Exact Learning Boolean Functions via the Monotone Theory

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Abstract

We study the learnability of boolean functions from membership and equivalence queries. We develop the Monotone Theory that proves

1) Any boolean function is learnable in polynomial time in its minimal DNF size, its minimal CNF size and the number of variables n.

In particular,

2) Decision trees are learnable.

Our algorithms are in the model of exact learning with membership queries and unrestricted equivalence queires. The hypotheses to the equivalence queries and the output hypotheses are depth 3 formulas.

1 Introduction

One of the main open problems in machine learning is whether boolean functions are learnable from membership and equivalence queries in polynomial time in the number of variables and their (disjunctive normal form) DNF sizes (the minimum number of terms in an equivalent formula in Disjunctive Normal Form). A more general line of research is whether all boolean functions are learnable in polynomial time using other representations, such as (conjunctive normal form) CNF, decision trees and multivariate polynomials. The latter problem for multivariate polynomials was recently solved by Schapire and Sellie [SS]. They show that any boolean function is learnable in polynomial time in n and its multivariate polynomial size.

Many results in the literature give evidence that learning boolean functions in different models in polynomial time in the DNF size and the number of variables is hard [AK91, AHP92, PR94]. On the other hand, many other results gave subclasses of boolean functions that are learnable in their DNF representations. Such subclasses include monotone DNF [A88] (DNF which contains

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no negated variables), read-twice DNF [AP192,PR294] (DNF where each variable occurs at most twice), Horn sentences [AFP92] (DNF where each term contains at most one negated variable), k-DNF [V84] (DNF where each term contains at most k literals, for a constant k), $O(\log n)$-term DNF [BR92], where n is the number of variables (DNF with $O(\log n)$ terms) and read-k sat-j DNF [AP292] (DNF where each variable appears at most k times and every assignment to the variables satisfies at most j terms, j and k are constants).

In this paper we develop new techniques for exact learning of boolean functions. We show

1. Any boolean function is learnable in polynomial time in its minimal DNF size, its minimal CNF size and the number of variables n.

2. Decision trees are learnable.

This solves the open problem of learnability of decision trees.

Learning in this model implies learning in Valiant's PAC model if membership queries are available [A88], [V84]. It also implies that these classes are polynomial time predictable with a polynomial mistake bound if membership queries are available [L88].

Our algorithms are in the model of exact learning with membership queries and unrestricted equivalence queires. The hypotheses to the equivalence queries and the output hypotheses are depth 3 formulas.

2 The Learning Model

The learning criterion we consider is *exact identification*. There is a function f, called the *target function*, which is a member of a class of functions C defined over the variable set $\{X_1, \ldots, X_n\}$. The goal of the learning algorithm is to halt and output a formula h that is logically equivalent to f.

In a *membership query* the learning algorithm supplies an assignment a to the variables in $\{X_1, \cdots, X_n\}$ as input to a *membership oracle* and receives in return the value of $f(a)$. We represent a as a vector of length n.

In an (unrestricted) *equivalence query* the learning algorithm supplies any function h from a class of functions $H \supseteq C$ as input to an *equivalence oracle* and the reply of the oracle is either "yes", signifying that h is equivalent to f, or a *counterexample*, which is an assignment b such that $h(b) \neq f(b)$.

For more about this model see [A88].

3 Results

We prove the following results.

Theorem 1. *For assignments $A = \{a_1, \ldots, a_t\}$ let $\Lambda(A)$ be the set of all boolean functions that can be represented as CNF with clauses in which each clause is falsified by some assignment* *in A. Then any $f \in \Lambda(A)$ is learnable in polynomial time in the number of variables, the DNF size of f, and t.*

Theorem 2. *Any boolean function is learnable in polynomial time in the number of variables, its DNF size, and its CNF size.*

Theorem 3. *The conjunction of any boolean function f with a boolean function $g \in \Lambda(A)$ is learnable in polynomial time in the number of variables, the DNF size of $f \wedge g$, the size of A, and the CNF size of f.*

The above theorems imply

Result 1. *The class of k-almost monotone DNF is the class of monotone DNF formulas that allow the existence of at most (constant) k non-monotone terms. This class includes the monotone DNF and the k-term DNF classes. Any k-almost monotone DNF is learnable in polynomial time in its DNF size and n. Also any conjunction of functions from this class is learnable in polynomial time in its DNF size and n.*

Proof. If we change k-almost monotone DNF to CNF by taking the conjunction of all possible disjunction of one literal from each term, we notice that every clause contains at most k negated variables.

Let $A = \{$ all vectors of Hamming weight $\leq k\}$. It is clear that any clause with at most k negated variables is falsified by some assignment in A. Now using Theorem 1 the result follows. A conjunction of any number of k-almost monotone DNF can also be represented as CNF where each clause has at most k negated variables.$\Box$

This result is a generalization of the learnability of monotone DNF and k-term DNF [A88].

