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Calculus Reform from a Constructivist Perspective

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Abstract

The calculus reform movement has grown to large proportions. More than 500 mathematics departments in the United States are currently implementing some level of calculus reform, affecting approximately 1/3 of all students enrolled in calculus. (Ganter, 1997)

Educators involved in the movement are looking for ways to improve undergraduate calculus instruction. With the traditional way calculus is taught, memorization and repeated practice of template problems is often the route to a passing grade. Reform activists are looking for ways to help students achieve higher levels of conceptual understanding.

In the following chapters, I will present a history of calculus reform; a philosophy of mathematics leading to a constructivist framework for mathematics education; a more precise definition of what reform is and how it fits the constructivist perspective; a summary of current efforts and research; and some ideas for the future based on the constructivist ideas.

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1. Introduction

A History of Disappointment

I have been a student and teacher of calculus for seven years. Since the beginning, I have believed there must be a better way. It all started in grade twelve when I first took calculus by distance education. One other student and I "met" with the teacher twice a week over a speaker phone during lunch hour. He talked, we listened. If we had any questions, we asked them back over the phone. Not only were the lectures dry, but while we were struggling to understand the fuzzy voice coming from the brown speaker box, students in the hallway were loudly enjoying their lunch hour. The entire course was lectured to us in this way, with one weekly visit by the teacher to our school. I did not understand any of the concepts, but I did learn how to differentiate any function and calculate several limits. Lesson learned: It takes more than just lecturing to teach calculus. It takes active involvement on the part of the students. A suitable environment also helps.

Not only were the class sessions rather disappointing, but resources were also limited. In fact, human resources were practically non-existent. One time I needed help with a concept. First I asked around to see if anyone in my school knew anything about calculus. One teacher pointed me to the distance education administration office to a woman there who supposedly knew calculus. When I approached her, she told me that it was not her responsibility to help me, and turned away. Disappointed, I then tried the next resource -- phoning the teacher from the other town, whose voice I was so familiar with from class. Have you ever tried to do mathematics over the phone? I can tell you right now that it is not easy. In fact, it did not help me at all. I was very close to giving up on trying to understand calculus.

The next day, my science teacher approached me. He had taken calculus in university and was willing to help me whenever I needed him. Someone had finally intervened. I was not pleased with the distance education concept, but I will never forget Mr. Friesen. Lesson learned: It takes caring, dedicated humans to teach calculus.

The next year I entered first year university, where I enrolled in Calculus I and II. There were many similarities between Calculus I and my grade 12 course with regards to content and teaching. In particular, the only difference seemed to be that I could actually see the lecturer. I still did not understand how or why a derivative could do all the things it did. I only knew how to find and use it for about five different types of problems. I thought I was so good at mathematics because my marks were fantastic. Little did I know that there is more to mathematics then memorizing formulas. After whizzing through first year calculus, I went on to major in mathematics. It took me four years to fully understand the calculus concepts, and to realize that understanding the concepts in any mathematical subject is the only way to get anywhere. Lesson learned: You can't just memorize math!

I started teaching calculus two years ago. I followed the same patterns as my calculus teachers had -- lecturing of methods. I still felt that there had to be a better way. It was obvious from students' questions and test results that they did not understand what they were doing, and did not want to. They were concerned with getting a good grade and getting out. That is when I found out about the calculus reform movement happening in North America, and in particular, the United States. I had to investigate.

I have been reading and learning about the movement ever since. Instructors are trying to solve the problems with calculus instruction that we have all known existed from the beginning. What is this reform movement about? Have they been successful in achieving their goals? Could they do better? Over the past two years I have tried to answer those questions and others for myself. This thesis represents my understanding and assessment of the calculus reform movement.

2. A Historical Review

Calculus reform efforts have roots dating back to the 1950's and 60's, for it was then that calculus achieved status as a freshman college course. Before then, it was the norm for collegiate mathematics to start with a year-long course in algebra and trigonometry. Calculus with analytic geometry was a sophomore level course. At that time, an engineering degree required five years of study.

According to Tucker and Leitzel (Tucker and Leitzel, 1995), it was in 1953 that the Mathematical Association of America (MAA) created the Committee on the Undergraduate Program (CUP) in response to "widespread interest in revision of the elementary courses in mathematics." Some key factors which drove the modernization of undergraduate mathematics reform at that time were:

1. The undergraduate curriculum had lagged behind the new analytical and algebraic framework for mathematics developed in the first half of the century.
2. Use of mathematics in engineering and science had been greatly enhanced from technological advances initially spurred by World War II and later by the advent of digital computers.
3. Society gave demand for large scientific enterprise in support of the Cold War.
4. There were large numbers of solidly trained, well-motivated potential mathematics majors (5% of the entering freshmen in 1965 expressed an interest in becoming mathematics majors).

In response to these factors, CUP developed plans for a course called Universal Mathematics, a year-long introduction to college mathematics for freshmen. Because Universal Mathematics had several goals in common with the current calculus reform movement, I will briefly describe Part 1 of the course, "Functions and Limits." This part of the course was an introduction to calculus. Topics included co-ordinate systems, lines, vectors, proportion, measurement theory, informal max/min problems, divided differences, polynomial differentiation with extensive applications, integrals, and logarithmic and exponential functions. Part 2 of the course, was Mathematics of Sets.

The text from Part 1 of Universal Mathematics had a dual style, giving intuitive and formal presentations of each topic on opposing pages. Both the intuitive and formal approaches were revolutionary for their time. The intuitive approach included conceptual visualization, applied interpretations, and both discrete and numerical components. The formal approach was more rigorous than even an honors calculus course today. The course also gave extensive applications. However, the Functions and Limits course never made it beyond a preliminary text. This was partly due to pressure from engineering and physics departments, who wanted students strong in techniques of calculus. Thus, the techniques course became the standard for freshmen in those disciplines instead. By the late 1950's, the United States was engaged in the technological Cold-War battle for supremacy in space, and increased production of engineers was a high priority in the national agenda for academia. For apparent reasons, the techniques approach to calculus became the winner in this period. (Tucker and Leitzel, 1995)

However, there was another motivating force in the 1950's for undergraduate mathematics reform. The undergraduate mathematics program had lagged behind research developments in the discipline, making the transition from undergraduate to graduate studies very difficult. In fact, many first year graduate students dropped out of their program. The rapidly increasing enrolment in higher education created concern in the mathematics community about producing sufficient numbers of Ph.D.s in mathematics. In response to this problem, around 1960, the MAA charged CUPM (now the Committee on the Undergraduate Program in Mathematics) to plan a modernized curriculum for undergraduate mathematics. This plan included more theory and course work in algebra and contemporary analysis so as to better prepare mathematics majors for graduate work. (Tucker and Leitzel, 1995)

During the 1960's, extensive curriculum studies on undergraduate mathematics were undertaken by the MAA with liberal support from the National Science Foundation. The NSF funded establishment of summer institutes to upgrade the mathematical knowledge of collegiate mathematics faculty, few of whom had a Ph.D. in 1960. Universities introduced honors calculus courses which reflected the formal approach in Universal Mathematics. With the techniques approach being used for engineers and the formal approach for mathematics majors, the interpretative approach of Universal Mathematics was almost totally neglected.

Two other (failed) reforms of the 1960's are given by Flashman (Flashman,1994). The first of these was the use of computers as an aid for instruction. The focus was on the computational power of computers, with programming an essential element of this attempted reform. One reason this reform failed was the time taken from calculus instruction for students to learn computer programming. The other effort of the 1960's reform movement Flashman (Flashman, 1994) mentions was the attempt to connect calculus with applied probability and statistics. The rationale for this was that the majority of students taking calculus would encounter concepts from probability and statistics more frequently in life situations than other calculus topics. This reform attempt did not succeed for reasons of time economics. It was generally believed then that to teach probability concepts required a discussion of discrete combinatorial probability, which would take too much time out of the course.

Although the reform of undergraduate mathematics in the 1960's may have gone too far towards theory and preparation for graduate study, it is still impressive in the dramatic change in curriculum that resulted. For example, in less than a decade, the theory of vector spaces moved from a graduate-level course to a second year undergraduate course.

However, universities and colleges soon discovered many problems with the "new calculus" focusing on techniques and formal approaches. The comparatively high standards led to high failure rates, even in special sections created for the client disciplines. Large sections and overworked faculty and TA's led to elimination of grading homework at many universities. Introduction to calculus in high school set the stage for superficial learning of the techniques with little understanding. Foreign-born graduate students increased in numbers and their language and cultural differences often provided a source for student complaints. All of these developments reinforced students' view of mathematics as an arbitrary hurdle. Also, instructors were unhappy about the content and treatment of calculus. According to Schoenfeld (Schoenfeld, 1995), many felt that the stereotypical "mimicry course" in calculus was superficial; the range of mathematics ability expected of students was too narrow; and calculus was of little intrinsic interest to and receiving little attention from tenured faculty.

One might ask why a reform movement was not started in the 1970's when problems like these first started to emerge. There are several reasons, foremost of which was the lack of attention to calculus instruction in the mathematics profession. At major research universities, senior faculty had limited involvement with calculus instruction. Even more influential, the advent of federal research grants fostered an environment in which universities were more concerned about recruiting research faculty and making them, not the students, comfortable.

According to Tucker and Leitzel (Tucker and Leitzel, 1995), between 1970 and 1980, the number of mathematics majors dropped by 60%. The hurdle for the mathematics community was to finally face the truth -- calculus instruction had serious problems that needed to be addressed. At the 1985 Joint Mathematics Meetings, Ron Douglas and Steve Mauere of the Sloan Foundation arranged a

The conference had workshops on content, instructional methods, and implementation. Participants in the content and methods workshops agreed that greater conceptual understanding, developed through a variety of approaches, should be a guiding theme. Some suggested approaches include using applications, numerical explorations with computers and calculators, and traditional algebraic methods. For this to happen, some topics would have to be omitted or de-emphasized, and in fact, integration techniques, related rates, l'Hopital's rule, and infinite series have been downplayed by many reform texts.

The participants in the workshop on methods supported the notion that *how* calculus is taught is as important as *what* is taught. Their report called for activities to make students active learners and to "help them develop the ability to apply what they have learned with flexibility and resourcefulness." The report encouraged co-operative learning, class discussions, open-ended problems, computer laboratories, and using writing.

The topic of the third section of the workshop, implementation or the politics of change, was key in determining the success of the reform efforts. As Douglas (Douglas, 1995) puts it, "The participants recognized that failing to deal with this issue had doomed many promising past efforts by keeping them small and isolated." The result of the implementation workshop was a plan in which new calculus courses would be independently developed at a carefully selected set of independent institutions, using the Tulane Conference recommendations as a starting point. It was suggested that the course developers meet to share ideas and carefully evaluate techniques. The report also recommended extensive dissemination efforts and an inclusive spirit, that is, the new calculus was to be property of all the mathematics community, and not just those directly involved with the project.

In summary, the Tulane conference participants proposed the following three themes as a blueprint for calculus reform (Tucker and Leitzel, 1995):

1. Focusing on a conceptual understanding that uses a variety of intuitive graphical and numerical approaches and is geared to the needs of average students. Since students were only memorizing algorithms, methods, and formulas in order to manipulate symbols, conceptual understanding was not a level that was being reached in the traditional calculus courses.
2. Emphasizing the importance of changing the modes of instructions and the use of technology to engage students as active learners. New modes suggested by the participants included computers in the classroom, group projects, open-ended problems, and class discussions.
3. Fostering an inclusive spirit in the reform initiative and emphasizing the importance of co-operation and broad dissemination at every stage. If each school did its own changes and there was no communication between schools, there would not be enough co-operative effort to get any large scale changes implemented.

Not long after the Tulane Conference, the National Research Council (NRC) created a calculus panel headed by Douglas, and the MAA created a calculus committee headed by Richard Anderson.

The timing of the workshop and its recommendations were also fortuitous in influencing the NSF to decided to return to funding of collegiate education with a calculus initiative in 1987. To energize the mathematics community for the new NSF initiative, Douglas's NRC panel organized a national colloquium in October 1987, entitled "Calculus for a New Century." More than 700 participants attended the sessions, at their own expense, to interact and express their opinions about the teaching of calculus. According to Tucker and Leitzel (Tucker and Leitzel, 1995), it was the first time the NRC auditorium was filled to capacity for a non-entertainment event! The conference proceedings were disseminated rapidly as MAA Notes 8, "Calculus for a New Century." The calculus reform movement was underway.

The NSF Calculus Initiative began awarding funding in 1988. A variety of efforts were supported, with 5 multi-year projects and 19 planning grants in the first year. One of the multi-year projects was the development of a new course and text, "Calculus in Context," with an applications approach. One was the development of a collection of open-ended student projects. Another was a co-operatively taught mathematics and physics course. The fourth project was technologically oriented, including the development of a user-friendly interface for students to use Maple to do calculus explorations. The final project was the creation of UME Trends, a journal for disseminating information about calculus reform and other collegiate educational issues.

