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Insurance Claims Modulated by a Hidden Brownian Marked Point Process*

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Abstract

Aimed at better modeling insurance claims in an economic environment driven by business cycles, a new Markov-modulated Poisson process model is proposed, and an algorithm is derived to estimate the hidden Markov process by using the observed information. Our method differs from existing ones in following ways: the new hidden process relevant to the Brownian motion of continuous states space can model more efficiently the cyclic state of the economic environment; our theory is based on a variation of the law of large numbers and is easy to understand, while ensuring its convergence; the Fourier expansion-based parameter estimation algorithm is much more flexible and can be more easily implemented than other algorithms. Simulation results not only demonstrate the practicality of our model and algorithm, but also show the efficiency and robustness of the estimation algorithm.

Keywords: Insurance Risk Models; Markov-modulated Poisson Processes; Brownian Motion; Reference Probability;

1 Background and Introduction

In recent years, there has been great interest in the applications of Markov-modulated processes in risk and ruin theory. These models include a hidden Markov process whose state space represents economic environments. The change of the economic environment can be attributed to variations in macroeconomic conditions, changes in political regimes or the impact of macroeconomic news and business cycles, etc. In this paper, we shall focus on an often encountered economic environment, which is driven by business cycles, and we shall present a new scheme to estimate the stochastic intensity of the Markov-modulated Poisson process. Before introducing our new method, we first review three kinds of models for describing business cycles in the literature.

Proposed in [16] and [1], the regime-switching model attracts mostly attention in risk theory. This model builds a continuous-time Markov chain, whose states represent different states of an economy, into the insurance risk model. Many papers on risk theory rely on the Markov regime-switching model. For example, [21] studies ruin probabilities, deficit at ruin and expected ruin time. [12] computes moments of claims under this model. [15] investigates European options under the model. An EM algorithm is recently proposed in [8] to estimate the hidden Markov chain based on the observed information.

In the regime-switching model, an economic environment is represented by a continuous-time, finite-state Markov process. It seems that these processes are excellent models for structural changes in political regimes and macroeconomic conditions, but they do not model short-term fluctuations in business cycles. The *switching* of the states for the business cycles will not be used in this paper.

The real business cycle (RBC) model in [14] is another economic model which attracts much attention, although it has little application in risk theory. A good review about the RBC can be found in [4]. Recent contributions in this field include papers such as [5], [17] and [20]. Specifically, used in [17], is an RBC model consisting of representative agents and firms. The agent's income consists of wages and rents received from selling labor and renting capital to firms. The representative firm rents labor and capital from the agent and combines them by using the constant returns to scale technology.

The technology shock follows the following exogenous stochastic process

$$z_{t+1} = \epsilon_t (z_t)^\rho,$$

where $\rho \in (-1, 1)$ and the innovation ϵ_t is assumed to be independently, identically, and normally distributed with zero mean and variance σ^2.

The assumed economic environment in the RBC models follows a multi-dimensional discrete-time continuous-state Markov process. This kind of model describes a small, but rudimental, economic world. There is a simulated market with diversified economic ingredients in this kind of model. Many economic relations among those ingredients are gathered in the models. Thus, among three kinds of models reviewed, it seems that the RBC model is the most suitable one for the real-life economic environment driven by business cycles. Its properties should be preserved in future models. For example, in many cases, the stochastic systems in the RBC models are stationary processes. That is, there are stochastic periods in these models, and the phase of the periodic process at any time is a random variable with uniform distribution. The disadvantage of the RBC models is that they include many variables, but none of them is significant. It is thus difficult to choose one variable to represent economic conditions when applying it to the Markov-modulated Poisson processes. Moreover, an economic environment driven by business cycles is not a discrete-time system.

Chaos models provide purely deterministic mechanisms to represent periodic, quasi-periodic, as well as chaotic fluctuations in an economic environment. There are two methods to create quasi-periodicity in this kind of models: delay differential equations as in the Kaldor-Kalecki model[13] and non-linear dynamics. A review of models for economic environments based on non-linear dynamics is provided in [10]. As a simple and representative example, the following business chaotic cycle model is adopted in [2]:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \mu(x^2 - 1)\frac{\mathrm{d}x}{\mathrm{d}t} + x = C\sin(\omega t),$$

Here x is the output in the economical system and ω is the excitation frequency. A few authors, such as those in [11] and [19], already recognized that economic environments are stochastic and tried to introduce stochastic ingredients to chaos models, but we have not seen any related works.

With the concept of excitation frequency, the quasi-periodic fluctuations of business cycles are emphasized in chaos models, which is in contrast to the inconspicuous period in RBC models and regime-switching models. The disadvantage of chaos models is their purely deterministic mechanism, which makes it difficult to apply stochastic theory.

The significant properties of macroeconomic environments are their positive long-term growth trend and short-term random fluctuations. Different from the positive long-term growth caused by changes in political regimes and other macroeconomic factors, the influence of business cycles on macroeconomy is mainly random. This randomness, according to the above discussion, should be modeled by a bounded continuous path, which is quasi-periodic and stable in the sense of stochastic processes. Therefore, the economic environment change driven by business cycles can be represented by a continuous-time, continuous-state Markov process. As in RBC models, this process should be a stationary process. It is known from stochastic calculus that a typical Markov process with a continuous path is closely related to Brownian motion. Consequently, in order to properly and precisely describe above properties, a simple but efficient model of this kind will be proposed in this paper. Specifically, we use the cosine of a Brownian motion to represent the impact of business cycles. The cosine function is selected here since it is one of the simplest periodic functions. The defined process is a continuous-state Markov process and keeps the excitation frequency as in [2]. The Markov-modulated Poisson process for an insurance claims process is then established by following the canonical method such as in [21], [8], [16] and [1]. An inhomogeneous Poisson process is further selected to model the arrivals of claims. The stochastic intensity of the claim arrivals changes over time according to the above hidden Markov process, which is adopted to represent the impact of business cycles. Similarly to the usual assumption in risk theory, the claim numbers can be observed but the hidden Markov process is not observable in practice. We shall develop an algorithm to estimate the intensity of the claim arrivals, by utilizing the observed information about the number of claims and the excitation frequency. Our algorithm is easy to implement in practice.

The rest of the paper is organized as follows. Section 2 describes the framework, defines auxiliary stationary processes, and demonstrates that the estimation of the intensity function is equivalent to finding the Fourier coefficients for that function. In section 3, we show that the ergodic properties of auxiliary stationary processes imply that those Fourier coefficients can be expressed as some conditional expectations; we then formulate the stochastic differential equations whose solutions are exactly the conditional expectations. Following these preparations, section 4 proves that the Fourier coefficients are the fixed points of some operator, which provides a practical algorithm to estimate the coefficients. Simulation results are then presented. Section 5 concludes.

2 Framework and Auxiliary Processes

In this section, we shall define the hidden Markov process X used to describe the economic environment, and the $X-$modulated Poisson processes N used to describe the arrivals of insurance claims. For the convenience of the following discussion, several auxiliary stationary stochastic processes are first introduced.