Result 2. *Any k-CNF, for $k = O(\log n)$, is learnable in polynomial time in its DNF size and n. In particular, a conjunction of any number of $O(\log n)$-term DNF is learnable in polynomial time in its DNF size and n. This implies that a conjunction of $O(\log n/\log\log n)$ number of $O(\log n)$-term DNF is learnable in polynomial time in n.*

Proof. An $(n,k)-$*universal set* is a set $\{b_1,\ldots,b_t\} \subseteq \{0,1\}^n$ such that every subset of k variables assumes all of its 2^k possible assignments in the b_i's. In [ABNR92,CZ93] an $(n,k)-$universal set is constructed of size $t = O(k2^{3k}\log n)$. This size is polynomial when $k = O(\log n)$.

We show that the above classes are subsets of $\Lambda(A)$ where A is any (n,k)-universal set. Let f be any k-CNF. Let $C = X_{i_1}^{c_1} \vee \cdots \vee X_{i_l}^{c_l}$, $l \leq k$ be a clause in f where $X^c = X$ when $c = 0$ and $\bar{X}$ when $c = 1$. Since A is an (n,k)-universal set there is $a \in A$ such that $a[i_j] = c_j$ for $j = 1,\ldots,l$ and this assignment falsifies C.

A conjunction of $O(\log n/\log\log n)$ number of $O(\log n)$-term DNF has DNF size at most $O(\log n)^{O(\log n/\log\log n)} = poly(n)$.$\Box$

This is a generalization of Blum and Rudich's result for learning $O(\log n)$-term DNF [BR92].

Result 3. *A (constant) depth $2t+1$ $(k_1,\ldots,k_{2t})$-formula is a formula that is a conjunction* *of at most k_{2t} formulas, each of which is a disjunction of at most k_{2t-1} formulas, each of which is a $(k_1, \ldots, k_{2t-2})$-formula. A ()-formula is a term. The class of $(k_1, \ldots, k_{2t})$-formulas, where $k_1 = k_3 = \cdots = k_{2t-1} = O((\log n)^{1/t})$ and $k_2 = k_4 = \cdots = k_{2t} = O((\log n / \log\log n)^{1/t})$ is learnable in polynomial time in n. Also any circuit of depth t and fanin $O((\log n)^{1/t})$ for the OR gates (no restriction on the AND-fanin) is learnable in polynomial time in its DNF size and n.*

Proof. It is not hard to demonstrate that the above classes are subclasses of $O(\log(n))$-CNF.$\Box$

The only nontrivial large depth formulas (depth more than 2) that are proved to be learnable in the literature are read-once formulas over different basis [BHH92a, BHH92b, BHHK91]. Linial, Mansour and Nisan [LMN93] showed that constant depth circuits are PAC learnable with membership queries in time $n^{poly(\log n)}$ under the uniform distribution. The above formula is the first nontrivial class of constant depth formulas that are learnable and have no read restrictions.

Result 4. *Any boolean function is learnable in polynomial time in its decision tree size and n. This implies that any decision tree of polynomial size is learnable in polynomial time in n.*

Proof. The size of any decision tree of a boolean function f is polynomial in its DNF and CNF size. By Theorem 2 the result follows.$\Box$

Several polynomial time learning algorithms have been given for learning certain restricted subclasses of decision trees. Ehrenfeucht and Haussler [EH89] gave an algorithm for PAC- learning (without membership queries) decision trees of constant rank in polynomial time and gave a PAC-learning algorithm for general polynomial size decision trees in time $n^{O(\log n)}$, (see also [Bl92,R87]). Kushilevitz and Mansour [KM91] gave a polynomial time PAC learning algorithm with membership queries for decision trees under the uniform distribution. Hancock [H93] gave a polynomial time algorithm for PAC learning read-k decision trees. Our result implies PAC learning of decision trees with membership queries under any distribution.

Result 5. *A CDNF boolean function is a boolean function whose CNF size is polynomial in its DNF size. Any CDNF boolean formula is learnable in polynomial time in its DNF size and n. In particular, a polynomial size DNF that has a polynomial size CNF is learnable in polynomial time in n. In the former case the learner need not know any information about the DNF and the CNF size of the target formula.*

Proof. Follows immediately from Theorem 2.$\Box$

Ehrenfeucht and Haussler [EH89] showed that polynomial size DNF that have polynomial size CNF are PAC learnable under any distribution in time $n^{O(\log^2 n)}$.

Result 6. *Decision trees over terms are decision trees where each node contains a term. Decision trees of constant depth over terms are learnable in polynomial time in n.*

Proof. We show that depth k decision trees over terms have DNF and CNF size at most $(2n)^k$. Then using theorem 2 the result follows. Any depth k decision tree (over variables) is a k-DNF and k-CNF. Therefore if f is a decision tree over terms of depth k then it can be written as $f = \bigvee_{i=1}^{r} D_i$ where

$$D_i = T_{i,1} \wedge \cdots \wedge T_{i,j_i} \wedge \overline{T_{i,j_i+1}} \wedge \cdots \overline{T_{i,m_i}}$$

$m_i \leq k$ and $T_{i,l}$ are terms. The DNF size of D_i is at most $n^{m_i} \leq n^k$. Since the depth of the tree is k we have that the number of leaves in the tree is at most 2^k and therefore the DNF size of f is at most $(2n)^k$. Duality implies the CNF size of f is at most $(2n)^k$.$\square$

This is a generalization of the learnability of k-term DNF, k-clause CNF and k-term decision lists, [A88,R87]. It also implies that the class of multivariate polynomials with (not necessarily monotone) k terms is learnable.