In the second round of funding, in 1989, six larger projects for courses, most with texts, and twelve smaller projects were funded. This is when the most successful reform courses and textbooks began, including the Harvard Consortium, Project CALC at Duke, Calculus and Mathematica at the University of Illinois, and the Oregon State Project. I will go into more detail about some of these projects in a later section. Also in 1989, a consortium of 26 Midwest liberal arts colleges was funded to produce a set of books and resource materials for enriching calculus courses using traditional calculus texts. Over 6500 copies of the resulting five-volume set have been acquired by faculty.

Starting in 1991, the NSF program shifted its focus from the development of pilot projects to dissemination and large-scale implementation efforts. It also began more funding of pre- and post-calculus courses involving high schools, colleges, and universities. (Tucker and Leitzel, 1995)

The NSF initiative ended as a separate program in 1994. The 127 awards given fell largely into 7 categories: implementation (27), primary curriculum development (23), planning grants (20), technology-based calculus laboratories and software (15), extensions to pre- and post-calculus courses (15), dissemination and conferences (14), and projects and supplementary materials/ co-operative learning (13).

After all of this, one could say that success has been attained in that calculus reform initiatives have been and are still being implemented throughout North America. It has also been attained in that the resulting changes in faculty and student attitudes and student successes are quite positive (see research section). And finally, success has been obtained in making calculus reform efforts well known. To what factors can we attribute these successes?

Douglas (Douglas, 1995) concentrates on two factors for the success of the calculus reform movement. One factor he gives is the growing emphasis on undergraduate education. This has provided support and encouragement to faculty to take more interest in teaching calculus. This emphasis is also what caused the NSF to reassume responsibility of undergraduate mathematics education. The support that the NSF provided has been an influential driving force of the calculus reform movement. One might even venture to say that the movement's success can be largely attributed to the NSF. According to Tucker and Leitzel (Tucker and Leitzel, 1995), its success depended on the extensive set of dissemination and information sharing activities that the NSF, the MAA, and the major projects sponsored. Recall that this broad dissemination was one of the original goals of the participants of the Tulane Conference.

The second reason Douglas gives for the success of reform concerns breakthroughs in technology. Graphing calculators appeared just after the Tulane Conference. There have been vast improvements in computers and software allowing instructors and students to use them in the classroom without spending too much time learning how. Also, mathematicians are using computers and software in their own research which illustrates better than anything how technology is influencing the way mathematics is used. This all means that it is easier and more useful to incorporate technology into the classroom than anyone predicted. And although it is possible to reform calculus without the addition of technology, one of the aspects of reform that has succeeded is that very use. Many feel it is important to keep mathematics, as a discipline, up to date with technology of the times, and reform certainly allows for that. On the other hand, traditional calculus really does not.

I believe it is worthwhile to quickly compare calculus reform today to the attempted reform of the 1960's. The failed efforts of that decade contrasts with current successes. Why?

One reason that Universal Mathematics failed was the need for increased production of engineers to engage in the Cold War battle for technological supremacy. This meant the techniques method of calculus dominated. Currently, any such battle has fizzled and educators realize it is time for change in mathematics education once again.

Another reason why reforms of the 1960's did not work was that there was a shortage of Ph.D.s in mathematics. Now, such graduates are abundant enough that increased production of them is no longer a concern.

Thirdly, computers failed as a learning aid in the 1960's. Now they are being successfully used in the classroom as an excellent aid for instruction. Very little time is taken from calculus instruction with the new and wonderful software and technologies.

The final failure of the 1960's effort was the addition of Applied Probability and Statistics to the first year calculus course. These topics took too much time away from other calculus topics. However, new topics are not part of current reform. On the contrary, the number of topics covered is being decreased, and some topics are being de-emphasized in the name of deeper conceptual understanding of the topics that are covered.

Having now summarized the history of the calculus reform movement, the next question to ask is, what attributes should a reformed calculus course have, and what attributes does it actually have? In the next section we will discuss theories of mathematics, learning, and education to generate some ideas about how one might improve calculus instruction. In the following section, we will look at what the calculus reform movement is actually doing in the United States.

3. A Theoretical Framework

In this section I will describe a theoretical foundation for mathematics, education, and learning. I will start with Bloom's Taxonomy of Educational Objectives, which describes the different levels of cognition students can reach. I will then proceed to look at a philosophy of mathematics, which leads naturally into a philosophy of learning mathematics.

Taxonomy of Educational Objectives

I begin by asking the question, "What should be different about a student's behavior and beliefs after they take a calculus course?" Notice here the focus is on the student, as it is the student who is undergoing the learning process. What are the educational objectives for the student? Obviously, a teacher wants more for students than the ability to recite derivatives and integrals like a machine. A deeper, "conceptual understanding" would be desirable. To clarify this objective, I turn to the educational guru, Benjamin S. Bloom, and what is now commonly known as "Bloom's Taxonomy."

Benjamin S. Bloom published Taxonomy of Educational Objectives in 1956. I take the following from Bloom's 1994 paper, "Reflections on the Development and Use of the Taxonomy" from Bloom's Taxonomy, A Forty-Year Retrospective. Bloom hoped that his taxonomy would help improve the exchange of ideas and materials among testers and other individuals concerned with educational research and curriculum development. For instance, developing a precise definition and classification of such vaguely defined terms as "thinking" and "problem solving" would enable a group of schools to discern the similarities and differences among the goals of their instructional programs. There was agreement that such a theoretical framework might best be obtained through a system of classifying the goals of the educational process. Two reasons for this were that educational objectives provided the basis for building curricula and tests, and they represented the starting point for much educational research at the time.

Bloom felt that although the objectives could be specified in an almost unlimited number of ways, the student *behaviors* involved in these objectives could be represented by a relatively small number of classes. Thus, the taxonomy was intended as a classification of student behaviors that represent educational objectives. It was assumed that essentially the same classes of behavior could be observed in any subject-matter at different levels of education (elementary, high school, college).

Bloom recognized that the *actual behaviors* of students after they have completed a unit of instruction may differ in degree as well as kind from the *intended behaviors* specified by the objectives. The emphasis in the Taxonomy was on obtaining evidence on the extent to which desired and intended behaviors have been learned by the student.

The Taxonomy contains six major classes of educational behaviors:

1. Knowledge
2. Comprehension
3. Application
4. Analysis
5. Synthesis
6. Evaluation

A very brief description of each of these classes is given, along with an example involving the concept of the integral.

Knowledge is the least complex of the classes of behaviors. It includes those behaviors and test situations that emphasize remembering, either by recognition or recall, of ideas, material, or phenomena. In the learning situation, a student is expected to store certain information, and the behavior expected later is the recall of this information. It is recognized that Knowledge is involved in the more complex classes of the Taxonomy, but the Knowledge category differs from the others in that the major psychological process involved here is remembering. Subclasses of Knowledge include Knowledge of Specific Facts, Terminology, Conventions, Trends, Classifications and Categories, Methodologies, Principles and Generalizations, and Theories and Structures. For example, if a student is able to calculate the definite integral of a given function similar to one he or she has calculated before, he or she would be demonstrating Knowledge.

Comprehension is the largest general class of intellectual abilities and skills emphasized in schools. The Comprehension class includes those objectives, behaviors, and responses that represent an understanding of the literal message contained in a communication. In reaching such an understanding, a student may change the communication in his or her mind, or in overt responses to some parallel form more meaningful to him or her. The subclasses of Comprehension include Translation (from one form to another), Interpretation (of literal meaning), and Extrapolation (of some consequences). If a student comprehends the idea of the definite integral, then he or she should be able to integrate a new function using the substitution rule, if he or she is told to use it. Another example of Comprehension would be the ability to translate a word problem asking for an integral into a mathematical problem.

Application is a super-class of Comprehension in that one must comprehend the method, theory, principle, or abstraction applied. There is a distinction between mere Comprehension and Application. A demonstration of Comprehension shows that the student *can* use an abstraction when its particular use is specified. A demonstration of Application shows that he or she *will* use it correctly, given an appropriate situation in which *no* mode of solution is specified. Application includes such behaviors as applying principles, theorems, and other abstractions to new situations, and applying mathematical laws to practical situations. In this case, a student would be able to apply the substitution rule, even if it is not stated explicitly as a possible approach. He or she would also be able to see the applicability of the integral in a problem that is not stated in the context of the integral.

Analysis emphasizes the breakdown of the material into its constituent parts and detection of the relationships between the parts and the ways they are organized. Analysis can be divided into three levels. At the first level, a student is expected to identify or classify the elements of a communication. At the second level, he or she is required to make explicit the relationships among the elements, to determine their connections and interactions. At the third level, the student recognizes the arrangement and structure which hold the communication together. To clarify, at the analysis level, students are expected to be able to distinguish fact from hypothesis, to identify conclusions and supporting statements, or to distinguish relevant from extraneous material. For example, a problem might be given in too much generality, and a student at the Analysis level could extract the important information, apply the integral where necessary, and reach a conclusion about the solution.

Synthesis is defined as the putting together of elements and parts so as to form a whole. In Synthesis, a student must draw upon elements from many sources and put these together into a structure not clearly there before. There are three subcategories of Synthesis. One is the production of a unique communication, such as writing an essay or giving a talk. Another is producing a plan or a proposed set of operations, for example proposing ways of testing hypotheses or integrating the results of an investigation to solve a problem. If a student has achieved this level of educational behavior, then he or she should be able to see the relationships between the definite and indefinite integrals; the integral as an area or volume; and numerical integration. Furthermore, he or she should be able to communicate this understanding. That is, he or she should be able to understand the relationships between algebraic, graphical, and numerical approaches, as well as be able to express these relationships verbally.

Evaluation is the final level of educational behavior. It is defined as making judgements about the value of ideas, works, solutions, methods, materials, etc. for some purpose. Evaluation represents an end process in dealing with cognitive behaviors. Judgements are made in terms of internal and external criteria. Here, not only pure mathematical understanding is demonstrated, but also the relationships the integral has to domains other than mathematics and how it can be "integrated" into them!

With Bloom's definitions of the six levels of cognitive behaviors, we can clearly see that most traditional introductory calculus courses do not reach much beyond the second level -- Comprehension. The common complaint that students are mindlessly implementing symbolic algorithms with little or no understanding reflects this very idea. Very few Application problems are given, and of those that are, most are carbon copies of each other. Analysis, Synthesis, and Evaluation are levels of cognition that no ordinary first year calculus students reach. One of the objectives of calculus reform is to get past the second level, with an ideal goal of attaining the level of Evaluation.

Philosophy of Mathematics

I have tried to briefly explain what students should be learning, or achieving, in an introductory calculus course. Now I ask, "How can teachers help students attain these objectives?" In order to fully understand the perspective taken in this paper, it is necessary to understand the underlying philosophies of mathematics and mathematics education. Thus, let me begin with the philosophy of mathematics.

> "The philosophy of mathematics is the branch of philosophy whose task is to reflect on, and account for the nature of mathematics. The philosophy of mathematics addresses such questions as: What is the basis for mathematical knowledge? What is the nature of mathematical truth? What characterizes the truths of mathematics? What is the justification for their assertion? Why are the truths of mathematics necessary truths?" (Ernest, 1991)

There are two dominant epistemological perspectives of mathematical philosophy -- absolutism and fallibilism. Absolutism assumes that the role of the philosophy of mathematics is to provide a systematic and absolutely secure foundation for mathematical knowledge, that is for mathematical truth. (Ernest, 1991) An absolutist would argue that mathematics forms a body of knowledge that is certain truth. Mathematics is value-free, objective, and superhuman. Mathematics can only be *discovered* by humans.

At the other end of the spectrum, a fallibilist would argue that mathematics is a process *and* a product of human inquiry, which is corrigible and revisable. Mathematics is a value-laden social construction, which is always changing. This is the perspective I have chosen to take.

The weaknesses of absolutism are irrefutable. First of all, the absolutist claims that mathematics consists of certain and unchallengeable truths. Yet a number of contradictions have been derived in mathematics, such as Russell's Paradox. Furthermore, Godel's Incompleteness Theorem states that proof is not adequate to show all truths.

Another problem with absolutism is that any mathematical system depends on a set of assumptions. For example, in Peano arithmetic there are the definitions of whole numbers and axioms like $x + 0 = x$. To qualify as *true* knowledge, these assumptions require a warrant for their assertions. However, in mathematics, the only warrant for assertion is proof. Trying to establish their certainty by proving them leads to an infinite regression, for there is no way of eliminating the assumptions, and without proof, the assumptions remain fallible beliefs, not certain knowledge. Furthermore, the establishment of mathematical truths, that is the deduction of theorems from a set of axioms, requires the assumption of the rules of inference of logic itself. The above argument applies equally well against the assumption of logic.