Consider a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\mathcal{P}$ is the real-world probability measure. We suppose that $(\Omega, \mathcal{F}, P)$ is rich enough to model the randomness of the observations process and the hidden state process. A continuous-time continuous-state hidden Markov process on $(\Omega, \mathcal{F}, \mathcal{P})$, $X := \{X_t\}$, with the state space being $[-1, 1]$, is defined by

$$X_t = \cos(\omega B_t + \theta).$$

Here ω is the excitation frequency, $B := \{B_t\}$ is a standard Brownian motion, and θ, a random variable with uniform distribution on $[0, \pi)$, which is independent of B. The uniform distribution of θ ensures that X is a stationary stochastic process[1].

Let N_t denote the number of claim arrivals over time $(0, t]$. It is assumed that $N := \{N_t\}$ is a Poisson process on $(\Omega, \mathcal{F}, \mathcal{P})$, whose stochastic intensity is given by the following equation.

$$\lambda_t := \lambda(X_t). \tag{1}$$

Here $\lambda : [-1, 1] \to \Re^+$ is a continuous function. We will show that estimation of λ is equivalent to estimation of Fourier coefficients for another relevant function $\tilde{\lambda}$.

[1]Please see Appendix A for a proof.

It is known from the definition of $\{X_t\}$ that the stochastic intensity can be written as $\lambda(X_t) = \lambda(\cos(\omega B_t + \theta))$. We define stochastic processes $\tilde{B} := \{\tilde{B}_t\}$,

$$\tilde{B}_t = B_t + \frac{\theta}{\omega}.$$

Then $\lambda(X_t) = \lambda(\cos(\omega\tilde{B}_t))$. For conciseness, we consider the continuous periodic function $\tilde{\lambda} : \Re \to \Re^+$ given by

$$\tilde{\lambda}(\psi) := \lambda(\cos(\omega\psi)). \tag{2}$$

The period of $\tilde{\lambda}(\psi)$ is $2\pi/\omega$. Suppose that the Fourier series of $\tilde{\lambda}$ is

$$\sum_{k=0}^{\infty} a_k \cos(k\omega\psi) = a_0 + \sum_{k=1}^{\infty} \frac{a_k}{2} \left(\exp(\mathrm{i}k\omega\psi) + \exp(-\mathrm{i}k\omega\psi)\right), \tag{3}$$

where i is the imaginary unit and

$$\begin{aligned} a_0 &= \tfrac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} \tilde{\lambda}(\psi)\mathrm{d}\psi, \\ a_k &= \tfrac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} \tilde{\lambda}(\psi)\exp(\mathrm{i}k\omega\psi)\mathrm{d}\psi \\ &= \tfrac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} \tilde{\lambda}(\psi)\exp(-\mathrm{i}k\omega\psi)\mathrm{d}\psi, \quad k = 1, 2, \ldots. \end{aligned} \tag{4}$$

Since λ is continuous, it is then known that the Fourier series (3) of $\tilde{\lambda}$ exists and converges uniformly to the continuous periodic function $\tilde{\lambda}$ on $\Re$.

Define stochastic processes $X^{(k)} := \{X_t^{(k)}\}$ as

$$X_t^{(k)} = \exp(\mathrm{i}k\omega\tilde{B}_t), \quad k \in Z. \tag{5}$$

Then $\lambda(X_t) = \lambda(\cos(\omega\tilde{B}_t)) = \tilde{\lambda}(\tilde{B}_t)$, and we can easily establish the following Lemma.

Lemma 1. $X^{(k)}$ and $\lambda(\cdot)$ satisfy the following properties:

1) For each $k \in Z$, $X^{(k)}$ is a wide-sense stationary stochastic process.

2) $X_t^{(0)} = 1$ and $EX_t^{(k)} = 0$ for $k \neq 0$.

3) $EX_t^{(m)}X_t^{(k)} = \begin{cases} 1, & k = -m, \\ 0, & k \neq -m. \end{cases}$

For $s \neq t$, $k \neq -m$, we have $EX_s^{(m)} X_t^{(k)} = 0$.

4)

$$\lambda(X_t) = a_0 X_t^{(0)} + \sum_{m=1}^{\infty} \frac{a_m}{2} \left(X_t^{(m)} + X_t^{(-m)} \right). \tag{6}$$

5) For each $k \in Z$, the stochastic process $X^{(k)}$ is ergodic. That is,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T X_t^{(k)} \mathrm{d}t = EX_0^{(k)} = \begin{cases} 1, & k = 0, \\ 0, & k \neq 0. \end{cases}$$

Here the convergence is in mean square.

Proof. 1), 2) and 3) are obvious. (6) of 4) follows from $X_t^{(0)} \equiv 1$ and

$$\lambda(X_t) = \tilde{\lambda}(\tilde{B}_t) = a_0 + \sum_{m=1}^{\infty} \frac{a_m}{2} \left(X_t^{(m)} + X_t^{(-m)} \right).$$

As for 5). Firstly, $X_t^{(0)} = 1$, the result is obvious for $k = 0$. For $k \neq 0$, $EX_t^{(k)} = 0$, then $E\frac{1}{T}\int_0^T X_t^{(k)} = 0$. Because

$$X_t^{(k)} = \exp(\mathrm{i}k\omega B_t + \mathrm{i}k\theta),$$

the mean square of the time average is

$$\begin{aligned} \left\| \tfrac{1}{T} \textstyle\int_0^T X_t^{(k)} \right\|^2 &= \tfrac{1}{T^2} E \textstyle\int_0^T \exp(-\mathrm{i}k\omega B_s)\mathrm{d}s \int_0^T \exp(\mathrm{i}k\omega B_t)\mathrm{d}t \\ &= \tfrac{1}{T^2} \textstyle\int_0^T \int_0^T E\exp(\mathrm{i}k\omega(B_t - B_s))\mathrm{d}s\mathrm{d}t, \end{aligned}$$

where $\exp(-\mathrm{i}k\omega B_s)$ is the complex conjugate of $\exp(\mathrm{i}k\omega B_s)$.

Since $E\exp(\mathrm{i}k\omega(B_t - B_s)) = \exp(-k^2\omega^2|t-s|/2)$,

$$\begin{aligned} \left\| \tfrac{1}{T} \textstyle\int_0^T X_t^{(k)} \right\|^2 &= \tfrac{1}{T^2} \textstyle\int_0^T \left(\int_0^t \exp\left(-\frac{k^2\omega^2(t-s)}{2}\right) \mathrm{d}s + \int_t^T \exp\left(-\frac{k^2\omega^2(s-t)}{2}\right) \mathrm{d}s \right) \mathrm{d}t \\ &= \tfrac{2}{T^2 k^2 \omega^2} \textstyle\int_0^T \left(1 - \exp\left(-\frac{k^2\omega^2(t-s)}{2}\right) + 1 - \exp\left(-\frac{k^2\omega^2(t-s)}{2}\right) \right) \mathrm{d}t. \end{aligned}$$

Therefore, as $T \to 0$,

$$\left\| \frac{1}{T} \int_0^T X_t^{(k)} \right\|^2 \leq \frac{4}{Tk^2\omega^2} \to 0. \tag{7}$$

This is the required result. $\square$

Lastly, to make it convenient to establish our main results, we consider the right-continuous, complete versions of the filtrations

$$\begin{aligned}
\mathcal{F}^X &:= \{\mathcal{F}_t^X\}, && \mathcal{F}_t^X := \sigma\{\theta, B_u | u \in [0,t]\} \ , \\
\mathcal{F}^N &:= \{\mathcal{F}_t^N\}, && \mathcal{F}_t^N := \sigma\{N_u | u \in [0,t]\} \ , \\
\mathcal{G} &:= \{\mathcal{G}_t\}, && \mathcal{G}_t := \mathcal{F}_t^X \vee \mathcal{F}_t^N \ .
\end{aligned}$$

Here, $\mathcal{F}^N$ is observable while $\mathcal{F}^X$ is unobservable[8][9] . Suppose the Doob-Meyer decomposition (see [7]) for N is

$$N_t = \int_0^t \lambda(X_u)\mathrm{d}u + V_t,$$

where $V := \{V_t\}$ is a $(\mathcal{P}, \mathcal{G})$-martingale and $V_0 = 0$.

