, but still have a very narrowly defined Q-region if spiraling is sufficient.

In §4.2.1, it will be shown that for most real flows, $\Pi_3$ will be large, and consequently $\omega_z \approx \Omega_0$ when $Q = 0$, similar to the Lamb-Oseen case.

### 4.2 A matter of accounting: vortex circulation

For real flows, the Q-criterion is a compelling tool for identifying vortex cores, since viscous effects dampen velocities near the vortex centers, leading to regions that behave approximately as solid bodies in rotation (Smits, 2010). However, unlike the idealized Rankine model, real flows do not have discrete dividing lines wherein solid body rotation and vorticity stop. Consequently, determining the amount of vorticity in a vortex is both ambiguous and challenging. To the author’s knowledge, there is no vortex definition that allows for a full accounting of the vorticity associated with a vortex, and nor should there be. For all but the simplest flows, a criterion capable of fully determining a vortex’s contribution to the vorticity field would invariably extend into regions dominated by different phenomena, and thus, such a criterion would poorly represent the vortex itself.

In the following subsections, extensions to the Burgers and Lamb-Oseen vortex models will be made using the Q-criterion. These extensions will allow for a more complete account-
Figure 4.3: Burgers vortex. Top left: the quotient of the local vorticity, and the angular velocity about the origin; \( \omega_z/\Omega_0 = 2 \) indicates perfect solid-body rotation. Profiles are shown for two different \( \Pi_3 = \Gamma_{\text{tot}}/\nu \) cases. The two profiles can be collapsed onto a single curve by plotting against a non-dimensionalized variable: \( \Pi_2 = r^2/r_0^2 \). By not collapsing the curve, it can be seen that when \( Q = 0 \), \( \omega_z \) and \( \Omega_0 \) need not be equal. Top right: relationship between \( \omega_z/\Omega_0 \) and \( \Pi_3 \) when \( Q = 0 \). Bottom left: the vorticity field with overlaid streamlines. \( Q = 0 \) is indicated by dashed green lines, \( \cdots \). The red line \( \cdots \) indicates the \( \omega_z/\Omega \) profile (top left) associated with the vorticity field. Bottom right: the relationship between the \( u_r-u_\theta \) quotient and \( \Pi_3 \) when \( Q = 0 \).

\[ \text{ing of the total circulation (} \Gamma_{\text{tot}} \text{) based solely the vorticity contained in the Q-regions.} \]

### 4.2.1 An analytical accounting of the Burgers vortex using the Q-criterion

To investigate if a relationship between the total vorticity (4.17) and the Q-region exists, the Q-field for a Burgers vortex is derived analytically.
To compute the Q-field analytically, the deformation tensor is first computed:

\[ u_{i,j} = S_{ij} + R_{ij} \]  \hspace{1cm} (4.19)

Where \( S_{ij} \) is the symmetric rate of strain tensor, and \( R_{ij} \) is the skew-symmetric rate of rotation tensor. In cylindrical polar coordinates, the components of the rate of strain tensor are:

\[
S_{rr} = S_{11} = \frac{\partial u_r}{\partial r} \\
S_{\theta\theta} = S_{22} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \\
S_{zz} = S_{33} = \frac{\partial u_z}{\partial z} \\
S_{r\theta} = S_{12} = S_{21} = S_{\theta r} = \frac{1}{2r} \left( \frac{\partial u_r}{\partial \theta} + r \frac{\partial u_\theta}{\partial r} - u_\theta \right) \\
S_{rz} = S_{13} = S_{31} = S_{zr} = \frac{1}{2} \left( \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) \\
S_{\theta z} = S_{23} = S_{32} = S_{z\theta} = \frac{1}{2} \left( \frac{1}{r} \frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} \right). \]  \hspace{1cm} (4.20)

Similarly, the components of the rate of rotation tensor are:

\[
R_{rr} = R_{11} = 0 \\
R_{\theta\theta} = R_{22} = 0 \\
R_{zz} = R_{33} = 0 \\
R_{r\theta} = R_{12} = -R_{21} = -R_{\theta r} = \frac{1}{2} \left( \frac{1}{r} \frac{\partial (ru_\theta)}{\partial r} - \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \\
R_{rz} = R_{13} = -R_{31} = -R_{zr} = \frac{1}{2} \left( \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) \\
R_{\theta z} = R_{23} = -R_{32} = -R_{z\theta} = \frac{1}{2} \left( \frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \right). \]  \hspace{1cm} (4.21)

Combining (4.20) and (4.21) to recreate the deformation tensor (4.19), and noting that for
the Burgers vortex model (4.14) \( Q \) (4.2) simplifies to:

\[
Q = - \left( u_{1,1}^2 + u_{2,2}^2 + u_{3,3}^2 + 2u_{1,2}u_{2,1} \right), \tag{4.22}
\]

\( Q \) can be expressed as:

\[
Q = -\frac{\Gamma_{\text{tot}}}{4\nu \pi^2 r^4} \left[ 2\nu \Gamma_{\text{tot}} \left( \exp \left( -\frac{\gamma r^2}{4\nu} \right) - 1 \right)^2 + \gamma r^2 \exp \left( -\frac{\gamma r^2}{4\nu} \right) \right] - \frac{3\gamma^2}{2}. \tag{4.23}
\]

Clearly, the complexity of (4.23) obscures any revelations. By applying the Buckingham Pi theorem (Buckingham, 1914) to non-dimensionalize (4.23), a simpler formulation is revealed:

\[
\Pi_1 = \left( -\frac{\Pi_3}{4\pi^2 \Pi_2^2} \right) \left( \exp \left( -\frac{\Pi_2}{4} \right) - 1 \right) \left( \exp \left( -\frac{\Pi_2}{4} \right) (2 + \Pi_2) - 2 \right) - \frac{3}{2} \tag{4.24}
\]

where the Pi groups are:

\[
\Pi_1 = \frac{Q}{\gamma^2} \tag{4.25}
\]

\[
\Pi_2 = \frac{\gamma r^2}{\nu} = \frac{r^2}{r_0^2} \tag{4.26}
\]

and

\[
\Pi_3 = \frac{\Gamma_{\text{tot}}}{\nu}. \tag{4.27}
\]

By setting \( \Pi_1 = 0 \), (4.24) can be written as:

\[
\Pi_3 = \frac{\sqrt{6\pi \Pi_2 e^{\Pi_2}/4}}{\sqrt{\left( e^{\Pi_2}/4 - 1 \right) \left( \Pi_2 - 2e^{\Pi_2}/4 + 2 \right)}}. \tag{4.28}
\]

Equation (4.28) has an asymptote at \( \Pi_2 \approx 5.03572 \). By reformulating (4.18) in terms of \( \Pi_2 \),

\[
\frac{\Gamma(r)}{\Gamma_{\text{tot}}} = 1 - \exp \left( -\frac{\Pi_2}{4} \right), \tag{4.29}
\]

and substituting in the asymptotic value of \( \Pi_2 \), the maximum amount of circulation associ-
ated with the Q-defined core region, $Q \geq 0$, of a Burgers vortex is $\Gamma_v/\Gamma_{tot} \approx 0.7153$. From Fig. 4.4, it can be deduced that for the majority of fluids (where the kinematic viscosity $\nu$ is very small), (4.29) will approach the asymptotic 0.7153 value.

![Figure 4.4: Flooded isocontours of $\Pi_1$, with $\Pi_1 = 0$ denoted in green (−).](image)

### 4.2.2 An analytical accounting of the Lamb-Oseen vortex using the Q-criterion

Consistent with the treatment of the Burgers vortex model, the Q-criterion was analytically applied to the Lamb-Oseen vortex, and the analytical expression for $Q$ was subsequently simplified using a Buckingham Pi analysis. For the Lamb-Oseen case, the fraction of the total circulation contained up to an arbitrary radius was identical in form to the Burgers vortex case (4.29). With $Q = 0$, the only non-trivial solution (i.e., when $r$ and $t \neq 0$) for the Lamb-Oseen case occurred when $\Pi_2 \approx 5.03572$. Consequently, when the Q-criterion was applied to the Lamb-Oseen vortex, $\Gamma_v/\Gamma_{tot} \approx 0.7153$, similar to the Burgers case. However, unlike the Burgers vortex, for the Lamb-Oseen case, the 0.7153 ratio is a constant, i.e., independent of the viscosity. In Appendix C, the derivation for the Lamb-Oseen case is given.

---

3For the Lamb-Oseen vortex, $\Pi_2 = r^2/r_0^2 = r^2/(\nu t)$; for the Burgers case, $\Pi_2 = r^2/r_0^2 = \gamma r^2/(\nu)$. 49
4.3 Burgers and Lamb-Oseen vortex models applied to flat plate flows

In §4.2, it was shown that applying the Q-criterion to the idealized Burgers and Lamb-Oseen vortices will capture at most 71.53% of the total vorticity of the associated vortices. In this section, it will be shown that this ratio provides a good approximation for real vortices when applied to the flat plate wakes.

4.3.1 A manual accounting of vorticity

Before the 0.7153 ratio can be assessed for the flat plate geometries, it is necessary to estimate the amount of vorticity not captured by the Q-criterion. As discussed in §4.2, standard vortex definitions fail to capture all of the vorticity associated with a given vortex. Consequently, a graphical user interface (GUI) was created that allowed for the manual selection of vortices; a brief description of the GUI, along with the source code, is presented in Appendix E.

For the purposes of estimating $\Gamma_v / \Gamma_{tot}$, the circulation was estimated using the integrated vorticity in the manually selected regions; the estimation of $\Gamma_{tot}$ is denoted as $\Gamma_{tot}^*$. Figure 4.5 shows a standard output from the GUI for phase averaged closed ends data. In magenta is the Q-identified region for the separating vortex, and in green is the user selected region; Lagrangian streamlines (reference frame moving with the vortex center) are also displayed in gray. To achieve consistent results, a standard methodology was developed. In particular, user defined vortex boundaries were chosen such that they would begin at the saddlepoint and then follow a vorticity iso-contour, which was chosen to be equal to 4% of the maximum vorticity in the field; the vorticity iso-contours are seen in Fig. 4.5 as solid red and blue lines. Setting the vorticity level lines to be less than 4% often resulted in jagged profiles that were clearly subject to noise, and for level lines greater than 4%, the isobars would encroach upon the Q-regions.
Figure 4.5: Prototypical GUI snapshot for a phase-averaged vorticity field with overlaid Lagrangian streamlines (reference frame moving with the vortex center). User selected region is marked by solid green circles (●) along the border, and open green circles within the selected area (○). Magenta lines (—) indicate the Q-identified region, while solid blue (—) and red lines (—) indicate isobars equal to 4% of the maximum negative and positive vorticity, respectively.

4.3.2 Results

The analytical Lamb-Oseen and Burgers vortex models have the same (maximum) $\Gamma_v/\Gamma_{\text{tot}}$ ratios; this result lends additional confidence to the models, and suggest that the ratio can be applied to both steady, and time-varying vortices, provided that the vortex tubes are essentially parallel. However, unlike the Lamb-Oseen case, when the Q-criterion is applied to the Burgers model, $\Gamma_v/\Gamma_{\text{tot}}$ is not constant, varying with changes in $\Pi_3$. For flat plate flows, changes in $\Pi_3$ are approximately equivalent to changes in $Re$. Indeed, for flat plate flows,
\( \Gamma_{\text{tot}} \) will generally scale with, and be of similar order to \( cU_{\infty} \), where \( c \) is the characteristic diameter, and \( U_{\infty} \) is the free-stream velocity \( (U_{\infty}) \), thus \( \Pi_3 = \Gamma_{\text{tot}} / \nu \propto Re \). As noted in §4.1.3, for the Burgers case, as \( \Pi_3 \) becomes very small, limits in the Q-criterion’s ability to identify vortex cores are revealed. Extending this observation to real flows suggests that the Q-criterion will struggle to distinguish cores for very week vortices, or cores in creeping flows \( (Re \ll 1) \).

For the flat plate flows, the shed vortex circulations were found to be of order \( 10^{0} \) and \( 10^{-1} \) for CE and OE (scaling with \( cU_{\infty} \)), and the fluid kinematic viscosity of air is of order \( 10^{-5} \). By referring to Fig. 4.4, it is clear that the Burgers vortex model will predict that \( \approx 71\% \) of the vorticity found using the user defined integration regions of §4.3.1 will be located within the \( Q = 0 \) bounds. For the Lamb-Oseen case, the \( \Gamma_v / \Gamma_{\text{tot}} \) ratio is constant, and will also predict \( \approx 71\% \).

For both closed and open ends, vortex strengths were computed by integrating over the vorticity field using \( Q \) and user-selected regions as boundaries\(^4\). Twenty different phase averaged snapshots were assessed to assure that the results were consistent with minimal variance. For closed ends, the average ratio was found to be \( \Gamma_v / \Gamma_{\text{tot}}^* = 0.70 \), with a standard deviation of \( 7.7 \times 10^{-3} \). For open ends, the average ratio was \( \Gamma_v / \Gamma_{\text{tot}}^* = 0.71 \), with a standard deviation of \( 7.1 \times 10^{-3} \).

To assess the sensitivity of the results to noise, the thresholding value of \( Q \) was varied. From Fig. 4.6, it can be seen that the Burgers model accurately predicts \( \Gamma_v / \Gamma_{\text{tot}}^* \), even when the thresholding value of \( Q \) is changed from zero. Moreover, this figure suggests that extrapolation can be used to minimize the sensitivity to experimental uncertainty in estimating \( Q = 0 \).

Finally, the sensitivity to the vorticity iso-contour levels, used for calculating \( \Gamma_{\text{tot}}^* \), were tested. For open ends, allowing the iso-contour levels to vary between \( 0.5 - 4\% \) had a very minimal impact on the estimate; at \( 0.5\% \) the average ratio dropped to \( \Gamma_v / \Gamma_{\text{tot}}^* = 0.70 \) with

\[ \Gamma_{\text{tot}}^* \] was estimated using vorticity iso-contours set to \( 4\% \) of the maximum vorticity.
a standard deviation of $8.6 \times 10^{-3}$. For closed ends, the impact of the level lines was more noticeable. From Fig. 4.7, the impact of the $\%$-vorticity iso-contours levels on $\Gamma^*_\text{tot}$ can be seen; at 0.5%, $\Gamma_v/\Gamma^*_\text{tot}$ dropped to 0.67 with a standard deviation of $6.6 \times 10^{-3}$. Fig. 4.7 shows a linear trend, and it’s therefore expected that even as the $\%$-vorticity level approaches zero, $\Gamma_v/\Gamma^*_\text{tot}$ will remain greater than 0.66. Because both the Lamb-Oseen and Burgers vortex models assume isolated vortices, the greater change in $\Gamma^*_\text{tot}$ for closed ends is thought to be related to increased mixing and interactions between the forming vortices and separated shear layers (see Chapter 5). Throughout this study, $\%$-vorticity levels were kept at 4% for both the CE and OE cases. It was observed that lower $\%$-vorticity often contained noisy data, and higher $\%$-vorticity resulted in regions that encroached on the Q-identified regions.

![Figure 4.6](image)

Figure 4.6: $\Gamma_v/\Gamma^*_\text{tot}$ as a function of $Q/Q_{\text{max}}$; ▲ CE, ○ OE, with analytical model as the red line (—).

### 4.4 A brief comparison of different methods for educating vortex centers

In this chapter, several methods for identifying vortices were described. Using the Q-identified regions, it was shown that total circulation can be approximated. However, further insight into the vortex dynamics is gained by characterizing the location and rate of displace-
ment of the vortex. Since a vortex is a finite and deformable region of correlated rotational motion, its location is defined by an effective center. The center’s rate of displacement then corresponds to the convective velocity of the vortex. Upon initial glance, critical point theory (Strogatz, 1994) would appear up to the task of identifying these effective centers. After all, critical point theory can be used to reveal foci that are enclosed by streamlines which are connected to saddle points. However, to utilize critical point theory in this manner requires a priori knowledge of the streamlines in a Lagrangian reference frame that moves with the vortex centers.

In this section, three methods for defining the centers that are Galilean invariant are presented and compared. Subsequently, the effective centers will be used to estimate the convective velocities of the vortices.