There are further more subtle arguments given by Ernest (Ernest, 1991), but I believe the above satisfactorily demonstrates that an absolutist view will not do. I therefore accept the opposing fallibilist perspective as the "guiding light" for this paper. To summarize this perspective, let us turn to what some of the great mathematical philosophers have been saying all along:

"Mathematical knowledge resembles *empirical* knowledge – that is, the criterion of truth in mathematics just as much as in physics is success of our ideas in practice, and that mathematical knowledge is corrigible and not absolute." (Putnam, 1975)

"There are no authoritative sources of knowledge, and no 'source' is particularly reliable. Everything is welcome as a source of inspiration, including 'intuition' . . . But nothing is secure, and we are all fallible." (Popper, 1979)

"Why not honestly admit mathematical fallibility, and try to defend the dignity of *fallible* knowledge from cynical skepticism, rather than delude ourselves that we shall be able to mend invisibly the latest tear in the fabric of our 'ultimate' intuitions." (Lakatos, 1962)

In particular, humans do not discover mathematical truths. Instead, we invent mathematics to serve us. It is a tool to help us cope with social and physical constraints in our world. Mathematics is not empirical truth, but rather a social creation, which exists within ourselves. Logical foundations, the rules of inference, and the resulting statements of mathematics are all fallible, and we modify all these components of mathematics to reach as viable conclusions as we can.

An Educational Framework

Now that I have outlined a basis for a philosophy of mathematics, the next step is to build a corresponding philosophy of mathematics education. What is the purpose of a philosophy of mathematics education? It is to be hoped that such a philosophy will answer questions such as: What is considered mathematical knowledge? How do students learn mathematics? How can educators help students learn mathematics? What is the goal of mathematics education?

The perspective I have chosen to apply in this thesis is a version of constructivism. Within the fallibilist perspective it is assumed that mathematics is not certain. What one might believe to be true today could become false tomorrow. This may seem hard to digest at first but consider, for instance, that ancient mathematicians believed that all numbers were "rational," including pi. This was a mathematical truth. Legend has it that he who tried to say otherwise was thrown over the side of a boat for heresy. When it was finally accepted that not all numbers could be expressed as a ratio, the existence of "irrational" numbers became a new mathematical truth. This uncertain nature of truth leads to an inevitable conclusion – even if there is a "reality" out there, no human can ever know it, for what we believe is truth now might not actually be truth.

If there is no certainty in mathematics and if humans can never know "reality," why do we all agree upon the same qualities of mathematics, and why does mathematics serve us so consistently? I need to redefine "truth" before I can proceed to answer these questions. The traditional absolutist philosophy is based on two assumptions about truth:

1. True knowledge must be independent of the knower.
2. A fully structured and knowable world "exists."

These assumptions cause a dilemma – even if someone succeeded in describing exactly how the world really is, he or she would not know they had done so because the description must have in some way been derived from his or her experience. All properties of the world that one observes must first pass through the senses and are related to previous experience. Thus, perceived knowledge can not be independent of the knower. (Von Glasersfeld, 1984, 1994)

If one assumes that all knowledge is a human construction, and thus fallible, a new notion of "truth" can be defined. This is done by retaining the assumption that a reality does exist and affects us through our senses and eliminating the first assumption. Instead, one can use a different way of relating knowledge to reality that does not imply a match or correspondence. Instead, consider Darwin's concept of "fit." Darwin's theory of survival of the fittest is based on the principle of constraints. The organisms alive at any particular moment of evolutionary history, and their ways of behaving, are the result of cumulative accidental variations. The influence of the environment is limited to the elimination of nonviable variants (Von Glasersfeld, 1984.) Those species that are not as adaptable to the environment die out. The common phrase "survival of the fittest" reflects this concept.

Just as in Darwin's concept of evolution, the experiential world acts as a testing ground for viability of ideas. (Von Glasersfeld, 1984) Generally, an individual's knowledge is useful and viable if it stands up to physical and social experience and helps make predictions and cause or prevent certain phenomena. However, if an individual experiences a new phenomenon that does not fit with knowledge from previous experiences, or if he or she experiences a conflicting phenomenon, the knowledge does not serve this useful purpose. To cope with the new or conflicting experience, the individual must modify his or her knowledge to "fit" it. Thus the only parts of the real world that enter into experience are its constraints.

It is easy enough to see how the physical world might constrain beliefs, or knowledge. For example, one could attempt to walk through a desk, but would never succeed because of the physical constraint that does not allow walking through solid objects. Thus an individual's experience constrains his or her knowledge to include not having the ability to walk through desks. In turn, humans use the term "solid" to refer to such objects. This viable knowledge is also referred to as "objective knowledge."

However, sometimes an individual only experiences a conflict between his or her knowledge and that which is *socially* accepted to be "true." For example (Von Glasersfeld, 1987), suppose a student hears the word "mermaid" for the first time, and is told that it is a creature with a woman's head and torso and the tail of a fish. The individual does not need to have met such a creature to imagine it, provided he or she is familiar with the concepts of "woman," "fish," and the other words used in the explanation. However, if the student is not told that the fish's tail *replaces* the woman's legs, then he or she might imagine a fish-tailed biped. The student might read stories about mermaids or discuss them for some time without needing to adjust his or her image. The deviant notion of a mermaid's physique could only be corrected if he or she got into a situation where the image of a creature with legs and a fish's tail came into explicit conflict with a picture, or with what speakers of the language said about mermaids. That is, the concept of the mermaid would only be modified if some social context made it necessary.

Thus, also included in the realm of objective knowledge is knowledge that is *socially* accepted to be true. (Ernest, 1991) Such knowledge is also often referred to as "taken-as-shared," meaning that knowers act *as if* they share the same interpretation of meaning. (As in the mermaid example, this is not necessarily the case.) Social acceptance can occur through publication and scrutiny by many individuals (such as an advanced theorem), through general agreement (such as the "fact" that one cannot divide by zero), or perhaps through social negotiation. Thus, the objectivity of mathematics lies in public, intersubjective agreement. Knowing mathematics is seen as a matter of being able to participate in mathematical practice. Mathematical knowledge does not belong to an individual. It includes the shared conventions and rules of language usage, and it includes so-called "facts." It includes theories, axioms, conjectures, and both formal and informal proofs. Although this definition of objectivity goes against traditional notions of its attributes, such as its enduring and immutable nature, we have already established that mathematics is fallible, so these attributes are immediately dismissed.

On the other hand, subjective knowledge is an individual's personal creation or construction. It is largely internalized. The term constructivism originates from the idea that each individual *constructs* his or her understanding through experiences, and furthermore, the character of these experiences is influenced by his or her cognitive view. (Confrey, 1990) The connection between objective and subjective knowledge is this: Objective knowledge is internalized or learned, and reconstructed to become an individual's subjective knowledge. Using this knowledge, individuals can then cope with the world around them, both physically and socially. Sometimes an individual will also use his or her subjective knowledge to create and publish new knowledge, which can in turn become socially accepted, that is, objective. This forms the cycle of the genesis of knowledge.

It is important to consider exactly how a learner modifies his or her knowledge to fit new or conflicting experiences. As Confrey (Confrey, 1990) puts it, ". . . the reflective process, wherein a construct becomes the *object* of scrutiny itself, is essential." Reflection exerts an influential force on the development of the individuality of any learner. (Confrey, 1992) Mathematics is not built from sensory data, but from human activity such as counting, ordering, comparing, etc. To create a language of mathematics, an individual must reflect on that activity, learning to carry it out in his or her imagination, and to name and represent it in symbols and images. However, reflection in itself is not enough. One must also be able to use reflection to modify and stabilize a current construction. In other words, one must be able to apply the reflection to lead to a viable construction. This is the process of abstraction. Reflection (the mind observing its own actions) and abstraction (making conclusions and predictions based on these observations) constitute the cycle by which individuals modify knowledge to fit experience.

Having discussed learning from a theoretical point of view, I now ask, how does it apply to mathematics education? First note that research in this field has just begun. In fact, some believe there is no such thing as "constructivist teaching" because constructivism describes knowledge development whether or not there is a teacher present or teaching is taking place. There is no simple function that maps teaching methodology onto constructivist principles. (Simon, 1995) However, I believe that constructivism can contribute to the development of useful theoretical frameworks for mathematics pedagogy. I will now look at what some researchers have posed as possible applications of constructivism to pedagogy.

First of all, a constructivist must recognize the importance of a student's own actions in solving problems, and his or her engagement with a variety of tools and materials in such problem-solving activity. (Confrey, 1992) As stated before, physical interaction with objects is one type of experience that has a significant impact on what one learns. The particular choice of tools and materials is important, as different choices can lead to a variety of different mental constructions in students.

A constructivist teacher also needs to recognize the importance of social interaction, usually in the form of communication. Communicating one's ideas to others helps the communicator to further develop his or her own constructions. Furthermore, social interaction allows the society and culture to exert their effects on an individual's knowledge, making it more viable within that framework.

Social scrutiny is necessary for subjective knowledge to become objective, and communication of some type is necessary for objective knowledge to become subjective. Lastly, humans engage in cultural exchanges routinely throughout their lifespan, so development of communicative abilities serves an important purpose beyond the school environment.

A constructivist must also recognize that a student modifies his or her mental constructions in reaction to a new experience that does not fit into his or her current scheme, or to some conflict with previous experience. Therefore, providing new or conflicting situations in an educational setting could prove invaluable. Cobb, Yackel, and Wood (Cobb, Yackel, and Wood, 1992) suggest viewing teaching as an activity in which the teacher guides students' constructive efforts, thereby coaxing them into objective mathematical ways of knowing. Learning could be viewed as an active, constructive process in which students attempt to resolve problems that arise as they participate in the mathematical practices of the classroom. Such a view emphasizes that the learning-teaching process is interactive in nature and involves the implicit and explicit negotiations of mathematical meanings.

However, a teacher must be careful about how he or she develops these situations and guides the student towards discovery. Conflict will not always arise in an individual's beliefs, even if the opportunity for conflict is there. Learning can only occur when the learner has the necessary mental equipment available to make use of the new experiences. (Ginsburg, 1979) When the requisite cognitive structure is present, he or she can learn and come to understand. When the structure is absent, new experience has only superficial effects. The student might transform the experience into a form that can be readily assimilated, thus not learning what is intended (E.g., learns something "false"). On the other hand, he or she might learn a specific response that has no strength or stability and cannot be generalized (E.g., rote memorization of calculus "rules").

A final change that might occur, should one adopt a constructivist perspective, is a shift in emphasis from the student's "correct" replication of what the teacher does to the student's successful organization of his or her own experience. In mathematics, to be able to find the derivative of a function is no doubt useful, but is in itself no demonstration of mathematical knowledge. Performing such a calculation goes no deeper than the first level of Bloom's taxonomy, Knowledge. Mathematical understanding cannot be reduced to a stock of retrievable facts, but concerns the ability to compute new results, i.e., it is the product of reflection and abstraction. A teacher should therefore adopt strategies that challenge students to achieve deeper conceptual understanding.

I have given some general suggestions about pedagogy. However, one might ask, is there a recipe for successful teaching? Constructivism commits one to teaching students how to create more viable constructions. However, since each individual's constructions are subjective, a teacher cannot *know* what any individual student's mathematical constructions really are. Thus, Confrey (Confrey, 1990) suggests teachers must build *models* of students' understanding of mathematics. Decentering, the ability to see a situation as perceived by another human being, is attempted with the assumption that the constructions of others have integrity and sensibility within their individual frameworks. Confrey (Confrey, 1992) recommends to do this by creating many varied ways of gathering evidence for judging the strength of a student's constructions. The result will be that a teacher creates a "case study" of each student. Even if the student's constructions appear different from the teacher's, they hold a reasonable level of validity for the student, and therefore should be taken into consideration and used to adapt instruction suitably. Proper use of these models could assist students in restructuring their views to be more adequate from the students' and the teacher's perspectives. (Confrey, 1990) Of course, this implies that an ideal teacher has time to judge each individual's constructions, which is not realistic in a large class. Simon (Simon, 1995) proposes a promising alternative to Confrey. His idea is called the Mathematics Teaching Cycle.

He describes a teacher whose teaching is directed by his or her conceptual goals for the students. The lesson goal and design are based on relating two factors: the teacher's mathematical understanding and the teacher's hypotheses about the students' understanding. Because each student's knowledge is unique and internal, the teacher can only guess the nature of the students' understandings from his or her interpretations of their behaviors.

The learning goals provide direction for what Simon (Simon, 1995) calls the "hypothetical learning trajectory." This refers to the teacher's prediction of the path by which learning *might* proceed. It assumes that "an individual's learning has some regularity to it, that the classroom community constrains mathematical activity often in predictable ways, and that many of the students in the same class can benefit from the same mathematical task." The creation and ongoing modification of the trajectory is the central piece of the model. The teacher constantly modifies the goals and resulting activities based on communication with and observation of the students.