Jump processes and continuous martingales are always orthogonal, but B_t and N_t are not independent under the real-world probability measure $\mathcal{P}$. This makes it difficult for us to calculate many relevant expectations. To overcome this problem, we consider the reference probability measure $\mathcal{P}^\dagger$ under which N is a Poisson process with unit intensity and is independent of X. In this situation,

$$Q_t := N_t - t \tag{8}$$

is a martingale under $\mathcal{P}^\dagger$. Define the process $\Lambda := \{\Lambda_t\}$ as the solution to the following stochastic differential equation.

$$\Lambda_t = 1 + \int_0^t \Lambda_{u-} \left(\lambda(X_u) - 1\right) \mathrm{d}Q_u. \tag{9}$$

Similarly to [8], we define $\mathcal{P}$ by setting

$$\Lambda_t = \left.\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}\mathcal{P}^\dagger}\right|_{\mathcal{G}_t} . \tag{10}$$

Then under $\mathcal{P}$,

$$V_t = N_t - \int_0^t \lambda(X_u)\mathrm{d}u$$

is a martingale.

3 Main Results

Two theorems will be established in this section. The first theorem shows that the ergodic properties of $X^{(k)}$, $k \in Z$, imply that the Fourier coefficients of $\tilde{\lambda}(\cdot)$ can be represented by stochastic integrations of $X^{(k)}$. It provides a statistical method for calculating the Fourier coefficients in (4). The second theorem formulates stochastic differential equations for those stochastic integrations of $X^{(k)}$. With the latter theorem, all relevant stochastic integrations can be computed by utilizing the Fourier coefficients in (4).

Theorem 1. We have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T E\left[X_u^{(k)} \middle| \mathcal{F}_T^N \right] \mathrm{d}N_u = \begin{cases} a_0, & k = 0, \\ a_k/2, & k \neq 0. \end{cases} \tag{11}$$

Here the convergence is in mean square.

Proof. Let the real part of $X_t^{(k)}$ be R_t, which is a bounded stochastic process. According to the definition of $X_t^{(k)}$, both $X_t^{(k)}$ and its real part R_t have continuous paths. Write $R_M := \sup\{R_t; t \geq 0\} \leq 1$ and $\lambda_M := \sup\{\lambda(x); x \in [-1, 1]\}$. For any given $T \geq 0$, we define a random variable D_T,

$$D_T := \int_0^T R_t \mathrm{d}V_t = \int_0^T R_t \mathrm{d}N_t - \int_0^T R_t \lambda(X_t) \mathrm{d}t. \tag{12}$$

We have $D_0 = 0$ and $\{D_T : T \geq 0\}$ is a right-continuous and left-limit martingale. Then

$$\begin{aligned} (D_T)^2 &= 2\int_0^T D_{t_-} \mathrm{d}D_t + \sum_{0<t\leq T} (\Delta D_t)^2 \\ &= 2\int_0^T D_{t_-} \mathrm{d}D_t + \sum_{0<t\leq T} R_t^2 \Delta N_t \\ &= 2\int_0^T D_{t_-} \mathrm{d}D_t + \int_0^T R_t^2 \mathrm{d}N_t \\ &= 2\int_0^T D_{t_-} R_t \mathrm{d}V_t + \int_0^T R_t^2 \mathrm{d}V_t + \int_0^T R_t^2 \lambda(X_t) \mathrm{d}t. \end{aligned} \tag{13}$$

The first two terms of the above expression are martingales and their expectations are 0. Then $E(D_T)^2 = E\int_0^T R_t^2\lambda(X_t)\mathrm{d}t$ and hence,

$$\begin{aligned}
&\left\| \tfrac{1}{T}\left(\int_0^T R_t \mathrm{d}N_t - \int_0^T R_t\lambda(X_t)\mathrm{d}t\right)\right\|^2 \\
&= E(D_T)^2/T^2 = E\int_0^T R_t^2\lambda(X_t)\mathrm{d}t/T^2 \\
&\le ER_M^2\lambda_M/T \to 0.
\end{aligned} \tag{14}$$

The same conclusion can be established for the imaginary part of $X_t^{(k)}$. Therefore,

$$\lim_{T\to\infty}\frac{1}{T}\left(\int_0^T X_t^{(k)}\mathrm{d}N_t - \int_0^T X_t^{(k)}\lambda(X_t)\mathrm{d}t\right) = 0 \tag{15}$$

in the L^2 sense.

It follows by (5) that $X_t^{(k)}X_t^{(m)} = X_t^{(k+m)}$. Then we have from 4) of Lemma 1 that

$$X_t^{(k)}\lambda(X_t) = a_0X_t^{(k)} + \sum_{m=1}^{\infty}\frac{a_m}{2}\left(X_t^{(k+m)} + X_t^{(k-m)}\right). \tag{16}$$

It follow from (16) and 3) of Lemma 1 that

$$\begin{aligned}
&\left\|\frac{1}{T}\int_0^T X_t^{(k)}\lambda(X_t)\mathrm{d}t - \frac{1}{T}\int_0^T\left(a_0X_t^{(k)} + \sum_{m=1}^{n}\frac{a_m}{2}\left(X_t^{(k+m)} + X_t^{(k-m)}\right)\right)\mathrm{d}t\right\|^2 \\
&\le \sum_{|m|>n}\sum_{|l|>n}\left|\frac{1}{T^2}\int_0^T\int_0^T\frac{a_m a_l}{2}EX_t^{(k+m)}X_s^{(k+l)}\mathrm{d}t\mathrm{d}s\right| \\
&= \sum_{|m|>n}\frac{|a_m|^2}{4}\left\|\frac{1}{T}\int_0^T X_t^{(k+m)}\mathrm{d}t\right\|^2.
\end{aligned} \tag{17}$$

As $X_t^{(k+m)}(t\ge 0, m\in Z)$ is uniformly bounded, that is, there is a constant C such that $\left|X_t^{(k+m)}\right| \le C$, $m\in Z$, we have $\left\|\frac{1}{T}\int_0^T X_t^{(k+m)}\mathrm{d}t\right\|^2 \le C^2$. Moreover, the Parseval's equality implies $\lim\limits_{n\to\infty}\sum\limits_{|m|>n}|a_m|^2 \to 0$. These facts and (17) imply