Result 7 . *Any conjunction of the formulas in results 1-6 is learnable in polynomial time in its DNF size and n. For example, a conjunction of CDNF boolean functions with $O(\log n)$-CNF is learnable in polynomial time in its DNF size and n.*

Proof. Follows from theorem 3 and the fact that the conjunction of a boolean function in $\Lambda(A)$ with a boolean function in $\Lambda(B)$ is in $\Lambda(A \cup B)$.$\square$

All of the above results are for boolean functions. In a companion paper [B93] we study the learnability of concept classes over nonboolean domains. We define DNFs, CNFs and decision trees over any alphabet and show conditions for which the above results are true for any concept. In particular, we show that

Result 8. *Any polynomial size decision tree over any alphabet is learnable and any function $f : \Sigma^n \rightarrow \Sigma$ over any alphabet Σ is learnable in polynomial time in its minimal decision tree size and the number of variables.*

This paper is organized as follows. In section 4 we give the monotone theory of boolean functions and prove some preliminary results. In section 5 we present the first algorithm and prove Theorem 1. In section 6 we present the second algorithm and prove Theorem 2. In section 7 we show how to combine the two algorithms into one general algorithm and prove Theorem 3.

4 The Monotone Theory

In this section we give the notation, definitions, and preliminary results, that we need to write the algorithms and prove their correctness.

For a vector x representing an assignment to $\{X_1, \ldots, X_n\}$, $x[i]$ is the i-th entry of x, i.e., the assignment x makes to the variable X_i. For two vectors we write $y \leq x$ if '$y[i] = 1$ implies $x[i] = 1$' for all i. For an assignment $a \in \{0,1\}^n$ we define $x \leq_a y$ if and only if $x + a \leq y + a$. Here $x + a$ is the group addition in $GF(2)^n$ (bitwise XOR).

A boolean function f is called a-monotone if $f(x+a)$ is monotone. The *(minimal) a-monotone boolean function $\mathcal{M}_a(f)$* of f is defined as follows:

$$\mathcal{M}_a(f)(x) = \begin{cases} 1 & (\exists y \leq_a x)\ f(y) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

We write $\mathcal{M}$ for $\mathcal{M}_0$. The following are properties of $\mathcal{M}_a$.

Lemma 1 . We have

1. $\mathcal{M}_a(f) = \mathcal{M}(f(x+a))(x+a)$.
2. $\mathcal{M}_a(f \vee g) = \mathcal{M}_a(f) \vee \mathcal{M}_a(g)$.

3. $\mathcal{M}_a(f \wedge g) \Rightarrow \mathcal{M}_a(f) \wedge \mathcal{M}_a(g)$.
4. f is a-monotone iff $\mathcal{M}_a(f) = f$.
5. $f \Rightarrow \mathcal{M}_a(f)$.
6. If $f(a) = 1$ then $\mathcal{M}_a(f) \equiv 1$.
7. For a DNF $f = \bigvee_{i=1}^{k} (X_{e_{i,1}}^{c_{i,1}} \wedge \cdots \wedge X_{e_{i,\delta_i}}^{c_{i,\delta_i}})$ (here $X^0 = X$ and $X^1 = \bar{X}$) we have

$$\mathcal{M}_a(f) = \bigvee_{i=1}^{k} \bigwedge_{j:c_{i,j}=a[e_{i,j}]} X_{e_{i,j}^{c_{i,j}}}.$$

   If all $c_{i,j} \neq a[e_{i,j}]$ for some term, then $\mathcal{M}_a(f) = 1$.

proof. The proof is given in the Appendix.$\Box$
Another interesting property of the a-monotone boolean function of f is
Lemma 2 . We have

$$f \equiv \bigwedge_{a \in \{0,1\}^n} \mathcal{M}_a(f).$$

Proof. If $\bigwedge_{a \in \{0,1\}^n} \mathcal{M}_a(f)(x_0) = 1$ then for every a there exists $y \leq_a x_0$ such that $f(y) = 1$. If we choose $a = x_0$ then since $y + a \leq 0$ we must have $y = a = x_0$ and then $f(x_0) = 1$. This implies that $\bigwedge_{a \in \{0,1\}^n} \mathcal{M}_a(f) \Rightarrow f$. Since by (5) in lemma 1, $f \Rightarrow \mathcal{M}_a(f)$ we have $f \Rightarrow \bigwedge_{a \in \{0,1\}^n} \mathcal{M}_a(f)$.$\Box$

Using lemma 2 we define the *Monotone dimension of a class* of boolean functions C.