The development of a hypothetical learning process and the development of learning activities are interconnected. Generating ideas for learning activities depends on the teacher's hypotheses about the development of the students' thinking and learning. Likewise, generation of hypotheses of student conceptual development depends on the nature of the anticipated activities.

In light of the constructivist perspective, I observe here that each teacher will have different interpretations of the students' understanding, the activities that should be used, and the direction the learning might take. Furthermore, each class will react differently, and each student will generate a different construction. Thus, although an explicit teaching method cannot be described, at least one can use this model as a framework by which to structure teaching.

I have outlined a few guiding principles for mathematics education which result from the learning theory. However, I have made no explicit recommendations for mathematics education. Instead, I will now look at what calculus reform is doing, or attempting to do, and investigate whether or not its attempts "fit" this constructivist framework

4. What Constitutes Calculus Reform?

In the second section, I outlined some general philosophies guiding calculus reform efforts. These included aiming for greater conceptual understanding, varying teaching methods to engage students as active learners, and encouraging communication between faculties and between schools. But what does this all really mean? How much innovation is necessary to have a calculus course qualify as a "reformed" course that produces real change? Do these reformations reflect the constructivist view? These are questions I hope to answer in this chapter. To do so, I have broken down reform changes into two specific categories: course content and pedagogy.

Changes in Course Content

Tucker and Leitzel (Tucker and Leitzel, 1995) state, "the overall focus on raising students' conceptual understanding, problem-solving skills, analytic ability, and transferability of calculus skills to work in other disciplines has led to general changes not in the list of topics and techniques covered, but in how these topics are developed." This new focus means reform calculus courses often spend more time on fundamental concepts and less time on symbolic manipulation. In some cases, the computation is relegated to a computer algebra system (CAS) such as MAPLE or DERIVE. In others, computation is merely reduced and replaced by emphasis on deeper conceptual understanding. As noted by Gordon et al. (Gordon et al., 1994) in the Report of the Content Workshop for the conference "Preparing for a New Calculus," in no reform calculus project "is the emphasis on having students perform long lists of similar exercises to gain dexterity at finding limits, derivatives or integrals or to produce graphs of functions."

How is the emphasis on greater conceptual understanding achieved? One of the foremost principles is what the Calculus Consortium at Harvard (CCH) calls The Rule of Five. It started as The Rule of Three -- approach each topic from numerical, analytic, and graphical viewpoints. Then, because of Deborah Hughes-Hallet of the CCH, it grew into The Rule of Four with the addition of verbal approaches. She said (Hughes-Hallet, Gleason, et al., 1994 video):

> "Our project is based on the belief that three aspects of calculus -- graphical, numerical, and analytical -- should all be emphasized throughout. We call this approach 'The Rule of Three' . . . Students are repeatedly confronted with the graphical and numerical meanings of what they are doing. Besides encouraging understanding, this approach gives students with weak manipulative skills a chance to grasp the concepts behind calculus while strengthening their backgrounds. . . Indeed, it is now clear that 'The Rule of Three' should become 'The Rule of Four' -- verbal should be included with graphical, numerical, and analytical."

With the addition of writing mathematics as another useful approach to aid understanding of mathematics, it should really be called the Rule of Five. Let me give an example of how to apply The Rule of Five to the idea of the derivative. This concept is covered in chapter 2 of the Harvard Consortium's <u>Calculus</u>. (Hughes-Hallet, Gleason, et al., 1994) The first approach the authors take is a numerical approach, with the application of measuring the average velocity of an object, given its position at evenly spaced moments in time. Once the simple concept of average velocity is understood, the text goes on to show how instantaneous velocity can be estimated using the idea of a limit. No formal definition of a limit is yet given. The text then guides the student through a short exercise on how to visualize velocity as the slope of a curve. This is not the extent of the graphical approach, but rather just a quick introduction to it.

Now that the student has an understanding of how to calculate average velocity and estimate instantaneous velocity, the formal algebraic method is given. This is the second approach taken in the text. Average velocity is defined as change in position divided by change in time. Instantaneous velocity at $t = a$ is the limit as h approaches 0 of the average velocity between a and $a + h$. The text then proceeds to develop the definition more formally. This is the same approach taken by most traditional texts.

The third approach is the graphical or visual approach. The student can look at the derivative as the slope of a curve and the slope of a tangent line. One whole section of the chapter on derivatives is dedicated to finding the derivative of a function given only its graph, or only a numerical chart of values. Notice that it is acceptable to jump between approaches -- the developers believe that the more integrated the idea becomes for the student, the better.

The chapter concludes with notes on second derivatives, approximations and local linearity, and differentiability. The text was actually developed with The Rule of Three in mind, so the final two approaches (verbal and written) are left to the instructor. These can easily be covered with a project or assignment on the concept of the derivative, including a written and oral aspect.

Although the Rule of Five is a term used by the CCH, the concept of varied approaches to each topic is not foreign to many reform efforts. Reform activists have realized that covering more topics in little depth is not as beneficial as striving for understanding, even if it means that fewer topics can be covered.

What else, besides a broader study of each topic, is added to reform calculus courses? In modest reform courses, where calculator and/or computer activities in problem sessions complement lectures from a traditional text, there is little change in the content. The technology-based activities add graphical and numerical experiences to enhance understanding. Activities to develop mathematical reasoning usually involve applied models. However, in search of greater conceptual understanding, most efforts agree on emphasizing the following content areas (Schoenfeld, 1996 and Tucker and Leitzel, 1995):

1. Functions and their representations.
2. Limits and continuity from an intuitive point of view.
3. Parametric representations of curves.
4. The derivative as a function giving the relative rate of change or the slope of a tangent. Using numerical methods to approximate the derivative.
5. The integral as an accumulation providing a means of calculating a total change, area, volume, and other geometrical and physical quantities. Integration by parts and straightforward substitution, using integral tables for

more complex integrals. Using numerical methods to approximate the definite integral.

Covering each concept in depth, such as outlined for the Harvard Consortium, takes more time than covering each concept the traditional way (algebraically only, without aiming for understanding). For example, Hughes-Hallet, Gleason, et al., (Hughes-Hallet, Gleason, et al., 1994 video) recommend spending at least three to five weeks on the first chapter of their text (A Library of Functions), which is all precalculus material. Thus, since reform calculus courses are so much richer in actual content, they are often leaner in terms of the number of topics in the syllabus. (Smith, 1996) So the next fair question to ask is, "What is de-emphasized?"

As suggested in the Tulane conference, the content of many calculus reform courses was developed starting with a clean slate. After discussions with mathematicians, engineers, physicists, chemists, biologists, and economists, Tulane conference participants omitted some topics that just could not be justified. This led to changes that Tucker and Leitzel (Tucker and Leitzel, 1995) and Schoenfeld (Schoefeld, 1996) summarize nicely:

1. Most courses streamline the discussion of techniques of integration, sometimes eliminating trigonometric substitutions almost completely.
2. Many downplay symbolic computation skills and curve-sketching techniques. Graphing calculators and CAS have largely eliminated the need for such repetitious activities.

3. Related rates are often missing.
4. Limit theorems and calculus theorems such as the mean value and intermediate value theorem are also downplayed.
5. Epsilon-delta arguments about limits and continuity are usually eliminated. (Schoenfeld,1996)
6. Tests for convergence of series beyond comparison with geometric series are often eliminated. (Schoenfeld,1996)

However, one must not make the mistake of putting too much weight on what reform courses are omitting. These omissions are justified, and additions have been made to content in their place. For example, methods of integration do not teach students anything about integration – except how to apply some memorized tricks and rules. Curve sketching by hand is a time-consuming process, which can easily be done with a calculator. Related rates are just template problems that apply nothing more than the chain rule. The intermediate value and mean value theorems become intuitively obvious with the addition of graphical approaches and technology-aided sketching.

At this point I pose the question, do these changes reflect constructivist philosophies? In most respects, the answer is affirmative. Reform calculus allows students to study mathematics from many different perspectives. Experiencing each topic from different points of view allows for a more connected, synthesized understanding. This means students can reach higher levels of learning in Bloom's Taxonomy. Each time a topic is reviewed from a different angle, students have the chance to reflect on their previous understanding, and modify it as necessary. They also have the chance to make further conclusions and predictions based on these new experiences, that is, to abstract their ideas.

Expressing each topic orally and in writing enables students to construct meaning from the content, rather than simply memorize the methods of computation. This also points the way to success in achieving the upper echelons of the Taxonomy. Furthermore, it gives students the chance to test the viability of their understanding in a social context. As argued in the previous chapter, public agreement is necessary for one's subjective knowledge to become objective. Listening to other students express their understanding of topics also helps the classroom's objective knowledge to become internalized within each individual.

Finally, allowing technology as an aid in performing symbolic manipulations allows for more time to be spent towards higher levels of learning in Bloom's Taxonomy and on helping students construct an understanding of the deeper meaning behind each concept.

Are there any tenets of constructivism that have been ignored with these changes in content? There is not much more that can be gained from content revision from a constructivist point of view. The constructivist philosophy primarily attempts to explain *how* students learn what they learn. Teaching more *depth* of content, which reform does, is the chief result that can be achieved. However, perhaps it would be beneficial to include physical models and applications. As stated in the previous chapter, the physical environment can lead to growth of new, objective knowledge.

Obviously, changing content is not enough to make the calculus reform movement work. More importantly, calculus reformers have attempted to change *how* calculus is taught. I will now deal with this very issue.

Changes in Pedagogy

Although there is broad agreement concerning general instructional strategies for these new courses, there are substantial differences in specific implementations. The broad agreement includes certain distinguishing features of reform calculus from traditional calculus counterparts. In this section we will discuss these general differences, giving actual examples in the later chapter on specific projects.

Reform calculus is aimed at increasing learning in "the average student." Specifically, it is not aimed at teaching mathematics majors. It is assumed that the students who achieve excellence in traditional courses would do so, and learn just as much, no matter how the course was taught. Thus, the goal of new instructional methods is to reduce tedious calculations and to try to involve all students more directly in the learning process, so they can find meaning in the mathematics rather than memorize the formulas and theorems. One can already see traces of a constructivist framework here, but let me delve more deeply into the specific ideas reform educators have been attempting to implement before making any conclusions. Besides the changes in content, I have found there to be seven pedagogical principles or strategies used in multiple reform calculus projects. These are:

1. De-emphasizing lecturing, replacing it with active learning on the part of the students.
2. Putting more responsibility on the students for their learning.
3. Encouraging students and promoting self-confidence.
4. Promoting co-operative learning.
5. Using technology appropriately as a tool.
6. Including modeling and applications to drive learning.
7. Employing a variety of assessment techniques.

I will now discuss what each of these principles means in practical terms and analyze whether, as teachers' roles, they fit into a constructivist framework. The first principle is well expressed by the phrase "Telling is not teaching, and listening is not learning." Active involvement of students must replace passive reception of "the word" from an all-knowing lecturer. (Smith, 1996) One of the basic goals of calculus reform is to involve the students more directly in their learning. Some reform projects have given up lecturing altogether, replacing it with classroom and laboratory activities. These activities often involve experimentation, discovery, and open-ended problems (those with more than one correct answer or multiple paths to a correct solution). This does not mean requiring less rigor, making calculus "easier," or marking "easier." In fact, most students who have taken a reform course agree that more work is required and that it is "harder" than a traditional course. The idea is not to simplify the calculus, but rather to improve the way it is learned and retained.

It seems obvious that promoting more active learning on the students' part fits into a constructivist framework because it allows for each student to construct meaning from the mathematics. One of the basic tenets of constructivism is that individuals construct knowledge for themselves on the basis of their experiences. Knowledge cannot be transmitted to them from others or absorbed from the environment, such as through a lecture. (Gregg, 1995) Contrary to lecturing, experimentation, discovery, and open-ended problems invite students to build the concepts themselves. These methods also require more reflection and abstraction then enduring a lecture.

Reform students must also take more responsibility for their learning. Students are required to read their textbooks (something which few students do in traditional courses), and to read mathematics more critically. (Smith, 1996) They are also expected to work independently or in small groups with less prescriptive guidance from the teacher and textbook, to analyze problems, and to explain their solutions. Many activities lead more naturally to written responses such as reports and journals than to tests. (Smith, 1996) It is safe to say that following this principle reflects the constructivist view because, as in the previous principle, reform efforts are requiring constructive activities on the part of the students. Furthermore, it shows that reform efforts are striving to bring students to higher levels in Bloom's Taxonomy.