$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{T}\int_0^T X_t^{(k)}\lambda(X_t)\mathrm{d}t \\
=\ &\lim_{T\to\infty}\frac{1}{T}\int_0^T a_0X_t^{(k)}\mathrm{d}t + \sum_{m=1}^{\infty}\frac{a_m}{2}\lim_{T\to\infty}\frac{1}{T}\int_0^T\left(X_t^{(k+m)} + X_t^{(k-m)}\right)\mathrm{d}t
\end{aligned} \tag{18}$$

in the L^2 sense. It follows from (18) and 5) of Lemma 1 that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T X_t^{(k)}\lambda(X_t)\mathrm{d}t = \begin{cases} a_0, & k=0, \\ a_k/2, & k\neq 0. \end{cases} \tag{19}$$

This and (15) mean that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T X_t^{(k)}\mathrm{d}N_t = \begin{cases} a_0, & k=0, \\ a_k/2, & k\neq 0. \end{cases} \tag{20}$$

The limits in (19) and (20) are constants, hence for each $S\in(0,\infty)$, we have

$$E\left[\lim_{T\to\infty}\frac{1}{T}\int_0^T X_t^{(k)}\mathrm{d}N_t\bigg|\,\mathcal{F}_S^N\right] = \lim_{T\to\infty}\frac{1}{T}\int_0^T X_t^{(k)}\mathrm{d}N_t. \tag{21}$$

Because $\left\|E\left[Z|\,\mathcal{F}_S^N\right]\right\| \leq \|Z\|$ for any square integrable random variable Z, we have

$$\begin{aligned} &\lim_{S\to\infty}\left\|E\left[\frac{1}{S}\int_0^S X_t^{(k)}\mathrm{d}N_t - \lim_{T\to\infty}\frac{1}{T}\int_0^T X_t^{(k)}\mathrm{d}N_t\bigg|\,\mathcal{F}_S^N\right]\right\| \\ \leq\ &\lim_{S\to\infty}\left\|\frac{1}{S}\int_0^S X_t^{(k)}\mathrm{d}N_t - \lim_{T\to\infty}\frac{1}{T}\int_0^T X_t^{(k)}\mathrm{d}N_t\right\| = 0. \end{aligned}$$

This and (21) imply that

$$\begin{aligned} &\lim_{T\to\infty}\frac{1}{T}\int_0^T X_t^{(k)}\mathrm{d}N_t \\ =\ &\lim_{S\to\infty} E\left[\frac{1}{S}\int_0^S X_t^{(k)}\mathrm{d}N_t\bigg|\,\mathcal{F}_S^N\right] \\ =\ &\lim_{S\to\infty}\frac{1}{S}\int_0^S E\left[X_t^{(k)}\Big|\,\mathcal{F}_S^N\right]\mathrm{d}N_t. \end{aligned}$$

The desired result (11) follows by the above equation and (20). □

In Theorem 1, the process $E[X_t^{(k)}|\mathcal{F}_T^N]$ is a function of the observable process $\{N_u : 0\leq u\leq T\}$. To derive an explicit form of this function, we need to rewrite $E[X_t^{(k)}|\mathcal{F}_T^N]$ in terms of the reference probability $\mathcal{P}^\dagger$. By the Bayes' rule[6, 8], we have

$$E[U|\mathcal{F}_t^N] = \frac{E^\dagger[\Lambda_t U|\mathcal{F}_t^N]}{E^\dagger[\Lambda_t|\mathcal{F}_t^N]} \tag{22}$$

for a random variable $U \in \mathcal{G}_t$. Let

$$f_t^k := E^\dagger[\Lambda_t X_t^{(k)} | \mathcal{F}_t^N], \tag{23}$$

$$g_{s,t}^{(k,l)} := E^\dagger[X_s^{(k)} \Lambda_t X_t^{(l)} | \mathcal{F}_t^N], \tag{24}$$

here $k, l \in Z$ and $0 \le s \le t$. We obtain that $f_t^0 = E^\dagger[\Lambda_t | \mathcal{F}_t^N]$, $f_t^k = f_t^{-k}$ and $g_{s,s}^{(k,l)} = f_s^{k+l}$. The following theorem provides a convenient way[2] to compute f_t^k, $g_{s,t}^{(k,l)}$ and $E[X_t^{(k)} | \mathcal{F}_T^N]$.

Theorem 2.

1) $f_t^k, k \in Z$, satisfies the following stochastic integral equation,

$$\begin{aligned} f_t^k &= f_0^k - \int_0^t \frac{\omega^2 k^2}{2} f_u^k \mathrm{d}u \\ &\quad - \int_0^t \left((1 - a_0) f_{u-}^k - \sum_{m=1}^{\infty} \frac{a_m}{2} \left(f_{u-}^{k+m} + f_{u-}^{k-m} \right) \right) \mathrm{d}Q_u, \end{aligned} \tag{25}$$

with the initial condition being

$$f_0^0 = 1, \quad f_0^k = 0, \ k \neq 0.$$

2) $g_{s,t}^{(k,l)}$ satisfies the following stochastic integral equation,

$$\begin{aligned} g_{s,t}^{(k,l)} &= g_{s,s}^{(k,l)} - \int_s^t \frac{\omega^2 l^2}{2} g_{s,u}^{(k,l)} \mathrm{d}u \\ &\quad - \int_s^t \left((1 - a_0) g_{s,u-}^{(k,l)} - \sum_{m=1}^{\infty} \frac{a_m}{2} \left(g_{s,u-}^{(k,l+m)} + g_{s,u-}^{(k,l-m)} \right) \right) \mathrm{d}Q_u, \end{aligned} \tag{26}$$

with the initial condition being $g_{s,s}^{(k,l)} = f_s^{k+l}$.

3) $E[X_t^{(k)} | \mathcal{F}_T^N]$ can be represented as

$$E[X_t^{(k)} | \mathcal{F}_T^N] = \frac{E^\dagger[X_t^{(k)} \Lambda_T | \mathcal{F}_T^N]}{f_T^0} = \frac{g_{t,T}^{(k,0)}}{f_T^0}.$$

Here $E^\dagger[X_t^{(k)} \Lambda_T | \mathcal{F}_T^N]$ is given by the following stochastic integral,

$$E^\dagger[X_t^{(k)} \Lambda_T | \mathcal{F}_T^N] = f_t^k - \int_t^T \left((1 - a_0) g_{t,u-}^{(k,0)} - \sum_{m=1}^{\infty} \frac{a_m}{2} \left(g_{t,u-}^{(k,m)} + g_{t,u-}^{(k,-m)} \right) \right) \mathrm{d}Q_u.$$

[2] Please see Appendix B for another quick way to compute $\frac{1}{T} \int_0^T E \left[X_u^{(k)} \middle| \mathcal{F}_T^N \right] \mathrm{d}N_u$, but the corresponding algorithm is numerical unstable.