Definition 1 . Let C be a class of boolean functions. We define the monotone dimension $d = Mdim(C)$ of C to be the minimal number of assignments $a_1, \ldots, a_d$ such that for any $f \in C$ we have

$$f \equiv \bigwedge_{i=1}^{d} \mathcal{M}_{a_i}(f).$$

A set of assignments $\{a_1, \ldots, a_d\}$ that satisfies the above equivalence for all $f \in C$ is called an *M-basis* of C (d need not be minimal).

The following lemma shows a simple way to find the M-basis and Mdim of a class.

Lemma 3 . Let C be a class of boolean functions. Then $Mdim(C)$ is the minimal number of assignments $a_1, \ldots, a_d$ such that for every $f \in C$, there exist d monotone boolean functions $M_1, \ldots, M_d$ such that

$$f \equiv M_1(x + a_1) \wedge \cdots \wedge M_d(x + a_d).$$

A set $\{a_1, \ldots, a_d\}$ that satisfies the above equivalence for any $f \in C$ (with possibly different M_i) is an M-basis for C.

Proof. Let $a_1, \ldots, a_d$ be assignments that satisfy the conditions in the lemma. Let

$$g = \bigwedge_{i=1}^{d} \mathcal{M}_{a_i}(f).$$

If we show that for every $f \in C$, $g \equiv f$, then $Mdim(C) \leq d$ and $\{a_1, \ldots, a_d\}$ is an M-basis for C.

Since by (5) in lemma 1 $f(x) \Rightarrow \mathcal{M}_a(f)$ for any a we have $f(x) \Rightarrow g(x)$. Now by (3) and (4) in lemma 1,

$$\mathcal{M}_{a_i}(f) \Rightarrow \mathcal{M}_{a_i}(M_i(x+a_i)) \Rightarrow M_i(x+a_i),$$

and therefore $g(x) \Rightarrow f(x)$.$\Box$

Lemma 4. A set of assignments $A = \{a_1, \ldots, a_d\}$ is an M-basis for C if and only if every boolean function in C can be represented as CNF such that every clause in this CNF is falsified by some assignment in A.

In particular, if $f = c_1 \wedge \cdots \wedge c_s$ is any CNF of f and each clause c_i is falsified by some $a \in A$ then A is an M-basis for $\{f\}$ and

$$f = \bigwedge_{a \in A} \mathcal{M}_a(f).$$

Proof. If $A = \{a_1, \ldots, a_d\}$ is an M-basis for C then by lemma 3 every function $f \in C$ can be written as $f = M_1(x+a_1) \wedge \cdots \wedge M_d(x+a_d)$ for some monotone M_i. For $1 \le i \le d$ let c be a monotone clause in the monotone CNF representation of M_i. Since $c(x+a_i)$ is a clause in the CNF representation of $M_i(x+a_i)$ and $c(a_i+a_i) = c(0_n) = 0$ (here 0_n is the zero assignment), this clause is falsified by $a_i \in A$.

Now suppose every boolean function $f \in C$ can be represented as CNF, $f = c_1 \wedge \cdots \wedge c_s$ such that every clause c_i in this CNF is falsified by some assignment $a_{j(i)}$ in A. Then it is clear that $M_i' = c_i(x + a_{j(i)})$ is monotone and f can be written as

$$f = M_1(x+a_1) \wedge \cdots \wedge M_d(x+a_d)$$

where

$$M_r(x) = \bigwedge_{\{i | j(i) = r\}} M_i'(x)$$

is monotone. By lemma 3, A is an M-basis for C.$\Box$

For a function f, the DNF size (CNF size) of f, $size_{DNF}(f)$, $(size_{CNF}(f))$, is the minimal number of terms (clauses) over all possible DNF (CNF) formulas of f. For a decision tree T the *decision tree size* of T, $size_{DT}(T)$, is the number of leaves in the tree. The decision tree size of a boolean function f is

$$size_{DT}(f) = \min_{T \equiv f} size_{DT}(T).$$

Lemma 5 . We have

1. $size_{DNF}(\mathcal{M}(f)) \le size_{DNF}(f)$.
2. $size_{DT}(f) \ge size_{DNF}(f) + size_{CNF}(f)$.
3. $Mdim(\{f\}) \le size_{CNF}(f)$.

Proof. Part 1 follows from (7) in lemma 1. Part 2 follows from the fact that in a decision tree each leaf that is labeled with 1 corresponds to one term in the corresponding DNF representation of the tree and each leaf that is labeled with 0 corresponds to one clause in the corresponding CNF representation of the tree. Part 3 follows from lemma 4.$\Box$

The Λ-Algorithm
$\{a_1,\ldots,a_t\}$ is an M-basis for C.