### 4.4.1 Vortex centroid method

One method for estimating a vortex’s center is to determine the mean position of the vortex core, weighted by local vorticity, i.e. the centroid of vorticity. In three-dimensions, the
vorticity centroid \((x_0, y_0, z_0)\) can be written as:

\[
\int_\Omega\int_\Omega\int_\Omega \left[ (x, y, z) - (x_0, y_0, z_0) \right] \langle \omega \rangle \, dV = 0
\]

\[
(x_0, y_0, z_0) = \frac{\int_\Omega \int_\Omega \int_\Omega (x, y) \langle \omega \rangle \, dV}{\int_\Omega \int_\Omega \int_\Omega \langle \omega \rangle \, dV}.
\] (4.30)

It’s interesting to note that unlike other centroidal calculations (such as mass centroids), the vorticity centroid loses all meaning if the integration domain is allowed to extend to infinity. This is because at infinity, the integrated vorticity will be zero \((\int_\Omega \int_\Omega \int_\Omega \langle \omega \rangle \, dV = 0)\), whereas there is no requirement that integrated mass (for example) go to zero for an infinite control volume. Mathematically, Saffman (1992) demonstrates this shortcoming by rewriting vorticity as:

\[
\omega_i = x_{i,j} \omega_j + x_i \omega_{j,j} = \delta_{ij}\omega_j + x_i \omega_{j,j} = \omega_i + x_i \omega_{j,j} = 0
\] (4.31)

after which, the divergence theorem can be applied and the total vorticity can be expressed in volume and surface integral forms as:

\[
\int_\Omega \int_\Omega \int_\Omega (x_i \omega_j) \, dV = \oint_{\partial_c} x_i \omega_j \delta_{jp} n_p \, ds,
\] (4.32)

where \(\delta_{jp}\) comes from the scalar product \(^5\) of the vorticity field \((\omega_j)\) with the unit vectors \((n_p e_p)\) along, and normal to, the control volume’s border \((\partial_c)\). Because vortex tubes must terminate on themselves, a boundary, or a free surface, if the domain is extended to infinity,

---

\(^5\) For an orthonormal basis, the scalar product of two vectors can be expressed as:

\[
v \cdot w = (v_i \mathbf{e}_i) \cdot (w_j \mathbf{e}_j) = v_i w_j \delta_{ij} = v_i w_i,
\] (4.33)

where,

\[
\delta_{ij} = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } i \neq j
\end{cases}
\] (4.34)
\(\mathbf{n} \cdot \mathbf{\omega}\) will be equal to zero locally (since the vortex tubes at infinity will run parallel to the domain boundary) and the integrated vorticity will be zero. Consequently, Saffman (1992) argued that an alternative definition for vortex centroids ought to be used if the closed volume is bounded by irrotational flow (i.e., the vortices are contained inside the volume, thus forming closed loops). However, defining the centroid of a vortex element between two planes cutting through the same vortex obviates this problem and (4.30) stays finite and can be applied to real and theoretical flows.

Naively applying (4.30) to a field containing multiple vortices will result in a centroid that’s weighted by each vortex. Consequently, to characterize vortex positions using (4.30), centroid calculations should be performed individually for each vortex, with the integration domains being defined by the vortex cores. The motivation for limiting the integration domains to the cores is because vortex cores will most closely approach solid-body rotation, and will therefore have approximate centers of rotation. Fluid elements outside of the cores will not necessarily rotate about these common centers, and therefore their associated vorticity should be excluded from the centroid calculations. In this way, the vorticity of any fluid particles that are overly influenced by other physical phenomena will not bias the centroid. Due to its ability to identify the regions of flow that behave most similarly to solid bodies under rotation, the Q-criterion will be used to define the cores. As a final note, for the 2D flow flat plate flows, (4.30) becomes an area integral, since \(\frac{d}{dz} = 0\).

When the centroids were first calculated for the two flat plate flows, it was found that they were frequently poor approximations. Comparing the centroids to the vorticity fields would often reveal centroids that were far upstream of the associated vortex foci. It was subsequently determined that a bias was being introduced by vortex tails, which were also being included in the Q-identified integration regions. These vortex tails would appear during the formation process, and would reflect the fact that the forming vortices had not fully separated themselves from their feeding shear layers. Because the fluid elements in the vortex tails have different centers of rotation than the fluid elements in the main vortex
bodies, it was decided that they should be omitted from the centroid calculations, despite being identified by the Q-criterion.

To remove the vortex tails, the Q-identified vortices were subjected to a preprocessing step. The preprocessing involved treating the vortices as though they were solid bodies with uniform “temperatures”. Anything outside of the Q-identified regions were treated as being isothermal with zero temperature. Subsequently, the “heat” in the vortices would be allowed to diffuse. After a set amount of time, the boundary conditions would be changed, the Q-identified region would be treated as insulating, and heat would again be allowed to diffuse. As a final step, areas that were below a certain “temperature” level were filtered out. It is very important to note that this preprocessing step was not informed by any presumed physics of the flow, despite the fact that it utilizes the concept of diffusion. Instead, this model was based on the intuition that “skinny” bodies will experience greater decreases in temperature than larger bodies. Indeed, if the “temperature” of the vortex is assumed to be constant throughout the body, the initial rate of heat loss will be the same at all points along the vortex border. However, because the bulk of the “heat” is not contained in the vortex tail, the vortex tail will experience greater losses in temperature than the main body, similar to how inland bays are subject to greater temperature fluctuations than the larger bodies of water that feed them. As a final note, it should be made clear that this filtering was only ever used for the centroid calculations.

Referring to Fig. 4.8, phase averaged vorticity fields with overlaid Lagrangian streamlines (reference frame moving with the vortex center) can be seen. For each phase in Fig. 4.8, the Lagrangian reference frames were calculated using the convective speeds of the bottom half-plane vortices (the vortices with positive vorticity); further information on the vortex convection speed calculations can be found in §4.4.5. From Fig. 4.8, it can be seen that the vortex centroids and the vortex foci closely line up. Furthermore, it can be seen that even when the vortex has not fully separated, the filtering process has reduced the bias introduced by the tail, and the vortex centroid does not appear egregiously upstream.
4.4.2 Pressure minima method

For steady, incompressible, inviscid flow, it can be shown that the centrifugal force balances the pressure force:

\[ \frac{-u^2}{R} = \frac{-1}{\rho} \frac{\partial p}{\partial n}, \]  

where \( u \) is the velocity tangent to the streamlines, \( n \) represents the local normal to the streamline, and \( R \) is the streamline radius of curvature. Consequently, for circular flows with no radial velocity, (4.35) dictates that \( p \propto r^2 \) and the pressure distribution becomes identical to the solid-body Rankine vortex pressure distribution. Ahlborn et al. (1998) also demonstrates a relationship (4.36) between curvature and pressure by considering the vorticity transport equation for 2-dimensional cylindrical wakes with negligible convective and diffusive terms, and a constant rate of vorticity production:

\[ \zeta^2 \approx \frac{U_\infty^2}{\sqrt{2}R^2} (1 - C_{p_b}), \]  

where \( \zeta \) is the average vorticity found within a vortex of radius \( R \) accumulated over half a shedding cycle.

For real, turbulent flows, the connection between the curvature and pressure becomes more complicated. None the less, as can be seen in Fig. 4.8, pressure minima appear to be strong indicators for vortex centers in the thin flat plate wakes.

To determine the minima, pressure fields were estimated using a Green’s function solution to the pressure Poisson equation; a brief explanation can be found in Appendix D, and a more detailed account in (de Plessix, 2015). It was quickly realized that a preprocessing step was necessary, as the pressure minima were slaved to land on the coarse PIV grid points, resulting in discontinuous jumps in the minima positions. To create more continuous pressure fields, piecewise-cubic interpolation was preformed on the pressure fields, followed by grid refinement.
4.4.3 Q-maxima method

From §4.1, it was shown that as Q becomes greater, flow rotates more and more as though it were a solid body. With vortices in real, viscous flows being characterized by cores that behave approximately as solid-bodies under rotation (Smits, 2010), it is unsurprising that the Q-maxima correspond nicely with vortex centers, as seen in Fig 4.8.

Similar to the pressure minima method for vortex center identification (§4.4.2), determining the Q-maxima from discrete PIV data involved applying piecewise-cubic interpolation to the Q-field, and subsequently a grid refinement.

4.4.4 Comparison of the centroid, pressure minima, and Q-maxima methods for determining vortex centers

From Fig 4.8 and Fig 4.9, all three methods deliver convincing results. However, by looking at the phase-to-phase \( x/c \) and \( y/c \)–displacements, the centroid method yields the smoothest paths with the smallest displacement variances. Indeed, for the centroid method, the standard deviation for the \((x/c, y/c)\)–displacements are \((0.035, 0.011)\). Meanwhile for the pressure-minima method, the standard deviations are \((0.057, 0.022)\), and for the Q-maxima method, \((0.072, 0.023)\).

Throughout the rest of this thesis, any information relating to vortex centers will be determined using the centroid method.

4.4.5 Comparison of the centroid, pressure minima, and Q-maxima methods for determining vortex convection speeds

To calculate the vortex convection speeds in the streamwise direction \( (c_x) \), the vortex centers of Fig. 4.9 were used. In particular, least-squares fitted polynomials were determined for the center paths, which were subsequently differentiated with respect to time. Finally, the vortex convection speed profiles were averaged for the negative and positive vortices to reduce the

59
impact of noise. From (Cantwell and Coles, 1983), it is anticipated that the vortex convection speeds will be well approximated by a 3rd order polynomial. As such, 4th order polynomials were fit to the vortex center paths. From Fig. 4.10, the vortex convection speed as a function of streamwise location is shown for the closed end case.

From Lyn et al. (1995), the vortex convective speed for square and circular cylinders is correlated by:

$$\frac{c_x}{U_\infty} = 0.45 \left( 1 + \frac{U_0}{U_\infty} \right),$$

(4.37)

where $U_0$ is the mean centerline velocity in the streamwise direction, taken far enough downstream such that $\partial U_0/\partial x \approx 0$. For closed ends, extrapolation of $U_0/U_\infty$ predicts that the streamwise centerline velocity will level off at $U_0/U_\infty \approx 0.43$. After leveling off, its anticipated that both the mean centerline velocity, and the convective velocities will begin increasing again (due to viscous diffusion) and eventually approach $U_\infty$ far downstream, outside of the measured domain. Where the convection profile initially levels off, (4.37) predicts $c_x/U_\infty \approx 0.64$. From Fig. 4.10, it can be seen that the centroid method for calculating $c_x/U_\infty$ aligns best with this estimate, peaking at $c_x/U_\infty \approx 0.66$. From (Fage and Johansen, 1927), $c_x/U_\infty$ was found to be 0.765, however the experiment of Fage and Johansen (1927)
had a blockage ratio of 0.09, as opposed to 0.057 for the closed end case. Additionally, the centroid method produced smoother convection speed profiles. And as a final note, when the $c_x/U_\infty$ profiles for the top half-plane (negative), and bottom half-plane (positive) vortices were compared, the centroid method produced the most consistent results, which is critical for a flow that is symmetric in the mean. For a more refined PIV grid, it is anticipated that the pressure minima and Q-maxima methods will improve. Moreover, with a sufficiently refined grid, these methods might be preferable. In particular, the centroid method cannot be used near the domain boundaries, since the method will fail if the entire core is not contained within the measurement domain. Also, the pressure minima and Q-maxima methods are not biased by the vortex tails, and therefore do not require any filtering, thus improving their objectivity.

The Lagrangian streamlines of Figure 4.8 were generated using the centroid method $c_x/U_\infty$ profile of Fig. 4.10, with the reference frames attached to the bottom half-plane (positive) vortex centroids. From Fig 4.8, clear separatrices can be seen enclosing the bottom vortices, as well as spiraling streamlines that converge on the identified centroids.

Finally, it should be noted that a dispersion relation could also be used to estimate the convective speeds. In particular, downstream where $\partial c_x/\partial x \approx 0$, the convective speed should
be equal to the wavelength, multiplied by the shedding frequency (where the wavelength can be determined from the spatial correlation function). However, accurate estimates would necessitate better spatial resolution i.e., the PIV measurement grid was too coarse.

Figure 4.10: Vortex convection speeds estimated from phase averaged vortex paths for the closed end case. Centroid method, •; pressure minima method, —; Q-maxima method —.

4.5 Concluding remarks

In this chapter, the Q-criterion’s ability to identify vortex cores was assessed. When the Q-criterion was applied to the Burgers and Lamb-Oseen vortex models, it was analytically proven that only 71.53% of the associated vorticity would fall within the Q-identified boundaries. Subsequently, real flows were assessed, and it was found that the Q-criterion’s performance remained consistent, with 70% of the vorticity being captured for CE, and 71% for OE. In addition, three different methods (centroid, pressure minima, and Q-maxima) for calculating vortex centers, as well as vortex convection speeds, were compared and assessed. The pressure minima and Q-maxima conditions were found to be particularly sensitive to the resolution of the PIV measurements, showing a lot of scatter and variance. The centroid method was thus preferred, as calculations of derivatives (for example, to calculate the vortex convection speed) were much more suitable.
Chapter 5

An accounting of circulation transport in the flat plate wake

From Chapter 3, an in-depth accounting of the flat plate wakes was provided. Nonetheless, a more detailed consideration of the vortex formation process is expected to provide a better understanding of the mechanisms contributing to the wake dynamics. In particular, the rate at which vorticity is transported to the wake and contributes to the shed vortex circulation is of interest. Consequently, the following chapter will present a reinvigorated investigation into the vorticity field. First, several heuristic vortex shedding models will be detailed. Next, two different modes of vorticity decay will be identified. Subsequently, a detailed overview of three different methods for estimating vorticity transport rates will be established, and a complete hermetic accounting of the vorticity field will be realized. Finally, additional evidence for increased turbulent mixing in the closed end wake will be provided.

5.1 Vortex shedding models

To lay the ground work for the analysis of the flat plate flows, details of accepted vortex shedding models are presented for comparison. The vortex shedding models of Gerrard (1965), Roshko (1954b), and Ahlborn et al. (1998) will be summarized. Additionally, the
sheding topology for the flat plate flows will be described.

5.1.1 Gerrard’s vortex shedding model

Gerrard’s description of vortex shedding has become a reference for conceptualizing the shedding process. For turbulent bluff body flows, instabilities in the separated shear layers initially grow in amplitude leading to the roll up of the shear layer. Through the roll up process, vorticity in concentrated in a forming vortex. The forming vortex acts through induction on the opposing shear layer (of opposite signed vorticity), drawing it across the wake. Gerrard (1965) proposes that as the opposing shear layer gets drawn across the wake centerline it becomes engulfed in either i) the forming vortex, ii) the shear layer feeding the forming vortex, or iii) the formation region of the wake. Opposite signed vorticity engulfed in the feeding shear layer serves to sever the forming vortex. Meanwhile, vorticity engulfed in the formation region will contribute to the formation of the next vortex. A schematic of vortex shedding about a circular cylinder is shown in Fig. 5.1. From Fig. 5.1, the red dots indicate saddle points within the flow topology, and the attached separatrices distinguish the associated vortices from the remaining wake. In the case of forming vortices, the appearance of a saddle points coincide with the severing/shedding events.
Figure 5.1: Vortex shedding model, closely recreated from Williamson (1996), for flow about a circular cylinder. Saddlepoints are denoted by red circles (●).
5.1.2 Roshko’s bluff body wake model

The Helmholtz-Kirchhoff free streamline model, described in Lamb (1945) and Roshko (1954a), was revolutionary in that it yielded theoretical results to bluff body flows subject to non-zero drag forces; previous models had failed in this respect (D’Alembert’s paradox). To achieve this result, the Helmholtz-Kirchhoff model distinguished itself by allowing for separated flow. Fundamentally, the model was a potential flow model, making use of conformal mappings. From the model, a central streamline converges on a stagnation point and subsequently splits into two diverging streamlines, referred to as the free streamlines. The free streamlines bound a region of motionless flow that extends to, and reconverges at, infinity in the streamwise direction. Thus, the free streamlines can be thought of as being similar to separation streamlines breaking over a flat plate, although this parallel is not perfect. Indeed, for the Helmholtz-Kirchhoff and Roshko models, the free/separation streamlines reconverge at infinity, whereas for real flows that are 2D in the mean, the mean separation streamlines reconverge at a finite distance behind the plate. Specifically, these converge at the saddle point, marking the end of the base region recirculation. Within the motionless region (wake), the pressure is chosen to be equal to the free stream pressure ($P_\infty$), thus slaving the velocities along the free streamlines ($U_s$) to be equal to the free stream velocity ($U_\infty$). Moreover, the velocity discontinuity across the free streamlines results in two infinite vortex sheets. By allowing an infinite wake, the model introduces an asymmetry not found in typical bluff body potential flow models. Namely, for all practical purposes the Helmholtz-Kirchhoff model only has one stagnation point, instead of two, located on the upstream face of the theoretical plate (i.e., ignoring the stagnation point at infinity). Thus, at the stagnation point, an upstream pressure maximum is established, thereby introducing drag into the system. Despite its novel approach, by setting the motionless wake’s pressure to be equal to the free stream pressure, the model severely underestimates the drag.

