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Fracture in functionally gradient materials: static and dynamic analyses

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FRACTURE IN FUNCTIONALLY GRADIENT MATERIALS
STATIC AND DYNAMIC
ANALYSES

by

REZA BABAYI

A DISSERTATION
SUBMITTED TO THE FACULTY OF GRADUATE STUDIES
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE
DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MECHANICAL ENGINEERING

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Abstract

A descriptive summary is a table of contents in paragraph form; it is a general map for readers. -Michael Alley

A theoretical and numerical treatment of a finite crack in a medium with spatially varying material properties is provided in this work. The variation of material properties is in a direction perpendicular to the crack surfaces. At first, response of an interfacial layer which is made of functionally gradient materials, subjected to an anti-plane shear impact load is considered. Laplace and Fourier transforms are applied to reduce this mixed boundary value problem to a system of dual integral equations which in turn will be reduced to a standard Fredholm integral equation of the second kind. The Fredholm integral equation is solved in the Laplace transform plane numerically. The time inversion is accomplished by a numerical scheme. The dynamic stress intensity factor is found to either increase or decrease with the crack length to layer thickness depending on the relative magnitudes of the material properties of the adjoining layer.

Next, a numerical treatment of an interfacial crack subjected to an in-plane mechanical loading is provided. Unlike earlier studies which considered the crack encountered as open, the current investigation studies cracks in an essentially compressive environment in which the crack faces are in contact and frictional effects play an important role. A simple and efficient, iterative finite element technique for solving frictional contact problems under small deformations is described. Stress in-
Intensity factors and energy release rates are calculated by using numerical crack flank displacement and two term parameter techniques. Numerical examples are provided to verify the technique and to show the effect of the thickness of the interfacial layer, the coefficient of friction, and the material properties upon the stress intensity factors and energy release rates of the crack.

Frictional contact problem of cracks in functionally gradient materials under combined mechanical and thermal loadings is studied. Both steady-state and transient thermal stresses are considered. Due to the nonuniform temperature distribution in the transient thermal field, the possibility of heat transfer across the crack surfaces in the contact region exists. The heat transfer across the crack surfaces results in a two-way coupling between the thermal and mechanical fields and will cause the problem to be a great deal more difficult. Stress intensity factors are calculated. The effect of the coefficient of friction, crack length, and material properties of the interfacial layer on the stress intensity factors in the mixed mode is studied. From the results it is revealed that the stress intensity factors are reduced considerably when functionally gradient material is used as an interfacial layer instead of homogeneous materials.

Finally, a nonlinear theory on the statics of multilayered shells, including transverse effects and delamination of general shapes, is studied. Delaminations are included by introducing new vectors which we name as conjugate directors. The approach is purely kinematical, the displacement field is assumed to belong to a certain finite parameter family of functions while the exact three-dimensional kinematic relations and constitutive equations are used. Discontinuity between two layers are considered as a planar delamination within a laminate whose boundary is defined by
an arbitrary function. A relatively weak interface between the two plies is hypoth-
esized. A planar delamination then would not kink into the adjacent plies but be
constrained to move in its own plane.
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It has often been pointed out that assistance in compiling a dissertation of this nature comes in so many forms that it is impossible to thank all those concerned. Nevertheless, my eternal gratitude goes to Dr. Stanislaw A. Lukasiewicz, my supervisor, for his conscientious guidance, support, encouragement, and suggestions. My heartfelt thanks to Dr. Marcelo Epstein for his invaluable contribution in preparing this dissertation.

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To my mother, Efat Kiaei,
heroine and coach.
She taught me about strategy and achieving.
And to my father Abbas Ali Babayi,
friend and cheering section.
His definitions of ethics and integrity
are to be emulated.
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Nomenclature

A

area of the body / determinant of the three-dimensional metric tensor

$A_i$

constants of integral

$\hat{A}_i$

natural base vectors

$A_{i,j}$

Three-dimensional metric tensor

$B_i$

constants of integral

$B^{\alpha \beta KL}$

stress integral, eqn (7.32)

$C$

boundary curve of the reference surface

$C_{ij}$

transverse wave speed of material

$D_i^{ijkl..L}$

see eqn (7.52)

$E$

Young's modulus

$E_c$

Young's modulus of ceramic

$E_m$

Young's modulus of metal

$E_i$

Young's modulus of the i-th layer

$B_i^{ijkl}$

elastic modulus of the I-th layer in the shell theory

$EVW$

external virtual work of the entire body

$EVW_i$

external virtual work of the i-th layer

$F$

a general single valued function

$F$

vector of external forces

$F_i$

vector of external forces acting on the i-th layer

$F_i$

a path independent integral

FGMs

functionally gradient materials

$G$

shear modulus

$G_i$

shear modulus of the i-th layer

$G$

elastic energy release rate per tip

$G_c$

critical value of $G$

$G_T$

total elastic energy release rate per tip

$G_I, G_{II}, G_{III}$

elastic energy release rate per tip for modes I, II, and III, respectively

$G_{Ic}, G_{IIc}, G_{IIIc}$

critical value of $G_I, G_{II}, G_{III}$, respectively

$H(t)$

Heaviside unit step function

$IVW$

internal virtual work of the entire body

$IVW_i$

internal virtual work of the i-th layer

$J_0$

the zero order Bessel function of the first kind

$J_i$

the first order Bessel function of the first kind

$J$

line integral (path independent)

$J_{Ic}$

critical value of $J$

$J_{ac}$

Jacobean matrix
\( J_c \)  typical nodal point in the contact surface  
\( J_t \)  typical nodal point in the target surface  
\( K \)  stress intensity factor  
\( K_I, K_{II}, K_{III} \)  stress intensity factors for modes I, II, and III, respectively  
\( K_{Ic}, K_{IIc}, K_{IIIc} \)  critical value of \( K_I, K_{II}, K_{III} \)  
\( K^*_I, K^*_II \)  normalized stress intensity factors for modes I and II, respectively  
\( K_3 \)  normalized stress intensity factor for mode III  
\( K \)  stiffness matrix  
\( K_i \)  stiffness matrix of the \( i \)-th layer  
\( K_c \)  contact stiffness matrix  
\( K_{cd} \)  contact stiffness matrix considering the frictional effects  
\( K^*_N \)  effective normalized stress intensity factor  
\( L \)  Lagrangian function  
\( L_i \)  boundary of the \( i \)-th layer  
\( M^{a,b,K} \)  stress integral eqn (7.32)  
\( N \)  total number of nodal points / number of layers  
\( N^{a,b} \)  stress integral, eqn (7.32)  
\( O_I \)  origin of layer I  
\( O'_A \)  origin of interface A  
\( P \)  Laplace transform variable  
\( P_{ij} \)  energy momentum tensor  
\( \sigma \)  dead load per unit area of the outer surface of the undeformed shell  
\( \sigma^I \)  stress integral, eqn (7.32)  
\( P_{ij} \)  energy momentum tensor  
\( P^*(x) \)  Jacobi polynomials  
\( Q \)  heat rate generation per unit volume / point off the reference surface  
\( Q^{\alpha,IK} \)  stress integral eqn (7.32)  
\( R \)  vector of external forces combined with the frictional force  
\( \vec{R} \)  position vector of point \( Q \)  
\( S \)  strain energy density / area of the undeformed reference surface  
\( S_{cr} \)  Fourier transform variable  
\( S_I \)  critical value of the \( S \)  
\( S_{min} \)  boundary surface of the \( I \)-th layer  
\( S^\alpha_I \)  minimum value of the \( S \)  
\( S^\alpha_{II} \)  stress integral, eqn (7.32)  
\( T \)  temperature  
\( U \)  strain energy per unit volume  
\( U \)  vector of the nodal point displacements
\( U_i \) vector of the nodal point displacements for the i-th layer
\( V \) volume of the shell
\( V_c \) volumetric ratio of ceramic
\( V_I \) volume of the I-th layer
\( V_m \) volumetric ratio of metal
\( W_i \) weighting coefficients
\( X_i \) body force per unit volume in the i direction
\( X_m \) \((m=1,2,3)\) Cartesian coordinates
XYZ Cartesian coordinate system
\( Y \) vector of unknown displacements and Lagrange multipliers
\( a \) half of the crack length / determinant of the reference surface metric
\( a_c \) critical crack size
\( a_1 \) effective half crack length
\( a_{ij} \) coefficients defining strain energy density
\( \bar{a}_a \) natural base vectors on the reference surface
\( a_{\alpha\beta} \) metric tensor of the reference surface
\( b \) plate dimension
\( c \) specific heat capacity
\( \tilde{c}_A \) conjugate director of the A-th interface
\( e_{\alpha\beta A}, (e_{\alpha\beta A})_I \) see eqn (7.50)
\( ds \) element of the arc length
\( d\tilde{I} \) director of the I-th layer
\( dV \) volume of an element
\( dY \) plastic zone ahead of the crack tip
\( e_{\alpha\beta}, e_{\alpha\beta K}, e_{\alpha\beta KL} \) see eqn (7.50)
\( (e_{\alpha\beta})_I, (e_{\alpha\beta K})_I \) see eqn (7.50)
\( (e_{33})_I \) see eqn (7.50)
\( f \) a general single value function
\( h_0 \) convective heat transfer coefficient
\( h \) thickness
\( h_i \) thickness of the i-th layer
\( h_I \) length of the I-th layer
\( i=1,2,3 \) subscripts corresponding to x, y, and z direction, respectively
\( i,j,k,l \) tensorial indices with range 1,2,3
\( i = \sqrt{-1} \) complex number
\( j=1,2,3 \) subscripts corresponding to x, y, and z direction, respectively
\( k, k_i \) thermal conductivity
\( n_c \) total number of nodal points in contact
\( n_i \) number of iteration
\( \hat{n}_I \) unit normal to the boundary surface of the I-th layer
unit vector tangent to the target surface
outward unit vector normal to the target surface
porosity of materials
dead load per unit area of the boundary surface of the undeformed shell
vector of the heat flux
polar coordinates
position vector of P
time
displacement components in x and r direction, respectively
displacement vector of P
displacement components in y and ϑ direction, respectively
plate dimension
interface coordinates
displacement component in z direction for the i-th layer
Rectangular coordinates
reference surface coordinates
layer coordinates
denotes the value of ( ) at y=0 unless otherwise specified
denotes Laplace transform of ( )(x, y, t) unless otherwise specified
denotes Fourier transform of ( )(x, y, P) unless otherwise specified
partial derivative operator
two-dimensional Laplace operator
two-dimensional Laplace operator
transpose of matrix ( )
tensorial indices with range 1,2
linear thermal expansion coefficient
linear thermal expansion coefficient for the i-th layer
linear thermal expansion coefficient for ceramic
linear thermal expansion coefficient for metal
vector which is included to the system of equations to consider
the frictional effects
material constant
crack angle with respect to the horizontal line
material constant
shearing strain components
parameter which is controlling the stability and accuracy of the Miller method
specific surface energy
shearing strain component in Polar coordinates
Shearing strain components in Rectangular coordinates.

Material constant.

Parameter which is controlling the stability and accuracy of the Miller method.

$\delta_{ij}$ Kronecker delta.

$\delta t$ Time step.

$\delta ()$ First variation of ($$).

$\varepsilon_{ij}$ Components of the strain tensor.

$\varepsilon_r, \varepsilon_\theta$ Radial and tangential strain components in Polar coordinates.

$\varepsilon_x, \varepsilon_y, \varepsilon_z$ Normal strain in $x, y, \text{ and } z$ directions, respectively.

$\zeta, \eta$ Lagrange multipliers.

$\lambda$ Eigenvalue.

$\lambda_i$ Lame’s coefficient for the i-th layer.

$\mu$ Coefficient of the friction.

$\mu_i$ Shear modulus of the i-th layer.

$\nu$ Poisson’s ratio.

$\nu_c$ Poisson’s ratio for ceramic.

$\nu_m$ Poisson’s ratio for metal.

$\rho$ Mass density.

$\rho_i$ Mass density of the i-th layer.

$\sigma_{ij}$ Components of the stress tensor.

$\sigma_{cr}$ Critical normal stress.

$\sigma_f$ Fracture stress.

$\sigma_{r}, \sigma_{\theta}$ Normal stresses in radial and tangential directions, respectively.

$\sigma_n$ Compressive normal traction.

$\sigma_x, \sigma_y, \sigma_z$ Normal stresses parallel to $x, y, \text{ and } z$ axes.

$\sigma_Y$ Yield stress.

$\tau$ Shear stress.

$\tau_{r\theta}$ Shear stress in Polar coordinates.

$\tau_t$ Tangential traction.

$\tau_{xy}, \tau_{xz}, \tau_{yz}$ Shear stress components in rectangular coordinates.

$\Phi^*(P)$ Laplace transform of function $\Phi(t)$.

$\chi$ Airy stress function.

$\Psi^*$ Solution of the Fredholm integral equation.

$\Gamma$ Curve surrounding the notch tip.

$\Delta_{ct}$ Relative displacement vector in the tangential direction for the nodal points in contact without considering the frictional effects.

$\Delta_{u}, \Delta_{v}$ Relative displacements of the nodal points along the tangential and normal directions, respectively.
Chapter 1

Introduction

The man of science appears to be the only person who has something to say just now,
and the only man who does not know how to say it. -Sir James Barrie

1.1 Prologue

New technologies constantly generate new demands for exotic materials to be used in severe environments. The rapid development of aerospace industries during the last three decades has required the creation of new materials to survive thermal and mechanical loads. The aerospace industry has usually led industry in the application of new materials and it is intended that the positive results presented herein will continue that trend. There are many engineering applications in which two parts of the same material or two parts of different materials are attached together via a third material known as adhesive. Ideally, adhesive and adherent materials should have identical properties, a condition which is approximately fulfilled in the fusion welding of metal-to-metal. With the usual type of metal to metal ceramic adhesive the ideal is not even approximately achieved, and the adhesive is much weaker than the adherent. The discrepancies in thermal expansion rates and other material properties between the materials cause strength problems at the interface.

At room temperature, and under predominantly tensile loading, stress in ceramics is proportional to strain up to the point of fracture, so that ceramics are brittle
materials. The non-homogeneity of the mechanical and thermal properties of adhesives and adherents cause intensification of thermal and mechanical stresses in the interfaces. Therefore, a nominally uniform stress system will induce nonuniform stresses in the material. Non-uniformity of stress will also arise at surface imperfections, on internal voids. The accumulation of these individual effects causes the nominal tensile breaking stress to be a small fraction of the very high stress necessary to break the strong primary atomic bonds, on the hypothesis of uniform load distribution among the atoms.

In recent years, the concept of so-called functionally graded materials (FGMs) was introduced and applied to the development of structural components. In aerospace technology, multi-material bodies comprised of ceramics and high-toughness materials, such as metals, have been used. In an attempt to solve the aforementioned problem, a functionally graded material (FGM) is, therefore, devised in such a manner that material composition is continuously varied with location in order to smooth the strain discontinuity. By properly grading the composition of the constituents, the thermal stresses in the FGM can be minimized. Since FGMs are inherently non-homogeneous, and they are also thermally non-homogeneous in that the material properties change over wide ranges of temperatures to that which the material is designed to be subjected to, special care is needed to analyze these kinds of materials.

The FGM is still in the material research and development stage. To design the material, the prompt establishment of an analytical methodology is urged. For the analysis of such a complex material behavior, rapidly developing computational mechanics is best suited. Due to the limitations that the available analytical models have, the general elastic and thermoelastic behavior for the FGMs is still not fully
established. Accordingly, there is a need to study the static and dynamic behavior of FGMs, under mechanical and thermal loadings, to satisfy the requirement for engineering uses.

1.2 Scope and outline of this dissertation

The stress intensity factors and energy release rate are of fundamental importance in the prediction of brittle failure using linear elastic fracture mechanics principles. They are functions of the cracked geometry, material properties, and the associated loading. Although only relatively simple geometries are solvable by analytical methods, (usually this implies a body of infinite extent) it is always desirable to have an analytical solution to determine the response of a system to the external mechanical and thermal loads. Analytical solutions furnish a very safe and sound base of comparison for the experimental and numerical results. On the other hand, numerical procedures are necessary when stress intensities and energy release rates are desired for more general configuration and loading. The finite element method has become one of the most popular and generally used numerical method of structural analysis.

In an effort to solve some of the above problems especially those problems related to functionally gradient materials, the author prepared the content of this dissertation. In the theoretical section of this dissertation, a general fundamental solution is provided to the singularity behavior of a crack in an interface with spatially varying elastic properties under anti-plane shear impact load. The analysis is based on the use of an integral transform technique coupled with the solution of a Fredholm integral equation. This integral equation is solved numerically. The dynamic local stress
intensity factor of the crack is investigated in detail. The effect of the interface upon the dynamic stress intensity factor of the crack is examined and discussed. The current interfacial layer model indicates the sensitivity of the stress intensity factor to the location of the crack and the elastic properties of the media. This is an important finding and should be seriously considered in future modelling and characterization of the phenomenon.

In a compressive environment, the crack faces are either in contact over the whole length of the crack or a part of it. In such cases, frictional effects play an important role. Literature on the subject of the general contact is very rich; however, there are very few analytical solutions available for simple contact problems. To solve more complicated problems, numerical techniques are necessary. Since existing methods try to solve contact problems in general form, they are usually very time consuming. Therefore, one of the objectives of this work is to provide an efficient, simple, and iterative technique to reduce the computation time required to solve the frictional contact problem of a crack under small deformations. Non-homogeneity of the material properties are included in the present algorithm.

Under usual conditions of heat exchange, the rate of temperature change is small in comparison with the speed of sound in the material. Thus, at any given moment, the thermal stresses in an elastic body can be determined on the basis of the instantaneous values of the temperature field. There is no need to consider the inertia forces corresponding to the motion of the particles during the varying thermal expansion. When the mechanical coupling terms in the heat conduction equation and the inertia terms in the equations of motion are disregarded, the formulation of the thermoelastic problem is said to be quasi-static. However, with the existence of unknown
contact surfaces, the temperature and displacement fields are no longer independent of each other. The possibility of heat flow across the crack in the contact region exists. The resulting two-way coupling between the thermal and mechanical fields will cause the problem to be a great deal more difficult. Frictional contact problems of cracks in functionally gradient materials under combined thermal and mechanical loads are considered in this dissertation. Numerical examples are provided and time dependent stress intensity factors are calculated. The variation in stress intensity factors due to changes in material properties of the interfacial layer is studied as well.

Analytical modelling of laminated composites containing delamination has received very little or no attention in the past. This is mainly because the problem is extremely complicated since it is basically three-dimensional in nature. The presence of cracks or surfaces of discontinuity adds additional difficulties to the development of a tractable theory of laminated composites. Overall, in the study of delamination problems, most of the analyses performed to date have been restricted to relatively simple models or to general three-dimensional finite element models. A non-linear theory on the statics of multi-layered shells, including transverse effects and delamination of general shapes, is studied in this work. Delaminations are included by introducing conjugate directors. The displacement field is assumed to belong to a certain finite parameter family of functions while the exact three-dimensional kinematic relations and constitutive equations are used. Stresses and strains, rather than stress resultants and associated kinematic variables, are used in formulating the principle of virtual work, from which the field equations and relevant boundary conditions are obtained. The simplest kinematic hypothesis which still accounts for
transverse effects is a piecewise linear displacement field. For this reason, and for the sake of brevity, the equations resulting from such a first order theory are given here in detail.

1.3 Organization of the text

A literature review is provided in Chapter 2. It contains the works on the subject of linear fracture mechanics with an emphasis on composite materials, functionally gradient materials, and frictional contact problems. By no means should this review be considered as complete.

Chapter 3 gives fundamental definitions and basic formulations of linear fracture mechanics. It introduces modes of fractures, stress intensity factors, energy release rates, fracture criterion, and mathematical formulations governing the stress and displacement fields of functionally gradient materials.

An analytical treatment of a finite crack subjected to an anti-plane shear impact load in a functionally gradient material is presented in Chapter 4. The analysis is based on the use of an integral transform technique. The boundary value problem is reduced to the solution of dual integral equations. Dual integral equations are solved by reducing the problem to the Fredholm integral equation.

Chapter 5 provides a comprehensive numerical treatment of a finite crack in an interfacial layer made of functionally gradient materials under in-plane mechanical loading conditions. The primary objective of this chapter is to study the variation of stress intensity factors and energy release rates in an essentially compressive loading condition. In this chapter a simple and efficient, iterative finite element technique
for solving frictional contact problems under small deformations is also described.

Chapter 6 utilizes the numerical procedure developed in Chapter 5 to solve frictional contact problem of cracks in functionally gradient materials under thermally induced stresses. Coulomb friction law is taken into account. Both steady-state and transient thermal fields are considered and finally, dynamic stress intensity factors are plotted versus time.

A non-linear theory of multi-layered shell of variable thickness with finite number of delaminations at its interfaces is studied in Chapter 7. To include interface dilemma to the general governing equations, it is assumed that each layer is separate from its adjacent layer. One can think of each empty space between two layers as a two-dimensional set of directed straight non-material line segments both ends of which describe smooth surfaces in three-dimensional Euclidean space. The general non-linear equilibrium equations expressed in terms of the stress resultants, couples, and unknown directors are obtained by using the principle of virtual work together with the continuity conditions at interfaces. In order to comprehend more easily this general non-linear theory, a multi-layered beam which is laterally loaded is studied and numerical results are obtained and discussed in detail.

A general discussion is presented in Chapter 8. This chapter makes a conclusion based on the results which are given in Chapters 4 through 7.
Chapter 2

Literature review

There is a certain method in this madness. - Horace

2.1 Introduction

A problem of considerable, and increasing importance within the fields of mechanical, aeronautical, civil, and marine engineering is the failure of structures due to brittle fracture. This type of failure has been observed to occur under both constant and cyclic loading conditions. Fracture mechanics has evolved as a result of attempts to understand and prevent such failures.

Historically, fracture mechanics developed along parallel lines at the microscopic and macroscopic levels. The latter, in the form of solid mechanics, provided numerous contributions to the mathematical theory of elasticity as applied to structures containing crack-like defects. Fracture mechanics, in the broadest sense of the concept includes that part of the science of strength of materials and structures which relates to a study of the carrying capacity of a body both with and without consideration of initial cracks. It also relates to the study of various laws governing crack development. One notes right away that while such disciplines as the classical theory of elasticity, the theory of plasticity and the theory of creep are clearly formulated and well studied, the mechanics of fracture is still far from complete. The main stages of the development of fracture mechanics will be discussed in the following
review work.

2.2 Literature survey

Despite the creation of such magnificent architectural structures as the pyramids of Egypt, man has been grappling with the problem of strength of materials since ancient times. What knowledge was gleaned has been passed on from generation to generation; however, as an art form rather than a science.

The founder of fracture mechanics is rightfully considered Galileo Galilei. He stated that "the breaking load is directly proportional to its cross-sectional area" (This result in a slightly modified form is still used today in engineering strength design particularly in an homogeneous state of stress). In general, the first stage of the investigation on fracture mechanics is associated with the names of Galileo Galilei, Robert Hooke, Charles Augustine de Coulomb, Barre de Saint Venant, Otto Mohr and is characterized by extensive studies of deformation properties of solids, and by the phenomenological development of various failure criteria, termed strength theories. These theories state that fractures occur at the moment when a certain combination of parameters, such as stress and strain, reaches a critical value. At present, several strength theories (the maximum principal stress theory, the maximum shearing theory, the distortion energy theory or the octahedral shearing stress theory, to mention a few) are applied in strength design, depending on the type of material and service conditions. As significant as the study of strength theory may be within the framework of this approach, it is inadequate for a number of reasons. For example, numerous case studies of early failure of structures at stresses less than
the yield stress were a clear indication of the inadequacy of the concepts of strength as a material constant. The work of A. A. Griffith, G. I. Taylor, E. O. Orowan, G. R. Irwin, and others demonstrates a new direction in strength analysis, a direction central to a detailed study of the fracture process itself. Since fractures develop as a result of real defects existing in a structure, the consideration of cracks present in a structure and the determination of their effects on strength are essential in the evaluation of strength.

At present the significance of investigations on fracture mechanics is far beyond the scope of the problem of carrying capacity. First of all, the study of the fracture process is of interest in its own right. Monitoring the fracture process and knowing its laws are of great importance in engineering. To slow down the process of crack growth in structures and buildings is desirable, whereas to facilitate the fracture in every possible way, is necessary when cutting materials.

The most important aspect of crack theory is the identification of the conditions for a local fracture at a particular point on the crack contour. This is as important in solving the problem of crack growth as it is in the choosing the correct criteria for the onset of yielding in a volume element.

The condition for a local fracture is simply formulated in the theory of so-called quasi-brittle cracks where the largest dimension of the region of irreversible strains at a particular point on the crack contour is small compared with the crack length and with the distance from this point to the nearest boundary of the body. The simplest variant of this condition was proposed by Irwin [1, 2, 3, 4, 5] based on the physical and mathematical ideas of Griffith [6, 7], Neuber [8], and Westergaard [9, 10]. It assumes that the coefficient of stress singularity at the point under consideration at
the moment of local fracture (and crack extension at that point) is equal to a certain material constant. The stresses are then calculated on the assumption that the body is perfectly elastic.

The above mentioned coefficient represents a certain function of the external loads, crack length and body geometry. Therefore, the condition for local fracture on the contour of a crack provides, in principle, a means for determining its development and, in particular, for finding a combination of external loads separating stable and unstable regions.

Later several models were proposed for the fracture mechanism at the tip of a quasi-brittle crack. However, all known models (there are some 10 models at present), which differ in details of the description of local rupture at the tip of a brittle crack, are equivalent in the sense that they always reduce to the Griffith-Irwin condition [11].

A general approach to the description of crack extension in an arbitrary continuous media was proposed by Rice [12, 13] and Cherepanov [15]. In the works of Eshelby and Rice, the idea is to introduce a line integral which has the same value for all paths surrounding the tip of a crack in the two-dimensional strain field of an elastic or deformation-type elastic-plastic material. Appropriate integration path choice serves both to relate the integral to the near tip deformations and, in many cases, to permit its direct evaluation. This averaged measure of the near tip field leads to approximate solutions for several strain-concentration problems. Cherepanov [11] studied the dynamic cracks in elastic solids and quasi-static cracks in elastic-plastic

1Note that the expression for the path-independent integral (which is now commonly termed the Rice integral or J-integral) had been previously obtained by Eshelby [14].
and rigid-plastic solids by making use of the energy conservation law and the physical concept of the fracture energy.

Since the subject of this work is in perfectly brittle or elastic fracture mechanics, especially in plane elasticity, from now on only those works which are related to brittle fracture mechanics will be discussed. The reader who is interested in the subject of elastic-plastic fracture mechanics may refer to the work of Parton and Morozov [16] which to my knowledge, is some of the best research done in this area and provides an extensive cross-reference to the reader. In spite of certain limitations of linear fracture mechanics, the range of problems reliably solvable with its help is sufficiently broad. The development of this theory reduces in great part to an accumulation of the fund of solved problems of elasticity theory for cracks of different shape in various bodies.

2.3 Solution methods and techniques

The analysis of crack problems in plane elasticity has intrigued mathematicians for nearly sixty years. Inglis [17] found the solution to a single crack in an infinite sheet with the use of elliptic coordinates. Since then, many sophisticated mathematical approaches have been applied to a variety of crack configurations and loading conditions. It is easy to see the mathematical interest in an area where solution techniques span such diverse topics as analytic function theory, integral equations, transform methods, conformal mapping, body force method, finite differences, finite elements, boundary elements, and asymptotic methods to mention a few.
2.3.1 Complex and conformal mapping

Muskheilishvili's work [18] on the complex form of the two-dimensional equations due to Kolosoff [19] has undoubtedly had a major influence on the development of analytical techniques for solving plane crack problems. As a result of his work, particularly the formulation of the problems of “linear relationship” using analytic continuation arguments, the analyst is provided with considerable insight into the mathematical character of solutions in terms of analytic function theory. With this insight, mathematic singularities due to geometry or loading can usually be anticipated and frequently the structure of the singularity can be predicted. This feature is invaluable in the choice of solution method.

Using conformal mapping to solve crack problems in plane elasticity has attracted many mathematicians and engineers. The earliest application of conformal mapping to two-dimensional crack problems can be considered Kolosoff’s solution [20] which is for an elliptic hole in an infinite region using the mapping \( z = ccosh(\eta) \). Next, Muskheilishvili [21] used mapping of a circular region to solve this problem in terms of Cauchy integrals. In spite of the considerable interest in crack problems generated by Griffith’s theory [7] and the subsequent refinements of the complex variable theory by Muskheilishvili, remarkably few solutions were available prior to the late 1950’s. Neuber [22] found solutions for hyperbolic notches in a region of infinite extent. Outstanding crack solutions in this period include Sneddon and Elliot’s solution [23] for varying pressure inside a crack by transform theory. An interesting account of the evolution of several of these earlier solutions is contained in a survey by Sneddon [24].

The application of the method of polynomial mapping approximation to compli-
icated configurations involving cracks was first carried out by Bowie [25] to find the solution for radial cracks emanating from a circular hole in an infinite sheet. Later Bowie and Neal [26] presented a modified mapping-collocation technique which combined modified versions of conformal mapping and boundary collocation arguments. Several applications of modified mapping-collocation technique were presented by Bowie [27]. Among those applications are oblique edge crack in a rectangular panel and radial cracks emanating from a circular cut-out in a rectangular panel.

Isida [28] presented a general method of analysis of internal cracks in isotropic homogeneous elastic media based on the Laurent series expansions of the complex potentials. These complex potentials were consistent with the single valuedness of displacements as well as stresses and strains. The method was applied to longitudinal shear, plane extension and classical plate bending problems.

2.3.2 Integral transforms

Among the earliest papers leading to a revival of the interest in crack problems in the classical theory of elasticity were those in which the solution of the relevant boundary value problem was obtained by the systematic use of the theory of integral transforms. In this connection reference can be made to Sneddon and Elliott [23]. Interest in this method of problem solving has been maintained over the years. By using Fourier transforms and dual integral equations, Sneddon [29] was able to solve different crack problems in two-dimensional elastic medium. Among them was the problem of a crack subjected to varying pressure along its surfaces. Sneddon [24] presented the solution to the half-plane problem for a penny-shaped crack, which is considered a three-dimensional case of Griffith crack, by using the Hankel transform. In this work
an extensive bibliography was included. An approximate three-dimensional theory of plates with application for crack problems was proposed by Hartranft and Sih [30]. Using a variational principle, a system of equations was derived for the theory of extension and bending of elastic plates. The system of equations was formulated based on the generalized transverse displacement and two other functions which represent the distribution of the transverse shear stresses in the plane of the plate. With the aid of Fourier transforms, the boundary conditions of the crack problem led to a set of dual integral equations which could, in turn, be reduced to the solution of a single Fredholm equation of the second kind. Finally, asymptotic expansions of the stresses near the end points of the crack were carried out.

The reflection and refraction of elastic waves by a crack of finite width in an infinitely extended medium were discussed by Sih [31]. An integral transform method was used to obtain the detailed structure of the crack-front stress and displacement fields which were the pre-requisites for assessing the strength degradation for bodies containing flaws. While the dynamic stress singularity at the crack tip was the same as that of the static one, the magnitude of the local stress field governed by the dynamic stress-intensity factors was altered. These factors were found to depend on Poisson’s ratio, the crack geometry, and the wave length or frequency of the travelling waves.

2.3.3 Asymptotic method

Asymptotic methods play an important role in all branches of applied mathematics in the evaluation of the solutions of problems depending on a parameter with a certain range. Usually the solution is obtained first in a more or less explicit form, and is
then simplified by the appropriate asymptotic expansions when the parameter tends to one or both limiting values of its interval. On the other hand, in elasticity one often encounters problems for which no explicit solution is available for arbitrary values of the parameter. Numerical techniques may enable us to obtain accurate numerical solutions for specific values of the parameter. Such numerical methods, however, are often less convenient when discussing the asymptotic behaviour of the solution when the parameter tends to a limiting value of its range.

In some problems it has proved advantageous to consider from the outset the two cases of the limiting values of the parameter. This advantage occurs only, of course, if the problem takes a simpler form for both limiting values of the parameter, (although these forms are naturally quite different.) The more difficult problem of a complete solution is evaded. It is hoped that the solutions for both ends of the range of the parameter are each valid in a sufficiently wide neighbourhood, thus enabling a satisfactory interpolation for intermediate values of the parameter. This approach has been advocated strongly in a book edited by Sih [32]. An asymptotic analysis of finite deformations near the tip of an interface-crack was presented by Herrmann [33]. In this work, he investigated the behaviour of a traction-free crack at the interface of two semi-infinite slabs bonded together under the conditions of plane strain. A determination of the mathematical form of the deformation and stresses near the crack-tip, consistent with the fully non-linear equilibrium theory of compressible elastic solids, was found by an asymptotic treatment of the deformation. Each slab was assumed to be hyperelastic, homogeneous, and isotropic with Knowles-Sternberg type asymptotic conditions on its strain-energy density. It was shown that under these conditions, the interface-crack problem admits solutions in which oscillatory
singularities do not occur. This suggested that it is the approximations made by the linear theory which produce these singularities.

A procedure for analysing a class of transient elastodynamic crack problems was presented by Georgiadis and Brock [34]. These problems modelled certain experimental situations which inferred fracture toughness values for materials under stress-wave loadings. They presented exact expressions for the elastodynamic stress intensity factor at the tip of a long external crack in a strip-like body whose lateral (upper and lower) boundaries were parallel to the crack line. The loadings that were used had an arbitrary time dependence, but were spatially uniform. The problem analysis was based on integral transforms and asymptotic use of the Wiener-Hopf technique. The character of the local stress distribution at the base of a crack in a stretched plate was examined for both symmetrical and antisymmetrical loading by Williams [35]. In this paper, he solved the problem by using biharmonic Airy-stress function. He considered \( \chi = r^{\lambda+1} F(\theta, \lambda) \) in which \( \chi \) is a biharmonic stress function and \( \lambda \) is an unknown eigen value which can be calculated from characteristic equation obtained from applying boundary conditions on the stress and displacement relations. The elastic stresses were found to vary as the inverse square root of the radial distance from the point of the crack, hence approaching mathematically infinite values at the point itself. This square-root singularity was formulated by Inglis [17] and Westergaard [10]. Later on Williams [36] investigated a similar problem where the plate was subjected to either symmetrical or antisymmetrical bending instead of stretching.
2.3.4 **Body force method**

Many two and three-dimensional crack problems have been solved by the body force method (Sih [37]). Generally, the body force method can be classified into two different approaches - the stress method and the resultant force method - depending on the boundary conditions one chooses to satisfy directly. For the stress method, the actual stress boundary conditions are satisfied. For the resultant force method, the force boundary conditions, that is, the resultant force values in certain sections along the boundary, are satisfied. Isida et al. developed the resultant force method for calculating the stress intensity factor for arbitrarily shaped cracks in an infinite plate ([38, 39]), in an infinite plate with an elliptical hole ([40]). Finally, Mori et al. [41] obtained crack opening displacement of bent or branched cracks by applying the body force method.

2.3.5 **Numerical techniques**

Since the work of Irwin [3], many numerical solutions and techniques have appeared in the last three decades following the great surge of interest in crack problems. Irwin’s concept of stress intensity related the stress distribution local to the crack tip to the earlier Griffith [7] energy concept. This stimulated the recent growth in the field of fracture mechanics. Considerable credit for many of the solutions must also be given to the substantial growth of computer technology in this same period. Few, if any, of the effective numerical techniques used today would have been considered feasible thirty years ago in the age of the desk calculator.

The great variety of numerical techniques for computing crack stresses has played a healthy role in ensuring reliability of numerical results. This variety has been es-
sential in a problem area where all techniques tend to be difficult. Most key problems have been duplicated by several techniques. The common accuracy requirement is an error toleration of less than two percent. This has minimized gross errors from appearing in the literature and forced careful refinements of such approaches as finite elements, etc.

One approach for the numerical determination of the amplitude of the stress field in the immediate vicinity of the crack tip, the stress intensity factors (SIF), which has received considerable attention because of its ability to treat very general geometric and loading conditions is the finite element technique. Many articles and a number of books have been written about various aspects and applications of the finite element method (FEM). Since stresses at the crack tip are singular, the straight forward application of the finite element technique with no special attention given to the stress singularity is not able to solve those kind of problems accurately. In an attempt to eliminate some of these undesirable features, an alternative approach has been developed by Barsoum [42]. In this paper, quadratic isoparametric elements which embody the inverse square root singularity were used in the calculation of stress intensity factors of elastic fracture mechanics. Examples of the plane eight noded isoparametric element showed that it had the same singularity as other special crack tip elements, and still includes the constant strain and rigid body motion modes. Similarly, this method was used by Henshell and Shaw [43]. In both situations, square-root singularity was achieved by placing the mid-side node near the crack tip at the quarter point.