1) $S_i \leftarrow \emptyset, H_i \leftarrow 0$ for $i=1,\ldots,t$.
2) EQ($\bigwedge_{i=1}^t H_i) \rightarrow v$; If the answer is "YES" then stop.
3) $I \leftarrow \{i|H_i(v)=0\}$.
4) For each $i \in I$ do
    $v_i \leftarrow v$;
    Walk from v_i toward a_i while keeping $f(v_i)=1$
    $S_i \leftarrow S_i \cup \{v_i+a_i\}$.
5) $H_i \leftarrow M_{DNF}(S_i)(x+a_i)$ for $i=1,\ldots,t$.
6) Goto 2.

Figure 1: *An algorithm that learns f from $size_{DNF}(f)Mdim(C)n^2$ membership queries and $size_{DNF}(f)Mdim(C)$ equivalence queries.*

Let a be an assignment. We define $M_{DNF}(a)=\bigwedge_{a[i]=1} X_i$ for $a \neq 0_n$ and $M_{DNF}(0_n)=1$. For a set of assignments S we define $M_{DNF}(S)=\vee_{a\in S} M_{DNF}(a)$ if S is not empty and $M_{DNF}(\emptyset)=0$.

For an assignment a and a variable X_j we define the assignment $b=a_{\neg X_j}$ where $b[i]=a[i]$ for $i\neq j$ and $b[j]=1+a[j]$.

5 Learning A Class With Known M-basis

In this section we prove that any boolean function f in a class C is learnable from $size_{DNF}(f)$ $Mdim(C)$ n^2 membership queries and $size_{DNF}(f)Mdim(C)$ equivalence queries.

5.1 The Λ-Algorithm

The algorithm is given in Figure 1. In the algorithm the command EQ($\wedge_{i=1}^t H_i$)$\rightarrow$ v asks an equivalence query on the formula $\wedge_{i=1}^t H_i$. If the answer is "YES" then the algorithm stops. Otherwise the equivalence oracle returns the counterexample v. The routine *Walk from v toward a while keeping $f(v)=1$* takes two assignments v and a. It tries to flip the bits of v that differ from the corresponding bits of a, while preserving the property that $f(v)=1$. More specifically, it executes the following loop: While there exists a variable X_j such that $v[j]\neq a[j]$ and $f(x_{\neg X_j})=1, v \leftarrow v_{\neg X_j}$. This procedure runs with at most n^2 membership queries.

5.1.1 Proof of correctness

Before we prove the correctness of the algorithm we prove the following.

Proposition A. Let f be a boolean function. If y is an assignment such that $f(y)=1$ and for any i where $y[i]=1$ we have $f(y_{\neg X_i})=0$, then for any DNF $\vee_{i=1}^{s}T_i$ of f there exists a term T_{i_0} such that

$$\mathcal{M}(T_{i_0})=M_{DNF}(y).$$

Proof. If $y=0_n$ then there exists a term $T=\bar{X}_{i_1}\cdots\bar{X}_{i_k}$ in f and $\mathcal{M}(T)=1=M_{DNF}(0_n)$. Suppose $y=(y[1],\ldots,y[n])\neq 0_n$. Since $f(y)=1$ we have $\vee_{i=1}^{s}T_i(y)=1$ and therefore there exists $T_{i_0}(y)=1$. Let $T_{i_0}=X_1\wedge X_2\cdots\wedge X_{m_1}\wedge\bar{X}_{m_1+1}\wedge\cdots\wedge\bar{X}_{m_2}$ (without loss of generality). Since $T_{i_0}(y)=1$ we have $y[1]=y[2]=\cdots=y[m_1]=1$ and $y[m_1+1]=\cdots=y[m_2]=0$. If $y[i]=1$ for some $i>m_2$ then $T_{i_0}(y_{\neg X_i})=1$ and therefore $f(y_{\neg X_i})=1$ which is a contradiction. Therefore, $y[i]=0$ for $i>m_2$ and we have

$$\mathcal{M}(T_{i_0})=X_1\wedge\cdots\wedge X_{m_1}=M_{DNF}(y).\Box$$

Now, let $f\in C$ and $f=\bigvee_{i=1}^{s}(X_{e_{i1}}^{c_{i1}}\wedge\cdots\wedge X_{e_{i\delta_i}}^{c_{i\delta_i}})$ where $s=size_{DNF}(f)$. Recall that $X^0=X$ and $X^1=\bar{X}$. Since $\{a_1,\ldots,a_t\}$ is an M-basis for C we have

$$f=\bigwedge_{i=1}^{t}\mathcal{M}_{a_i}(f). \tag{1}$$

Let $a_i=(a_i[1],\ldots,a_i[n])$. By (7) in lemma 1,

$$\mathcal{M}(f(x+a_k))=\bigvee_{i=1}^{s}\bigwedge_{j:a_k[e_{ij}]=c_{ij}}X_{e_{ij}} \tag{2}$$

has at most s terms (If all $a_k[e_{i,j}]\neq c_{i,j}$, then the i-th term is 1). Let

$$\mathcal{T}_k=\{\bigwedge_{j:a_k[e_{ij}]=c_{ij}}X_{e_{ij}}|i=1,\ldots,s\}.$$

The set $\mathcal{T}_k$ contains the terms of $\mathcal{M}(f(x+a_k))$. Notice that

$$\left(\bigwedge_{T\in\mathcal{T}_i}T\right)(x+a_i)=\mathcal{M}_{a_i}(f).$$

Let $H_{i,\delta},S_{i,\delta},(i=1,\ldots,t),v_\delta,I_\delta$ and $v_{i,\delta}$ for $i\in I_\delta$ be the variables $H_i,S_i,(i=1,\ldots,t),v,I$ and v_i for $i\in I$, respectively, of the algorithm in the δ-th iteration. We prove by induction that