To remedy the drag inconsistency, Roshko (1954b) devised a new model, known as the notched-hodograph model. Similar to the Helmholtz-Kirchhoff model, the notched-
hodograph model is a conformal mapped potential flow with separated shear layers bordering a motionless wake that extends to infinity in the streamwise direction. Again, the separation streamlines and free streamlines are one and the same, and \( P_s = P_0 - \frac{1}{2} \rho U_s^2 \) (where \( P_0 \) is the stagnation pressure). Importantly, the free streamlines of the notched-hodograph model have velocities equal to \( U_s = kU_\infty \), where \( k \) is a constant. With the freestream pressure necessarily being equal to the motionless wake’s pressure, the base-pressure coefficient can be written as:

\[
C_{P_b} = \frac{P_s - P_\infty}{\frac{1}{2} \rho U_\infty^2} = 1 - \frac{U_s^2}{U_\infty^2} = 1 - k^2. \tag{5.1}
\]

The model can be completely prescribed from the base pressure coefficient, the Strouhal number, and the Reynolds number.

For the purposes of this study, the rate at which vorticity is carried downstream by the shed vortices is of particular interest. From Roshko’s model, the rate at which vorticity is produced can be calculated by determining the rate of shear layer vorticity transport, and is equal to \( k^2U_\infty^2/2 \). Additionally, the rate of vorticity production can be related to the associated rate of circulation accumulation in an infinite control volume immediately downstream of the bluff body:

\[
\frac{1}{U_\infty^2} \frac{d\Gamma}{dt} = \frac{1 - C_{P_b}}{2}. \tag{5.2}
\]

Over a single shedding cycle, if it is assumed that all vorticity associated with a given shear layer rolls up to form a vortex, then the theoretical amount of circulation contained in the shed vortex, \( \Gamma_{tot} \), can be calculated by integrating (5.2) over a full period:

\[
\frac{\Gamma_{tot}}{U_\infty c} = \frac{1 - C_{P_b}}{2 S_t}, \tag{5.3}
\]

and (5.2) can be rewritten as:

\[
\frac{1}{U_\infty^2} \frac{d\Gamma}{dt} \equiv S_t \left( \frac{\Gamma_{tot}}{U_\infty c} \right). \tag{5.4}
\]

However, the experimental work of Fage and Johansen (1928) indicates that only half (ap-
proximately) of the generated vorticity will be subsequently found in the shed vortices. Accordingly, for the calculation of shed vortex strengths, Roshko modified (5.3) by a multiplicative factor, $\epsilon \approx 0.5$.

### 5.1.3 Ahlborn’s bluff body wake model

Similar to Roshko, Ahlborn et al. (1998) developed a theoretical model for flow about a thin flat plate, normal to oncoming, 2D flow. Notably, the model devised by Ahlborn et al. (1998) can be obtained directly from the vorticity transport equation:

\[
\frac{\partial \omega_i}{\partial t} = \omega_j u_{i,j} - u_j \omega_{i,j} - \epsilon_{ijk} \left( \frac{1}{\rho} p, k \right)_j + \nu \omega_{i,jj} \quad (5.5)
\]

Critical to the model is the pressure term,

\[- \epsilon_{ijk} \left( \frac{1}{\rho} p, k \right)_j = - \frac{1}{\rho} \epsilon_{ijk} (p, kj) + \frac{1}{\rho^2} \epsilon_{ijk} (\rho, j p, k) \quad (5.6)\]

which identically vanishes for constant density flows, since the curl of the gradient of a scalar field is always zero ($- \frac{1}{\rho} \epsilon_{ijk} (p, kj) = 0$), and the gradient of the density will also be zero ($\frac{1}{\rho^2} \epsilon_{ijk} (\rho, j p, k) = 0$). Ahlborn observed that if the solid body is treated as a region of infinite density flow, the resulting density singularity at the solid body’s boundary introduces vorticity generation into the model. Defining a control volume that encloses the top half plane of the flow field, and bisects the flat plate (as seen in Fig. 5.2), the vorticity generation associated with an individual separated shear layer is:

\[
\frac{d}{dt} \int_A \omega_i \, dA = - \int_A \epsilon_{ijk} \left( \frac{1}{\rho} p, k \right)_j \, dA. \quad (5.7)
\]

Subsequently applying Green’s theorem, the rate of circulation accumulation in the wake is:

\[
\frac{d\Gamma}{dt} = \frac{d}{dt} \int_A \omega_i \delta_{ip} n_p \, dA = - \int_A \epsilon_{ijk} \left( \frac{1}{\rho} p, k \right)_j \delta_{ip} n_p \, dA = - \oint_{\partial c} \frac{1}{\rho} p, i \delta_{ij} b_j \, ds. \quad (5.8)
\]
where $\delta_{ip}$ comes from the scalar product of (5.7) with the control volume’s outward facing normal unit-vector ($n_pe_p$), and $\delta_{ij}$ from the scalar product of the pressure gradient ($p_{,i}$) with the unit vectors ($b_{j}e_j$) along, and tangent to, the control volume’s border ($\partial c$).

From (5.8), the importance of Alhborn’s density discontinuity can be readily observed. If density were constant, (5.8) would be over the gradient of a single valued scalar vector field, i.e., over the conservative vector field $p_{,i}$ (scaled by the constant density), and would necessarily integrate to zero. However, by introducing a discontinuity, a finite production of vorticity is made possible:

\[
\frac{d\Gamma}{dt} = -\left( \int_a^b \frac{1}{\rho} p_{,i} \delta_{ij} b_j \, ds + \int_b^a \frac{1}{\rho} p_{,i} \delta_{ij} b_j \, ds \right) = -\frac{1}{\rho} (p_a - p_b), \tag{5.9}
\]

where the integration limits ($a$ and $b$) are on the surface of the “flat plate” at the mid-chord (Fig. 5.2). Within the infinitely dense flat plate, the integral goes to zero. As a final step, Ahlborn et al. (1998) assumes that the streamline connected to the stagnation point $a$ is lossless and steady ($p_a = p_\infty + 1/2 \rho U_\infty^2$). Thus, the magnitude of (5.9) can be expressed as:

\[
\frac{1}{U_\infty^2} \frac{d\Gamma}{dt} = \frac{(1 - C_{Pb})}{2}, \tag{5.10}
\]

where $C_{Pb} = \frac{p_b - p_\infty}{\frac{1}{2} \rho U_\infty^2}$. Making identical assumptions as in §5.1.2, (5.10) can be integrated over one shedding cycle to give the strength of a single shed vortex:

\[
\frac{\Gamma_{tot}}{U_\infty c} = \frac{1 - C_{Pb}}{2 S_t}, \tag{5.11}
\]

and therefore,

\[
\frac{1}{U_\infty^2} \frac{d\Gamma}{dt} \equiv S_t \left( \frac{\Gamma_{tot}}{U_\infty c} \right). \tag{5.12}
\]

Despite vastly different approaches to the same problem, the model of Ahlborn et al.
(1998) produces practically identical results to that of Roshko (1954b) (assuming that all generated vorticity subsequently contributes to the shed vortex strengths\(^1\)). The most notable difference between the two approaches is that Roshko’s model assumes that the pressure is constant along the back face of the flat plate, and can be prescribed from the base pressure, while the model of Ahlborn et al. (1998) assumes that only the pressure at the mid-chord is known (also prescribed from the base pressure).

\[ \begin{align*}
U_\infty & \quad x \\
\end{align*} \]

Figure 5.2: Control volume for the Ahlborn et al. (1998) model in red.

5.1.4 The flat plate topology

From Fig. 5.3, the streamline topology for phase averaged turbulent flow about a thin flat plate is shown, with three distinct regions being identified: i) a wake “bubble” whose streamlines begin and terminate on the flat plate, ii) a forming vortex, and iii) a separated vortex.

A separated shear layer supplies the forming vortex (ii) with circulation. As the forming vortex gains strength and advects downstream, a “bubble” (i) on the backface of the plate begins to grow. This “bubble” can be conceptualized by first imagining a cavity of recirculated flow (i.e., two stationary vortices behind the plate) for a steady, low-Reynolds number scenario. While the forming vortex advects downstream, new fluid pathways within the cavity region are created (thus forming the “bubble”) as the separated shear layer begins

\(^1\)Ahlborn et al. (1998) does not include a multiplicative factor to account for differences between vorticity generation at the flat plate edges and subsequent shed vortex strengths, unlike Roshko (1954b).
to be drawn inwards (Gerrard, 1965; Perry et al., 1982; Williamson, 1996). As the forming vortex continues to advect downstream, the bubble will grow until it covers the entire back face of the plate, at which point it will interrupt the opposing shear layer (which is supplying the forming vortex with vorticity). This interruption demarks the moment at which the forming vortex is said to have shed, and can be most easily identified by the appearance of a new saddle point. Visually, this is seen in Fig. 5.3, where the saddle points attached to (ii) clearly cuts off the circulation to the shed vortex (iii). Note that unlike the circular cylinder (Fig. 5.1), the “bubble” begins and terminates at fixed points (the edges of the plate), whereas for circular cylinders, the equivalent points oscillate in their locations.
Figure 5.3: Vortex shedding model for flow about a flat plate cylinder. Saddlepoints are denoted by red circles (●).
5.2 Vorticity cancellation and annihilation

Before delving into an analysis of the flat plate vortex shedding, it’s important to establish some terminology, namely, vorticity cancellation and vorticity annihilation. For the purposes of this thesis, vorticity cancellation will refer to apparent decreases in vorticity due to averaging, whereas annihilation will reflect decreases that have physical underpinnings.

5.2.1 Vorticity cancellation (type-1)

Vorticity cancellation of the first type is the predictable result of temporal averaging. From Fig. 5.4, a simple vortex shedding schematic is seen. Averaging over a typical shedding cycle, vorticity contained within the gray hatched region will be “canceled” out during the averaging process by the opposing and oppositely signed vortex street. Consequently, without any preprocessing steps, the interpretation of quantities like the average vorticity, vorticity flux, etc., become ambiguous in this region.

5.2.2 Vorticity cancellation (type-2)

Vorticity cancellation of the second type occurs when vorticity is engulfed into an oppositely signed vortex. The vortex shedding model of Gerrard (1965) (discussed in §5.1.1) briefly details one way by which this could occur, specific to the vortex shedding process. From Fig. 5.4, the hatched blue circle illustrates this type of engulfment, representing a region of vorticity that has been enveloped by an oppositely signed vortex. Through the energy cascade, a transfer of energy across scales is inevitable. And thus, there will also exist smaller engulfed eddies, represented by the blue dots in the right most vortex of Fig. 5.4.

Similar to vorticity cancellation of the first type, type-2 cancellation will impact time averaged vorticity quantities, as well as circulation calculations performed on the instantaneous, phase averaged, and average fields.
5.2.3 Vorticity annihilation

Vorticity, being a solenoidal/divergence free quantity \(^2\) cannot have any sources or sinks within the fluid. Moreover, Kelvin’s theorem shows (Batchelor, 1967) that for constant density fluids,

\[
\frac{\text{D} \Gamma}{\text{D} t} = \nu \int_{\partial c} u_{i,jj} \delta_{ip} b_p \, ds,
\]

(5.13)

where \(\nu u_{i,jj}\) is the momentum diffusion, and \(b_p \mathbf{e}_p\) represents unit vectors that are along, and tangent to, the control volume’s border (\(\partial c\)). In other words, any decay of vorticity within a constant density fluid (i.e., excluding any physical boundary conditions) will be due to viscous effects which enable the transfer of vorticity through the control volume’s borders, regardless of whether or not fluid elements pass through the border.

It is commonplace to observe decreases in vorticity that cannot be explained by diffusive losses across a control volume’s border. A simple thought experiment can be constructed that demonstrates this type of vorticity loss. Utilizing the Lamb-Oseen vortex model, and substituting the velocity and vorticity fields (4.12, 4.13) into the vorticity transport equation for plane two-dimensional flow,

\[
\partial_t \omega_z + u \frac{\partial \omega_z}{\partial r} + u_\theta \frac{1}{r} \frac{\partial \omega_z}{\partial \theta} = \nu \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \omega_z}{\partial r} \right) + \nu \frac{1}{r^2} \frac{\partial^2 \omega_z}{\partial \theta^2},
\]

(5.14)

it can be seen that the rate of change of vorticity is uniquely determined by the diffusive term \(\left( \nu \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \omega_z}{\partial r} \right) \right)\), which comes from the Laplacian \(^3\) of the vorticity field. In other words, (5.14) becomes a homogeneous linear PDE \((\partial_t \omega_z - \nu \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \omega_z}{\partial r} \right) = 0)\), and the principle of superposition applies. Superimposing two oppositely signed Lamb-Oseen vortices next to each other and allowing time to advance reveals a decay in the strength of each vortex, as the two vortices diffuse into each other and their oppositely signed vorticity fields begin

---

\(^2\)The divergence of the curl of a vector field is necessarily zero \((\omega_{i,i} = \partial_i (\epsilon_{ijk} u_{k,j}) = 0)\).

\(^3\)Letting \(P(r, \theta) = U(r \cos \theta, r \sin \theta)\), where \(U(x, y)\) is a function expressed in cartesian coordinates, the Laplacians of \(P(r, \theta)\) and \(U(x, y)\) can be expressed as \(\frac{1}{r} \frac{\partial^2 P}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial P}{\partial r} \right) = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\).
to overlap. Vorticity that is lost in this way will be categorized as vorticity annihilation. Experiments by Huang and Keffer (1996) and Huang et al. (1996) suggest that vorticity lost in this manner might not truly be lost. Indeed, the works of Huang and Keffer (1996) and Huang et al. (1996) demonstrated an instance wherein vorticity appeared to have been “lost” due to annihilation, but reemerged (albeit, after having been reorganized) at a later point in the flow.

In this study, measured vorticity decay that cannot be explained by mathematical cancellation will be attributed to annihilation. However, no attempt will be made to explain the physics of annihilation, beyond the hypothesis that annihilation increases with mixing, such that the small pockets of oppositely signed, engulfed vorticity (which contribute to type-2 cancellation), will eventually contribute to annihilation, as they continue to merge and diffuse into their surroundings.

![Figure 5.4](image_url)

Figure 5.4: Schematic showing the different types of vorticity cancellation and annihilation.

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4The vorticity transport equation does not include a generation or dissipation term (vorticity cannot be created or destroyed within the interior of a fluid), hence the works of Huang and Keffer (1996) and Huang et al. (1996) demonstrate the subtlety in the interpretation of what is meant by annihilation, cancellation, and mixing.
Figure 5.5: Schematic showing a theoretical semi-infinite control volume. The red plane indicates the PIV domain, the arrows denote the oncoming free stream flow, the black plane represents a thin flat plate, and the blue plane illustrates an arbitrary integration plane through which the rate of vorticity transport can be calculated.

5.3 Vorticity transport in the flat plate wakes

An accurate accounting of vorticity transport in the flat plate wakes is of significant interest. Such an accounting could be used to further validate the models of Roshko and Ahlborn, and provide important insights into wake mixing and turbulence. Consequently, this section introduces three different methods for estimating rates of vorticity transport in the flat plate wakes, with the results being compared and assessed in §5.3.4.