Proof. 1) The initial condition follows by the 2) of Lemma 1. By applying the Itô formula to $X_t^{(k)} = \exp(\mathrm{i}k\omega\tilde{B}_t)$, we have

$$\mathrm{d}X_t^{(k)} = \mathrm{i}\omega k X_t^{(k)}\mathrm{d}\tilde{B}_t - \frac{\omega^2 k^2}{2}X_t^{(k)}\mathrm{d}t. \tag{27}$$

From this and (9), by applying the product rule to $\Lambda_t X_t^{(k)}$, we obtain

$$\Lambda_t X_t^{(k)} - \Lambda_0 X_0^{(k)} = \int_0^t \Lambda_u \mathrm{d}X_u^{(k)} + \int_0^t X_{u_-}^{(k)}\mathrm{d}\Lambda_u$$
$$= \int_0^t \mathrm{i}\omega k\Lambda_u X_u^{(k)}\mathrm{d}\tilde{B}_u - \int_0^t \frac{\omega^2 k^2}{2}\Lambda_u X_u^{(k)}\mathrm{d}u + \int_0^t \Lambda_{u_-}\left(\lambda(X_u) - 1\right)X_{u_-}^{(k)}\mathrm{d}Q_u.$$

Because $\lambda(\cdot)$ is continuous and $X_t^{(k)}$, $k \in Z$, have continuous paths, we have $\lambda(X_u)X_{u_-}^{(k)} = \lambda(X_u)X_u^{(k)}$. Similarly to (16), we have

$$\Lambda_t X_t^{(k)} = \Lambda_0 X_0^{(k)} + \int_0^t \mathrm{i}\omega k\Lambda_u X_u^{(k)}\mathrm{d}\tilde{B}_u - \int_0^t \frac{\omega^2 k^2}{2}\Lambda_u X_u^{(k)}\mathrm{d}u$$
$$- \int_0^t (1-a_0)\Lambda_{u_-}X_{u_-}^{(k)}\mathrm{d}Q_u + \int_0^t \sum_{m=1}^{\infty}\frac{a_m}{2}\left(\Lambda_{u-}X_{u_-}^{(k+m)} + \Lambda_{u_-}X_{u_-}^{(k-m)}\right)\mathrm{d}Q_u.$$

This completes the proof by taking the $\mathcal{F}^N$-optional projection under $\mathcal{P}^\dagger$ on both sides of the above equation.[3]

2) Similarly to 1), for $t \geq s$, we have

$$\Lambda_t X_t^{(l)} = \Lambda_s X_s^{(l)} + \int_s^t \mathrm{i}\omega l\Lambda_u X_u^{(l)}\mathrm{d}\tilde{B}_u - \int_s^t \frac{\omega^2 l^2}{2}\Lambda_u X_u^{(l)}\mathrm{d}u$$
$$- \int_s^t (1-a_0)\Lambda_{u_-}X_{u_-}^{(l)}\mathrm{d}Q_u + \int_s^t \sum_{m=1}^{\infty}\frac{a_m}{2}\left(\Lambda_{u-}X_{u_-}^{(l+m)} + \Lambda_{u_-}X_{u_-}^{(l-m)}\right)\mathrm{d}Q_u.$$

Multiplying the above equation by $X_s^{(k)}$ gives

$$X_s^{(k)}\Lambda_t X_t^{(l)} = \Lambda_s X_s^{(k+l)} + \int_s^t \mathrm{i}\omega l X_s^{(k)}\Lambda_u X_u^{(l)}\mathrm{d}\tilde{B}_u - \int_s^t \frac{\omega^2 l^2}{2}X_s^{(k)}\Lambda_u X_u^{(l)}\mathrm{d}u$$
$$- \int_s^t X_s^{(k)}\Lambda_{u_-}\left((1-a_0)X_{u_-}^{(l)} - \sum_{m=1}^{\infty}\frac{a_m}{2}\left(X_{u_-}^{(l+m)} + X_{u_-}^{(l-m)}\right)\right)\mathrm{d}Q_u.$$

[3] Under $\mathcal{P}^\dagger$, $\tilde{B}$ and N are independent. Then $\Lambda_u X_u^{(k)}\mathrm{d}\tilde{B}_u$ vanishes with respect to the $\mathcal{F}^N$-optional projection. As $\mathcal{G}_u$ and $\{N_s - N_u : s \geq u\}$ are independent under $\mathcal{P}^\dagger$, we have $E[\Lambda_u X_u^{(k)}|\mathcal{F}_t^N] = E[\Lambda_u X_u^{(k)}|\mathcal{F}_u^N]$, and so on.

The required result follows by taking the $\mathcal{F}^N$-optional projection under $\mathcal{P}^\dagger$ of both sides of the above equation.

3) The required result can be easily established from the definition of f_t^k, $g_{s,t}^{(k,l)}$, 1), 2) and the Bayes' formula (22). $\square$

Remark 1. The corresponding stochastic differential equation formulation of (25) is

$$\mathrm{d}f_t^k = -\frac{\omega^2 k^2}{2} f_t^k \mathrm{d}t - \left((1-a_0) f_{t-}^k - \sum_{m=1}^{\infty} \frac{a_m}{2} \left(f_{t-}^{k+m} + f_{t-}^{k-m} \right) \right) \mathrm{d}Q_t,\ k \in Z,$$

with the initial condition being

$$f_0^0 = 1, \quad f_0^k = 0,\ k \neq 0.$$

Remark 2. To ensure the numerical stability of our algorithm, we define

$$\tilde{g}_{s,t}^{(k,l)} := g_{s,t}^{(k,l)} + g_{s,t}^{(k,-l)}. \tag{28}$$

Then

$$\begin{aligned} \tilde{g}_{s,t}^{(k,l)} &= \tilde{g}_{s,s}^{(k,l)} - \int_s^t \frac{\omega^2 l^2}{2} \tilde{g}_{s,u}^{(k,l)} \mathrm{d}u \\ &\quad - \int_s^t \left((1-a_0) \tilde{g}_{s,u-}^{(k,l)} - \sum_{m=1}^{\infty} \frac{a_m}{2} \left(\tilde{g}_{s,u-}^{(k,l+m)} + \tilde{g}_{s,u-}^{(k,l-m)} \right) \right) \mathrm{d}Q_u. \end{aligned} \tag{29}$$

The corresponding stochastic differential equation formulation of (29) is

$$\mathrm{d}\tilde{g}_{s,t}^{(k,l)} = -\frac{\omega^2 l^2}{2} \tilde{g}_{s,t}^{(k,l)} \mathrm{d}t - \left((1-a_0) \tilde{g}_{s,t-}^{(k,l)} - \sum_{m=1}^{\infty} \frac{a_m}{2} \left(\tilde{g}_{s,t-}^{(k,l+m)} + \tilde{g}_{s,t-}^{(k,l-m)} \right) \right) \mathrm{d}Q_t,$$

with the initial condition being $\tilde{g}_{s,s}^{(k,l)} = f_s^{k+l} + f_s^{k-l}$. Hence we have

$$E^\dagger[\Lambda_T X_t^{(k)} | \mathcal{F}_T^N] = \frac{\tilde{g}_{t,T}^{(k,0)}}{2}. \tag{30}$$

4 Estimation Algorithm and Simulation Results

In many papers about nonlinear stochastic models, see, for instance, [3],[18], it is usually necessary to estimate parameters basing on unobserved data, but the estimation of parameters and the estimation of unobserved data are dependent on each other. In the literature, EM algorithms are often adopted to solve this dilemma by using separate E-steps and M-steps. In the situation discussed above, the calculations of the stochastic integrations in Theorem 2, and the Fourier coefficients in Theorem 1, are also dependent on each other. Instead of using EM algorithms, in this section, we shall solve this dilemma using a fixed point algorithm.