1. v_δ is a positive counterexample.
2. I_δ is not empty.
3. $M_{DNF}(v_{i,\delta}+a_i)\in\mathcal{T}_i\backslash\{M_{DNF}(r)|r\in S_{i,\delta-1}\}$ for $i\in I_\delta$.
4. $\{M_{DNF}(r)|r\in S_{i,\delta}\}\subseteq\mathcal{T}_i$ for $i=1,\ldots,t$.
5. $H_{i,\delta}\Rightarrow\mathcal{M}_{a_i}(f)$ for $i=1,\ldots,t$.

We prove 1-5 for the $\delta+1$ iteration. The same proof also applies for the first iteration. Since by induction hypothesis 5, $H_{i,\delta} \Rightarrow \mathcal{M}_{a_i}(f)$, we have

$$\bigwedge_{i=1}^{t} H_{i,\delta} \Rightarrow \bigwedge_{i=1}^{t} \mathcal{M}_{a_i}(f) = f. \tag{3}$$

(Also true for $\delta = 1$). Therefore $v_{\delta+1}$ is a positive counterexample. This proves 1 for $\delta+1$.

Since $\bigwedge_{i=1}^{t} H_{i,\delta}(v_{\delta+1}) = 0$ the set $I_{\delta+1}$ is not empty. This proves 2 for $\delta+1$. Now we have for all $i \in I_{\delta+1}$

$$H_{i,\delta}(v_{\delta+1}) = 0 \tag{4}$$

and since $v_{\delta+1}$ is a positive counterexample,

$$f(v_{\delta+1}) = 1. \tag{5}$$

From (4) and (5) and by the definition of $H_{i,\delta}$ we have for all $i \in I_{\delta+1}$

$$M_{DNF}(S_{i,\delta})(v_{\delta+1} + a_i) = 0,$$

and

$$f((v_{\delta+1} + a_i) + a_i) = 1. \tag{6}$$

By step 4 in the algorithm we have for every $i \in I_{\delta+1}$

$$v_{i,\delta+1} + a_i \le v_{\delta+1} + a_i, \tag{7}$$

$f(v_{i,\delta+1}) = f((v_{i,\delta+1} + a_i) + a_i) = 1$ and for every j where $(v_{i,\delta+1} + a_i)[j] = 1$,

$$f((v_{i,\delta+1})_{\neg X_j}) = f((v_{i,\delta+1} + a_i)_{\neg X_j} + a_i) = 0. \tag{8}$$

Since $M_{DNF}(S_{i,\delta})(x)$ is monotone, (6) and (7) imply that

$$M_{DNF}(S_{i,\delta})(v_{i,\delta+1} + a_i) = 0. \tag{9}$$

By proposition A, (8) implies that there exists a term T in the DNF of $f(x+a_i)$ such that $\mathcal{M}(T) = M_{DNF}(v_{i,\delta+1} + a_i)$. Therefore,

$$M_{DNF}(v_{i,\delta+1} + a_i) \in \mathcal{T}_i. \tag{10}$$

From (9) and (10) and the induction hypotheses 4, we get 3 for $\delta+1$. Now since $S_{i,\delta+1} = S_{i,\delta} \cup \{v_{i,\delta+1} + a_i\}$ for $i \in I_{\delta+1}$ and $S_{i,\delta+1} = S_{i,\delta}$ for $i \notin I_{\delta+1}$, then by induction hypothesis 4 and (10) we get 4 for $\delta+1$.

By 4 for $\delta+1$ we have

$$\begin{aligned} H_{i,\delta+1}(x+a_i) &= \bigvee_{s \in S_{i,\delta+1}} M_{DNF}(s) \\ &\Rightarrow \bigvee_{T \in \mathcal{T}_i} T = \mathcal{M}(f(x+a_i)). \end{aligned}$$

This proves 5 for $\delta+1$. This completes the proof of 1-5.