5.3.1 Method 1: Semi-infinite control volume

As a first approach towards the characterization of vorticity transport in the flat plate wakes, the average rate of vorticity transport can be estimated from the vorticity transport equation. From Fig. 5.5, a theoretical semi-infinite control volume (CV) is illustrated. Assuming that flow enters the CV through the \( yz \)-plane, and that the CV is so large that there is no flow entering or exiting any other faces of the CV, then the rate of change of vorticity within the CV is determined by the transport of vorticity though the initial \( yz \)-plane. By assuming that the flow is 2D in the mean, the analysis can be preemptively restricted to the \( z \)-component of
the vorticity transport equation (5.5). For baroclinic flows with negligible viscous diffusion, the $z$-component of the vorticity transport equation can be expressed in its integral form as:

$$\frac{d}{dt} \iiint_V \omega_z \, dV = -\iiint_V u_j \omega_{z,j} \, dV + \iiint_V \omega_j u_{z,j} \, dV.$$  \hspace{1cm} (5.15)

Noting that $(u_j \omega_{z,j} = (\omega_z u_j)_j - \omega_z u_{j,j})$ and $(w_j u_{z,j} = (u_z \omega_j)_j - u_z \omega_{j,j})$, where the right most terms go to zero (due to continuity and vorticity being solenoidal), the divergence theorem can be applied to the right hand side of (5.15):

$$\frac{d}{dt} \iiint_V \omega_z \, dV = -\oint_{\partial c} \omega_z u_i \delta_{ip} n_p \, ds + \oint_{\partial c} u_z \omega_i \delta_{ip} n_p \, ds,$$

$$\frac{d}{dt} \iiint_V \omega_z \, dV = -\oint_{\partial c} \omega_z u_i \delta_{ip} n_p \, ds + \oint_{\partial c} u_z \omega_i \delta_{ip} n_p \, ds,$$

(5.16)

where $n_p$ represents the components of the volume’s outward facing normal unit-vectors ($n_p e_p$). Because the flow has been assumed to be 2D in the mean ($\frac{\partial}{\partial z} \left( \frac{1}{T} \int \omega_z \, dt \right) = 0$), the average rate of vorticity transport through $yz$-planes at arbitrary streamwise ($x/c$) locations (shown as an arbitrary blue plane in Fig. 5.5) becomes:

$$\frac{1}{T} \int \frac{d}{dt} \iiint_V \omega_z \, dV \, dt = \Delta_z \int \omega_z u_x \, dy - \Delta_z \int u_z \omega_x \, dy,$$

(5.17)

where the mean two-dimensionality has been used to transform the right hand side $yz$-surface integrals into line integrals along $y$, scaled by the hight ($\Delta_z$) of the CV. Focusing on the left hand side of (5.17), Stokes theorem can be applied, and vorticity in the $xy$-planes can be

$^5$In the following derivation, the velocity vector components will be expressed using subscripts to limit confusion between the $z$-velocity and vorticity e.g., $u = (u_x, u_y, u_z)$ instead of $u = (u, v, w)$. 

77
expressed in terms of circulation:

\[
\frac{1}{T} \int \frac{d}{dt} \int \int \int \int \omega_z \, dV \, dt = \frac{1}{T} \int \frac{d}{dt} \left( \int \frac{d}{dz} \int \Gamma \, dz \right) \, dt = \frac{1}{T} \int \left( \int \frac{d}{dt} \frac{d}{dz} \right) \, dz = \int \frac{\Delta z}{\Delta t} \, dy = \Delta z \frac{\Delta \Gamma}{\Delta t}
\]

(5.18)

Thus (5.17) can be rewritten as:

\[
\Delta z \frac{\Delta \Gamma}{\Delta t} = \Delta z \int \omega_z u_x \, dy - \Delta z \int u_z \omega_x \, dy,
\]

(5.19)

Although the left hand side of (5.19) is mathematically correct, its interpretation can be misleading; circulation only exists about surfaces, not volumes. Therefore, it is best to continue interpreting this term as being the average rate of vorticity accumulation within the CV.

Expressing the velocity and vorticity fields in terms of double and triple decompositions:

\[
u = U^* + u'' = U + \langle u \rangle + u''
\]

(5.20)

\[
\omega = \Omega^* + \omega'' = \Omega + \langle \omega \rangle + \omega''
\]

where, \( \langle \cdot \rangle \) indicates phase averaging, (5.19) can be rewritten as:

\[
\Delta z \frac{\Delta \Gamma}{\Delta t} = \Delta z \int \left( \frac{\Omega^* u_x}{\omega_z} + \omega'' u_x \right) \, dy - \Delta z \int \left( \frac{U^* \Omega^*}{\omega_z} + \Omega'' \omega_x \right) \, dy.
\]

(5.21)

Note that the choice to split the field based on the phase average (as opposed to the POD) was deliberate. For mean two-dimensional flow, \( U_z \) and \( \Omega_x \) are zero, by definition. Moreover, \( \langle u_z \rangle \) and \( \langle \omega_x \rangle \) will be uncorrelated with the condition of the phase, and will also be zero. Therefore, \( \Omega^*_x \) and \( U^*_z \) will both be zero instantaneously, greatly simplifying the analysis. Additionally, (5.21) assumes that the residual and phase averaged fields and uncorrelated
linearly. Equation 5.21 can subsequently be expressed as:

$$\Delta_z \frac{d\Gamma}{dt} = \Delta_z \int \Omega_z U_x \, dy + \Delta_z \int \langle \omega_z \rangle \langle u_x \rangle \, dy + \Delta_z \int \left( \omega'_z u'_x - u'_z \omega'_x \right) \, dy,$$

(5.22)

where $\Delta_z \int \omega'_z u'_x \, dy$ and $\Delta_z \int u'_z \omega'_x \, dy$ represent the turbulent diffusion and tilting, respectively. Developing the turbulent diffusion and tilting terms, it can be shown that,

$$\Delta_z \int \left( u''_x \omega''_z - \omega''_x u''_z \right) \, dy = -\frac{\Delta_z}{2} \left| u''_z \right|_{y_{\text{min}}}^{y_{\text{max}}} + \Delta_z \int u''_x \frac{\partial u''_y}{\partial x} \, dy - \frac{\Delta_z}{2} \left| u''_z \right|_{y_{\text{min}}}^{y_{\text{max}}} + \Delta_z \int u''_z \frac{\partial u''_y}{\partial z} \, dy.$$

(5.23)

From (5.23), $\Delta_z \int u''_x \frac{\partial u''_y}{\partial x} \, dy$ was found to be negligible. Meanwhile, $\Delta_z \int u''_z \frac{\partial u''_y}{\partial z} \, dy$ was not calculable from the planar PIV data, but was assumed to be negligible due to the mean two-dimensionality of the flow. Thus, the average rate of vorticity accumulation in a CV downstream of some arbitrary $yz$-plane can be estimated as:

$$\Delta_z \frac{d\Gamma}{dt} \approx \Delta_z \int \Omega_z U_x \, dy + \Delta_z \int \langle \omega_z \rangle \langle u_x \rangle \, dy - \frac{\Delta_z}{2} \left( u''_x + u''_z \right) \left| y_{\text{max}} \right|_{y_{\text{min}}}.$$

(5.24)

Similarly, the average rate of circulation accumulation within the PIV integration domain can be estimated as:

$$\frac{d\Gamma}{dt} \approx \int \Omega_z U_x \, dy + \int \langle \omega_z \rangle \langle u_x \rangle \, dy - \frac{1}{2} \left( u''_x + u''_z \right) \left| y_{\text{max}} \right|_{y_{\text{min}}}.\quad (5.25)$$

Intuitively, if the entire PIV domain is used, the average rate of change of circulation will be zero due to the canceling effect of oppositely signed vortex streets entering the same integration domain. However, by separating the domain about the symmetry line into two parts (an upper half-plane and a lower half-plane), finite mean rates of circulation accumulation on the half-planes can be calculated.

From Fig. 5.6, profiles for the rates of circulation accumulation are shown. The dashed lines represent contributions from the combined mean and phase averaged fields (first two
terms on the right hand side of (5.25)), while the solid lines also include the effects of turbulence (i.e., the rightmost term of (5.25)). Consequently, the difference between the solid and dashed lines is the contribution of the turbulent diffusion and tilting terms. It can be seen that for CE not only are the turbulence levels greater, but they appear to be roughly constant (OE turbulent contributions show an initial growth). In both cases, the turbulent contributions do not change the character of the curves. For the CE case, the near constant turbulent contribution is consistent with observations made in Section 3.2.5 of Chapter 3 that align with a backflow impingement of coherent motions on the leeward face of the flat plate, contributing to increased incoherence in the near wake.

Figure 5.6: Rate of (absolute) circulation transport for the top and bottom half planes (averaged) of the PIV measurement domain for CE (left), and OE (right). Solid profiles were calculated using the averaged vorticity transport equation (5.25) and the dashed profiles were similarly calculated, less the turbulent diffusion and turbulent tilting terms

\[ \frac{1}{2} \left( \frac{\overline{u''_x}}{\overline{u''_x}} + \frac{\overline{u''_z}}{\overline{u''_z}} \right) \left( \left| y \right|_{y_{max}} \right) \]

5.3.2 Method 2 - Infinite control volume

The rates of circulation transport (5.25) of §5.3.1 provide important insights into the wake dynamics as they demonstrate that turbulent diffusion and tilting contribute appreciably to the rates of streamwise vorticity transport. However, the method of §5.3.1 cannot account for mathematical vorticity cancellation of the first and second types (§5.2.1 and §5.2.2).
To lessen the impacts of cancellation, an alternative method was constructed based on a common definition for flux:

\[
\text{Flux} = \int_{\partial c} (\cdot) u_i \delta_{ip} n_p \, ds. \tag{5.26}
\]

Equation 5.26 expresses the rate of transport of some arbitrary scalar, \((\cdot)\), through some surface \((s)\), where \(n_p\) represents the components of the surface’s outward facing normal unit-vector \((n_p e_p)\). Setting the integration surface to be a \(yz\)-plane (the blue surface in Fig. 5.5), the rate of vorticity transport can be calculated for each component of the vorticity, and at each \(x/c\) location within a CV. In the case of mean two-dimensional flow passing through a \(yz\)-plane, the rate of \(z\)-vorticity flux is:

\[
\text{Flux}_{\omega_z} = -\int_{\partial c} \omega_z u_x \, ds. \tag{5.27}
\]

If vorticity passing through the integration surface of (5.27) is assumed to have been dumped into an infinitely large CV, immediately downstream of the integration surface, then the flux can be written to expresses the rate at which vorticity accumulates within this CV:

\[
\frac{d}{dt} \iiint_V \omega_z \, dV = -\int_{\partial c} \omega_z u_x \, ds. \tag{5.28}
\]

Applying Stokes theorem, the left hand side of (5.28) can be rewritten in terms of \(xy\)-plane circulation:

\[
\frac{d}{dt} \int \Gamma \, dz = -\int_{\partial c} \omega_z u_x \, ds. \tag{5.29}
\]

As a final step, (5.29) can be time averaged. Making full use of the assumed mean two-dimensionality (similar to §5.3.1), the rate of circulation accumulation within the PIV integration domain can be written as:

\[
\frac{d\Gamma}{dt} = -\int \overline{\omega_z u_x} \, dy. \tag{5.30}
\]
Thus, for the flat plate planar PIV data, the mean rate of vorticity transport at different \( x/c \) locations is calculated by integrating over a single slice of the \( yz \)-plane, at \( z = 0 \) (thin blue line of Fig. 5.5).

Without modification, calculating rates of vorticity transport/rates of circulation accumulation in the flat plate wake using (5.30) is impacted by vorticity cancellation, similar to the method of §5.3.1. Indeed, selecting an integration line in the \( xy \)-plane that is parallel to the flat plate and extends to (±) infinity will result in a trivial solution, with the average rate of vorticity accumulation going to zero. The cancellation can be minimized by restricting the integration line to be \( y = [0, \pm \infty] \) (i.e., by considering the half planes separately, as was done in §5.3.1). Nonetheless, this flux will still reflect contributions from any oppositely signed vorticity that crosses the symmetry line, or becomes engulfed in an opposing vortex.

It was hypothesized that a more accurate accounting of the rate of vorticity transport (that would minimize type-1 and 2 vorticity cancellation) could be obtained by splitting the entire vorticity field into two parts, with the vorticity of each part being associated with one of the vortex streets. Noting that the coherent contributions to the vorticity are well captured by the low order model, the mean and coherent velocity fields were combined (i.e., \( \mathbf{u}^* (x, t) = \mathbf{U} (x) + \mathbf{u}_c (x, t) \)), and their resultant vorticity fields (\( \omega^* \)) were used as the basis for the split:

\[
\omega^+ (x, t) = \{ \omega (x, t) : \omega^* (x, t) > 0 \},
\]

\[
\omega^- (x, t) = \{ \omega (x, t) : \omega^* (x, t) \leq 0 \},
\]

such that,

\[
\omega (x, t) = \omega^+ (x, t) + \omega^- (x, t).
\]

In this way, the contributions of positive and negative vorticity to the rates of vorticity
transport/rates of circulation accumulation could be treated separately:

\[
\frac{d\Gamma^+}{dt} = - \int \omega^+_z u_x \, dy \\
\frac{d\Gamma^-}{dt} = - \int \omega^-_z u_x \, dy.
\]  

(5.33)

Expanding (5.33) (where only the positive case is shown for brevity):

\[
\frac{d\Gamma^+}{dt} = - \int \omega^+_z u_x \, dy \\
= - \int \left( \frac{\Omega^+_z U_x}{\text{Term 1}} + \frac{\Omega^+_z u'_x}{\text{Term 2}} + \frac{\omega^+_z U_x}{\text{Term 3}} + \frac{\omega^+_z u'_x}{\text{Term 4}} + \frac{\Omega^+_z u''_x}{\text{Term 5}} + \frac{\omega^+_z u''_x}{\text{Term 6}} + \frac{\omega''_z U_x}{\text{Term 7}} + \frac{\omega''_z u'_x}{\text{Term 8}} + \frac{\omega''_z u''_x}{\text{Term 9}} \right) dy.
\]  

(5.34)

From Fig. 5.7, the average $\frac{1}{U_\infty} \frac{d\Gamma}{dt}$ profiles are shown for CE and OE. Additionally, the $\frac{1}{U_\infty} \frac{d\Gamma}{dt}$ profiles associated with the combined mean and coherent fields are shown as dashed lines, and can be calculated using the first four terms of (5.34). Similar to the method of §5.3.1, the impact of the incoherent fluctuations are non-negligible.

Figure 5.7: Rate of circulation transport calculated along the PIV measurement domain using method-2; CE (left), and OE (right). Solid profiles were calculated from the total field, and the dashed profiles were similarly calculated, less any contributions from the incoherent field.
5.3.3 Method 3 - Vortex circulation

For a 2D bluff-body wake, the models of Ahlborn et al. (1998); Roshko (1954a) (§5.1.2, 5.1.3) demonstrate that the rate at which vorticity is generated on the bluff body’s surface can be approximated by the circulation that enters the wake over a single shedding cycle, multiplied by the shedding frequency, \( \frac{d\Gamma}{dt} = f_{sh} \Gamma \). If it is assumed that all the vorticity entering the wake subsequently rolls up and contributes to the strengths of the shedding vortices, then the rate of vorticity transport due to the motion of individual vortices convecting across the observation plane (blue line in Fig. 5.5) should match \( \frac{d\Gamma}{dt} \equiv f_{sh} \Gamma_{tot} \) (where \( \Gamma_{tot} \) reflects the total circulation associated with a given vortex). Under these assumptions, the transport of vorticity can be estimated at any streamwise location where the local vortex strength is known. Regrettably, the determination of vortex strengths (and by extension, the associated vorticity transport) is complicated by the difficulty of properly identifying vortices, and their associated vorticity. In the present analysis, the method outlined in Chapter 4 was used to estimate the strengths of shed vortices. Briefly, the circulation of shed vortices was determined by first identifying vortex cores from phase averaged velocity fields using the Q-criterion (Hunt et al., 1988). Next, initial estimates for the strengths (\( \Gamma_v \)) of the vortices were calculated by integrating the vorticity fields over the Q-identified regions (\( A_v \)). Finally, the strengths were divided by corrective factors (0.70, CE; 0.71, 0E) (§4.3.2) to account for vorticity that was not identified by the Q-criterion, but still associated with the vortices, i.e., \( \Gamma_{tot} \).

From Fig. 5.8 the estimated rates of circulation transport are shown; circles were generated using the combined mean and coherent phase averaged fields, while triangles were generated using the total phase averaged fields. It should be noted that the phase averaged vortex strengths, \( \langle \Gamma_v \rangle \), were not calculated when parts of the vortices fell outside of the PIV domain (i.e., far downstream), or within the formation region where the strengths of vortices are ambiguous due to the presence of vortex tails.

Interestingly, Fig. 5.8 shows a difference (albeit small) between the phase averaged mean
plus coherent field results, and the phase averaged total field results. Theoretically, the two cases should be similar, provided that the coherent field captures all motions (or non-linear couplings of motions) that are strongly correlated with the shedding frequency.

5.3.4 Methods for estimating circulation transport rates - comparisons and results

Figure 5.9 provides a comparison for the different methods of calculating the vorticity flux. Differences between methods 1 and 2 suggest that the mathematical cancellation of vorticity in the mean is indeed non-negligible. To further verify this assessment, profiles for method 2’s integrands (5.33) were plotted, as seen in Fig. 5.10. Indeed, Fig. 5.10 illustrates that in regions when methods 1 and 2 don’t align, there is additional vorticity transport that has either convected or diffused across the symmetry line. This additional vorticity will be mathematically canceled (type-1) using the RANS half plane approach (method 1). Figures 4.5 and 5.4 provide supplementary visual intuition for this transport.