In many engineering fracture mechanics applications, the stress singularity near a crack tip varies as \( r^{\lambda - 1} \), where \( r \) is the distance from the crack tip and \( \lambda \) is the order
of singularity. For example, in case of a crack terminating at the interface of a bi-material composite, the value of $\lambda$ can be complex (this means oscillatory stresses and displacements) or real depending on the angle between the crack and the interface, and the material properties of the two composites (see Bogy [44]). In the case of a kinked crack, the order of the stress singularity arising at the knee is $\pm f(\theta)$ where $f(\theta) > 0$ (see Williams [45]). This singularity can interact with the singularity at the kink tip (which is of the square root type) as the kinking length becomes smaller. Therefore, a proper modelling of this singularity improves the accuracy of the results. A two-dimensional $\lambda$ singularity was obtained by Abdi [46]. In his work, it was shown that the singularity of stresses near the tip of a crack in an elastic bi-material could be obtained by using degenerate triangular elements, the shape functions of which were derived from classical isoparametric elements. Later, a three-dimensional six-noded prism (wedge) finite element which contained a singularity of order $\lambda$ was developed by Abdel Wahab and Roeck [47]. They adopted the interpolation functions of the displacement and coordinates such that they varied proportionally to the power $\lambda$ of the distance from the crack front, along the crack surface and to the distance in the perpendicular direction. Leung and Su [48] suggested a semi-analytical method to determine the stress intensity factor of two-dimensional crack problems. In this work, the fractal geometry concept and the two-level finite element method were employed to automatically generate an infinitesimal mesh and transform these large numbers of degrees of freedom around the crack tip to a small set of generalized coordinates. By taking advantage of the same stiffness of two-dimensional element with a similar shape, one transformation of the stiffness for the first layer of mesh was enough for all layers. The vibration behaviour of cracked plates was investigated by Qian et al. [49].
A finite element model of cracked plates was consequently established. Applying the FEM to a simply-supported square plate and a cantilever plate with a through crack, the eigenfrequencies were determined for different crack lengths. Cracked plates subjected to large deformations were analysed using the finite element technique of Alwar and Thiagarjan [50]. The crack closure which takes place in the compression zone was considered. The numerical results regarding SIF were presented for cracked plates with and without crack closure. The total Lagrangian approach was used for the formulation of the problem.

2.4 Composite materials

Layered composites have found application in many high performance engineering structures because of their ability to absorb energy. Williams [51] was the first to perform an asymptotic analysis of the elastic fields at the tip of an interfacial crack. His solution indicated the oscillatory nature of the resulting stress and displacement fields. Erdogan [52], using a complex variable formulation, examined the case of two half-planes bonded to each other along a finite number of straight line segments and evaluated the stress distributions near the ends of the cracks. Similar to Williams [51] work, he found that violent oscillations occur in the stresses near the ends of the cracks. England [53] showed that the solution to the problem of a single line crack opened by equal and opposite normal pressures between two bonded dissimilar half-planes, was physically inadmissible since it predicted that the upper and lower surfaces of the crack should wrinkle and overlap near the ends of the crack. Rice and Sih [54] showed how the complex variable method combined with
eigenfunction expansion used by Sih and Rice [55] could be applied to formulate the problem of bonded dissimilar elastic planes containing cracks along the bond. Solutions were given in closed form for a number of extensional problems of an isolated complex force - a force vector having components in the \( x \) and \( y \) directions - applied at an arbitrary location on each side of the crack surface. Hutchinson et al. [56] considered a crack paralleling a bonded plane interface between two dissimilar isotropic elastic solids. They showed that if the influence of external loading and geometry on the interface crack is known, then this information can immediately be used to generate the stress intensity factors for the sub-interface crack. Conditions for cracks to propagate near and parallel to, but not along, an interface were also derived.

Elastic fracture mechanics concepts for a crack on the interface between dissimilar solids were re-examined by Rice [57]. Using function theory Rice gave a derivation of the form of stress and displacement fields in the vicinity of the crack tip, equivalent to complete Williams expansions of both inner and outer type. The complex stress intensity factor, \( K \), associated with an elastic interface crack for which contact was ignored, was discussed. Specifically, its validity as a crack tip characterizing parameter was noted for cases of small scale nonlinear material behaviour and/or small scale contact zones at the crack tip. The anti-plane strain problem of two dissimilar anisotropic composite wedges of arbitrary angles that are bonded together along a common edge was considered by Ma and Hour [58]. The surfaces of the wedge could be subjected to traction-traction, traction-displacement or displacement-displacement boundary conditions. They studied the dependence of the order of the stress singularity on the wedge angles and material constants. It was found that the order of the stress singularity was always real for anti-plane dissimilar an-isotropic wedge prob-
lems. This is quite different for the in-plane case in which a complex type of stress singularity might exist. A crack impinging on an interface joining two dissimilar materials may arrest or advance by either penetrating the interface or deflecting into the interface. The competition between deflection and penetration was examined by He and Hutchinson [59] when the materials on either side of the interface were elastic and isotropic. The maximum energy release rate for the deflected crack was compared with the maximum energy release rate for a penetrating crack.

2.5 Time-dependent fracture mechanics

Although the stationary crack problem in nonhomogeneous materials has received considerable attention, only a few articles were devoted to the crack propagation along the interface of adjoining materials with different elastic properties.

The two-dimensional problem of a finite crack along the interface between two dissimilar solids loaded by a plane wave was considered by Qu [60]. Through use of the Fourier transform method, the boundary value problem of wave scattering was reduced to a vectorial Cauchy singular integral equation for the dislocation density on the crack face. A Jacobi polynomial technique was then used to solve the integral equation numerically. Crack opening displacements and stress intensity factors were obtained for various incident frequencies and incident angles. It was found that the crack faces interpenetrate each other near the crack-tips, and the crack-tip singular fields were oscillatory. The oscillatory index was the same as that for an interface crack under static loading which can be expressed by the second Dundurs bi-material constant. The combined effects of high crack-tip speed, and the prox-
imity of a bond-plane on the elastodynamic stress-intensity factor were investigated by Chen et al. [61]. The model-problem that he considered concerned the steady propagation of a crack of length 2a, parallel to a bond-plane with a half-plane of different material properties. By using a moving coordinate system and applying Fourier transform techniques and superposition methods, the mixed boundary-value problem was reduced to a dual singular integral equation with Cauchy-type kernels which was solved numerically using the method of Erdogan and Gupta [62]. Li and Tai [63] considered the elastodynamic response of a four-layered composite with an interface crack under anti-plane shear impact load. Laplace and Fourier transforms were applied to reduce this mixed boundary value problem to a Cauchy-type singular integral equation of the first kind in Laplace transform plane, which was solved numerically. A Laplace inversion technique developed by Miller and Guy [64] was then used to get the solution in the physical plane. Finally, the elastodynamic stress intensity factors were obtained as functions of time, geometrical parameters and material properties.

Sih and Chen [65] studied moving cracks in layered composites. In this work, a three-layered composite model was used with a crack moving in the centre layer. The material properties of the middle layer differed from those of the surrounding layers. Both in-plane extensional and out-of-plane shear loading were considered. Making use of the Galilean transformation and Fourier sine and cosine transforms, the dynamic crack tip stress intensity factors were evaluated numerically from the standard Fredholm integral equations. The intensity of the local dynamic stress were found to either increase or decrease with the crack length to layer thickness depending on the relative magnitudes of the adjoining layer material properties.
The crack speed tended to amplify the effect of material non-homogeneity.

2.6 Fracture and functionally gradient material

For cracks in functionally gradient materials, stress intensity factors are affected by the material gradients. Moreover, the fracture modes of the cracks in FGMs are inherently mixed. There are typically both normal and shear traction ahead of the crack tips because of the non-symmetry in the material properties.

To characterize material, fracture toughness data is required. To obtain the fracture toughness data, stress intensity factors for specimens subjected to variable external loads are needed. In an attempt to find stress intensity factors in FGMs, Delale and Erdogan [66] studied the crack problem for two bonded dissimilar homogeneous elastic half-planes. It was assumed that the interfacial region could be modelled by a very thin layer of homogeneous material, even though the formulation given was rather general. In the particular model used the elastic properties of the interfacial layer were assumed to vary continuously from those of the two semi-infinite planes. The layer was assumed to have a series of collinear cracks parallel to the nominal interface. In modes I and II stress intensity factors, the energy release rate and the direction of a probable crack growth were calculated. Erdogan [67] investigated the singular nature of the crack-tip stress field in a non-homogeneous medium having a shear modulus with a discontinuous derivative. The problem was considered for the simplest possible loading and geometry, namely the anti-plane shear loading of two bonded half spaces in which the crack was perpendicular to the interface. It showed that the square-root singularity of the crack-tip stress field was
unaffected by the discontinuity in the derivative of the shear modulus.

Jin and Noda [68] and Noda and Jin [69] studied the crack problems in non-homogeneous solids under thermal loading. In these studies, the material properties were selected so that the crack-tip stresses have a square root singularity and the stress intensity factor concept is well defined. Jin and Noda [70] studied singular stress and heat flux fields at the tip of a crack in a general non-homogeneous material. Plastic stress singularity was also considered. They found that the crack-tip field singularities and angular distributions were the same as those in the homogeneous material provided that the properties of the material were continuous and piece-wise differentiable and the material properties did not vanish at the crack tip.

Erdogan et al. [71] considered the plane elasticity problem for two bonded half-planes containing a crack perpendicular to the interface. The primary objective of the paper was to study the effect of very steep variations in the material properties near the diffusion plane on the singular behaviour of the stresses and stress intensity factors. Of particular interest was the examination of the nature of stress singularity near a crack tip terminating at the interface where the shear modulus had a discontinuous derivative. The linear elasticity problem for an interface crack between two bonded half planes was reconsidered by Delale and Erdogan [72]. It was assumed that one of the half planes was homogeneous and the other was non-homogeneous in such a way that the elastic properties were continuous throughout the plane and had discontinuous derivatives along the interface. The problem was formulated in terms of a system of integral equations and the asymptotic behaviour of the stress state near the crack tip was determined. The results led to the conclusion that the singular behaviour of stresses in the non-homogeneous medium was identical to that
in a homogeneous material provided the spacial distribution of material properties was continuous near and at the crack tip.

Atkinson [73] calculated energy release rates for cracks propagating in media with spatially varying elastic moduli. This variation is in a direction perpendicular to the crack growth direction. Results were given for transient problems of semi-infinite cracks in infinite media for certain special forms of the variation in shear modulus. It was shown by the use of a certain path independent integral that a simple formula for the energy release rate could be obtained for general variations in elastic moduli provided these variations were in a direction perpendicular to the crack. Wang and Meguid [74] provided a theoretical and numerical treatment of a finite crack propagating in an interfacial layer with spatially varying elastic properties under anti-plane loading condition. The theoretical formulations governing the steady state solution were based upon the use of an integral transform technique. The resulting dynamic stress intensity factor of the propagating crack was obtained by solving the appropriate singular integral equations using Chebyshev polynomials for different non-homogeneous materials. Numerical examples were provided to verify the technique.

2.7 Fracture under compressive loading

In all the aforementioned studies, it was considered that the crack faces were open and free of frictional traction. In most applications of fracture mechanics, the cracks encountered are open meaning that the crack faces are separated and free of traction. Exceptions arise in tribology, geophysics, the study of compression failure of brittle
materials and many other applied fields where cracks propagate in an essentially compressive environment. In such cases, the crack faces are in contact either over the whole length of the crack or a part of it. Frictional effects play an important role. In this regard, Comninou and Dundurs [75] considered a crack in a field of pure bending. The crack was closed over part of its extension, and the two crack faces in contact were allowed to slip under frictional constraints. The solution was constructed under the assumption that the applied loading was monotonically increasing. In another paper Comninou and Dundurs [76] studied a crack in a linearly varying field of normal stress that was kept constant, and applied shearing traction that increased with time. Eventually, this led to slip progressing in the closed part of the crack. If the crack laid entirely in the compressive part of the normal stress field, the problem could be solved in closed form. It was easy to get results for this and also for shearing traction that started to decrease and eventually led to backslip.

Chao and Rau [77] studied the partial contact problem of an arc crack in an infinite isotropic elastic solid under uniaxial loading at infinity. They included the effect of friction on the crack surfaces and considered the contact crack surfaces in slip and no-slip contact conditions. Their formulation of the problem was based on integral equations. They found that $K_{I}$ at the closed crack tip changed significantly as the friction coefficient varied. The size of contact zone was also affected by both the loading orientation and the coefficient of friction. Sungha [78] investigated several situations involving partially or fully closed cracks in the presence of frictional slip. In this work, the Coulomb law of dry friction was used to model the phenomenon of slip and stick.

The analysis of two- and three-dimensional elastic contact problems involving
cracks using the boundary element method was proposed by Shubin [79]. In this work, he employed the classical contact algorithm using a load incremental iterative approach to treat frictional contact problems. Wei and Bremaecker [80] studied the direction of initiation of the crack under compression using the maximum energy release rate criterion. The mathematical formulation of the criterion was approached by way of constrained optimization, and the solution was proven to exist uniquely. The numerical implementation was based on a finite element scheme. An iterative method was employed to handle the material and geometric non-linearities.

Dundurs and Comninou [81] gave the solution to an elasticity problem which in itself was not of much practical significance. However, its simplicity allowed for ease in following potentially complicated issues such as history of loading, residual stresses left after loading, and the distinction between weak and strong friction. The solution could, thus, serve as a basis for modelling other more realistic situations, and its purpose was mainly educational. The problem of a crack terminating at an interface between two materials which was governed by Coulomb's law of friction was studied by Wijeyewickrema et al. [82] using Mellin integral transforms. Depending on the relative slip directions of the two wedges that were created by the crack, both the case of the two wedges moving in opposite directions and that of the two wedges moving in the same direction were treated. The characteristic equations which yield the order of the crack-tip singularity were obtained in terms of the Dundurs constants, the inclination of the crack and the coefficient of friction.

Solutions to the contact problem which have been formulated using the classical theory of elasticity have been rather limited to cases involving simple geometry and loading configurations. In order to overcome these limitations, most compact
problems are currently being treated using computational methods with the finite element method being the most appealing.

Finite element incremental contact analysis with various frictional conditions was presented by Okamoto and Nakazawa [83]. Their method was based on the finite element method and load incremental theory. The geometric and the static boundary conditions on contact surfaces were treated as additional conditions independent of stiffness equations. As a result, the algorithm for calculation was simplified and only that part of the simultaneous equations related to the contact surfaces at each step was required to be solved instead of the overall stiffness equations. An iterative procedure for finite element stress analysis of frictional contact problems was presented by Rahman et al. [84]. Their paper provided simple iterative finite element techniques for solving frictional contact problems without the need for any particular constitutive model or special element at the contact region. The scheme was demonstrated by annualizing a mechanical joint in orthotropic wood. Bathe and Chaudhary [85] presented a solution procedure for the analysis of planar and axisymmetric contact problems involving sticking, frictional sliding and separation under large deformations. The contact conditions were imposed using the total potential of the contact forces with the geometric compatibility conditions which led to contact system matrices and force vectors.

A mixed variational statement and corresponding finite element model were developed by Heyliger and Reddy [86] for an arbitrary plane body undergoing large deflections -large displacements, large rotations and small strains- under external loads using the updated Lagrangian formulation. The mixed finite element formulation allowed the nodal displacements and stresses to be approximated independently. A
review of contact algorithms under the aspects of mathematical exactness and practical applicability was given by Bohm [87]. The basic assumptions and possibilities of some of the main algorithms were displayed, and recommendations were given for a synthesis of different approaches. Numerical experiments and comparisons showed the efficiency of existing programs.

Woo et al. [88] suggested a simple model to solve problems of partially closed crack. In his paper, the length of the closed part of the crack, and the SIF value of the other crack tip had been calculated by using a collocation method. Two symmetric finite element methods for the solution of two-dimensional elastic contact problems were presented by Pascoe and Mottershead [89]. Overlapping of the meshes in the contact region was prevented by the inclusion of displacement and force constraints, which were based on the finite element shape functions and Coulomb's friction law. As a result of the application of these constraints, the stiffness matrix, displacement vector and force vector became augmented with additional terms. The effects of sliding friction was included by iteration with friction forces added to the augmented force vector in one method and normal gap terms added to the force vector in the other method.

Zang and Gudmundson [90] presented a numerical method for the solution of two-dimensional crack problems including the effects of crack kinks and frictional contact between crack faces. The method was based on an integral equation for the resultant forces along a crack. Coulomb friction between contacting crack surfaces were taken into account. The numerical implementation was demonstrated by considering the surface and sub-surface piece-wise straight line cracks in a half-plane. Theocaris and Panagiotopoulos [91] used the boundary integral method to study cracks having
a given geometry by taking into account unilateral contact and friction phenomena between the two sides of the crack. Numerical examples concerning the calculation of stress intensity factors under the unilateral contact and friction interface conditions illustrated the developed method.

A finite element algorithm for incremental analysis of large three-dimensional frictional contact problems of linear elasticity was presented by Zboinski [92]. He briefly presented a theoretical basis for the incremental description of the frictional contact problem of two thermoelastic bodies moving together. Then, the general finite element algorithm of the problem was described, which was based on the variational formulation. The development and implementation of a variational inequalities approach to treat the general frictional contact problem was presented by Refaat and Meguid [93]. They used quadratic programming and Lagrange's multipliers to solve the frictional contact problem and identify the candidate contact surface. Montenegro et al. [94] studied the contact problem in cracks subjected to a strong gradient in the surface direction of the crack, with compressive loads acting on part of it. The approach was based on geometrical considerations and used the Weight function method to obtain the effective crack length and mode I crack tip stress intensity factor. The method was illustrated for an infinite cracked plate with linear and quadratic distributions of a monotonically increasing load.

Ju et al. [95] developed a contact element based on the penalty function method for frictional contact problems in finite element analysis. The advantage of using this algorithm was that the contact element stiffness matrix was symmetric, even for frictional contact problems with a large sliding mode. This element could simulate sticking, sliding and separation modes in frictional contact analysis.
Existing methods for the analysis of contact problems deal with the inequality constraints arising from contact conditions by means of an implicit iteration on all constraints. Eterovic and Bathe [96] presented a formulation for contact problems with friction for large deformations where all inequality constraints were enforced explicitly. To replace inequality constraints with equivalent equality constrained they used a method presented by Mangasarian [97]. Although Mangasarian’s method is completely correct, the extension of that method to the contact problem was totally wrong. The same mistake was done by Watson and Haftka [98] in replacing Kuhn-Tucker’s necessary optimality conditions with an equivalent nonlinear system of equality conditions.

2.8 Stress intensity factor

The stress intensity factor is of fundamental importance in the prediction of brittle failure using linear elastic fracture mechanics (LEFM) principles. It is a function of both the cracked geometry and the associated loading. It is common practice to present K solutions in dimensionless form, normalized with respect to an appropriate infinite-sheet solution. The problem facing a designer is to strike a balance between time, cost and accuracy in selecting a suitable method for determining stress intensities. In a relatively short introductory text, it is clearly not feasible nor desirable to cover each method in detail. In the following, some of the more useful methods of evaluating stress intensity factors are cited.

Analytical solutions are those which lead to explicit expressions for stress intensity factors. Only relatively simple geometries are solvable by analytical methods which
implies a body of infinite extent. Among analytical methods are those which make use of Westergaard’s stress functions [4, 99, 100, 101], Muskhelishvili’s complex stress function [18], Williams’s stress function [35], Conformal mapping [102, 103, 104, 105, 106, 107, 26], Green’s function [108, 109], Integral transforms [110, 111, 112, 113, 114, 115, 116], and Weight function techniques [117, 118, 119].

Alternating methods, sometimes referred to as the Schwartz alternating technique, have been useful in determining stress intensity factors for a number of two and three-dimensional cracks [120, 121, 122, 123]. This technique involves knowing the solution of, usually, two auxiliary problems and is most useful in determining the effect of a single stress-free boundary near to or intersecting with the crack. When applied to crack problems, one auxiliary solution will be for a loaded crack in an infinite plane and the other will be for a plane containing a boundary subjected to an arbitrary stress distribution. The combination of these two solutions leads to the final result.

Experimental methods for finding stress intensity factors may either use a known relationship between a measurable quantity (e.g. compliance or fatigue crack growth rate) and the stress intensity factor, or involve direct measurements on a model (e.g. by photoelasticity). For the compliance method, the reader is referred to the Irwin and Kies [124]. The method has been used for many problems [125, 126, 127]. Of the optical methods for determining stress intensity factors, photoelasticity has been most used. An assessment of this technique in relation to crack tip stress fields was made by Schroedl et al. [128]. The use of photoelasticity in fracture mechanics emphasizing a technique for estimating both flaw shapes and SIF distributions for complex, three-dimensional cracked body problems was done by Smith[129]. A
comprehensive guide to the application of the optical method of caustics to opaque engineering materials was provided by Wallhead and Edwards [130]. Practical recommendations were given which facilitate the measurement of stress intensity factors at considerably higher accuracies. For more detailed discussion on analytical and experimental methods of evaluating stress intensity factors, the reader is referred to the works of Cartwright and Rooke [108] and Sih [131].

Numerical procedures are necessary when stress intensities are desired for more general configuration and loading. The finite element method has become one of the most popular and generally used numerical method of structural analysis. It is difficult to determine the time of the first application of the method, particularly since some ideas of this method used in the analysis of framed structures were substantiated by J. C. Maxwell and A. Castigliano back in the last century. However, the beginning of the extensive development and application of the finite element method can be dated from the middle of the fifties for two reasons. First, at that time electronic computers came into use, and without these the realization of the method is impracticable as it involves large systems of algebraic linear equations. Second, at that time the results of particular problems of continuum mechanics became known and the analytic relationships between nodal forces and displacements were formulated for simple structural elements such as a triangle [132] and a rectangle [133]. Thus, the finite element method appears to be a bridge between strength of materials and elasticity theory. For more discussion about finite element method, the reader is referred to [134].

To calculate stress intensity factor, Barsoum [42] used quadratic isoparametric elements which embody the inverse square root singularity. Using examples of the
plane eight noded isoparametric element showed that it had the same singularity as other special crack tip elements, and still included the constant strain and rigid body motion modes. Stress intensity factors for bi-material bodies were calculated by Smelser [135] by using numerical crack flank displacement data. The discontinuity displacement method to calculate the SIFs considering the contact problem of arc crack surfaces under uniaxial tension and pressure was provided by Hua and Yu [136]. A path-independent line integral, J, was derived for axisymmetric cracks under non-axisymmetric loading conditions by Kuo [137]. Relationships between J and stress intensity factors were also presented for linear elastic fracture problems. Numerical example were carried out on cracked pipes under bending and torsion. Leung and Su [48] calculated a mode I stress intensity factor by using fractal two-level finite element methods. They compared their results with those available in the literature and the results were satisfactory.

Dong [138] represented stress intensity factors, $K_I$ and $K_{II}$, by path integrals of the first stress invariant and its partial derivative. In this work, $K_{III}$ was represented by a path integral of the anti-plane displacement as well as its partial derivative. An energy release rate equation for the crack problem with centrifugal loads was presented by Lee [139]. Using the concept of shape design sensitivity, the final analytical equation was derived by the energy principle and the material derivative. The equation was equivalent to the existing J-integral. Finite element method was applied to the resulting equations. Lim and Lee [140] evaluated stress intensity factors for a crack normal to bi-material interface using isoparametric finite elements. Numerical testing was carried out with the eight-noded and six-noded crack tip elements. A displacement extrapolation method was developed by Zhu and Smith [141] to obtain
crack tip singular stresses and stress intensity factors using only nodal displacements in the first layer of elements around the crack tip. A method for calculating stress intensity factors based on an evaluation of J-integral by the virtual crack extension method was presented by Li [142]. Expressions for calculating $K_I$ and $K_{II}$ by using the displacements and the stiffness derivative of the finite element solution and asymptotic crack tip displacements were derived. Zehnder and Hui [143] calculated SIFs for a finite crack in an infinite plate under bending and shearing loads by assuming Kirchhoff plate theory. In both cases, the crack was oriented at an arbitrary angle to the axis of loading. The two-term parameter technique, was introduced by Rhee and Ernst [144] for the computation of the energy release rate in specimens made of composite materials. The mode II energy release rate calculated by the two parameter technique, was compared with that determined by using the crack closure method. A slant edge crack and an embedded central slant crack under uniform tension load were considered by Kuang and Chen [145]. Comparisons of these two mixed-mode crack problems were made between results using a modified linear displacement extrapolation method and those obtained by Barsoum's extrapolation method.

### 2.9 Thermal stresses and fracture

The deformation of a body is associated with a change in the heat content and, consequently, with a change in the body temperature. A deformation varying in time leads to a change in the temperature field and conversely, a change in the temperature produces a strain. The internal energy of the body, therefore, becomes
a function of the deformation and the temperature. The branch of science dealing with these coupled processes is called *thermoelasticity*.

The theory of thermal stresses which developed almost simultaneously with the theory of elasticity employed the classical heat conduction equation. It did not contain any terms related to the deformation of the body. Knowing the distribution of the temperature resulting from the solution of the heat conduction equation, the displacement equations of elasticity theory were solved. The latter contained known terms involving temperature gradients.

Certain simplified assumptions concerning the conditions for heat conduction can be made in connection with the formulation of the coupled thermoelastic problem. When the external heat sources cause non-uniform heating, it can be assumed that the temperature field is independent of the strains caused by it. Then, the mechanical coupling term in the heat conduction equation can be omitted. If, however, the temperature changes in the elastic body are not produced by an external heat source but are due to the strains, the irreversible process is accompanied by thermoelastic energy dissipation and the mechanical coupling term must be retained in the equation. Moreover, in view of the small temperature changes associated with the deformations caused by the external forces, it is possible to omit the coupling terms in the equation of motion. In either of these cases, the thermoelastic problem is uncoupled and the field of deformations is independent of the temperature field. For a comprehensive review of the fundamental equations of the thermoelasticity, reader is referred to the works of Kovalenko [146], Nowacki [147], and Hetnarski [148, 149, 150].

Literature in thermoelasticity is very rich. Considering recent developments in thermoelasticity, only some of those works which are related to the linear fracture
mechanics and contact problems are listed.

Thermal stresses around a crack that lies in an elastic layer sandwiched between two dissimilar elastic half-planes under uniform heat flow were examined by Itou [151]. Surfaces of the crack were assumed to be thermally insulated. The thermoelastic problem was reduced to that of solving a pair of dual integral equations. The stress intensity factors were calculated numerically. Sala and Abe [152] studied two-dimensional electrothermal crack problems. Steady temperature distribution near the tip of a crack was analysed for a homogeneous isotropic conductive material with the steady direct current (dc). A path-independent integral was introduced to calculate stress intensity factor $K_{II}$. A parametric study of thermally induced stresses in tri-layered media was analysed by Kwon et al. [153]. A finite element method was employed and the effect of the material property was examined on the behaviour of the system. In this work, they found that the interface normal stresses were of a local nature.

Zhang and Norio [154] studied basic, singular solutions of plane thermoelasticity for an infinite medium with a crack. Thermal stresses and thermal stress intensity factors under the action of a point thermal inclusion or a point heat source were evaluated by using the complex function method and some quadrature techniques. Kaczynski [155] studied the state of thermal stresses for a periodic two-layered elastic space weakened by an interface thermally insulated Griffith crack and loaded by concentrated line heat sources. Two examples were presented and stress intensity factors were determined. Bonding strength of dissimilar material joints with an interface crack was evaluated by Seo et al. [156]. In this work, the critical J-value was used as the parameter for the bonding strength criterion of dissimilar material joints.
with an interface crack. The effect of several factors (Young's modulus, temperature variations, crack length etc.) in J-value was discussed. Analytical solutions and numerical examples for thermoelastic interface crack problems in dissimilar anisotropic media were performed by Chao and Chang [157]. Heat flux as well as traction on the crack surface were used as thermal and mechanical loading. Their research indicated that the nature of singularities in the stress field at the crack tips was unaltered by the presence of an applied heat flux. Rao and Hasebe [158] studied axially symmetric stress distribution in the neighbourhood of a penny-shaped crack situated in an infinite isotropic elastic solid under general surface loadings and general surface temperature. The equations of equilibrium of an elastic solid conducting heat was solved using Hankel transforms and Abel's integral operator of the second kind. Crack opening displacement and stress intensity factors were expressed in terms of the prescribed surface temperature functions.

Hilbert problems were derived by Lee and Park [159] to evaluate thermal stress intensity factors for a partially insulated crack subjected to vertically uniform heat flow in an infinite bonded dissimilar material. Both mode I and II thermal stress intensity factors were obtained.

Edge cracks in a non-homogeneous half plane under thermal loading was studied by Jin and Noda [160]. All material properties were supposed to be exponentially dependent on the distance from the boundary of the plate. By using the Fourier transform, the problem was reduced to a singular integral equation which was solved numerically. The steady thermal stresses in a hollow circular cylinder and a hollow sphere made of a functionally gradient material was presented by Obata and Noda [161]. The aim of this research was to understand the effect of the composition
on stresses and to design the optimum FGM hollow circular cylinder and hollow sphere.

Kozlov et al. [162] studied plane quasi-static thermoelastic problems for domains of arbitrary shape with a slit. Special attention was paid to the case of point heat sources. Three types of boundary conditions were considered on the sides of the slit: ideal contact, zero temperature, and heat isolation. For each of them, the leading terms of the asymptotic of tensile and shear stress-intensity factors were obtained. Choi et al. [163] studied the transient thermal stress problem for a cladded medium containing an under clad crack with in the two-dimensional framework of uncoupled, quasi-static thermoelasticity. The cladded medium was modelled such that a thin cladding was bonded to a substrate via a transitional layer. The cladding and the transitional layer were represented as homogeneous strips. Transient thermal stress intensity factors were evaluated. A coupled transient thermoelastic behaviour of an axial-cracked hollow circular cylinder and an edge-cracked subjected to a sudden heating was investigated by Chen and Kuo [164, 165]. It was shown that surface heating may induce the compressive thermal stress near the inner surface of the cylinder which in turn may force the cracked surfaces to close together. The normalized stress intensity factor for the crack tip of the cylinder was obtained. Yeo and Barber [166] conducted stability analysis for thermoelastic contact by using linear perturbation methods for one-dimensional and simple two-dimensional geometries. Since analytical solutions were very complicated, the finite element method was used to reduce the stability problem to an eigenvalue problem.

Kim et al. [167] presented a numerical method to compute the stress intensity factor for the cracked body subjected to a thermal transient loading. The method
was developed on the basis of the Green's function concept and Duhamel's theorem. The transient thermoelastic behaviour of a two-layer annular circular cylinder with an internal edge crack subjected to a sudden heating was investigated by Chen and Kuo [168]. Assuming that the existence of the crack does not alter the temperature distribution, they divided the problem into two parts and solved them by the principle of superposition. The contact length and contact pressure of the real cracked cylinder were obtained by the elimination finite element scheme. Fattah et al. [169] studied the fracture problem for an elastic strip with an embedded and edge crack under transient thermal stresses. A ramp function cooling rate was used at the boundary, which was more realistic than the step function. The problem was treated as quasi-static.

Cherepanov [11] studied dynamic cracks in elastic solids, quasi-static cracks in elastic-plastic and rigid-plastic solids, and the problem of the crack extension in dissipating viscoelastic bodies. He applied continuum mechanics to the crack propagation processes. The crack extension was governed by an additional condition at the crack-tip. Stress intensity factors for two thermal shock problems in transversely isotropic cylinders which contain either an external annular crack or a penny-shaped crack was discussed by Noda et al. [170]. The thermal stress field was analysed by means of the potential function method for transversely isotropic solids. Numerical calculations of the stress intensity factors were carried out for a beryllium oxide cylinder which possesses transverse isotropy. Finally, Lee and Sim [171] calculated stress intensity factor for an edge-cracked plate subjected to thermal shock by using Bueckner's weight function method. It was shown that thermal shock stress intensity factor had maximum values with variation in time and crack length and that there
was a critical crack length.
Chapter 3

Fundamental definitions and basic formulations in linear fracture mechanics

_The great tragedy of science -the slaying of a beautiful hypothesis by an ugly fact._

-T. H. Huxley

3.1 Introduction

A convenient and reasonably rigorous definition of fracture mechanics is the *applied mechanics of crack growth*. Fracture mechanics does not tell us anything about fracture processes, but it does provide the necessary descriptive and analytic framework for its study. Present day fracture mechanics deals largely with macroscopic aspects of crack growth. Although, fracture surfaces are assumed to be smooth, microscopically they are actually very irregular. In the course of analysis, several other simplified assumptions have also been made. For instance, the development of the concept of stress intensity factor was based on the assumption that the material is a linear elastic continuum. Various modifications to basic theory can be made to account for the actual behavior of real materials, and much recent work is concerned with the rigorous analysis of situations involving gross plasticity. In early fracture mechanics, semi-empirical derivations often had to be used because rigorous solutions were not available. This leads to a lack of confidence in the utility of fracture mechanics concepts in practical engineering. However, the situation has improved
recently.

Fracture mechanics has helped to quantify the rather elusive concept of *toughness*. It can now be defined as *resistance to crack growth*. Notice that the type of loading and environment are not specified. Early work in fracture mechanics was focused on brittle fractures. However, the development of the important concept of stress intensity factor $K$, as a single parameter description of the elastic stress field in the vicinity of a crack tip, encouraged the application of fracture mechanics to virtually any type of crack growth. Fracture mechanics only deals with cracked materials; it can not help with situations involving uncracked materials. However, a majority of structures contain cracks which either are introduced during the manufacturing or are initiated early in the life of the structures. These types of cracks are frequently the source of service failures.

The logical basis of fracture mechanics theory for static loadings is the premise that for crack growth to occur two conditions are necessary and sufficient. Firstly, sufficient stress must be available at the crack tip to force crack growth and, secondly, sufficient energy must flow to the crack tip to supply the work done in the creation of new surfaces. Initially, it was believed that only the first was required; however, Inglis’ solution for an elliptical hole indicated that the elastic stress tends to infinity as a crack tip is approached, leading to the paradoxical conclusion that a cracked body can not support any load. This paradox was resolved by Griffith, who used an energy balance approach based on surface energy to explain the fracture behavior of glass. The energy balance approach was extended by Orowan and Irwin to include the energy associated with plastic deformation adjacent to the new crack surfaces.
3.2 The usage of fracture mechanics.

Fracture mechanics has led to the emergence of new design concepts, and efforts are being made to incorporate these in standards and codes of practice. In particular small cracks may be accepted provided that the component is still fit for its design purpose; indeed the attempted repair of a defect can sometimes be more harmful than its acceptance. The introduction of the concept of the use of a flawed component has no counterpart in conventional design. The only way the relationship between stress level and resistance to crack growth can be used is by specifying a flaw size, and this, by its very nature forms no part of the normal design process. In other words, continuity between the design method for flawed or unflawed components is lacking.

In practice, two levels of usage tend to emerge. The first, covering perhaps 90 percent of practical problems, is in a simple go or no-go prediction. The crudest assessment of K and rough knowledge of material properties such as \( K_{IC} \) may well be adequate to indicate the acceptability of a particular postulated or detected flaw.

At the other extreme very careful examination of local stress fields (including residual and thermal stresses) three dimensional effects, and adjacent defects may be necessary to estimate K. Equally, material properties such as \( K_{IC} \) may be affected by degradation through welding, some form of metallurgical embrittlement, anisotropy and material variability. The high technology industries have tended to establish very thorough procedures for quality control, including inspection and nondestructive testing as a necessary workshop complement to the usage of design methods based on fracture mechanics. If a small defect is accepted then knowledge of its growth rate,
generally by fatigue, will allow the logical definition of a re-inspection cycle time that will be adequate to detect a larger crack before it has grown large enough to cause a fracture. Similarly, if there is uncertainty about the size of a crack that may have escaped detection, then survival of an overload test demonstrates that cracks above a certain size cannot exist, or they would have caused a fracture. Hence, at operating load a margin exists between possible and acceptable crack size. Fracture mechanics can assist in the selection of the optimum material or heat treatment needed for a particular job. For example, the right balance between strength and toughness can be specified, and reasonable absolute values can be established for a particular purpose, thus avoiding either costly early failure or the use of unnecessarily expensive material.

In high technology problems, where perhaps an actual defect has been detected, knowledge is often found to be barely adequate to make a final assessment of the margins on which acceptance or rejection of a very expensive piece of hardware may be based [172].