Now, 1-5 shows that at each iteration for some $i=1,\ldots,t$ we add one new term of $\mathcal{T}_i$ to the formula H_i. Since $|\mathcal{T}_i|\le s$ after some number λ of iterations of the algorithm such that $\lambda\le st\le size_{DNF}(f)Mdim(C)$ we will have

$$H_{i,\lambda}=\left(\bigwedge_{T\in\mathcal{T}_i} T\right)(x+a_i)=\mathcal{M}_{a_i}(f),$$

and $\bigwedge_{i=1}^t H_{i,\lambda}=f$. This completes the proof of the correctness of the algorithm.$\square$

5.1.2 The Complexity of the Λ-Algorithm

It is clear that the algorithm runs with $n^2 size_{DNF}(f)Mdim(C)$ membership queries and $size_{DNF}(f)$ $Mdim(C)$ equivalence queries. The factor n^2 in the number of the membership queries is due to the fact that the procedure *Walk* might ask n^2 membership queries each iteration.

6 Learning Boolean Functions

In this section we give an algorithm that learns any boolean function in polynomial time in the number of variables, the DNF size of f and the CNF size of f with $size_{DNF}(f)size_{CNF}(f)n^2$ membership queries and $size_{DNF}(f)size_{CNF}(f)$ equivalence queries.

6.1 The CDNF-Algorithm

To prove the correctness of the algorithm, we need the following proposition.

Proposition B. Let f be a boolean function and $f=\bigwedge_{i=1}^s c_i$ be a minimal size CNF for f. Let $r<s$ and $\{a_1,\cdots,a_t\}$ be a set of assignments such that for every c_i, $i\le r<s$, there exists $a_{j(i)}$ such that $c_i(a_{j(i)})=0$. Let H_i, $i=1,\ldots,t$ be a boolean function that satisfies $H_i\Rightarrow\mathcal{M}_{a_i}(f)$. If for an assignment a we have

$$\left(\bigwedge_{i=1}^t H_i\right)(a)=1\quad\text{and}\quad f(a)=0,$$

then there exists a clause $c_i, i>r$, that satisfies $c_i(a)=0$.

Proof. Since $(\bigwedge_{i=1}^t H_i)(a)=1$ we also have $\wedge_{i=1}^t\mathcal{M}(f(x+a_i))(a+a_i)=1$. If $c_{i_0}(a)=0$ for some $i_0\le r$, then by (3) and (4) in lemma 1

$$\begin{aligned}1 &= \mathcal{M}(f(x+a_{j(i_0)}))(a+a_{j(i_0)})\\ &\Rightarrow \left(\bigwedge_{i=1}^s\mathcal{M}(c_i(x+a_{j(i_0)}))\right)(a+a_{j(i_0)})\\ &\Rightarrow \mathcal{M}(c_{i_0}(x+a_{j(i_0)}))(a+a_{j(i_0)})\\ &\Rightarrow c_{i_0}(a)=0\end{aligned}$$

The CDNF-Algorithm

1) $t \leftarrow 0$
2) EQ(1) $\rightarrow v$; If the answer is "YES" then stop.
3) $t \leftarrow t+1, H_t \leftarrow 0, S_t \leftarrow \emptyset, a_t \leftarrow v$
4) EQ($\bigwedge_{i=1}^{t} H_i$) $\rightarrow v$; If the answer is "YES" then stop.
5) $I \leftarrow \{i | H_i(v) = 0\}$.
6) If I is empty then Goto 3.
7) For each $i \in I$ do
    $v_i \leftarrow v$;
    Walk from v_i toward a_i while keeping $f(v_i) = 1$
    $S_i \leftarrow S_i \cup \{v_i + a_i\}$.
8) $H_i \leftarrow M_{DNF}(S_i)(x + a_i)$ for $i = 1, \ldots, t$.
9) Goto 4.

Figure 2: *An algorithm that learns f from $size_{DNF}(f) size_{CNF}(f) n^2$ membership queries and $size_{DNF}(f) size_{CNF}(f)$ equivalence queries.*

A contradiction. Therefore $c_i(a) = 1$ for all $i \leq r$. Since $f(a) = 0$ we must have $c_i(a) = 0$ for some $i > r$.$\Box$

Now to prove the correctness of the CDNF-algorithm notice that steps 4-9 are identical, except for step 6, to the steps of the Λ-algorithm. We have added lines 1-3 and 6 to find an M-basis for $\{f\}$. Let $H = \bigwedge c_i$ be a minimal size CNF for f. By proposition B each negative counterexample a will falsify a new clause c_i of H. After at most $size_{CNF}(f)$ negative counterexamples we will have: for every clause c_i in $H \equiv f$ there exists $a_{j(i)}$ that satisfies $c_i(a_{j(i)}) = 0$. Now by lemma 4, $\{a_i\}$ is an M-basis of $\{f\}$ and therefore the Λ-algorithm learns f.$\Box$

7 Learning Conjunctions of Classes

In this section we combine the two algorithms into one algorithm that learns a conjunction of any two classes such that the first is learnable using the Λ-algorithm and the second is learnable using the CDNF-algorithm.

The algorithm is given in Figure 3.

Notice that we can build an M-basis for $f_1 \wedge f_2$ from the union of each M-basis for f_1 and f_2. Since A is an M-basis for C_1 and $f_1 \in C_1$, A is also an M-basis for f_1. Therefore we start with the M-basis A and build a basis for f_2 from negative counterexamples.