Given any initial guess γ for a Fourier coefficient, a better estimate γ^* can be derived by using Theorems 1 and 2 and the sample path of N_t. Then γ^* can be generically regarded as a function of γ:

$$\mathcal{T}(\gamma) = \gamma^*.$$

Theorems 1 and 2 indicate that the true values of Fourier coefficients correspond to a fixed point vector of the operator $\mathcal{T}$. By utilizing a suitable fixed point algorithm in functional analysis, we shall design a numerical algorithm for computing the Fourier coefficients a_i in (4). Furthermore, as an illustration to the application of our scheme in risk theory, the feasibility and efficiency of the proposed algorithm will be demonstrated by a numerical simulation at the end of this section.

As in any numerical algorithm, we assume, without loss of generality, the number of nonzero coefficients a_i in (4) is finite. That is, for some sufficiently large number L,

$$a_i = 0, \quad |i| > L.$$

Here L can be determined according to the approximation accuracy requirement. Since it is impossible for us to establish the sample path of the Poisson process N_t over the whole time region $(0, +\infty)$, we assume that there exists a large enough number T, and that $\{0 < t_1 \leq \ldots \leq t_M \leq T\}$ represents a sample path[4] of the Poisson process N_t over $(0, T]$. That is,

$$N_t = m, \quad t_m \leq t < t_{m+1}, \quad m = 0, \ldots, M,$$

where $t_0 = 0$ and $t_{M+1} = T$.

The following iteration method provides a way to calculate a_i for $i = 0, 1, \ldots, L$.

Algorithm 1.

Step 1. Given the accuracy parameter ϵ and an initial value

$$\gamma_0 = (\gamma_{-L}^{(0)}, \ldots, \gamma_0^{(0)}, \ldots, \gamma_L^{(0)})^T \in \Re^{2L+1},$$

where $\gamma_{-i}^{(0)} = \gamma_i^{(0)}$. Let $n = 0$.

Step 2. Let $\gamma_i^{(n)} = 0$ for $|i| > L$. Define $(2L+1) \times (2L+1)$ symmetric matrices $B = (b_{i,j})\,, C = (c_{i,j})$ through

$$b_{i,j} = \begin{cases} 1 - \gamma_0^{(n)} - \frac{1}{2}\omega^2 i^2, & i = j \\ -\frac{1}{2}\gamma_{i-j}^{(n)}, & i \neq j \end{cases}$$

$$c_{i,j} = \begin{cases} \gamma_0^{(n)}, & i = j \\ \frac{1}{2}\gamma_{i-j}^{(n)}, & i \neq j \end{cases}$$

respectively, where $i, j = -L, \ldots, 0, \ldots, L$.

Step 3. Compute the eigenvalues $\mu_1, \ldots, \mu_{2L+1}$ and corresponding unitary eigenvectors $q_1, \ldots, q_{2L+1}$ of B. Define $H = (q_1, \ldots, q_{2L+1})$, from which we

[4]For the intensity $\lambda(X_t)$, the traditional construction of the sample path is very slow. Please see Appendix C for our new construction.

get

$$B = H\,\mathrm{diag}(\mu_1, \ldots, \mu_{2L+1})\,H^T.$$

Here H^T is the transpose of H.

Step 4. Let $f(0) = (f_{-L}(0), \ldots, f_0(0), \ldots, f_L(0))^T \in \Re^{2L+1}$, with $f_0(0) = 1$ and $f_i(0) = 0$ for $i \neq 0$. Let $m = 1$.

Step 5. Set

$$f(t_m - 0) = H\,\mathrm{diag}(\,\exp(\mu_1\delta_m), \ldots, \exp(\mu_{2L+1}\delta_m))\,H^T\,f(t_{m-1}),$$

and $f(t_m) = C\,f(t_m - 0)$, where $\delta_m = t_m - t_{m-1}$.

Step 6. For $k = -L, \ldots, L$, let

$$\tilde{g}^{(k)}(t_m, t_m) = \left(\tilde{g}^{(k,-L)}(t_m, t_m), \ldots, \tilde{g}^{(k,L)}(t_m, t_m)\right)^T \in \Re^{2L+1},$$

with $\tilde{g}^{(k,i)}(t_m, t_m) = f_{k+i}(t_m) + f_{k-i}(t_m)$.

Step 7. For $l = 1, \ldots, m-1$ and $k = -L, \ldots, L$, set

$$\tilde{g}^{(k)}(t_l, t_m - 0) = H\,\mathrm{diag}(\,\exp(\mu_1\delta_m), \ldots, \exp(\mu_{2L+1}\delta_m))\,H^T\,\tilde{g}^{(k)}(t_l, t_{m-1}),$$

and $\tilde{g}^{(k)}(t_l, t_m) = C\,\tilde{g}^{(k)}(t_l, t_m - 0)$.

Step 8. If $m <= M$, let $m \leftarrow m + 1$ and go to Step 5. Otherwise, we have $\gamma_0^{(n+1)} = \frac{M}{T}$ and

$$\gamma_i^{(n+1)} = \frac{1}{T}\sum_{m=1}^{M} \frac{\tilde{g}^{(i,0)}(t_m, t_{M+1} - 0)}{f_0(t_{M+1} - 0)}.$$

Here $\gamma_{n+1} = (\gamma_{-L}^{(n+1)}, \ldots, \gamma_0^{(n+1)}, \ldots, \gamma_L^{(n+1)})^T \in \Re^{2L+1}$.

Step 9. If $\|\gamma_{n+1} - \gamma_n\| \geq \epsilon$, then let $n \leftarrow n + 1$ and go to Step 2. Otherwise, stop and output γ_{n+1} as an estimation to $(a_{-L}, \ldots, a_0, \ldots, a_L)^T$.

The following Theorem ensures the correctness of Algorithm 1.

Theorem 3. Suppose that $\mathcal{T}$ is contractive almost surely and T tends to $+\infty$. Then for any initial γ_0, $\gamma_i^{(n)}$ converges almost surely to the Fourier coefficient a_i, $\forall i \in Z$, when n tends to $+\infty$.

Proof. Because $\mathcal{T}$ is a contraction operator almost surely, for any initial γ_0, as n tends to infinity, $\gamma^{(n)}$ converges almost surely to the unique fixed point of $\mathcal{T}$. The proof completes if we can show that $a_i, i \in Z$, are fixed points of $\mathcal{T}$. That is, if $\gamma_i^{(n)} = a_i, i \in Z$, then $\gamma_i^{(n+1)} = a_i$ almost surely.

As a matter of fact, in this case, $f_i(t)$ is equal to f_t^i defined in (23).