3.3 Brittle fracture

Fracture may be classified into two types: brittle and ductile. In many cases, brittle fractures give rise to fast growth of a crack or cracks in the body. Metals, which are usually ductile, can behave in a brittle manner leading to fast crack propagation, even when the applied stress is less than the general yield stress of the uncracked region. It is possible in some cases to have limited plasticity near the crack tip, and such failures are referred to as quasi-brittle failures. Brittle fracture is often thought
to refer to failures which occur catastrophically without any excessive deformation of the material. However, this definition is limited, it does not explain whether the fracture is propagated at a stress below or above the yield stress. In order to avoid ambiguity, let us define ideal brittle fracture as the failure due to propagation of a crack or cracks when the applied stress and the stresses near the crack tip are less than the yield stress. In practice most brittle materials may show very limited plasticity at the crack tip. Plasticity does not help, but instead resists the propagation of a crack in brittle fracture.

### 3.4 Ductile fracture

On the other hand, in a ductile fracture, plastic deformation always helps in the final fracture. Plastic instability or necking, creep and fatigue in metals are ductile fractures, since they are aided by plastic deformation. It is interesting to note that a specimen which has failed due to fatigue appears very similar to a specimen which has failed due to brittle fracture. An inexperienced engineer may often be unable to distinguish a brittle fracture from a fatigue failure. However, close examination of the fracture surface should reveal the type of fracture. In fatigue, the initiation and initial growth of cracks occurs due to plastic deformation and it is therefore classified as a ductile fracture [173].

### 3.5 Modes of crack propagation

Before we discuss the stress field near the crack tip, it may be useful at this stage to describe three loading situations that become important in the analysis of different
types of stresses. Figure 3.1 shows the three modes and directions of crack propagation. These modes are generally identified by subscripts I, II, and III. Mode I, which is also called the Opening mode, refers to cases in which the in-plane loading is symmetrical with respect to the crack plane; Mode II, usually called the Sliding mode, refers to cases in which the in-plane loading is skew symmetrical to the crack plane; Mode III, also referred to as the Tearing mode, is those cases in which the loading is anti-plane shear. These three modes essentially describe three independent kinematic movements of the upper and lower crack surfaces with respect to each other and are sufficient to describe all possible modes of crack propagation in an elastic material.

Figure 3.1: Mods of crack deformation.
3.6 Stress intensity factors

The plane problem of elasto-statics in the absence of body force for the homogeneous isotropic solid, either in plane strain or idealized plane stress, reduces to the specification of a biharmonic function

\[ \nabla^2 \nabla^2 \chi(r, \theta) = 0 \]

(3.1)

\[ \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \]

in which \( \chi \) is Airy stress function and \( \nabla^2 \) is Laplacian operator in polar coordinates. \( r \) and \( \theta \) are shown in Figure 3.2. The polar stress components can be determined according to the relations

![Figure 3.2: General fracture mechanics geometry.](image)

\[ \sigma_r = \frac{1}{r} \frac{\partial \chi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2} \]
\[ \sigma_\theta = \frac{\partial^2 \chi}{\partial r^2} \]
\[ \tau_{r\theta} = \frac{1}{r^2} \frac{\partial \chi}{\partial \theta} - \frac{1}{r} \frac{\partial^2 \chi}{\partial r \partial \theta} \]  

Equation 3.1 is a reduced form of the stress compatibility equations, and Equation 3.2 may be shown to satisfy the equations of equilibrium in the absence of body forces. Williams [35] solved the above plane problem by using the eigenfunction expansion method. Omitting the intermediate relations, the stresses and displacements in the vicinity of the crack tip can be expressed in terms of the Cartesian and polar coordinate system. \( r \) and \( \theta \) are measured from the tip of the crack. (See Figure 3.2)

\[ \sigma_y = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} [1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2}] + \frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} + O(r^0) \]
\[ \sigma_z = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} [1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2}] - \frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} [2 + \cos \frac{\theta}{2} + \cos \frac{3\theta}{2}] + O(r^0) \]
\[ \tau_{r\theta} = \frac{K_I}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} + \frac{K_{II}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} [1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2}] + O(r^0) \]
\[ u = \frac{K_I}{2E} \sqrt{\frac{\pi}{2\pi}} (1 + \nu) [(2\kappa - 1) \cos \frac{\theta}{2} - \cos \frac{3\theta}{2}] \]
\[ + \frac{K_{II}}{2E} \sqrt{\frac{\pi}{2\pi}} (1 + \nu) [(2\kappa + 3) \sin \frac{\theta}{2} + \sin \frac{3\theta}{2}] + O(r) \]
\[ v = \frac{K_I}{2E} \sqrt{\frac{\pi}{2\pi}} (1 + \nu) [(2\kappa + 1) \sin \frac{\theta}{2} - \sin \frac{3\theta}{2}] \]
\[ + \frac{K_{II}}{2E} \sqrt{\frac{\pi}{2\pi}} (1 + \nu) [-(2\kappa - 3) \cos \frac{\theta}{2} + \cos \frac{3\theta}{2}] + O(r) \]  

where \( \sigma_x, \sigma_y, \) and \( \tau_{xy} \) are normal and shear components of the stress tensor, respectively. \( u \) and \( v \) are displacement components in Cartesian coordinate system.

In the polar coordinate system the above expressions can be written as
\[
\sigma_r = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} (1 + \sin^2 \frac{\theta}{2}) + \frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} (1 - 3\sin^2 \frac{\theta}{2}) + O(r^0)
\]
\[
\sigma_\theta = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} - \frac{K_{II}}{\sqrt{2\pi r}} (3\sin \frac{\theta}{2} \cos^2 \frac{\theta}{2}) + O(r^0)
\]
\[
\tau_{r\theta} = \frac{K_I}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos^2 \frac{\theta}{2} + \frac{K_{II}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} (1 - 3\sin^2 \frac{\theta}{2}) + O(r^0)
\]
\[
u_r = \frac{K_I}{2E\sqrt{2\pi}} (1 + \nu) [(2\kappa - 1) \cos \frac{\theta}{2} - \cos \frac{3\theta}{2}]
\]
\[
+ \frac{K_{II}}{2E\sqrt{2\pi}} (1 + \nu) [-(2\kappa - 1) \sin \frac{\theta}{2} + 3\sin \frac{3\theta}{2}] + O(r)
\]
\[
u_\theta = \frac{K_I}{2E\sqrt{2\pi}} (1 + \nu) [-(2\kappa + 1) \sin \frac{\theta}{2} + \sin \frac{3\theta}{2}]
\]
\[
+ \frac{K_{II}}{2E\sqrt{2\pi}} (1 + \nu) [-(2\kappa + 1) \cos \frac{\theta}{2} + 3\cos \frac{3\theta}{2}] + O(r)
\]

where \(u_r\) and \(u_\theta\) are radial and tangential components of displacement field. \(E\) is Young's modulus and \(\nu\) is Poisson's ratio.

\[
\kappa = \begin{cases} 
\frac{3 - \nu}{1 + \nu} & \text{if plane stress} \\
3 - 4\nu & \text{if plane strain}
\end{cases}
\]

Similarly the asymptotic singular field around the crack tip for anti-plane deformation of the medium can be expressed as

\[
\tau_{zz} = -\frac{K_{III}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} + O(r^0)
\]
\[
\tau_{yz} = -\frac{K_{III}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} + O(r^0)
\]
\[
w = -\frac{2K_{III}}{G} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2} + O(r)
\]

where \(w\) is anti-plane displacement. \(\tau_{zz}\) and \(\tau_{yz}\) are anti-plane shear stresses and \(G\) is the shear modulus.
In anti-plane deformation, expressions in polar coordinate system are the same as expressions in Cartesian coordinate system.

\( K_1, K_{II}, \) and \( K_{III} \) in the foregoing expressions, are defined as the stress intensity factors for the three respective modes. The stress intensity factor depends on the geometry of the body and the loading configuration. It should be noted that the above expressions were derived for both plane strain and plane stress systems.

### 3.7 Energy release rate

When a body is loaded, the movement of the applied loads does work on the body, which is stored in the form of strain energy. It is possible to express the energy stored per unit volume of material (\( U \)) in terms of stress components alone. Then, for a general three-dimensional loading system \( U \) can be written as

\[
U = \frac{1}{2E} \left[ \sigma_z^2 + \sigma_y^2 + \sigma_z^2 - 2\nu(\sigma_z\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_z) \right] + 2(1 + \nu)(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2) \tag{3.6}
\]

Crack instability is associated with the stationary value of the total energy curve. Beyond this point the energy release during an incremental crack extension exceeds the energy required to create new crack surfaces. The value of \( \frac{\partial U}{\partial a} \) defines the strain energy release rate for an incremental crack extension. \( a \) is half of the crack length. The elastic energy release rate per tip \( G \) is defined by

\[
G = \frac{\partial U}{\partial a} \tag{3.7}
\]
It must be noted that the word rate has no reference to time. \( G \) is also termed as the crack extension force. If the fracture occurs via Mode I (Opening mode) then it is expressed as \( G_I \), and similarly \( G_{II} \) and \( G_{III} \) for the other modes.

![Diagram of energy terms and energy release rate with respect to semi-crack size.](image)

Figure 3.3: Variations of energy term and \( G \) with respect to semi-crack size.

It is interesting at this stage to observe the variation of the energy terms and the energy release rate with respect to crack size. Figure 3.3 schematically shows that a fracture commences when the total energy reaches a point of stationary at \( a = a_c \), which is referred to as the critical crack size for a given value of applied stress. In other words, if \( a < a_c \), the system is stable, while if \( a > a_c \), the system is unstable. In Figure 3.3, it can be observed that when the energy release rate reaches a certain value, fracture occurs. The value of \( G \) at fracture is called the critical energy release rate: \( G_c \). For a very brittle material, this value of \( G_c \) is equal to
twice the thermodynamic surface energy i.e.

\[ G_c = 2\gamma_0 \]

In other words, when crack extension occurs, the energy release is \(2\gamma_0\) multiplied by the new crack surface. The factor 2 represents the creation of two surfaces. An advantage of \(G_c\) is that it can include all supplementary energy dissipating terms. For example, it can include the energy dissipated as a result of the limited plasticity at the crack tip [173].

### 3.8 Some basic fracture criterion

![Diagram of a central crack in a plate](image)

**Figure 3.4:** A central crack in a plate.

#### 3.8.1 Griffith concept

The Griffith concept [7, 6] of imperfection instability in a solid was the first step towards predicting the fracture strength of solids. The basic idea behind his theory
is that a crack will begin to propagate if the elastic energy released by its growth is greater than the energy required to create the fractured surfaces. As a model, Griffith considered the problem of a crack of length 2a in a plate under tension \( \sigma \) as in Figure 3.4. He then found that the critical stress \( \sigma_{cr} \) required for crack growth is

\[
\sigma_{cr}a^{\frac{1}{2}} = \left( \frac{2E\gamma}{\pi} \right)^{\frac{1}{2}}
\]

(3.8)

where \( E \) is the Young's modulus and \( \gamma \) the specific surface energy. Since the quantity \( (2E\gamma/\pi) \) contains only material constants, the factor \( \sigma_{cr}a^{1/2} \) should be an intrinsic material parameter. Twice the specific surface energy \( \gamma \) is equal to the critical elastic energy release rate \( G_{ic} \), i.e., \( G_{ic} = 2\gamma \). The experiments Griffith performed on glass show that the values of \( \sigma_{cr}a^{1/2} \) were indeed the same over a wide range of crack lengths. The concept of crack energy release leads to serious drawbacks in carrying out the mathematical details for cracks in a combined stress field. The energy release concept assumes the direction of crack propagation to be known as a priori. Hence, the Griffith theory can only treat problems with the crack lying in a plane normal to the applied stress as in Figure 3.4. A simple question, such as what will be the direction of crack propagation if the crack was inclined at an angle \( \beta \) to the loading axis, can not be answered satisfactorily. In such a case, Equation 3.8 is no longer valid [174].

3.8.2 Stress intensity factor approach

Irwin [3, 2], unlike Griffith's crack extension criterion based on energy principles, interpreted the stress field near a crack tip at fracture as a means of characterizing the resistance of a material to crack propagation. From Equation 3.3, the stress field
near a crack tip for Mode I can be written as

\[
\begin{align*}
\sigma_y &= \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left[ 1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right] + O(r) \\
\sigma_x &= \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left[ 1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right] + O(r) \\
\tau_{xy} &= \frac{K_I}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} + O(r) 
\end{align*}
\]  

(3.9)

In the above expression \( K_I \) depends on the loading configuration and the geometry of the system. In general, the stress intensity factor in the first mode can be expressed as

\[ K_I = \sigma \sqrt{\pi a f(a, \alpha)} \]  

(3.10)

where \( f(a, \alpha) \) is a function of crack size, \( a \), and \( \alpha \) is a characteristic dimension of the specimen, and \( \sigma \) is the remote uniform normal stress.

The principle of the stress intensity factor approach is that at the onset of cracking, the stress field near a crack tip is constant. In other words, whatever the geometry and the loading configuration of a specimen are, the stress field near the crack tip is always the same at fracture. If expression 3.9 is closely examined in the light of the above statement, it is obvious that at fracture, \( K_I \) reaches a unique value for a given material. This value of the stress intensity factor is defined as the critical stress intensity factor, \( K_Ic \). Thus, \( K_Ic \) is a material property, and is sometimes defined as the fracture toughness. Therefore, the expression 3.10 at fracture (\( \sigma = \sigma_f \)), takes the form

\[ K_{Ic} = \sigma_f \sqrt{\pi a f(a, \alpha)} \]  

(3.11)
Let us consider two examples which illustrate the interpretation of the above expression. Figures 3.5 and 3.6 show two loading configurations for the same material: (a) a central crack in an infinite body, subjected to a uniform stress; (b) an edge crack in a semi-infinite body, subjected to a uniform stress. One can very easily verify the following expressions for $K_{IC}$

$$K_{IC} = \begin{cases} \sigma_{f,a} \sqrt{\pi a} & \text{infinite body} \\ 1.12 \sigma_{f,b} \sqrt{\pi a} & \text{semi-infinite body} \end{cases}$$

where subscripts a and b refer to respective loading systems. Since $K_{IC}$ is a material property, the fracture stress for the infinite body is 1.12 times the fracture stress for the semi-infinite body. That is,

$$\sigma_{f,a} = 1.12 \sigma_{f,b}$$

Thus, it can be concluded that the fracture stress takes a value, depending on the
Figure 3.6: A surface crack in a semi-infinite body.

loaded configuration and geometry of the body, such as to make $K_I$ equal to $K_{IC}$ at fracture. It is important not to confuse $K_I$ with $K_{IC}$. $K_I$, as mentioned previously is to be dependent on the configuration of the system, but $K_{IC}$ (the value of $K_I$ at fracture) is a material property and is independence of the configuration of the system.

In this approach, it can be accounted for limited plasticity at the crack tip by suitably modifying expression 3.11. The principle of this approach is based on the stress field near the crack tip. Therefore, the yielded zone near the crack tip should be examined. Irwin [175] estimated that the plastic zone which extends a distance, $d_Y$, ahead of the crack tip, as shown in Figure 3.7. Curve A shows the stress, $\sigma_Y$, if the material is very brittle. Let $\sigma_Y$ be the yield stress. The plastic zone is a result of the tendency to produce further yielding if $\sigma_Y$ is not allowed to increase beyond $\sigma_Y$. The area of the shaded section is approximately equal to $(d_Y - r_Y)\sigma_Y$ ($r_Y$ is shown
Figure 3.7: The plastic zone ahead of the crack tip.

in Figure 3.7). The argument is that the curve A ($\sigma_y < \sigma_Y$) is displaced a distance of $(dy - r_Y)$ to the new position. Therefore,

$$\int_0^{r_Y} \sigma_y \cdot dr - \sigma_Y r_Y = \sigma_Y(dy - r_Y)$$

(3.12)

since $\sigma_y = \frac{K_I}{\sqrt{2\pi r}}$ we have

$$\int_0^{r_Y} \frac{K_I}{\sqrt{2\pi r}} \cdot dr = \sigma_Y dy$$

(3.13)

that is,

$$\sigma_Y dy = \sqrt{\frac{2}{\pi}} K_I \sqrt{r_Y}$$

since $\sigma_Y = \frac{K_I}{\sqrt{2\pi r_Y}}$ we have

$$dy = 2r_Y$$

(3.14)
At fracture $K_f = K_{lc}$ therefore

$$r_Y = \frac{K_{lc}^2}{2\pi \sigma_Y^2}$$

In order to apply the linear elastic analysis to this system, where $r_Y$ is small compared with the crack size, an equivalent or notional crack is defined and is taken to be equal to $2(a + r_Y)$. Thus, $K_{lc}$ for an infinite body, may be modified as

$$K_{lc} = \sigma_f \sqrt{\pi (a + r_Y)}$$

where

$$r_Y = \frac{K_{lc}^2}{2\pi \sigma_Y^2} \approx \frac{\sigma_f^2 \pi a}{2\pi \sigma_Y^2} = a \frac{\sigma_f^2}{2\sigma_Y^2}$$

Therefore we have

$$K_{lc} = \sigma_f \sqrt{\pi a \left(1 + \frac{\sigma_f^2}{2\sigma_Y^2}\right)}$$

For very brittle materials $\sigma_Y \gg \sigma_f$, hence the unmodified expression holds correct. It is apparent, for materials with limited plasticity, that if expression 3.17 is not used to estimate the fracture toughness, or $K_{lc}$, of the material, the result is to underestimate the $K_{lc}$ value.

It must be emphasized that expression 3.17 can only be applied to materials with limited plasticity. The expression is not reliable at high values of $\sigma_f/\sigma_Y$. In the extreme case when $\sigma_f = \sigma_Y$, gross yielding occurs and this theory does not apply.

### 3.8.3 The J integral criterion

In section 3.8.1 the strain energy release rate for a small crack extension on the basis of assumed linear elastic behavior was considered. If the plastic zone dimensions are
no longer negligibly small, the energy release rate, $G$, will not be valid. To overcome this difficulty Rice [13] presented a line integral which has the same value for all integration paths surrounding a class of notch tips in two-dimensional deformation field of linear or nonlinear materials.

Consider a homogeneous body of linear or nonlinear elastic material free of body forces and subjected to a two-dimensional deformation field (plane strain, generalized plane stress, anti-plane strain) so that all stresses $\sigma_{ij}$ depend only on two Cartesian coordinates $x_1(=x)$ and $x_2(=y)$. Suppose the body contains a notch of the type shown in Figure 3.8, having flat surfaces parallel to the x-axis and a rounded tip denoted by the arc $\Gamma_1$, straight crack is a limiting case. Define the strain energy per unit volume $U$ by

$$U = U(x, y) = U(\varepsilon) = \int_0^\varepsilon \sigma_{ij} \cdot d\varepsilon_{ij}$$

(3.18)
where \( \varepsilon = [\varepsilon_{ij}] \) is the infinitesimal strain tensor. Now consider the integral \( J \) defined by

\[
J = \int_{\Gamma} (U dy - T \frac{\partial u}{\partial x}) ds
\]

(3.19)

where \( \Gamma \) is a curve surrounding the notch tip, the integral being evaluated in a counterclockwise sense starting from the lower flat notch surface and continuing along the path \( \Gamma \) to the upper flat surface. \( T \) is the traction vector defined according to the outward normal along \( \Gamma \), \( T_i = \sigma_{ij}n_j \), \( u \) is the displacement vector, and \( ds \) is an element of arc length along \( \Gamma \). To prove path independence, consider any closed curve \( \Gamma^* \) enclosing an area \( A^* \) in a two-dimensional deformation field free of body forces. An application of Green’s theorem gives

\[
\int_{\Gamma^*} (U dy - T \cdot \frac{\partial u}{\partial x}) ds = \int_{A^*} \left[ \frac{\partial U}{\partial x} - \frac{\partial}{\partial x_j} (\sigma_{ij} \frac{\partial u_i}{\partial x}) \right] \cdot dx dy
\]

(3.20)

Differentiating the strain energy per unit volume,

\[
\frac{\partial U}{\partial x} = \frac{\partial U}{\partial \varepsilon_{ij}} \frac{\partial \varepsilon_{ij}}{\partial x} = \sigma_{ij} \frac{\partial \varepsilon_{ij}}{\partial x} \quad [\text{by equation 3.18}] \\
= \frac{1}{2} \sigma_{ij} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \sigma_{ij} \frac{\partial}{\partial x_j} \left( \frac{\partial u_i}{\partial x} \right) \\
= \frac{\partial}{\partial x_j} (\sigma_{ij} \frac{\partial u_i}{\partial x}) \quad (\text{since } \frac{\partial \sigma_{ij}}{\partial x} = 0)
\]

(3.21)

the area integral in equation 3.20 vanishes identically, and thus

\[
\int_{\Gamma^*} (U dy - T \cdot \frac{\partial u}{\partial x}) ds = 0
\]

(3.22)

for any closed curve \( \Gamma^* \). Thus, the \( J \) integral taken around the closed contour ABCDEF shown in Figure 3.9 is zero. Since no contribution arises from CD and AF on the crack surfaces (i.e. \( T=0 \) and \( dy=0 \)), the integral along ABC must be equal
Figure 3.9: Closed contour at crack tip.

and opposite to that along DEF. Therefore, the J integral taken along an unclosed contour between unloaded crack surfaces is \textit{path independent}.

Evaluation of J integral for the case of linear elasticity leads to [176]

\[ J = G \]  (3.23)

Thus J characterizes energy release rate during a small crack extension, which may also be valid in the presence of significant crack tip plasticity. There will be a critical value of J, termed \( J_{fc} \), at which crack extension occurs. Since this also holds for the purely elastic case, it follows that

\[ J_{fc} = G_{fc} \]  (3.24)

which implies that a failure criterion associated with significant plasticity can be used to determine \( G_{fc} \) and vice versa. Since J is path independent one may choose the most convenient path, usually the specimen boundary.
3.8.4 Energy-momentum tensor

A more general path independent integral can be defined by using the energy-momentum tensor concept which was first presented by Eshelby [14]. To define energy-momentum tensor, suppose that we have a set of quantities \( u_i(X_m) \) depending on the independent variables \( X_m \) and a function

\[
L = L(u_i, u_{i,j}, X_m)
\]  

which depends on \( u_i \) and its first derivatives \( u_{i,j} = \partial u_i / \partial X_j \) and also explicitly on the \( X_m \). \( L \) can be regarded as a Lagrangian function. Therefore, \( u_i \) are the displacements, \( u_{i,j} \) the displacement gradients, \( X_m, m = 1, 2, 3 \) are Cartesian coordinates, \( i = 1, 2, 3 \) and if time is explicitly present \( X_4 = t \). \( L \) generates the field equations governing the displacements \( u_i \) via the Euler equations

\[
\left( \frac{\partial}{\partial X_j} - \frac{\partial L}{\partial u_{i,j}} \right) \frac{\partial L}{\partial u_i} = 0
\]

The notation \( \partial L / \partial X_i \) shall be used to denote the gradient of \( L \), so that \( \partial L / \partial X_i dX_i \) is, to order \( dX_i \), the numerical value of \( L \) at \( X_i + dX_i \) minus its numerical value at \( X_i \). From it we must distinguish the explicit partial derivative of \( L \) with respect to \( X_i \) when its other arguments \( u_i, u_{i,j} \) and the remaining \( X_m \) are held constant. We shall denote it by \( \partial L / \partial X_i \)_{\text{exp}} so that

\[
\left( \frac{\partial L}{\partial X_i} \right)_{\text{exp}} = \frac{\partial L(u_i, u_{i,j}, X_m)}{\partial X_i} \bigg|_{u_i, u_{i,j}, X_m \neq i} \quad \text{Const.}
\]

the components of the gradient of \( L \) are thus

\[
\frac{\partial L}{\partial X_i} = \frac{\partial L}{\partial u_i} u_{i,i} + \frac{\partial L}{\partial u_{i,j}} u_{i,j} + \left( \frac{\partial L}{\partial X_i} \right)_{\text{exp}}
\]
or
\[
\frac{\partial L}{\partial X_i} = \left(\frac{\partial L}{\partial u_i} - \frac{\partial}{\partial X_j} \frac{\partial L}{\partial u_{i,j}}\right)u_{i,j} + \frac{\partial L}{\partial X_j}\left(\frac{\partial L}{\partial u_{j,i}} + \left(\frac{\partial L}{\partial X_i}\right)_{\text{exp}}\right)
\]  
(3.28)

From 3.26 the first term on the right of equation 3.28 vanishes and equation 3.28 can be rewritten as
\[
\frac{\partial P_{ij}}{\partial X_j} = -\left(\frac{\partial L}{\partial X_i}\right)_{\text{exp}}
\]  
(3.29)

where
\[
P_{ij} = \frac{\partial L}{\partial u_{i,j}} u_{i,j} - L \delta_{ij}
\]  
(3.30)

is the energy-momentum tensor we are seeking for [14]. Having defined the energy-momentum tensor the following integral is defined
\[
F_l = \int_A P_{ij} \cdot dA_j
\]  
(3.31)

where A is a surface surrounding the crack tip. Applying the divergence theorem to equation 3.31, we have
\[
F_l = \int_A P_{ij} \cdot dA_j = \int \int_V P_{ij} \cdot dV
\]  
(3.32)

where V is the volume surrounding the crack tip. This integral will be path independent provided \(\partial P_{ij}/\partial X_j = 0\) and from equation 3.29 this is seen to be so provided
\[
\left(\frac{\partial L}{\partial X_i}\right)_{\text{exp}} = 0
\]

Therefore, if Lagrangian is not an explicit function of \(X_i\), the integral \(F_l\) is a path independent integral.

Although Equations 3.25 to 3.30 are defined with the possibility of three Cartesian coordinates and time; the Lagrangian considered in equation 3.25 does not
involve time explicitly. The field equations given by equation 3.26 correspond to the equations governing specific problems in which time has been transformed away. For example the question of the time will be eliminated for problems of transient stresses on stationary cracks by using the Laplace transform. Therefore, in this case, an $L$ will be chosen for which equation 3.26 corresponds to the transformed equations governing the problem. Then the integral $F_t$ will be a path independent integral, provided only that $L$ does not involve $X_t$ explicitly. Using the energy-momentum concept, Atkinson [73] solved some steady-state and transient crack motion in a strip.

3.8.5 The strain-energy-density concept

A theory of fracture based on the field strength of the local strain-energy-density was proposed by Sih [174]. The theory requires no calculation on the energy release rate and thus possesses the inherent advantage of being able to treat all mixed mode crack extension problems. Unlike the conventional theory of $G$ and $K$, which measures only the amplitude of the local stresses, the fundamental parameter in this theory, the \textit{strain-energy-density} factor $S$, is also direction sensitive. The difference between $K$ (or $G$) and $S$ is analogous to the difference between a scalar and vector.

Referring to Figures 3.10a and 3.10b, the Griffith-Irwin theory can be viewed as a scalar theory in that it specifies only the critical value of a scalar $G_{fc}$ (or $K_{fc}$) at incipient fracture. The direction of crack propagation is always pre-assumed to be normal to the load. Moreover, the crack front must be straight so that $G$ or $K$ does not vary along the leading edge of the crack. In addition, a scalar theory cannot yield the correct material parameter if two or more stress intensity factors are present along the crack border. The $S$ factor senses the direction of least resistance by
attaining a stationary value with respect to the angle $\theta$ as indicated in Figure 3.10b. The stationary value of $S_{\text{min}}$ can be used as an intrinsic material parameter whose value at the point of crack instability $S_{\text{cr}}$ is independent of the crack geometry and loading. In the general context, the Griffith-Irwin theory is the special case when $\theta = 0$ and the director $S$ coincides with the x-axis.

Substituting from 3.3 and 3.5 into 3.6 yields the quadratic form for the strain-energy-density function

$$U = \frac{1}{r}(a_{11}K_I^2 + 2a_{12}K_I K_{II} + a_{22}K_{II}^2 + a_{33}K_{III}^2) + \cdots$$  \hspace{1cm} (3.33)

Note that the higher order terms in $r$ have been neglected and that the strain-energy-density function near the crack possesses a $1/r$ energy singularity. Hence the quadratic

$$S = (a_{11}K_I^2 + 2a_{12}K_I K_{II} + a_{22}K_{II}^2 + a_{33}K_{III}^2) + \cdots$$  \hspace{1cm} (3.34)
represents the amplitude or the intensity of the strain-energy-density field and it varies with polar angle $\theta$. The coefficients $a_{ij}(i, j = 1, 2, 3)$ are given by

$$
a_{11} = \frac{1}{16G}(3 - 4\nu - \cos \theta)(1 + \cos \theta)
$$
$$
a_{12} = \frac{1}{16G}2\sin \theta[\cos \theta - (1 - 2\nu)]
$$
$$
a_{22} = \frac{1}{16G}[4(1 - \nu)(1 - \cos \theta) + (1 + \cos \theta)(3\cos \theta - 1)]
$$
$$
a_{33} = \frac{1}{4G}
$$

For two-dimensional problems where the crack extends in the xy-plane, the stress-intensity factors do not vary along the crack front and $S$ depends only on one variable, namely the angle $\theta$. In three dimensions, $K_I$, $K_{II}$, and $K_{III}$ may occur simultaneously and they can also vary from point to point on the crack border.

**Fundamental hypotheses on crack initiation and direction**

Since the strain-energy-density factor $S$ has some attributes of the intensity of a force field associated with the type of potential, it is natural to inquire on the relationship between $S$ and the potential energy in the system. If the cracked body is subjected to traction only; then the potential energy is equal to the negative of the strain energy $^1$.

$P$ stands for the potential energy per unit volume of the element located at a distance $r$ from the crack border as shown in Figure 3.11. Making use of the relation $P = -U$ and $U = S/r$, the potential energy per unit volume becomes

$$
P = -\frac{S}{r}
$$

Two fundamental hypotheses of crack extension are:

$^1$The opposite holds for displacement loading conditions.
Hypothesis (1): *The crack will spread in the direction of maximum potential energy density.*

Hypothesis (2): *The critical intensity $S_\sigma$ of this potential field governs the onset of crack propagation.*

Note that for crack propagation to take place in the xy-plane the direction of maximum potential energy density must be found. In two-dimensional problems, the direction of crack propagation can be determined by a single variable $\theta$ and hence hypothesis (1) can be satisfied by the application of the calculus of variations. A necessary condition for the potential energy $P$ to have a stationary value is that

$$\frac{\partial P}{\partial \theta} = 0, \quad \text{at} \quad \theta = \theta_0$$

(3.37)

The value of $\theta_0$, which makes $P$ a maximum, determines the angle of the plane along...
which the crack spreads and can be found by further requiring that

\[
\frac{\partial^2 P}{\partial \theta^2} < 0 \quad \text{at} \quad \theta = \theta_0
\]  

(3.38)

which is a position of unstable equilibrium. Rewriting the conditions in equations 3.37 and 3.38 in terms of the strain-energy-density function tenders

\[
\frac{\partial S}{\partial \theta} = 0 \quad , \quad \frac{\partial^2 S}{\partial \theta^2} > 0 \quad \text{at} \quad \theta = \theta_0
\]  

(3.39)

which are the necessary and sufficient conditions for S to be a minimum. Hypothesis (1) is equivalent to the assumption that crack initiation will start in a radial direction along which the strain-energy-density is a minimum. The above criterion is based on the local density of the energy field in the crack tip region and requires no special assumption on the direction in which the energy released by the separating crack surfaces is computed, as in the Griffith theory and others. This removes the fundamental difficulties involved for computing energy release rate in mixed mode problems. Any fracture criterion based on a single stress parameter such as \( K_I \) alone will not be sufficient to describe the problem of mixed mode fracture [174].

3.9 Functionally gradient materials (FGMs)

In 1984, the initial concept of functionally gradient materials (FGMs) was proposed. The original purpose of these unique composite materials was for the development of super heat-resistant materials for the propulsion system and airframe of the space craft. A FGM is a composite that smoothly transitions from one material at one surface to another material at the opposite surface. Usually, metals and ceramics are the materials that are combined in a controlled manner to optimize a specific
property. The properties (functions) of FGMs can be preselected by choosing the appropriate composition and layer thickness. FGM composite materials are used as thermal barrier coatings and in joining ceramics to ceramics or to metals. It is evident that FGM composite materials hold the key for many applications requiring structural materials that can withstand high temperatures and that are wear- and corrosion-resistant. In addition, FGM applications have been extended from structural to five other functional areas: electronic, chemical, optical, nuclear, and biological.

3.9.1 Mathematical formulations

Consider the plane elasticity problem shown in Figure 3.2 where a non-homogeneous body contains a crack. For the purpose of examining the nature of crack tip singularity, and for studying the effect of material non-homogeneity on the stress intensity factors, it is assumed that the elastic and thermal properties of the non-homogeneous material can be expressed by continuous and piecewise differentiable functions of coordinates x and y as follows:

\[
\begin{align*}
\lambda &= \lambda(x, y) \\
G &= G(x, y) \\
\rho &= \rho(x, y) \\
\alpha &= \alpha(x, y) \\
k &= k(x, y)
\end{align*}
\]

(3.40)

where \(\lambda\) and \(G\) are Lame's coefficients and \(\rho\) is the mass density. \(\alpha\) and \(k\) are the linear expansion coefficient and the thermal conductivity of the material, respectively.
Equations of motion for plane elasticity without considering the effect of body forces, in Cartesian coordinate system, can be written as:

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = \rho \frac{\partial^2 u}{\partial t^2}
\]

\[
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = \rho \frac{\partial^2 v}{\partial t^2}
\]  

(3.41)

in which \(u\) and \(v\) are components of displacement vector in the \(x\) and \(y\) direction and \(t\) is time.

The Duhamel-Neumann constitutive equations for a body under mechanical and thermal loadings can be written as:

\[
\sigma_x = \lambda(\varepsilon_x + \varepsilon_y) + 2G\varepsilon_x - (3\lambda + 2G)\alpha T
\]

\[
\sigma_y = \lambda(\varepsilon_x + \varepsilon_y) + 2G\varepsilon_y - (3\lambda + 2G)\alpha T
\]

\[
\tau_{xy} = G\gamma_{xy}
\]  

(3.42)

where \(\varepsilon_x\), \(\varepsilon_y\), and \(\gamma_{xy}\) are normal and shear strain components, respectively, and \(T\) is the temperature distribution in the body. Equations 3.42 are constitutive equations of the isothermal theory of elasticity (Hooke's law) augmented by temperature term \(-(3\lambda + 2G)\alpha T\). The strain-displacement relations in small deformation theory can be expressed as:

\[
\varepsilon_x = \frac{\partial u}{\partial x}
\]

\[
\varepsilon_y = \frac{\partial v}{\partial y}
\]

\[
\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}
\]  

(3.43)
By expressing stresses in terms of displacements through the constitutive and strain-displacement relations, Equations 3.41 reduce to the following differential equations:

\[
(\lambda + 2G) \frac{\partial^2 u}{\partial x^2} + \frac{\partial(\lambda + 2G)}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial \lambda}{\partial x} \frac{\partial u}{\partial y} + \lambda \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial G}{\partial y} \frac{\partial u}{\partial y} \\
+ \frac{\partial G}{\partial y} \frac{\partial v}{\partial x} + G \frac{\partial^2 u}{\partial y^2} + G \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial [(3\lambda + 2G)\alpha]}{\partial x} T \\
-(3\lambda + 2G)\alpha \frac{\partial T}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}
\]

\[
(\lambda + 2G) \frac{\partial^2 v}{\partial y^2} + \frac{\partial(\lambda + 2G)}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial \lambda}{\partial y} \frac{\partial v}{\partial x} + \lambda \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial G}{\partial x} \frac{\partial v}{\partial x} \\
+ \frac{\partial G}{\partial x} \frac{\partial u}{\partial y} + G \frac{\partial^2 u}{\partial x^2} + G \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial [(3\lambda + 2G)\alpha]}{\partial y} T \\
-(3\lambda + 2G)\alpha \frac{\partial T}{\partial y} = \rho \frac{\partial^2 v}{\partial t^2}
\]  \hspace{1cm} (3.44)

By dropping inertia terms from the right-hand side of equations 3.44, the equations of motion will be reduced to the equilibrium equations in the quasi-static or static state of loading.