Comparing methods 2 and 3, it can be seen that when the incoherent contributions are retained, the resultant vorticity transport estimates align quite convincingly, both in
magnitude and slope. The implication of this alignment is that the strengths and movements of shedding vortices overwhelmingly account for the convection of vorticity in the wakes. Although this result was not unexpected, it is nevertheless important, in that it completes the hermetic accounting of vorticity transport, and lends confidence to method-2. The newfound confidence in method-2 is a welcome result, due to the simplicity with which the method can be implemented, and the fact that it can be applied near the edges of the PIV domain, as well as within the formation region (unlike method-3 (§5.3.3)).

Interestingly, methods 2 and 3 don’t overlap nearly as well when the incoherent terms are omitted from the models. Although the misalignment is most noticeable for CE, this could be a red herring. Indeed, from §5.3.3 (method 3), it was noted that differences in results using the total phase averaged fields, and the combined mean and coherent phase averaged fields, likely indicate that there are additional motions (or non-linear couplings of motions) that are important, but not captured by the current definitions of the coherent field. Because method 3 is linear, whereas method 2 is not, even small changes in the coherent field’s definition could have amplified impacts on method 2. It is speculated that with a limited number of additional modes, methods 2 and 3 will align when using the combined mean and coherent fields. Consequently, a compelling avenue for future research would be the utilization of methods 2 and 3 for the discovery of new motions that are important to the flat plate wake dynamics.

For the measured PIV data, an ancillary benefit of being able to calculate vorticity transport rates in the formation region is that these calculations can also be used to assess the applicability of Roshko and Ahlborn’s models (§5.1.2 and §5.1.3). In particular, by extrapolating the near-field vorticity transport profiles back to $x = 0$, the vorticity production rates at the flat plate edges can be estimated, and subsequently compared to $\frac{1}{\rho_\infty} \frac{d\Gamma}{dt} = \frac{1-C_p}{2}$ (Equations 5.3 and 5.11). With base pressure coefficients of $-1.22$ (CE) and $-0.93$ (OE), the models of Roshko and Ahlborn predict vorticity production rates of 1.11 and 0.965, respectively. Using the total field, the extrapolated ($2^{nd}$ order polynomial) production rates
of method-2 (Fig. 5.11) show agreement with these estimates within 7.0% (CE) and 7.7% (OE); for method-1, the estimates are within 12% (CE) and 0.52% (OE).

Method’s 2 and 3 shed light on the results of Fage and Johansen (1928), who found that only a fraction of the generated vorticity will subsequently be found in individual, shed vortices. Notably, the results of these methods indicate that mathematical cancellation cannot explain the decay, leaving only annihilation (§5.2.3) as a viable pathway. From §5.2.3, it was speculated that annihilation increases with mixing. Using the total field, methods 2 and 3 show greater losses/annihilation of vorticity transport for CE, than OE. Thus, the hypothesis relating mixing to annihilation aligns with the increased incoherent contributions in the CE wake, as seen using method 1 (§5.3.1), as well as the indicators of mixing presented in Chapter 3, such as the vorticity thickness. Additional evidence for greater mixing in the

---

6For the method-1 extrapolation, CE points less than $x/\ell_R < 0.38$ were omitted.
Figure 5.10: Profiles at different $x/\ell_R$ locations for the vorticity transport integrands from method 2 i.e., $\frac{c}{U_\infty}(-\omega_z^{+/\text{-}}u_x)$, using the total field. CE (top) and OE (bottom). Profiles taken at different $x/\ell_R$ locations have been offset for visual clarity.

CE wake will be provided in the subsequent section.

5.4 Indicators of increased turbulence in the closed end wake

From Chapter 3, evidence indicating increased turbulence in the CE wake was presented. The results of §5.3.1 provided further evidence to this effect. In the following sections, additional analysis indicating increased wake turbulence in the closed ends case will be presented.
5.4.1 Centrifugal instabilities in the flat plate wakes

Curvature and turbulence have long been understood to be connected. Indeed, as early as 1880, Rayleigh (1880) concluded that for axis-symmetric flows (flows that are functions of $r$, $z$, but not $\theta$), the necessary and sufficient condition for centrifugal stability in the presence of disturbances is that the square of the circulation not decrease with increasing $r$. Subsequent work has revealed that the intuition gained from Rayleigh’s circulation criterion is applicable to real curved flows as well. Indeed, from the work of Liou (1994), centrifugal forces may either act to enhance, or stabilize disturbances. For curved shear layer flows, two instabilities are possible, namely, Kelvin-Helmholtz and Taylor-Görtler instabilities, producing spanwise and streamwise vortices, respectively. For centrifugally stable flows, the centrifugal forces generated by the streamline curvature works against the growth of disturbances, stabilizing the flow. When centrifugal stability is achieved, Taylor-Görtler vortices are dampened out, and the spatial development of Helmholtz instabilities slows down as curvature is increased. Conversely, for centrifugally unstable curved flows, along with the generation and growth of Taylor-Görtler vortices, the growth-rate of Helmholtz instabilities increases with curvature. In short, depending on if the flow is centrifugally stable or unstable, increased curvature will
further stabilize, or destabilize the flow.

Figure 5.12 shows the mean separation streamlines for the CE and OE cases. The separation streamlines were calculated as the isolines along which \( \int_0 U \, dy = 0 \), i.e., \( m = 0 \) (Castro and Haque, 1987; Mohammed-Taifour and Weiss, 2006). Perpendicular to the separation streamlines are several lines, colored by the mean velocities tangent to the separation streamlines. In the mean, it is evident that Rayleigh’s circulation criterion is not universally satisfied throughout the flow. Consequently, amplification of Kelvin-Helmholtz and Taylor-Görtler vortices is anticipated, with greater amplification in the CE case, due to its increased curvature. This is in agreement with the greater CE TKE and incoherent Reynolds stresses, as observed in Chapter 3. As an area for future work, a spanwise PIV plane, perpendicular to the flow, could be used to investigate the Taylor-Görtler vortices.

Figure 5.12: Left: the recirculation streamline \( \int_0 U/U_\infty \, d(\gamma/c) = 0 \) (green) with profiles of velocity tangent to the streamline curvature. Middle and Right: the \( U/U_\infty \) and \( V/U_\infty \)-velocity fields. Top, CE; bottom, OE.
5.4.2 Streamline trajectories as an indicator of mixing

As an additional, qualitative measure of mixing, averaged cycle-to-cycle streamlines were investigated; it was speculated that increased mixing would be noticed in the streamline paths.

Investigating all streamlines, each averaged over every shedding cycle, would have been impractical. Consequently, only streamlines that were closely related to the separation streamlines were investigated. As noted in §5.4.1, separation streamlines are typically calculated as the isolines along which $\int_0 U \, dy = 0$ (Castro and Haque, 1987; Mohammed-Taifour and Weiss, 2006). Although this definition is practical to implement, it fails to provide meaningful information after the point at which the separation streamlines reconverge (the recirculation length). To better capture the streamline dynamics past the recirculation length, only the streamlines whose paths monotonically increased in the streamwise direction and were closest to the axis of symmetry upon entering the PIV domain were considered. Additionally, to differentiate the averaged cycle-to-cycle streamlines, the streamlines were colored based on i) their local rate of vorticity transport, calculated at every $x/\ell_R$ location as $\frac{d\Gamma}{dt} = -\int_{-Y}^Y \omega_z u_x \, dy$, ii) the shift mode strength ($a_\Delta$), and iii) the flapping mode strength ($a_f$) \footnote{Streamlines colored by flapping mode strength were not provided for OE as the flapping motion is most readily observed in the OE extended field.}, with each metric being averaged over each shedding cycle. The rates of vorticity transport integration limits $[-Y,Y]$ were chosen such that they omitted any vorticity that was less than 5% of the maximum vorticity within the cycle averaged shear layers, for each $x/\ell_R$ location.

From Fig. 5.13, the cycle averaged streamlines can be seen. Immediately it is evident that the CE streamline paths experience much greater variance for similar shift mode strengths. Indeed, comparing CE to OE, the OE streamlines display very little variance for similar shift mode values, with paths that smoothly transition as the shift mode changes. The increased CE cycle-to-cycle variance is consistent with increased mixing.
As an additional indicator of mixing, it is noted that the CE case experiences consistently greater decay of vorticity transport; this is in line with observations made in §5.3.4. Focusing attention towards streamlines that are close to the average recirculation length $x/\ell_R \approx 1$, it can be seen that for both cases, vorticity decay increases when the shift mode is strongest, i.e., when the streamline curvature is greatest and the recirculation bubble is smallest. Under these circumstances, the greater decay aligns with greater mixing by means of curvature enhanced instabilities (§5.4.1), as well as observation made in Chapter 3, Section 3.2.5 that related increased coherent and incoherent Reynolds stresses to shortened formation lengths.

Of final note, a relationship between the shift and flapping mode can be seen for CE. In particular, it is noted that when the shift mode is strongest, so too is the flapping mode. This observation appears to be consistent with the phase portraits of Fig. 3.9 of Chapter 3.

![Figure 5.13: Cycle averaged streamlines colored based on i) local rates of cycle averaged vorticity transport, $\frac{1}{U_\infty} \frac{d\Gamma}{dt}$, (left), cycle averaged $a_\Delta$ (top-middle, bottom-right), and cycle averaged $a_f$ (top-left). CE (top), OE (bottom).](image-url)
5.5 Concluding remarks

In this chapter, three different methods for approximating rates of vorticity transport in mean 2D wakes were described. The first method made use of the averaged vorticity transport equation, integrated over half-planes that were split along the axis of symmetry. The second method employed a standard flux definition, integrated over the entire plane, that considered the transport of positive and negative vorticity separately. Finally, the third method made use of shed vortex strengths, as well as the flow Strouhal number, to estimate vorticity transport rates.

From the three methods, the second and third showed strong agreement with each other, while the first method deviated, largely due to increased mathematical vorticity cancellation. However the first method still yielded important insights. Namely, the first method provided a measure for turbulent diffusion and tilting of vorticity. Meanwhile, the alignment of the second and third methods reaffirmed the intuitive notion that the wake vorticity is predominantly transported by the shed vortices. Additionally, unlike the first method, the second and third methods minimized the impacts of mathematical cancellation of vorticity, thereby providing insight into the nebulous phenomena of vorticity decay/annihilation in bluff body wakes, which was speculated to be related to turbulence and mixing. The hypothesized link between vorticity annihilation and turbulence/mixing was in agreement with observation; it was shown that the CE wake not only had greater vorticity decay, but also had greater indicators of turbulent mixing, such as turbulent vorticity diffusion and tilting (as measured by the first method).

Finally, an area of future work was identified, namely, utilizing the second and third methods to discover new motions, or couplings of motions, that contribute appreciably to the wake dynamics.
Chapter 6

Synthesis

Flow about two thin flat plates with different end conditions (with and without end plates), normal to a uniform stream, was investigated. In both cases, the wetted span-to-chord ratios were large enough that the flow at center span could be assumed to be 2D in the mean; this was confirmed through measurement. For closed ends, the Reynolds number was 20000 and for open ends, 6600. Despite differences in $Re$, it was shown that for open ends, important mean field quantities such as Reynolds stresses do not change for $Re > 6600$; on this basis, the two fields were compared. As motivation for this investigation was the flat plate’s sensitivity to end conditions, with different end conditions resulting in two different dynamically stable wakes. For this investigation, the primary objectives were to i) compare and characterize the open and closed end flat plate flows, ii) investigate the characteristics of the vortex streets of the two flows, including vortex strength (circulation), trajectories, and convective speeds, and iii) investigate and compare different methods for estimating rates of vorticity transport and to complete a hermetic accounting of the vorticity transport in the two wakes. This chapter will detail and synthesize the important findings and results of this research, and provide suggestions for future work.
6.1 Conclusions

Two dynamically stable wake flow solutions were shown to exist for the flat plate flows, dependent on end condition. Despite both flows retaining their mean two-dimensionality, important differences in the wake structures, mean field properties, and spectral signatures were observed.

Many structural differences were observed between the two fields, despite having streamline topologies that were qualitatively similar. First, differences in recirculation lengths were noted, with the mean recirculation length of the closed ends case being shorter. The wake mean structure was found to not scale with the recirculation lengths, and different scalings were present in the streamwise and flow-normal directions. Within the base regions of the flows, the major contributions to the Reynolds stress fields were from the coherent motions (i.e., vortex formation and shedding); the structures of the Reynolds stress fields were determined by these dynamics as well. It was noted that for the closed ends case, Reynolds stress maxima were located within the recirculation region, and outside for open ends. In general, coherent Reynolds stresses were greater within the closed ends recirculation region. These differences demonstrated that the formation and recirculation regions are not synonymous, and highlighted the need to better understand the formation process.

Comparisons of the velocity spectra and auto-correlations taken at key points in the wakes indicated important differences in the cycle-to-cycle shedding events of the two geometries. Indeed, for open ends, a more rapid decay of the auto-correlations suggested greater variance in the shedding. Utilizing proper orthogonal decomposition, an energetic exchange between a fundamental harmonic pair describing the shedding and a slowly varying base flow (referred to as the drift mode) was noted. Phase portraits of the shedding and drift modes yielded a paraboloid, wherein the cusp of the paraboloid demarcated moments of shedding suppression (an unsteady solution for the wake flow). For open ends, the phase space trajectories regularly approached and encircled the paraboloid cusp, consistent with the observed auto-correlation decays. For closed ends, complete shedding suppression was
never observed, reflecting decreased variance in the wake. Consequently, the drift motion accounted for 7.4% of the open ends TKE, compared to the closed end’s 2.6%. Finally, the strength of the shedding events (and by extension the drift motion) were observed to influence the wake broadness, with greater curvature of the separating shear layers, and narrower shed vortex trajectories during strong shedding cycles. These differences not only indicate that the phase space solution trajectories are different, but that the turbulence levels (which can be thought of as stochastic forcing functions, perturbing the stable limit cycle trajectories) play an important role as a feedback mechanism within the wakes.

Looking at circular or square cylinders, the slow drift and harmonic motions are usually sufficient to describe the dynamics of the vortex shedding. However, for the flat plate an additional motion, referred to as the flapping mode, was found to be important to the wake dynamics. Notably, the flapping motion was shown to have a strong influence on the shed vortex trajectories, resulting in in-tandum, flow-normal deflections of the vortices, as compared to their mean paths. These deflections aligned with Biot-Savart induction, induced by imbalances in the shed vortex strengths of the the upper and lower vortex streets, thereby changing the kinetic energy content of the wake. Consequently, a tacit energetic link between the drift motion and the flapping motion (by extension, the first harmonic) was hypothesized. In agreement with this hypothesis, it was noted that the asymmetric shear layer/vortex trajectory deflections were weakest during periods of vortex suppression. This link was most apparent for the closed ends, wherein the flapping motion contributed comparatively more to the wake’s turbulent kinetic energy, resulting in greater deflections that were noticeable further upstream. Thus, the flapping motion, as well as its apparent energetic couplings, distinguishes the flat plate from typical bluff body geometries, such as the circular and square cylinders.

To improve investigations into the vortex formation and convection process, limitations in the Q-criterion’s ability to capture all vorticity contributing to a vortex’s strength were examined and a corrective factor was proposed. Notably, applying the Q-criterion to two
standard vortex models (Lamb-Oseen and Burgers) revealed that the Q-criterion captures at most 71.53% of a vortex’s vorticity. This percentage was experimentally checked for the two flow geometries, and good agreement was found, where 70% and 71% of the vorticity was described by Q-identified regions for the closed and open end flows, respectively. In addition, vorticity centroids, calculated over the Q-identified regions, were used to estimate vortex center locations, and vortex convection speeds. Compared to centers and convective speeds achieved using pressure minima and Q-maxima locations, it was found that the centroid method performed best, although the differences were slight.