Reform activists also hope that students' self-confidence about their understanding of calculus concepts and their mathematical reasoning skills will increase as a result of making them more active learners. This tenet of calculus reform is sometimes called "positive incentive." It is based on the psychological theory that students will respond better to positive reinforcement (success, or good try) than to negative punishment (wrong, you fail). This principle implies that educators should not be aiming their teaching towards the "cream of the crop," but rather to the ordinary students. As suggested at a July 1992 NSF-hosted conference, "the student should become aware of the beauty of calculus . . ." (Leitzel and Dossey, 1996) rather than be afraid of it. In this case I do not see a fit to or a conflict with constructivist ideas. Of course it is beneficial to reward good work, because it motivates students to do more good work. However, using positive incentive does not affect *how* a student learns, which is all that constructivism can tell us.

The next three principles are really pedagogical strategies used to aid learning. To reiterate, these strategies are: promoting co-operative learning, using technology, and driving learning with modeling and applications.

The first strategy, co-operative learning, can occur within informal or formal learning groups, or in study teams. In each case, small groups of students (3-5) work together to accomplish tasks. Researchers report that students working in small groups tend to learn more of what is taught and retain it longer than when the same content is presented in other instructional formats. (Davis, 1993) Interactions within a group keep students focused on the task at hand. A group can solve a problem that few students could manage alone. The team effort reflects real world situations. And explaining a concept to another student requires a deep understanding of it.

Typically in calculus reform classes, this group learning occurs in problem or laboratory sessions while working on projects or in-depth problem sets. However, co-operative behavior must be modeled, practiced, and supervised in the classroom, or it will never be achieved successfully. (Smith, 1996) Janet Ray of Seattle Central Community College states that "the hardest part of teaching a reform course is to learn to structure and manage the classroom for effective learning." (Smith, 1996) Tucker and Leitzel (Tucker and Leitzel, 1995) found in their survey of reform calculus courses that 42% of respondents made substantial use of co-operative learning, while another 34% used the method, but infrequently.

Co-operative learning is another example of a constructivist fit. In the previous chapter, it was argued that social interaction helps the communicator to further develop his or her own constructions. Explaining one's own ideas to the other students allows one to test their viability in a social context. Listening to other students in the group can lead to objective knowledge becoming subjective for individuals in the group.

Technology often plays a role in reform calculus courses, but should not be mistaken as an if-and-only-if component. Reform can occur with or without technology, and if it is used, it can be used productively or not. Several reformers argue technology should only be used as a tool to assist students in learning. The goal is for computers and calculators to help enable students to see the interplay between graphical, numerical, and algebraic interpretations of discrete and continuous models and methods. (Smith, 1996) The C^4L Project uses computer software as a means to stimulate students to undertake the needed mental constructions. (Tucker and Leitzel, 1995) In Tucker and Leitzel's survey (Tucker and Leitzel, 1995), 80% of respondents said calculators were used in a substantial way, and the use of computer algebra software (CAS) was also widespread.

Just as use of technology can be productive or not, it can also reflect constructivism or not. If it is used properly, it can provide alternative perspectives on topics, as in the previously-stated goal. This could aid students in constructing connections between concepts leading to successful organization of their own experiences. In particular, it could foster the processes of reflection and abstraction (making conclusions and predictions based on concepts learned). Physical representations, such as graphs or pictures, can also help students build viable constructions about the mathematics. However, merely looking at pictures or calculating derivatives using CAS does not mean any of the above will take place. Along with the use of technology, there must be direction. Otherwise, use of technology does not constitute constructive learning.

Modeling and applications can also help to promote deeper understanding of calculus concepts. Furthermore, it can motivate the curriculum for students. Well-designed projects challenge students to use all the tools at hand -- pencil and paper, calculator, computer, and most importantly, prior experience with mathematical concepts and techniques. Tucker and Leitzel (Tucker and Leitzel, 1995) report that 65% of respondents used substantial modeling and applications.

Once again, there is physical interaction with the mathematics, which can help students construct mental representations. I also see a direct example of abstraction taking place when students are required to model real-world situations and apply their knowledge to these models. Thus, constructivism harmonizes with use of modeling and applications.

The final difference between traditional and reformed calculus courses is how students are assessed. Teachers need to be able to assess the broad spectrum of understandings that their students need to develop. (Schoenfeld, 1996) In reform calculus, timed tests are de-emphasized because of their inadequacy in discovering how much students can really do. The purpose of this traditional mode of assessment was to take a "content inventory" of what students know. Such tests had the advantage of being relatively reliable – if students took the same test twice or alternate versions of a test, their scores were essentially the same. However, the calculus community has come to recognize the inadequacy of the content inventory that ignores fundamental aspects of mathematical performance. (Schoenfeld, 1996)

For that reason, large number of alternate assessment techniques are being developed. They have the advantage of being more consistent with new goals for instruction, but also the disadvantage that reliability has had to be redefined. In place of tests, writing assignments (such as journals or reports), oral presentations, and challenging problem sets have been used. Tucker and Leitzel (Tucker and Leitzel, 1995) found that almost half the survey respondents reported substantial use of projects and 35% reported substantial use of writing, mostly through write-ups of projects and in-depth problems. Schoenfeld (Schoenfeld, 1996) describes some further assessment devices including open-ended problems, student-constructed tests or test items, observation at the whole class level and at the level of individual students, interviews, portfolios, and student self-assessment.

Do these new methods of testing fit the constructivist view? They certainly require activity on the students' part, as well as reflection and abstraction. Going beyond testing facts not only emulates constructivist principles, but also requires students to demonstrate achievement of higher levels of Bloom's Taxonomy. Thus, I do agree that these "higher forms" of testing do fit.

There are further constructivist ideas that are not reflected by any of the principles. In particular, I found no mention in reform literature of providing conflicting experiences or situations for students as an aid to learning. This is one of the chief ways new knowledge is created, or previous knowledge is modified. I believe that this area of calculus pedagogy should be further investigated.

Also, nowhere in reform calculus publications have I found anything about building models of students' understanding. With these models, one can adapt the approach taken to each concept around the students' interpretations. The Mathematics Teaching Cycle, or some similar method, could prove invaluable in guiding the direction each class takes. If a teacher is unaware of his or her students' understanding, he or she would not know which experiences would best serve as new or conflicting for them. The teacher would also have difficulty in selecting models and applications, and guiding co-operative learning within student groups.

5. The Projects

Project CALC: Calculus as a Laboratory Course

Project CALC at Duke University in Durham, North Carolina is a curriculum development project funded by the National Science Foundation. The developers, Lawrence Moore and David Smith, began implementing it in 1989. Since then, "it has produced materials for a three-semester reformed calculus course that emphasizes real-world problems, hands-on activities, discovery learning, writing and revision of writing, teamwork, intelligent use of available tools, and high expectations of students." (Moore and Smith, 1996) More than forty schools in the United States have used their materials.

Foundation

What motivated Project CALC? In 1989, Joe Garofalo outlined some beliefs commonly held by secondary mathematics students, based on his experience as a teacher. Three of the beliefs were: (Bookman, 1996)

1. Most mathematics problems can be solved by direct application of the facts, rules, formulas and procedures shown by the teacher or given in the textbook.
2. Formulas are important, but their derivations are not.
3. The teacher and the textbook [are] authorities in, and dispensers of, mathematical knowledge.

Also, in 1989, The National Council of Teachers of Mathematics published its Curriculum and Evaluation Standards for School Mathematics that called for major changes in the content and pedagogy of mathematics instruction. The Standards suggested that "the teacher's role shift from dispensing information to facilitating learning" and that students should be encouraged "to become self-directed learners who routinely engage in constructing, symbolizing, applying and generalizing mathematical ideas."

Moore and Smith wanted to respond to these concerns as well as the growing dissatisfaction of students and faculty with calculus instruction. They began with the philosophy that instructors should concentrate on what students are learning and how students' behaviors are changing, rather than what is being taught. They believe that teachers cannot change students' beliefs, only their behaviors. In particular, "it makes little difference what the 'teacher' *says* or how well he or she says it. What matters is what the students *do*." (Smith and Moore, 1990)

Thus, Project CALC was designed to address the needs of all students of calculus and ideally, to reach desired student outcomes. That is, rather than focusing on how the instructor should teach, they focused on changes in student behaviors that should result from participating in the course. These desired outcomes are reflected by the following set of goals given by Moore and Smith: (Moore and Smith, 1996)

exploration exercises with pencil, paper, and graphing calculator.

2. Interesting problems outside mathematics are used as motivation to drive the mathematical development.
3. Graphical and numerical approaches are emphasized at least as much as symbolic ones. For example, the derivative is based on local linearity and the integral on regular left-hand sums.
4. Theory is downplayed significantly. Students discover many truths of calculus and work through plausible reasons that support these truths.
5. Most of the major topics of a traditional course are included, but with significant changes of order and altered emphases. The text starts with an extensive treatment of the concept of function. There are no specific chapters for techniques of integration and differentiation.

Moore and Smith (Moore and Smith, 1996) describe the format of the course as follows. Project CALC students at Duke meet three times a week for 50 minutes in a classroom with a demonstration computer or calculator projection. In class, time is divided among lectures, discussion, and working in groups. Lecturing is limited to a brief introduction of new topics and responses to questions for more information.

The students also participate in a computer laboratory session once a week for two hours. Here, students work in pairs to explore real-world problems with real data, conjecture and test their conjectures, discuss their work with each other, and write up their results and conclusions on a technical word processor. The laboratory reports are submitted almost every week. Three to five of the reports per semester are "formal" submissions -- that is, the instructor reviews them and allows resubmission for a grade. The others are "fill-in-the-paragraph" style reports, often finished in the laboratory period, and submitted only once. These laboratory sessions are the central component to the course; they shape the contents and the approach of the text and the format for many classroom activities. Students are also assessed using quizzes, tests, and homework.

Comparison

As can be seen, Project CALC fits into the scheme of calculus reform. It seems to be almost a perfect fit, with every aspect of reform discussed in Chapter 4 being addressed except for the goal of encouraging students and promoting self-confidence in mathematics. This goal can be achieved by individual instructors.

Does Project CALC correspond with constructivism? Since it is such a good example of a reform course, analysis of it would be very similar to that of the general reform course given in Chapter 4. In particular, the content and pedagogical changes do synchronize with constructivist philosophies for the most part. Lacking is an attempt to supply conflicting or contrasting situations. Also, there does not seem to be strong emphasis in adapting teaching to what the students are doing, even though the developers claim that the focus is on the students' actions. Instead, it appears they believe that if the students do the allotted mathematics exercises, in co-operative situations, the action will entail the necessary learning automatically. Moore and Smith designed these exercises based on the belief that they should require deeper thought. However, it is questionable whether the exercises work or not, and Moore and Smith have not indicated modifying them in response to students' reactions. Overall, the project has been quite successful as studies in the next section will show, but it has also had its downfalls.

C^{4}L: Calculus, Concepts, Computers, and Cooperative Learning

C^{4}L is the one reform project with a solid learning theory behind it. In fact, the creators, Ed Dubinsky, Keith Schwingendorf, and David Mathews of Purdue University claim that they used a constructivist basis when developing the calculus project. The purpose of their model was "to propose a description of specific mental constructions that a learner might make in order to develop her or his understanding of the concept." (Asiala et al., 1996)

Foundation

The following summary of their version of constructivism is taken from Dubinsky (Dubinsky, 1989) and Asiala et al., (Asiala et al., 1996). Dubinsky rejects the idea that mathematics is learned spontaneously by looking at diagrams or listening to a speaker. He also does not believe it is learned by doing examples, extracting common features and important ideas from these experiences, and organizing that information in the mind. Instead, learning mathematics is an active *process* in which the learner constructs the tools and concepts rather than just using them. Mathematics knowledge is not something you have, but rather something you do.

As a result, when a learner is faced with a problem situation, he or she reconstructs what he or she has previously constructed in order to deal with the present situation. In some cases, this reconstruction can lead to inconsistency or produce tools that are less powerful than the individual has previously used. On other occasions, it can produce something that is more powerful, sophisticated, and effective. In these cases, growth of the individual's mathematical knowledge is said to have taken place. Thus, in the problem of knowing, there are two issues: learning a concept and accessing it when needed.

How does mathematical knowledge grow? According to Dubinsky, it is a spiraling process. *Actions* (such as calculations) are done on *objects* (such as numbers). For example, adding three to a number is an action. A student who has an action concept of the function $x+3$ can do very little with it besides compute values at specific points and manipulate the formula.

However, if the learner repeatedly adds three to various numbers, he or she can become aware of this action and interiorise it, resulting in a *process*, adding three. Now the student can reflect on and describe the function $x+3$ without actually performing the actions on it. Processes can be constructed by composing two processes, such as adding three and squaring to give $(x+3)^2$, or by reversing a given process, for example to obtain the process of subtracting three or $x-3$. When an individual reflects on operations applied to a particular process, he or she becomes aware of the process as a totality. The process can then be encapsulated to become a new object. In this case that object would be the expression $x+3$.