Since \( \lambda, G, \) and \( \alpha \) are continuous and piece wise differentiable functions of space positions and are limited at the crack tip, the governing equation for the singular solution near the crack tip becomes:

\[
(\lambda + 2G) \frac{\partial^2 u}{\partial x^2} + (\lambda + G) \frac{\partial^2 v}{\partial x \partial y} + G \frac{\partial^2 u}{\partial y^2} = 0
\]

\[
(\lambda + 2G) \frac{\partial^2 v}{\partial y^2} + (\lambda + G) \frac{\partial^2 u}{\partial x \partial y} + G \frac{\partial^2 v}{\partial x^2} = 0
\]  \hspace{1cm} (3.45)

The singular solution to the homogeneous material satisfies the same equation. Therefore, it is also the solution to the non-homogeneous material in every differen-
tiable pieces. The continuity in displacements and traction across the weak physics
discontinuity lines are maintained by this solution, as long as the material proper-
ties are continuous. Therefore, the singularity and the angular distribution of the
crack-tip stress field for the non-homogeneous material are identical to those in the
homogeneous material (equations 3.3). The same argument is correct for the ideal-
ized plane stress situation.

The basic equations which govern the anti-plane deformation behavior of the
medium can be expressed in a Cartesian coordinate system as:

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = \rho \frac{\partial^2 w}{\partial t^2} \tag{3.46}$$

where w is the anti-plane displacement. The constitutive equations are:

$$\tau_{xx} = G\gamma_{xx}$$
$$\tau_{yz} = G\gamma_{yz} \tag{3.47}$$

where $\gamma_{xx}$ and $\gamma_{yz}$ are shear strain components, respectively. The strain-displacement
relations in small deformation theory are:

$$\gamma_{xx} = \frac{\partial w}{\partial x}$$
$$\gamma_{yz} = \frac{\partial w}{\partial y} \tag{3.48}$$

Substituting from Equations 3.47 and 3.48 into Equation 3.46, we will have

$$G\nabla^2 w + \frac{\partial G}{\partial x} \frac{\partial w}{\partial x} + \frac{\partial G}{\partial y} \frac{\partial w}{\partial y} = \rho \frac{\partial^2 w}{\partial t^2} \tag{3.49}$$

Since G is continuous and piecewise differentiable function of space positions and
is limited at the crack tip, the governing equation for the singular solution near the

$$G\nabla^2 w = 0 \tag{3.50}$$

crack tip (in case of quasi-static or static equilibrium condition) becomes:
which is the same as the governing equation for the homogeneous materials.

From the above discussion, it is evident that the fracture criterion, namely, Griffith's theory or energy release rate concept; Irwin's theory or stress intensity factor, J integral idea, energy-momentum tensor, and strain-energy-density concepts can be used in the framework of functionally gradient materials without any further assumption or correction. Material properties in the equations are no longer constant and they are functions of space positions.
Chapter 4

Elastodynamic response of an interface crack in FGMs under anti-plane shear impact load

If a man look sharply and attentively, he shall see fortune.

-William Shakespeare

4.1 Introduction

This chapter provides a theoretical and numerical treatment of a finite crack subjected to an anti-plane shear impact load in a medium with spatially varying elastic properties. This variation is in a direction perpendicular to the crack surfaces. The elastic properties of the interfacial layer are assumed to vary continuously between that of two dissimilar homogeneous bonding layers. Laplace and Fourier transforms are applied to reduce this mixed boundary value problem to a system of dual integral equations which in turn will be reduced to a standard Fredholm integral equation of the second kind. The Fredholm integral equation is solved in the Laplace transform plane numerically. The time inversion is accomplished by a numerical scheme. A Laplace inversion technique developed by Miller and Guy [64] is combined with a quadrature method to get the solution in the physical plane. Numerical results are presented to illustrate the effect of the geometric configuration and the mechanical properties of the interfacial layer upon the singularity behavior of the crack. The dynamic stress intensity factor is found to either increase or decrease with the crack
length to the layer thickness depending on the relative magnitudes of the material properties of the adjoining layer [177].

4.2 Formulation

Let a crack lie in the \( y=0 \) plane of a right handed Cartesian coordinate system \((x,y,z)\) and be bound between lines \( x = \pm a \) as in figure 4.1. A functionally gradient material layer is bounded between two homogeneous dissimilar half-planes 1 and 4. For the current problem it is convenient to divide the interfacial layer into two regions, namely material number 2 with a thickness of \( h_1 \) and material number 3 with a thickness of \( h_2 \) as shown in figure 4.1. The material properties of layers 1 and 4 \((G_1, \rho_1, G_4, \rho_4)\) are constant. \( G \) is the shear modulus and \( \rho \) is the mass density of the material. Assuming that the variation of the shear modulus and the mass density in the interfacial layer are described by:

\[
G_2 = G_0 \exp(\beta y) \\
\rho_2 = \rho_0 \exp(\beta y) \\
G_3 = G_0 \exp(\beta y) \\
\rho_3 = \rho_0 \exp(\beta y) 
\]

(4.1)

from the continuity conditions for material properties at \( y = h_1 \) and \( y = -h_2 \) constants \( G_0, \beta, \) and \( \rho_0 \) can be calculated as:

\[
\beta = \frac{1}{h_1 + h_2} \ln\left(\frac{G_1}{G_4}\right) \\
G_0 = G_1 \exp(-\beta h_1)
\]
In the above formulations it is assumed that the ratio $\frac{\alpha_2}{\alpha_4}$ is equal to $\frac{\alpha_1}{\alpha_3}$. For anti-plane shear motion, the components of the displacement in $x$, $y$, and $z$ directions are given by:

$$u_k = v_k = 0, \quad w_k = w_k(x, y, t)$$

(4.3)

subscripts $k=1,2,3,4$ refer to layers 2 and 3 and the half-planes 1 and 4, and $t$ is the time. The basic equations which govern the anti-plane deformation behavior of the medium can be expressed in a Cartesian coordinate system as:

$$\frac{\partial \tau_{xzk}}{\partial x} + \frac{\partial \tau_{yzk}}{\partial y} = \rho_k \frac{\partial^2 w_k}{\partial t^2}$$

(4.4)

where

$$\tau_{xzk} = G_k \frac{\partial w_k}{\partial x}$$
where $\tau_{yxk}$ and $\tau_{yxk}$ are the nonzero components of the shear stresses. The equations of motion in terms of displacement components for the two half-planes 1 and 4 are

\[
\nabla^2 w_i = \frac{1}{C_{2i}^2} \frac{\partial^2 w_i}{\partial t^2}
\]

where $i=1,4$ and $C_{2i} = \sqrt{\frac{G_i}{\rho_i}}$ is the transverse wave speed of the material and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the two-dimensional Laplacian operator. For the functionally gradient materials in layers 2 and 3 these equations can be written as:

\[
G_j \nabla^2 w_j + \frac{\partial G_j}{\partial y} \frac{\partial w_j}{\partial y} = \rho_j \frac{\partial^2 w_j}{\partial t^2}
\]

where $j=2,3$. Substitution of $G_j$ and $\rho_j$, from equations 4.1 into equation 4.7 gives

\[
\nabla^2 w_j + \beta \frac{\partial w_j}{\partial y} = \frac{1}{C_{2j}^2} \frac{\partial^2 w_j}{\partial t^2}
\]

with $C_{2j} = \sqrt{\frac{G_j}{\rho_0}}$ being the local transverse wave speed of the interfacial layers in $y = 0$ plane. If we assume that the crack is subjected to an anti-plane shear impact load at $t = 0$, the boundary conditions take the following forms:

\[
\tau_{yx2}(x,0,t) = \tau_{yx3}(x,0,t) = -\tau_0 H(t), \quad |x| \leq a
\]

\[
w_2(x,0,t) - w_3(x,0,t) = 0, \quad |x| \geq a
\]

where $\tau_0$ is constant and $H(t)$ denotes the Heaviside unit step function. Since the shear stress and the displacement are continuous across the interfaces, the following conditions must be satisfied

\[
w_1(x,h_1,t) = w_2(x,h_1,t)
\]
Let the Laplace transform pair be written as:

\[ w_3(x, -h_2, t) = w_4(x, -h_2, t) \]  \hspace{1cm} (4.12)

\[ \tau_{yz1}(x, h_1, t) = \tau_{yz2}(x, h_1, t) \]  \hspace{1cm} (4.13)

\[ \tau_{yz3}(x, -h_2, t) = \tau_{yz4}(x, -h_2, t) \]  \hspace{1cm} (4.14)

\[ \tau_{yz2}(x, 0, t) = \tau_{yz3}(x, 0, t) \]  \hspace{1cm} (4.15)

### 4.3 Analysis

Let the Laplace transform pair be written as:

\[ \Phi^*(P) = \int_0^\infty \Phi(t) \exp(-Pt) \cdot dt \]  \hspace{1cm} (4.16)

\[ \Phi(t) = \frac{1}{2\pi i} \int_{Br} \Phi^*(P) \exp(Pt) \cdot dP \]  \hspace{1cm} (4.17)

where the integral in equation 4.17 is taken over the Bromwich path. Applying equation 4.16 to equations 4.6 and 4.8, the deflection \( w^*_i \) in the transform domain depends only on the space variables \( x \) and \( y \).

\[ \nabla^2 w^*_i = \frac{P^2}{C_{2i}^2} w^*_i \]  \hspace{1cm} (4.18)

\[ \nabla^2 w^*_j + \beta \frac{\partial w^*_i}{\partial y} = \frac{P^2}{C_{2j}^2} w^*_i \]  \hspace{1cm} (4.19)

where \( i=1,4 \) and \( j=2,3 \). In addition, if the Fourier cosine transform is employed on the variable \( x \), equations 4.18 and 4.19 are reduce to ordinary differential equations as follows:

\[ \frac{d^2 w^*_i}{dy^2} - \left( \frac{P^2}{C_{2i}^2} + S^2 \right) w^*_i = 0 \]  \hspace{1cm} (4.20)
\[
\frac{d^2 w_{i}^*}{dy^2} + \beta \frac{dw_{i}^*}{dy} - \left( \frac{p^2}{c_{2i}^2} + s^2 \right) w_{i}^* = 0
\] (4.21)

which can be readily solved. With the aid of the Fourier inversion theorem, a solution in terms of \( P \) is found

\[
w_{i}^*(x, y, P) = \frac{2}{\pi} \int_0^\infty \left[ A_i(S, P) \exp(-\alpha_i y) + B_i(S, P) \exp(\alpha_i y) \right] \cos(Sx) \cdot dS
\] (4.22)

\[
w_{j}^*(x, y, P) = \frac{2}{\pi} \int_0^\infty \left[ A_j(S, P) \exp(\alpha_j y) + B_j(S, P) \exp(-\alpha_j y) \right] \cos(Sx) \cdot dS
\] (4.23)

The parameters \( \alpha_i, \alpha_{j1}, \) and \( \alpha_{j2} \) are given by:

\[
\alpha_i = \sqrt{\left( S^2 + \frac{p^2}{c_{2i}^2} \right)}
\] (4.24)

\[
\alpha_{j1} = \frac{-\beta - \sqrt{\left( \beta^2 + 4(S^2 + \frac{p^2}{c_{2j}^2}) \right)}}{2}
\] (4.25)

\[
\alpha_{j2} = \frac{-\beta + \sqrt{\left( \beta^2 + 4(S^2 + \frac{p^2}{c_{2j}^2}) \right)}}{2}
\] (4.26)

where \( i=1,4 \) and \( j=2,3 \). The displacement condition that \( w_1 \) and \( w_4 \) vanish at infinity requires that \( B_1(S, P) \) and \( A_4(S, P) \) be zero. Solving for the appropriate shear stresses in the transformation domain, we will have:

\[
\tau_{y_i} = \frac{2}{\pi} \int_0^\infty \alpha_1 A_1(S, P) \exp(-\alpha_1 y) \cos(Sx) \cdot dS
\]

\[
\tau_{y_2} = \frac{2}{\pi} \int_0^\infty [\alpha_{21} A_2(S, P) \exp(\alpha_{21} y) + \alpha_{22} B_2(S, P) \exp(\alpha_{22} y)] \cos(Sx) \cdot dS
\]

\[
\tau_{y_3} = \frac{2}{\pi} \int_0^\infty [\alpha_{31} A_3(S, P) \exp(\alpha_{31} y) + \alpha_{32} B_3(S, P) \exp(\alpha_{32} y)] \cos(Sx) \cdot dS
\]

\[
\tau_{y_4} = \frac{2}{\pi} \int_0^\infty \alpha_4 B_4(S, P) \exp(\alpha_4 y) \cos(Sx) \cdot dS
\] (4.27)
Making use of equations 4.22, 4.23, and 4.27, the conditions in equations 4.11 through 4.15 lead to the following equations:

\[ A_1 \exp(-\alpha_1 h_1) = A_2 \exp(\alpha_2 h_1) + B_2 \exp(\alpha_3 h_1) \]
\[ A_3 \exp(-\alpha_1 h_2) + B_3 \exp(-\alpha_2 h_2) = B_4 \exp(-\alpha_3 h_2) \]
\[ -\alpha_1 A_1 \exp(-\alpha_1 h_1) = \alpha_2 A_2 \exp(\alpha_2 h_1) + \alpha_3 B_2 \exp(\alpha_3 h_1) \]
\[ \alpha_4 B_4 \exp(-\alpha_4 h_2) = \alpha_3 A_3 \exp(-\alpha_3 h_2) + \alpha_3 B_3 \exp(-\alpha_3 h_2) \]
\[ \alpha_2 A_2 + \alpha_3 B_2 = \alpha_3 A_3 + \alpha_3 B_3 \quad (4.28) \]

Using equations 4.9 and 4.10 together with equation 4.28, we will end up with a pair of dual integral equations as follows:

\[ \int_0^\infty A(S, P) \cos(Sx) \cdot dS = 0, \quad |x| \geq a \quad (4.29) \]
\[ \int_0^\infty S \Phi(S, P) A(S, P) \cos(Sx) \cdot dS = \frac{\pi r_0}{2G_0 P}, \quad |x| \leq a \quad (4.30) \]

Under the boundary conditions, the unknowns in equations 4.22 and 4.23 may be expressed in terms of a single function \( A(S, P) \)

\[ A_1(S, P) = g_3(S, P) A_2(S, P) \]
\[ A_3(S, P) = g_3(S, P) A_2(S, P) \]
\[ A_4(S, P) = 0 \]
\[ B_1(S, P) = 0 \]
\[ B_2(S, P) = g_1(S, P) A_2(S, P) \]
where 

\[ B_3(S, P) = g_4(S, P)A_2(S, P) \]

\[ B_4(S, P) = g_5(S, P)A_2(S, P) \]

\[ A_2(S, P) = \frac{1}{1 - g_3(S, P) + g_1(S, P) - g_4(S, P)}A(S, P) \quad (4.31) \]

and \( F_1(S, P) \) in equation 4.30 is a known function and it is given by

\[ F_1(S, P) = \frac{\alpha_{21} + \alpha_{22}g_1(S, P)}{S(g_4(S, P) - g_1(S, P) + g_3(S, P) - 1)} \quad (4.33) \]

To change dual integral equations 4.29 and 4.30 to the canonical form, we will introduce the following change of variables

\[ A(S, P) = \frac{B(S, P)}{S} \quad (4.34) \]
substitute equation 4.34 into 4.29 and 4.30

\[
\int_0^\infty B(S, P) \frac{\cos(Sx)}{S} \cdot dS = 0, \quad |x| \geq a
\]

\[
\int_0^\infty F_1(S, P) B(S, P) \cos(Sx) \cdot dS = \frac{\pi \tau_0}{2G_0 P}, \quad |x| \leq a
\]

or

\[
\int_0^\infty B(S, P) \frac{\cos(Sx)}{S} \cdot dS = 0, \quad |x| \geq a
\]

\[
\int_0^\infty B(S, P) \cos(Sx) \cdot dS = P_0(x), \quad |x| \leq a
\]  (4.35)

where

\[
P_0(x) = \frac{\pi \tau_0}{2G_0 P} - \int_0^\infty [F_1(S, P) - 1] B(S, P) \cos(Sx) \cdot dS
\]

Now consider the following variables

\[S = \frac{r}{c}\]

\[x = qc\]

\[B(S, P) = r^{\frac{1}{2}} Q(r, P)\]

\[g(q) = c \left( \frac{2}{\pi q} \right)^{\frac{1}{2}} P_0(qc)\]

\[
\cos(Sx) = \cos(qc) = \left( \frac{\pi qr}{2} \right)^{\frac{1}{2}} J_{-\frac{1}{2}}(qr)
\]  (4.36)

substituting equation 4.36 into equation 4.35, we will have the following canonical form of dual integral equations.

\[
\int_0^\infty Q(r, P) J_{-\frac{1}{2}}(qr) \cdot dr = 0, \quad |q| \geq \frac{a}{c}
\]  (4.37)

\[
\int_0^\infty rQ(r, P) J_{-\frac{1}{2}}(qr) \cdot dr = g(q), \quad |q| \leq \frac{a}{c}
\]  (4.38)
Following Sih and Chen [178] and applying the method of Copson [179], the unknown \( A(S,P) \) in the dual integral equations 4.29 and 4.30 can be solved as follows

\[
A(S,P) = \frac{\pi \eta a^2}{2G_0 P} \int_0^1 \sqrt[3]{\psi^*(\zeta, P)} J_0(Sa\zeta) \cdot d\zeta
\]  

(4.39)

in which \( \psi^*(\zeta, P) \) is governed by a Fredholm integral equation of the second kind

\[
\psi^*(\zeta, P) + \int_0^1 \psi^*(\eta, P) K(\zeta, \eta, P) \cdot d\eta = \sqrt[3]{\zeta}
\]  

(4.40)

The kernel \( K(\zeta, \eta, P) \) takes the form

\[
K(\zeta, \eta, P) = \sqrt[3]{\zeta} \int_0^\infty \left[ F_1(Sa, P) - 1 \right] J_0(S\zeta) J_0(S\eta) \cdot dS
\]  

(4.41)

where \( J_0 \) is the zero order Bessel function of the first kind.

### 4.4 Stress intensity factor

Since the coefficient \( A(S,P) \) is known, the entire stress field can be obtained. Using integration by parts \( A(S, P) \) can be written as

\[
A(S, P) = \frac{\pi \eta a}{2G_0 P} \left[ \psi^*(1, P) J_1(Sa) - \int_0^1 \zeta J_1(Sa\zeta) \frac{d}{d\zeta} \left( \frac{\psi^*(\zeta, P)}{\sqrt[3]{\zeta}} \right) \cdot d\zeta \right]
\]  

(4.42)

with \( J_1 \) being the first order Bessel function of the first kind. The integral in equation 4.42 is bounded at crack tip \( x = a \) and the singular behaviour of the stresses is governed by the leading term containing \( \psi^*(1, P) \). The significant quantities for the crack problem, are the stresses just ahead of the crack tip which can be specified by the stress intensity factor. At \( y = 0 \) shear stress \( \tau_{xy} \) can be written as

\[
\tau_{xy}(x, 0, P) = \frac{2}{\pi} G_0 \int_0^\infty \left[ \alpha_{21} A_2(S, P) + \alpha_{22} B_2(S, P) \right] \cos(Sx) \cdot dS
\]  

(4.43)
substitute from equation 4.31 into equation 4.43 we will have

\[ \tau^*_{xy}(x, 0, P) = \frac{2}{\pi} G_0 \int_0^\infty \frac{\alpha_{21} + \alpha_{22} g_1(S, P)}{1 - g_3(S, P) + g_1(S, P) - g_4(S, P)} A(S, P) \cos(Sx) \cdot dS \] (4.44)

To obtain asymptotic behavior of shear stress near the crack tip, one can consider only the first term of \( A(S, P) \) in equation 4.42. Substitution of this part into equation 4.44 gives

\[ \tau^*_{xy}(x, 0, P) = \frac{\tau_0 a}{P} \int_0^\infty \frac{\alpha_{21} + \alpha_{22} g_1(S, P)}{1 - g_3(S, P) + g_1(S, P) - g_4(S, P)} \Psi^*(1, P) \]

\[ J_1(Sa) \cos(Sx) \cdot dS \] (4.45)

For large \( S \) Bessel function \( J_1 \) can be approximated by:

\[ J_1(Sa) \approx \left( \frac{2}{\pi Sai} \right)^{\frac{1}{2}} \cos(Sa - \frac{3\pi}{4}) \quad \text{for large } S \] (4.46)

Moreover, it can be shown that

\[ \frac{\alpha_{21} + \alpha_{22} g_1(S, P)}{1 - g_3(S, P) + g_1(S, P) - g_4(S, P)} \approx -1 \quad \text{for large } S \] (4.47)

Substituting from equations 4.46 and 4.47 into equation 4.45 we will have

\[ \tau^*_{xy}(x, 0, P) \approx -\Psi^*(1, P) \frac{\tau_0 a}{P} \sqrt{\frac{2}{\pi a}} \int_0^\infty \frac{\cos(Sa - \frac{3\pi}{4}) \cos(Sx)}{\sqrt{S}} \cdot dS \] (4.48)

Knowing that

\[ \cos\left(\frac{3\pi}{4} - Sa\right) \cos(Sx) = \frac{1}{2} \left[ -\frac{\sqrt{2}}{2} \cos(S(x - a)) - \frac{\sqrt{2}}{2} \sin(S(x - a)) \right] \]

\[ + \frac{1}{2} \left[ -\frac{\sqrt{2}}{2} \cos(S(x + a)) + \frac{\sqrt{2}}{2} \sin(S(x + a)) \right] \] (4.49)

and

\[ \int_0^\infty \frac{\cos(q)}{\sqrt{q}} \cdot dq = \int_0^\infty \frac{\sin(q)}{\sqrt{q}} \cdot dq = \sqrt{\frac{\pi}{2}} \] (4.50)
the shear stress in equation 4.48 can be written as

\[ \tau_{x_2}(x, 0, P) \approx \frac{\Psi^*(1, P)\tau_0\sqrt{\pi a}}{P} \cdot \frac{1}{\sqrt{2\pi(x - a)}} \]  \hspace{1cm} (4.51)

From the above equation the stress intensity factor can be defined in the Laplace plane as

\[ K_{III}^*(P) = \frac{\Psi^*(1, P)}{P} \cdot \tau_0\sqrt{\pi a} \]  \hspace{1cm} (4.52)

and can be inverted in accordance with equation 4.17 and written as

\[ K_{III}(t) = \frac{\tau_0\sqrt{\pi a}}{2\pi i} \int_{\gamma} \frac{\Psi^*(1, P)}{P} \exp(Pt) \cdot dP \]  \hspace{1cm} (4.53)

in a physical space.

### 4.5 Numerical examples and results

In this section, the effects of the pertinent parameters upon the dynamic stress intensity factor \( K_3 = \frac{K_{III}}{\tau_0\sqrt{\pi a}} \) are examined, with \( \tau_0 \) being the known uniform shear stress along the crack surfaces. First, the Fredholm integral equation in Laplace space is solved numerically using a combination of Gauss-quadrature and Chebyshev polynomials. To change limits of the integral in equation 4.40, consider the following change of the variable

\[ \eta = \frac{1}{2}(Q_0 + 1) \]

\[ d\eta = \frac{1}{2}dQ_0 \]  \hspace{1cm} (4.54)

substituting equation 4.54 into equation 4.40 we will have

\[ \Psi^*(\zeta, P) + \int_{-1}^{1} \Psi^*(\frac{1}{2}(Q_0 + 1), P)K(\zeta, \frac{1}{2}(Q_0 + 1), P) \cdot \frac{1}{2}dQ_0 = \sqrt{\zeta} \]  \hspace{1cm} (4.55)
considering
\[ \Psi^*(\frac{1}{2}(Q_0 + 1), P) = \frac{1}{\sqrt{1 - Q_0^2}} \Psi^*_0(Q_0, P) \] (4.56)
equation 4.55 becomes:
\[ \Psi^*(\zeta, P) + \int_{-1}^{1} \frac{1}{\sqrt{1 - Q_0^2}} \Psi^*_0(Q_0, P) K(\zeta, \frac{1}{2}(Q_0 + 1), P) \cdot \frac{1}{2} dQ_0 = \sqrt{\zeta} \] (4.57)
the above equation can be solved, for any \( \zeta \) and \( P \). For the sake of simplicity we chose
\[ \zeta = \frac{P + 1}{2} \] (4.58)
using equations 4.56 and 4.58, equation 4.57 can be written as
\[ \frac{1}{\sqrt{1 - P^2}} \Psi^*_0(P, P) + \int_{-1}^{1} \frac{1}{\sqrt{1 - Q_0^2}} \Psi^*_0(Q_0, P) K(\frac{1}{2}(P + 1), \frac{1}{2}(Q_0 + 1), P) \cdot \frac{1}{2} dQ_0 = \sqrt{\frac{P + 1}{2}} \] (4.59)
To solve equation 4.59 we need to have the kernel \( K(\frac{1}{2}(P + 1), \frac{1}{2}(Q_0 + 1), P) \) which is given by equation 4.41. To integrate this equation, consider the following change of variable
\[ S = \frac{1 + q_0}{1 - q_0} \]
\[ dS = \frac{2}{(1 - q_0)^2} dq_0 \] (4.60)
substitute 4.60 into 4.41 we will end up with
\[ K(\zeta, \eta, P) = \sqrt{\zeta} \int_{-1}^{1} \frac{1 + q_0}{1 - q_0} \left[ F_1(\frac{1 + q_0}{(1 - q_0)a}, P) - 1 \right] J_0(\frac{1 + q_0}{1 - q_0} \zeta) \]
\[ \cdot J_0(\frac{1 + q_0}{1 - q_0} \eta) \cdot \frac{2}{(1 - q_0)^2} \cdot dq_0 \] (4.61)
and finally by substituting $\eta$ and $\zeta$ from equations 4.54 and 4.58 into the above equation the kernel $K$ can be written as

$$K(\frac{1}{2}(\varepsilon + 1), \frac{1}{2}(Q_0 + 1), P) = \sqrt{\frac{(\varepsilon + 1)(Q_0 + 1)}{4}} \int_{-1}^{1} \frac{1 + q_0}{1 - q_0} d\alpha$$

$$[F_{\alpha}(\frac{1 + q_0}{(1 - q_0)\alpha}, P) - 1]J_0(\frac{1 + q_0 \varepsilon + 1}{2})$$

$$J_0(\frac{1 + q_0 Q_0 + 1}{2}) \frac{2}{(1 - q_0)^2} \cdot dq_0 \quad (4.62)$$

To integrate equation 4.62 the six-point Gauss-quadrature method is used and finally to solve integral equation 4.59 Chebyshev polynomials are used.

Figures 4.2 to 4.4 display the results of $\Psi^*(1, P)$ as a function of $C_{21}/Pa$ where $C_{21}$ represents the shear wave speed in layer 1. Figure 4.2 shows that $\Psi^*(1, P)$
increases with an increase in the ratio of the shear modulus of the two bounding planes, namely $G_1/G_4$. In this case the crack is in the middle plane of the interfacial layer ($h_1 = h_2$) and the ratio of the crack length to the thickness is equal to one ($a/h_1 = 1$). Figure 4.3 shows the variation of $\Psi^*(1, P)$ with $C_{21}/Pa$ for two limiting cases. A considerable increase in $\Psi^*(1, P)$ can be seen as $h_1$ decreases. Finally, Figure 4.4 depicts the variations of $\Psi^*(1, P)$ with respect to $C_{21}/Pa$ for different values of $a/h_1$ when the crack is laying at $y = 0$. In this case $\Psi^*(1, P)$ increases with the increase in the crack-length ratio $\frac{a}{h_1}$.

To find the dynamic stress intensity factor, we should accomplish the Laplace inversion of equation 4.53 numerically. We will have succeeded in our mission of providing feasible numerical techniques if we present a simple, general technique for
obtaining the values of $\Phi(t)$ in equation 4.16, given the value $\Phi^*(P)$. Two numerical methods for solving the above problem will be discussed here, namely,

- Laplace inversion using quadrature technique
- Laplace inversion using Miller method

### 4.5.1 Laplace inversion using quadrature technique

The essential and completely classical idea guiding steps in our program for the numerical inversion of the Laplace transform is that of replacing an integral by a finite sum. This quite simple and reasonable approximation procedure reduces the solution of the linear integral equation

$$\Phi^*(P) = \int_0^\infty \Phi(t) \exp(-Pt) \cdot dt$$
Figure 4.5: Approximations for $\Phi(t) = \exp(-t) \sin t$ using quadrature method

to that of a system of linear algebraic equations. The basic assumptions we made are two fold: first, that this is a well-posed problem in the sense that an exact, and hence unique, determination of $\Phi^*(P)$ would lead to an exact, and hence unique, determination of $\Phi(t)$; second, that $\Phi(t)$ is sufficiently smooth to permit the approximation methods we employ. Using Legendre polynomial the above linear integral equation will reduce to the following algebraic equation

$$\sum_{i=1}^{N} W_i x_i^{P-1} \Phi(-\ln(x_i)) = \Phi^*(P)$$

where $W_i$ are weighting coefficients and $x_i$ are zeros of the Legendre polynomials. Letting $P$ assumes $N$ different values, say $P = 1, 2, \ldots, N$, equation 4.63 yields a linear system of $N$ equations in the $N$ unknowns, $\Phi(t_i)$ in which $t_i = -\ln(x_i)$. This system of equations provides approximate values of $\Phi(t)$ at the values $t_i = -\ln(x_i)$. It is not
difficult to show that the roots of Legendre polynomials are uniformly distributed over $[0,1]$ to a higher and higher degree of regularity as $N \to \infty$. Unfortunately, the logarithms of the $1/x_i$ do not possess the same equidistribution property over $[0,1]$. Thus, the $t_i$ values tend to cluster around $t = 0$. Hence, we face a problem if we wish to determine $\Phi(t)$ over a more extensive range. To show this effect consider the Laplace transform defined by $\Phi^*(P) = \frac{1}{(P+1)^2+1}$. The known inverse is $\Phi(t) = \exp(-t)\sin t$. Figure 4.5 shows the numerical inversion of the $\Phi^*(P)$ obtained by using the quadrature method. It can be seen from this figure that although this method is very accurate, it is not able to cover the whole range of $t$ and therefore, it does not give a smooth curve for function $\Phi(t)$.

4.5.2 Laplace inversion using Miller method

To overcome the above difficulty Miller and Guy [64] presented a method which we will outline as follows:

Consider the Laplace transform of $\Phi(t)$ defined by 4.16 and assume that $\Phi^*(P)$ is known or can be computed at discrete points along the real $P$-axis. The variable of integration may be changed by the substitution

$$ x = 2\exp(-\delta t) - 1 $$

(4.64)

where $\delta$ is a real positive number. If this equation is solved for $t$, then

$$ t = -\left(\frac{1}{\delta}\right)\ln\left(\frac{1+x}{2}\right) $$

(4.65)

and a new function $\Phi_I$ is defined over $(-1,1)$ by

$$ \Phi_I(x) = \Phi\left(-\left(\frac{1}{\delta}\right)\ln\left(\frac{1+x}{2}\right)\right) = \Phi(t) $$

(4.66)
Substitution of equation 4.64 into equation 4.16 and some algebraic manipulation give

\[ \Phi^*(P) = \left( \frac{1}{2\delta} \right) \int_{-1}^{1} \left( \frac{1+x}{2} \right)^{j-1} \Phi_I(x) \cdot dx \]  \hspace{1cm} (4.67)

Miller and Guy [64] expanded \( \Phi_I(x) \) over \((-1, 1)\) in an infinite series by using Jacobi polynomials \( P_n(x) \) as follow

\[ \Phi_I(x) = \sum_{n=0}^{\infty} C_n P_n^{(\gamma)}(x) \]  \hspace{1cm} (4.68)

If the coefficients \( C_n \) are known then \( \Phi_I(x) \) is known, which implies that \( \Phi(t) \) can be calculated for any \( t = t_0 \) by means of equation 4.66.

After performing some calculation the coefficients \( C_n \) can be calculated from the following equations

\[ \delta \Phi^*[(\gamma + 1 + k)\delta] = \sum_{m=0}^{k} \frac{k(k-1) \cdots [k-(m-1)]}{(k+\gamma+1)(k+\gamma+2) \cdots (k+\gamma+1+m)} C_m \]  \hspace{1cm} (4.69)

This result is true for \( k = 0, 1, \cdots \), and for \( k = 0 \) the right side of this expression is replaced by \( C_0/(\gamma + 1) \). The coefficient \( C_0 \) is determined by allowing \( k = 0 \) and knowledge of \( \Phi^*(P) \) at \( P = (\gamma + 1)\delta \). For \( k = 1 \) the coefficient \( C_1 \) is determined from the value (calculated) of \( C_0 \) and \( \Phi^*(P) \) at \( P = (\gamma + 2)\delta \). In a similar manner the remaining coefficients \( C_2, C_3, \cdots \) can be determined.

If \( N \) coefficients are calculated then \( \Phi_I(x) \) may be approximated by

\[ \Phi_I(x) \approx \sum_{n=0}^{N} C_n P_n^{(\gamma)}(x) \]  \hspace{1cm} (4.70)

Since \( x = 2 \exp(-\delta t) - 1 \), the Jacobi polynomials may be expressed as functions of \( t \) directly. From equation 4.66 it then follows that

\[ \Phi(t) \approx \sum_{n=0}^{N} C_n P_n^{(\gamma)}(2\exp(-\delta t) - 1) \]  \hspace{1cm} (4.71)
As it can be seen from equation 4.71, $\Phi(t)$ can be found for a very large range of $t$. This is the biggest advantage of this method in comparison with the quadrature technique. However in this method there is a crucial parameter $\delta$ which controls the stability and accuracy of the method and varies from one problem to another. To see this problem, again consider the Laplace transform defined by $\Phi^*(P) = 1/((P + 1)^2 + 1)$. Numerical inversion of the $\Phi^*(P)$ is shown in Figure 4.6. Although this method is able to provide a very smooth solution for a wide range of parameter $t$, it is obvious that the parameter $\delta$ is very crucial in the accuracy of the method. As one can see for different values of $\delta$ there are different approximations to the original function which is a disadvantage of this method.

Figure 4.6: Approximations for $\Phi(t) = \exp(-t)\sin t$ using Miller method
To overcome this problem, first we use a quadrature method based on shifted Legendre polynomials and then we combine this method with that which was proposed by Miller and Guy to find the correct $\delta$. With this procedure we accomplish the Laplace inversion of equation 4.62. The mode III normalized dynamic stress intensity factor ($SIF$), $K_3(t)$, is obtained for different geometrical parameters $a$, $h_1$, and $h_2$ and different material properties of the two half-planes 1 and 4. Figures 4.7 to 4.9 shows the variation of $K_3(t)$ with respect to $C_{21}t/a$. In all those figures, the general feature of the curves is that the stress intensity factor rises rapidly with time, reaches a peak, then decreases in magnitude to reach its static value.
4.6 Conclusion

A theoretical and numerical treatment of a finite crack subjected to an anti-plane shear impact load in a functionally gradient material is presented. The analysis is based upon the use of an integral transform technique. The boundary value problem is reduced to the solution of the dual integral equations. Dual integral equations are solved by reducing the problem to the Fredholm integral equation. The Fredholm integral equation is solved numerically in the Laplace space by using a combination of Gauss-quadrature and Chebyshev polynomials. Using an asymptotic solution we find the relation between the solution of Fredholm integral equation at $P = 1$ and the stress intensity factor in Laplace space. Finally, Laplace inversion is accomplished by the use of Gauss-quadrature and Jacobi polynomials and the dynamic stress intensity factor.
factor is calculated for different geometrical parameters and material properties. In general, the dynamic stress intensity factor is found as a function of the crack length, distance from bounding layers and material properties of the composite. As the crack distance $h_1$, from the half-plane 1, which possesses larger shear modulus, decreases the dynamic stress intensity factor increases.
Chapter 5

Evaluation of stress intensity factors and energy release rate for a crack in FGMs under frictional contact load

*The harder you work, the luckier you get.*

-Gary Player

5.1 Introduction

This chapter provides a comprehensive numerical treatment of a finite crack in an interfacial layer with spatially varying elastic properties under in-plane mechanical loading conditions. The elastic properties of the interfacial layer are assumed to vary continuously between those of the two homogeneous bonding layers. The primary objective of this chapter is to study the variation of stress intensity factors and energy release rates in an essentially compressive loading condition. In the most recent studies of fracture mechanics, it is considered that the crack faces are open and free of frictional traction. Although, in most applications of fracture mechanics, the cracks encountered are open, meaning that the crack faces are separated and free of traction, exceptions arise in tribology, geophysics, the study of compression failure of brittle materials and many other applied fields, where cracks propagate in an essentially compressive environment. In such cases, the crack faces are in contact over the whole length of the crack or a part of it, and frictional effects play an important
role. Although frictional-contact of a crack in an isotropic homogeneous material has received some attention, it is author’s knowledge that there is no article which is devoted to the frictional-contact of a crack in functionally gradient materials. It is therefore, the objective of this work to provide a comprehensive numerical investigation of the behavior of an interfacial crack in an interfacial layer with spatially varying elastic properties under plane shear and compressive loads. Stress intensity factors are calculated using crack flank displacement which correlates the displacements at nodal points of the finite element mesh with those of asymptotic solutions near the crack tip. The separation of mixed-mode energy release rates has been achieved by a two-term parameters technique which uses only the current stress and displacement distributions to compute the energy release rate components. Numerical examples are provided to show the accuracy and the feasibility of the method. and comparisons done with existing results in the literature. The effect of material properties, coefficient of friction, and thickness of interfacial layer on the stress intensity factors, energy release rate, and stress and displacement distributions are studied [180] and [181].