Λ+CDNF-Algorithm
$A = \{a_1, \ldots, a_r\}$ is an M-basis for C_1.

1) $t \leftarrow r, H_i \leftarrow 0, S_i \leftarrow \emptyset, i = 1, \ldots, t$
2) EQ($\bigwedge_{i=1}^{t} H_i$) $\rightarrow v$; If the answer is "YES" then stop.
3) $I \leftarrow \{i | H_i(v) = 0\}$.
4) If I is empty then
   $t \leftarrow t + 1, H_t \leftarrow 0, S_t \leftarrow \emptyset$.
   $a_t \leftarrow v$.
   Goto 2.
5) For each $i \in I$ do
   $v_i \leftarrow v$;
   Walk from v_i toward a_i while keeping $f(v_i) = 1$
   $S_i \leftarrow S_i \cup \{v_i + a_i\}$.
6) $H_i \leftarrow M_{DNF}(S_i)(x + a_i)$ for $i = 1, \ldots, t$.
7) Goto 2.

Figure 3: *An algorithm that learns* $f = f_1 \wedge f_2 \in C_1 \wedge C_2$ *from* $size_{DNF}(f)(Mdim(C_1) + size_{CNF}(f_2))n^2$ *membership queries and* $size_{DNF}(f)(Mdim(C_1) + size_{CNF}(f_2))$ *equivalence queries.*

8 Appendix

In this appendix we prove lemma 1.

1.

$$\begin{aligned}\mathcal{M}_a(f)(x_0)=1 &\iff (\exists y \leq_a x_0) f(y)=1\\ &\iff (\exists y+a \leq x_0+a) f((y+a)+a)=1\\ &\iff (\exists z \leq x_0+a) f(z+a)=1\\ &\iff \mathcal{M}(f(x+a))(x_0+a)=1\end{aligned}$$

2.

$$\begin{aligned}\mathcal{M}_a(f \vee g)(x_0)=1 &\iff (\exists y \leq_a x_0)(f(y)=1 \vee g(y)=1)\\ &\iff (\exists y \leq_a x_0) f(y)=1 \vee (\exists y \leq_a x_0) g(y)=1\\ &\iff \mathcal{M}_a(f)(x_0)=1 \vee \mathcal{M}_a(g)(x_0)=1\\ &\iff \mathcal{M}_a(f)(x_0) \vee \mathcal{M}_a(g)(x_0)=1\end{aligned}$$

3.

$$\begin{aligned}\mathcal{M}_a(f \wedge g)(x_0)=1 &\iff (\exists y \leq_a x_0)(f(y)=1 \wedge g(y)=1)\\ &\Rightarrow (\exists y \leq_a x_0) f(y)=1 \wedge (\exists y \leq_a x_0) g(y)=1\\ &\iff \mathcal{M}_a(f)(x_0)=1 \wedge \mathcal{M}_a(g)(x_0)=1\\ &\iff \mathcal{M}_a(f)(x_0) \wedge \mathcal{M}_a(g)(x_0)=1\end{aligned}$$

4. We have f is a-monotone if and only if $f(x+a)$ is monotone. Let $g(x)=f(x+a)$ and $z=y+a$. Since g is monotone

$$\begin{aligned}\mathcal{M}_a(f)(x_0)=1 &\iff (\exists y \leq_a x_0) f(y)=1\\ &\iff (\exists z \leq x_0+a) f(z+a)=1\\ &\iff (\exists z \leq x_0+a) g(z)=1\\ &\iff g(x_0+a)=1\\ &\iff f(x_0)=1\end{aligned}$$

5. Since $x_0 \leq_a x_0$,

$$f(x_0)=1 \Rightarrow (\exists y \leq_a x_0) f(y)=1 \iff \mathcal{M}_a(f)(x_0)=1.$$

6. Since $a \leq_a x_0$ for any x_0

$$f(a)=1 \Rightarrow (\forall x_0)(\exists y \leq_a x_0) f(y)=1 \Rightarrow (\forall x_0)\mathcal{M}_a(f)(x_0)=1 \iff \mathcal{M}_a(f) \equiv 1.$$

7. First notice that the minimal (in the order $<$) assignment that satisfies $t=X_1 \wedge \cdots \wedge X_r \wedge \bar{X}_{r+1} \wedge \cdots \wedge \bar{X}_s$ is $(1_r 0_{n-r})$, where $\xi_r \in \{0,1\}$ is the assignment in $\{0,1\}^r$ such that all

of its entries are ξ. Also any assignment that satisfies t is greater than $(1_r 0_{n-r})$. Therefore $\mathcal{M}(X_1 \wedge \cdots \wedge X_r \wedge \bar{X}_{r+1} \wedge \cdots \wedge \bar{X}_s) = X_1 \wedge \cdots \wedge X_r$.

Now 7 follows immediately from 1 and 2.$\square$

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