Vorticity generation at the plate edges, as well as the accumulation of vorticity within shed vortices, were subsequently investigated. This investigation revealed increased turbulent diffusion and tilting of vorticity within the closed ends wake, which aligned with observations made throughout the study, including the higher drag/lower base pressure (in agreement with stronger mixing of high-momentum fluid), as well as increased vorticity thickness and shear layer curvature (leading to strengthened Taylor-Görtler and Kelvin-Helmholtz instabilities). By limiting the impact of mathematical cancellation on the vorticity transport rate estimates, insights into vorticity annihilation in the wake, as well as comparisons to estimates proposed by Ahlborn et al. (1998) and Roshko (1954b) were realized. The comparison of methods showed that the convection of vorticity almost entirely accounts for the wake transport of vorticity. Extrapolating the vorticity transport rate estimates back to the flat plate boundary \( x = 0 \) revealed strong agreement with generation rates proposed by the Ahlborn et al. (1998) and Roshko (1954b) models, which only require knowledge of the base pressure \( \left( \frac{1}{U_\infty^2} \frac{d\Gamma}{dt} = \frac{\left(1-C_Pb\right)}{2} \right) \). Finally, decreases in the streamwise transport of vorticity (vorticity annihilation) were shown to be greater for closed ends, and are thought to be related to increased mixing, in agreement with the closed end’s increased turbulence, vorticity thickness, and shear layer curvature. With vorticity decay being a phenomena that affects all bluff body flows, the methods developed to distinguish cancellation from annihilation, as well as the observations linking decay to mixing, are of general interest for understanding the mechanisms underlying
the redistribution of vorticity.

Although it remains unclear why different mean wakes arise for the flat plate flows (as opposed to other standard bluff bodies, such as the circular and square cylinders), it is speculated that the open ends creates a constant pressure end condition, leading to the pressurization of the base region (increasing the base pressure). Meanwhile, for the closed ends case, the end plates are thought to impose a Neumann like boundary condition, such that the boundary pressure levels are controlled by the flow. These differences are reflected in the base pressures and are of importance; any changes to the base pressures will be felt in all other aspects of the flow.

6.2 Recommendations for future work

Throughout this study, several areas deserving of future research/investigation were identified; these areas are detailed below:

1. Investigate energy dynamics within the flat plate wakes. This study focused heavily on the transport and dynamics of momentum and vorticity. However, a full investigation warrants a deeper look into the energy field and energy transport. Of particular interest is the stability of the separated shear layers, and how they affect energetic exchanges between the coherent motions, such as the tacit energetic link that appears to exist between the drift and flapping motions. With the flapping motion not being observed in other cylinder flows, these exchanges are of great interest, and could provide insight into why the flat plate has two stable solutions, instead of one.

2. Investigate whether methods 2 and 3 for estimating rates of vorticity transport (§5.3.2 and §5.3.3) can be used to identify additional motions that are important to the wake dynamics. Vorticity transport rates using method 3 were shown to change slightly, depending on whether or not phase averaging was preformed on the low-order models (mean plus coherent fields), or the total fields. If it is assumed that the low-order model
captures all contributions that are correlated with the shedding, then these difference should not exist. Differences using method 2 were also observed when comparing the total field and low-order model results, and it is expected that a more complete low-order model will similarly lead to alignment for method 2. Thus, these methods could be used to tease out additional motions, or non-linear couplings of motions, that contribute meaningfully to the flat plate wake dynamics.

3. **Investigate the presence of Taylor-Görtler vortices using a spanwise PIV plane, perpendicular to the oncoming flow.** The increased vorticity annihilation for the closed ends case was attributed to greater mixing in the closed end wake. And the increased mixing is believed to be fueled (in part) by curvature enhanced instabilities such as Taylor-Görtler vortices. Utilizing a spanwise PIV plane, the existence of these vortices could be confirmed or refuted. Not only would this contribute to the present understanding of the flat plate wakes, but it would provide insights that can be generalized for other bluff body flows, since shear layer curvature, mixing, and vorticity decay are not exclusive to the flat plate.

4. **Investigate methods for creating reduced order dynamical models describing the flat plate flows.** Not included in this thesis was a lengthy investigation into SINDy (Sparse Identification of Non-Linear Dynamics) (Brunton et al., 2016), and its ability to predict future flow states. Ultimately, the SINDy method showed promise for simplified, low-Reynolds number circular cylinder wakes. However this promise quickly evaporated when the method was applied to the highly turbulent flat plate flows. Nonetheless, a dynamic model describing the flat plate wake remains an important area for future research.
Bibliography


Appendix A

POD representations of the flat plate flows

This appendix presents the first seven POD modes for the symmetrically and asymmetrically split fluctuating velocity fields, associated with the closed, open, and extended open ends fields. Taking advantage of the flow symmetries, the symmetric and asymmetrically split fluctuating velocity fields were defined as:

\[
\begin{align*}
    u'_s (x,y,t) &= \frac{[u' (x,y,t) + u' (x,-y,t)]}{2} \\
    u'_a (x,y,t) &= \frac{[u' (x,y,t) - u' (x,-y,t)]}{2} \\
    v'_s (x,y,t) &= \frac{[v' (x,y,t) - v' (x,-y,t)]}{2} \\
    v'_a (x,y,t) &= \frac{[v' (x,y,t) + v' (x,-y,t)]}{2} \\
    w'_s (x,y,t) &= \frac{[w' (x,y,t) + w' (x,-y,t)]}{2} \\
    w'_a (x,y,t) &= \frac{[w' (x,y,t) - w' (x,-y,t)]}{2}.
\end{align*}
\]
As a matter of convention, modes that are symmetric in $\Phi^u$ and asymmetric in $\Phi^v$ are referred to as the symmetric modes, whereas modes that are asymmetric in $\Phi^u$ and symmetric in $\Phi^v$ are referred to as the asymmetric modes.
Closed ends - symmetric

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<th>$\Phi_n^v$</th>
<th>$\Phi_n^w$</th>
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<th>Spectrum</th>
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<td>0.013% of TKE</td>
<td>$10^{-10}$</td>
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</tbody>
</table>

Figure A.1: Closed ends (symmetric): Flooded isocontours of spatial modes $\Phi_n^u$, $\Phi_n^v$, and $\Phi_n^w$, as well as pseudo-streamlines and PSD of the modal temporal coefficients, $a_n$. Spatial modes are orthonormal ($\langle \Phi_n, \Phi_m \rangle = \delta_{nm}$), and $\int PSD(a_n) df = \overline{a_n^2}$. 

109
Closed ends - asymmetric

Figure A.2: Closed ends (asymmetric): Flooded isocontours of spatial modes \( \Phi_u^n \), \( \Phi_v^n \), and \( \Phi_w^n \), as well as pseudo-streamlines and PSD of the modal temporal coefficients, \( a_n \). Spatial modes are orthonormal \( ((\Phi_n, \Phi_m) = \delta_{nm}) \), and \( \int PSD(a_n)df = \overline{a_n} \).
Open ends - symmetric

Figure A.3: Open ends (symmetric): Flooded isocontours of spatial modes $\Phi^u_n$, $\Phi^v_n$, and $\Phi^w_n$, as well as pseudo-streamlines and PSD of the modal temporal coefficients, $a_n$. Spatial modes are orthonormal $(\Phi_n, \Phi_m) = \delta_{nm}$, and $\int PSD(a_n) df = \overline{a_n^2}$.  

111
Figure A.4: Open ends - asymmetric. Flooded isocontours of spatial modes $\Phi_n^u$, $\Phi_n^v$, and $\Phi_n^w$, as well as pseudo-streamlines and PSD of the modal temporal coefficients, $a_n$. Spatial modes are orthonormal ($\langle \Phi_n, \Phi_m \rangle = \delta_{nm}$), and $\int PSD(a_n)df = \overline{a_n^2}$. 

<table>
<thead>
<tr>
<th>$\Phi_n^u$</th>
<th>$\Phi_n^v$</th>
<th>$\Phi_n^w$</th>
<th>Streamlines</th>
<th>Spectrum</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.5% of TKE</td>
<td>25% of TKE</td>
<td>0.07% of TKE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6.6% of TKE</td>
<td>18% of TKE</td>
<td>0.07% of TKE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.99% of TKE</td>
<td>0.78% of TKE</td>
<td>0.019% of TKE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.37% of TKE</td>
<td>0.63% of TKE</td>
<td>0.023% of TKE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.034% of TKE</td>
<td>0.016% of TKE</td>
<td>0.89% of TKE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.094% of TKE</td>
<td>0.082% of TKE</td>
<td>0.54% of TKE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.22% of TKE</td>
<td>0.28% of TKE</td>
<td>0.16% of TKE</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Open ends (extended field) - symmetric

<table>
<thead>
<tr>
<th>$\Phi_u^n$</th>
<th>$\Phi_v^n$</th>
<th>$\Phi_w^n$</th>
<th>Streamlines</th>
<th>Spectrum</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.5% of TKE</td>
<td>0.34% of TKE</td>
<td>0.031% of TKE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$y/e$</td>
<td>$y/e$</td>
<td>$y/e$</td>
<td>$y/e$</td>
<td>$y/e$</td>
</tr>
<tr>
<td>-2</td>
<td>0</td>
<td>2</td>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>1.3% of TKE</td>
<td>1% of TKE</td>
<td>0.014% of TKE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.02% of TKE</td>
<td>0.02% of TKE</td>
<td>1.1% of TKE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.31% of TKE</td>
<td>0.73% of TKE</td>
<td>0.024% of TKE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.33% of TKE</td>
<td>0.67% of TKE</td>
<td>0.019% of TKE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.38% of TKE</td>
<td>0.4% of TKE</td>
<td>0.013% of TKE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.56% of TKE</td>
<td>0.091% of TKE</td>
<td>0.0057% of TKE</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure A.5: Open ends extended field (symmetric): Flooded isocontours of spatial modes $\Phi_u^n$, $\Phi_v^n$, and $\Phi_w^n$, as well as pseudo-streamlines and PSD of the modal temporal coefficients, $a_n$. Spatial modes are orthonormal ($\langle \Phi_n, \Phi_m \rangle = \delta_{nm}$), and $\int PSD(a_n)df = a_n^2$. 

113
Open ends (extended field) - asymmetric

<table>
<thead>
<tr>
<th>$\Phi_u^n$</th>
<th>$\Phi_v^n$</th>
<th>$\Phi_w^n$</th>
<th>Streamlines</th>
<th>Spectrum</th>
</tr>
</thead>
<tbody>
<tr>
<td>5% of TKE</td>
<td>25% of TKE</td>
<td>0.037% of TKE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.1% of TKE</td>
<td>21% of TKE</td>
<td>0.039% of TKE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.63% of TKE</td>
<td>0.85% of TKE</td>
<td>0.0023% of TKE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0078% of TKE</td>
<td>0.0066% of TKE</td>
<td>1.1% of TKE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.45% of TKE</td>
<td>0.53% of TKE</td>
<td>0.0084% of TKE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0031% of TKE</td>
<td>0.0045% of TKE</td>
<td>0.89% of TKE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.18% of TKE</td>
<td>0.45% of TKE</td>
<td>0.0048% of TKE</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure A.6: Open ends extended field (asymmetric): Flooded isocontours of spatial modes $\Phi_u^n$, $\Phi_v^n$, and $\Phi_w^n$, as well as pseudo-streamlines and PSD of the modal temporal coefficients, $a_n$. Spatial modes are orthonormal $((\Phi_n, \Phi_m) = \delta_{nm})$, and $\int PSD(a_n) df = a_n^2$. 

114
Appendix B

Transition modes

Figure B.1: Flooded isocontours for spatial function $u$ and $v$ components together with phase portraits and spectra of temporal coefficients for the transition mode, $a_t$. Phase portraits show $a_t$ vs. $a_2$ ($a_1 \approx 0$) with modal coefficients scaled by $\sqrt{\lambda_n}$. Red lines indicate a regression. CE (top), OE (middle), OE extended (bottom). Spatial modes are orthonormal: $(\Phi_n, \Phi_m) = \delta_{nm}$.

Figure B.1 shows the $u,v$ spatial functions, the $a_t-a_2$ phase portrait and spectra of the temporal coefficient $a_t$ for the first of the mode pairs (modes 7 and 8) observed in the three
POD spaces. These modes are interpreted as transition modes. Their spatial distributions are similar to those of the flapping mode, but the spectra show energetic contributions associated with the first harmonic. Their energetic contribution to the TKE is 2.5% (CE), 2.8% (OE) and 1.5% (OE-extended), respectively.

As can be inferred from the $a_1-a_2$ phase portrait, a relationship between the transition and fundamental harmonic pairs exists. For the three fields, a regression analysis suggests a relationship of the type $a_t \approx (A^2 + c_2)(a_1 + a_2)$, with $c_2$ a constant. Implied is thus a tacit link to $a_\Delta$ through $a_\Delta \approx c_0 + c_1 A^2$. Note that the regression lines are not indicative of the trajectories in phase space. Recalling that $A$ is nearly constant during a cycle, the trajectories are contained in the $a_1 - a_2$ plane and are thus normal to the $a_t - a_2$ plane. These transition modes thus appear to describe an exchange of energy between modes relating three distinct motions. While the details describing the energy exchange are beyond the present scope, inclusion of these modes appears important in considering the wake dynamics.
Appendix C

Q-criterion applied to Lamb-Oseen vortex

For the Lamb-Oseen vortex, the Q-criterion (4.2) becomes:

\[ Q = -u_{i,j} u_{j,i} = -(u_1^2 + u_2^2 + 2u_{1,2}u_{2,1}). \]  \hspace{1cm} (C.1)

Utilizing (4.12), Q can be expressed as:

\[ Q = -\left[ \frac{\Gamma_{\text{tot}}}{\pi r^2} \left( \frac{\Gamma_{\text{tot}} \left( \exp \left( -\frac{r^2}{4\nu t} \right) - 1 \right)}{2\pi r^2} + \frac{\Gamma_{\text{tot}} \exp \left( -\frac{r^2}{4\nu t} \right)}{4\nu \pi t} \right) \left( \exp \left( -\frac{r^2}{4\nu t} \right) - 1 \right) \right]. \]  \hspace{1cm} (C.2)

Non-dimensionalizing (C.2) yields:

\[ \Pi_1 = -\frac{\Pi_3^2}{2\pi^2} \left[ \left( \exp \left( -\Pi_2/4 \right) - 1 \right) + \frac{\Pi_2 \exp \left( -\Pi_2/4 \right)}{2} \right] \left( \exp \left( -\Pi_2/4 \right) - 1 \right) \]  \hspace{1cm} (C.3)

where the Pi groups are,

\[ \Pi_1 = Qt^2 \]  \hspace{1cm} (C.4)
\[ \Pi_2 = \frac{r^2}{\nu t} = \frac{r^2}{r_0^2} \]  \hspace{1cm} (C.5)
and

\[ \Pi_3 = \frac{t \Gamma_{\text{tot}}}{r^2}. \tag{C.6} \]

From (C.3), it can be seen that \( \Pi_1 = 0 \) for the trivial cases when \( \Pi_3 = 0 \) \((t = 0)\), and \( \Pi_2 = 0 \) \((r = 0)\). Additionally, it is easily shown that \( \Pi_1 \) is zero for the non-trivial case when \( \Pi_2 \approx 5.03572 \). Thus, for the non-trivial case, the fraction of the total circulation contained within the \( Q = 0 \) region is \( \Gamma/\Gamma_{\text{tot}} = 1 - \exp(-\Pi_2/4) = 0.7153 \).

Of additional note: consistent with results presented graphically in Fig. 4.2,

\[ \frac{\omega_z}{\Omega_0} = \frac{\omega_z}{u_\theta/r} = \frac{\Pi_2 \exp(-\Pi_2/4)}{2(1 - \exp(-\Pi_2/4))} = 1, \tag{C.7} \]

when \( Q = 0 \).
Appendix D

Pressure field estimation

Although the study of the thin flat plates did involve pressure measurements, the measurements were only preformed on the leeward faces of the plates, by means of embedded pressure taps. Therefore, a full (measured) accounting of the pressure fields was not available for analysis. However, this lack of pressure data was far from exceptional, as pressure measurement devices are intrusive, and would have created new, unwanted dynamics in the wakes. Fortunately, for incompressible flows, the pressure field can easily be estimated from the velocity field. Indeed, from the Navier-Stokes equation,

$$\frac{\partial}{\partial t} u_i = -u_{i,j}u_j - p_i/\rho + g_i + \nu u_{i,jj}$$ (D.1)

the divergence can be taken,

$$\frac{\partial}{\partial t} u_{i,i} = -(u_{i,jj}u_j + u_{i,j}u_{j,i}) - p_{,ii}/\rho + g_{i,i} + \nu u_{i,jji}$$ (D.2)

and continuity applied,

$$u_{i,i} = 0$$ (D.3)
yielding the Poisson pressure equation:

\[ p_{ii} = -\rho u_{i,j} u_{j,i}, \]  

(D.4)

an elliptic PDE that can be numerically solved to give the instantaneous pressure fields. Briefly, using the PIV velocity data, the pressure was calculated using an iterative over-relaxation process for a second-order finite difference scheme. From the momentum balance equation, \( \frac{dp}{dx} \) or \( \frac{dp}{dy} \) can be determined and subsequently used to satisfy the edge boundary conditions (i.e., a Neumann type boundary conditions is applied to the edges). To specify the corner values, Dirichlet boundary conditions were applied. In effect, reference points where chosen that extended far outside of the measurement domain. In addition, regions outside of the measurement domain were assumed inviscid. After, the unsteady Euler equation was evoked, which reduced to the standard Bernoulli equation, and the corner pressures were determined, integrating from the reference points to the corners. A more detailed accounting of the pressure field approximation process can be found in (de Plessix, 2015).
Appendix E

Vortex selection code

The following section includes the functions that were used to manually define the vortex cores, as described in §4.3.1. All non-proprietary code has been included with the exceptions of bluewhitered.m and tspo_ga.m. The omitted MATLAB scripts can be accessed online\(^1\). Additionally, the author would like to thank Conrad Bingham for his collaboration on the TrimVor.m code.