5.2 Formulation

A plane crack of length 2a in a layer of height \( h = h_1 + h_2 \) is considered as illustrated in Figure 5.1. The crack is under plane loads in a functionally gradient material having Young’s modulus \( E \) and Poisson’s ratio \( \nu \) which are varying with coordinates \( y \). The surrounding materials are homogeneous and isotropic and the material properties \( E_2, \) and \( \nu_2 \) are constant through the layers. As seen in Figure 5.1, \( x \) and \( y \) form a
set of rectangular Cartesian coordinates fixed in the middle of the interfacial layer.

The displacement components corresponding to the coordinate system have the subscripts $i$ as $u_{ij}$ where $i=1$ and 2 refer respectively to $x$ and $y$ directions and $j=1$ and 2 refer respectively to the interfacial layer and the surrounding material. Assuming that the variation of the Young's modulus $E$ and the Poisson's ratio $\nu$ of the interfacial layer are described by the following relations:

\begin{align}
E &= E_0 + E_1 \left(\frac{2y}{h}\right)^2 \\
\nu &= \nu_0 + \nu_1 \left(\frac{2y}{h}\right)^2
\end{align}

where $E_0$, $E_1$, $\nu_0$, and $\nu_1$ are material constants. From the continuity conditions for material properties at $y = \pm \frac{h}{2}$, constants $E_1$ and $\nu_1$ can be calculated as

$$E_1 = E_2 - E_0$$
\[ \nu_1 = \nu_2 - \nu_0 \]  

(5.2)

\( E_0 \) and \( \nu_0 \) are material constants similar to Young's modulus and Poisson's ratio in isotropic homogeneous materials. Other forms of material distributions for interfacial layer will be introduced in section 5.5. The basic equations which govern the elastostatic behavior of the surrounding materials can be expressed in a Cartesian coordinate system as:

\[ \sigma_{ki,k} + X_i = 0 \]  

(5.3)

where \( X_i \) are body forces per unit volume in \( x \) and \( y \) directions, respectively. \( \sigma_{ki} \) are stress components and "\( , \)" denotes the partial differential operator. By expressing stresses in terms of displacements through the constitutive and strain-displacement relations for isotropic Hookean materials, equation 5.3 reduces to the following elliptic partial differential equation:

\[ (\lambda_2 + G_2)e_{i,i} + G\alpha_{i2,kk} + X_i = 0, i = 1, 2 \]  

(5.4)

where \( e_2 = u_{k2,k} \) and

\[
\lambda_2 = \frac{E_2\nu_2}{(1 - 2\nu_2)(1 + \nu_2)} \\
G_2 = \frac{E_2}{2(1 + \nu_2)} 
\]  

(5.5)

\( \lambda_2 \) and \( G_2 \) are Lame's constants for surrounding materials and "\( , kk \)" is the two-dimensional Laplacian operator. Moreover, the internal and external virtual work for surrounding materials can be written as:

\[
IVW_t = t \int_{A_2} \sigma_{ij}\delta e_{ij} \cdot dA \\
EVW_t = t \int_{A_2} X_i\delta u_i \cdot dA + t \int_{L_2} q_i\delta u_i \cdot dL 
\]  

(5.6)
where $t$ is the thickness of the plane which is constant through the body. $q_i$ are components of the external forces acting on the body. $A_2$ and $L_2$ are area and boundary of the surrounding materials.

For functionally gradient materials in layer one, equilibrium equations in term of stress, strain-displacement relations, and constitutive equations remain unchanged. The only equations which are changed are equilibrium equations in terms of displacements (i.e. eqn 5.4) which can be written as:

$$
(\lambda + 2G)u_{11,11} + (\lambda + G)u_{21,12} + G_2(u_{11,2} + u_{21,1}) + Gu_{11,22} = 0
$$

$$
(\lambda + 2G)u_{21,22} + (\lambda + G)u_{11,12} + (\lambda + 2G)_2u_{21,2} + \lambda_2u_{11,1} + Gu_{21,11} = 0
$$

(5.7)

(5.8)

where:

$$
\lambda = \frac{E\nu}{(1 + \nu)(1 - 2\nu)}
$$

$$
G = \frac{E}{2(1 + \nu)}
$$

$E$ and $\nu$ are defined by Equation 5.1. The internal and external virtual work of the interfacial layer can be written as:

$$
IVW_i = t \int_{A_i} \sigma_{ij} \delta \epsilon_{ij} \cdot dA
$$

$$
EVW_i = t \int_{A_i} X_i \delta u_i \cdot dA + t \int_{L_i} q_i \delta u_i \cdot dL
$$

(5.9)

where $A_1$ and $L_1$ are the area and the boundary of the interfacial layer.

Note that in Equation 5.9 $E$ and $\nu$ are not constant and they are functions of $y$.

Using Equations 5.6 and 5.9, the internal and external virtual work for the entire body can be written as:

$$
IVW = IVW_i + IVW_i
$$
\[ EVW = EVW_s + EVW_i \]  

(5.10)

### 5.2.1 Open crack

If loading and configuration are in such a way that the crack faces are open then, there is no need to include contact conditions to the internal virtual work of the system. The principle of virtual work is stated as:

\[ IVW = EVW \]  

(5.11)

By using standard finite element procedure, Equation 5.11 can be written as:

\[ KU = F \]  

(5.12)

where \( K, U, \) and \( F \) are the stiffness matrix, displacement vector and external forces for the entire body.

Equation 5.12 together with usual boundary conditions, will give the solution of the system.

### 5.2.2 Closed crack

Most elasticity solutions for cracks are aimed at understanding fractures under generally tensile conditions. Sometimes a specific boundary value problem leads to contradictory results because the crack faces are seen to overlap after the solution is constructed. However, in some applications and perhaps most notably in geophysics, cracks need be considered in an imperatively compressive environment. This means that the cracks are partially, if not fully, closed and the friction between the crack faces is liable to play an important role in the ensuing phenomena. The accurate numerical simulation of the response of two elastic bodies in contact with one another
under external load remains one of the most challenging problems of computational solid mechanics due to several inherent complications. With the application of loads to the bodies in contact, the actual surface on which these bodies meet is generally unknown. In addition, with the friction forces being modelled by nonlinear relationships, mathematical modelling of contact problems involves a system of inequalities or nonlinear equations. Solutions to the contact problem of cracks which have been formulated using the classical theory of elasticity have been rather limited to cases involving simple geometry and loading configurations (Dundurs and Comninou [81]).

In order to overcome these limitations, most compact problems are currently being treated using computational methods, with the finite-element method being the most appealing (Refaat and Meguid [93]). Contact problems range from frictionless contact in small-strain elastic analysis, to contact with friction in general large-strain inelastic analysis. Although conceptually related, these cases differ significantly in the way that they may be formulated and solved. Since we are considering crack problems in the small deformation theory, we do not need to study contact problems in their general form, which is highly time consuming. This chapter is attending to provide a simple, efficient, and iterative finite-element technique to solve frictional contact problems under the small deformation theory of elasticity. An algorithm proposed by Bathe and Chaudhary [85], which is suitable to solve frictional contact problems under large deformation analysis, is modified in such a way that it becomes suitable and efficient to solve contact problems under small deformation theory.

Figure 5.2a shows schematically a generic body which has a crack before deformation. One surface of the crack is arbitrarily denoted as contact surface and the other surface is denoted as target surface. n, is the unit vector tangent to the target
surface and \( n_s \) is the outward unit vector normal to the target surface. It is always possible to mesh the body in such a way that the number of nodal points in the contact segment can be equal to the number of nodal points in the target segment. Moreover, as it can be seen from Figure 5.2a, coordinates of contact and target points can be chosen to match one by one. For example, \( X_{J_c} = X_{J_t} \) in which \( c \) denotes the contact point and \( t \) stands for the target point. Now consider the body after deformation. The crack faces can not interpenetrate during the deformation. This condition is often called Signorini's condition. It is evident that there is no need to consider nodes without penetration because they do not violate the constraint. A node with penetration such as the node at \( X_{J_c} + u(X_{J_c}) \) in Figure 5.2b represents only a potential penetration. It is geometrically clear that node \( J_c \) must be
in contact with the face defined by the nodes at \( X_{Jt} + u(X_{It}) \) and \( X_{Jc} + u(X_{Jc}) \). Since the body experiences small deformations without loosing required accuracy it can be assumed that, to satisfy the no penetration condition, normal components of the displacement vector at point \( Jc \) are equal to the normal components of the displacement vector at point \( Jt \). This condition can be formally stated as:

\[
u(X_{Jc})\cdot n_s = \nu(X_{Jt})\cdot n_s \tag{5.13}\]

If the contact problem is frictionless, which is not true in most of the physical problems, Equation 5.12 and condition 5.13 together with usual boundary conditions are enough to solve this contact problem. The following steps are required to accomplish the solution.

- (1) Solving Equation 5.12 without considering contact conditions.
- (2) For all nodal points on the surfaces of the crack checking whether or not penetration occurred.
- (3) If penetration happened, adding condition 5.13 to the internal virtual work of the system (Equ 5.10) by using Lagrange multipliers as follows:

\[
IVW = IVW_s + IVW_i + [(\delta u(X_{Jc}) - \delta u(X_{Jt}))\cdot n_s]T\eta \tag{5.14}
\]

where \( \eta \) is the vector of Lagrange multipliers.
- (4) Using the principle of virtual work and standard finite element procedure, Equations 5.11, and 5.13 give a system of linear algebraic equations as follows:

\[
\begin{bmatrix}
K & K_c \\
K_c^T & 0
\end{bmatrix}
\begin{bmatrix}
U \\
\eta
\end{bmatrix} =
\begin{bmatrix}
F \\
0
\end{bmatrix} \tag{5.15}
\]
where

\[ K_c = \text{contact stiffness matrix, for the effect of contact conditions} \ (2N \times nc) \]

\[ N = \text{total number of nodal points and} \]

\[ nc = \text{total number of nodal points in contact} \]

- (5) Solving Equation 5.15 together with usual boundary conditions.
- (6) Calculating stresses at the contact nodes. Terminating the program if all normal stresses at contact nodal points are negative, otherwise removing those nodal points which have positive normal stresses from Equation 5.14 and repeating steps 3 to 6.

The above steps can be used to solve the frictionless contact problem of a crack.

5.2.3 The friction law

Sliding of some points on crack faces is possible. Such sliding with the existence of friction is subjected to the chosen friction law. Coulomb’s dry-friction law is used here and can be described as follows. Consider the particles initially in contact. If \( \tau \) represents the developed tangential traction along the contact surfaces, it is assumed that there is no relative motion between two adjacent particles on the contact and the target in contact, as long as \( |\tau| \leq \mu |\sigma_n| \). \( \sigma_n \) is the compressive normal traction. Once the developed tangential traction exceeds frictional capacity, the tangential traction is set equal to \( \mu |\sigma_n| \), that can actually be resisted, and the motion will start and continue as long as the tangential traction does not drop below
the frictional capacity \( \mu | \sigma_n | \). To apply the above constraint, which introduces non-linearity effects even in linear elasticity problems, the author developed an iterative method as follows:

- (1) Solving Equation 5.12 without considering contact conditions.
- (2) For all nodal points on the surfaces of the crack checking whether or not penetration occurred.
- (3) If penetration happened, adding condition 5.13 to the internal virtual work of the system, Equation 5.10, to get Equations 5.14 and 5.15.
- (4) Solving Equation 5.15 which leads to the solution of frictionless contact of the crack problem.
- (5) Calculating stresses at the contact nodes. Going to the next step if all normal stresses at contact nodal points are negative, otherwise removing those nodal points which have positive normal stresses from Equation 5.14 and repeating steps 3 to 5.
- (6) For all nodal points on the contact calculating relative displacement in the tangential direction \( n_t \) and storing it in a vector for example \( \Delta_{ct} \).
- (7) Introducing a vector \( \beta \) which has the dimension equal to that of \( \Delta_{ct} \). At the onset of iteration, setting all elements of \( \beta \) equal to zero.
- (8) Considering the following constraints in the tangential direction \( n_t \) for the generic contact and target points \( J_c \) and \( J_t \).

\[
(u(X_{J_c}) - u(X_{J_t})).n_r = \beta(J)\Delta_{ct}(J)
\] (5.16)
Note that $\beta = 0$ means that no relative movement between the nodes in contact is allowed in the tangential direction.

- (9) Adding virtual work of the frictional forces to the internal virtual work of the system, Equation 5.14, by using Lagrange multipliers as follows:

$$IVW = IVW_s + IVW_i + [(\delta u(X_{Je}) - \delta u(X_{Je}) \cdot n_e)^T \eta$$
$$+ [(\delta u(X_{Je}) - \delta u(X_{Je}) \cdot n_e)^T \zeta$$

(5.17)

where $\zeta$ similar to $\eta$ is the vector of Lagrange multipliers.

- (10) Using the principle of virtual work together with constraints 5.13 and 5.16 a system of linear algebraic equations can be written as:

$$[ Jac ] [ Y ] = [ R ]$$

(5.18)

where

$$[ Jac ] = [ K \quad K_{cd} ]$$
$$[ K_{cd}^T \quad 0 ]$$

$$[ Y ] = [ U \quad \eta \quad \zeta ]$$

$$[ R ] = [ F \quad 0 \quad \beta(J) \Delta_{cd}(J) ]$$

- (11) Solving Equation 5.18 together with usual boundary conditions.
• (12) Calculating tangential and normal stresses $\tau$ and $\sigma$ for all contact nodal points. If $|\tau_{\cdot Jc}| \geq \mu |\sigma_{\cdot Jc}|$ for each contact point $Jc$, adding $d\beta$ to the $\beta(J)$ as follows:

$$\beta(J)^{i+1} = \beta(J)^{i} + d\beta \tag{5.19}$$

Equation 5.19 gives the displacement incremental vector $\beta$ for the next iteration $(i + 1)$.

• (13) Checking for convergence. Convergence is reached whenever

$$\beta^{i+1} - \beta^{i} = 0 \tag{5.20}$$

• (14) Terminating if condition 5.20 is reached, otherwise repeating steps 11 through 14.

Three distinguishing characters of the above iterative method are:

• (1) Jacobean matrix $Jac$ is calculated once and it remains constant during the iterations.

• (2) Jacobean matrix $Jac$ is symmetric and needs only one-half of the storage space which is very crucial in a large scale structural analysis.

• (3) The elements of incremental displacement vector $\beta$ are changing from zero to one $0 \leq \beta(J) \leq 1$. Since elements of $\beta$ are changing monotonically, it is guaranteed that after maximum $ni = 1/d\beta$ iterations convergence will be reached. Where $d\beta$ is the user defined incremental step size.
Such a computational efficiency renders this algorithm highly suitable for analyzing the frictional contact problem of cracks under compressive loads.

5.3 Stress intensity factor

The stress intensity factors (SIFs) are key parameters in linear elastic fracture mechanics. There are many methods which are devoted to evaluation of these parameters. For a comprehensive review of these methods, the reader is referred to a work by Cartwright and Rooke [108]. Among those methods is the crack tip displacement method which correlates the displacements at nodal points of the finite-element mesh with those at the crack tip which are given by:

\[
\begin{align*}
  u &= \frac{K_I}{2E}\sqrt{\frac{\tau}{2\pi}}(1 + \nu)[(2\kappa - 1)\cos \frac{\theta}{2} - \cos \frac{3\theta}{2}] \\
  &\quad + \frac{K_{II}}{2E}\sqrt{\frac{\tau}{2\pi}}(1 + \nu)[(2\kappa + 3)\sin \frac{\theta}{2} + \sin \frac{3\theta}{2}] + O(r)
\end{align*}
\]

\[
\begin{align*}
  v &= \frac{K_I}{2E}\sqrt{\frac{\tau}{2\pi}}(1 + \nu)[(2\kappa + 1)\sin \frac{\theta}{2} - \sin \frac{3\theta}{2}] \\
  &\quad + \frac{K_{II}}{2E}\sqrt{\frac{\tau}{2\pi}}(1 + \nu)[-(2\kappa - 3)\cos \frac{\theta}{2} + \cos \frac{3\theta}{2}] + O(r)
\end{align*}
\]

From the above equations one can find:

\[
\begin{align*}
  |\Delta v| &= \frac{K_I}{G}(1 + \kappa)\sqrt{\frac{\tau}{2\pi}} \\
  |\Delta u| &= \frac{K_{II}}{G}(1 + \kappa)\sqrt{\frac{\tau}{2\pi}}
\end{align*}
\]

(5.21)

where

\[
\kappa = \begin{cases} 
  3 - 4\nu & \text{if plane strain} \\
  \frac{3-\nu}{1+\nu} & \text{if plane stress}
\end{cases}
\]
\(\Delta u\) and \(\Delta v\) are motions of one crack face with respect to another in tangential and normal directions at \(\theta = \pm \pi\) and radial distance \(r\) from the crack tip. From Equation 5.21 it is observed that the magnitude of \(\Delta v\) or \(\Delta u\) is linear in the square root of \(r\), and a plot of \(\Delta v\) or \(\Delta u\) versus \(\sqrt{r}\) will yield a straight line through the origin whose slope is proportional to the stress intensity factor \(K_I\) or \(K_{II}\). Alternatively, a plot of \(|\Delta u|/\sqrt{r}\) versus \(\sqrt{r}\) gives a constant which is also proportional to the stress intensity factor. This furnishes a convenient method for obtaining stress intensity factors from crack flank displacement data. The numerical crack displacement data is used to compute the magnitude of \(\Delta v\) or \(\Delta u\) for a number of radial positions from the crack tip. A plot of \(|\Delta v|/\sqrt{r}\) or \(|\Delta u|/\sqrt{r}\) versus \(\sqrt{r}\) is made and the best straight line, consistent with the above mentioned restrictions, is fit through the data [135]. This procedure has been carried out through this chapter to calculate stress intensity factors for different problems.

### 5.4 Energy release rate

There are several methods available for computing the energy release rate numerically. One of these is the virtual crack extension method, which calculates the total energy release rate by using the potential energy change due to a slight change of the crack length. Another is the crack closure method which makes it possible to calculate separate energy release rate components. A third approach for computing the energy release rate is to use stresses and displacements near the crack tip. This chapter will use a simple method, called the two-term parameter technique, which was introduced by Rhee and Ernst [144]. According to Irwin [4], for an elastic struc-
ture, the energy absorption during a crack extension of size, \( \delta a \), is equal to the work required to close the same amount of the crack. Crack extension occurs when the energy release rate equals the energy required for the crack to grow. This statement can be expressed mathematically as follows:

\[
G_T = \lim_{\delta a \to 0} \frac{1}{2\delta a} \int_0^{\delta a} \sigma_n(r, 0) \Delta u_n(\delta a - r, \pi) \cdot dr
+ \lim_{\delta a \to 0} \frac{1}{2\delta a} \int_0^{\delta a} \tau_t(r, 0) \Delta u_t(\delta a - r, \pi) \cdot dr
\]  
(5.22)

where \( G_T \) is the total energy release rate, and \( \sigma_n \) and \( \tau_t \) are the crack tip peel and shear stresses, respectively. The symbols \( \Delta u_t \) and \( \Delta u_n \) represent the relative sliding and opening displacements between points on the crack faces and \( \delta a \) is the crack extension size. The first term on the right hand side of the above equation represents the mode I energy release rate, \( G_I \), and the last term represents the mode II energy release rate, \( G_{II} \). The crack tip stress and displacement distribution along the crack plane are approximated by the following series (Eftis, et al. [182]):

\[
\sigma_n(r, 0) = \frac{a_0}{\sqrt{r}} + \sum_{n=1}^{\infty} a_n r^{\frac{n+1}{2}}
\]

\[
\tau_t(r, 0) = \frac{b_0}{\sqrt{r}} + \sum_{n=1}^{\infty} b_n r^{\frac{n+1}{2}}
\]

\[
\Delta u_t(r, \pi) = c_0 \sqrt{r} + \sum_{n=1}^{\infty} c_n r^{\frac{n+1}{2}}
\]

\[
\Delta u_n(r, \pi) = d_0 \sqrt{r} + \sum_{n=1}^{\infty} d_n r^{\frac{n+1}{2}}
\]  
(5.23)

Substituting from Equation 5.23 into Equation 5.22 and making the transformation of the variable \( s = r/\delta a \) gives:

\[
G_I = \frac{1}{2} a_0 d_0 \int_0^{1} \left( \frac{1 - s}{s} \right)^{1/2} \cdot ds
\]
\[ + \lim_{\delta a \to 0} \frac{1}{2} d_0 \sum_{n=1}^{\infty} a_n (\delta a)^{n/2} \int_0^1 s^{1/2}(s - 1)^{(n-1)/2} \cdot ds \]
\[ + \lim_{\delta a \to 0} \frac{1}{2} a_0 \sum_{n=1}^{\infty} a'_n (\delta a)^{n/2} \int_0^1 s^{(n+1)/2}(s - 1)^{-1/2} \cdot ds \]
\[ + \lim_{\delta a \to 0} \frac{1}{2} \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} a_m d_n (\delta a)^{(m+n)/2} \int_0^1 s^{(n+1)/2}(s - 1)^{(m-1)/2} \cdot ds \quad (5.24) \]

\[ G_{II} = \frac{1}{2} \hat{b}_0 c_0 \int_0^1 \left( \frac{1 - s}{s} \right)^{1/2} \cdot ds \]
\[ + \lim_{\delta a \to 0} \frac{1}{2} c_0 \sum_{n=1}^{\infty} b_n (\delta a)^{n/2} \int_0^1 s^{1/2}(s - 1)^{(n-1)/2} \cdot ds \]
\[ + \lim_{\delta a \to 0} \frac{1}{2} b_0 \sum_{n=1}^{\infty} c_n (\delta a)^{n/2} \int_0^1 s^{(n+1)/2}(s - 1)^{-1/2} \cdot ds \]
\[ + \lim_{\delta a \to 0} \frac{1}{2} \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} b_m c_n (\delta a)^{(m+n)/2} \int_0^1 s^{(n+1)/2}(s - 1)^{(m-1)/2} \cdot ds \quad (5.25) \]

In the above equations, the integrals under the summation sign are finite values and the terms having \( \delta a \) vanish in the limit as \( \delta a \) goes to zero. Only the integral of the first term does not vanish in the limit and is equal to \( \pi/2 \). In the limit Equations 5.24 and 5.25 becomes:

\[ G_{I} = \frac{\pi}{4} a_0 d_0 \]
\[ G_{II} = \frac{\pi}{4} \hat{b}_0 c_0 \quad (5.26) \]

Therefore, in order to evaluate \( G_{I} \) and \( G_{II} \), the appropriate determination of \( a_0 \), \( b_0 \), \( c_0 \), and \( d_0 \) is required. These constants can be obtained accurately by using the two-term parameter technique. For instance, to obtain the value of \( b_0 \), only the first two terms in the equation related to \( \tau_I \) are taken into account

\[ \tau_I(r, 0) = \frac{b_0}{\sqrt{r}} + b_1 \quad (5.27) \]
Thus a plot of \( \tau_\ell(r,0)\sqrt{r} \) versus \( \sqrt{r} \) yields a straight line whose intercept on the ordinate is \( b_0 \). Similarly, we can calculate \( a_0, c_0, \) and \( d_0 \). Since finding constants \( a_0, b_0, c_0, \) and \( d_0 \) graphically is not an accurate procedure, the least square method which makes use of minimization techniques and gives required constants as a solution of a linear algebraic equation system is used.

5.5 Numerical examples and results

In this section, the feasibility and accuracy of the proposed algorithm is examined by solving different examples and compare these results to existing known solutions. In all the following examples, isoparametric 6-node triangular elements are used and at the crack tips, singularity is modelled by moving middle points to quarter distance from the crack tips.

5.5.1 Example 5.1

As a first example a rectangular plate with a length of \( 2b = 8(\text{in}) \), a width of \( 2w = 5(\text{in}) \) and a thickness of \( t = 1(\text{in}) \) as shown in the Figure 5.3 is considered. The crack length, which is laying at the centre of the plate, is \( 2a = 1(\text{in}) \). Material properties are: \( E = 30 \times 10^6(\text{psi}) \) (Young’s modulus), \( \nu = 0.3 \) (Poisson’s ratio), and \( \mu = 0.3 \) (coefficient of the friction). Applied load \( F_y = -10000(\text{Ib}) \) is a concentrated load acting in the middle of the plate on the upper boundary of the body \( (x = 0, y = w) \) and boundary conditions are assumed to be clamped at \( x = \pm b \) and free
at $y = \pm w$. With the above loading and boundary conditions, crack surfaces will experience compressive stresses. Therefore the crack will be fully closed. To compare the results of the present algorithm (P.A.), the same problem is solved by using a commercial finite element package, ANSYS5.2, which applies a penalty method with implicit contact constraint iterations. This is achieved using the contact element CONTACT48 with $K_N = 100E$ and $K_T = E$ which is suggested by the program's manual for good accuracy. Figure 5.4 shows the finite-element mesh of the first example. Convergence study is done by using different mesh size and the final results are shown in Figures 5.5 to 5.8.

Figure 5.5 shows variation of the normal displacement for contact and target points along the crack surfaces. As it can be seen from this figure, the normal displacements of contact and target points which are calculated by the present algo-
Figure 5.4: Finite element mesh for the example 5.1.

...
Figure 5.5: Normal displacement for contact and target points along crack surfaces.

Figure 5.6: Tangential displacement for contact nodal points along crack surfaces.
Figure 5.7: Tangential displacement for contact and target points along crack surfaces (crack tips are not included in this figure).

Figure 5.8: Absolute values of tangential Lagrange multiplier $\zeta$ and $\mu$ times normal Lagrange multiplier $\mu\eta$. 
the $\mu$ the smaller the sliding portion.

Finally, the variation of average absolute values of the tangential Lagrange multipliers $\zeta$ and $\mu$ times normal Lagrange multiplier $\eta$ are shown in figure 5.8. It can be seen that the average of $|\zeta| = \mu |\eta|$ for those points along the crack faces which are in the slipping area and the average of $|\zeta| < \mu |\eta|$ for those points which are in the sticking area.

5.5.2 Example 5.2

As a second example, a rectangular cracked plate subjected to a pair of moments is considered as shown in Figure 5.9. The sheet with a length of $2b$ and a width of $2w$ contains a crack with a length of $2a$ located centrally. In this work we consider $a/b = 0.1$ and $w/b = 3$. This problem was solved by Montenegro et al. [94]. Their approach was based on geometrical considerations and used the weight function method to obtain an effective crack length and mode I crack tip stress intensity factor $K_I$. Due to pure bending load conditions the crack will be partially closed. The open part of the crack is called the effective crack length $2a_1$ and the mode I stress intensity factor for the open part of the crack is $K_{N1}^{eff}$ which is normalized with respect to $\sigma \sqrt{\pi \times 1}$. In this example $a = 1$.

<table>
<thead>
<tr>
<th></th>
<th>This work</th>
<th>Montenegro ... [94]</th>
<th>Diff. (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>0.6608</td>
<td>0.6667</td>
<td>0.88</td>
</tr>
<tr>
<td>$K_{N1}^{eff}$</td>
<td>0.5298</td>
<td>0.5443</td>
<td>2.6</td>
</tr>
</tbody>
</table>

Table 5.1 shows a comparison of the normalized effective crack tip stress intensity
factor $K_N^{eff}$ and effective half crack length $a_1$ from the present algorithm with those obtained by Montenegro et al. [94]. As it can be seen from Table 5.1 there is good agreement between the results of these two works and the differences are in the acceptable range for engineering usage.

5.5.3 Example 5.3

This example is related to a functionally gradient material. Again consider Figure 5.3 and note that the material which occupies the space between $-\frac{b}{2} \leq y \leq \frac{b}{2}$ has properties described by equation 5.1 and surrounding materials are steel with Young's module $E_2 = 30e+6$ psi and Poisson's ratio $\nu_2 = 0.3$. Figure 5.10 shows the variation of normalized material properties, $E/\max(E)$ and $\nu/\max(\nu)$ for interfacial layer. As it can be seen from this figure material properties $E$ and $\nu$ have their highest value
at the boundaries of interfacial layer $h = \pm 0.5$ and they posses their minimum at the crack tip.

Figure 5.10: Variation of normalized material properties of interfacial layer for example 5.3

Tables 5.2 - 5.4 show the effect of material properties, coefficient of friction, and the thickness of the interfacial layer on the stress intensity factor $K_{II}(\text{psi}\sqrt{\text{in}})$ and energy release rate $G_{II}(\text{Ibf-in/in})$. Table 5.2 shows the effect of $E_0/E_2$ on the stress intensity factor $K_{II}$ and energy release rate $G_{II}$ for the case where $h = 1(\text{in})$, $\mu = 0.15$, and $\nu_0 = 0.25$. With the increase of Young's modulus $E_0$ at the crack tip, stress intensity factor and energy release rate will increase. This variation is not linear.

Table 5.3 shows the change in the stress intensity factor and energy release rate with respect to the coefficient of friction while the ratio of $E_0/E_2 = 0.1$ and the
thickness $h = 1\text{ (in)}$ are kept constant. With the increase of the coefficient of the friction the stress intensity factor will decrease until $\mu = 0.3$ which gives $K_{II} = 0$. Finally, the change of the stress intensity factor and energy release rate with respect to the interfacial layer's thickness for the case where $E_0/E_2 = 0.1$ and $\mu = 0.15$ are shown in Table 5.4. As is expected with the increase of the thickness of the interfacial layer, stress intensity factor and energy release rate decrease.

Table 5.3: The variation of $K_{II}$ and $G_{II}$ versus $\mu$.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>0</th>
<th>0.15</th>
<th>0.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_{II}$</td>
<td>171.27</td>
<td>46.69</td>
<td>0.00</td>
</tr>
<tr>
<td>$G_{II}$</td>
<td>0.0103</td>
<td>6.7413e-4</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Table 5.4: The variation of $K_{II}$ and $G_{II}$ versus $h$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>0.3</th>
<th>1</th>
<th>1.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_{II}$</td>
<td>78.95</td>
<td>46.69</td>
<td>31.76</td>
</tr>
<tr>
<td>$G_{II}$</td>
<td>0.0022</td>
<td>6.7413e-4</td>
<td>3.2479e-4</td>
</tr>
</tbody>
</table>

5.5.4 Example 5.4

Consider a double edge crack in a homogeneous plate as shown in Figure 5.11. The crack length is $a$ and the half width of the plate is $w$. It is assumed that the plate is
subject to uniform tension \( \sigma \) on the boundaries \( y = \pm w \). The analytical expression for the mode I stress intensity factor in this case is expressed as:

\[
K_I = \sigma \sqrt{\pi a f(a/w)}
\]  

(5.29)

where the factor \( f(a/w) \) is a function of the crack length \( a \) and the plate width \( w \).

The theoretical factor \( f \) is found in refs [183] and [32] as follows:

\[
f(a/w) = \frac{1.12 - 0.61(a/w) + 0.13(a/w)^3}{\sqrt{1 - a/w}}
\]  

(5.30)

For a strip with a centre crack and subject to uniform uniaxial tension, Figure 5.12, the stress intensity factor is also expressed similarly in Equation 5.29. The \( f(a/w) \) chosen in ref. [32] is

\[
f(a/w) = \frac{1.0 - 0.5(a/w) + 0.326(a/w)^2}{\sqrt{1 - a/w}}
\]  

(5.31)

It should be noted that the above expressions are true only for homogeneous materials and there are no similar analytical relations for functionally gradient materials.
Table 5.5 shows the normalized stress intensity factor $f(a/w) = K_I/\sigma \sqrt{\pi a}$ for both double edge crack plate and centre crack plate with $a/w = 0.5$. As it can be seen from this table, excellent agreement exists between theoretical results and present algorithm.

Table 5.5: Normalized stress intensity factor (homogeneous material distribution)

<table>
<thead>
<tr>
<th></th>
<th>$f(a/w) = K_I/\sigma \sqrt{\pi a}$</th>
<th>Present</th>
<th>Theoretical</th>
<th>Error(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Double edge</td>
<td>0.5</td>
<td>1.1873</td>
<td>1.1755</td>
<td>1.00</td>
</tr>
<tr>
<td>Centre crack</td>
<td>0.5</td>
<td>1.1738</td>
<td>1.1759</td>
<td>0.18</td>
</tr>
</tbody>
</table>

Now consider a functionally gradient material with a thickness of $h$ sandwiched between two elastic planes with elastic properties $E_1 = 2.1 \times 10^{10}$ Pa, $\nu_1 = 0.25$, $E_2 = 2.1 \times 10^{11}$ Pa, and $\nu_2 = 0.3$, respectively, as shown in Figure 5.13. The layer of height $h$ contains a crack with a length of $2a = 0.05$ m that is situated in the mid-plane and is parallel to the interface. The loading and boundary conditions are assumed to be:

\[ v = 0 \quad \text{at} \quad y = -w = -0.05 \quad \text{m} \]
\[ \sigma = \sigma_0 \quad \text{at} \quad y = +w = +0.05 \quad \text{m} \]

The normalized stress intensity factor $K = K_I/\sigma_0 \sqrt{\pi a}$ is calculated at the crack tip for different distributions of material properties of the interfacial layer. These distributions are summarized as follows:
Figure 5.13: Geometry and boundary condition for example 5.3

- **Linear**
  
  \[ E = E_0 + \bar{E}(2y) \]

  where
  
  \[ E_0 = \frac{E_1 + E_2}{2} \]
  \[ \bar{E} = E_2 - E_0 \]

- **Quadratic**
  
  \[ E = E_0 + \bar{E}(2y) + \bar{E}(2y)^2 \]

  where
  
  \[ E_0 = \frac{3E_2 + E_1}{4} \]
  \[ \bar{E} = \frac{E_2 - E_1}{2} \]
  \[ \bar{E} = \frac{E_1 - E_2}{4} \]
- **Cubic**

  
  \[ E = E_0 + \bar{E} \left( \frac{2\mu}{h} \right) + \bar{E} \left( \frac{2\mu}{h} \right)^3 \]

  where

  \[ E_0 = \frac{E_1 + E_2}{2} \]

  \[ \bar{E} = \frac{3}{4}(E_2 - E_1) \]

  \[ \bar{E} = \frac{E_1 - E_2}{4} \]

- **Exponential**

  
  \[ E = E_0 \exp(\alpha \frac{2\mu}{h}) \]

  where

  \[ \alpha = \frac{1}{2} \ln \left( \frac{E_2}{E_1} \right) \]

  \[ E_0 = E_2 \exp(-\alpha) \]

Normalized stress intensity factor \( K = K_I/\sigma_0 \sqrt{\pi a} \) is shown in table 5.6. In all four cases the normalized stress intensity factor will decrease with an increase in the thickness of the interfacial layer \( h \). It means that the smoother the transition between layers 1 and 2 the smaller stress intensity factor. From table 5.6 it can be seen that the exponential function for material distributions is better than linear, quadratic, or cubic distribution of material properties of interfacial layer. The above conclusion by no means is complete, and for each problem extensive examination is needed. However, this example suggests that it is possible to reach a better design of interface layers by changing material properties of the interfacial layer properly.
Table 5.6: Normalized stress intensity factor (functionally gradient materials)

<table>
<thead>
<tr>
<th></th>
<th>$K = K_I/\sigma_0\sqrt{\pi a}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$h=0.025$</td>
</tr>
<tr>
<td>Linear</td>
<td>1.35</td>
</tr>
<tr>
<td>Quadratic</td>
<td>1.50</td>
</tr>
<tr>
<td>Cubic</td>
<td>1.38</td>
</tr>
<tr>
<td>Exponential</td>
<td>1.16</td>
</tr>
</tbody>
</table>

5.6 Conclusion

A numerical treatment of a finite crack in functionally gradient materials under mixed mode loading conditions including the effect of the frictional contact between crack faces is presented. The method is based on finite-element analysis. Coulomb friction between contacting crack surfaces is taken into account. A simple and efficient, iterative finite-element technique for solving frictional contact problems under small deformations is described. Using numerical crack flank displacement and the two-term parameter technique, the stress intensity factor and the energy release rate were calculated. Finding the linear part of the displacement, or stress curve versus the square root of the distance of nodal points from the crack tip is the biggest disadvantage of the correlation method, and a small error in the correlation of data will produce a large error in calculating the stress intensity factor and the energy release rate. A number of case studies are provided to validate the feasibility and the accuracy of the developed algorithm.
Chapter 6

Frictional contact problems of cracks in FGMs
under thermally induced stresses

Now is no time to think of what you do not have. Think of what you can do with what there is.           -Ernest Hemingway

6.1 Introduction

In the last two chapters, response of an interface crack in FGMs under dynamic and static mechanical loading was studied. Dynamic stress intensity factor (in Chapter 4), static stress intensity factors, and energy release rates (in Chapter 5) were calculated. It was shown, through different examples, that FGMs, as an interface material was better than composite materials with homogeneous layers. However, functionally graded materials to be used as, *inter alia*, superheat-resistant materials have some attractive applications in furnace liners, space structures, fusion reactors, and electronic component packaging. As was mentioned in previous chapters, FGMs consist of two distinct material phases, such as ceramic and metal alloys, and are a mixture of them in that the composition of each changes continuously along one direction. The change in microstructure induces chemical, material, and microstructural gradients, which makes functionally graded materials different in behavior from homogeneous materials and traditional composite materials [184].