For $t \in (t_{m-1}, t_m)$, it follows from Theorem 2 that $f(t)$ satisfies the following ordinary differential equation,

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t) = B\, f(t),$$

with the initial condition being $f(t)|_{t=t_{m-1}} = f(t_{m-1})$. Consequently, we have

$$f(t) = H\,\mathrm{diag}(\exp(\mu_1(t-t_j)), \ldots, \exp(\mu_{2L+1}(t-t_j)))\, H^T\, f(t_{m-1}),$$

for $t \in [t_{m-1}, t_m)$. So

$$f(t_m - 0) = H\,\mathrm{diag}(\exp(\mu_1 \delta_{j+1}), \ldots, \exp(\mu_{2L+1}\delta_{j+1}))\, H^T\, f(t_{m-1}).$$

Since t_m is a jump time of N_t, it follows from Theorem 2 and the integral rule for the Poisson process that

$$f(t_m) = C\, f(t_m - 0).$$

Similarly, we have that Steps 6 and 7 determine the solution of (29). Therefore, for $i \neq 0$, it follows from (22), (29) and (30) that

$$\gamma_i^{(n+1)} = \frac{2}{T} \sum_{m=1}^{M} \frac{\tilde{g}^{(i,0)}(t_m, t_{M+1} - 0)}{f_0(t_{M+1} - 0)} = \frac{2}{T} \int_0^T Z_t^{(i)} \mathrm{d}N_t.$$

This, and Theorem 1, ensure that when T tends to $+\infty$, the difference between a_i and $\gamma_i^{(n+1)}$ tends to 0. As for $i = 0$, it is obvious that

$$\gamma_0^{(n+1)} = N_T/T = \int_0^T Z_t^{(0)} \mathrm{d}N_t/T.$$

And hence when $T \to +\infty$, $\gamma_0^{(n+1)}$ tends to a_0. $\square$

Theorem 3 tells us that Algorithm 1 is globally convergent as long as $\mathcal{T}$ is a contraction operator. If the initial point is near to the true Fourier coefficients or the fixed point of $\mathcal{T}$, $\mathcal{T}$ would very often be contractive. In the following, the feasibility and efficiency of Algorithm 1 will be illustrated through two simulated numerical examples.

Example 1. We set $\omega = 1$, $a_0 = 2$, $a_1 = 1$, that is, $\lambda(x) = 2 - x \geq 0, x \in [-1, 1]$. To show the influence of different initial values and the simulating time T on the final output of Algorithm 1, we consider several combinations of the initial vector and the simulation time length. The results are shown in Table 1.

initial a_0	initial a_1	T	Max iteration	final a_0	final a_1
0	1	20	30	1.700000	0.796579
0	1	100	30	1.908750	0.909401
0	1	400	30	1.990000	0.945139
0	1	1000	30	1.981000	0.987998

Table 1. Influence of initial values and the simulating time.

Example 2. As an illustration to the efficiency of Algorithm 1, we examine in this example the accuracy of Algorithm 1 under different a_0s and a_1s. Here we set $T = 400$, the max iteration is 30. Table 2 presents the obtained results.

true a_0	true a_1	final a_0	final a_1
10	3	9.662500	3.596574
3	3	3.250000	2.819720
3	-2	2.795000	-2.086164

Table 2. Efficiency of Algorithm 1 with respect to different a_0s and a_1s.

It is easy to see from above examples that Algorithm 1 is not only practical, but also efficient and robust. It can get accurate results for different initial values, and the simulation time T does not need to be very long. Therefore, our numerical algorithm improves existing methods for similar problems, such as those in [8] and [9]. Moreover, Algorithm 1 is simple, stable and flexible when compared with current algorithms. For example, because we transform the original problem to a fixed point problem, and we just adopt the classical scheme in Algorithm 1 to find the fixed point, the efficiency of our method will be surely improved if we use another, better technique for finding the fixed point.

5 Conclusion

When observing macroeconomic environments, one can often observe continuous random fluctuations, occasionally broken by big jumps. The large jumps are caused by the impact of macroeconomic news, changes in political regimes, and so on. The continuous random fluctuations are mainly driven by business cycles. In the fields of risk and ruin theory, Markov chains are accepted as a suitable model for describing the jumps in macroeconomic environments. The cosine of a Brownian motion is used to represent continuous random fluctuations in this paper. Then an inhomogeneous Poisson process, which is adopted to model the arrivals of insurance claims in the literature, is extended to describe the macroeconomic environment driven by business cycles. The new feature is that we use the cosine of a Brownian motion as the intensity function of the Poisson process. Based on the observed information of the number of claims, the problem for estimating unobservable parameters of Brownian motions is solved in this paper.

The estimation of unobservable parameters, which in fact is the intensity function, is equivalent to finding the Fourier coefficients of the intensity function. Following this observation, we establish two theorems. Theorem 1 shows that the Fourier coefficients can be represented by some stochastic integration with the integrand being the hidden unobservable Markov process; Theorem 2 reformulates the Fourier coefficients as the solution of stochastic integrals described by the observable information. Unlike previous methods, a fixed-point type algorithm is then designed to compute these Fourier coefficients, instead of directly solving the stochastic integration equation. Two simulation examples show that our numerical algorithm is practical, efficient and robust.

As for further improvement of our model, there are at least two problems that can be investigated in the future. Firstly, the excitation frequency is a given constant in this paper; it might be better to estimate it together with other parameters. Secondly, it is worthwhile to examine the impact of truncation errors of the Fourier coefficients on the final numerical accuracy.

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Appendix A.

Proposition A.1. The uniform distribution of θ implies that X is a stationary stochastic process.

Proof. In fact, X_t satisfies the following stochastic differential equation,

$$\mathrm{d}X_t = -\omega \sin(\omega B_t + \theta)\mathrm{d}B_t - \frac{\omega^2}{2} X_t \mathrm{d}t.$$

Because $(-\omega \sin(\omega B_t + \theta))^2 = \omega^2(1 - X_t^2)$, X_t is a time-homogeneous diffusion process with the generator

$$(\mathcal{A}f)(x) = \frac{\omega^2(1-x^2)}{2} f''(x) - \frac{\omega^2 x}{2} f'(x).$$

Using the adjoint operator of $\mathcal{A}$, we can write the Fokker-Plank equation, which describes the probability density $p(t,x)$ of random variable X_t, as

$$\frac{\partial}{\partial t} p(t,x) = \frac{\omega^2}{2} \frac{\partial^2((1-x^2)p(t,x))}{\partial x^2} + \frac{\omega^2}{2} \frac{\partial(xp(t,x))}{\partial x}.$$

Clearly, $p(t,x) = \frac{1}{\pi\sqrt{1-x^2}}$ is a sloution to the above equation.

On the other hand, the distribution of θ being a uniform distribution on $[0, \pi)$ means that the probability density function X_0 is

$$f(x) = \frac{1}{\pi\sqrt{1-x^2}}, \quad x \in (-1, 1).$$

Hence, The uniform distribution of θ implies that X is in its steady-state distribution.

Moreover, X is a time-homogeneous Markov process, then X is a stationary stochastic process. □

Appendix B.

A quick way to compute $\frac{1}{T}\int_0^T E\left[\left. X_u^{(k)} \right| \mathcal{F}_T^N\right] \mathrm{d}N_u$.

The following method should be quicker than Algorithm 1, but its corresponding alogrithm is numerical unstable.