It is critical in this constructivist theory that one is aware of the mathematical operations that he or she performs, and reflects on their meaning. Interiorisation and encapsulation occur in part as a result of reflecting on the problem situation and methods of dealing with the problem – both successful and unsuccessful. This type of reflection parallels the reflection described in the previous section on philosophy of mathematics education.

Once constructed, objects and processes can be organized together to form a *schema*. For example, functions can be formed into sets, operation on these sets can be introduced, and properties of the operations can be checked. Encapsulation and schematization resemble the previously described notion of abstraction.

Dubinsky and the other members of the project claim that C^4L was a direct result of the pedagogical implications of this learning theory foundation. The primary goal of their teaching is to help students construct appropriate processes and objects and to get them to reflect on and use these constructions in a social context. They believe that this will enable students to more successfully deal with problem situations that arise in mathematics. Hence, the primary role of the teacher is not to lecture, explain, or otherwise attempt to transfer mathematical knowledge, but rather to create situations for students to foster their making the necessary mental constructions to learn mathematical concepts.

Description

The program is embodied in the Activity-Class-Exercise (ACE) learning cycle. At Purdue, each ACE unit, which lasts about a week, begins with computer activities to be worked on in cooperative groups. The purpose of these activities is to provide students with an experience base. There are several mathematical concepts that they have an opportunity to construct at the action level. These concepts include the use of simple propositions, the formation of pairs and triples of numbers, the action of picking out an indexed term of a given sequence, the calculation of the dot product, and the algorithm for computing the length of a vector in three dimensions. Programming activities can also be used to help students to understand functions as processes. Research suggests that students who write code to implement the point-wise sum, product, and composition of functions tend to make progress in developing a process conception of function. (Asiala et al., 1996) Writing programs in which the processes are inputs and/or outputs of the program is used to help get students to encapsulate the processes into objects. The current method employed is to ask students to write a set of computer programs that implements a mathematical concept and then to apply the code to specific situations.

The computer activities are followed by classroom tasks consisting of problems and questions posed by the instructor for students to reflect on and work on in small groups. On occasion, the instructor will provide definitions, explanations, and overviews to tie together what the students have been thinking about.

The final stage of the cycle consists of exercises for students to work on outside class. These are relatively traditional exercises, contrary to other reform projects. The purpose of the exercises is to reinforce the ideas students have constructed, to use the mathematics they have learned, and to begin thinking about situations that will be studied later.

Comparison

How does C^4L compare to other reform projects? It is true that they de-emphasize lecturing, replacing it with computer use and cooperative learning. They also put more responsibility for learning on the students. For example, the laboratory problems are intended to take longer than the laboratory period and students are expected to complete them on their own time along with the exercises. Many of the problems and examples are real-life examples or models. However, the project differs from other projects in that the implementers do not mention placing importance on promoting self-confidence or employing alternative assessment techniques. Furthermore, they do not indicate using multiple representations to aid students in developing what they would call schemas, that is, organized understanding of an area of mathematics. Finally, there is no indication that the content of the course has been modified.

How well does the C^4L project reflect the notion of constructivism described in the earlier chapter? Obviously, the framework it is built on resembles my notion closely. However, the pedagogical interpretation differs somewhat. The C^4L project has limited itself to a subset of the suggestions that arose in the earlier section on constructivism. In particular, the C^4L project has recognized the importance of reflection and abstraction. They have allowed interaction with physical objects (calculators and computers). Social interaction is a key part of the project. The developers also indicate the need to provide experiences for students that will guide them toward discovery. They meet this need by providing demonstrations and problems, which can certainly produce conflict if they are carefully chosen. However, C^4L does not appear to aim for higher cognitive levels of understanding -- it uses traditional homework exercises following a traditional curriculum. Also, it does not include any alternative testing strategies and does not approach each topic from various viewpoints. Overall, although it has a strong foundation, its implementation leaves something to be desired.

The Calculus Consortium at Harvard

The Calculus Consortium at Harvard (CCH) is the most widespread reform project, and also one of the more conservative ones. In fact, the textbook is the most popular calculus textbook of all texts sold in the United States. As of 1994, the CCH materials were being used at over 350 institutions of all types.

Foundation

Although CCH does not claim to be based on any learning theory foundation, the developers did begin with some basic motivating factors. The developers, Gordon and Hughes-Hallet (Gordon and Hughes-Hallet, 1994) describe the situation that motivated the project well:

"Unfortunately, very few of the students who have taken a traditional calculus course [regard] calculus as one of the greatest intellectual achievements of western civilization. All too often, they have instead seen calculus as:

- a collection of poorly understood rules and formulas involving manipulations,
- a set of artificial problems that provide little feel for the power of calculus,
- a collection of poorly understood and rarely appreciated theoretical results that were memorized and regurgitated."

In early discussions and planning, the members of CCH decided that the primary problem with the traditional calculus approach was that it focused almost exclusively on symbolic manipulation. Other approaches to the subject were usually neglected or omitted entirely. However, understanding of mathematical concepts is often better conveyed by geometrical images and numerical approaches. Thus, the underlying philosophy behind CCH came to be known as the Rule of Three: Every topic in calculus should be approached geometrically, numerically, and symbolically. In addition, CCH soon realized that much of mathematics is also conveyed verbally, so their approach became the Rule of Four instead.

In order to implement the Rule of Four, the designers of CCH found it was necessary to redesign both the content and the overall focus of introductory calculus. The result was an attempt at a course that focuses heavily on developing mathematical thinking and less on developing manipulative skills. The course is highly problem driven including applications in areas such as probability, biology, economics, and finance as well as the usual applications from the physical sciences and engineering.

In designing the syllabus, Sudholz (Sudholz, 1995) and the other developers followed three principles:

1. Start from scratch. CCH wanted to start from scratch, deciding which topics should and should not be left out. They consulted other departments to find out what topics were important to them. The client disciplines (E.g., physics, engineering, business, and computer science) were interested in seeing students learn numerical methods, mathematical modeling, and how to work in teams.
2. Show students what calculus can do, not what it can't do. CCH believes that in a beginning college course, students should experience the power of calculus, not the special cases in which it fails. For example, students should explore Newton's method without emphasis on cases in which it does not work, and learn error approximations, but not an exhaustive treatment of error estimates.

3. Finally, be realistic about students' abilities and the amount of time they will spend on calculus. In traditional calculus, we cover so many topics so fast, that students end up with little understanding of anything. It is better to teach fewer topics well.

The growing availability of sophisticated technology also influenced CCH. In 1987, Lynn Steen reported on the results of a survey he had conducted of calculus final exams from all types of institutions: 90% of all questions could be answered using widely available computer algebra systems. (Gordon and Hughes-Hallett, 1994) He reasoned that we should not be teaching to machines, but to humans. The existence of cheap, powerful calculating machines forced the CCH developers to reassess what is important for students to know and be able to do.

In response to this issue, they assumed students would have availability of some minimum technology that could graph a function, locate zeros of a function, perform numerical integration, and display a slope field of a differential equation. However, they decided to focus on the mathematical ideas, with technology being only supportive.

Description

The main reform aspect and focus of the CCH project is the textbook. Part of what makes the CCH text different from traditional texts is the order in which topics are covered, and the amount of time spent on each. It begins with a long chapter on functions. CCH members view this chapter as the "key to the course" (Hughes-Hallett, Gleason, 1994 video) and they recommend spending three to five

strong, but after an hour and a half he is so exhausted that he has to stop. The data Jeff collected is summarized below:

Time spent running (min)	0	15	30	45	60	75	90
Speed (mph)	12	11	10	10	8	7	0

(a) Assuming that Roger's speed is always decreasing, give upper and lower estimates for the distance Roger ran during the first half hour.
(b) Give upper and lower estimates for the distance Roger ran in total.
(c) How often would Jeff have needed to measure Roger's pace in order to find lower and upper estimates within 0.1 mile of the actual distance that he ran?

p. 309, Ex. 8: A car starts at noon and travels with the velocity shown in Figure 6.4 (a curve is sketched on a grid). A truck starts at 1 p.m. from the same place and travels at a constant velocity of 50 mph.

(a) How far away is the car when the truck starts?
(b) During the period when the car is ahead of the truck, when is the distance between them greatest, and what is that greatest distance?
(c) When does the truck overtake the car, and how far have both traveled then?

We must remember that these text and content changes are cosmetic compared to the changes in pedagogy. What does CCH recommend on how to teach their course? Firstly, they suggest attending a workshop or minicourse that they offer. In the case of the workshops, this is a week-long training session. In contrast, the mini-course is one day. I attended a two-day version of this workshop, and did not gain much in the way of teaching methods. It seemed as though the emphasis was entirely on the text. The hosts only outlined a couple of pedagogical approaches in the workshop as well as in the video "Calculus Consortium based at Harvard Workshop." Their web site only dedicates three lines to pedagogy.

Collaborative learning is one of these approaches. They are not specific about how or when to use group work – it can be done in every class, once a week, or sporadically throughout the term. Group assignments can also be used to promote teamwork.

The workshop leaders also suggest technology can be beneficial as a supportive aid in learning. However, mathematics comes first, and they even state that the course can be taught without technology. They do not go into any depth about *how* to use the technology. At Northern Arizona University, a computer laboratory using Mathematica accompanies all sections of first and second semester calculus. (Hagood, 1995) Students spend from one to two hours a week completing a laboratory in the presence of an experienced laboratory aide. These laboratories generally parallel the CCH materials, covering some topics in depth and reinforcing other topics that are covered more thoroughly in class. The laboratories attempt to attain high levels of Bloom's Taxonomy, with problems phrased using words such as investigate, analyze, compare, and discover.

Comparison

As can be seen, the Calculus Consortium based at Harvard reflects many of the trends in calculus reform. This may be in part because it has been a leader in the movement and many other projects have been developed with knowledge of CCH in mind.

How does CCH compare to other reform projects? I think CCH shows definite strength in the areas of content development and textbook style. Furthermore, the multi-representational approach is admirable. The developers also hope students will see the beauty of mathematics and will gain self-confidence in their mathematical abilities. The project promotes cooperative learning and use of technology. It also includes some modeling and applications

However, the project does not shun lecturing – in fact it has nothing to say about the matter. Active learning also goes unmentioned. Gordon and Hughes-Hallet seem to agree that students should achieve deeper conceptual understanding, but do not place a lot of the responsibility for achieving this on the shoulders of the students. Instead, they credit the text and syllabus for this. They also leave assessment techniques to each individual teacher. If one were to lecture the CCH course in a traditional manner, with traditional assessment techniques, it could end up being more like a traditional course than a reformed one. Ignoring these other facets of reform is a weakness that some other projects overcome.

Do their choices reflect the constructivist perspective? Although CCH does not overtly recognize the necessity of reflection and abstraction, their multi-representational approach does help encourage these processes. The project also allows for interaction with computers, but it does not require it nor does it recommend how to effectively use technology. Cooperative learning provides the necessary social interaction. As for producing conflicting experiences, I have seen signs of this in their challenging problem sets. However, once again, the developers did not seem to aim for this goal.

Constructivism focuses on the student's learning process. In CCH, the focus is not on student outcomes. The focus is on what should be done to the text, the content, and somewhat to the teacher, to make students learn. The developers outlined problem areas in calculus teaching and learning, and proposed some corresponding solutions, ignoring others without reason. How did they choose their new methods, problems, and approaches? It appears to have been a subjective process, that is, they did what "made sense." I believe their work could be strengthened if they had a pedagogical foundation for their reforms. This would lead to more of a focus on what the students are doing and not on what everyone else is!

6. Research on Reform Efforts

There have been a growing number of studies on calculus reform efforts in the past few years. In this section, I will outline the results of some such studies. As the field is still in its infancy, the sample of studies is small, some would say inadequate. Very few of the 110 projects funded by the NSF included a plan to evaluate outcomes of their work. The caliber of the studies also varies. Finally, many of the studies were carried out by those who were biased towards reform in the first place.

Because of the large-scale changes that calculus reform has undertaken, it is difficult to assess it in standard ways. For example, common final exams for traditional and reform students address only what is common to the courses. (Ostebee, 1990) Most reformers are more interested in what is uncommon in their courses: do students really benefit from seeing mathematics from graphical, numerical, and algebraic viewpoints? How do the new courses affect students' algebraic facility, their ability to use calculus ideas, and their attitudes toward mathematics? Do students really achieve better conceptual understanding? How will they fare in later mathematics courses? (Ostebee, 1990) How do faculty attitudes change as a result of teaching reform courses?