Stress intensity factors are affected by gradients of the material properties. More-
over, the fracture modes of the cracks in FGMs are inherently mixed, i.e. there are typically both normal and shear traction ahead of the crack tips due to nonsymmetry in the material properties. Most previous works on FGM crack configurations have concentrated on the exponential variation of material properties of the interface layer (i.e. $E = E_0 \exp(\beta y)$ and $\alpha = \alpha_0 \exp(\gamma y)$, $k = k_0 \exp(\delta y)$ where $E_0$, $\beta$, $\alpha_0$, $\gamma$, $k_0$, and $\delta$ are material constant which are calculated from continuity conditions) [67, 72, 68]. The reason for choosing an exponential function is obvious as it makes an analytical solution of plane elasticity for some simple problems possible. Since finite element analysis is used here, different distributions for material properties of the interfacial layer can be considered.

Generally, this chapter provides a numerical treatment of a finite crack in an interfacial layer with spatially varying elastic and thermal properties. Unlike earlier studies which considered the cracks encountered as open, the current investigation studies cracks in an essentially compressive environment in which the crack faces are in contact. A finite element method which was described in the previous chapter is employed to solve frictional contact problems under small deformations. Both steady-state and transient thermal stresses are considered and numerical examples are provided. Stress intensity factors (SIFs) are calculated and the variation in SIFs due to change in material properties of the interfacial layer is studied [185] and [186].

6.2 Governing equations

In the classical theory of thermal stresses the solutions are obtained in two steps. First, the temperature $T(x, y, t)$ is established; it is assumed, therefore, that the
temperature is independent of deformations. Second, the temperature $T(x, y, t)$ is used in finding displacements, strains, and stresses in the body. This chapter is restricted to the theory of elasticity for small deformations with the inclusion of thermal effects.

6.2.1 Heat conduction equations

The heat conduction process is assumed to obey Fourier's law which is:

$$ q_i = -k T_{,i}, \quad i = 1, 2 $$

(6.1)

where $k$ is the thermal conductivity of the material, $q_i$ is the vector of the heat flux, and $T_{,i}$ is the temperature gradient. Assuming the existence of heat sources in the body which generate heat at the rate of $Q$ per unit of time and unit of volume, and knowing that the total rate of change of internal energy is $\rho c \partial T / \partial t$ where $\rho$ is density and $c$ is the specific heat, the balance of energy is:

$$ (k T_{,i})_i + Q = \rho c \partial T / \partial t $$

(6.2)

The above equation is true for both homogeneous layers with constant thermal conductivity and nonhomogeneous layers with variable thermal conductivity. The heat conduction in Equation 6.2 is solved with appropriate initial and boundary conditions. The initial conditions specify the field of temperature at a prescribed moment of time. The boundary conditions usually belong to the following five types:

- a) Given surface temperature of the body:

$$ T(p, t) = f(p, t) $$

(6.3)
where \( p \) is a point on the surface, \( t \) is the time, and \( f(p, t) \) is a prescribed function.

\( \bullet \) b) Given heat flux:

\[
q_n(p, t) = -kT_n = g(p, t)
\]  \hspace{1cm} (6.4)

where \( n \) indicates a normal to the surface, and \( g(p, t) \) is a prescribed function. If the body is exposed to the radiation of an external heat source of temperature \( T_1 \) then this condition, following the Stefan-Boltzmann law, becomes:

\[
kT_n = C(T_1^4 - T^4)
\]  \hspace{1cm} (6.5)

where \( C \) is a constant coefficient.

\( \bullet \) c) Insulated surface is a special case of type (b):

\[
T_n = 0
\]  \hspace{1cm} (6.6)

\( \bullet \) d) Convection boundary condition:

\[
kT_n = h_0(T_1 - T(p, t))
\]  \hspace{1cm} (6.7)

where \( h_0 \) and \( T_1 \) denote the boundary conductance and the temperature of the surrounding medium, respectively.

\( \bullet \) e) The contact of two bodies:

\[
T_1(p, t) = T_2(p, t)
\]  \hspace{1cm} (6.8)

\[
k_1T_{1,n} = k_2T_{2,n} \quad \text{(at } p \text{ and } t) \]  \hspace{1cm} (6.9)
where \( p \) is a point of the surface of contact, \( n \) is a common normal to the surface of contact and subscripts 1 and 2 refer to the first and the second body, respectively.

### 6.2.2 Basic equations of the thermal stresses

The basic equations which govern the quasi-static thermoelastic behavior of homogeneous materials in a two-dimensional space can be expressed in a Cartesian coordinate system as:

\[
\sigma_{ki,k} + X_i = 0 \tag{6.10}
\]

where \( X_i \) is body forces per unit volume in \( x \) and \( y \) directions, respectively and "\( \cdot \)" denotes the partial differential operator.

\[
\sigma_{ki} = 2G_l\varepsilon_{ki} + \lambda_l\varepsilon_{kj}\delta_{ki} - n_lT\delta_{ki} \tag{6.11}
\]

where \( l \) is the number of layers with homogeneous material properties and \( \delta_{ki} \) is the Kronecker delta. \( \lambda_l \) and \( G_l \) are Lame's constants for layers \( l \).

\[
\lambda_l = \frac{E_l\nu_l}{(1-2\nu_l)(1+\nu_l)}
\]

\[
G_l = \frac{E_l}{2(1+\nu_l)}
\]

\[
n_l = (3\lambda_l + 2G_l)\alpha_l \tag{6.12}
\]

and \( \alpha_l \) being the coefficient of linear thermal expansion. Equations 6.11 represents the Duhamel-Neumann equations. It can be seen that they are constitutive equations of the isothermal theory of elasticity (Hooke's law) augmented by temperature
terms \(-n_i T \delta_{ki}\) [148, 150]. By expressing stresses in terms of displacements through the constitutive and strain-displacement relations for isotropic Hookean materials, Equation 6.10 reduces to the following elliptic partial differential equations:

\[(\lambda_i + G_i)e_{ij} + Gu_{kk,ij} + X_i - n_i T_i = 0, \quad i = 1, 2\]  

(6.13)

where \(e_i = u_{kk,ij}\) and "\(kk\)" is the two-dimensional Laplacian operator. \(i\) represents coordinates \(x\) and \(y\). Moreover, the internal and external virtual work of the body can be expressed as follows:

\[IVW_i = t_0 \int_{A_i} \sigma_{ij} \delta \epsilon_{ij} \cdot dA\]

\[EVW_i = t_0 \int_{A_i} X_i \delta u_i \cdot dA + t_0 \int_{L_i} q_i \delta u_i \cdot dL\]  

(6.14)

where \(t_0\) is the thickness of the plane which is constant through the body. \(q_i\) are components of the external forces acting on the body. \(A_i\) and \(L_i\) are the area and the boundary of the homogeneous materials.

For functionally gradient materials, equilibrium equations in terms of stress, strain-displacement relations, and constitutive equations remain unchanged. The only equations which are changed are equilibrium equations in terms of displacements (i.e. Eq 6.13) which can be written as:

\[(\lambda + 2G)u_{1i,11} + (\lambda + G)u_{2i,12} + G_2(u_{1i,2} + u_{2i,1}) + Gu_{1i,22}\]

\[-n T_{1i} + X_i = 0\]  

(6.15)
where 1, 2, and i stand for x, y coordinates and the nonhomogeneous layer, respectively.

\[
\lambda = \frac{E\nu}{(1 + \nu)(1 - 2\nu)}
\]
\[
G = \frac{E}{2(1 + \nu)}
\]
\[
n = (3\lambda + 2G)\alpha
\]

It should be noted that \(E, \nu, \) and \(\alpha\) are no longer constants and they are functions of coordinate \(y\). The internal and external virtual work of the interfacial layer can be written as:

\[
IVW_i = t_0 \int_{A_i} \sigma_{ij} \delta e_{ij} \cdot dA
\]
\[
EVW_i = t_0 \int_{A_i} X_i \delta u_i \cdot dA + t_0 \int_{L_i} q_i \delta u_i \cdot dL
\]

where \(A_i\) and \(L_i\) are area and the boundary of the nonhomogeneous layer. Adding Equations 6.14 and 6.18, the internal and external virtual work of the entire body can be written as:

\[
IVW = IVW_i + IVW_i
\]
\[
EVW = EVW_i + EVW_i
\]
Note that in Equations 6.18 and 6.19 $E$, $\nu$, and $\alpha$ are not constant and they are functions of $y$.

If temperature distribution is uniform throughout the body (see section 6.4), the solution procedure is exactly the same as the procedure described in sections 5.2.1 and 5.2.2. On the other hand, if temperature distribution is not uniform, the solution procedure outlined in the sections 5.2.1 and 5.2.2 should be slightly modified as follows:

- (1) Solving Equation 6.2 together with usual boundary and initial conditions without considering contact conditions.

- (2) Following the procedure outlined in sections 5.2.1 and 5.2.2 and finding which pair of nodal points in the surfaces of the crack are in contact.

- (3) Adding contact boundary conditions to the usual boundary conditions and again solving Equation 6.2 to get temperature distribution in the body.

- (4) Repeating steps 2 and 3 until the nodal points in contact become the same after two successive iterations.

- (5) To consider the effect of friction, follow the procedure outlined in sections 5.2.3 till convergence is reached.

It should be noted that Coulomb’s dry-friction law, which was discussed in detail in section 5.2.3, is utilized throughout the chapter.
6.3 Thermal stress intensity factors

It can be shown that the displacement and stress distributions in the vicinity of a crack tip in FGMs are the same as those in homogeneous media (see 3.9.1). Generally, the displacement distributions in the vicinity of the crack tip subjected to mode I or mode II loading are given by:

\[
\begin{align*}
    u &= \frac{K_I}{2G} \sqrt{\left(\frac{r}{2\pi}\right)} \cos\left(\frac{\theta}{2}\right)(\chi - 1 + 2\sin^2\left(\frac{\theta}{2}\right)) \\
    v &= \frac{K_I}{2G} \sqrt{\left(\frac{r}{2\pi}\right)} \sin\left(\frac{\theta}{2}\right)(\chi + 1 - 2\cos^2\left(\frac{\theta}{2}\right)) \\
\end{align*}
\] (6.20)

\[
\begin{align*}
    u &= \frac{K_{II}}{2G} \sqrt{\left(\frac{r}{2\pi}\right)} \sin\left(\frac{\theta}{2}\right)(\chi + 1 + 2\cos^2\left(\frac{\theta}{2}\right)) \\
    v &= \frac{K_{II}}{2G} \sqrt{\left(\frac{r}{2\pi}\right)} \cos\left(\frac{\theta}{2}\right)(\chi - 1 - 2\sin^2\left(\frac{\theta}{2}\right)) \\
\end{align*}
\] (6.21)

where \(u\) and \(v\) are the displacements of a point with a radial and tangential distance \(r\) and \(\theta\), respectively. \(r\) and \(\theta\) are polar coordinates attached to the tip of the crack [145]. \(K_I\) and \(K_{II}\) are the thermal stress intensity factors for mode I and mode II, respectively. The parameters in these equations are: \(G\) = shear modulus; \(\nu\) = Poisson's ratio; \(\chi = (3 - 4\nu)\) plane strain; and \(\chi = (3 - \nu)/(1 + \nu)\) for plane stress.

There are many methods which are devoted to evaluation of stress intensity factors. For a comprehensive review of these methods, the reader is referred to a work by Cartwright and Rooke [108]. Among these methods is the crack tip displacement method which correlates the displacements at nodal points of the finite element mesh with those at the crack tip which are given by Equations 6.20 and 6.21. This method was clearly explained by Smelser [135] and is used here.
6.4 Statement of problem I

A typical electronic package consists of a semiconductor (chip) attached to a substrate material by an adhesive as depicted in Figure 6.1. Kwon et al [153] modelled an electronic package by a trimaterial configuration. They considered that the material properties of the three layers were different from each other but constant through each layer. In that work they did not consider the existence of the crack in the interface layer, but they concluded that the normal and shear stresses had local properties and the normal stresses at the interfaces could be compressive.

![Diagram of a typical electronic package](image)

Figure 6.1: A typical electronic package

The geometrical configuration and the coordinate system for the problem are shown in Figure 6.2. Material properties of layers one and three are assumed to be constant through those layers and material properties of the interfacial layer, layer two, are assumed to be functionally graded. Two cases are studied here. In Case I, variation of Young's modulus $E$, Poisson's ratio $\nu$, and the thermal coefficient of expansion $\alpha$ in the interfacial layer are described by:

$$E = E_0 \exp(\beta_0 y)$$
Figure 6.2: Geometry of the problem

\[ \nu = \nu_0 \exp(\gamma_0 y) \]  
\[ \alpha = \alpha_0 \exp(\delta_0 y) \]

where \( E_0, \nu_0, \alpha_0, \beta_0, \gamma_0, \) and \( \delta_0 \) are material constants which are calculated from continuity conditions at \( y = h_1 \) and \( y = -h_2 \). More realistic expressions for FGMs are considered in Case II. Following Kingery [187] and Obata and Noda [161], the material properties of the FGM as a mixture of ceramic and metal may be expressed as:

\[ E = \frac{E_0(1 - p)}{1 + \frac{2(5 + 8w)(37 - 8\nu_0)}{6(1 + \nu_0)(23 + 8\nu_0)}} \]
\[ \nu = \nu_0 \]
\[ \alpha = \alpha_0 \]

where \( p \) is the porosity and:

\[ E_0 = E_c \frac{E_c + (E_m - E_c)V_m^2}{E_c + (E_m - E_c)(V_m^2 - V_m)} \]
\[ \nu_0 = \nu_m V_m + \nu_c V_c \]
\[
\alpha_0 = \frac{\alpha_m K_m V_m + \alpha_c K_c V_c}{K_m V_m + K_c V_c}
\]

(6.24)

\[
K_m = \frac{E_m}{2(1 - \nu_m)} \quad K_c = \frac{E_c}{2(1 - \nu_c)}
\]

\[
V_c = 1 - V_m
\]

where \( V_m \) denotes the volumetric ratio of metal and subscripts \( c \) and \( m \) show the properties of ceramic and metal, respectively. The volumetric ratio of metal \( V_m \) is expressed as follows:

\[
V_m = \begin{cases} 
(1 - \frac{y-h_3}{h_2})^d & \text{if layer 3 is metal} \\
(1 + \frac{y-h_2-h_3}{h_2})^d & \text{if layer 3 is ceramic} 
\end{cases}
\]

in this chapter the porosity \( p \) is equal to zero and parameter \( d \) is equal to one.

6.4.1 Numerical results for problem I

In this section, the feasibility of the proposed algorithm is examined. The non-homogeneity of the mechanical and thermal properties through the layers which are experiencing uniform temperature causes thermal stresses on the trimaterial configuration of Figure 6.2. The effect of these properties on system behavior is studied. The isoparametric six-node triangular elements are used in the following examples and singularity is modelled at the crack by moving middle points to a quarter distance from the crack tips [42]. Figure 6.3 shows the finite element mesh of an electronic package. Due to symmetry, half of the model is meshed and a simply support boundary condition is assumed at \( y = 0 \). At \( x = 0 \) the boundary condition is coming from symmetry.
Figure 6.3: Boundary conditions and finite element mesh of the model.

Material properties and geometry of layers one and three are shown in table 6.1. The crack length which lies between layers one and two is $2a = 1$ (mm). The geometry of layer two, i.e. interfacial layer, is equal to that of layer one and layer three but material properties of this layer are not constant and vary throughout the layer.

Table 6.1: Material properties and the geometry of layers one and three.

<table>
<thead>
<tr>
<th></th>
<th>E(GPa)</th>
<th>$\alpha$(1/°C)</th>
<th>$\nu$</th>
<th>n(MPa/°C)</th>
<th>Thickness (mm)</th>
<th>Length(mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Layer 1</td>
<td>110</td>
<td>10e-6</td>
<td>0.3</td>
<td>2.750</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>Layer 3</td>
<td>15.1</td>
<td>300e-6</td>
<td>0.28</td>
<td>10.295</td>
<td>1</td>
<td>10</td>
</tr>
</tbody>
</table>

Figures 6.4 and 6.5 show the normalized distribution of material properties of the interfacial layer for Cases I and II (see section 6.4), respectively. Although Young's modulus $E$ in Case II is varying linearly, the coefficient of thermal expansion $\alpha$ and parameter $n$ are not given by linear relations.
Figure 6.4: Normalized distribution of material properties of layer two, Case I.

To appreciate the concept of functionally graded materials, we will also consider Case III in which material properties of the interfacial layer are constant and their values are between those of layers one and three; they are given in Table 6.2.

**Table 6.2: Material properties and the geometry of layer two in Case III.**

<table>
<thead>
<tr>
<th>E(GPa)</th>
<th>$\alpha(1/°C)$</th>
<th>$\nu$</th>
<th>$n(MPa/°C)$</th>
<th>Thickness (mm)</th>
<th>Length (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Layer 2</td>
<td>80</td>
<td>200e-6</td>
<td>0.29</td>
<td>38.095</td>
<td>1</td>
</tr>
</tbody>
</table>

Thermal stresses induced in the system are due to a uniform change in temperature ($\Delta T$) which is equal to 100°C. In compressive loading conditions the crack faces are closed and therefore the stress intensity factor $K_I$ is zero. If two faces of the crack are sliding against each other, the second mode stress intensity factor $K_{II}$ becomes
important and the magnitude of $K_{II}$ is proportional to this relative displacement.

Figure 6.6 shows the distribution of the relative displacements in the tangential direction along the crack faces for different distributions of material properties in the interfacial layer. This figure very clearly shows the advantage of using functionally graded materials for the interfacial layer. Relative displacements produced in FGMs are much smaller than those produced in the interfacial layer with constant material properties. Therefore, the stress intensity factor $K_{II}$ is reduced by using FGMs as an interfacial layer. If connecting two dissimilar layers, it is better to use FGMs as an interfacial layer.

Figure 6.7 and 6.8 show the shear and normal stress distributions along the surfaces of the crack for Case I and Case II, respectively. As is expected normal
stresses along the crack faces are negative and equal for both contact and target surfaces. The equality of the normal stresses along the crack length for the contact and target faces shows the stability of the numerical procedure. Using different distributions for material properties of the interfacial layer, normal stresses at the crack faces (interface between the interfacial layer and layer one) are calculated and the results are shown in Figure 6.9. The same pattern can be seen from the other stress components.

Since, in the compressive environment, the crack faces are in contact over the whole length of the crack or a part of it, the frictional effects play an important role. In this work the Coulomb law of dry friction is used to model the phenomenon of slip and stick. Table 6.3 shows the effect of the coefficient of friction $\mu$ on normalized
Figure 6.7: Normal and shear stress distributions along the surfaces of the crack, Case I.

Figure 6.8: Normal and shear stress distributions along the surfaces of the crack, Case II.
stress intensity factor $K_{II}^* = \frac{K_{II}}{B_1 a_1 \Delta T \sqrt{a}}$ for Case I and Case II. In both cases the stress intensity factor decreases with the increase of the coefficient of the friction. However, the behavior of the two cases changes qualitatively with the increase of the coefficient of the friction. In Case I the normalized stress intensity factor has a higher value for a small value of $\mu$. With the increase of $\mu$ this situation becomes reversed and the stress intensity factor has a higher value in Case II.

Table 6.3: The variation of $K_{II}^*$ versus $\mu$ for Case I and Case II.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case I $K_{II}^*$</td>
<td>0.3455</td>
<td>0.3332</td>
<td>0.1472</td>
<td>0.0608</td>
<td>0.0263</td>
</tr>
<tr>
<td>Case II $K_{II}^*$</td>
<td>0.3077</td>
<td>0.2586</td>
<td>0.1872</td>
<td>0.1483</td>
<td>0.0867</td>
</tr>
</tbody>
</table>

If we exchange material properties of layer three with those of layer one or in
Table 6.4: Normalized stress intensity factors $K_f^*$ and $K_{II}^*$

<table>
<thead>
<tr>
<th></th>
<th>Case I</th>
<th>Case II</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_f^*$</td>
<td>0.2635</td>
<td>0.2552</td>
</tr>
<tr>
<td>$K_{II}^*$</td>
<td>0.0616</td>
<td>0.0567</td>
</tr>
</tbody>
</table>

other words, if we exchange layer one with layer three and layer three with layer one, the normal stresses at the interfaces change from negative to positive. It means that the surfaces of the crack are no longer in contact. Table 6.4 shows the normalized stress intensity factor for both cases when the crack surfaces are open. Similar to the second mode, the normalized stress intensity factor for first mode can be defined as $K_f^* = \frac{K_f}{E \sigma_0 \Delta T^{1/2}}$. As it can be seen from this table the normalized stress intensity factor has a higher value in Case I for both mode I and II.

6.5 Statement of problem II

Pure mechanical loading was used in Chapter 5 to solve different problems of linear fracture mechanics for simple and functionally gradient materials. In section 6.4 an electronic package was studied under pure thermal loading. Since the temperature rise was assumed to be uniform (independent of position) the thermal problem was independent of the mechanical problem. On the other hand, if there is a nonuniform temperature distribution, the possibility will exist of heat flow across the crack in the contact region and the resulting two-way coupling between the thermal and mechanical fields will cause the problem to be a great deal more difficult. Moreover, if the temperature distribution is time dependent, the problem will become of more interest and of course very difficult as well. In this case, the procedure outlined at the
end of section 6.2.2 will be repeated for each time step. In the following, an interfacial crack under combined mechanical and thermal loads is considered. Mechanical load and boundary conditions are independent of time, whereas temperature distribution in the body is time dependent.

![Diagram of the geometry and mechanical loading and boundary conditions of problem II.]

Figure 6.10: Geometry and mechanical loading and boundary conditions of problem II.

The geometrical configuration, mechanical loading, boundary conditions, and the coordinate system of problem II are shown in Figure 6.10. It is assumed that material properties of the surrounding layers (layers one and three) are the same and constant. For the interfacial layer (layer two) two cases are considered. In Case I, variation of the Young's modulus $E$, Poisson's ratio $\nu$, mass density $\rho$, thermal conductivity $k$, specific heat capacity $c$, and the thermal coefficient of expansion $\alpha$ in the interfacial layer are described by:
where $E_1, \nu_1, \rho_1, k_1, c_1,$ and $\alpha_1$ are material constants which are calculated from continuity conditions at $y = \pm h/2$. Material properties of the interfacial layer at $y = 0$ are given in Table 6.5.

Table 6.5: Material properties of the interfacial layer at $y=0$.

<table>
<thead>
<tr>
<th>$E_0 (GPa)$</th>
<th>$\nu_0$</th>
<th>$\rho_0 (Kg/M^3)$</th>
<th>$K_{10} (W/m^oC)$</th>
<th>$c_0 (J/Kg^0K)$</th>
<th>$\alpha_0 (1/oC)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>21</td>
<td>0.25</td>
<td>6000</td>
<td>25.37</td>
<td>0.08</td>
<td>12e-7</td>
</tr>
</tbody>
</table>

In Case II, it is assumed that the material properties of the interfacial layer are constant and they are equal to those of the surrounding layers, layers one and three.

### 6.5.1 Numerical results for problem II

Consider a cracked plate with a length of $2b=0.2$ (m), a width of $2w=0.125$ (m), and a unit thickness as shown in Figure 6.10. The crack length is $2a=0.025$ (m) and it is assumed to be at the centre of the plate. The thickness of the interfacial layer is $h=0.025$ (m). A linear distribution of normal stress is applied at $y = 0.0525(m)$ with
the $\sigma = 100(MPa)$ as shown in the Figure 6.10. Mechanical boundary conditions are assumed to be clamped at $y=-0.0625$ (m) and free at the other three boundaries. Thermal initial and boundary conditions are as follows:

\[ T(x,y) = 0 \quad \text{for } t < 0 \]
\[ T(\pm b,y) = 0 \quad \text{for } t \geq 0 \]
\[ T(x,+w) = 0 \quad \text{for } t \geq 0 \]
\[ T(x,-w) = -1 \quad \text{for } t \geq 0 \]

In the contact zone, perfect contact is assumed. It means that the temperature at the contact surface is equal to the temperature at the target surface.

Table 6.6: Material properties of layers one and three.

<table>
<thead>
<tr>
<th>E(GPa)</th>
<th>$\nu$</th>
<th>$\rho(Kg/M^3)$</th>
<th>$K_t(W/m^\circ C)$</th>
<th>$c(J/Kg^\circ C)$</th>
<th>$\alpha(1/\circ C)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>210</td>
<td>0.3</td>
<td>7800</td>
<td>45.37</td>
<td>0.114</td>
<td>12e-6</td>
</tr>
</tbody>
</table>

Material properties of the surrounding layers which are constant and are the same for both layers are shown in Table 6.6. Figure 6.11 shows the normalized distribution of material properties for the interfacial layer, layer two.

Finite element mesh of the cracked plate is shown in Figure 6.12. Due to the lack of symmetry the whole model is meshed. The mesh includes 958 elements in which the first layer elements around the crack tip are triangular quarter point elements and the total degree of freedom is 3998. Several calculations were done using different meshes focused at the crack tip to test the convergence and mesh dependence. Only the final results and mesh are shown here. To solve the differential equation of the temperature field, which is time dependent, a $\theta$ family of approximation which
Figure 6.11: Normalized distribution of material properties of the interfacial layer, problem II Case I.

approximates a weighted average of the time derivative of a dependent variable at two consecutive time steps by linear interpolation of the values of the variable at the two time steps is used. For an unconditionally stable scheme, $\theta = 1/2$ is chosen. A time step $\delta t = 10^{-3}$ is enough to solve this problem accurately.

The temperature distributions along the surfaces of the crack, contact and target surfaces, at time $t = 25\delta t$ are presented in Figure 6.13. It can be seen that the crack surfaces are partially closed and the open part of the crack, $2a_1$, in functionally gradient material (Case I) is smaller than that of the homogeneous material (Case II). These results are shown in Table 6.7. In the open part of the crack, the temperature difference between contact and target surfaces is higher in FGM than that of homogeneous material.
Figure 6.12: Finite element mesh and mechanical boundary conditions for problem II.

The normal displacement distributions along the surfaces of the crack at time $t = 25\delta t$ (Sec) are plotted in Figure 6.14. As expected, the same value can be observed for the open part of the crack, $2a_1$, which is shown in Table 6.7.

For sufficiently large time, transient solutions reduce to the corresponding steady-state solutions. Figures 6.15 and 6.16 contain plots of the steady-state temperature and normal displacement distributions along the surfaces of the crack, respectively. The same pattern can be seen in these figures which were described in Figures 6.13 and 6.14. The open length of the crack from the steady-state solution is shown in Table 6.7. It can be concluded that the open length of the crack tends to increase
Figure 6.13: Temperature distributions along the surfaces of the crack at time $25\delta t$.

Table 6.7: Open length of the crack after deformation.

<table>
<thead>
<tr>
<th></th>
<th>FGM $(25\delta t)$</th>
<th>FGM (Steady-state)</th>
<th>Simple material $(25\delta t)$</th>
<th>Simple material (Steady-state)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2a_1$ (m)</td>
<td>0.01637</td>
<td>0.01772</td>
<td>0.01906</td>
<td>0.02093</td>
</tr>
</tbody>
</table>

as time increases.

The curves in Figures 6.17 and 6.18 show the change in temperature distribution for nodal points 10 and 20 with respect to time $t$. Location of nodal points 10 and 20 are shown in Table 6.8. They are two nodal points along the crack surfaces. Nodal point 10 is in the open side of the crack and, as it can be seen from Figure 6.17, the distance between target and contact increases as time increases. On the other hand, nodal point 20 is in the closed part of the crack, from the beginning of the
Figure 6.14: Normal displacement distributions along the surfaces of the crack at time $25\delta t$.

Figure 6.15: Steady-state temperature distributions along the surfaces of the crack.
Figure 6.16: Steady-state normal displacement distributions along the surfaces of the crack.

Table 6.8: Location of the nodal points 10, 15, 17, and 20 along the crack faces.

<table>
<thead>
<tr>
<th>Node</th>
<th>10</th>
<th>15</th>
<th>17</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>x (m)</td>
<td>-0.0075</td>
<td>0.0000</td>
<td>0.0039</td>
<td>0.0075</td>
</tr>
</tbody>
</table>

cooling process until steady-state solution is reached. Figures 6.17 and 6.18 show that FGMs are more resistant to the change of temperature.

Figures 6.19 and 6.20 display the numerical results of the normal displacement distribution for nodal points 15 and 17 with respect to time t. Location of nodal points 15 and 17 are listed in Table 6.8. All the curves decrease with time. Figure 6.19 depicts the response of the homogeneous material to the applied mechanical and thermal loadings. On the other hand, the variation of the normal displacement
Figure 6.17: Temperature distribution for nodal point 10 versus time.

Figure 6.18: Temperature distribution for nodal point 20 versus time.
Figure 6.19: Normal displacement distribution for nodal point 15 and 17 versus time for Case II.

Figure 6.20: Normal displacement distribution for nodal points 15 and 17 versus time for Case I.
Figure 6.21: Distribution of normalized SIF versus time, mode I.

Figure 6.22: Distribution of normalized SIF versus time, mode II.
versus time for the FGMs is shown in Figure 6.20. Both figures show that, at the beginning of the cooling process, surfaces of the crack at nodal point 17 are closed. As time increases this part tends to be open. After comparing the results in Figures 6.19 and 6.20, it is interesting to note that the time required to open the surfaces of the crack at nodal point 17, for functionally gradient materials, is much greater than that of homogeneous materials. Figure 6.21 shows a plot of normalized stress intensity factor $K^*_{I}(t) = K_I(t)/(E\alpha\Delta T \sqrt{a})$ versus $t$ for homogeneous and functionally gradient materials. Both curves increase with time. The variation of the normalized stress intensity factor $K^*_{II}(t) = K_{II}(t)/(E\alpha\Delta T \sqrt{a})$ for the second mode plotted versus time is shown in Figure 6.22. In contrary to mode I, the normalized stress intensity factor in the second mode decreases as time increases. From Figures 6.21 and 6.22, it is realized that a FGM is more suitable for use as an interfacial layer under mechanical and thermal loadings than ordinary materials because of its ability to reduce stress intensity factors.

### 6.6 Generalizations and concluding remarks

A numerical treatment of a finite crack in functionally graded materials under steady-state thermal loading conditions (problem I) and transient thermal loading conditions (problem II) is presented. A typical trilayer electronic package is considered as a model. Because of a mismatch in mechanical and thermal properties, thermoelastic stresses develop in multi-layered media subjected to uniform temperatures. The objective of the present work is to reduce these thermoelastic stresses by using functionally graded materials as an interfacial layer. Coulomb friction between con-
tacting crack surfaces is taken into account. A simple and efficient, iterative finite element technique for solving frictional contact problems under small deformations is used. Numerical examples are provided to validate the feasibility and the stability of the developed algorithm. Stress intensity factors are calculated by using numerical crack flank displacement. The effect of the coefficient of friction \( \mu \) on the stress intensity factor in the second mode is studied. With the increase of \( \mu \), the stress intensity factor decreases in both cases, exponential and linear distributions of material properties. However, an interesting result, which seems to be very important for designing functionally graded materials, is obtained. For a small value of the coefficient of friction, the stress intensity factor possesses a higher value in Case I and for a relatively large value of \( \mu \) the stress intensity factor possesses a higher value in Case II. In the second problem, a central crack in an interfacial layer under mechanical and thermal loading is considered. Transient and steady-state response of the system are studied and the dynamic stress intensity factors for modes I and II are plotted. It is revealed that the stress intensity factors are reduced considerably when functionally gradient material is used as an interfacial layer instead of homogeneous material.
Chapter 7

Delamination in composite shells

*The mark of a good idea is that it's so simple you can't claim credit for it.*

-M. Singer

7.1 Introduction

Failure analysis of laminated composites has steadily gained importance with the increasing use of such materials in high-performance structures, notably in the aerospace industry. One major obstacle to achieving the full weight saving potential of advanced composite materials in large, highly strained structures is the tendency of these materials to delaminate, either during their manufacturing process or while in use. Delamination, or a debonding between the plies of such a laminate, represents one of the weakest failure modes in a laminated composite and is often responsible for the loss of its stiffness, strength, and fatigue life.

Analytical modeling of laminated composites including delamination has received very little attention in the past. This is mainly because the problem is complicated since it is basically three-dimensional in nature. The presence of cracks or surfaces of discontinuity adds further difficulties to the development of a tractable theory of laminated composites. Overall, in the study of delamination problems, most of the analyses performed to date have been restricted to relatively simple models or to general three-dimensional finite element models. In the author's opinion, however,
the general problem of modeling growth of delamination of arbitrary shape is, as yet, not well understood. The basic principles for stress analysis around a stationary delamination are reasonably well known and these set the stage for development of more complex models and various efficient approaches towards solving realistic problems of practical interest. In this chapter a non-linear theory on the statics of multi layered shells, including transverse effects and delamination with an arbitrary shape, is studied. The approach adopted in [188] is modified and delaminations are included by introducing new vector variables [189]. At first, all layers are considered as separate layers and continuity conditions are applied by using Lagrange multipliers. This approach is purely kinematical. Thickness-wise, the displacement field is assumed to belong to a certain finite parameter family of functions while the exact three-dimensional kinematic relations and constitutive equations are used. Stresses and strains, rather than stress resultants and associated kinematic variables, are used in formulating the principle of virtual work, from which the field equations and relevant boundary conditions are obtained. Stress resultants appear only formally in the equations and their number and nature is a direct consequence of the choice of the functions for the displacement field. In particular, if these functions are polynomials in the thickness coordinate, the stress integrals turn out to be moments of various orders. The simplest kinematic hypothesis which still accounts for transverse effects is a piecewise linear displacement field. For this reason, and for the sake of brevity, the equations resulting from such a first order theory are given here in detail. Discontinuity between two layers is considered as a planar delamination between two layers and its boundary is defined by \( F(x, y) = 0 \). In the general case, this delamination would then propagate under externally applied loads, the boundary of which
can then be defined to be $H(x,y) = 0$. Here, a relatively weak interface between the two plies is hypothesized. A planar delamination then would not kink into the adjacent plies but be constrained to move in its own plane.

7.2 Method of approach

7.2.1 Directors and conjugate directors

Consider a multilayered shell of variable thickness with finite number of delaminations at its interfaces as pictured in Figure 7.1. It is convenient to think of each layer as a two-dimensional set of directed straight material line segments both ends of which describe smooth surfaces in three-dimensional Euclidean space as shown in Figure 7.2. As in [188] these directed material segments are called directors $d_I$ ($I =$...
1, ..., N). Where N denotes the number of material layers. All directors are assumed to have the same general orientation, with no tip to tip or tail to tail contact between directors of adjacent layers. To include the surface of discontinuity or interface delamination to the general governing equations, it is helpful to assume that each layer is separate from its adjacent layer. Now we think of each empty space between two layers as a two-dimensional set of directed straight non-material line segments both ends of which describe smooth surfaces in three-dimensional Euclidean space. These directed non-material segments may be called conjugate directors \( \vec{c}_A \) (\( A = 1, \ldots, N-1 \)).

![Figure 7.2: Layer coordinates for a 4-layer shell.](image)
7.2.2 Some introductory remarks on the theory of surfaces

From the geometrical point of view the shell is characterized, first of all, by its reference surface (middle surface in the classical theory of shells). It is therefore justified to give at the beginning some information concerning the theory of surfaces.