Briefly, this code creates a graphical user interface (GUI) that allows users to determine integrated vorticity from manually selected regions of the vorticity field. To assist the user in the boundary selection, the GUI displays the vorticity field, as well as vorticity iso-contours that are equal to some fraction of the absolute maximum vorticity in the field. Thus, the user can use the vorticity iso-contours to avoid the inclusion of noisy data. Typical use of the GUI involves selecting points that begin and end at the saddle points, and then quickly merge with the vorticity iso-contours. The GUI has five different buttons: Select, Delete, Clear, Circ, and Finish. When Select is chosen, the user can freely click on the vorticity field and define an integration region, with each click generating a green dot to visually indicate the chosen point. When the Delete button is chosen, user clicks will delete the

\(^1\)bluewhitered.m can be accessed at: https://www.mathworks.com/matlabcentral/fileexchange/4058-bluewhitered

\(^2\)tspo_ga.m can be accessed at: https://www.mathworks.com/matlabcentral/fileexchange/21196-open-traveling-salesman-problem-genetic-algorithm
nearest points in the user defined integration region. The *Clear* button returns the GUI to
its initial state, and deletes any user selected points. When the user is satisfied with their
integration region, they can press the *Circ* button. The *Circ* button will fill in the user
defined region with additional open-green dots (to indicate the selected integration area),
display the unfiltered Q-identified region of the associated vortex, display to vortex centroid
as described in §4.4.1, and calculate the vorticity contained within the user defined region
and the Q-region. Additionally, important quantities, such as the circulation ratio between
the user-defined region, and the Q-region, are displayed in the MATLAB command window.
Finally, the *Finish* button will save and then close the figure. A sample frame from the GUI
can be seen in Fig 4.5.

To use the code, users must input their \(x, y\)-spatial grids, and the \(u, v\)-velocity fields (where \(x\) corresponds to the streamwise direction, and \(y\) is normal to the flow). The spatial
grids must be 2D matrices (\(I \times J\)), while the velocity fields are 3D with dimensions (\(I \times J \times N_t\)), where \(N_t\) is the number of snapshots. Additionally, an estimation of the streamwise
vortex convection speed must be included. Because PIV data typically includes thousands
of snapshots, the user has the option to specify the snapshots of interest in a numbered
array. Additional (optional) arguments allow the user to change the vorticity iso-contour
levels, the amount of filtering applied to the Q-identified regions (as described in Chapter 4,
Section 4.4.1), the value of the Q threshold, and the directory locations for saving the output
figures.

### E.1 GUI - Front End

```matlab
function [vortex_circ_userSelect, vortex_circ_unfiltered, ...
  vortex_circ_filtered, xCentroid_filtered, yCentroid_filtered] = ...
  ManualVortexID(x_g, y_g, u_g, v_g, u_conv, ind, varargin)

%% Manual Vortex Identification
```

122
% Following script allows for the manual selection of vortex boundaries.
% Users can submit their entire velocity data and control which timesteps
% are examined using the "ind" input.

% Inputs

% x_g: spatial grid (JxI)
% y_g: spatial grid (JxI)
% u_g: u-velocity field (JxIxNt)
% v_g: v-velocity field (JxIxNt)
% u_conv: estimate of the vortex convection velocity (u-component)
% ind: the indices of the timesteps you wish to investigate ex: ind = [1,3]
% (Optional) contour_perc: percent of max vorticity that we would like to
draw a contour for (default: 3)
% (Optional) Qcut: cutoff value for Q-criterion (default: 10E-8)
% (Optional) thresh: used for filtering vortex tails - higher value
% results in greater filtering (default: 0.2)
% (Optional) iter: used for filtering vortex tails - higher value
% results in greater filtering (default: 25)
% (Optional) pub_dir: path to folder you wish to save .fig files to ...
% (default: [])
% (Optional) vis_dir: path to folder you wish to save .eps files to ...
% (default: [])

% Outputs

% vortex_circ_userSelect: circulation of user selected vortex
vortex

% vortex_circ_unfiltered: circulation of Q-identified vortex w/o filtering
% vortex_circ_filtered: circulation of Q-identified vortex w/ filtering
% xCentroid_filtered: location of vortex centroid within Q-identified ...
% vortex
% yCentroid_filtered: location of vortex centroid within Q-identified ...
% vortex

% Important notes:

% This code assumes that the grid spacing is constant!

% For a fully separated vortex, the filtered circulation and the unfiltered
% circulation should be identical. The filtering process was designed to
% remove the vortex tails, and nothing else. This ensures that the centroid
% is always calculated properly (i.e. not biased heavily by the vortex
% tail).

%% Handle/parse input arguments
% parse varargin
p = inputParser;
addRequired(p,'x_g');
addRequired(p,'y_g');
addRequired(p,'u_g');
addRequired(p,'v_g');
addRequired(p,'u_conv');
addRequired(p,'ind');
addOptional(p,'contour_perc', 3);
addOptional(p,'Qcut', 10E-8);
addOptional(p,'thresh', 0.2);
addOptional(p,'iter', 25);
addOptional(p,'pub_dir', []);
addOptional(p,'vis_dir', []);
parse(p,x_g, y_g, u_g, v_g, u_conv, ind, varargin{:})

contour_perc = p.Results.contour_perc/100;
Qcut = p.Results.Qcut;
thresh = p.Results.thresh;
iter = p.Results.iter;
pub_dir = p.Results.pub_dir;
vis_dir = p.Results.vis_dir;

%% Main
% define the size of the data
num_snapshots = length(ind);
J = size(x_g, 1);
I = size(x_g, 2);

% Only look at desired data
u_g = u_g(:,:, ind);
v_g = v_g(:,:, ind);

% calculate grid spacing (assumes all grid spacing constant)
dx = abs(x_g(1,2) - x_g(1,1));
dy = abs(y_g(2,1) - y_g(1,1));

% Predefine matrices
vort = zeros(J, I, num_snapshots);
Q = zeros(J, I, num_snapshots);
Qbin = zeros(J, I, num_snapshots);
vortexLabel_unfiltered = zeros(J, I, num_snapshots);
vortexLabel_filtered = zeros(J, I, num_snapshots);
vortex_circ_userSelect = zeros(1, num_snapshots);
vortex_circ_unfiltered = zeros(1, num_snapshots);
vortex_circ_filtered = zeros(1, num_snapshots);
xCentroid_filtered = zeros(1, num_snapshots);
yCentroid_filtered = zeros(1, num_snapshots);

for ii = 1:num_snapshots
    % Calculate vorticity field
    [vort(:,:,ii), -] = curl(x_g, y_g, u_g(:,:,ii), v_g(:,:,ii));
    % Calculate Q-field
    Q(:,:,ii) = QCriterion(u_g(:,:,ii), v_g(:,:,ii), dx, dy);
    % filter the Q-field
    for xIndex=1:I
        for yIndex=1:J
            if Q(yIndex, xIndex, ii) > Qcut
                Qbin(yIndex, xIndex, ii) = 1; % set all Q-criterion ...
                values to "on or off"
            end
        end
    end
    % create labels for each vortex
    [vortexLabel_unfiltered(:,:,ii), -] = vortexLabelling(Qbin(:,:,ii), ...
    vort(:,:,ii), I, J, 0, thresh, iter); % identify the unique ...
    vortices, even when two vortices of opposite sign are attached
    [vortexLabel_filtered(:,:,ii), -] = vortexLabelling(Qbin(:,:,ii), ...
    vort(:,:,ii), I, J, 1, thresh, iter); % identify the unique ...
    vortices, even when two vortices of opposite sign are attached

    % Run GUI for manual vortex selection
    [vortex_circ_userSelect(1, ii), vortex_circ_unfiltered(1, ii), ...
    vortex_circ_filtered(1, ii), xCentroid_filtered(1, ii), ...
    yCentroid_filtered(1, ii)] = UserVortexDefine(x_g, y_g, ...
    vort(:,:,ii), u_g(:,:,ii), v_g(:,:,ii), u_conv, ind(ii), ...
    vortexLabel_filtered(:,:,ii), vortexLabel_unfiltered(:,:,ii), ...
    contour_perc, pub_dir, vis_dir);
end
function [vortex_circ_userSelect, vortex_circ_unfiltered, ...
    vortex_circ_filtered, xCentroid_filtered, yCentroid_filtered] = ...
    UserVortexDefine(x_g, y_g, vort, u_g, v_g, u_conv, frame_num, ...
    vortexLabel_filtered, vortexLabel_unfiltered, contour_perc, pub_dir, ...
    vis_dir)

%% Manual Vortex Identification
% Author: Eric Braun - eabraun@ucalgary.ca
% Date: 10/31/2019

% The following code creates a gui that allows the user to manually
% specify the vortex core of interest. Upon pressing "calculate circ"
% button, the plot will update and show the user selected region, as well
% as the associated region based on the unfiltered Q-criterion (the magenta
% outline). The centroid will also be plotted, and can be seen as the
% magenta star.

% ---------------------------------------------------------------
% Inputs
% ---------------------------------------------------------------
% x_g: spatial grid
% y_g: spatial grid
% vort: vorticity field
% uf_g: fluctuating u-velocity field
% vf_g: fluctuating v-velocity field
% u_conv: estimate of the vortex convection velocity (u-component)
% frame_num: the indices of the timesteps you wish to investigate
% vortexLabel_filtered: labels of filtered vortex
% vortexLabel_unfiltered: labels of unfiltered vortex
% contour_perc: percentage of max vorticity that you want to draw contour
% for
% pub_dir: path to folder you wish to save .fig files to
% vis_dir: path to folder you wish to save .eps files to

% Outputs

% vortex_circ_userSelect: circulation of user selected vortex
% vortex_circ_unfiltered: circulation of Q-identified vortex w/o filtering
% vortex_circ_filtered: circulation of Q-identified vortex w/ filtering
% xCentroid_filtered: location of vortex centroid within Q-identified vortex
% yCentroid_filtered: location of vortex centroid within Q-identified vortex

%% Initial GUI Setup

set(0,'defaulttextinterpreter','default')
set(0,'DefaultTextFontSize', 10)
set(0,'DefaultTextFontname', 'CMU Serif')
set(0,'DefaultAxesFontSize', 10)
set(0,'DefaultAxesFontName','CMU Serif')
set(0,'defaultuicontrolunits','normalized')

% Create our figure

f = ... 
figure('Visible','off','units','normalized','outerposition',[0.5000 0.0370 0.5000 0.9630]);

% gui_data will hold anything that we want to be able to pass into our
% push functions and callback functions.

gui_data.add_data = 1; % if data.add_data = 1 when user selectes ...
points,
% they will be added to the list. If data.add_data = 0, then we will subtract the % nearest selected point

gui.data.x_pts = []; % will hold our selected points
gui.data.y_pts = [];
finish = 0; % when user sets finish = 1, figure closes, data % is stored, and the code progresses

gui.data.region = [];

gui.data.x_g = x_g;

 gui.data.y_g = y_g;

 vortex_circ_userSelect = [NaN];

 xCentroid_unfiltered = [NaN];

 yCentroid_unfiltered = [NaN];

 vortex_circ_unfiltered = [NaN];

 xCentroid_filtered = [NaN];

 yCentroid_filtered = [NaN];

 vortex_circ_filtered = [NaN];

 gui.data.vort = vort;

 gui.data.vortexLabel_filtered = vortexLabel_filtered;

 gui.data.vortexLabel_unfiltered = vortexLabel_unfiltered;

guidata(f,gui.data) % guidata gets passed into any callback and ...

    pushbutton functions

% Following is used to define the figure dimension

num_buttons = 5; % number of buttons

button.w = f.Position(3)/6; % width of buttons

button.h = f.Position(4)/15; % height of buttons

button.s = button.w/3; % spacing between buttons

button.loc_y = button.h/10; % Position of button on screen

button.loc_x = 0.5 - (num_buttons*(button.w) + ...  
               (num_buttons-1)*button.s)/2;
% Create our buttons!
hsselect = uicontrol('Style','pushbutton','String','Select',...
    'Position',[button_loc_x, button_loc_y, button_w, button_h], ...
    'Callback',{@selectbutton_Callback});

hdelete = uicontrol('Style','pushbutton','String','Delete',...
    'Position',[button_loc_x + button_w + button_s, button_loc_y, ...
                button_w, button_h], ...
    'Callback',{@deletebutton_Callback});

hclear = uicontrol('Style','pushbutton','String','Clear',...
    'Position',[button_loc_x + 2*(button_w + button_s), ...
               button_loc_y, button_w, button_h], ...
    'Callback',{@clearbutton_Callback});

hcirc = uicontrol('Style','pushbutton','String','Circ',...
    'Position',[button_loc_x + 3*(button_w + button_s), ...
               button_loc_y, button_w, button_h], ...
    'Callback',{@circbutton_Callback});

hfinish = uicontrol('Style', 'pushbutton', 'String', 'Finish', ...
    'Position', [button_loc_x + 4*(button_w + button_s), ...
                 button_loc_y, button_w, button_h], ...
    'Callback', {@finishbutton_Callback});

% create figure axes
ha = axes('Units', 'Normalized', 'Position', [button_w, ...
    10*button_loc_y+button_h, 1-2*button_w, 1 - ...) 1.5*(10*button_loc_y+button_h)]);
title([['Vorticity field - frame number: ', num2str(frame_num)]]) % ...
    title for the axes
align([hselect,hdelete],'none', 'Center'); % this code doesn't really seem to do anything lol

%% Plot the vorticity plot
nlev = 20;
clim = max(abs(vort(:)));
clev = linspace(floor((-clim)*100)/100, ceil((clim)*100)/100, nlev);

% show the contour that cuts off the bottom 1% of vorticity
min_contour = min(vort(:))*contour_perc;
max_contour = max(vort(:))*contour_perc;

% Make the UI visible.
f.Visible = 'on';

% plot everything
hold on
cf = contourf(x_g,y_g, vort, clev, 'edgecolor','none', ...
  'ButtonDownFcn',{@choose_data,ButtonDownFcn});
mincf = contour(x_g,y_g,vort, [min_contour, min_contour], 'b', ...
  'pickableparts', 'none', 'LineWidth', 3);
maxcf = contour(x_g,y_g,vort, [max_contour, max_contour], 'r', ...
  'pickableparts', 'none', 'LineWidth', 3);
ss = streamslice_mod(x_g, y_g, u_g - u_conv, v_g);
set(ss,'color','k', 'pickableparts', 'none')
misc_ax = plot(x_g(1,:), zeros(1,size(x_g,2)), '--', 'Color', 'k', ...
  'pickableparts', 'none');
caxis([min(clev), max(clev)]);
colormap(bluewhitered(nlev, 'soft'));
xlabel('x/c')
ylabel('y/c')

%% GUI Functions
% Note: for plotting, ff = figure handle, and aa = axis handle

function selectbutton_Callback(source, eventdata)
    data = guidata(source);
    data.add_data = 1;
    guidata(source,data)
end

function deletebutton_Callback(source, eventdata)
    data = guidata(source);
    data.add_data = 0;
    guidata(source,data)
end

function clearbutton_Callback(source, eventdata)
    data = guidata(source);
    hh = gcbo; % Handle of object whose callback is executing
    ff = gcf; % our main figure
    aa = ff.Children(end); % since figure is always defined before
    % the push buttons, this will always ...
    % grabbing the
    % correct handle for the axis
    set(ff, 'CurrentAxes', aa) % this sets the current axis properly