Reports I have chosen to include come from evaluations of Project CALC and the Calculus Consortium at Harvard, studies comparing traditional versus reform calculus at Baylor University and U.S. Naval Academy, a Purdue C^4L study on subsequent calculus enrollments, a University of Michigan at Dearborn study on computer use, and a Clemson University study on calculator use. I will also include general comments given by Tucker and Leitzel in Assessing Calculus Reform Efforts and Keith in "How do Students Feel about Calculus Reform, and How Can We Tell?"

Final Report: Evaluation of Project CALC

Jack Bookman of Duke University designed and implemented an evaluation of Project CALC from 1989 - 1993. The following is a summary of some of the results. (Bookman, 1994) For more specifics, please refer to his paper, "Evaluation of Project CALC 1989 - 1993."

Year Two (1990 - 1991)

During year one, the project implementers focused on getting things underway in a few calculus sections and ironing out the wrinkles. No evaluations were attempted until year two. During that year, the evaluator, Bookman, taught both an experimental Project CALC (PC) and traditional (TR) course. He also observed another PC class and a TR class once a week, held a dinner discussion on alternate weeks with a regular group of students from each of the observed classes, read students' comments about both classes, and held extensive conversations with faculty teaching PC classes. All students involved in the PC classes were self-selected volunteers.

In the Spring session of year two (equivalent to Winter session at the University of Calgary), PC students were asked the question "What are the strongest aspects of this course?" Responses that occurred most often were (1) that the students must understand the material rather than just memorize it and (2) that it was interesting to see the connections to the real world. When asked what they did not like about the course, the most common answers were (1) the reader, (2) the 'subjective' grading, and (3) that the material was too vague and complicated and not explained clearly (single quotations are used for student phrasing).

When TR students were asked the same questions, they gave the strongest aspects of the course as (1) it 'forces students to learn lots of material,' (2) the 'basics of calculus were taught,' (3) none, (4) it 'fulfills a requirement,' and (5) 'the teacher.' What did traditional students say they did not like? (1) the textbook, (2) difficult tests, (3) that there was too much material, and (4) that the course was boring.

From classroom observations, Bookman felt it was clear that

> "the level of attention and concentration of the Project CALC students was much higher. In every traditional class observed, at least some of the students fell asleep and most of them started packing their books before the lecture was finished. These behaviors were not observed in Project CALC classes. In fact, often the class ran five minutes overtime without the students even noticing."

The faculty found that teaching the PC course took much more of their time than the regular course. They had difficulty learning how to grade writing, how to increase individual accountability, and how to make up appropriate tests.

Most of the faculty liked the laboratories although there were suggestions on how to improve them. Some teachers thought more lecturing would be appropriate, especially to connect and put closure on the ideas developed by the students while working in groups.

In the spring of 1991 a five-question test of problem solving was developed and administered to all students of the evaluator, 23 of whom were PC students and 42 of whom were TR. It was hypothesized that PC students would do better because the PC course was designed to provide students with many more problem-solving opportunities than the TR course. The questions were novel to both groups and were based on the goals of Project CALC stressing application and understanding of concepts over technique. Out of a possible 20 points, PC students averaged 9.7 and TR students averaged 7.1. The difference of the means was significant at the .01 level. Because the test was skewed toward Project CALC's goals, this is not a surprising finding. However, it does indicate that Project CALC had made some progress towards meeting its goals.

Year Three (1991 - 1992)

In year three, the main focus of evaluation activities was a second version of the problem-solving test and a five-part written test to sophomores and juniors who had taken the two courses. The five parts included measures of attitude toward mathematics, skills, writing, problem solving, and conceptual understanding. Once again, the students enrolled in Project CALC calculus were volunteers.

The results of the problem solving test were consistent with those from the previous year. The mean for PC students versus TR students was 10.77 versus 8.45. While the performance of neither group can be considered very good, the difference in means was highly statistically significant.

For the five-part test, there were 41 PC students and 46 TR students participating. Thirty-five items gathered from various course evaluations written by both PC and TR students were each recoded so that for "positive" items, strongly disagree was given a score of 1 and strongly agree was given a score of 4. Thus, the total score could range from 35 (completely negative) to 140 (completely positive). The mean for PC students was 110.232, for TR students, 92.315. The attitude difference favoring PC students was highly statistically significant.

The skills test involved 10 questions on computation and basic skills. Each item was scored from 0 to 3, 0 meaning no idea; 1 for some idea; 2 for a good start; 3 for an almost correct or correct solution. The mean for PC students was 18.634, for TR students, 21.989. The difference favoring TR students was just shy of being statistically significant.

For the writing test, students were given 20 minutes to write a short essay illustrating the concept of the derivative by describing the difference between average velocity and instantaneous velocity. Responses were graded from 0 to 4 by two independent scorers. The mean and standard deviation for PC students were 1.963 and 1.272, for TR students, 1.772 and 1.02. The difference favoring PC students was not statistically significant.

On the problem-solving section, students were given two problems. The first involved depreciation. Neither group had seen a problem in this context. The second problem involved finding a solution to a polynomial of degree four, or explaining why no solution exists. Each problem was scored from 0 to 4. The results were:

	PC		TR	
	mean	SD	mean	SD
problem 1	1.866	1.577	1.304	0.946
problem 2	1.244	1.488	1.457	1.433
total	1.555	1.272	1.380	0.916

The final section of the five-part test tested conceptual understanding. Students were given one hour to work 10 items. The questions were scored on the 0 to 3 scale used for the skills section. The mean for the PC group was 11.524, and for the TR group, 9.641. The scores were not impressive for either group, perhaps because the students had already spent two hours on the previous four parts of the test.

Year Four (1992 - 1993)

The evaluation in the fourth and final year of the study focused on the question: Do experimental students do better in and take more subsequent courses that require calculus? Information from the registrar was gathered on PC and TR students who took Calculus I and II in the Fall 90 and Spring 91 semesters respectively (equivalent to Fall and Winter at the University of Calgary).

For each student, the following information was requested for analysis:

1. SAT scores – math, verbal, major
2. grades in Calculus I and II
3. overall GPA at Duke
4. the student's grades in subsequent courses requiring calculus.

An average mathematics related grade (MSGR) was computed for each student who had at least two math-related courses beyond the level of Calculus III and/or any science or economics class. It was found on average that TR students were better by .2 of a four point GPA. This is a statistically significant difference, which did not vanish even when SAT scores and performance in Calculus I and II were controlled for.

To better understand these results, seven pairs of students matched by major and SAT scores, with one student from the experimental course and one from the traditional course, were interviewed. In general, the PC students said they were better in conceptual understanding than computational skill and that they did not have enough practice, especially with techniques of integration. The sentiment of the TR students was that their course was harder than high school and had lots of rote work. They liked the familiar structure of the course. In further mathematics courses, the PC students reported that the course had helped them understand the mathematics in further courses, but that these courses were not in the format of Project CALC courses. TR students did not report any problems. When asked what they had learned in Calculus I and II, the PC students' responses were that they learned applications of math, how to express their mathematical ideas in writing, where the main ideas in calculus come from, and that mathematics is useful. The TR students indicated they learned the 'usual stuff,' the basic 'concepts' and 'a million ways to plug along and do it.'

In January 1994, the grades of the same group of subjects, now seniors, were examined again. The results of the analysis were similar to that of the previous year: there was about a .2 difference in GPA favoring the TR students over the PC students that remained statistically significant even after controlling for SAT scores. When the grades in just the most mathematically related courses – engineering, physics, computer science, and mathematics – were examined, there was virtually no difference between the average grades of the PC and TR students. On average, PC students took about one more such course than TR students. The difference was statistically significant.

The evidence gathered indicates that PC students are learning more about how mathematics is used. On the other hand, students in the PC course did not feel that enough emphasis was placed on computational skill. There remain some unresolved issues. No cost-benefit analysis was conducted. No evaluation has been done at Project CALC's other test sites.

The Calculus Consortium at Harvard

Although the CCH Project has not done such a large-scale, quantitative study, it has solicited general feedback from implementers. According to Deborah Hughes-Hallett, in the CCH workshop video (Hughes-Hallet, Gleason, et al., 1994), the results have been quite positive. She claims that

> "many students with weak algebraic skills report understanding mathematics for the first time. Mathematics has become more accessible to them. Math is more intriguing and students are interested in both the questions and answers. Students experience a great boost in confidence when they realize they can figure out how to solve problems."

In the video, Hughes-Hallet claims there is also a greater persistence rate from students who take calculus using the CCH text. For example, at SUNY - Stony Brook and University of Michigan, more students continued into second term calculus after having taking the first term through CCH:

SUNY - Stony Brook		Enrollment 1st term	Continued 2nd term
1990 - 1991	Traditional	408	53 %
1991 - 1992	Traditional	424	53 %
1992 - 1993	Consortium	358	63 %

University of Michigan		Enrollment 1st term	Continued 2nd term
1992 - 1993	Traditional	1438	53 %
1993 - 1994	Consortium	300	59 %

Hughes-Hallett adds that none of these results were obtained from a properly controlled study, but that they are encouraging. She also claims that there are fewer D's and F's in CCH courses, and a general shift up in grades, probably because weak algebraic skills do not pull students down. For example, at SUNY - Stony Brook the grades increased when the CCH text was introduced:

Grade	1991 - traditional	1992 - consortium
A, A-, B+, B	31.9 %	46.0 %
B-, C+, C, C-	46.3 %	33.5 %
D+, D, F, Incomplete	21.7 %	19.9 %

How do students do in follow-up courses? In one study at the University of Arizona, the grades earned by the two groups in Calculus III and IV were virtually identical. However, 45 % of the consortium students completed Calculus I - III within four semesters whereas only 33 % of traditional students did. Also, 23 % of the consortium students completed Calculus I - IV within four semesters whereas only 18 % of the traditional students did.

Finally, the CCH developers reported on student reactions about their text. For the most part, these reactions were extremely positive – particularly from those students with weak algebraic skills who found graphical and numerical ways to solve problems. There were more complaints from advanced placement students or exceptional students because it is not quicker and easier for them. Faculty reactions resembled those of Project CALC – there was more preparation time required for the course on the part of the teacher.

Comparison Studies

Tucker and Leitzel (Tucker and Leitzel, 1995) provide brief descriptions of two well-controlled studies which compare student performance in reform and traditional first-year calculus. These were undertaken by Baylor and the U.S. Naval Academy in 1993 - 1994 involving the CCH materials. In both studies, first-year calculus was split into reform and traditional sections with much or all of the final exam being common to both groups.

At Baylor, the results were controlled for ability by classifying the students in advance into three categories based on high school grades and SAT scores. The common part of the final exam consisted of 20 multiple-choice questions based on a traditional course. The traditional students in all three categories did better on two of the questions. The reform students in all three categories did better on 11 questions and better in two out of three categories on the other seven questions. For students in the weakest category, the reform courses produced only marginally better overall scores, but for the average and stronger students, the reform students averaged a full grade higher in the multiple-choice test. Interestingly enough, the correlation between ability category and performance in calculus, reform or traditional, was extremely high. According to Tucker and Leitzel (Tucker and Leitzel, 1994),

> "The Baylor faculty had been split about 50-50 on the value of reform at the beginning of the 1993-94 academic year. After seeing this data, the faculty voted unanimously to switch all sections to the reform text in fall 1994."

At the Naval Academy, students were assigned in a random fashion to reform and traditional sections. The final had 10 common questions, four graphical in nature and six traditional. Traditionally-taught students did a bit better on one question. The reform students did a bit or a lot better on the other nine.

Purdue C^4L Study

At Purdue, some analyses of subsequent enrollment in calculus and post-calculus courses have been done. The study compares students whose first calculus course was a C^4L course with those whose first calculus course was non-reformed. Allowing for choice of major, predicted GPA, and other factors, the study found that the average number of subsequent semesters of calculus and the average number of post-calculus courses were greater by a statistically significant amount for students in the reformed course. (Tucker and Leitzel, 1994)

University of Michigan - Dearborn

This study is outlined in <u>Priming the Calculus Pump: Innovations and Resources.</u> (Höft and James, 1990) At the University of Michigan - Dearborn, computer laboratories were implemented in the course which was traditional in other ways. The laboratory sessions took place once a week for 80 minutes. They began with 20 minutes of lecturing and then the students had an hour to work on the computers. The computer assignments were meant to get students to explore a topic and to elicit cooperation between students. The students worked in pairs, with conversation between pairs also being encouraged. The problems usually went beyond what was covered in lectures and required some ingenuity. One week after each laboratory session, each team of two students submitted a laboratory report with solutions, graphs, references to theorems, and a description of what was accomplished during the session.

At the time of the study (1989), students were aware of the pilot sections when they signed up for the course. At the end of the semester, they were given a questionnaire to solicit their reactions to the new laboratory format. The responses indicated that the students in the laboratory sections were more interested in the material, more responsive, and asked more questions than their counterparts in traditional sections. The instructors also observed that during the lectures they seemed to be more attentive and seemed to concentrate harder.