Let us consider the system of curvilinear coordinates \( x^a, (\alpha = 1,2) \) on a surface, as shown in Figure 7.3. The lines \( x^1 = \text{const} \) and \( x^2 = \text{const} \) constitute two families of curves on the surface. Every point \( P \) on the surface can be considered as the intersection point of two coordinate lines \( x^1 \) and \( x^2 \). Let \( \tilde{r}(x^a) \) denotes the radius vector from a fixed origin of the Cartesian co-ordinate \( X_i \) \((i = 1,2,3)\) to a generic
point on the surface as a vector function of the surface coordinates:

$$\vec{r} = \vec{r}(x^1, x^2)$$

(7.1)

We have the following relations between the coordinates $x^\alpha$ and the Cartesian coordinates $X_i$

$$x^\alpha = x^\alpha(X_1, X_2, X_3).$$

Let us consider now the vector $\vec{r} + \Delta \vec{r}$ corresponding to the point $P'$ of the line $x^2 = \text{const}$. Now consider the ratio $\Delta \vec{r} / \Delta x^1$. If $\Delta x^1 \to 0$, we obtain the partial derivative of the vector with respect to the coordinate $x^1$

$$\vec{a}_1 = \lim_{\Delta x^1 \to 0} \frac{\Delta \vec{r}}{\Delta x^1} = \frac{\partial \vec{r}}{\partial x^1} = \vec{r}_1$$

(7.2)

the direction of the vector $\partial \vec{r} / \partial x^1$ follows the direction of the line $x^2 = \text{const}$. at the point P. the second vector $\vec{a}_2 = \partial \vec{r} / \partial x^2$ is directed tangentially to the line $x^1 = \text{const}$.

The vectors $\vec{a}_\alpha$ are called the natural base vectors associated with the coordinate system $x^\alpha$. The plane S given by two vectors $\vec{a}_1, \vec{a}_2$ is a plane tangential to the surface at point P. If the lines of the system of coordinates cross each other with the angle $\phi$, then from the scalar product of the vectors $\vec{a}_\alpha$ we have $\vec{a}_1 \cdot \vec{a}_2 = | \vec{a}_1 | \cdot | \vec{a}_2 | \cos \phi = a_{12}$; then

$$\cos \phi = \sqrt{\frac{a_{12}^2}{a_{11} \cdot a_{22}}}$$

(7.3)

where we denoted

$$a_{\alpha \beta} = \vec{a}_\alpha \cdot \vec{a}_\beta$$

(7.4)
Since $\sin^2 \phi = 1 - \cos^2 \phi$ from 7.3 it can be shown that
\[
\sin \phi = \sqrt{\frac{a}{a_{11} \cdot a_{22}}}
\]
where
\[
a = a_{11}a_{22} - a_{12}^2 = \det[a_{\alpha \beta}]. \tag{7.5}
\]

Let us calculate now the square of the length of the line element on the surface, which is given by the two points $P(x^1, x^2)$, $P'(x^1 + dx^1, x^2 + dx^2)$. If we consider $d\vec{r}$ as the increase of the radius vector $\vec{r}$ by moving from the point $P$ to $P'$, the square of the length of the line element is defined by the scalar product
\[
d s^2 = d\vec{r} \cdot d\vec{r} = \bar{a}_\alpha \cdot \bar{a}_\beta dx^\alpha dx^\beta = a_{\alpha \beta} dx^\alpha dx^\beta \tag{7.6}
\]

the relation 7.6 is called the first fundamental quadratic form of the surface. The components $a_{\alpha \beta}$ are functions of the coordinates $x^\alpha$. The first quadratic form of the surface will be used in the determination of the strains in the reference surface of the shell. The quantities $a_{\alpha \beta}$ are called the components of the metric tensor of the surface.

The covariant derivative of the vector $\bar{u} = \bar{a}_\alpha u^\alpha$ is defined in the following way. Let us notice that while moving from the point $P$ to the point $P'$ we change not only the components of the vector $u^\alpha$ but also the base vector $\bar{a}_\alpha$. We have
\[
\dot{u}_\beta = \bar{a}_\alpha \frac{\partial u^\alpha}{\partial x^\beta} + u^\alpha \frac{\partial \bar{a}_\alpha}{\partial x^\beta} \tag{7.7}
\]
writing
\[
\frac{\partial \bar{a}_\alpha}{\partial x^\beta} = \frac{\partial^2 \bar{r}}{\partial x^\alpha \partial x^\beta} = \frac{\partial \bar{a}_\beta}{\partial x^\alpha} = \Gamma^\gamma_{\alpha \beta} \bar{a}_\gamma
\]

\[\text{1} \text{The summation convention is understood for diagonally repeated indices}\]
we obtain

\[ \ddot{u}_\beta = \ddot{a}_\alpha \frac{\partial u^\alpha}{\partial x^\beta} + u^\alpha \Gamma^\gamma_{\alpha\beta} \ddot{a}_\gamma = (\frac{\partial u^\gamma}{\partial x^\beta} + u^\alpha \Gamma^\gamma_{\alpha\beta}) \ddot{a}_\gamma = u^\gamma |_\beta \ddot{a}_\gamma \]  

(7.8)

where \( \Gamma^\gamma_{\alpha\beta} \) are the Christoffel symbols. The expression \( u^\gamma |_\beta = u^\gamma_\beta + u^\alpha \Gamma^\gamma_{\alpha\beta} \) gives the components of the covariant derivatives of the vector field \( \dot{u} \). For more information regarding to the theory of surface, the reader is addressed to [190].

7.2.3 Reference surface

The reference surface of the laminated shell is assumed to be a smooth surface contained completely within a single layer. To each point \( P \) of the reference surface there corresponds a combination of material lines and non-material lines, generally zig-zag, composed of the \( N \) directors and \( N-1 \) conjugate directors continuously connected to \( P \). This line will be called the directrix at \( P \). A multilayered shell may thus be regarded as a two-dimensional collection of directrices. Since \( x^\alpha (\alpha = 1, 2) \) is a parametrization of the reference surface, this parametrization refers also naturally to the directrix field and therefore to the director/conjugate director fields of all layers.

7.2.4 Undeformed multilayered shell

Consider an isotropic non-homogeneous multi-layered shell of variable thickness as shown in Figure 7.4. Let us use the system of curvilinear coordinates \( x^1, x^2 \) lying on the reference surface of the shell. Assume that there exists the relation \( X_i = X_i(x^\alpha) \) between the Cartesian coordinates \( X_i \) and the curvilinear coordinates \( x^\alpha \). The position of an arbitrary point \( Q \) in undeformed shell is defined by the radius
Figure 7.4: Definition of layer coordinates for a 4-layer shell.

vector $\bar{R}(x^a)$ as:

$$\bar{R} = \bar{r}(x^a) + z^I e_I(x^a) + w^A e_A(x^a)$$  \hspace{1cm} (7.9)

where $z^I$ ($I = 1, ..., N$), are layer coordinates and $w^A$ ($A = 1, ..., N-1$), are interfacial layer or conjugate layer coordinates abiding by the following definition:

Every point in the shell body belong to one and only one directrix (alternating directors and conjugate directors). A set of non-dimensional layer coordinates, $z^I$ ($I = 1, ..., N$), is defined for each point of the directrix at P in the following manner:
• Starting at \( P \) and following the director, the first point of layer \( I \) encountered is called the origin, \( O_I(P) \), of that layer corresponding to \( P \). Thus the local origin of any layer is either the \textit{tail} or the \textit{tip} of its director which is part of the directrix at \( P \), except for the layer containing the reference surface, for which the origin is \( P \) itself.

• At layer \( I \), the value of \( z^J \) is evaluated as follows: (i) for \( J = I \): \( |\vec{d}_I| \) measures length along \( \vec{d}_I \), starting from zero at the origin \( O_I(P) \) and increasing in the positive sense of \( \vec{d}_I \); (ii) for \( J \neq I \): \( z^J \) is a constant equal to the last value attained by \( z^J \) in layer \( J \) when travelling from \( P \) towards layer \( I \), or zero if layer \( J \) is not traversed in that trajectory.

• For the interfacial layer above the reference surface the value of \( w^A \) is either zero or plus one.

• For the interfacial layer below the reference surface the value of \( w^A \) is either zero or minus one.

In both situations, zero denotes that interfacial layer \( A \) is not traversed in a monotonic continuous trajectory from point \( P \) to an arbitrary point \( Q \) within the shell space.

For simplicity, it can be assumed that in the reference configuration there is no gap between adjacent layers. In other words the layers are lying on each other but there is no connectivity between them. The above assumption can be translated into a mathematical formula as follows:

\[
\varepsilon_A = 0 \quad A = 1, \ldots, N - 1 \quad (7.10)
\]
substituting equation 7.10 into equation 7.9, the position vector of an arbitrary point Q from layer I in the undeformed shell can be written as:

$$\tilde{R} = \tilde{r}(z^\alpha) + z'^I \tilde{d}_I(z^\alpha)$$  \hspace{1cm} (7.11)

Let us define now the components of the metric tensor for a point Q in layer I of the shell space corresponding to this triple of coordinates \((x^1, x^2, z^I)\). With a similar treatment as in the previous section, we obtain the base vectors for a point Q in layer I as follows:

$$\tilde{A}_\alpha = \tilde{R}_{,\alpha} = \tilde{r}_{,\alpha} + z^K \tilde{d}_{K,\alpha} = \tilde{a}_\alpha + z^K \tilde{a}_{K,\alpha}$$

$$\left(\tilde{A}_3\right)_I = \left(\frac{\partial \tilde{R}}{\partial z^I}\right) = \tilde{d}_I$$  \hspace{1cm} (7.12)

The components of the metric tensor for point Q in layer I are defined by the scalar product of the base vectors at that point as follows:

$$A_{\alpha\beta} = \tilde{A}_\alpha \cdot \tilde{A}_\beta = \tilde{a}_\alpha \cdot \tilde{a}_\beta + z^K (\tilde{a}_\alpha \cdot \tilde{a}_{K,\beta} + \tilde{a}_\beta \cdot \tilde{a}_{K,\alpha}) + z^K z^K \tilde{d}_{K,\alpha} \cdot \tilde{d}_{L,\beta}$$

$$\left(A_{3\alpha}\right)_I = \left(A_{33}\right)_I = \tilde{A}_\alpha \cdot \tilde{d}_I = \tilde{a}_\alpha \cdot \tilde{d}_I + z^K \tilde{d}_{K,\alpha} \cdot \tilde{d}_I$$

$$\left(A_{33}\right)_I = \tilde{d}_I \cdot \tilde{d}_I = h_I^2$$  \hspace{1cm} (7.13)

where \(\tilde{a}_\alpha \cdot \tilde{a}_\beta = a_{\alpha\beta}\) is the metric tensor of the reference surface and \(h_I\) is the length of the vector \(\tilde{d}_I\) (i.e., the thickness of layer I measured along the director).

For the particular case of a straight directrix normal to the reference surface and for layers of constant thickness, equations 7.13 reduce to the familiar expressions of thin shell theory.
7.2.5 Transformation of surface and volume elements

Consider a small area in the reference surface as shown in Figure 7.5. The magnitude of this area is:

\[ dS = |\bar{a}_1 \times \bar{a}_2| \, dx^1 dx^2 = |\bar{a}_1| |\bar{a}_2| \sin \phi \, dx^1 dx^2 \]  

(7.14)

where

\[ |\bar{a}_1| = \sqrt{\bar{a}_1 \cdot \bar{a}_1} = \sqrt{a_{11}} \]

\[ |\bar{a}_2| = \sqrt{\bar{a}_2 \cdot \bar{a}_2} = \sqrt{a_{22}} \]  

(7.15)

substituting from equations 7.5 and 7.15 into equation 7.14, we obtain:

\[ dS = \sqrt{a} dx^1 dx^2 \]  

(7.16)
following the same procedure, the magnitude of an area element at point \( Q \) in the layer \( I \) can be written as:

\[
d_I S = \left| \vec{A}_1 \times \vec{A}_2 \right| dx^1 dx^2 = \sqrt{rA} dx^1 dx^2 \tag{7.17}\]

where \( rA \) denotes the determinant of \( A_{\alpha\beta} \) evaluated at the point \( Q \).

By definition, the volume element \( dV_I \) at point \( Q \) in the layer \( I \) is given by:

\[
dV_I = \epsilon_{ijk} R^i_1 R^j_2 R^k_3 dx^1 dx^2 dx^I \tag{7.18}\]

where \( \epsilon_{ijk} \) is the permutation symbol and \( \vec{R} \) is the position vector of the point \( Q \) which is given by equ 7.11. "",3" denotes the partial derivative with respect to \( x^I \).

By definition, Jacobian matrix at point \( Q \) is given by:

\[
J = \begin{pmatrix}
R^1_1 & R^1_2 & R^1_3 \\
R^2_1 & R^2_2 & R^2_3 \\
R^3_1 & R^3_2 & R^3_3
\end{pmatrix} \tag{7.19}
\]

therefore

\[
J^T J = \begin{pmatrix}
R^1_1 & R^2_1 & R^3_1 \\
R^2_1 & R^2_2 & R^2_3 \\
R^3_1 & R^3_2 & R^3_3
\end{pmatrix}^T \begin{pmatrix}
R^1_1 & R^1_2 & R^1_3 \\
R^2_1 & R^2_2 & R^2_3 \\
R^3_1 & R^3_2 & R^3_3
\end{pmatrix} = \begin{pmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{pmatrix} \tag{7.20}
\]

where \( (\cdot)^T \) denotes transpose of \( (\cdot) \) and determinant of \( J \) can be written as:

\[
det(J) = \sqrt{det(J^T J)} = \sqrt{rA} \tag{7.21}\]

where \( A \) denotes determinant of \( A_{ij} \) at point \( Q \).

From equations 7.18 and 7.21 one can conclude that
\[ dV_I = \varepsilon_{ijk} R_i^1 R_i^2 R_i^3 dx^1 dx^2 dx^3 = \det(J) dx^1 dx^2 dx^3 = \sqrt{A} dx^1 dx^2 dz^I \] (7.22)

substituting for \( dx^1 dx^2 \) from equation 7.16 into the equation 7.22, we obtain:

\[ dV_I = \sqrt{A} dS dz^I \] (7.23)

where \( S \) denotes the area of the undeformed reference surface.

7.3 Deformation of the shell

7.3.1 The law of variation of displacements across the thickness of the shell

Let the position of a point on the reference surface of the shell be determined by the curvilinear coordinates \( x^1, x^2 \). Further, as the result of deformation, let the arbitrary point \( P \) on the reference surface undergo the displacement \( \bar{u} \) and moves to the point \( P^* \). This vector will be a function of \( x^1, x^2 \), that is, a function of the particular point on the reference surface, and its knowledge is equivalent to determining the deformation of the reference surface.

At the point \( P \) erect a directrix and consider a second point \( Q \) in the layer I of the shell and located on this directrix at a distance \( z^I \bar{d}_I \) from the reference surface. After deformation, point \( Q \) will be located at \( Q^* \) as shown in Figure 7.6. Using vector notation, the position vector \( \bar{R}^* \) of the point \( Q^* \) in the deformed shell can be written as:

\[ \bar{R}^* = \bar{r} + \bar{u} + z^I \bar{d}_I + w^A \bar{e}^*_A \] (7.24)
where $\vec{u}$ denotes the displacement of point $P$ of the reference surface. $\vec{d}_f$ and $\vec{e}_A$ are the deformed counterparts of $\vec{d}_f$ and $\vec{e}_A$, assumed to remain straight after deformation which is equivalent to assume piecewise linear variation of all displacement components across the thickness. The above assumption is the only assumption which we make in the present theory. The exact three-dimensional kinematic relations and constitutive equations are used in conjunction with the principle of virtual work, eliminating the need for any further approximations or assumptions in the theory.
7.3.2 Strain-displacement equations

At any point \( Q^* \) of layer I the base vectors of the deformed body are obtained as:

\[
\begin{align*}
\vec{A}_\alpha^* &= \vec{R}_\alpha^* = \vec{a}_\alpha + \vec{u}_\alpha + z^K \vec{d}_{K,\alpha} + w^A \vec{c}_{A,\alpha} \\
(\vec{A}_3^*)_I &= \vec{R}_3^* = \vec{d}_I
\end{align*}
\]

(7.25)

the components of the metric tensor for the point \( Q^* \) in the layer I can be written as:

\[
A^*_\alpha\beta = a_{\alpha\beta} + \vec{a}_\alpha \cdot \vec{u}_\beta + \vec{a}_\beta \cdot \vec{u}_\alpha + \vec{u}_\alpha \cdot \vec{u}_\beta \\
+ z^K [\vec{a}_\alpha \cdot \vec{d}_{K,\beta} + \vec{a}_\beta \cdot \vec{d}_{K,\alpha} + \vec{u}_\alpha \cdot \vec{d}_{K,\beta} + \vec{u}_\beta \cdot \vec{d}_{K,\alpha} + w^A \vec{d}_{A,\alpha} \cdot \vec{c}_{A,\beta}] \\
+ w^A \vec{d}_{A,\alpha} \cdot \vec{c}_{A,\beta} \\
+ w^A \vec{d}_I \cdot \vec{d}_I \\
(\vec{A}_3^*)_I = (\vec{A}^*_3)_I = \vec{a}_\alpha \cdot \vec{d}_I + \vec{u}_\alpha \cdot \vec{d}_I + z^K \vec{e}_{K,\alpha} \cdot \vec{d}_I + w^A \vec{c}_{A,\alpha} \cdot \vec{d}_I \\
(\vec{A}_3^*)_I = (\vec{A}^*_3)_I \cdot (\vec{A}^*_3)_I = \vec{d}_I \cdot \vec{d}_I
\]

(7.26)

Using equations 7.13 and 7.26, the components of the Lagrangian strain tensor for a point \( Q \) in layer I, can be obtained as:

\[
\epsilon_{\alpha\beta} = \frac{1}{2} (A^*_{\alpha\beta} - A_{\alpha\beta}) = \frac{1}{2} [\vec{a}_\alpha \cdot \vec{u}_\beta + \vec{a}_\beta \cdot \vec{u}_\alpha + \vec{u}_\alpha \cdot \vec{u}_\beta] \\
+ \frac{1}{2} z^K [\vec{a}_\alpha \cdot (\vec{d}_{K,\beta} - \vec{d}_{K,\beta}) + \vec{a}_\beta \cdot (\vec{d}_{K,\alpha} - \vec{d}_{K,\alpha}) + \vec{u}_\alpha \cdot \vec{d}_{K,\beta}] \\
+ \vec{u}_\beta \cdot \vec{d}_{K,\alpha} + w^A \vec{d}_{A,\alpha} \cdot \vec{c}_{A,\beta} + w^A \vec{d}_{A,\beta} \cdot \vec{c}_{A,\alpha}]
\]
It has to be noted that indices I and L are range from 1 to N while indices A and B are vary from 1 to N-1.

7.4 Governing equations

The discussion so far has dealt with the geometry of the shell and its deformation regardless of the forces which cause that deformation in the shell space. To relate deformations to the forces which cause those deformations, the following procedure is adopted.

7.4.1 Virtual work

Internal virtual work in a three-dimensional body is expressed as:

$$ IVW = \int_V \sigma^{ij} \delta \varepsilon_{ij} dV \quad (i, j = 1, 2, 3) $$

where $\sigma^{ij}$ is the symmetric or second Piola-Kirchhoff stress tensor, $\varepsilon_{ij}$ is the Lagrangian strain tensor and $V$ designates the volume of the undeformed body. Since the continuity constraints at interfaces within the multilayered shell should be imposed, we have to add constraints to the internal virtual work by using Lagrange
multipliers. So the expression for the internal virtual work can be written as:

\[
IVW = \sum_{i=1}^{N} \int_{V_i} \sigma_{ij} \delta e_{ij} dV_i + \int_{S_{inA}} \lambda^A \delta \bar{c}^A dS_{inA} + \int_{S_{inA}} \delta A^A \bar{c}^A dS_{inA} \tag{7.29}
\]

where \( \lambda^A \) is Lagrange multiplier and \( S_{inA} \) is the \( A \text{th} \) interface between layer \( A \) and \( A+1 \). It should be noted that \( A \) is range from 1 to \( N-1 \). Rewriting expression 7.29

\[
IVW = \sum_{i=1}^{N} \int_{V_i} (\sigma^\alpha_{\beta \alpha} \delta e_{\alpha \beta} + 2\sigma^\alpha \delta e_{\alpha \beta} + \sigma^{33} \delta e_{33}) dV_i + \int_{S_{inA}} \lambda^A \delta \bar{c}^A dS_{inA}
\]

\[
+ \int_{S_{inA}} \delta A^A \bar{c}^A dS_{inA} \tag{7.30}
\]

Introducing eqn 7.27 into eqn 7.30, we obtain

\[
IVW = \int_{S} \left\{ N^\alpha\beta (\bar{a}_\alpha + \bar{u}_\alpha) \cdot \delta \bar{c}_\beta \\
+ M^\alpha\beta K[(\bar{a}_\alpha + \bar{u}_\alpha + w^A \bar{c}^A_{\alpha \alpha}) \cdot \delta \bar{c}^K_{\beta \beta} + \bar{a}_\beta \cdot \delta \bar{c}_\alpha + w^A \bar{c}^A_{\beta \alpha} \cdot \delta \bar{c}_\beta]
+ B^\alpha\beta KL \bar{a}^K_{\alpha \beta} \cdot \delta \bar{c}_\beta + N^\alpha\beta [w^A (\bar{a}_\alpha + \bar{u}_\alpha) \cdot \delta \bar{c}^A_{\alpha \beta} + w^A \bar{c}^A_{\beta \alpha} \cdot \delta \bar{c}_\beta]
+ N^\alpha\beta w^A w^B \bar{c}^A_{\beta \alpha} \cdot \delta \bar{c}^B_{A,\beta} \\
+ S^\alpha I[(\bar{a}_\alpha + \bar{u}_\alpha + w^A \bar{c}^A_{\alpha \alpha}) \cdot \delta \bar{d}_I + \bar{a}_I \cdot \delta \bar{u}_\alpha + w^A \bar{d}^A_I \cdot \delta \bar{c}^A_{\alpha \alpha}]
+ Q^\alpha IK [\bar{d}^I \cdot \delta \bar{c}^A_{\alpha \alpha} + \bar{d}^A_{\alpha \alpha} \cdot \delta \bar{d}^I_I + \bar{B}^I \cdot \delta \bar{d}^I_I] dS
+ \int_{S} \sqrt{\frac{m^A A}{a}} \lambda^A \delta \bar{c}^A dS + \int_{S} \sqrt{\frac{m^A A}{a}} \bar{c}^A \delta \lambda^A dS \right\} dS \tag{7.31}
\]

where \( S \) is the area of the undeformed reference surface and \( A, B = 1, ..., N - 1 \). New notations which were introduced in the above equation have the following definitions:

\[
N^\alpha\beta = \sum_{i=1}^{N} \int \frac{A}{a} \sigma^\alpha\beta dz^I
\]

\[
M^\alpha\beta K = \sum_{i=1}^{N} \int \sqrt{\frac{A}{a}} \sigma^\alpha\beta_z K dz^I
\]

\[
B^\alpha\beta KL = \sum_{i=1}^{N} \int \sqrt{\frac{A}{a}} \sigma^\alpha\beta_z K dz^I
\]
\[ S^{\alpha I} = \int \sqrt{\frac{A}{a} \sigma^{\alpha \gamma} dz^I} \]
\[ Q^{\alpha IK} = \int \sqrt{\frac{A}{a} \sigma^{\alpha \gamma} x^I dz^I} \]
\[ \tilde{P}^I = \tilde{d}_I \int \sqrt{\frac{A}{a} \sigma^{33} dz^I} \]
\[ A = \text{det}[A_{ij}] \]
\[ a = \text{det}[a_{\alpha \beta}] \]
\[ m_A A = \text{det}[A_{\alpha \beta}] \quad \text{at } A^{th} \text{ interface} \]

The first three relations of the above equations represent stress resultants of various kinds.

Consider now the virtual work of the external forces. Following the work presented in [188], only two types of loading are included (Figure 7.7) as follows:

- Dead load \( \tilde{P} \), acting on the outer surface of the shell.
- Dead load \( \tilde{q} \), acting on the boundary surface of the shell.

The virtual work of the first type of loading is given by:

\[ EVW_1 = \int_0^S \tilde{P} \cdot (\delta \tilde{u} + \tilde{z}^I \delta \tilde{d}_I + \tilde{w}^A \delta \tilde{c}_A^I) dS \]

where the left subscript "o" refers to the outer surface. Knowing that

\[ \frac{d_\sigma S}{dS} = \sqrt{\frac{\sigma a}{a}} \]

where \( \sigma a = \text{det}[A_{\alpha \beta}] \) evaluated at the outer surface. The virtual work of the first type of loading can be written as:

\[ EVW_1 = \int_S \sqrt{\frac{\sigma a}{a}} \tilde{P} \cdot (\delta \tilde{u} + \tilde{z}^I \delta \tilde{d}_I + \tilde{w}^A \delta \tilde{c}_A^I) dS \]
where $S$ denotes the area of the reference surface. The virtual work of the second type of loading can be written as:

$$EVW_2 = \sum_{i=1}^{N} \int_{S_i} \bar{q} \cdot (\delta \bar{u} + z^K \delta \bar{d}_K + w^A \delta \bar{e}_A^A) dS_i$$

Equation 7.36 may be written as:

$$EVW_2 = \oint_C \sum_{i=1}^{N} \int_{S_i} \bar{q} \cdot (\delta \bar{u} + z^K \delta \bar{d}_K + w^A \delta \bar{e}_A^A)[\bar{n}_I \cdot (\bar{A}_a \times \bar{d}_I)] \frac{d\alpha}{dC} dz'dC$$

where $C$ is the boundary of the reference surface and the circle on the first integral is used to emphasize that $C$ is a closed curve. $\bar{n}_I$ is the outward unit normal to $S_I$. 

Figure 7.7: Shell loading.
7.4.2 Constitutive equations

For a given elastic material a definite relation exists between the stress tensor \( \sigma \) and the strain tensor \( \epsilon \); symbolically we write:

\[
\sigma^{ij} = F^{ij}(\epsilon_{kl})
\]

(7.38)
such a relation as (7.38) is called a constitutive equation for a given material. Since \( \epsilon_{kl} \) is a function of kinematic variables \( \bar{u}, \bar{d}_I^a, \) and \( \bar{c}_A^a, \) a functional relationship between the stress and the kinematic variables can be written as:

\[
\sigma^{ij} = F^{ij}[\epsilon_{kl}(\bar{u}, \bar{d}_I^a, \bar{c}_A^a)] = \Phi^{ij}(\bar{u}, \bar{d}_I, \bar{c}_A)
\]

(7.39)

by substituting equation 7.39 into equation 7.32, the various stress resultants become functions of the kinematic variables.

7.4.3 Equilibrium equations

The equations of equilibrium expressed in terms of the stress resultants, couples, and unknown directors and displacements can be obtained by using the principle of virtual work together with the continuity conditions at interfaces as follows:

\[
IVW = EVW = EVW_1 + EVW_2
\]

\[
\lambda^A = 0 \quad \text{for} \quad S_{disA} \subset inAS
\]

(7.40)

where \( S_{disA} \) denote discontinuous surfaces at the \( A^{th} \) interface.

Using Green's theorem and eqn (7.40), the equilibrium equations are obtained as:

\[
-N^\alpha(\bar{\alpha}_\beta + \bar{\alpha}_\beta + w^A \epsilon_{A,\beta}) + M^{\alpha\beta I} \bar{d}_I^\beta + S^\alpha I \bar{d}_I)\| \alpha = \sqrt{\frac{\alpha}{a}} \bar{P}
\]

(7.41)
\[-[M^{\alpha\beta I}(\ddot{a}_\alpha + \ddot{u}_\alpha + w^A \delta^I_{A,\alpha}) + B^{\alpha\beta KI} \ddot{d}^K_{K,\alpha} + Q^{\beta KI} \ddot{d}^K_R]\]
\[+ S^{\alpha I}(\ddot{a}_\alpha + \ddot{u}_\alpha + w^A \delta^\alpha_{A,\alpha} + Q^{\alpha IK} \ddot{d}^K_{K,\alpha} + \ddot{P} I = \sqrt{\frac{6\alpha}{a} z^F_0 \bar{P}} \]
\[(7.42)\]

\[-[N^{\alpha\beta}(\ddot{a}_\alpha + \ddot{u}_\alpha + w^B \delta^\beta_{B,\alpha}) \nu^A + M^{\alpha\beta K} \ddot{d}^K_{K,\alpha} w^A + S^{\beta I} \ddot{P} I w^A]\]
\[+ \sqrt{\frac{\mu A}{a}} \lambda^A = \sqrt{\frac{6\alpha}{a} \bar{P} \nu^A} \]
\[(7.43)\]

\[\lambda^A = 0 \quad \text{for} \quad S_{\text{dis}A} \subset \text{in} A S \]
\[(7.44)\]

where \(\ddot{\cdot}\) denotes covariant differentiation with respect to the undeformed metric.

Equation (7.41-7.44) represent a total number of \((7N-1)\) equations. Three equations for three components of \(\ddot{u}\), \(3N\) equations for \(3N\) components of \(\ddot{d}^I\), \((3N - 3)\) equations for \((3N - 3)\) components of \(\ddot{c}^A\), and finally \((N - 1)\) equations for \((N - 1)\) unknown Lagrange multipliers \(\lambda^A\). It should be noted that although equations (7.41) and (7.43) are similar in appearance, a closer investigation shows that they are different in nature.

7.4.4 The natural boundary conditions

The differential equations (7.41-7.44) do not yet determine completely the state of stress in a shell as long as they are not subjected to boundary conditions. That means that certain number of relations between the forces, moments, displacements or functions of these quantities are specified at the edges of the shell. The boundary conditions of a shell can be expressed by means of displacements or internal forces.

In the case of conditions expressed by the displacements we require the displacements
\( \bar{u} \) and their derivatives \( \bar{u},_\alpha \) to have the values given in advance at the edge of the shell. When the forces are given at the edge, the forces obtained from the general solution should fulfil the given conditions. In the case of mixed boundary conditions we can use both the equations expressed in terms of displacements and forces.

The boundary conditions associated with equations (7.41-7.44) are:

\[
\begin{align*}
&\begin{bmatrix}
N^{\alpha\beta}(\bar{a}_\beta + \bar{u},_\beta + w^A \bar{c}_{A,\beta}) + M^{\alpha\beta I}\bar{d}_{I,\beta} + S^{\alpha I}\bar{d}_I \nu_\alpha \\
= \sum_{I=1}^N \int_S \bar{q} [\bar{n}_I \cdot (\bar{A}_\alpha \times \bar{d}_I)] \frac{\partial \bar{u}}{\partial C} d\bar{z}_I
\end{bmatrix} \\
&\text{or} \\
\bar{u} & \text{prescribed}
\end{align*}
\]  
(7.45)

\[
\begin{align*}
&\begin{bmatrix}
M^{\alpha\beta I}(\bar{a}_\alpha + \bar{u},_\alpha + w^A \bar{c}_{A,\alpha}) + B^{\alpha\beta KI}\bar{d}_{K,\alpha} + Q^{\beta KI}\bar{d}_K \nu_\beta \\
= \sum_{K=1}^N \int_S \bar{q} [\bar{n}_K \cdot (\bar{A}_\alpha \times \bar{d}_K)] \frac{\partial \bar{u}}{\partial C} d\bar{z}_K
\end{bmatrix} \\
&\text{or} \\
\bar{d}_I & \text{prescribed}
\end{align*}
\]  
(7.46)

\[
\begin{align*}
&\begin{bmatrix}
N^{\alpha\beta}(\bar{a}_\alpha + \bar{u},_\alpha + w^B \bar{c}_{B,\alpha}) w^A + M^{\alpha\beta K}\bar{d}_{K,\alpha} w^A + S^{\beta I} \bar{d}_I w^A \nu_\beta \\
= \sum_{I=1}^N \int_S \bar{q} w^A [\bar{n}_I \cdot (\bar{A}_\alpha \times \bar{d}_I)] \frac{\partial \bar{c}_A}{\partial C} d\bar{z}_I
\end{bmatrix} \\
&\text{or} \\
\bar{c}_A & \text{prescribed}
\end{align*}
\]  
(7.47)

where \( \nu \) is the unit normal to the boundary curve on the reference surface.