    % if the user presses clear and the vortex bound has already
    % been created, delete the points within the boundary and its edges
    % as well as the centroid and Q border.
    if ~isempty(findobj(aa.Children, 'flat', 'displayname', ...
        'vort.edge'))
edge_plt = findobj(aa.Children, 'flat', 'displayname', ...
    'vort.edge');
edge_plt.XData = [];
edge_plt.YData = [];

inon_plt = findobj(aa.Children, 'flat', 'displayname', ...
    'vort.inon');
inon_plt.XData = [];
inon_plt.YData = [];

Q_plt = findobj(aa.Children, 'flat', 'displayname', 'Q');
Q_plt.XData = [];
Q_plt.YData = [];
Q_plt.ZData = [];

Cent_plt = findobj(aa.Children, 'flat', 'displayname', ...
    'centroid');
Cent_plt.XData = [];
Cent_plt.YData = [];

end

% Delete points
if isempty(data.x_pts)
    disp('No data to clear')
else
    data.x_pts = [];
    data.y_pts = [];
    pt.slect_plt = findobj(aa.Children, 'flat', 'displayname', ...
        'selected_pts');
    pt.slect_plt.XData = data.x_pts;
    pt.slect_plt.YData = data.y_pts;
    guidata(source, data)
end
vortex_circ_userSelect = NaN;
vortex_circ_unfiltered = NaN;
vortex_circ_filtered = NaN;
xCentroid_filtered = NaN;
yCentroid_filtered = NaN;
xCentroid_unfiltered = [NaN];
yCentroid_unfiltered = [NaN];
end

function choose_data_ButtonDownFcn(source, eventdata)
    data = guidata(source);
    hh = gcbo; % Handle of object whose callback is executing
    ff = gcf; % our main figure
    aa = ff.Children(end); % since figure is always defined before
        % the push buttons, this will always ...
        % grab the
        % correct handle for the axis
    set(ff, 'CurrentAxes', aa) % this sets the current axis properly

    if ~isempty(findobj(aa.Children, 'flat', 'displayname', ...
        'vort.edge'))
        edge_plt = findobj(aa.Children, 'flat', 'displayname', ...
            'vort.edge');
        edge_plt.XData = [];
        edge_plt.YData = [];

        inon_plt = findobj(aa.Children, 'flat', 'displayname', ...
            'vort.inon');
end
inon.plt.XData = [];  
inon.plt.YData = [];  

Q_plt = findobj(aa.Children, 'flat', 'displayname', 'Q');  
Q_plt.XData = [];  
Q_plt.YData = [];  
Q_plt.ZData = [];  

Cent_plt = findobj(aa.Children, 'flat', 'displayname', 'centroid');  
Cent_plt.XData = [];  
Cent_plt.YData = [];  

end  

% Delete points
if data.add_data == 0 && isempty(data.x_pts)  
disp('No data to delete')  
elseif data.add_data == 0 && ~isempty(data.x_pts)  
x_del = eventdata.IntersectionPoint(1);  
y_del = eventdata.IntersectionPoint(2);  

% remove closest point to user selected point  
euclid_dist = ((data.x_pts - x_del).^2 + (data.y_pts - y_del).^2).^(1/2);  
[~, ind] = min(euclid_dist);  
data.x_pts(ind) = [];  
data.y_pts(ind) = [];  
pt.slct_plt = findobj(aa.Children, 'flat', 'displayname', 'selected_pts');  
pt.slct_plt.XData = data.x_pts;  
pt.slct_plt.YData = data.y_pts;  
guidata(source, data)  
end
% Add points
if data.add_data == 1
    data.x_pts = [data.x_pts, eventdata.IntersectionPoint(1)];
    data.y_pts = [data.y_pts, eventdata.IntersectionPoint(2)];
    display(['Data selected: x = ', num2str(data.x_pts(end)) ' ... |
            y = ' num2str(data.y_pts(end))])
end

% if we don't do the following loop, each time we place a new
% point, a new scatter plot will be created, which is annoying!
% We will name our scatter plot "selected_pts" so that it is
% easy to find. If the scatter plot does not already exist, we
% will create it and name it.
if ~isempty(findobj(aa.Children, 'flat', 'displayname', ...
    'selected_pts'))
    pt_slct_plt = findobj(aa.Children, 'flat', ...
        'displayname', 'selected_pts');
    pt_slct_plt.XData = data.x_pts;
    pt_slct_plt.YData = data.y_pts;
    drawnow
    guidata(source,data)
else
    scatter(data.x_pts(end), data.y_pts(end), 40, 'g', ...
        'filled', 'pickableparts', 'none', 'displayname', ...
        'selected_pts')
    guidata(source,data)
end

vortex_circ_userSelect = NaN;
vortex_circ_unfiltered = NaN;
vortex_circ_filtered = NaN;
xCentroid_filtered = NaN;
yCentroid_filtered = NaN;
xCentroid_unfiltered = [NaN];
yCentroid_unfiltered = [NaN];
end

function circbutton_CallBack(source, eventdata)
    data = guidata(source);
    hh = gcbo; % Handle of object whose callback is executing
    ff = gcf; % our main figure
    aa = ff.Children(end); % since figure is always defined before
    % the push buttons, this will always ...
        % grab the
    % correct handle for the axis
    set(ff, 'CurrentAxes', aa) % this sets the current axis properly

    % solve the traveling salesman problem to get the border
    if isempty(data.x`pts) || length(data.x`pts) <3
        disp('Not enough points')
    else
        xy = [data.x`pts', data.y`pts'];
        tsm_result = tspo.ga('XY', xy, 'showProg', 0, 'showResult', ... 0, 'showWaitbar', 0);
        data.region = tsm_result.optRoute;
        data.region = [tsm_result.optRoute, tsm_result.optRoute(1)];

        % get the border and interior points
        [in, on] = inpolygon(data.x`g, data.y`g, ...
            data.x`pts(data.region), data.y`pts(data.region));
        inon = logical((in + on)>0); % find all the border and ...
            interior points

        % Now that we have the border, we will calculate the vorticity.
        % Note, this assumes grid spacing is constant. Also, this does
        % not account for the fact that the border locations might cut
% through a given grid cell
dA = (data.x.g(1,2) - data.x.g(1,1))*(data.y.g(end,1) - ...
    data.y.g(end-1,1));
vortex_circ_userSelect = sum(data.vort.inon)*dA;

% Next, using vortexLabel.filtered and ...
    vortexLabel_unfiltered, we will determine
% which vortex our user defined vortex is likely to be ...  
    associated with. We will
% then use the region from vortexLabel_filtered and ...
    vortexLabel_unfiltered to calculate the
% centroid. This centroid will be the same centroid as was
% previously calculated in the vortexCentroid code. When we
% display the centroid on the graph, WE WILL USE THE FILTERED
% CENTROID!!!

% Multiply inon by data.vortexLabel_filtered or ...
    data.vortexLabel_unfiltered. Since inon is just 1s and
% 0s, we can determine which vortex number in
% data.vortexLabel_filtered
% sticks around the most (i.e. isn't multiplied by zero) to
% figure out which vortex is overlapped the most.

% preform operation for filtered and unfiltered case
for jj = 1:2
    if jj == 1
        vortexLabel = data.vortexLabel_unfiltered;
    else
        vortexLabel = data.vortexLabel_filtered;
    end

    overlapped_vort = vortexLabel.*inon;
max_ind = 0;
max_overlap = 0;
for ii = 1:max(vortexLabel(:))
    max_overlap_inter = sum(overlapped_vort(:) == ii);
    if max_overlap_inter > max_overlap
        max_ind = ii;
    end
end
[ind, ~] = find(vortexLabel(:) == max_ind);
Q = double(vortexLabel == max_ind);

% now we will recalculate the vortex centroid for the ... filtered
% case
xLength = length(data.x_g(1,:));
yLength = length(data.y_g(:,1));

xCentroid = 0;
yCentroid = 0;
sumA = 0;
circulation = 0;
vort_area = 0;
% calculate centroid

% Note, the below calculation of circulation is the ... same as if
% we had written vortex_circ_filtered = ...
    sum(data.vort(ind))*dA;
% Also note: vortexDefine was written such that it ... assumes that
% the grid spacing is constant.
for xIndex=1:xLength
    for yIndex=1:yLength

if vortexLabel(yIndex,xIndex) == max_ind
    xCentroid = xCentroid * sumA + data.x.g(yIndex,xIndex) * data.vort(yIndex,xIndex) * dA;
    yCentroid = yCentroid * sumA + data.y.g(yIndex,xIndex) * data.vort(yIndex,xIndex) * dA;
    sumA = sumA + data.vort(yIndex,xIndex) * dA;  % equivalent to calculating the mass for an element, where mass = Vorticity
    xCentroid = xCentroid / sumA;
    yCentroid = yCentroid / sumA;
    circulation = circulation + data.vort(yIndex,xIndex) * dA;
    vort_area = vort_area + dA;
end
end

if jj == 1
    vortex_circ_unfiltered = circulation;
    xCentroid_unfiltered = xCentroid;
    yCentroid_unfiltered = yCentroid;
    Q_unfiltered = Q;
else
    vortex_circ_filtered = circulation;
    xCentroid_filtered = xCentroid;
    yCentroid_filtered = yCentroid;
    Q_filtered = Q;
end
end

% Display circulation information
out_txt = ['**************************', newline, ...
'User Selected Circulation: ', ...
num2str(vortex_circ_userSelect), ...
newline, ...
newline, 'Q-Identified Circulation (unfiltered): ', ...
num2str(vortex_circ_unfiltered), ...
newline, 'Circulation Ratio (Q-Unfilt/User): ' ...
num2str(vortex_circ_unfiltered/vortex_circ_userSelect), ...
...
newline, ...
newline, 'Q-Identified Circulation (filtered): ', ...
num2str(vortex_circ_filtered), ...
newline, 'Circulation Ratio (Q-Filt/User): ' ...
num2str(vortex_circ_filtered/vortex_circ_userSelect), ...
...
newline, ...
'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'*'
Q_plt.XData = data.x_g;
Q_plt.YData = data.y_g;
Q_plt.ZData = Q_unfiltered;

Cent_plt = findobj(aa.Children, 'flat', 'displayname', ...
    'centroid');
Cent_plt.XData = xCentroid_filtered;
Cent_plt.YData = yCentroid_filtered;

% if plots don't exist, create them
else
    plot(data.x_pts(data.region), data.y_pts(data.region), ...
        'g', 'pickableparts', 'none', 'displayname', ...
        'vort_edge')
    scatter(data.x_g(inon), data.y_g(inon), 'g', ...
        'pickableparts', 'none', 'displayname', 'vort_inon')
    contour(data.x_g, data.y_g, Q_unfiltered, [0,1], 'm', ...
        'LineWidth', 5, 'pickableparts', 'none', ...
        'displayname', 'Q');
    scatter(xCentroid_filtered, yCentroid_filtered, 500, ...
        'm', 'filled', 'p', 'pickableparts', 'none', ...
        'displayname', 'centroid')
end
end
end

function finishbutton_Callback(source, eventdata)
    finish = 1;
end

%%% End script
while finish == 0
    pause(1);
E.3 GUI - Vortex Labeling

function [vortexLabel, Qbin] = vortexLabelling(Qbin, Vort, I, J, trim, ... thresh, iter)

   % Author: Eric Braun - eabraun@ucalgary.ca
   % ---------------------------------------------------------------
   % Inputs
   % ---------------------------------------------------------------
   % Qbin - areas of identified vorticity
   % Vort - matrix of calculated vorticity
   % I, J - number of x and y spatial points
   % trim - whether or not to filter the q-criterion identified vortex
   % thresh: used for filtering vortex tails. Higher threshold filters more
% iter: % Used for filtering vortex tails. Higher iter results in more ... 

diffusion and therefore more filtering

% ---------------------------------------------------------------------
% outputs
% ---------------------------------------------------------------------
% vortexLabel - matrix of identified unique vortices
% Qbin - the Q-field associated with vortexLabel

% first, trim vortex using diffusion method (thanks Conrad)
if trim == 1
    Qbin = TrimVor(Qbin,thresh,iter);
end

% Following finds all regions of vorticity and assigns separate regions 
% of vorticity different numbers. The following function gets 
% held up when we have connected regions of opposite signed vorticity. 
% later code in this function will address this issue.
vortexLabel=bwlabeln(Qbin);

% find total number of distinct regions
num_regions = max(max(vortexLabel));

label_num = num_regions;
for ii = 1:num_regions
    index = vortexLabel == ii;
    vortIndividual = Vort.*index;

    % Next, separate vortex into positive and negative regions. This is 
    % a bit convoluted. But basically, the following will parse the vortex
    % into regions of positive and negative vorticity. Next, it will
    % check if by parsing the field, we get new separate regions in
% each field (unlikely but possible). Using label_num, we define
% the newly created regions with numbers that have not previously
% been used. Finally, we combine the fields back together.
vortPosIndex = vortIndividual > 0;
vortNegIndex = vortIndividual < 0;

if sum(sum(vortPosIndex)) > 0 && sum(sum(vortNegIndex)) > 0 % only ...
  run loop if vortex contains both + and - vorticity
  vortexLabel_pos = bwlabeln(vortPosIndex);
  vortexLabel_pos = vortexLabel_pos + (vortexLabel_pos > 0)*label_num;
  label_num = max(max(vortexLabel_pos));
  vortexLabel_neg = bwlabeln(vortNegIndex);
  vortexLabel_neg = vortexLabel_neg + (vortexLabel_neg > 0)*label_num;
  label_num = max(max(vortexLabel_neg));
  vortexLabel(vortPosIndex) = vortexLabel_pos(vortPosIndex);
  vortexLabel(vortNegIndex) = vortexLabel_neg(vortNegIndex);
end
end

% the above section rewrites some of the vortex identifying numbers,
% resulting in a set of numbers that sometimes skip points. The
% following will correct this problem.
vortex_identifiers = unique(vortexLabel);
vortex_identifiers_old = vortex_identifiers(2:end); % skip zero
vortex_identifiers_new = 1:size(vortex_identifiers_old,1);
for i = 1:I
  for j = 1:J
    if vortexLabel(j,i) ≠ 0
      vortexLabel(j,i) = ...
      vortex_identifiers_new(vortex_identifiers_old == ...
E.4 Subfunction - Vortex Filtering

```matlab
function [ out ] = TrimVor( in, thresh, iter )
    [numY,numX]=size(in);
    out=in;
    for i=1:iter
        out = conv2(out, (1/9)*ones(3,3), 'same');
        out(in==0)=0;
    end
    out(in==0)=nan;
    out2=out;
    for i=1:iter
        for y=1:numY
            for x=1:numX
                if ~isnan(out(y,x))
                    ym=max([1 y-1]);
                    yp=min([numY y+1]);
                    xm=max([1 x-1]);
                    xp=min([numX x+1]);
                    num=0;
                    add=0;
                    if ~isnan(out(ym,x))
                        add=add+out(ym,x);
                    end
                    num=num+1;
                end
            end
        end
    end
```
23 end
24 if \( \neg \) \text{isnan}(\text{out}(\text{yp}, x))
25 add = add + \text{out}(\text{yp}, x);
26 num = num + 1;
27 end
28 if \( \neg \) \text{isnan}(\text{out}(y, \text{xm}))
29 add = add + \text{out}(y, \text{xm});
30 num = num + 1;
31 end
32 if \( \neg \) \text{isnan}(\text{out}(y, \text{xp}))
33 add = add + \text{out}(y, \text{xp});
34 num = num + 1;
35 end
36 \text{out2}(y, x) = \frac{\text{add}}{\text{num}};
37 end
38 end
39 end
40 \text{out} = \text{out2};
41 end
42 \text{out}(\text{out} < \text{thresh}) = 0;
43 \text{out}(\text{out} > 0) = 1;
44 \text{out}(\text{isnan} (\text{out})) = 0;
45 end

E.5 Subfunction - Q-Criterion

1 \textbf{function} \ [Q] = \text{QCriterion}(U, V, \text{dx}, \text{dy})
2 \% \text{Author: Eric Braun} - eabraun@ucalgary.ca
3 I = \text{size}(U, 2);
J = size(U,1);

Q = ones(J, I);  % stores the gradients at each spatial point in grid ...
               % and all time steps

[dudx, dudy] = gradient(U, dx, dy);
[dvdx, dvdy] = gradient(V, dx, dy);
for i = 1:I
    for j = 1:J
        u_11 = dudx(j,i);
        u_22 = dvdy(j,i);
        u_12 = dudy(j,i);
        u_21 = dvdx(j,i);
        Q(j, i) = -1/2*(u_11^2 + u_22^2) - u_12*u_21;
    end
end