Clemson University

Clemson University began using high-level calculators in its first year calculus courses in 1988. It was largely a pioneering effort as no resources were available at the time with respect to such calculators. Faculty struggled with many issues including

> "syllabi, texts, assessment, how to get entering students started and proficient in their use of the calculators, and how to foster and maintain their active involvement with the course material, students' misconceptions about mathematics, the role of technology and learning, and what were appropriate uses of the calculators to help improve learning." (LaTorre, 1994)

They also consulted with the Faculty of Engineering, since 60% of Clemson's enrollment in single variable calculus is comprised of students majoring in engineering.

LaTorre (LaTorre, 1994) states that Clemson faculty have been unable to develop objective measures of the true learning that is actually taking place in calculator enhanced courses. Comparisons with traditional courses are not appropriate since the expectations are substantially different, and grade distributions are of no help either. However, during the first three years of the project, an external evaluator solicited student perceptions of the role of the calculators and the learning that was taking place in the new courses. The results favored calculator use. Most students agreed that the graphics calculator helped them understand the material, do more exploration and investigation in solving problems, and have better intuition abut the material. They did not think that learning to use the calculator detracted from learning the material or meant less material was covered in the course. In fact, they recommended that entering freshmen seek out courses using the graphics calculator. See LaTorre's 1993 article for more details.

General Indicators

Student Performance

After stating that "the value of a college course, and more generally a college education, are very difficult to quantify," Tucker and Leitzel (Tucker and Leitzel, 1995) present the results of their 1994 survey which attempted to assess calculus reform efforts. The study's survey asked faculty for their assessments of changes in students performance and enrollment arising from calculus reform. Many of the 62 responses noted that reform efforts had just started and it was too soon to discern any changes. However, a quarter of the respondents indicated that reform students were doing better in terms of passing rates and general performance. A quarter also felt that reform courses improved retention rates in calculus. A fifth of the respondents said that post-calculus enrollments seemed to have increased after calculus reform was instituted. The bachelor-degree level institutions noted a shift away from failures and A's. The university responses tend to attribute better retention and pass rates to the fact that reform calculus appears less threatening.

Half of the 62 institutions indicated they perceived improved conceptual understanding in students and a quarter of the respondents said that students' general problem-solving skills had increased. In a 1993 survey, many respondents at all but the doctoral level institutions indicated that their students were becoming more proficient in their attacks on and solutions of open-ended problems. Overall, they reported a noticeable increase in student skills to communicate and reason mathematically, and to attack contextual problems. Doctoral level respondents in this survey reported observing little if any changes in process abilities.

As for content, the respondents did not have solid evidence to indicate that their students were significantly different in terms of the amount of calculus content they had learned and retained. Although testing procedures had changed little at most institutions, several respondents indicated the need to lessen the heavy reliance on timed tests.

The information obtained by the NSF about the 110 projects it sponsored revealed similar trends to the survey by Tucker and Leitzel:

> "Evaluations conducted as part of the curriculum development projects revealed better conceptual understanding, higher retention rates, higher scores on common final exams, and higher confidence levels and involvement in mathematics for students in reform courses versus those in traditional courses; no significant loss of traditional skills was observed in reform students." (Ganter, 1997)

Student Attitudes

Overall, reports on student attitudes were positive. One quarter of the 62 respondents to the 1994 survey stated that students showed noticeably greater self-confidence in their mathematical reasoning ability and skill in attacking open-ended problems. One quarter remarked on the pleasure students showed in becoming proficient in the use of technology. Some wrote that students seemed to find mathematics more interesting and more useful. Most respondents commented that students spent more time on the course than in the past.

Students reported that technology use was more time-consuming, but they liked the portability of the graphing calculators. The role of technology also seemed to have unexpected benefits in students' attitudes toward mathematics. The power and diverse uses of technology became associated with mathematics in students' minds. Thus mathematics came to be seen as contemporary and as a tool for the future.

Most respondents reported that students had some difficulty in adjusting to cooperative learning, problem sets, writing, and projects. On the other hand, students adapted fairly quickly to technology use. Students found reform harder because of the necessity to read the book, problem sessions out of class, and the absence of a visible teacher spooning out information. Resistance came from advanced placement students and repeaters in calculus. Most of the students who fought reform to the end were those who relied on memorization or rejected the use of technology for visualization. However, by the end of the second semester, most students had adapted to the expectations of the course and overall there was widespread satisfaction.

Keith reported on student reactions from nearly 20 colleges and universities (Keith, 1995). She found that many students loved the freedom of the reform projects, the applications, the group work, the feeling of "really learning for the first time," and of being in control of the process, the problems, and the technology. On the other hand, students often spoke of the need for teachers to be instructed in the new methodologies, for answers to problems, for more instruction in the software, and for smaller classrooms. The first term was often hard on students, but many eventually became advocates who were prepared to promote the program to others.

Faculty Attitudes

According to Tucker and Leitzel (Tucker and Leitzel, 1994), at all types of institutions there is usually considerable skepticism and open opposition to reform initially. In time most faculty accept, or at least allow others to implement, calculus reform. However, there is often a minority of faculty who remain uninterested in reform or opposed to it. When beginning to implement reform efforts, faculty reported the most common hurdles to be difficulty in getting computer equipment, resistance of some faculty, lack of faculty interest in instructional issues, and the time-consuming nature of reform. All respondents to the survey reported that reform teaching takes more time to grade papers, meet with students, deal with the technology, and plan for class.

A common theme in responses to reform's impact on faculty was that those faculty actively engaged in teaching reform courses felt very good about what they were doing. Although limited changes had been seen to date, all the faculty felt that what they were doing was right for students and themselves as mathematics professors. The survey indicated a feeling of growth and professionalism among faculty at non-doctoral institutions. Many reported that calculus reform raised morale among the mathematics faculty.

Keeping in mind the subjective nature of many of these surveys and the difficulty in comparing reform calculus to its traditional counterpart, it still seems that reform is a step in the right direction. Results from student tests, and comments from students and faculty suggest that reform students are gaining more of a conceptual understanding than their traditional counterparts. The studies also suggest a good feeling towards calculus reform in both students and teachers. I next pose the question, where should the effort go now?

7. Where Should Reform Go Now?

I have analyzed some major reform efforts based on a constructivist perspective, specifically Project CALC, C^4L, and the Calculus Consortium at Harvard. From this analysis, I found strengths and weaknesses in all of the efforts. Project CALC developers Moore and Smith take a step in the right direction by focusing on students and what they do, and also succeed at implementing activity-based learning. However, they retain traditional content and have a weak foundation for their program. C^4L developers Dubinsky, Schwingendorf, and Mathews claim a strong constructivist basis, but they do not follow it through in terms of content and pedagogy. Hughes-Hallett and Gleason of the Calculus Consortium at Harvard have created an excellent content basis, but have ignored pedagogy and theoretical justification. Other efforts also demonstrate weaknesses in either foundation, content, or pedagogy.

There have been some studies on reform efforts. The general feeling in the research that has been done seems to be positive. Most reform activists believe students are achieving higher conceptual understanding, at least to some degree. Most believe the movement is heading in the right direction and are excited about the possibilities. However, the assessment techniques have been all but consistent and many efforts have gone unassessed.

Furthermore, many questions remain unanswered. What do the client disciplines think about reform calculus? They were a motivating force behind many of the content and pedagogy choices, yet few studies have solicited responses from them. Everyone agrees that there is a lot more work involved in teaching reform calculus than traditional. Is it worth it? How do reform efforts compare with each other? Which aspects of each effort are best, and which are less successful?

Before reform proceeds any further, what is needed is a much greater effort to evaluate calculus reform. Faculty universally agree upon the importance of evaluating the effect of reform efforts on students learning, faculty and student attitudes and content development. At this point, the mathematical community has little evidence to validate the expenditure of enormous amounts of time, energy, and financial resources. Without such evidence, how can they defend their work, understand the impact of various methods, and justify continuation of the struggle?

After such evaluation, if reform still seems the way to go, the next step should be a concerted, focused reform effort with a basis in research and a strong constructivist foundation. Such an effort should also follow through with this foundation in terms of content, pedagogy, and assessment. Furthermore, strong communication between faculty and schools at every stage must be encouraged, particularly in the areas of development and assessment. With such an effort, reform activists could do consistent, valuable research and help answer some unresolved questions.

Although the continued spread of calculus reform seems imminent, there are still potential pitfalls to avoid. In particular, the possibility of backlash still exists. Furthermore, reformers must not lose their focus on bettering calculus education for students everywhere. Competition between developers to come out in front could never be as successful as co-operation. Although it was fruitful to have many different efforts in the beginning, and to try many different methods, it is now time to narrow the gap between them.

Calculus educators have set an example for educators in all mathematical disciplines. Already reform is spreading to pre-calculus, algebra, differential equations, and multivariate calculus courses. Calculus reform has come far in the past ten years. Now, as a unified group, reform activists must continue to remember where they are going and not become lost.

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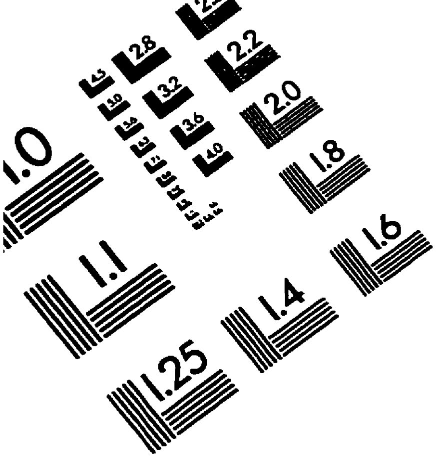

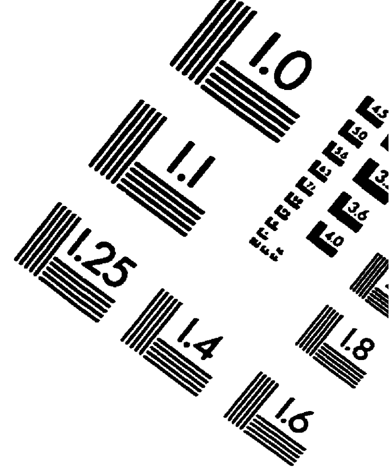

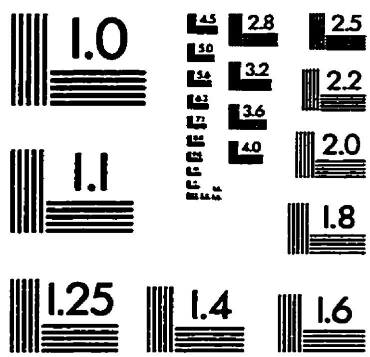

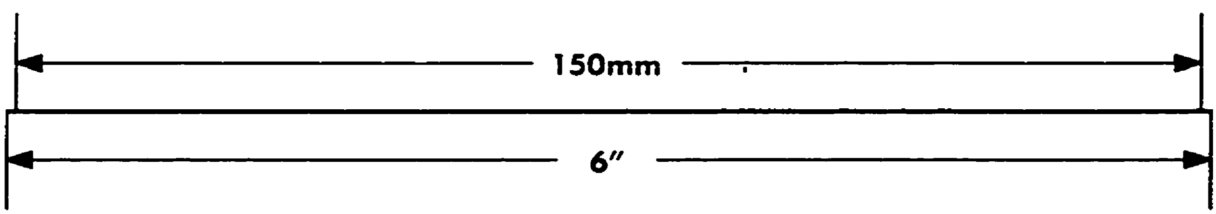

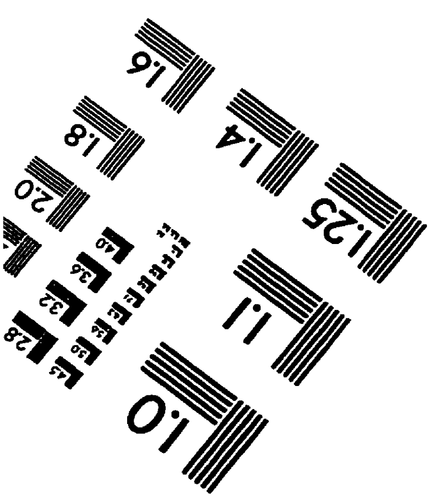

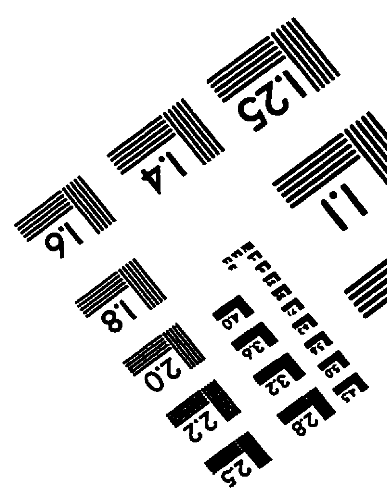

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