Let

$$Y_t^{(k)} = \frac{1}{t}\int_0^t X_u^{(k)} \mathrm{d}N_u,$$

and

$$h_t^{(k,l)} = E^{\dagger}\left[\left. \Lambda_t Y_t^{(k)} X_t^{(l)} \right| \mathcal{F}_t^N\right].$$

Then

$$E\left[\left. Y_t^{(k)} X_t^{(l)} \right| \mathcal{F}_t^N\right] = \frac{E^{\dagger}\left[\left. \Lambda_t Y_t^{(k)} X_t^{(l)} \right| \mathcal{F}_t^N\right]}{E^{\dagger}\left[\left. \Lambda_t \right| \mathcal{F}_t^N\right]} = \frac{h_t^{(k,l)}}{E^{\dagger}\left[\left. \Lambda_t \right| \mathcal{F}_t^N\right]}.$$

We wish to find $h_T^{(k,0)}$.

Proposition B.1. For $0 < s \le t$,

$$\begin{aligned} h_t^{(k,l)} &= h_s^{(k,l)} + \int_s^t \left(-\left(\frac{\omega^2 l^2}{2} + \frac{1}{u} \right) h_u^{(k,l)} + \frac{1}{u}(2 - \lambda(X_u))\Lambda_u X_u^{(k+l)} \right) \mathrm{d}u \\ &\quad + \int_s^t \left(-\frac{\lambda(X_{u-0})}{u} \Lambda_{u-0} X_{u-0}^{(k+l)} + \Lambda_{u-0}(1 - \lambda(X_{u-0})) Y_{u-0}^{(k)} X_{u-0}^{(l)} \right) \mathrm{d}Q_u \end{aligned}$$

Proof. For $0 < s \le t$, we have

$$Y_t^{(k)} - Y_s^{(k)} = -\int_s^t \frac{Y_u^{(k)}}{u} \mathrm{d}u + \int_s^t \frac{X_u^{(k)}}{u} \mathrm{d}N_u.$$

Similarly to the proof of Theorem 2, (9), (27) and the above equation imply the following equations.

$$Y_t^{(k)}\Lambda_t - Y_s^{(k)}\Lambda_s = \int_s^t Y_{u-}^{(k)} \mathrm{d}\Lambda_u + \int_s^t \Lambda_{u-} \mathrm{d}Y_u^{(k)} - \int_s^t \frac{X_{u-}^{(k)}}{u} \Lambda_{u-}(1 - \lambda(X_{u-})) \mathrm{d}N_u.$$

Then

$$Y_t^{(k)}\Lambda_t X_t^{(l)} - Y_s^{(k)}\Lambda_s X_s^{(l)} = \int_s^t X_u^{(l)} \mathrm{d}\left(Y_u^{(k)}\Lambda_u \right) + \int_s^t Y_u^{(k)}\Lambda_u \mathrm{d}X_u^{(l)}.$$

Therefore

$$
\begin{aligned}
Y_t^{(k)}\Lambda_t X_t^{(l)} \quad = \quad & Y_s^{(k)}\Lambda_s X_s^{(l)} + \mathrm{i}\omega l \int_s^t Y_u^{(k)}\Lambda_u X_u^{(l)} \mathrm{d}B_u \\
& + \int_s^t \left(-\left(\frac{\omega^2 l^2}{2} + \frac{1}{u} \right) + (1 - \lambda(X_u)) \right) Y_u^{(k)}\Lambda_u X_u^{(l)} \mathrm{d}u \\
& + \int_s^t \left(\frac{1}{u}\lambda(X_{u-}) X_{u-}^{(k+l)} - Y_{u-}^{(k)} (1 - \lambda(X_{u-})) X_{u-}^{(l)} \right) \Lambda_{u-} \mathrm{d}N_u .
\end{aligned}
$$

This completes the proof by taking the $\mathcal{F}^N$-optional projection on both sides of the above equation. □

By using the above proposition and the Fourier series (3) of $\lambda(X_t)$, we can obtain a fast scheme to compute $\frac{1}{T}\int_0^T E\left[X_u^{(k)} \middle| \mathcal{F}_T^N \right] \mathrm{d}N_u$.

Appendix C.

Construction of the sample path for the Poisson process with the intensity $\lambda(X_t)$.

Proposition C.1. Suppose Y is a random variable distributed uniformly on $(0, 1]$, Y is independent of the Brownian motion B. Let T be the random variable satisfying

$$\int_{\alpha}^{T+\alpha} \lambda(X_t)dt = -\ln(Y).$$

For any stopping time α, the next jumping time τ of Poisson process N_t is defined as

$$\tau = \inf\{t : N_t \neq N_\alpha, t > \alpha\}.$$

Then the distributions of T and τ are the same.

Proof. For any fixed $t \geq 0$ and given the sample path of X_t, the conditional density of τ is

$$f_\tau(t) = \lambda(X_{\alpha+t}) \exp\left(\int_{\alpha}^{\alpha+t} \lambda(X_s)\mathrm{d}s\right).$$

On the other hand, the density of $-\ln(Y)$ is

$$f(t) = I_{t\geq 0} \exp(-t).$$

Hence, we have under the given sample path of X_t,

$$\begin{aligned} P(T \leq t) &= P(\textstyle\int_{\alpha}^{t+\alpha} \lambda(X_t)dt > -\ln(Y)) \\ &= \textstyle\int_0^t \lambda(X_{\alpha+u}) \exp\left(\int_{\alpha}^{\alpha+u} \lambda(X_s)\mathrm{d}s\right) \mathrm{d}u, \end{aligned}$$

which implies that the density of T is the same of τ. □

To solve the stochastic equation

$$\int_{a}^{a+T} g(B_t)dt = C \tag{31}$$

in term of T, a fast numerical integral method is needed. For this purpose, we use a variant of the method introduced in *Platen E, Wagner W. On a Taylor formula for a class of Itô process.* **Prob Math Stat** *3, 1982, 37-51.*

The detailed algorithm for solving (31) can be described as follows.

Algorithm C.1.

Step 1. Given the accuracy parameter ϵ. Let $s = 0$, $m = \max_{x \in \Re} g(x)$, $b_s = B_a$, $I_s = 0$, $T_s = 0$ and $h = \epsilon^{2/3}$.

Step 2. If $I_s + mh \geq C$, goto step 5.

Step 3. Generate $W_s \sim N(0, h)$ and $U_s \sim N(0, h^3/12)$. Let $Y_s = U_s + W_s h/2$ and

$$I_{s+1} = I_s + f(b_s)h + f'(b_s)Y_s + f''(b_s)h^2/4, \quad b_{s+1} = b_s + W_s, \quad T_{s+1} = T_s + h.$$

Step 4. Let $s \leftarrow s + 1$. Go to step 2.

Step 5. If $I_s \geq C$, output T_s and $B_{a+T} = b_s$. Stop.

Step 6. Generate $W_s \sim N(0, \epsilon)$. Let

$$I_{s+1} = I_s + f(b_s)\epsilon, \quad b_{s+1} = b_s + W_s, \quad T_{s+1} = T_s + \epsilon.$$

Step 7. Let $s \leftarrow s + 1$. Go to step 5.