### 7.5 Shells with linearly elastic layers

In this section we shall be concerned with constitutive relations for so-called linear elastic Hookean media which may undergo large deflections. We shall employ the
Lagrangian strain and second Piola-Kirchhoff stress tensor as before. Adopting the notations used in [188], the constitutive equation for the $I^{th}$ layer can be written as:

$$
\sigma^{ij} = E^{ijkl}_{I} \epsilon_{kl}
$$

(7.48)

where in functionally gradient layers, the elastic module $E^{ijkl}_{I}$ depend on $z^{\alpha}$ and $z^{l}$. In this situation, strain components, eqns (7.27) can be written in short form as:

$$
\varepsilon_{\alpha \beta} = e_{\alpha \beta} + w^{A} c_{\alpha \beta A} + z^{K} e_{\alpha \beta K} + z^{K} z^{L} e_{\alpha \beta KL}
$$

$$
(e_{\alpha \beta})_{I} = (e_{\alpha \beta})_{I} + w^{A} (c_{\alpha \beta A})_{I} + z^{K} (e_{\alpha \beta K})_{I}
$$

$$
(e_{33})_{I} = (e_{33})_{I}
$$

(7.49)

where the e's and c's are surface tensor, as indicated by their Greek indices, defined by:

$$
\varepsilon_{\alpha \beta} = \frac{1}{2} [\ddot{a}_{\alpha} \cdot \ddot{u}_{\beta} + \ddot{a}_{\beta} \cdot \ddot{u}_{\alpha} + \ddot{u}_{\alpha} \cdot \ddot{u}_{\beta}]
$$

$$
\varepsilon_{\alpha \beta K} = \frac{1}{2} [\ddot{a}_{\alpha} \cdot (\ddot{d}_{K,\beta} - \ddot{d}_{K,\beta}) + \ddot{a}_{\beta} \cdot (\ddot{d}_{K,\alpha} - \ddot{d}_{K,\alpha}) + \ddot{u}_{\alpha} \cdot \ddot{d}_{K,\beta}
$$

$$
+ \ddot{u}_{\beta} \cdot \ddot{d}_{K,\alpha} + w^{A} \ddot{d}_{K,\alpha} \cdot \ddot{c}_{A,\beta} + w^{A} \ddot{d}_{K,\beta} \cdot \ddot{c}_{A,\alpha}]
$$

$$
\varepsilon_{\alpha \beta KL} = \frac{1}{2} [\ddot{d}_{K,\alpha} \cdot \ddot{d}_{L,\beta} - \ddot{d}_{K,\alpha} \cdot \ddot{d}_{L,\beta}]
$$

$$
c_{\alpha \beta A} = \frac{1}{2} [\ddot{a}_{\alpha} \cdot \ddot{c}_{A,\beta} + \ddot{a}_{\beta} \cdot \ddot{c}_{A,\alpha} + \ddot{u}_{\alpha} \cdot \ddot{c}_{A,\beta} + \ddot{u}_{\beta} \cdot \ddot{c}_{A,\alpha} + w^{B} \ddot{c}_{A,\alpha} \cdot \ddot{c}_{B,\beta}]
$$

$$
(c_{\alpha \beta A})_{I} = \frac{1}{2} \ddot{d}_{I} \cdot \ddot{c}_{A,\alpha}
$$

$$
(e_{33})_{I} = \frac{1}{2} [\ddot{a}_{\alpha} \cdot (\ddot{d}_{I}^{*} - \ddot{d}_{I}) + \ddot{u}_{\alpha} \cdot \ddot{d}_{I}]
$$

$$
(c_{33})_{I} = \frac{1}{2} \ddot{d}_{I} \cdot \ddot{c}_{33}
$$

(7.50)

Employing eqns (7.48 and 7.49) in conjunction with eqn (7.32), we see that:
\[ N^{\alpha \beta} = \sum_{l=1}^{N} \int \sqrt{\frac{A}{a}} \sigma^{\alpha \beta} dz^l = \sum_{l=1}^{N} \int \sqrt{\frac{A}{a}} E^{\alpha \beta k l} e_{k l} dz^l \]

\[ = \sum_{l=1}^{N} \int \sqrt{\frac{A}{a}} (E^{\alpha \beta \gamma \mu} e_{\gamma \mu} + 2E^{\alpha \beta \gamma 3}(e_{\gamma 3})_l + E^{\alpha \beta 33}(e_{33})_l) dz^l \]

or

\[ N^{\alpha \beta} = \sum_{l=1}^{N} \int \sqrt{\frac{A}{a}} [E^{\alpha \beta \gamma \mu}(e_{\gamma \mu} + w^A c_{\gamma \mu A} + z^K e_{\gamma \mu K} + z^K z^L e_{\gamma \mu KL}) + 2E^{\alpha \beta \gamma 3}(e_{\gamma 3})_l + w^A(c_{33 A})_l + z^K(e_{33 K})_l + E^{\alpha \beta 33}(e_{33})_l] dz^l \]

and finally:

\[ N^{\alpha \beta} = \sum_{l=1}^{N} [D_{\alpha \beta \gamma \mu}(e_{\gamma \mu} + w^A c_{\gamma \mu A}) + D_{\alpha \beta \gamma \mu K} e_{\gamma \mu K} + D_{\alpha \beta \gamma \mu KL} e_{\gamma \mu KL} + 2D_{\alpha \beta \gamma 3}(e_{\gamma 3})_l + w^A(c_{33 A})_l + 2D_{\alpha \beta \gamma 3 K}(e_{\gamma 3 K})_l + D_{\alpha \beta 33}(e_{33})_l] \]

(7.51)

where

\[ D_{ijkl...L} = \int z^I \sqrt{\frac{A}{a}} E^{ijkl...L} dz^l \]

(7.52)

Similarly:

\[ M^{\alpha \beta \gamma \mu} = \sum_{l=1}^{N} [D_{\alpha \beta \gamma \mu K}(e_{\gamma \mu} + w^A c_{\gamma \mu A}) + D_{\alpha \beta \gamma \mu KL} e_{\gamma \mu L} + D_{\alpha \beta \gamma \mu LM} e_{\gamma \mu LM} + 2D_{\alpha \beta \gamma 3 K}(e_{\gamma 3})_l + w^A(c_{33 A})_l + 2D_{\alpha \beta \gamma 3 L K}(e_{\gamma 3 L})_l + D_{\alpha \beta 33 K}(e_{33})_l] \]

(7.53)

\[ B^{\alpha \beta \gamma \mu} = \sum_{l=1}^{N} [D_{\alpha \beta \gamma \mu KL}(e_{\gamma \mu} + w^A c_{\gamma \mu A}) + D_{\alpha \beta \gamma \mu KLM} e_{\mu KL} + D_{\alpha \beta \gamma \mu LMN} e_{\mu LMN} + 2D_{\alpha \beta \gamma 3 KL}(e_{\gamma 3})_l + w^A(c_{33 A})_l + 2D_{\alpha \beta \gamma 3 N K L}(e_{\gamma 3 N})_l + D_{\alpha \beta 33 K L}(e_{33})_l] \]

(7.54)
\[ S^{\alpha I} = D^{\alpha\beta\gamma\lambda}_I (e_{\gamma\mu} + w^A c_{\beta\mu} w) + D^{\alpha\beta\gamma\lambda K}_I e_{\gamma\mu K} + D^{\alpha\beta\gamma\lambda K L}_I e_{\gamma\mu K L} + 2D^{\alpha\beta\gamma\lambda}_I (e_{\gamma J I} + w^A (c_{\beta J I}) I) + 2D^{\alpha\beta\gamma\lambda K}_I (e_{\gamma J K}) I + D^{\alpha\beta\gamma\lambda K L}_I (e_{33}) I \] (7.55)

\[ Q^{\alpha J K} = D^{\alpha\beta\gamma\lambda K}_I (e_{\gamma\mu} + w^A c_{\beta\mu}) + D^{\alpha\beta\gamma\lambda K L}_I e_{\gamma\mu L} + D^{\alpha\beta\gamma\lambda K L M}_I e_{\gamma\mu L M} + 2D^{\alpha\beta\gamma\lambda K}_I ((e_{\gamma J I} + w^A (c_{\beta J I}) I) + 2D^{\alpha\beta\gamma\lambda L K}_I (e_{\gamma J L}) I + D^{\alpha\beta\gamma\lambda K L}_I (e_{33}) I \] (7.56)

\[ \bar{P}^I = \bar{P}_I (D^{\alpha\beta\gamma\lambda}_I (e_{\gamma\mu} + w^A c_{\beta\mu} A) + D^{\alpha\beta\gamma\lambda K}_I e_{\gamma\mu K} + D^{\alpha\beta\gamma\lambda K L}_I e_{\gamma\mu K L} + 2D^{\alpha\beta\gamma\lambda}_I (e_{\gamma J I} + w^A (c_{\beta J I}) I) + 2D^{\alpha\beta\gamma\lambda K}_I (e_{\gamma J K}) I + D^{\alpha\beta\gamma\lambda K L}_I (e_{33}) I \] (7.57)

### 7.6 Small deformations of a layered beam

In order to comprehend more easily the theory that was described in the previous sections, we shall next consider the case in which a multi-layered beam is laterally loaded as shown in Figure 7.8. The reference surface is assumed to be in the middle of the first layer. The position vector of a point \( P \) on the reference surface can be written as:

\[ \bar{r} = x \bar{i} + y \bar{j} \] (7.58)

Similarly, the position vector of a point \( Q \) on the \( I^{th} \) layer before deformation can be written as:

\[ \bar{R} = x \bar{i} + y \bar{j} + z' \bar{d}_I \] (7.59)
Figure 7.8: Two-layered beam with an embedded delamination.

From the Figure 7.8, it can be seen that the unit vector $\bar{k}$ is the best candidate for directors $\bar{d}_f$. By choosing $\bar{d}_f = \bar{k}$, parameters $z^f$ are varying as:

$$-h \leq z^1 \leq h$$

$$0 \leq z^2 \leq h$$

(7.60)

By using equations (7.58) and (7.59), base vectors and metric tensor for the arbitrary points $P$ and $Q$ can be written as:

$$\bar{a}_1 = \vec{i}$$

$$\bar{a}_2 = \vec{j}$$

$$a_{11} = a_{22} = 1$$

$$a_{12} = a_{21} = 0$$

$$a = 1$$
\[ \vec{A}_1 = \vec{i} \]
\[ \vec{A}_2 = \vec{j} \]
\[ (\vec{A}_3)_I = \vec{k} \]
\[ A_{11} = A_{22} = 1 \]
\[ A_{12} = A_{21} = 0 \]
\[ (A_{13})_I = (A_{31})_I = (A_{23})_I = (A_{32})_I = 0 \]
\[ (A_{33})_I = 1 \]
\[ A = 1 \]

(7.61)

where \( \vec{i} \), \( \vec{j} \), and \( \vec{k} \) are unit vectors in the \( x \), \( y \), and \( z \) directions respectively. If deflections are small in comparison with the thickness of the beam, the higher order terms in the strain-displacement relations can be neglected. In this case, the strain components will be obtained as:

\[
\varepsilon_{\alpha\beta} = \frac{1}{2} [\vec{a}_\alpha \cdot \vec{u}_\beta + \vec{a}_\beta \cdot \vec{u}_\alpha] \\
+ \frac{1}{2} z^K [\vec{a}_\alpha \cdot (\vec{d}_{K,\beta} - \vec{d}_{K,\beta}) + \vec{a}_\beta \cdot (\vec{d}_{K,\alpha} - \vec{d}_{K,\alpha})] \\
+ \frac{1}{2} u^A [\vec{a}_\alpha \cdot \vec{c}_{A,\beta} + \vec{a}_\beta \cdot \vec{c}_{A,\alpha}] \\
(\varepsilon_{\alpha3})_I = \frac{1}{2} [\vec{a}_\alpha \cdot (\vec{d}_I - \vec{d}_I) + \vec{u}_\alpha \cdot \vec{d}_I] \\
+ \frac{1}{2} z^K [\vec{a}_{K,\alpha} \cdot \vec{d}_I - \vec{a}_{K,\alpha} \cdot \vec{d}_I] + \frac{1}{2} u^A \vec{c}_{A,\alpha} \cdot \vec{d}_I \\
(\varepsilon_{33})_I = \vec{d}_I \cdot \vec{d}_I - \vec{d}_I \cdot \vec{d}_I \\
(7.62)

It should be noted that:

\[ \vec{d}_I^* = \vec{d}_I + \vec{d}_I^* - \vec{d}_I = \vec{d}_I + \vec{e}_I \]
where $\ddot{e}_I$ is very small and its higher order can be neglected. Therefore

$$\ddot{d}_I \cdot \ddot{d}_I = (\ddot{d}_I + \ddot{e}_I) \cdot (\ddot{d}_I + \ddot{e}_I) \approx \ddot{d}_I \cdot \ddot{d}_I + 2\ddot{d}_I \cdot \ddot{e}_I$$

(7.63)

so that

$$(\varepsilon_{33})_I = \frac{1}{2} [d_I^* \cdot d_I^* - \ddot{d}_I \cdot \ddot{d}_I] \approx \ddot{d}_I \cdot \ddot{e}_I$$

as prescribed by 7.62.

Similarly, equilibrium equations can be written as:

$$- [N^{\alpha\beta}(\ddot{a}_a) + S^{\alpha I}d_I] \alpha = 0 \ddot{P}$$
$$- [M^{\alpha\beta}(\ddot{a}_a) + Q^{\beta KL}d_K] \beta + S^{\alpha I}\ddot{a}_a + \ddot{P}^I = 0 \ddot{z}^I 0 \ddot{P}$$
$$- [N^{\alpha\beta}(\ddot{a}_a)w^A + S^{\beta I}d_I w^A] \beta + \lambda^A = 0 \ddot{P}w^A$$

(7.64)

Using equation (7.62), the components of strain tensor in the Cartesian coordinate system, $xyz$, can be written as:

$$\varepsilon_{11} = \ddot{a}_1 \cdot \ddot{u}_{1,1} + \ddot{z}^K(\ddot{a}_1 \cdot \ddot{d}_{K,1}) + w^1 \ddot{a} \cdot \ddot{e}_{1,1}$$
$$\varepsilon_{22} = \varepsilon_{12} = \varepsilon_{21} = 0$$
$$(\varepsilon_{33})_I = \ddot{d}_I \cdot \ddot{d}_I - 1$$
$$(\varepsilon_{13})_I = \frac{1}{2} [\ddot{a}_1 \cdot (\ddot{d}_I^* - \ddot{d}_I) + \ddot{u}_{1,1} \cdot \ddot{d}_I + w^1 \ddot{e}_{1,1} \cdot \ddot{d}_I + \ddot{z}^K \ddot{d}_{K,1} \cdot \ddot{d}_I]$$
$$(\varepsilon_{23})_I = (\varepsilon_{32})_I = 0 \quad K, I = 1, 2$$

(7.65)

For a linear elastic material, the stress components are obtained as:

$$\sigma^{11} = D\varepsilon_{11} + B\varepsilon_{33}$$
$$\sigma^{22} = B(\varepsilon_{11} + \varepsilon_{33})$$
$$\sigma^{33} = D\varepsilon_{33} + B\varepsilon_{11}$$
where

\[
\begin{align*}
D &= \frac{E(1 - \nu)}{(1 + \nu)(1 - 2\nu)} \\
B &= \frac{E\nu}{(1 + \nu)(1 - 2\nu)} \\
H &= \frac{E}{(1 + \nu)} \quad (7.67)
\end{align*}
\]

\(E\) and \(\nu\) are Young’s modulus and Poisson’s ratio respectively.

Expanding the equilibrium equation, equation (7.64), the equilibrium equation for a beam element can be written as:

\[
\begin{align*}
-\left[N^{11}\bar{a}_1\right]_1 - \left[S^{11}\bar{d}_1\right]_1 - \left[S^{12}\bar{d}_2\right]_1 &= 0 \bar{P} \\
-\left[M^{11}\bar{a}_1\right]_1 - \left[Q^{11}\bar{d}_1\right]_1 - \left[Q^{12}\bar{d}_2\right]_1 + S^{11}\bar{a}_1 + \bar{P}_1 &= 0 \bar{P} h \\
-\left[M^{12}\bar{a}_1\right]_1 - \left[Q^{12}\bar{d}_2\right]_1 + S^{12}\bar{a}_1 + \bar{P}_2 &= 0 \bar{P} h \\
-\left[N^{11}\bar{a}_1w^1\right]_1 - \left[S^{12}\bar{d}_2w^1\right]_1 + \lambda^1 &= 0 \bar{P} w^1 \quad (7.68)
\end{align*}
\]

Using equations (7.65) and (7.66), stress resultant components are obtained as:

\[
\begin{align*}
N^{11} &= 3Dh\bar{a}_1 \cdot \bar{u}_{1,1} + 2Bh(\bar{d}_1 \cdot \bar{r}_1^e - 1) + Dh^2\bar{a}_1 \cdot \bar{r}_{1,1} + \frac{Dh^2}{2} \bar{a}_1 \cdot \bar{r}_{2,1} \\
&\quad + Dhw^1\bar{a}_1 \cdot \bar{e}_{1,1}^e + Bh(\bar{d}_2 \cdot \bar{r}_2^e - 1) \\
N^{12} &= N^{21} = 0 \\
M^{111} &= \frac{5Dh^3}{3} \bar{a}_1 \cdot \bar{d}_{1,1} + Dh^2\bar{a}_1 \cdot \bar{u}_{1,1} + \frac{Dh^3}{2} \bar{a}_1 \cdot \bar{d}_{2,1} + Dh^2w^1\bar{a}_1 \cdot \bar{e}_{1,1}^e \\
&\quad + Bh^2(\bar{d}_2 \cdot \bar{r}_2^e - 1)
\end{align*}
\]

\[
\sigma^{12} = \sigma^{21} = 0
\]

\[
(\sigma^{13})_l = H(\epsilon^{13})_l
\]

\[
(\sigma^{23})_l = (\sigma^{32})_l = 0 \quad (7.66)
\]
Substituting from equations (7.69) into equation (7.68) and considering the following notations

\[\ddot{u} = u\ddot{t} + w\ddot{k}\]
\[\ddot{d}_1 = d_{1x}\ddot{t} + d_{1x}\ddot{k}\]
\[\ddot{d}_2 = d_{2x}\ddot{t} + d_{2x}\ddot{k}\]
\[\ddot{c}_1 = c_{x}\ddot{t} + c_{x}\ddot{k}\]
\[
\lambda^1 = \lambda_x \tilde{t} + \lambda_x \tilde{k}
\] (7.70)

Equilibrium equations, in terms of displacements, can be written as:

\[
-\left[3Dh \frac{d^2 u}{dx^2} + Dh^2 \frac{d^2 d_{1x}}{dx^2} + \frac{Dh^2}{2} \frac{d^2 d_{2x}}{dx^2} + Dh \frac{d^2 c_x}{dx^2} + 2Bh \frac{d(d_{1x})}{dx} + Bh \frac{d(d_{2x})}{dx}\right] = 0
\]

\[
-\left[\frac{3Hh}{2} \frac{d^2 w}{dx^2} + \frac{Hh^2}{2} \frac{d^2 d_{1x}}{dx^2} + \frac{Hh^2}{4} \frac{d^2 d_{2x}}{dx^2} + \frac{Hh}{2} \frac{d^2 c_x}{dx^2} + \frac{Hh}{2} \frac{d(d_{1x})}{dx} + \frac{Hh}{2} \frac{d(d_{2x})}{dx}\right] = -P0
\]

\[
-\left[\frac{Dh}{2} \frac{d^2 u}{dx^2} + \frac{Dh}{2} \frac{d^2 d_{1x}}{dx^2} + \frac{Dh}{2} \frac{d^2 d_{2x}}{dx^2} + \frac{DH}{4} \frac{d^2 c_x}{dx^2} + \frac{DH}{2} \frac{d(d_{1x})}{dx} + \frac{Dh}{2} \frac{d(d_{2x})}{dx}\right] = -hP0
\]

\[
-\left[\frac{Dh}{2} \frac{d^2 u}{dx^2} + \frac{Dh}{2} \frac{d^2 d_{1x}}{dx^2} + \frac{Dh}{2} \frac{d^2 d_{2x}}{dx^2} + \frac{DH}{4} \frac{d^2 c_x}{dx^2} + \frac{DH}{2} \frac{d(d_{1x})}{dx} + \frac{Dh}{2} \frac{d(d_{2x})}{dx}\right] = -hP0
\]

\[
-\left[\frac{Dh}{2} \frac{d^2 d_{1x}}{dx^2} + \frac{Hh^2}{2} \frac{d^2 d_{1x}}{dx^2} + \frac{Hh^2}{4} \frac{d^2 d_{2x}}{dx^2} + \frac{Hh}{2} \frac{d^2 c_x}{dx^2} + \frac{Hh}{2} \frac{d(d_{1x})}{dx} + \frac{Hh}{2} \frac{d(d_{2x})}{dx}\right] = -P0
\]

\[
-\left[\frac{Dh}{2} \frac{d^2 d_{1x}}{dx^2} + \frac{Hh^2}{2} \frac{d^2 d_{1x}}{dx^2} + \frac{Hh^2}{4} \frac{d^2 d_{2x}}{dx^2} + \frac{Hh}{2} \frac{d^2 c_x}{dx^2} + \frac{Hh}{2} \frac{d(d_{1x})}{dx} + \frac{Hh}{2} \frac{d(d_{2x})}{dx}\right] = -P0
\] (7.71)
It should be noted that for a perfect bond between two layers, the conjugate director, $\mathbf{e}_1$, is zero everywhere along the interface. In this case the first six equations of (7.71) are enough to solve for six unknowns, $u, w, d_{1x}, d_{1z}, d_{2z}$, and $d_{2z}$. Lagrange multipliers $\lambda_x$ and $\lambda_z$ can be then calculated from the last two equations of (7.71). For a non-perfect bond, Lagrange multipliers $\lambda_x$ and $\lambda_z$ are zero, therefore, the whole set of equation (7.71) is enough to solve for eight unknowns, $u, w, d_{1x}, d_{1z}, d_{2z}, d_{2z}, c_z$, and $c_z$.

### 7.7 Numerical example

In this section, the feasibility and accuracy of the proposed theory is examined by solving an example of a layered beam which has a known discontinuity at its interfaces. The finite element procedure to obtain element stiffness matrix and force vector is outlined in Appendix I, and the numerical results are presented here for further discussion. As an example, a rectangular two-layered beam with a length of $l=1 \text{ (m)}$, a total thickness of $3h=0.03 \text{ (m)}$, and a unit width as shown in Figure 7.8 is considered. Layer one possesses the reference surface at $y=0$ and its thickness is $2h=0.02 \text{ (m)}$. At $y=h$ the second layer is attached to the first layer. Two cases are considered:

- Case I: it is assumed that the layers are connected to each other by a perfect bond
- Case II: it is assumed that there is a discontinuity surface between the two layers at:
  
  $$ y = h $$
Material properties are: \( E = 2.1 \times 10^{11} \) (Pa) (Young's modulus), \( \nu = 0.3 \) (Poisson's ratio). The applied load \( \vec{P}_0 = -1000\bar{k} \) (N/m) is a uniform load acting at \( y = 0.02 \) (m) (upper boundary of the layered beam) and boundary conditions are assumed to be clamped at \( x = 0, l \). With the above loading and boundary conditions, the interface between the layers will experience compressive stresses. To compare the results of the present theory, the same problem is solved by using a commercial finite element package, ANSYS 5.2, which utilizes a two-dimensional theory of elasticity (one dimension higher than the dimension of the proposed beam element). Figure 7.9 shows the two-dimensional finite element mesh of this example which was produced by ANSYS 5.2. To impose the no-penetration condition at the surface of discontinuity, the contact elements, CONTACT48, with \( K_N = 100E \) are used in this
Figure 7.10 shows the variation of the longitudinal displacement, \( u \), of the reference surface \((y=0)\) and the top surface \((y=2h)\) of the beam. As it can be seen from this figure, excellent agreement is obtained between the results of the present theory and the results of the two-dimensional theory of elasticity obtained by ANSYS5.2.

Figure 7.11 displays the results of \( u \) as a function of \( x \) at the target and contact surfaces \((y=h)\). Again agreement between the results of the present theory and those calculated by ANSYS5.2 can be observed from this figure.

Variations of the normal displacement \( w \) as a function of \( x \) at the reference, target, contact, and top surfaces of the layered beam are shown in Figures 7.12 and 7.13. The maximum difference between the results of the present theory and those of the two-dimensional theory of elasticity obtained by ANSYS5.2 is about 4% which is very small and acceptable in engineering practice.

Three curves in Figure 7.14 show the change of displacement distribution \( u \) at the target and contact surfaces together with the variation of the \( x \) component of conjugate director \( c_x \). At each point along the beam, \( c_x = u_{co} - u_{ta} \), where \( u_{co} \) and \( u_{ta} \) are displacement components at contact and target surface, respectively.

The \( x \) components of the director vectors \( d_1^x \) and \( d_2^x \) are shown in Figure 7.15 as a function of \( x \). There is no difference between these two variables in layer one and layer two, as it can be seen from this figure.

If we assume a perfect bond between layer one and two, there is no relative displacement between contact and target surfaces. From elementary strength of materials one can very easily verify that the normal internal force per unit length at the interface \((y=h)\) is constant and is equal to \( 20/27P_0 \) and since \( P_0 = -1000 \text{ (N/m)} \).
Figure 7.10: Longitudinal displacement distributions of the reference and top surfaces.

Figure 7.11: Longitudinal displacement distributions of the contact and target surfaces.
Figure 7.12: Normal displacement distributions of the reference and top surfaces.

Figure 7.13: Normal displacement distributions of the contact and target surfaces.
Figure 7.14: Displacement distributions in x direction.

Figure 7.15: Distributions of the director vectors in x direction.
therefore, \( \frac{20}{27} P_0 = -740.7 \) (N/m). Having this in mind, we calculated Lagrange multiplier \( \lambda_z \) along the interface by using the finite difference method. Figure 7.16 shows the variation of the Lagrange multiplier \( \lambda_z \). It is interesting to see that except for a short distance near the edges of the beam, the value of the Lagrange multiplier \( \lambda_z \) is constant and equal to -740.4.

Now consider that the applied load \( \bar{P}_0 = 1000k \) (N/m) is a uniform load acting at \( y=0.02 \) (m) (upper boundary of the beam) and the boundary conditions and geometry are as before. In this case the interface between two layers will experience tensile stresses. Therefore, the surfaces of discontinuity will be open. Figures 7.17 and 7.18 show the variation of the longitudinal and normal displacements of the top surface (\( y=h \)), respectively. The maximum difference between the results of the present theory and those of the two-dimensional theory of elasticity is about 2%.
Figure 7.17: Longitudinal displacement distributions of the top surfaces.

Figure 7.18: Normal displacement distributions of the top surfaces.
Figure 7.19: Displacement distributions in z direction.

Distributions of the normal displacement $w$ of the reference surface and the top surface together with conjugate director $c_z$ are shown in Figure 7.19. It is observed that the displacement $w$ at the top surface is equal to the summation of the displacement $w$ at the reference surface and conjugate director $c_z$. In other words, in this case, extension of the directors $d_{1z}$ and $d_{2z}$ does not play an important role in the deformation of the beam. Similar to the case of compressive loading, displacement distributions of the reference, target, contact, and the top surfaces are shown in Figures 7.20 - 7.22.

Finally, variations of directors $d_{1x}^2$ and $d_{2x}^2$ in the x direction are shown in Figure 7.23. Unlike the case of compressive loading (Figure 7.16), $d_{1x}$ and $d_{2x}$ are different at the discontinuity surface.
Figure 7.20: Displacement distributions in x direction.

Figure 7.21: Longitudinal displacement distributions of the reference, target, contact, and top surfaces.
Figure 7.22: Normal displacement distributions of the reference, target, contact, and top surfaces.

Figure 7.23: Distributions of the director vectors in x direction.
7.8 Conclusion

This chapter describes the development of a non-linear theory for multilayered shells, made of homogeneous and non-homogeneous elastic materials. By introducing conjugate directors, $\tilde{\varepsilon}_A$, delamination or surface of discontinuity, in composite laminates subjected to combined in-plane, lateral, and bending loading is analyzed. The entire treatment is relatively compact and simple, and analogous to the single-layer treatment of the classical shell theory. The approach adopted is purely kinematical. The displacement field is assumed to belong to a certain finite-parameter family of functions while the exact three-dimensional kinematic relations and constitutive equations are used.

To validate the feasibility and the accuracy of the developed theory, numerical examples are provided. Contact theory is used to prevent interpenetration between the faces of a delamination. The results of the present theory are compared with the results obtained from two dimensional finite element analysis and excellent agreement is observed.
Chapter 8

Concluding remarks

Great is the art of beginning, but greater is the art of ending. - H. W. Longfellow

8.1 Introduction

Fractures in functionally gradient materials under small and large deformations were presented in this monograph. Dynamic, quasi-static, and static responses of a two dimensional multilayer plane structure under mechanical and thermal loadings were studied. Emphasis was given to both analytical and numerical solutions. To prevent penetration of materials under compressive loads and to consider the effect of friction on sliding of the crack faces, a simple numerical method was devised and discussed in detail. Some results were compared with the solutions available in references. This study comprised some interesting phenomenons in functionally gradient materials which were not discussed in previous literature.

8.2 Summary

The main purposes of this dissertation were:

- To provide a theoretical solution to the singularity behavior of a finite crack in functionally gradient material under anti-plane shear impact load.
• To devise an efficient and simple iterative technique to reduce the computation time required for solving frictional contact problems of a crack under small deformations.

• To utilize the above algorithm for solving frictional contact problems of cracks in functionally gradient materials under combined mechanical and thermal, steady-state, and transient loads.

• To study the variation of stress intensity factors and energy release rates due to the change in the geometrical and material properties of the cracked body as well as the interfacial layer.

• To provide a nonlinear theory on the statics of multilayered shells, including transverse effects and interface delamination of the general shape.

8.3 Conclusion

In the theoretical section, a finite crack subjected to an anti-plane shear impact load in functionally gradient material was studied. The elastic properties of the interfacial layer were assumed to vary continuously between those of two dissimilar homogeneous bonding layers. Laplace and Fourier transforms were applied to reduce this mixed boundary value problem to a system of dual integral equations which in turn would be reduced to a standard Fredholm integral equation of the second kind. To find the dynamic stress intensity factor, the Laplace inversion was performed numerically by using Gauss-quadrature and Jacobi polynomials. In general, the dynamic stress intensity factor was found to be a function of the crack length, location
of the crack in the interfacial layer, and material properties of the surrounding layers as well as interfacial layer. Graphs of dynamic stress intensity factor versus time showed that SIF reached a peak after a very short time and declined toward the steady-state solution as time increased. With an increase in the crack length or in the ratio of the shear modulus of the bounding layers, the dynamic stress intensity factor increased. Finally, as the crack location approached the stiffer bounding layer, the stress intensity factor increased depending on the ratio of the shear modulus of the bounding layers.

To achieve the second objective, a simple and efficient iterative finite element technique was introduced. Signorini's condition of no penetration, was added to the internal virtual work of the system by using a Lagrange multiplier. With the existence of friction, sliding of the surfaces of the crack is subjected to the chosen friction law. Coulomb's dry-friction law was used in this monograph. By introducing displacement vector \( \beta \), the author provided a very simple iterative method to handle frictional problem of cracks under small deformations. Three distinct characteristics of this iterative method were:

- (1) Jacobean matrix \( \text{Jac} \) was calculated once and it remained constant during the iterations.

- (2) Jacobean matrix \( \text{Jac} \) was symmetrical and needed only half of the storage space which is very crucial in a large scale structural analysis.

- (3) The elements of incremental displacement vector \( \beta \) were changing from zero to one \( 0 \leq \beta(J) \leq 1 \). Since elements of \( \beta \) were changing monotonically, it was
guaranteed that after a maximum of \( n_i = 1/d\beta \) iterations, convergence would be reached. (\( d\beta \) is the user defined incremental step size.)

Such a computational efficiency rendered the proposed algorithm highly suitable for analyzing the frictional contact problems of cracks under compressive loads.

Like other new ideas, the proposed algorithm had to be tested. Since, in the author's knowledge, there was no work done on functionally gradient materials which provided the contact and frictional analysis, the above algorithm was tested for homogeneous materials. At first, a fully closed crack was considered and solved by applying the present method. The same problem was solved by using a commercial finite element package, ANSYS5.2, which applied a penalty method with implicit contact constraint iterations. Excellent agreement was obtained between the results of the present algorithm and those of the ANSYS5.2. This example showed that the Lagrange multipliers which were used to add the constraint to the total potential energy of the system, were equal to the normal and frictional forces. Next, a partially closed cracked plate subjected to a pair of moments was considered. The open part of the crack and the normalized stress intensity factor were calculated. There was good agreement between the results of this study and those of existing works.

Having examined the feasibility and accuracy of the above numerical algorithm, several examples were solved and the effect of different parameters on the stress intensity factor and energy release rates were studied. In general, with an increase in the coefficient of the friction, stress intensity factors and energy release rates decreased. On the other hand, the stress intensity factor increased as thickness of the interfacial layer went to zero. This means that the smoother the transition between
bounding layers, the lesser the stress intensity factor is. The normalized stress intensity factor was calculated for different distributions of the material properties of the interfacial layer. The results suggested that it was possible to reach a better design of interfacial layers by properly changing the material properties of the interfacial layer.

Frictional contact problems of cracks in functionally gradient materials subjected to uniform temperature distribution were considered to study a typical electronic package. The electronic package was modelled by a trilayer configuration with the non-homogeneous material as an interface layer. Stress and displacement distributions were obtained and stress intensity factors were calculated. The distribution of the relative displacements in the tangential direction along the crack faces very clearly showed the advantage of using functionally gradient materials as an interfacial layer. Finally, Transient and steady-state response of a central crack subjected to the thermal and mechanical loads were investigated. Geometry and boundary conditions of the problem was such that the surfaces of the crack were partially closed. The temperature, displacement, and time dependent stress intensity factor distributions were calculated. These results showed that crack surfaces were partially closed and that the open part of the crack in functionally gradient material was smaller than that of the homogeneous material. The variation of the normalized stress intensity factors for mode I and mode II were plotted versus time. In contrary to mode I, the normalized stress intensity factor in mode II decreased as time increased. From these results, it was considered that a functionally gradient material was more suitable to use as an interfacial layer than ordinary homogeneous materials because of its ability of reducing stress intensity factors.
The development of a nonlinear theory of multilayered shells, made of homogeneous and non-homogeneous elastic layers, were described. By introducing conjugate directors, $\bar{\varepsilon}_A$, delamination or surface of discontinuity, in composite laminates subjected to combined in-plane, lateral, and bending loading was analyzed. The general nonlinear equilibrium equations and boundary conditions expressed in terms of the stress resultants and unknown directors. Furthermore, constitutive relations for so-called linear elastic Hookean material were adopted and stress resultants were calculated for this kind of material. Finally, in order to comprehend more easily the theory which was described, a multilayered beam which was laterally loaded was considered. The feasibility and accuracy of the proposed theory was examined by providing numerical results for this layered beam which had surface of discontinuity. Contact theory was used to prevent interpenetration between the faces of a delamination. The same problem was solved by using a commercial finite element package, ANSYS5.2, which utilized a two-dimensional theory of elasticity (one dimension higher than the dimension of the proposed beam element). Excellent agreement was observed between these two results.

8.4 Recommendations for future work

The extent of the exploratory nature of this study suggests the following additional future work:

- Verification of the numerical results obtained for functionally gradient materials by performing some experimental work.
• Modification of the proposed algorithm to solve contact problems of cracked plates and shells under compressive mechanical and thermal loads.

• Extension of the method to handle other frictional laws rather than Coulomb's friction law.

• Investigation of the response of functionally gradient materials to thermal shock in the couple thermoelasticity regime.

• Extension of the presented shell theory to solve vibration and thermal problems of shells having the surface of discontinuity.
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Appendix A

One-dimensional second-order system of differential equations

Here we consider the finite element formulation of the one-dimensional second-order differential equations that arise in the linear elastic bending of beams, discussed in chapter 7. Variational method is adopted and for the sake of simplicity, linear shape functions are considered for all unknown functions as follows:

\[ V = N_i V_i \quad i = 1, 2 \]  \hspace{1cm} (A.1)

where \( V_i \) is the value of the function \( V \) at nodal point \( i \) and \( N_i \) is shape function defined as:

\[
N_1 = \frac{x_2 - x}{x_2 - x_1} \\
N_2 = \frac{x - x_1}{x_2 - x_1}
\]  \hspace{1cm} (A.2)

where \( x_1 \) and \( x_2 \) are shown in Figure A.1. To apply variational method for solving equation 7.71, the following integrals are needed:

\[
\int_{x_1}^{x_2} \left( \frac{dV}{dx} \right) \delta V \, dx = \frac{-1}{2} [V_1 \delta V_1 + V_1 \delta V_2 - V_2 \delta V_1 - V_2 \delta V_2]
\]

\[
\int_{x_1}^{x_2} V \delta V \, dx = \frac{1}{3} x_1 \delta V_1 + x_1 V_1 \delta V_2 + x_2 V_2 \delta V_1 + \frac{1}{3} x_2 V_2 \delta V_2
\]

\[
\int_{x_1}^{x_2} \delta V \, dx = \frac{1}{2} \delta V_1 + \frac{1}{2} \delta V_2
\]

\[
n = \frac{1}{l_i} [(x_2 + x_1)\left(\frac{x_2^2 - x_1^2}{2}\right) - x_2 x_1 l_i - \frac{x_2^3 - x_1^3}{3}] \]  \hspace{1cm} (A.3)
where \( l_i \) is the length of the \( i^{th} \) element. Using equations (A.1-A.3) and variational procedure, equation 7.71, for an arbitrary element, will reduce to:

\[
[K^e]\{\bar{U}\} = \{F^e\}
\]  

(A.4)

where \([K^e]\), \(\{\bar{U}\}\), and \(\{F^e\}\) are element stiffness matrix, displacement vector, and force vector, respectively, and can be written as:

\[
[K^e] = \begin{bmatrix}
K^e_{11} & K^e_{12} \\
K^e_{21} & K^e_{22}
\end{bmatrix}
\]

\[
[K^e_{11}] = \begin{bmatrix}
\frac{3Dh}{l_i} A & 0 & \frac{Dh^2}{l_i} A & BhG \\
0 & \frac{3Hh}{2l_i} A & \frac{Hh}{2} G & \frac{Hh^3}{2l_i} A \\
\frac{Dh^3}{l_i} A & \frac{Hh}{2} G^T & \frac{5Dh^3}{3l_i} A + HhK_{11} & 0 \\
BhG^T & \frac{Hh^2}{2l_i} A & 0 & \frac{5Hh^3}{6l_i} A + 2DhK_{11}
\end{bmatrix}
\]
It should be noted that the element stiffness matrix $K^e$ is a symmetric matrix.

\[ \{ \ddot{U} \}^T = \{ \ddot{U}_1 \quad \ddot{U}_2 \} \]

\{\ddot{U}_1\} = \{ u_1 \quad u_2 \quad w_1 \quad w_2 \quad d_{1x1} \quad d_{1x2} \quad d_{1z1} \quad d_{1z2} \} \\
\{\ddot{U}_2\} = \{ d_{2x1} \quad d_{2x2} \quad d_{2z1} \quad d_{2z2} \quad c_{x1} \quad c_{x2} \quad c_{z1} \quad c_{z2} \} \\

(A.5)
\begin{align*}
F_1^* &= -3Bh \\
F_7^* &= 3Bh \\
F_9^* &= -P_0 \frac{l_i}{2} \\
F_{10}^* &= -P_0 \frac{l_i}{2} \\
F_5^* &= -Bh^2 \\
F_6^* &= Bh^2 \\
F_7^* &= -hP_0 \frac{l_i}{2} + Dhl_i \\
F_8^* &= -hP_0 \frac{l_i}{2} + Dhl_i \\
F_9^* &= -\frac{Bh^2}{2} \\
F_{10}^* &= -\frac{Bh^2}{2} \\
F_{11}^* &= -hP_0 \frac{l_i}{2} + Dh \frac{l_i}{2} \\
F_{12}^* &= -hP_0 \frac{l_i}{2} + Dh \frac{l_i}{2} \\
F_{13}^* &= -Bh \\
F_{14}^* &= Bh \\
F_{15}^* &= -P_0 w^1 \frac{l_i}{2} \\
F_{15}^* &= -P_0 w^1 \frac{l_i}{2} \\
\end{align*}

(A.7)

where B, D, and H are defined in 7.67. It should be noted that in equation A.7 the notations \([ \ ]_{\alpha_1}\) and \([ \ ]_{\alpha_2}\) mean the value of function \([ \ ]\) at the point \(z_1\) or \(z_2\), respectively.